ELSEVIER

Theoretical Computer Science 290 (2003) 189–219

Theoretical Computer Science

www.elsevier.com/locate/tcs

A cartesian closed category in Martin-Löf's intuitionistic type theory

Silvio Valentini

Dipartimento di Matematica Pura ed Applicata, Università di Padova, via G. Belzoni n.7, I-35131 Padova, Italy

Received February 2000; received in revised form June 2001; accepted July 2001
Communicated by M. Nivat

Abstract

First, we briefly recall the main definitions of the theory of Information Bases and Translations. These mathematical structures are the basis to construct the cartesian closed category InfBas, which is equivalent to the category ScDom of Scott domains.

Then, we will show that all the definitions and the proof of all the properties that one needs in order to show that InfBas is indeed a cartesian closed category can be formalized within Martin-Löf's intuitionistic type theory. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 03B15; 03B20

Keywords: Intuitionistic type theory; Cartesian closed categories; Scott domains

1. Introduction

This paper is intended to be a continuation of [8] where the category InfBas of the Information Bases has been introduced and proved to be equivalent to the category ScDom of Scott domains. For this reason, here we only recall the main definitions and properties of information bases while for their philosophical motivations the reader is invited to refer to that paper.

The work in [8] stopped after the proof that the category InfBas is equivalent to the category ScDom of the (set based) Scott domains, besides being equivalent to the category NeighSys of the Neighbourhood Systems and InfSys of the Information Systems [9, 10]. Thus InfBas enjoys any categorical property which holds for the

E-mail address: silvio@math.unipd.it (S. Valentini).

PII: S0304-3975(01)00309-7

category ScDom and in particular it is a cartesian closed category. From a mathematical point of view this is a complete description of InfBas and one can be content with it, but the real reason of interest in working with InfBas instead that with ScDom stands on the fact that information bases can be completely formalized within a constructive framework. Indeed, Scott domains are a foundation of denotational semantics and adopting a fully constructive approach is more adequate since in this way all the results can be provided in terms of effective presentations. Of course, other approaches can be exploited for a constructive presentation of ScDom, and some of them are also closer to the original presentation by Dana Scott in [9, 10], but InfBas has an independent interest since it is a subcategory of the category of formal topologies (see [6]).

In this paper we will show how it is possible to build the category InfBas in a constructive framework by adopting Martin-Löf's intuitionistic type theory as ground theory for sets. This means, for instance, that we will carefully distinguish between sets, that is, inductive types, and collections since quantification is meaningful only over elements of a set (see [4]).

We will see that, in general, the main problem in the construction of InfBas is in finding the correct definitions and that most of the proofs are simple checks to be performed by using intuitionistic logic; this is the reason why we will only give a quick sketch of the proof for most of the theorems.

The notation that we use for type theory is mainly inspired by the one proposed in [5]. In the next section we will recall some basic facts and constructions in type theory; the reader who already knows Martin-Löf's type theory can probably jump directly to Section 3 and come back here when he meets some notation that he cannot recognize.

2. Preliminaries

In this section we recall some known facts on type theory, and introduce few new ones together with some new definitions that we will need in the following.

Given a set A, the set $\mathsf{List}(A)$ of the lists whose elements are in A can be formed (see [5, p. 75]). Its (canonical) elements are the empty list nil_A and, provided $a \in A$ and $l \in \mathsf{List}(A)$, the list $a \bullet l$. The set $\mathsf{List}(A)$ can be used to implement within type theory the collection of the finite subsets of the type A; hence in the following sections we will write also $\mathscr{P}_\omega(A)$ to mean $\mathsf{List}(A)$ when we will want to stress on considering a list on A as a finite subset of A. To identify the collection of the finite subsets of A with the set of the lists whose elements are in A is not really correct because the equality relation on the collection of the finite subsets of A is extensional whereas the one on the set $\mathsf{List}(A)$ is not; moreover, only some of the set-theoretic operations on finite subsets can be defined by using lists. Indeed, it is well known that to deal in a constructive way with finite sets is not straightforward and that all of the proposed approaches have some drawback (see [3, 12]). Anyhow the approach we suggested above is sufficient for the purposes of this paper.

Supposing U is a universe which contains the (code for the) set A, $a \in A$ and $l \in \mathsf{List}(A)$, the membership proposition[1] $a\varepsilon l$, whose recursive definition is

$$x\varepsilon\mathsf{nil} \equiv \bot,$$

$$x\varepsilon a \bullet l \equiv (x =^A a) \vee x\varepsilon l$$

can be solved in type theory by putting

$$a\varepsilon l \equiv \mathsf{L}_{\mathsf{rec}}(l, \bot, (x : A)(y : \mathsf{List}(A))(z : \mathsf{U})(x =^A a) \vee z),$$

where $(x : A)\ t$ denotes the term obtained by abstracting the variable x of type A from the term t.

