Type Theory via Exact Categories
Extended abstract

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Abstract

Partial equivalence relations (and categories of these) are a standard tool in semantics of type theories and programming languages, since they often provide a cartesian closed category with extended definability. Using the theory of exact categories, we give a category-theoretic explanation of why the construction of a category of partial equivalence relations often produces a cartesian closed category. We show how several familiar examples of categories of partial equivalence relations fit into the general framework.

1 Introduction

Partial equivalence relations (and categories of these) are a standard tool in semantics of programming languages, see *e.g.* [2, 5, 7, 9, 15, 17, 20, 22, 35] and [6, 29] for extensive surveys. They are usefully applied to give proofs of correctness and adequacy since they often provide a cartesian closed category with additional properties.

Take for instance a partial equivalence relation on the set of natural numbers: a binary relation $R \subseteq \mathbb{N}\times\mathbb{N}$ on the numbers which is only assumed to be symmetric and transitive (hence the name). Taking these as objects we then define the arrows: a map from R to R' is an equivalence class of partial recursive functions $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ which are defined on the domain of R and preserve the relation, in the sense that $n\ R\ n'$ always implies $f(n)\ R'\ f(n')$, and where equivalence is "extensionality" with respect to R':

$$f \sim f' \iff \forall n[n\ R\ n \Rightarrow f(n)\ R'\ f'(n)].$$

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The category $\mathbf{M}$, so defined, proved to be extremely useful as it provided a very natural model of polymorphism, see [26, 31, 39, 42].

Indeed, partial equivalence relations (PERs from now on) were introduced, following prior work on intuitionistic logic, in [40] (see also [41]), suggested by the results about algebraic lattices seen as topological spaces. One interesting category replaces $\mathbb{N}$ above by $\mathrm{P}(\mathbb{N})$, the powerset of $\mathbb{N}$, and the computable functions by continuous functions, which form the so-called graph model of the λ-calculus. Another one uses the *computable* continuous functions ($=$ enumeration operators of recursive function theory). Other λ-calculus models could be used as well. These are small categories. An interesting large category is obtained when one starts from the category of algebraic lattices and Scott continuous functions, and defines a category $\mathbf{PEQU}$ using the same idea of partial equivalence relations. The objects are pairs (L, R) where $R \subseteq L \times L$ is a symmetric, transitive relation on (the underlying set of) an algebraic lattice L and maps are equivalence classes $[f]$ of continuous functions which preserve the relation.

Like $\mathbf{M}$, the category $\mathbf{PEQU}$ proves to be cartesian closed as Scott recently discovered. To emphasize the similarity of the two situations, note that the cartesian closed structure of $\mathbf{PEQU}$ is given by:

- the terminal object is an(y) algebraic lattice L with the total relation $L \times L \subseteq L \times L$;
- the product of (L, R) and (L', R') is the algebraic lattice $L \times L'$ paired with partial equivalence relation defined as the pointwise product of R and R';
- the exponential of (L, R) and (L', R') is the algebraic lattice L'^{L} paired with the relation R'^{R}

which relates f and f' in L'^L if they define the same map from (L, R) into (L', R').

The reader who is familiar with the constructions for $\mathbf{M}$ will immediately note the similarity: instead of using the coding of pairing on the natural numbers and Kleene application, one uses products and exponentials of the underlying cartesian closed category $\mathbf{AlgLatt}$ of algebraic lattices. In the case of $\mathbf{M}$, the Kleene partial combinatory algebra on $\mathbb{N}$ gives a model of untyped computation for realizers and in the case of $\mathbf{PEQU}$ one can think of the category of algebraic lattices as providing a typed model of computation for realizers—and this suffices for cartesian closure. One of the aims of this paper is to provide a deeper understanding of this similarity.

In all cases the extension of the category (the monoid of partial recursive functions, the graph model, or the category of algebraic lattices and continuous functions) by means of PERs allows more extensive constructions of objects. For instance, in $\mathbf{M}$ one can define a "quotient" on the numbers by requiring that $n\ P\ n'$ if the n-th partial recursive function φ_n is the same as the n'-th, thus obtaining an ***object*** of partial recursive functions. Indeed, after doing it once one can continue and define an object of functionals on the partial recursive functions, and so on. The hierarchy obtained thus is precisely that of the partial effective operations, see [25].

Consider also as an example of added definability the category $\mathbf{PER}_{\mathrm{P}(\mathbb{N})}$ of PERs over the graph model of the lambda calculus mentioned above. Denote the diagonal relation on $\mathrm{P}(\mathbb{N})$ as G. The real numbers can be embedded into G by taking a real number to its neigbourhood filter of open rational intervals with suitable encodings. This allows us to embed $\mathsf{R}^{\mathbb{N}}$ into $G^{\mathbb{N}} \simeq G$. Given now an (ultra)filter ∇ in G one can define an equivalence relation on $\mathsf{R}^{\mathbb{N}}$ by

$$s \equiv_\nabla t \iff [\![s = t]\!] = \{n \in \mathbb{N} \mid s_n = t_n\} \in \nabla.$$

Hence, this gives an object R_∇ in the category of PERs on the graph model, and one can consider a fair amount of non-standard analysis inside this category.

