DSHW#10

$\boxed{Q1}$ G = Google matrix

(a) Prove the product of stochastic matrices is stochastic.

Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n\times n}$ be two stochastic matrices where

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} b_{ij} = 1$$

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = (a_{i1}b_{1j} + \cdots + a_{in}b_{nj})$$

For each row, compute sum of entries:

$$\sum_{j=1}^{n} (AB)_{ij} = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{ik}b_{kj} \right) = a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{in}b_{n1}$$
$$+ a_{i1}b_{12} + a_{i2}b_{22} + \cdots + a_{in}b_{n2}$$
$$+ \cdots$$
$$+ a_{i1}b_{1n} + a_{i2}b_{2n} + \cdots + a_{in}b_{nn}$$

$$= a_{i1}(b_{11} + b_{12} + \cdots + b_{1n}) + a_{i2}(b_{21} + b_{22} + \cdots + b_{2n})$$
$$+ \cdots + a_{in}(b_{n1} + b_{n2} + \cdots + b_{nn})$$

$$= \sum_{j=1}^{n} a_{ij} = 1$$

Therefore, (AB) is a stochastic matrix.

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(b) The inverse A^{-1} of a stochastic matrix is still stochastic

if the matrix is doubly stochastic and orthogonal.

$$\sum_{i=1}^{n} a_{ij} = 1 = \sum_{j=1}^{n} a_{ij}$$

$$AA^T = A^TA = I$$

i.e. $A^T = A^{-1}$

proof: A is stochastic

Since doubly stochastic, A^T is also stochastic.

Since orthogonal, i.e. $A^{-1} = A^T$, A^{-1} is stochastic. #.

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(C) $G = \alpha S + (1-\alpha)\frac{1}{n}ee' = \alpha S + (1-\alpha)E$ or $G = \alpha S + (1-\alpha)ev'$

- <u>Thm</u>: The largest eigenvalue of a stochastic matrix is 1.

<u>proof</u>. Let G be a stochastic matrix. (row sum = 1).

then $G\mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ is column vector whose entries are all 1's.

(By contradiction),

Suppose there's a $\lambda > 1$ and nonzero x s.t $Gx = \lambda x$.

Since the rows of G are non-negative and sum to 1,

each element of column vector (Gx) is a convex combination of components of $x \leq \max(x_1, \dots, x_n)$.

This implies each element of $(\lambda x) \leq \max(x_1, \dots, x_n)$. ①

However, $\lambda > 1$ makes $(\lambda x)_i > x_i \ \forall i$, which means at least there exist one component $(\lambda x)_i$ that is $> x_{max}$ for some i. ②

① and ② are contradictive.

We conclude that eigenvalues of G are all less or equal to 1.

$\Rightarrow \lambda_i \leq 1$.

What's more, since $G\mathbf{1} = \lambda \mathbf{1}$ $(x = \mathbf{1}) \Rightarrow \lambda = 1$

$\lambda = 1$ is one eigenvalue of G

$\therefore$ $\boxed{\text{The largest eigenvalue of } G \text{ is } \lambda_1 = 1.}$

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- <u>Thm</u> If the spectrum of the stochastic matrix S is $\{1, \lambda_2, \lambda_3, \dots, \lambda_n\}$, then the spectrum of the Google matrix $G = \alpha S + (1-\alpha) e v^T$ is $\{1, \alpha\lambda_2, \alpha\lambda_3, \dots, \alpha\lambda_n\}$.

<u>Proof</u>: Since S is stochastic, $(1, e)$ is an eigenpair of S.

Let $Q = (e\ X)$ be a non-singular matrix that has the eigenvector e as its 1st colm.

Let $Q^{-1} = \begin{pmatrix} y^T \\ Y^T \end{pmatrix}$, hence $Q^{-1}Q = \begin{pmatrix} y^T e & y^T X \\ Y^T e & Y^T X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}$

which gives us two Identities:

① $y^T e = 1$

② $Y^T e = 0$

Hence, appling similarity transformation to S:

$$Q^{-1} S Q = \begin{pmatrix} y^T e & y^T S X \\ Y^T e & Y^T S X \end{pmatrix} = \begin{pmatrix} 1 & y^T S X \\ 0 & \underline{\underline{Y^T S X}} \end{pmatrix}$$

Containing all the other eigenvalues of S: $\{\lambda_2, \dots, \lambda_n\}$

Hence, applying similarity transformation to G:

$$Q^{-1} G Q = Q^{-1}(\alpha S + (1-\alpha) e v^T) Q = Q^{-1} \alpha S Q + Q^{-1}(1-\alpha) e v^T Q$$

$$= \begin{pmatrix} \alpha & \alpha y^T S X \\ 0 & \alpha Y^T S X \end{pmatrix} + (1-\alpha) \begin{pmatrix} y^T e \\ Y^T e \end{pmatrix} (v^T e \ \ v^T X)$$

$$= \begin{pmatrix} 1 & \alpha y^T S X + (1-\alpha) v^T X \\ 0 & \underline{\underline{\alpha Y^T S X}} \end{pmatrix}$$

Containing all the other eigenvalues $\{\alpha\lambda_2, \dots, \alpha\lambda_n\}$.