By means of this proposition, we can easily define the order-relation $l \sqsubseteq m$ of *inclusion* between the lists l and m by putting

$$l \sqsubseteq m \equiv (\forall x \in A)\ x\varepsilon l \rightarrow x\varepsilon m.$$

Thus, $l \sqsubseteq m$ holds if and only if any element of l is also an element of m and hence

$$l \cong m \equiv (l \sqsubseteq m\ \&\ m \sqsubseteq l)$$

is an equivalence relation whose intended meaning is that the "finite subsets" l and m are extensionally equal.

By $\forall$-elimination, if $x\varepsilon l$ and $l \sqsubseteq m$ then $x\varepsilon m$ holds. Moreover, the following lemma is immediate by $\vee$-elimination.

Lemma 2.1. *Let $a \in A$ and $l \in \mathsf{List}(A)$. Then, if $a\varepsilon l$ then $a \bullet \mathsf{nil}_A \sqsubseteq l$.*

From now on, when we will refer to lists like subsets, we will write $\emptyset$ for the empty list nil_A and $\{a_1, \ldots, a_n\}$ for the list $a_1 \bullet \cdots \bullet a_n \bullet \mathsf{nil}_A$; hence $\{a\} \sqsubseteq l$ will be used for $a \bullet \mathsf{nil}_A \sqsubseteq l$.

Given two lists $l, m \in \mathsf{List}(A)$ we can define the operation @ which appends them one after the other. Its recursive definition is

$$\mathsf{nil}\ @\ m \equiv m,$$

$$(a \bullet l)\ @\ m \equiv a \bullet (l\ @\ m)$$

and it is solved in type theory by putting

$$l\ @\ m \equiv \mathsf{L}_{\mathsf{rec}}(l, m, (x : A)(y : \mathsf{List}(A))(z : \mathsf{List}(A))x \bullet z).$$

[1] We are going to distinguish among many membership relations. We will use the standard symbol $\in$ to mean the membership relation between an element and a set or a collection, the symbol ε to mean the membership relation between an element and a subset, which is not a set but a propositional function (see [7]) so that $a\varepsilon U$ means $U(a)$, and also the symbol ε, that we are introducing now, to mean the membership relation between an element and a list of elements which stands for a finite subset.

Supposing $l, m \in \mathsf{List}(A)$, it is straightforward to prove by induction on the construction of l that both $l \sqsubseteq (l \ @ \ m)$ and $m \sqsubseteq (l \ @ \ m)$ hold. Moreover, a proof by induction on the construction of l shows that, for any $x \in A$, $x\varepsilon(l \ @ \ m)$ if and only if $(x\varepsilon l)$ or $(x\varepsilon m)$ and hence, supposing $n \in \mathsf{List}(A)$, if $l \sqsubseteq n$ and $m \sqsubseteq n$, then $(l \ @ \ m) \sqsubseteq n$. Finally, also $l \ @ \ m \cong m \ @ \ l$ holds and thus $l \ @ \ m$ can be thought of as the union of the two "subsets" l and m. Thus, from now on, when the lists l and m will be used to denote two finite subsets we will write $l \sqcup m$ to mean $l \ @ \ m$. It is worth noting that intersection between "subsets" cannot be defined in type theory unless the equality proposition $=^A$ is decidable, which is a necessary condition to be able to define a map ϕ from $A \times \mathsf{List}(A)$ into the two-elements type Boole which codes the proposition $x\varepsilon l$ prop $[x : A, l : \mathsf{List}(A)]$, that is, such that $x\varepsilon l$ holds if and only if $\phi(x, l) =_{\mathsf{Boole}} \mathsf{true}$ holds (cf. [11]).

Supposing $f \in A \to \mathsf{List}(B)$ and $l \in \mathsf{List}(A)$, it is possible to define the operation of *list-indexed append* by the recursive equation

$$@_{x\varepsilon \mathsf{nil}_A} f(x) \equiv \mathsf{nil}_B,$$

$$@_{x\varepsilon a \bullet l} f(x) \equiv f(a) \ @ \ @_{x\varepsilon l} f(x),$$

which is solved in type theory by putting

$$@_{x\varepsilon l} f(x) \equiv \mathsf{L}_{\mathsf{rec}}(l, \mathsf{nil}_B, (x : A)(y : \mathsf{List}(A))(z : \mathsf{List}(B)) \ f(x) \ @ \ z).$$

If we suppose that $(\forall x \in A) \ x\varepsilon l \to f(x) \sqsubseteq m$ holds then it is possible to prove by induction on the construction of l that also $@_{x\varepsilon l} f(x) \sqsubseteq m$ holds. This is the reason why we generalize the previous notation also to the list-indexed append and write $\bigsqcup_{x\varepsilon l} f(x)$ to mean $@_{x\varepsilon l} f(x)$.