The lifting $R_\perp$ of a PER R is easily defined in each case as the constant computations which may fail to terminate. This was one example at the base of the so-called computational lambda-calculus, see [33, 34].

What we have illustrated has obviously to do with the possibility of taking quotients of relations, in particular any notion of extensional behaviour induces an equivalence on an appropriate object of ***codes***, hence one defines the desired object as a quotient on the codes (the principal example being that of the object of partial recursive functions). The use of quotients, as a further definitional device, appears also in type theory: for instance, they are in NUPRL [16], and *quotient types* have been added to various type theories, see *e.g.* [23, 24, 27, 32].

Although the construction of a category of partial equivalence relations is rather straightforward, there are problems:

1. There is no category-theoretic explanation of the reason it produces, in many useful cases, a cartesian closed category.

2. Even if the aim is to obtain sufficient definability by adding quotients, a category of PERs may well fail to be *exact* in the category-theoretic sense, *i.e.* to have *stable effective quotients* of equivalence relations.

The objective of the present paper is to provide a general approach to basic closure conditions of such constructions by giving what the authors feel is an appropriate categorical setting. The methodology proves to be very flexible and no doubt has more applications. What we find especially interesting is the natural embedding of several familiar categories into extensions with better closure properties. We believe the improved theory of types, and its accompaning logic, should have both mathematical applications and uses for semantics and the logic of programs.

2 Motivating Exact Categories

The notion of an exact category is algebraic in the language of categories and one can give the free exact completion of a category, or of a category with finite limits, or of a regular category. We shall follow the presentation [13] which was developed from a suggestion of A. Joyal, see also [11, 21].

Instead of diving into the diagrammatic definition of what a stable effective quotient is, for which we refer the reader to section 3, we shall try first to suggest which properties of the standard construction of a set of equivalence classes they single out. Given an equivalence relation $\sim$ on a set S, the quotient $S/\sim$ is the smallest solution to the problem of identifying elements $s \sim s'$. The property (that one then checks in proving the factorization theorem for set-functions), namely that

$$[x]_\sim = [x']_\sim \iff x \sim x',$$

can be restated as saying that the kernel equivalence relation induced by the canonical surjetion $S \longrightarrow S/\sim$ coincides with the given equivalence relation $\sim$. This makes a quotient of sets *effective*. Finally, it is a property of the logic that gives stability: any renaming of the equivalence classes $g\colon X \longrightarrow S/\sim$ is in bijection with the classes for an equivalence relation on $\{(x,s) \mid g(x) = [s]_\sim\}$.

In fact, it is the failure to produce an exact category which suggests where to search for a category-theoretic explanation of the construction of categories of PERs. In fact, we show that (in all the interesting cases) a category of PERs is a full reflective subcategory of an exact completion, and the reflector preserves products. Hence, the cartesian closed structure of the category of PERs is induced from that of the exact completion. This offers a solution to the general problem of where to look for the properties of a category of PERs: determine if and when an exact completion has a property, then check if this may be reflected down to the category of PERs.

For instance, there is a complete characterization of the case when an exact completion of a category with finite limits is cartesian closed, or locally cartesian closed, see [14]. And this fits precisely with the intuition that comes from models of computations. For instance, the condition that the category $\mathbf{C}$ with finite limits must satisfy in order that its exact completion $\mathbf{C}_{\text{ex}}$ be cartesian closed is (a slight strengthening of) the following condition of *weak cartesian closure*: for every pair of objects C and C' in $\mathbf{C}$ there is an object W and a map $e\colon W \times C \longrightarrow C'$ which is a *weak* evaluation, in the sense that, for every f, there is f' such that

$$\begin{tikzcd} W \times C \arrow[r, "e"] & C' \\ X \times C. \arrow[u, "f' \times \mathrm{id}_C"] \arrow[ur, "f"'] & \end{tikzcd}$$

There is no uniqueness requirement for f', just as there is no single number code for a recursive function.

In all interesting cases, a category of PERs is constructed over a category which satisfies (the appropriate strengthening of) the condition above, hence is cartesian closed as a reflective exponential ideal of an exact completion which is cartesian closed.

A remarkable byproduct of the result is that the construction of the appropriate category on which to take the exact completion is very general and of interest in itself. Also, in the process one sees that the exact completion of the category of (all) topological spaces and the effective topos (which is an exact completion of a category with finite limits, see [36]) share many similarities. In particular, they are both cartesian closed pretoposes for the same general reason. In particular, one also obtains a solution to a long standing problem in categorical topology of embedding the category of topological spaces into an exact cartesian closed category. See also [37] which obtains independently the same result with a rather different, non-elementary proof.

