NORTH CAROLINA STATE UNIVERSITY

Homework 1 for Chapter 2

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PROBLEM 1

PROBLEM 2 PAGE 119 PART A Let $g(t)$ be the approximation to f,

$$\|f-g\|_\infty = \sup_{1\le i\le N} |f(t)-g(t)|$$
$$g(t) = c = \frac{y+1}{2}$$

Therefore, the L_∞ approximation for $f(x)$ is $g(t) = \frac{y+1}{2}$, with error $\|f-g\|_\infty = \frac{y-1}{2}$.

PROBLEM 2 PAGE 119 PART B Let $g(t)$ be the approximation to f,

$$\|f-g\|_2 = (\sum_{i=1}^{N} |f(t_i)-g(t_i)|^2)^{\frac{1}{2}}$$
$$= \sqrt{(N-1)(c-1)^2+(c-y)^2} = \sqrt{Nc^2-2(N-1+y)c+(N-1)+y^2}$$

Solve this the function $h(c) = Nc^2-2(N-1+y)c+(N-1)+y^2$ for the minimum value and we get $c = \frac{N-1+y}{N}$.

Therefore, the L_2 approximation for $f(x)$ is $g(t) = \frac{N-1+y}{N}$, with error $\|f-g\|_2 = y^2(1-\frac{1}{N})-y(\frac{2N-2}{N})+\frac{N-1}{N}$

PROBLEM 2 PAGE 119 PART C As $N \to \infty$, the constant in the least square approximation goes to 1. It shows the least square approximation weights less on the outliers than the infinity approximation. Request more input.

PROBLEM 5 PAGE 119 PART A Define the following $\hat{f}(x) = 1 + cx$ and $f(x) = e^x$.

$$\begin{aligned}
\|\hat{f} - f\|_2^2 &= \int_0^1 |e^x - 1 - cx|^2 dx = \int_0^1 e^{2x} - 2e^x(1+cx) + (1-cx)^2 dx \\
&= \int_0^1 e^{2x} - (2e^x + 2ce^x x) + (1 - 2cx + c^2x^2) dx \\
&= \frac{1}{3}c^2 - c + \frac{e^2}{2} - 2e + \frac{5}{2}
\end{aligned}$$

Minimize the function $h(c) = \frac{1}{3}c^2 - c + \frac{e^2}{2} - 2e + \frac{5}{2}$ and the minimum reaches at $c = \frac{3}{2}$.

PROBLEM 5 PAGE 119 PART B Solve for the general case, $\max_{0\le x\le 1} |e^x - (1+cx)|$. Let $f_c(x) = e^x - (1+cx)$ and,

$$f_c'(x) = e^x - c$$

$c = 1$. $f_1(x)$ is monotonic increasing in the interval $[0,1]$ and the minimum is $f_1(0) = 0$. Then $|e_1(x)| = f_1(x)$. The maximum is at $x = 1$ and the max error is $e - 2$. For the case $c = \frac{3}{2}$, $f_{\frac{3}{2}}(x)$ is decreasing at $[0, ln\frac{3}{2}]$ and increasing at $[ln\frac{3}{2}, 1]$. The minimum is $f_{\frac{3}{2}}(ln\frac{3}{2}) < 0$. Therefore $\max e_2(x) = |f_{\frac{3}{2}}(x)|$ is either reached at the endpoint, $\{0,1\}$, or at $x = \ln\frac{3}{2}$. Plug in and the maximum of $e_2(x)$ at the interval $[0,1]$ is $e - 2.5$ at the endpoint $x = 1$.

PROBLEM 5 PAGE 119 PART C Define the following minimization problem,

$$\min_{0\le x\le 1} \|e^x - (1 + c_1x + c_2x^2)\|_2^2$$

$$\begin{aligned}
g(c_1, c_2) &= \int_0^1 (e^x - (1 + c_1x + c_2x^2))^2 dx \\
&= \frac{1}{3}c_1^2 + \frac{1}{2}(c_2 - 2)c_1 + \frac{1}{5}c_2^2 + \left(\frac{14}{3} - 2e\right)c_2 + \frac{1}{2}\left(5 - 4e + e^2\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g}{\partial c_1} &= \frac{1}{2}(c_2 - 2) + \frac{2}{3}c_1 = 0 \\
\frac{\partial g}{\partial c_2} &= \frac{1}{2}c_1 + \frac{2}{5}c_2 + \frac{14}{3} - 2e = 0
\end{aligned}$$

Solve this system of equations and get $c_1 = 164 - 60e = 0.9031$ and $c_2 = 80e - \frac{650}{3} = 0.7959$.

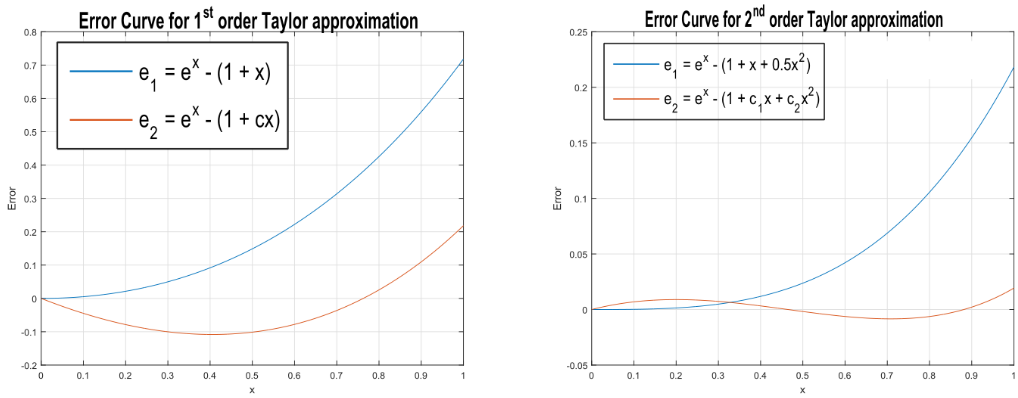

Figure 0.1: Error curves of $e_1(x)$ and $e_2(x)$

PROBLEM 33 PAGE 126 We use two different method to solve this problem.