Supposing $\cdot_A$ is a binary operation on the set A and Δ_A is a distinguished element of A, the operation $\odot$ on $\mathsf{List}(A)$ recursively defined by

$$\odot(\mathsf{nil}) \equiv \Delta_A,$$

$$\odot(a \bullet l) \equiv a \cdot_A \odot(l)$$

can be defined in type theory, by putting, for any $l \in \mathsf{List}(A)$:

$$\odot(l) \equiv \mathsf{L}_{\mathsf{rec}}(l, \Delta_A, (x : A)(y : \mathsf{List}(A))(z : A) \ x \cdot_A z).$$

Suppose now that $f : B \to A$ is a function from the set B into A and $l \in \mathsf{List}(B)$, then we will write $\odot_f(l)$ to mean the result of the application of the operation $\odot$ to the list $\mathsf{apply}(f, l) \in \mathsf{List}(A)$ obtained by applying the function f to any element in l. The recursive definition of $\mathsf{apply}(f, l)$ is

$$\mathsf{apply}(f, \mathsf{nil}_B) \equiv \mathsf{nil}_A,$$

$$\mathsf{apply}(f, b \bullet l) \equiv f(b) \bullet \mathsf{apply}(f, l),$$

which, supposing $l \in \mathsf{List}(B)$, is solved in type theory by putting

$$\mathsf{apply}(f,l) \equiv \mathsf{L}_{\mathsf{rec}}(l, \mathsf{nil}_A, (x:B)(y:\mathsf{List}(B))(z:\mathsf{List}(A))\ f(x) \bullet z).$$

Supposing $l \in \mathsf{List}(A)$, $P(x)$ prop $[x:A]$ and $R(x)$ prop $[x:\mathsf{List}(A)]$, we will use the following short-hands:

$$(\forall x \varepsilon l)\ P(x) \equiv (\forall x \in A)\ x \varepsilon l \to P(x),$$

$$(\exists x \varepsilon l)\ P(x) \equiv (\exists x \in A)\ x \varepsilon l\ \&\ P(x),$$

$$(\forall y \sqsubseteq l)\ R(y) \equiv (\forall y \in \mathsf{List}(A)) y \sqsubseteq l \to R(y),$$

$$(\exists y \sqsubseteq l)\ R(y) \equiv (\exists y \in \mathsf{List}(A)) y \sqsubseteq l\ \&\ R(y).$$

It is immediate to verify that the quantifiers so defined satisfy the usual intuitionistic rules of introduction and elimination for quantifiers (cf. [7]).

Moreover, supposing A set, $B(x)$ set $[x:A]$ and $c \in \Sigma(A,B)$, and recalling that $\mathsf{split}(\langle a,b\rangle, d) = d(a,b)$ prescribes the computational behaviour of the elimination constant split for the type $\Sigma(A,B)$ of the disjoint union of the family of sets $(B(x))_{x \in A}$ (see [5, p. 80]), we can set

$$\mathsf{fst}(c) \equiv \mathsf{split}(c, (x:A)(y:B(x))\ x),$$

$$\mathsf{snd}(c) \equiv \mathsf{split}(c, (x:A)(y:B(x))\ y)$$

in order to define, respectively, the first and the second projection for the elements of the set $\Sigma(A,B)$.

Finally, given a propositional function $F(x,y)$ prop $[x,y:S]$, we will need to consider the propositional function

$$C(n,F,x,y) \equiv (\exists z_1,\ldots,z_n \in S) F(x,z_1)\ \&\ \ldots\ \&\ F(z_n,y)$$

obtained by composition of the proposition F a certain number n of times. We can define it by induction on n if we work in a universe U which contains the propositional function F, provided we can solve the following equation:

$$C(0,F,x,y) = (x =^S y),$$

$$C(n+1,F,x,y) = (\exists z \in S) C(n,F,x,z)\ \&\ F(z,y).$$

To this aim we can solve first the equation

$$C'(0,F,x) = \lambda y.\ (x =^S y),$$

$$C'(n+1,F,x) = \lambda y.\ (\exists z \in S)\ \mathsf{Ap}(C'(n,F,x),z)\ \&\ F(z,y),$$

by putting

$$C'(n,F,x) \equiv \mathsf{R}_{\mathsf{Nat}}(n,$$
$$\lambda y.\ (x =^S y),$$
$$(u : \mathsf{Nat})(v : S \to \mathsf{U})\lambda y.\ (\exists z \in S)\ \mathsf{Ap}(v,z)\ \&\ F(z,y))$$

and then set

$$C(n,F,x,y) \equiv \mathsf{Ap}(C'(n,F,x),y).$$

We recall here also some properties of the type $\mathsf{Succ}(A)$, where A is any type, and of the type $S+T$ of the disjoint sum of the two types S and T that we will need in Sections 6.2 and 6.3 (see [5, pp. 103, 87]).