3 Defining exact categories

In order to recall the definition of an exact category from [8], we must introduce some preliminary notions in a category with pullbacks: an equivalence relation is a pair of parallel maps $X_1 \underset{r_2}{\overset{r_1}{\rightrightarrows}} X_0$ which is

(i) jointly monic: for every $x, y\colon Z \longrightarrow X_1$,

if $r_1 x = r_1 y$ and $r_2 x = r_2 y$, then $x = y$

(ii) reflexive:

$$\begin{tikzcd} & X_1 \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X_0 & X_0 \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] \arrow[u, "\exists r"] & X_0, \end{tikzcd}$$

(iii) symmetric:

$$\begin{tikzcd} & X_1 \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X_0 & X_1 \arrow[l, "r_2"] \arrow[r, "r_1"'] \arrow[u, "\exists s"] & X_0, \end{tikzcd}$$

(iv) transitive:

$$\begin{tikzcd}
 & & X_1 \arrow[dddll, "r_1"'] \arrow[dddrr, "r_2"] & & \\
 & & X_2 \arrow[u, dashed, "\exists t"'] \arrow[dl, "r_{12}"'] \arrow[dr, "r_{23}"] & & \\
 & X_1 \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \text{pb.} & X_1 \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X_0 & & X_0 & & X_0 .
\end{tikzcd}$$

Note that the three maps, r, s and t, which are required to exist, are unique under the relative commuting property because of joint monicity. Note also that, for any map $f\colon X \longrightarrow Y$, its *kernel pair* $K_f \overset{k_1}{\underset{k_2}{\rightrightarrows}} X$ defined by the pullback

$$\begin{tikzcd}
K_f \arrow[r, "k_1"] \arrow[d, "k_2"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

is always an equivalence relation.

A *quotient* of an equivalence relation $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$ is in general a coequalizer $f\colon X_0 \longrightarrow X$ of the parallel pair. It is said to be *effective* if its kernel pair is isomorphic to the given equivalence relation. It is *stable* if any pullback of it is a quotient. In the category of sets, all quotients are effective and stable but this fails in many other categories.

3.1 DEFINITION (see [8]) A category is called *exact* if it has finite limits and stable effective quotients of equivalence relations.

Examples of exact categories abound: the category of sets and functions (and, in fact, any topos); an abelian category (in fact, a category is abelian if and only if it is exact and additive); a category monadic over $\mathbf{Set}$; every category of algebras for an equational theory; the category of compact Hausdorff spaces and continuous functions; the category of sup-lattices and sup-preserving homomorphisms in a topos; and a slice of an exact category.

Note that in an exact category, every map can be factored as a quotient followed by a mono. In fact, one shows that quotients and monos form a stable factorization system, see [19]. Moreover, there are coequalizers also for *pseudo-equivalence relations*: a parallel pair $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$ which is reflexive, symmetric and transitive ("pseudo" because it may ***fail*** to satisfy the jointly-monic condition from the definition of an equivalence relation). This is a direct application of the factorization.

For every category $\mathbf{C}$ with finite limits, one can construct its free exact completion $\mathbf{C}_{\mathrm{ex}}$: we shall briefly recall the construction of $\mathbf{C}_{\mathrm{ex}}$ and its universal property here, referring the reader to [11, 13] for all the details.

3.2 DEFINITION (see [11, 13]) The objects of the category $\mathbf{C}_{\mathrm{ex}}$ are the pseudo-equivalence relations in $\mathbf{C}$ and the maps from $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$ into $Y_1 \overset{s_1}{\underset{s_2}{\rightrightarrows}} Y_0$ are equivalence classes $[f]$ of maps of $\mathbf{C}$ such that

$$\begin{tikzcd}
X_1 \arrow[r, dashed, "\exists g"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & Y_1 \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X_0 \arrow[r, "f"] & Y_0
\end{tikzcd}$$

(to be read componentwise as $f \circ r_i = s_i \circ g$, $i = 1, 2$) where we say $f' \sim f''$ iff

$$\begin{tikzcd}
X_1 \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & Y_1 \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\
X_0 \arrow[ur, dashed, "\exists h"] \arrow[r, shift left, "f'"] \arrow[r, shift right, "f''"'] & Y_0
\end{tikzcd}$$

(again, read $f^{(i)} = s_i \circ h$, $i = 1, 2$). Intuitively, f' and f'' induce the same map on the quotient.

3.3 THEOREM (see [11, 13]) *If* $\mathbf{C}$ *has finite limits, then* $\mathbf{C}_{\mathrm{ex}}$ *is exact and the assignment of the diagonal relation* $C \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} C$ *to objects* C *of* $\mathbf{C}$ *defines a fully faithful functor* $\mathrm{Y}\colon \mathbf{C} \longrightarrow \mathbf{C}_{\mathrm{ex}}$ *which preserves finite limits and is universal (in the 2-categorical sense) among limit preserving functors from* $\mathbf{C}$ *into an exact category.*

There is a useful characterization of the image of Y, and of those exact categories which appear as the exact completion of a category with finite limits. The objects in the image of Y are those P which are *regular projectives* of $\mathbf{C}_{\mathrm{ex}}$: for every quotient $q\colon X \longrightarrow X'$ in $\mathbf{C}_{\mathrm{ex}}$, and every map $x'\colon P \longrightarrow X'$ there exists a (not necessarily unique) map $x\colon P \longrightarrow X$ such that

$$\begin{tikzcd} P \arrow[d, "x"'] \arrow[dr, "x'"] & \\ X \arrow[r, "q"] & X'. \end{tikzcd}$$

$\mathbf{C}$ is equivalent to the full subcategory of $\mathbf{C}_{\mathrm{ex}}$ on the projectives. (In the set-theoretic intuition, a regular projective is a set for which choice principle holds.)