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Therefore, spectrum of $G = \{1, \alpha\lambda_2, \cdots, \alpha\lambda_n\}$ where $\{\lambda_2, \ldots, \lambda_n\}$ are eigenvalues of S.

Since the structure of Web makes it highly likely that $|\lambda_2| = 1$ or $|\lambda_2| \approx 1$,

$\boxed{\text{the second largest eigenvalue of } G \approx \alpha}$

(d) Show G is stochastic, irreducible, aperiodic and primitive.

~~Thm: sum of two stochastic matrices, is still stochastic.~~

~~proof: Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ be two stochastic matrices~~

~~where $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} b_{ij} = 1$~~

~~$(A+B)_{ij} = a_{ij} + b_{ij} \rightarrow$ row sum: $\sum_{j=1}^{n}(a_{ij}+b_{ij})$~~

~~$= \sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{n} b_{ij} = 2.$~~

$G = \alpha S + (1-\alpha)\frac{1}{n}ee'$ $\qquad S = H + \frac{1}{n}ae'$ $\qquad a_i = \begin{cases} 1 & \text{if row } i \text{ is zero row} \\ 0 & \text{o/w.} \end{cases}$

replacing zero row by $[\frac{1}{n}, \ldots, \frac{1}{n}]$ of H

row sum $= \begin{bmatrix} \alpha\sum_{j=1}^{n} s_{1j} \\ \vdots \\ \alpha\sum_{j=1}^{n} s_{nj} \end{bmatrix} + \begin{bmatrix} (1-\alpha) \\ \vdots \\ (1-\alpha) \end{bmatrix}$

$\frac{1}{n}ee' = \frac{1}{n}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}[1 \cdots 1] = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \ddots & \vdots \\ \frac{1}{n} & \cdots & \cdots & \frac{1}{n} \end{bmatrix}_{n\times n}.$

since S is stochastic, $\sum_{j=1}^{n} s_{ij} = 1 \; \forall i$.

row sum $= \begin{bmatrix} \alpha \\ \vdots \\ \alpha \end{bmatrix} + \begin{bmatrix} 1-\alpha \\ \vdots \\ 1-\alpha \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \rightarrow \boxed{G \text{ is stochastic}}.$

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Now, we want to show G is primitive, ie. $G^m > 0$ for some $m > 0$.

$$G = \alpha S + (1-\alpha)\frac{1}{n}ee'$$

Let $m = 1$, each entry of $G > 0$?

For matrix S, it replaces all the zero rows of H by $[\frac{1}{n}, \cdots, \frac{1}{n}]$. Therefore, $S_{ij} > 0$ $\forall i, j$

and S_{ij}, what's more, $\geq \frac{1}{n}$ by the structure of Web connection.

For $\frac{1}{n}ee' = E$ matrix, $e_{ij} = \frac{1}{n}$

$$\Rightarrow \quad g_{ij} = \alpha S_{ij} + (1-\alpha) e_{ij} \geq \alpha \frac{1}{n} + (1-\alpha)\frac{1}{n} = \frac{1}{n} > 0 \quad \#.$$

Therefore, $\boxed{G \text{ is primitive}}$.

Since primitive matrices are same as irreducible, aperiodic non-negative matrices, (Perron-Frobenius).

$\boxed{G \text{ is also irreducible, and aperiodic}}$

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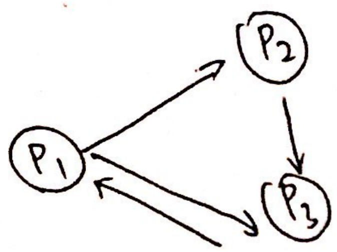

$$H = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = S \quad \text{(no zero rows).}$$

$$G = \alpha S + (1-\alpha)\frac{1}{n}ee' \quad (n=3) \quad (\alpha = 0.1)$$

$$= \begin{bmatrix} 0 & 0.05 & 0.05 \\ 0 & 0 & 0.1 \\ 0.1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.3 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\frac{1}{3} - \frac{1}{3}\alpha + \frac{1}{2}\alpha$$

$$\frac{3}{6} - \frac{2}{6}$$

$$\boxed{= \begin{bmatrix} \frac{1}{3} - \frac{1}{3}\alpha & \frac{1}{3} + \frac{1}{6}\alpha & \frac{1}{3} + \frac{1}{6}\alpha \\ \frac{1}{3} - \frac{1}{3}\alpha & \frac{1}{3} - \frac{1}{3}\alpha & \frac{1}{3} + \frac{2}{3}\alpha \\ \frac{1}{3} + \frac{2}{3}\alpha & \frac{1}{3} - \frac{1}{5}\alpha & \frac{1}{3} - \frac{1}{2}\alpha \end{bmatrix}}$$

$$= \begin{bmatrix} 0.3 & 0.35 & 0.35 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

$$\pi_{k+1}' = \pi_k' G \quad \text{where} \quad \pi = \begin{bmatrix} r(P_1) \\ r(P_2) \\ r(P_3) \end{bmatrix}$$

$$\pi_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \quad \pi_1' = \pi_0' G = [\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}] \begin{bmatrix} 0.3 & 0.35 & 0.35 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.3 \end{bmatrix} = \text{See ipynb/html.}$$

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