Let us suppose that A is any type; then the type $\mathsf{Succ}(A)$ is the type obtained by adding a new element to a copy of the type A. Its introduction rules are

$$1_{\mathsf{Succ}(A)} \in \mathsf{Succ}(A) \quad \frac{a \in A}{\mathsf{succ}(a) \in \mathsf{Succ}(A)}$$

and the elimination rule is

$$\frac{c \in \mathsf{Succ}(A) \quad d \in C(1_{\mathsf{Succ}(A)}) \quad e(x) \in C(\mathsf{succ}(x))\ [x : A]}{\mathsf{R}_{\mathsf{Succ}(A)}(c,d,e) \in C(c)}.$$

These rules allow to prove that any element of $\mathsf{Succ}(A)$ is equal to $1_{\mathsf{Succ}(A)}$ or to $\mathsf{succ}(a)$ for some $a \in A$, that is,

$$(\forall c \in \mathsf{Succ}(A))(c =^{\mathsf{Succ}(A)} 1_{\mathsf{Succ}(A)}) \vee (\exists a \in A)\ c =^{\mathsf{Succ}(A)} \mathsf{succ}(a)$$

and that

$$(\forall a \in A)\neg(1_{\mathsf{Succ}(A)} =^{\mathsf{Succ}(A)} \mathsf{succ}(a)).$$

The introduction rules for $S+T$ are

$$\frac{s \in S}{\mathsf{i}(s) \in S+T} \quad \frac{t \in T}{\mathsf{j}(t) \in S+T}$$

and the elimination rule is

$$\frac{c \in S+T \quad d(x) \in C(\mathsf{i}(x))\ [x : S] \quad e(y) \in C(\mathsf{j}(y))\ [y : T]}{\mathsf{D}(c,d,e) \in C(c)}.$$

In a way completely analogous to the previous case, these rules allow to prove that any element of $S+T$ is equal to $\mathsf{i}(s)$ for some element $s \in S$ or to $\mathsf{j}(t)$ for some $t \in T$, that is,

$$(\forall c \in S+T)(\exists s \in S)\ c =^{S+T} \mathsf{i}(s) \vee (\exists t \in T)\ c =^{S+T} \mathsf{j}(t)$$

and that

$$(\forall s \in S)(\forall t \in T)\neg(\mathsf{i}(s) =^{S+T} \mathsf{j}(t)).$$

3. Information base and translation

Information bases play the same role to present Scott domains than neighbourhood systems and information systems [9, 10] and it is even possible to show how to reconstruct the latter as suitable information bases [8]. Moreover, the definition of information base has an independent intuitive motivation, that has been inspired by the point-free approach to topology in [6] and that is discussed in detail in the appendix of [8].

An information base is a set S of tokens of information provided with an order relation $a \lhd_S b$ among tokens of information, whose intended meaning is that the information a is more precise than the information b, and a binary operation $\cdot_S$ of composition between tokens of information which respects such an order relation, that is, $a \lhd_S b$ and $c \lhd_S d$ yield $a \cdot_S c \lhd_S b \cdot_S d$. Moreover, a positivity predicate $\mathsf{Pos}_S(a)$ is defined on elements of S, meaning that the token of information a is consistent; the positivity predicate will play in Section 4.3 a main role in obtaining constructive proofs of the properties of the category of the information bases and it will be essential in Section 5 where we will give a constructive presentation of a generic (set-based) Scott domain. Here is the formal definition.

Definition 3.1 (*Information base*). An *information base* $\mathcal{S}$ is a structure

$$\langle S, \cdot_S, \Delta_S, \mathsf{Pos}_S, \lhd_S \rangle,$$

where S is a set, $\cdot_S$ is a binary operation between elements of S called *combination*, Δ_S is a distinguished element of S called *unit*, Pos_S is a property on elements of S called *positivity* predicate, and $\lhd_S$ is a binary relation between elements of S called *cover* relation, which satisfy the following conditions for all $a, b, c \in S$:

$$
\begin{array}{llll}
(\textit{properness}) & \mathsf{Pos}_S(\Delta_S) & & \\
(\textit{monotonicity}) & \dfrac{\mathsf{Pos}_S(a) \quad a \lhd_S b}{\mathsf{Pos}_S(b)}, & (\textit{positivity}) & \dfrac{\mathsf{Pos}_S(a) \to a \lhd_S b}{a \lhd_S b} \\
(\textit{unit}) & a \lhd_S \Delta_S, & & \\
(\textit{reflexivity}) & a \lhd_S a & (\textit{transitivity}) & \dfrac{a \lhd_S b \quad b \lhd_S c}{a \lhd_S c} \\
(\cdot\textit{-left}) & \dfrac{a \lhd_S b}{a \cdot_S c \lhd_S b} \; \dfrac{a \lhd_S b}{c \cdot_S a \lhd_S b} & (\cdot\textit{-right}) & \dfrac{a \lhd_S b \quad a \lhd_S c}{a \lhd_S b \cdot_S c}.
\end{array}
$$

In the following we are going to use some immediate consequences of the previous conditions. We will list them here.