3.4 THEOREM (see [11]) *An exact category $\mathbf{E}$ is a free completion if and only if the full subcategory on the regular projectives of $\mathbf{E}$ is closed under finite limits, and every object E of $\mathbf{E}$ is the target of a quotient map from a regular projective.*

It is interesting to compare this result with Peter Aczel's Constructive Set Theory, see [3]. It provides a standard set theoretic framework to develop constructive mathematics along the lines described by Errett Bishop, see [10]. The two main axioms of Aczel's theory require that every set is the image of a base and that bases are closed under binary products and subsets defined by equalities, *e.g.* $\{b \in B \mid f(b) = g(b)\}$, where f and g apply the base B into another base. (A *base* is Aczel's name for a set for which choice principle holds.)

It follows from 3.4 that, if $\mathbf{C}_{\mathrm{ex}}$ is cartesian closed, then $\mathbf{C}$ is, at least, weakly cartesian closed. Indeed, for given objects C and C' in $\mathbf{C}$, there is a (strong) evaluation $\mathrm{ev}\colon X \times \mathrm{Y}(C) \longrightarrow \mathrm{Y}(C')$ in $\mathbf{C}_{\mathrm{ex}}$. By 3.4, there is an object W in $\mathbf{C}$ (a regular projective in $\mathbf{C}_{\mathrm{ex}}$) and a quotient $q\colon \mathrm{Y}(W) \longrightarrow X$ in $\mathbf{C}_{\mathrm{ex}}$. Using preservation of products of Y, consider the composite in $\mathbf{C}_{\mathrm{ex}}$

$$\begin{tikzcd} \mathrm{Y}(W \times C) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Y}(C') \\ \mathrm{Y}(W) \times \mathrm{Y}(C) \arrow[r, "q \times \mathrm{id}"] & X \times \mathrm{Y}(C); \arrow[u, "\mathrm{ev}"'] \end{tikzcd}$$

since Y is full, it is of the form $\mathrm{Y}(e)$ for some map $e\colon W \times C \longrightarrow C'$ in $\mathbf{C}$. Again, by product preservation and full faithfulness of Y, one can easily check that e is a weak evaluation for C and C' in $\mathbf{C}$.

Note that weak cartesian closure can be expressed by saying that, for every pair of objects C and C', there is an object W and a natural surjection of functors:

$$\mathbf{C}(-, W) \twoheadrightarrow \mathbf{C}(- \times C, C').$$

As we said in the introduction, we need a strengthening of that weak condition of closure to state the characterization of when $\mathbf{C}_{\mathrm{ex}}$ is cartesian closed. We refer the reader to [14] for the technical details. Here we shall only try to convey the intuition leading to the statement. A non-expert can simply read the two statements in this section and continue to the next.

Since an exact category has pullbacks, one can say a little more about right adjoints to $\Leftrightarrow \times A$. In an arbitrary category $\mathbf{A}$ with binary products, the condition of closure requires a right adjoint for each functor $\Leftrightarrow \times A\colon \mathbf{A} \longrightarrow \mathbf{A}$. If $\mathbf{A}$ has also pullbacks, from the previous adjoint, one can construct a right adjoint for the functor $\Leftrightarrow \times A\colon \mathbf{A}/_{I} \longrightarrow \mathbf{A}/_{I \times A}$.

If $\mathbf{A}$ is $\mathbf{C}_{\mathrm{ex}}$ for some category $\mathbf{C}$ with finite limits, then the existence of those right adjoints yields information about $\mathbf{C}$. Note that, if C is an object of $\mathbf{C}$, then

$$\mathbf{C}_{\mathrm{ex}}/_{\mathrm{Y}(C)} \equiv (\mathbf{C}/_{C})_{\mathrm{ex}},$$

as follows immediately from 3.4. Hence, like before, from the existence of a right adjoint in $\mathbf{C}_{\mathrm{ex}}$, one can extract a weak representation

$$\mathbf{C}/_{I}(-, w) \twoheadrightarrow \mathbf{C}/_{I \times K}(- \times K, c)$$

for objects C and I, and $c\colon C \longrightarrow I \times K$ in $\mathbf{C}$. We expand the weak representability condition in the above in the following definition.

3.5 DEFINITION Let $\mathbf{C}$ be a category with finite limits and let K be an object of $\mathbf{C}$. A *weak simple product* of a map $c\colon C \longrightarrow I \times K$ with respect to K consists of a map $w\colon W \longrightarrow I$ (an object in the comma category $\mathbf{C}/_{I}$) and a map ε such that

$$\begin{tikzcd} W \times K \arrow[r, "\varepsilon"] \arrow[dr, "w \times K"'] & C \arrow[d, "c"] \\ & I \times K \end{tikzcd}$$

commutes. Moreover the pair of w and ε is *weakly universal* with the property: in any situation

$$\begin{tikzcd} X \times K \arrow[r, "f"] \arrow[dr, "x \times K"'] & C \arrow[d, "c"] \\ & I \times K \end{tikzcd}$$

there is a (not necessarily unique) map $f' \colon X \longrightarrow W$ over I such that $f = \varepsilon \circ (f' \times K)$.

