CSCI046 Notes: Runtime Analysis

1 Counting

Our goal is to count how many times something will happen in our code. This is often used as a proxy for how long it takes a program to run.

Example 1.

```
1  print('x')
2  print('x')
3  print('x')
4
5  for i in range(10):
6      print('y')
7
8  for i in range(10,20):
9      print('z')
10     print('z')
11     print('z')
```

1. What is the exact number of times that the letter `x` will be printed?

2. What is the exact number of times that the letter `y` will be printed?

3. What is the exact number of times that the letter `z` will be printed?

Example 2. Answer the questions below based on the following python code:

```
1  for i in range(10):
2      print('x')
3  for i in range(20):
4      print('x')
5
6  for i in range(10):
7      for j in range(20):
8          for k in range(30):
9              print('y')
10
11 print('z')
12 for i in range(10):
13     print('z')
14     for j in range(10):
15         print('z')
16         print('z')
17     for j in range(10):
18         print('z')
19 for i in range(10):
20     print('z')
```

1. What is the exact number of times that the letter `x` will be printed?

2. What is the exact number of times that the letter `y` will be printed?

3. What is the exact number of times that the letter `z` will be printed?

Example 3. Answer the questions below based on the following python code:

```
1  for i in range(n):
2      print('x')
3  for i in range(n*2):
4      print('x')
5
6  for i in range(n):
7      for j in range(n*2):
8          for k in range(n*3):
9              print('y')
10
11 print('z')
12 for i in range(n):
13     print('z')
14     for j in range(n):
15         print('z')
16         print('z')
17     for j in range(n):
18         print('z')
19 for i in range(n):
20     print('z')
```

1. What is the exact number of times that the letter `x` will be printed?

2. What is the exact number of times that the letter `y` will be printed?

3. What is the exact number of times that the letter `z` will be printed?

Example 4. Answer the questions below based on the following python code:

```
for i in range(n):
    for j in range(0,i):
        for k in range(0,j):
            print('x')

for i in range(n):
    for j in range(i,n):
        for k in range(j,n):
            print('y')

for i in range(n):
    for j in range(i,n):
        for k in range(i,j):
            print('z')
```

1. What is the exact number of times that the letter `x` will be printed?

2. What is the exact number of times that the letter `y` will be printed?

3. What is the exact number of times that the letter `z` will be printed?

2 Math: Big-O/Θ/Ω Notation

Key Ideas:

Definition 1. Let f, g be functions from $\mathbb{R}^+ \to \mathbb{R}^+$. Then,

1. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$, then we say $f = O(g)$.
2. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} > 0$, then we say $f = \Omega(g)$.
3. We say that $f = \Theta(g)$ if both $f = O(g)$ and $f = \Omega(g)$.

Intuitively, you should think of O as $\leq$, Ω as $\geq$, and Θ as $=$.

Example 5.

1. $f(x) = x$
   $g(x) = x^2$

2. $f(x) = x^2$
   $g(x) = x$

3. $f(x) = x^2 + 2x + 5$
   $g(x) = x$

4. $f(x) = x^2 + 2x + 5$
   $g(x) = x^2$

5. $f(x) = x^2 + 2x + 5$
   $g(x) = x^3$

Example 6. What happens when we double the size of the inputs? $f(x) = x$

Example 7. You should memorize the relationship between the following functions:

$$1 \qquad \log n \qquad n \qquad n \log n \qquad n^2 \qquad n^3 \qquad 2^n$$

Example 8. Complete each equation below by adding the symbol O if $f = O(g)$, Ω if $f = \Omega(g)$, or Θ if $f = \Theta(g)$. The first row is completed for you as an example.

f(n)		g(n)
1	$=$	$O(n)$
$3n \log n$	$=$	n^2
1	$=$	$1/n$
$\log_2 n$	$=$	$\log_3 n$
$\log n$	$=$	$\frac{1}{\log n}$
$5 \cdot 10^{30}$	$=$	$\log n$
$\log n$	$=$	$\log(n^2)$
2^n	$=$	3^n
$\frac{1}{n}$	$=$	$\sqrt{\frac{1}{n}}$
$\log n$	$=$	$(\log n)^2$

Example 9. Simplify the following expressions:

1. $O\Big(n^3 + n^2\Big)$

2. $O\Big(n^3 + 5n^2 \log n + \log n\Big)$

3. $O\Big(100000000000\Big)$

4. $O\Big(\log n + 100000000000\Big)$

5. $O\Big(\frac{1}{n} + \frac{1}{n^2}\Big)$