1. $a =_S b \equiv (a \lhd_S b \ \& \ b \lhd_S a)$ is an equivalence relation.
2. (*stability*) If $a \lhd_S c$ and $b \lhd_S d$, then $a \cdot_S b \lhd_S c \cdot_S d$.
3. The equivalence relation $=_S$ respects the structure of the information base, that is,
   - if $a =_S b$, then $\mathsf{Pos}_S(a)$ if and only if $\mathsf{Pos}_S(b)$;
   - if $a =_S b$ and $c =_S d$, then $a \lhd_S c$ if and only if $b \lhd_S d$;
   - if $a =_S b$ and $c =_S d$, then $a \cdot_S c =_S b \cdot_S d$.
4. $\Delta_S \cdot_S a =_S a =_S a \cdot_S \Delta_S$.
5. $a \lhd_S b$ if and only if $a \cdot_S \Delta_S \lhd_S b$ if and only if $\Delta_S \cdot_S a \lhd_S b$.
6. If $\Delta_S \lhd_S a$ then $b \lhd_S a \cdot_S b$.
7. The structure $(S_{/=_S}, \Delta_S, \cdot_S)$ is a commutative idempotent monoid; note however that in general the quotient $S_{/=_S}$ is not a set (see [3]).

Information bases can be used to construct domains in a similar way to what can be done by using information or neighbourhood systems. In fact, an element of a domain, which is a partial information on an abstract topic, can be identified with the subset of all the tokens of information that inherit to it. In the case of information bases, due to the topological interpretation of the cover relation and the positivity predicate, we call *formal point* any such subset of tokens of information. From now on we will write $\mathsf{Pt}(\mathcal{S})$ to mean the collection of all formal points of $\mathcal{S}$ equipped with the inclusion ordering (for details see [8]). Note that $\mathsf{Pt}(\mathcal{S})$ is a collection of subsets of S and hence it is never a set. Of course, two formal points coincide when they are extensionally equal, that is, when they contain the same tokens of information (see [7] for a complete description of the treatment of subsets and their equality within Martin-Löf's intuitionistic type theory).

Definition 3.2 (*Formal point*). Let $\mathcal{S}$ be an information base. Then, a *formal point* α of $\mathcal{S}$ is a subset of S which satisfies the following conditions for all $a, b \in S$:

$$\text{(i.1)}\ \Delta_S \varepsilon \alpha, \quad \text{(i.2)}\ \frac{a\varepsilon\alpha \qquad b\varepsilon\alpha}{a \cdot b \varepsilon \alpha}, \quad \text{(i.3)}\ \frac{a\varepsilon\alpha \qquad a \lhd_S b}{b\varepsilon\alpha},$$

$$\text{(ii)}\ \frac{a\varepsilon\alpha}{\mathsf{Pos}_S(a)}.$$

In Section 5 we will show a formalization inside type theory of the main result in [8], that is, the fact that any Scott domain with a constructive presentation is (isomorphic to) the collection of the points of a suitable information base. The reader who is curious to see the role of $\lhd$ and Pos in this construction can jump immediately there and come back here later to continue with the presentation of the category InfBas.

Not only Scott domains can be completely re-constructed by using information bases, but also their morphisms, that is, approximable functions [10]. Here we use *translations*. A translation F between the information bases $\mathcal{S}$ and $\mathcal{T}$ is a propositional function xFy prop $[x : S, y : T]$ which links a token of information a of $\mathcal{S}$ with all the tokens b

in $\mathcal{T}$ inherited to a partial information which is the translation of a partial information in a. The formal conditions are the following.

Definition 3.3 (*Translation*). Let $\mathcal{S}$ and $\mathcal{T}$ be information bases. Then a propositional function F between $\mathcal{S}$ and $\mathcal{T}$ is called a *translation* if, for all $a,c \in S$ and $b,d \in T$:

$$\text{(i.1)}\ aF\Delta_T, \qquad \text{(i.2)}\ \frac{aFb \qquad aFd}{aFb\cdot d},$$
$$\text{(i.3)}\ \frac{aFb \qquad b \lhd_T d}{aFd}, \quad \text{(i.4)}\ \frac{\mathsf{Pos}_S(a) \qquad aFb}{\mathsf{Pos}_T(b)},$$
$$\text{(ii)}\ \frac{a \lhd_S c \qquad cFb}{aFb}, \quad \text{(iii)}\ \frac{\mathsf{Pos}_S(a) \to aFb}{aFb}.$$

As usual, we will write $F \in \mathsf{Hom}(\mathcal{S},\mathcal{T})$ to mean that F is a translation between $\mathcal{S}$ and $\mathcal{T}$.