Note that in a category $\mathbf{C}$ with weak simple products, one can construct a weak exponential of A and B by taking a weak simple product W of

$$B \times A \xrightarrow{\text{snd}} A \xrightarrow{\sim} 1 \times A,$$

and a weak evaluation as the composite

$$W \times A \xrightarrow{\varepsilon} B \times A \xrightarrow{\text{fst}} B.$$

We shall present examples of weak simple products below, but before that we hasten to notice that weak simple products are a particular case of what one may well call *weak dependent products*: in a category $\mathbf{C}$ with pullbacks, a *weak dependent product* of an arrow $c \colon C \longrightarrow J$ along a map $\alpha \colon J \longrightarrow I$ is a weak representation

$$\mathbf{C}/_{I}(-, w) \longrightarrow \mathbf{C}/_{J}(\alpha^*(-), c).$$

It consists then of maps $w \colon W \longrightarrow I$ and $\epsilon \colon \alpha^*(w) \longrightarrow c$ in the comma category $\mathbf{C}/_{J}$ such that ϵ is weakly universal among maps from pullbacks of objects over I into c.

A weak simple product is a weak dependent product along a projection. Since a category with (strong) dependent products is locally cartesian closed, we say that a category is *weakly locally cartesian closed* if it has weak dependent products.

3.6 EXAMPLES One can construct an example of a category with weak simple products from a partial combinatory algebra A: the category $\mathbf{D}_A$ has objects the subsets $C \subseteq A^n$ of finite powers of A and a map $f \colon C \subseteq A^n \longrightarrow C' \subseteq A^m$ is an m-tuple of A-definable functions mapping C into C'. Given C and C', say both subsets of A for sake of simplicity, one can construct a weak exponential $e \colon W \times C \longrightarrow C'$ by taking

$$W = \{e \in A \mid \forall a \in C.\, e \cdot a \downarrow \wedge\, e \cdot a \in C'\} \qquad (1)$$

and defining e in the obvious way.

For the more general construction of weak simple products, consider $c \colon C' \longrightarrow I \times C$ and define

$$\begin{aligned} V = \{\langle i, e\rangle \mid i, e \in A \wedge \forall a \in C.& \\ \text{fst}(e \cdot a) = i \wedge \text{snd}(e \cdot a) = a\}& \end{aligned}$$

(where $\langle \cdot, \cdot \rangle$ is the pairing operation of the partial combinatory algebra) and let $w \colon V \to I = \text{fst}$ and $e \colon V \times C \to C'$ be the map $(a, a') \mapsto (\text{fst}(a)) \cdot a'$.

The presentation of the regular projectives in the effective topos as *partitioned assemblies* gives another example. Consider the monoid $\mathbf{P}$ of partial recursive functions and recall from [12] the category $\mathbf{PartAsm}$ whose objects are functions $\sigma \colon S \longrightarrow \mathbb{N}$ into $\mathbb{N}$ and maps are functions $f \colon S \longrightarrow S'$ such that

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\exists g"] & \mathbb{N} & \text{in } \mathbf{P} \\ S \arrow[u, "\sigma"] \arrow[r, "f"] & S'. \arrow[u, "\sigma'"'] \end{tikzcd}$$

One can construct a weak evaluation directly by taking ψ as the pullback

$$\begin{tikzcd} W \arrow[rr] \arrow[d, "\psi"'] & & {S'}^S \arrow[d] \\ \mathbb{N} \arrow[r, "\varphi_-"] & \mathbb{N}^{\mathbb{N}} \arrow[r] & \mathbb{N}^S. \end{tikzcd}$$

and define $\varepsilon \colon \psi \times \sigma \longrightarrow \sigma'$ by means of the standard evaluation function ${S'}^S \times S \longrightarrow S'$ of sets—compare that with (1) above.

A construction, similar to that of $\mathbf{PartAsm}$, can be applied to a weak cartesian closed category $\mathbf{X}$ to obtain a category $\mathcal{F}(\mathbf{X})$ with weak simple products. Let $\Gamma \colon \mathbf{X} \longrightarrow \mathbf{Set}$ be the functor taking global elements

$$\Gamma(X) = \mathbf{X}(1, X).$$

Define a category $\mathcal{F}(\mathbf{X})$ as consisting of pairs $(X, \sigma \colon S \longrightarrow \Gamma(X))$. A map between two such objects is a function $f \colon S \longrightarrow S'$ such that

$$X \xrightarrow{\exists g} X' \qquad \text{in } \mathbf{X}$$

$$\begin{tikzcd} \Gamma(X) \arrow[r, "\Gamma(g)"'] & \Gamma(X') \\ S \arrow[u, "\sigma"] \arrow[r, "f"] & S'. \arrow[u, "\sigma'"'] \end{tikzcd} \qquad (2)$$

