1)

Section A) $\min_{a,b} \; Var(S) = Var(aX+bY)$

$= cov(aX+bY, aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X, Y)$

$$\min_{a} \; Var(S) = a^2 \sigma_X^2 + (1-a)^2 \sigma_Y^2 + 2a(1-a)\rho\sigma_X\sigma_Y$$

F.O.D

$f'(a)=0$

$$\frac{d(Var(S))}{da} = 2\sigma_X^2 a + \sigma_Y^2 \cdot 2(1-a)\cdot(-1) + \rho\sigma_X\sigma_Y(2-4a) = 0$$

$$2\sigma_X^2 a + (a-1)2\sigma_Y^2 + 2\rho\sigma_X\sigma_Y - 4\rho\sigma_X\sigma_Y a = 0$$

$$a(2\sigma_X^2 + 2\sigma_Y^2 - 4\rho\sigma_X\sigma_Y) = 2\sigma_Y^2 - 2\rho\sigma_X\sigma_Y$$

$$\boxed{a^* = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}}$$

$$b^* = 1 - a^* = \frac{\sigma_X^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}$$

S.O.D

$f''(a) > 0$ check

$$\frac{d^2(Var(S))}{da^2} = 2\sigma_X^2 + 2\sigma_Y^2 - 4\rho\sigma_X\sigma_Y$$

$$= 2(\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y) > 0$$

$$= 2[\underbrace{(\sigma_X - \sigma_Y)^2}_{\geq 0} + \underbrace{2\sigma_X\sigma_Y(1-\rho)}_{\geq 0}]$$

$\because \rho \leq 1$, $1-\rho \geq 0$

For $0 \leq a \leq 1$ & $0 \leq b \leq 1$, it follows :

$$\rho \leq \min\left[\frac{\sigma_X}{\sigma_Y}, \frac{\sigma_Y}{\sigma_X}\right]$$