In the following we will often use the fact that, for any translation F, if aFb and cFd, then $a\cdot cFb\cdot d$ because aFb yields $a\cdot cFb$ and cFd yields $a\cdot cFd$ by (ii) together with $\cdot$-*left* and hence $a\cdot cFb\cdot d$ by (i.2).

Two translations $F,G \in \mathsf{Hom}(\mathcal{S},\mathcal{T})$ have to be considered equal if the propositional functions F and G hold for the same elements of S and T, that is, if F and G are extensionally equal. Thus, we put

$$F = G \equiv (\forall x : S)(\forall y : T)\ xFy \leftrightarrow xGy.$$

Of course, when a morphism will be defined, it will be necessary to check that its definition respects equality among morphisms, that is, that it does not depend on the particular representatives.

Given two translations $F \in \mathsf{Hom}(\mathcal{S},\mathcal{T})$ and $G \in \mathsf{Hom}(\mathcal{T},\mathcal{U})$ their *composition* is defined by putting, for any $s \in S$ and $u \in U$:

$$s(G * F)u \equiv \mathsf{Pos}_S(s) \to (\exists t \in T)\ sFt\ \&\ tGu.$$

It is immediate to verify that composition of translations is well defined and associative.

The identical translation Id_S of the information base $\mathcal{S}$ is simply the covering relation $\lhd_S$; in fact, it is immediate to see that the conditions on the cover relation comprise all of the requirements for $\lhd_S$ to be a translation. Moreover, conditions (i.3) and (ii) in Definition 3.3 of translation allow to show that Id_S is indeed the unit of the operation of composition between translations.

Thus, we have shown that Information Bases and Translations [2] form a category, which we call InfBas. As we already observed, this category is equivalent to the category ScDom of Scott domains (for a detailed proof see [8]). The key point in the proof is to show that the map Pt is a functor between InfBas and ScDom. In fact, a translation $F : \mathcal{S} \to \mathcal{T}$ can be easily lifted to an approximable function from $\mathsf{Pt}(S)$

[2] More pedantically, we should use equivalence classes of translations.

into $\mathsf{Pt}(T)$ by mapping any point α into the union of all the tokens of information which are the translation of some element a of α. Formally,

$$\mathsf{Pt}(F)(\alpha) \equiv \bigcup\{Fa : a\varepsilon\alpha\},$$

where $Fa \equiv \{b : aFb\}$.

We will also use the fact that, given any approximable function f from $\mathsf{Pt}(\mathcal{S})$ into $\mathsf{Pt}(\mathcal{T})$, the propositional function

$$sF_f t \equiv \mathsf{Pos}_S(s) \rightarrow t\varepsilon f(\uparrow s)\ [s : S, t : T],$$

where $\uparrow s \equiv \{u \in S \mid s \lhd_S u\}$ is a point of $\mathcal{S}$ whenever s is positive, is a translation between $\mathcal{S}$ and $\mathcal{T}$ such that $f = \mathsf{Pt}(F_f)$.

4. InfBas is a cartesian closed category

Since ScDom and InfBas are equivalent categories and ScDom is a cartesian closed category, then InfBas is also cartesian closed. Anyhow the proof of such a categorical equivalence cannot be completely formalized within type theory, mainly because Scott domains cannot be formalized therein. Thus, we have no constructive proof that InfBas is indeed a cartesian closed category.

In this section we will show how a terminal object, a cartesian product and an exponential object of two information bases can be defined within type theory.

4.1. Terminal information bases

First, we define a terminal object and a cartesian products in InfBas and then we will construct an exponential object.

Theorem 4.1 (*Terminal objects in* InfBas). *Any information base $\mathcal{T}$ such that for any $t \in T$*

$$(*)\ \mathsf{Pos}_T(t) \textit{ iff } \Delta_T \lhd_T t$$

is a terminal object in InfBas, *that is, for any information base $\mathcal{S}$, the relation*

$$sRt \equiv \mathsf{Pos}_S(s) \rightarrow \mathsf{Pos}_T(t)$$

is the unique translation between $\mathcal{S}$ and $\mathcal{T}$.