The category $\mathcal{F}(\mathbf{X})$ has finite limits: products are obvious, equalizers are obtained directly from those in $\mathbf{Set}$. Moreover, $\mathcal{F}(\mathbf{X})$ has weak simple products. Again, we only suggest how to construct a weak evaluation: given objects $(X, \sigma\colon S \longrightarrow \Gamma(X))$ and $(X', \sigma'\colon S' \longrightarrow \Gamma(X'))$, let $e\colon W \times X \longrightarrow X'$ be a weak evaluation in $\mathbf{X}$, and take the pullback

$$\begin{tikzcd} T \arrow[rr] \arrow[d, "\psi"'] & & S'^{S} \arrow[d] \\ \Gamma(W) \arrow[r, "\widehat{\Gamma(e)}"] & \Gamma(X')^{\Gamma(X)} \arrow[r] & \Gamma(X')^{S}. \end{tikzcd}$$

The weak evaluation is now defined similarly to the previous case. In fact, one can show, by similar constructions, that all the examples, $\mathbf{D}_A$, $\mathbf{PartAsm}$, and $\mathcal{F}(\mathbf{X})$, are also weakly locally cartesian closed.

The reader has probably realized that the category $\mathcal{F}(\mathbf{X})$ is a *quotient* category of the comma category $(\mathbf{Set}|\Gamma)$, and some of the constructions in $\mathcal{F}(\mathbf{X})$ resemble the corresponding ones in the comma category $(\mathbf{Set}|\Gamma)$. But it is important to notice that in the diagram (2), which defines maps in $\mathcal{F}(\mathbf{X})$, the map g is witnessing a property of f but does not take part in the structure; in particular, commutativity in $\mathbf{X}$ will never play a part in defining structure on $\mathcal{F}(\mathbf{X})$. The next proposition should provide a concrete explanation.

3.7 PROPOSITION

(i) *The category* $\mathbf{Top}$ *of topological spaces and continuous functions is equivalent to* $\mathcal{F}(\mathbf{AlgLatt})$.

(ii) *The category* $\mathbf{Top}_0$ *of* T_0*-spaces and continuous functions is equivalent to the full reflective subcategory of* $\mathcal{F}(\mathbf{AlgLatt})$ *on those pairs* $(L, \sigma\colon S \longrightarrow \Gamma(L))$ *with* σ *monic.*

Proof: For (ii): Given a T_0-space S_0, by the embedding theorem of [40], there is a subspace embedding $S_0 \hookrightarrow 2^{\mathrm{Open}(S_0)}$. By the extension theorem of [40], a map $f\colon S_0 \longrightarrow S_0'$ between T_0-spaces is continuous if and only if there is a continuous g such that

$$\begin{tikzcd} S_0 \arrow[r, "f"] \arrow[d, hook] & S_0' \arrow[d, hook] \\ 2^{\mathrm{Open}(S_0)} \arrow[r, "g"] & 2^{\mathrm{Open}(S_0')} \end{tikzcd}$$

commutes. From this, the result follows.
For (i): Given any topological space S, let S_0 be the quotient of S with respect to the equivalence

$$x \sim y \iff \forall U \subseteq_{\mathrm{open}} S[x \in U \Leftrightarrow y \in U]$$

and note that $\mathrm{Open(S)} \simeq \mathrm{Open(S_0)}$. It follows that, in a commutative diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, two heads] & S' \arrow[d, two heads] \\ S_0 \arrow[r, "f_0"'] & S_0', \end{tikzcd}$$

f is continuous if and only if f_0 is continuous. Using the result at the previous point, the conclusion follows. $\square$

The following theorem summarizes the characterization of when the exact completion of a category is (locally) cartesian closed.

3.8 THEOREM (see [14]) *Suppose* $\mathbf{C}$ *has finite limits. Then*

(i) $\mathbf{C}_{\mathrm{ex}}$ *is cartesian closed if and only if* $\mathbf{C}$ *has weak simple products* (*for all maps into a product*).

(ii) $\mathbf{C}_{\mathrm{ex}}$ *is locally cartesian closed if and only if* $\mathbf{C}$ *is weakly locally cartesian closed.*

4 Application to PERs

In this section we show how the general theory developed in the previous section applies to categories of partial equivalence relations.

4.1 THEOREM *The three categories* $\mathbf{Top}_{\mathrm{ex}}$, $(\mathbf{Top}_0)_{\mathrm{ex}}$, *and* $\mathbf{Eff}$ *are all locally cartesian closed.*

Proof: It follows immediately from the property of the respective categories of regular projectives, also for the last case since $\mathbf{Eff} \equiv \mathbf{PartAsm}_{\mathrm{ex}}$, see [11, 36]. $\square$

We should recall that the universal embedding $\mathrm{Y}\colon \mathbf{Top} \longrightarrow \mathbf{Top}_{\mathrm{ex}}$ preserves existing exponentials, therefore $\mathbf{Top}_{\mathrm{ex}}$ adds only those exponentials which could not be defined as topological spaces.