1. Lagrange Interpolation

$$i = 0,\ l_0(x) = \frac{x - x_1}{x_0 - x_1}\frac{x - x_2}{x_0 - x_2} = \frac{(x-11)(x-12)}{2}$$
$$i = 1,\ l_0(x) = \frac{x - x_0}{x_1 - x_0}\frac{x - x_2}{x_1 - x_2} = -(x-10)(x-12)$$
$$i = 2,\ l_0(x) = \frac{x - x_0}{x_2 - x_0}\frac{x - x_1}{x_2 - x_1} = \frac{(x-10)(x-11)}{2}$$
$$p(x) = \ln(10) * l_0(x) + \ln(11) * l_1(x) + \ln(12) * l_2(x)$$
$$p(11.1) = 2.406969856623995$$

   The relative error is 1.028×10^{-5}

2. Newton's Interpolation
   To solve the problem in Newton's form, first let's creating the divided difference table using the given points.

x_i	$f(x)$	$f[x_i, x_{i-1}]$	$f[x_i, x_{i-1}, x_{i-2}]$
10	ln(10)		
11	ln(11)	ln(11/10)	
12	ln(12)	ln(12/11)	0.5 ln(121/120)

   Then, the interpolation polynomial can be written as

$$p(x) = \ln(10) + (x-10)\ln(11/10) + 0.5(x-10)(x-11)\ln(121/120).$$

   therefore, $p(11.1) = 2.407882724933612$, the relative error for the interpolation is

$$\epsilon_{rel} = \frac{|p(11) - \ln(11.1)|}{\ln(11.1)} = 3.895 \times 10^{-4}$$

PROBLEM 36 PAGE 126 PART A We know that,

$$\begin{aligned}|E(x)| =& |e^x - p_n(f;x)| = |\frac{e^{\xi(x)}}{(n+1)!}\prod_{i=0}^{n}(x-x_i)| \\ =& \frac{e^{\xi(x)}}{(n+1)!}\prod_{i=0}^{n}|(x-\frac{i}{n})| \\ =& \frac{e^{\xi(x)}}{(n+1)!}\prod_{i=0}^{n}\sqrt{|(x-\frac{i}{n})(x-\frac{n-i}{n})|}\end{aligned}$$

We then show the hint is true by a simple maximization problem, $\forall i = 0 \cdots n$

$$\max_{0\le x\le 1}|(x-\frac{i}{n})(x-\frac{n-i}{n})|$$

Define $f(x) = (x-\frac{i}{n})(x-\frac{n-i}{n})$ and the minimum point is achieved at $x=\frac{1}{2}$. That is, $|f(x)|$ achieves maximum of $max = |(\frac{1}{2}-\frac{i}{n})(\frac{1}{2}-\frac{n-i}{n})| \le |(\frac{1}{2}-0)(\frac{1}{2}-1)| = \frac{1}{4}$ at $x=\frac{1}{2}$.

$$\max_{0\le x\le 1}|E(x)| \le \frac{e^1}{(n+1)!}\frac{1}{2^n}$$

Solve the following inequality,

$$\frac{e^1}{(n+1)!}\frac{1}{2^n} \le 10^{-6}$$

The smallest n is 7.

PROBLEM 36 PAGE 126 PART B For Taylor polynomial, the error can be written as:

$$|E(x)| = |e^x - p_n(f;x)| = |\frac{e^{\xi(x)}}{(n+1)!}x^{n+1}|$$

On the domain $[0,1]$, $\max_{0\le x\le 1}(|E(x)|) = \frac{e}{(n+1)!}$. So that only when $n \ge 10$, $\max_{0\le x\le 1}(|E(x)|) \le 1\times 10^{-6}$ According to this calculation result, we can see interpolation is a better approximation method than Taylor polynomial for the given problem. Thanks to the separation of the gird point, the interpolation approximation converge to the true function faster than Taylor series with a factor of 2^{-n}

PROBLEM 46 PAGE 128 We know that,

$$T_n(\cos\theta) = \cos(n\theta)$$

Let's $x = \cos\theta$, then we have

$$\frac{dT_n(x)}{d\theta} = \frac{dT_n(x)}{dx}\frac{dx}{d\theta}.$$

So that,

$$\begin{aligned}\frac{dT_n(x)}{dx} &= \frac{dT_n(x)}{d\theta}\Big/\frac{dx}{d\theta}\\ &= n\frac{\sin n\theta}{\sin\theta}\\ &= n\frac{\sin\left(n\cos^{-1}x\right)}{\sqrt{1-x^2}}\end{aligned}$$

So that

$$\begin{aligned}\frac{dT_n(0)}{dx} &= n\sin\left(\frac{n\pi}{2}\right)\\ &= \begin{cases}0 & \text{n is even,}\\ (-1)^{(n-1)/2} & \text{n is odd.}\end{cases}\end{aligned}$$

Programming Assignment

Problem 9 Page 137