Proof. The proof that R is a translation is straightforward. To verify its uniqueness, suppose $F \in \mathsf{Hom}(\mathcal{S}, \mathcal{T})$ and sRt, i.e. $\mathsf{Pos}_S(s) \rightarrow \mathsf{Pos}_T(t)$ or equivalently $\mathsf{Pos}_S(s) \rightarrow \Delta_T \lhd_T t$; then, by assuming $\mathsf{Pos}_S(s)$ we obtain $\Delta_T \lhd_T t$, but $sF\Delta_T$ holds and hence sFt follows by the conditions (i.3) and (iii) in the definition of translation; on the other hand supposing sFt and assuming $\mathsf{Pos}_S(s)$ we immediately obtain $\mathsf{Pos}_T(t)$. □

The easiest way to construct a terminal object $\mathbb{1}$ for the category InfBas within type theory is to use the one element set $\top$, whose only element is $*$, and to declare $*$ positive. We thus arrive at the following definitions:

$$\cdot_{\mathbb{1}} \equiv (x : \top)(y : \top)\ *,$$

$$\Delta_{\mathbb{1}} \equiv *,$$

$$\mathsf{Pos}_{\mathbb{1}} \equiv (x : \top)\ x =^{\top} x,$$

$$\lhd_{\mathbb{1}} \equiv (x : \top)(y : \top)\ x =^{\top} y.$$

Now the conditions *properness*, *monotonicity*, *positivity*, *reflexivity* and *transitivity* in Definition 3.1, which state that $\mathbb{1}$ is an information base, are easily verified by means of simple proofs within type theory. Moreover, the condition $(*)$ in Theorem 4.1 holds and hence $\mathbb{1}$ is a terminal object.

Note that the collection of the points of any terminal object has exactly one element. In fact, let us suppose that $\mathscr{T}$ is a terminal information base, that α and β are two elements of $\mathsf{Pt}(T)$ and that $a\varepsilon\alpha$; then $\mathsf{Pos}(a)$ holds and hence $\Delta_T \lhd_T a$, since $\mathscr{T}$ is a terminal information base; thus, $a\varepsilon\beta$ since $\Delta_T\varepsilon\beta$ because β is a point.

Finally, it is worth noting that there is a bijective correspondence between the translations from a terminal object to any information base and the points of such an information base. In fact, supposing $\mathscr{T}$ is a terminal information base, $\mathscr{S}$ is any information base and F is any translation between $\mathscr{T}$ and $\mathscr{S}$, we can associate to F the point

$$\alpha_F \equiv \{b \in S \mid \Delta_T F b\}$$

of $\mathscr{S}$. On the other hand, supposing α is any point of $\mathscr{S}$, we can associate it the translation

$$aF_\alpha b \equiv \mathsf{Pos}_T(a) \to b\varepsilon\alpha$$

and the correspondence is obviously bijective since

$$b\varepsilon\alpha_{F_\alpha} \text{ iff } \Delta_T F_\alpha b \text{ iff } \mathsf{Pos}_T(\Delta_T) \to b\varepsilon\alpha \text{ iff } b\varepsilon\alpha$$

and

$$aF_{\alpha_F} b \text{ iff } \mathsf{Pos}_T(a) \to b\varepsilon\alpha_F \text{ iff } \mathsf{Pos}_T(a) \to \Delta_T F b \text{ iff } \mathsf{Pos}_T(a) \to aFb \text{ iff } aFb,$$

where the third step is a consequence of the fact that if $\mathsf{Pos}_T(a)$ holds then $\Delta_T =_T a$.

4.2. Cartesian product of information bases

To define the cartesian product of information bases we will follow a hint from standard topology: a base for the product topology of two topological spaces is the cartesian product of the bases of the two topologies.

Proposition 4.2 (Cartesian product of information bases). *Let $\mathscr{S}$ and $\mathscr{T}$ be two information bases. Then,*

$$\mathscr{S} \times \mathscr{T} \equiv \langle S \times T, \cdot_{S\times T}, \Delta_{S\times T}, \mathsf{Pos}_{S\times T}, \lhd_{S\times T}\rangle,$$

where for any $c, d \in S \times T$:

$$c \cdot_{S\times T} d \equiv (\mathsf{fst}(c) \cdot_S \mathsf{fst}(d), \mathsf{snd}(c) \cdot_T \mathsf{snd}(d)),$$

$$\Delta_{S\times T} \equiv (\Delta_S, \Delta_T),$$

$$\mathsf{Pos}_{S\times T}(c) \equiv \mathsf{Pos}_S(\mathsf{fst}(c)) \ \& \ \mathsf{Pos}_T(\mathsf{snd}(c)),$$

$$c \lhd d \equiv \mathsf{Pos}_{S\times T}(c) \to (\mathsf{fst}(c) \lhd_S \mathsf{fst}(d) \ \& \ \mathsf{snd}(c) \lhd_T \mathsf{snd}(d))$$

is an information base.

Proof. All the verifications are straightforward proofs in type theory which use the rules for the type theoretic cartesian product ([5, p. 81]). It can be useful to observe that to prove the validity of $\cdot$-left and $\cdot$-right one has to use the fact that, for any $a, b \in S \times T$, $\mathsf{fst}(a \cdot_{S\times T} b) =_{\mathscr{S}} \mathsf{fst}(a) \cdot_S \mathsf{fst}(b)$ and $\mathsf{snd}(a \cdot_{S\times T} b) =_{\mathscr{T}} \mathsf{snd}(a) \cdot_T \mathsf{snd}(b)$. $\square$

Now, supposing $\mathscr{S}$ and $\mathscr{T}$ are two information bases, it is possible to show that $\mathscr{S} \times \mathscr{T}$ is their cartesian product.