To study partial equivalence relations in general, consider the case where a category $\mathbf{C}$ with finite limits has a stable factorization system $(\mathcal{E}, \mathcal{M})$. Recall from [19] that a *stable factorization system* is such that every map in $\mathbf{C}$ can be factored as a composite of a map in $\mathcal{E}$ and one in $\mathcal{M}$, that these collections are closed under pullbacks, and that they are *orthogonal*: there exists a unique diagonal filling in every commutative square such as sketched below

$$\begin{tikzcd} \cdot \arrow[r, "\in\mathcal{E}"] \arrow[d] & \cdot \arrow[d] \arrow[dl, dotted] \\ \cdot \arrow[r, "\in\mathcal{M}"'] & \cdot \end{tikzcd}$$

Define $\mathrm{PER}(\mathbf{C}, \mathcal{M})$ as the full subcategory of $\mathbf{C}_{\mathrm{ex}}$ on those pseudo-equivalence relations $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$ such that the pairing $\langle r_1, r_2\rangle : X_1 \longrightarrow X_0 \times X_0$ is in $\mathcal{M}$.

4.2 EXAMPLES When $\mathbf{C} = \mathbf{D}_A$, for the choice of the partial combinatory algebra A as the Kleene structure on $\mathbb{N}$ (or the graph model), and when the elements of $\mathcal{M}$ are the inclusions, the category $\mathrm{PER}(\mathbf{C}, \mathcal{M}) \subset \mathbf{C}_{\mathrm{ex}}$ is $\mathbf{M}$ (and for the graph model $\mathbf{PER}_{\mathrm{P}(\mathbb{N})}$). Next consider in $\mathbf{Top}_0$ the factorization system where the elements of $\mathcal{E}$ are the surjective countinuous functions and $\mathcal{M}$ are the subspace inclusions. The category $\mathbf{PEQU}$ is precisely $\mathrm{PER}(\mathbf{Top}_0, \mathcal{M})$. Similarly, taking the elements of $\mathcal{M}$ as the inclusions in $\mathbf{PartAsm}$ (and $\mathcal{E}$ accordingly), the category $\mathrm{PER}(\mathbf{PartAsm}, \mathcal{M})$ is equivalent to the category of assemblies, see [12].

More generally, in a category of the form $\mathcal{F}(\mathbf{X})$, there is a canonical factorization system: the collection $\mathcal{M}$ consists of those maps $m: (X, \sigma: S \longrightarrow \Gamma(X)) \longrightarrow (X, \sigma': S' \longrightarrow \Gamma(X))$ such that m is 1-1 and the diagram

$$\begin{tikzcd} \Gamma(X) \arrow[r, "\mathrm{id}"'] & \Gamma(X) \\ S \arrow[u, "\sigma"] \arrow[r, tail, "m"] & S' \arrow[u, "\sigma'"'] \end{tikzcd}$$

commutes. The collection $\mathcal{E}$ consists then of the surjective functions. For instance, when $\mathbf{X} = \mathbf{AlgLatt}$ and $\mathcal{F}(\mathbf{X}) \equiv \mathbf{Top}$, the collection $\mathcal{M}$ coincides with subspace inclusions.

In general, we have the following theorem, which covers all the above examples.

4.3 THEOREM *Suppose $\mathbf{C}$ is a category with finite limits and a stable factorization system $(\mathcal{E}, \mathcal{M})$. The inclusion $\mathrm{PER}(\mathbf{C}, \mathcal{M}) \hookrightarrow \mathbf{C}_{\mathrm{ex}}$ has a left adjoint which preserves products, and commutes with pullbacks along maps in the subcategory.*

Proof: Given a pseudo-equivalence relation of the form $X_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} X_0$, consider the factorization

$$\langle r_1, r_2\rangle : X_1 \xrightarrow{e \in \mathcal{E}} R \xrightarrow{m \in \mathcal{M}} X_0 \times X_0$$

with respect to the given system, and let $R \overset{m_1}{\underset{m_2}{\rightrightarrows}} X_0$ be the two components of m. It is easy to see using orthogonality that these parallel maps form an equivalence relation in $\mathrm{PER}(\mathbf{C}, \mathcal{M})$ and that this defines the reflection for $X_1 \rightrightarrows X_0$. The reflector preserves products because $\mathcal{E}$ is closed under products, and it is an easy exercise in computing limits in the exact completion to see that it preserves the special pullbacks. $\square$

4.4 COROLLARY *The categories $\mathbf{PEQU}$, $\mathbf{PER}_{\mathrm{P}(\mathbb{N})}$, and $\mathbf{M}$ are all locally cartesian closed.*

By the same token, replacing $\mathbf{Top}_0$ by $\mathbf{Top}$, we obtain a locally cartesian closed category $\mathrm{PER}(\mathbf{Top}, \mathcal{M}) \subset \mathbf{Top}_{\mathrm{ex}}$ for $\mathcal{M}$ the collection of subspace inclusions.

Also categories of cuPERs (=complete uniform PERs) of [1, 2, 4] fall within this framework. Suppose the partial combinatory algebra A is that of a directed-complete poset containing the poset of its partial continuous endofunctions as a retract. One can take the full subcategory $\mathbf{C}_A \subset \mathbf{D}_A$ on the *complete* subsets of A, *i.e.* closed under directed sups. The inclusion has a reflector which preserves finite limits. Hence $\mathbf{C}_A$ is weakly locally cartesian closed, and one finds the category of complete PERs of [1, 2, 4] as a reflective subcategory of $(\mathbf{C}_A)_{\mathrm{ex}}$ exactly like the other cases. Complete uniform PERs are then found as the σ-domains in $(\mathbf{C}_A)_{\mathrm{ex}}$, see [38].

5 On the natural numbers

In looking for important types in the categories discussed in this paper, one can consider the question of finding a natural number object. Recall a definition from [28]: a triple consisting of an object N and a pair of maps $1 \xrightarrow{0} N \xrightarrow{s} N$ is a *weak natural number object* if it is weakly initial among inductive structures $1 \xrightarrow{a} A \xrightarrow{h} A$, *i.e.* there is a (not necessarily unique) map $g\colon N \longrightarrow A$ and a commutative diagram

$$\begin{tikzcd} & N \arrow[r, "s"] \arrow[d, "g"] & N \arrow[d, "g"] \\ 1 \arrow[ur, "0"] \arrow[dr, "a"'] & & \\ & A \arrow[r, "a"] & A. \end{tikzcd}$$

We are interested in the case when the ambient category is cartesian closed with pullbacks.

5.1 PROPOSITION *Suppose* $\mathbf{A}$ *is cartesian closed and has equalizers. If* $\mathbf{A}$ *has a weak natural number object, then it has also a natural number object.*

Proof: Suppose N is a weak natural number object in $\mathbf{A}$. Define the object of endomorphisms of N taking the intersection $I \overset{m}{\rightarrowtail} N^N$ of the two equalizers

$$I_1 \rightarrowtail N^N \underset{-\circ s}{\overset{s\circ -}{\rightrightarrows}} N^N$$

$$I_2 \rightarrowtail N^N \underset{-\circ 0}{\overset{0!}{\rightrightarrows}} N$$

where ! is the unique map into the terminal object. Then consider the *generic* endomorphism given as the equalizer

$$K \rightarrowtail N \times I \underset{p_1}{\overset{\mathrm{ev}(\mathrm{id}\times \mathrm{m})}{\rightrightarrows}} N$$

where p_1 is the first projection from a product, and apply the right adjoint Π_I to the functor $- \times I\colon \mathbf{A}/_N \longrightarrow \mathbf{A}/_{N\times I}$ to determine the *intersection* of them all. This is the interpretation of the formula

$$\forall f \in N^N[(f \circ s = s \circ f \wedge f0 = 0) \to f(n) = n].$$

It is easy to see that the object obtained in this way is a (strong) natural number object. □

It is also easy to see that $\mathrm{Y}\colon \mathbf{C} \hookrightarrow \mathbf{C}_{\mathrm{ex}}$ preserves a weak natural number object. Therefore, this joins forces with the $\mathcal{F}$-construction to give the following:

5.2 COROLLARY *Suppose* $\mathbf{X}$ *is weakly cartesian closed, suppose moreover* $\mathbf{X}$ *has a weak natural number object. Then* $\mathcal{F}(\mathbf{X})$ *has a weak natural number object. Hence* $(\mathcal{F}(\mathbf{X}))_{\mathrm{ex}}$ *has a natural number object.*

6 Concluding remarks

We have shown that the construction of the category of partial equivalence relations is very fruitful for some general reasons that depend on the properties of the categories involved. In particular, *forcing* a category with weak exponentials to be exact by means of a completion process is a correct approach to the production of extensional models of functions, and relates directly to the classical hierarchy of partial effective functionals. In the discussion, the underlying construction of the *exact completion* of a category has emerged, and we believe that, in many cases, this may helpfully replace the related category of PERs.

As an instance of this, consider a characterization of categorical models of combinatory logic as given in [30], where it is essentially shown that categorical models of combinatory logic appear as objects U together with a map $u\colon U \longrightarrow U^U$ such that all functions $u\circ$–: $\hom(U^n, U) \longrightarrow \hom(U^n, U^U)$ are onto, which is not a simple condition. Given a combinatory algebra A, one can take the category $(\mathbf{D}_A)_{\mathrm{ex}}$ and find the regular projective $A \subseteq A$ (=object of $\mathbf{D}_A$) with a quotient map $u\colon A \longrightarrow A^A$, so that projectivitity of A will enforce projectivity of its powers, hence that all functions $u \circ$–: $\hom(A^n, A) \longrightarrow \hom(A^n, A^A)$ are onto.

On the topological side, since $\mathbf{Top}$ has stable disjoint sums, these are inherited by $\mathbf{Top}_{\mathrm{ex}}$, see [11], which is therefore a locally cartesian closed pretopos. This result is new and of importance to topology.

The presentation of the category of topological spaces suggested by the embedding theorem and the extension theorem, and given in 3.7, suggests a different approach to mathematical structures introduced for the study of computations: for instance, one can now take $\mathcal{F}(\mathbf{CDS})$, where $\mathbf{CDS}$ is the category of concrete data structures and algorithms. The exact completion $\mathcal{F}(\mathbf{CDS})_{\mathrm{ex}}$ seems to embed fully the category of hypercoherences, see [18].

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