Theorem 4.3. *Let $\mathscr{S}$ and $\mathscr{T}$ be two information bases. Then the propositional functions Π_S between $\mathscr{S} \times \mathscr{T}$ and $\mathscr{S}$ and Π_T between $\mathscr{S} \times \mathscr{T}$ and $\mathscr{T}$ defined by putting, for any $c \in S \times T$, $s \in S$ and $t \in T$:*

$$c\, \Pi_S\, s \ \equiv\ \mathsf{Pos}_{S\times T}(c) \to (\mathsf{fst}(c) \lhd_S s),$$

$$c\, \Pi_T\, t \ \equiv\ \mathsf{Pos}_{S\times T}(c) \to (\mathsf{snd}(c) \lhd_T t)$$

are translations. Moreover, if $\mathscr{W}$ is an information base, $F \in \mathsf{Hom}(\mathscr{W}, \mathscr{S})$ and $G \in \mathsf{Hom}(\mathscr{W}, \mathscr{T})$, then the propositional function $\langle F, G\rangle$ between $\mathscr{W}$ and $\mathscr{S} \times \mathscr{T}$ defined by putting, for any $c \in S \times T$ and $w \in W$:

$$w\, \langle F, G\rangle\, c \equiv \mathsf{Pos}_W(w) \to (w\, F\, \mathsf{fst}(c) \,\&\, w\, G\, \mathsf{snd}(c))$$

is a translation and, for any translation $H \in \mathsf{Hom}(\mathscr{W}, \mathscr{S} \times \mathscr{T})$, the following equations hold:

$$\Pi_S * \langle F, G\rangle = F,$$

$$\Pi_T * \langle F, G\rangle = G,$$

$$\langle \Pi_S * H, \Pi_T * H\rangle = H.$$

Proof. It is easy to see that Π_S, Π_T and $\langle F, G\rangle$ are translations. To prove the validity of the first equation note that if $w\, F\, s$ then $w\, \langle F, G\rangle\, (s, \Delta_T)$, since $w\ G\ \Delta_T$ holds, and hence $w\ \Pi_S * \langle F, G\rangle\, s$, since $(s, \Delta_T)\, \Pi_S\, s$.

A formal proof in type theory of the other inclusion is the following: suppose that $w\,\Pi_S * \langle F,G\rangle\, s$, that is, $\mathsf{Pos}_W(w) \to (\exists c \in S\times T)\, w\,\langle F,G\rangle\, c \,\&\, c\,\Pi_S\, s$, and assume $\mathsf{Pos}_W(w)$; then $(\exists c \in S\times T)\, w\,\langle F,G\rangle\, c \,\&\, c\,\Pi_S\, s$. Now from $w\,\langle F,G\rangle\, c$, that is, $\mathsf{Pos}_W(w) \to w\,F\,\mathsf{fst}(c) \,\&\, w\,G\,\mathsf{snd}(c)$, by using again the assumption $\mathsf{Pos}_W(w)$, we deduce both $w\,F\,\mathsf{fst}(c)$ and $w\,G\,\mathsf{snd}(c)$, which, by using for the third time the assumption $\mathsf{Pos}_W(w)$, show that $\mathsf{Pos}_S(\mathsf{fst}(c))$ and $\mathsf{Pos}_T(\mathsf{snd}(c))$, that is, $\mathsf{Pos}_{S\times T}(c)$, which allows to conclude $\mathsf{fst}(c) \lhd_S s$ from $\mathsf{Pos}_{S\times T}(c) \to \mathsf{fst}(c) \lhd_S s$, that is, $c\,\Pi_S\, s$. Thus, $w\,F\,s$ follows from $w\,F\,\mathsf{fst}(c)$ and $\mathsf{fst}(c) \lhd_S s$ and hence the result is obtained by $\exists$-elimination and condition (iii) in Definition 3.3 of translation. The proof of validity of the second equation is completely similar.

To prove the validity of the third equation suppose $w\,\langle \Pi_S * H, \Pi_T * H\rangle\, c$ and assume $\mathsf{Pos}_W(w)$. Then it is straightforward to prove that there exists $d \in S\times T$ such that $w\,H\,d \,\&\, d\,\Pi_S\, \mathsf{fst}(c)$; but, by assuming $\mathsf{Pos}_{S\times T}(d)$, $d\ \Pi_S\ \mathsf{fst}(c)$ implies $\mathsf{fst}(d) \lhd_S \mathsf{fst}(c)$ which shows $d \lhd_{S\times T} (\mathsf{fst}(c), \Delta_T)$, by discharging the assumption $\mathsf{Pos}_{S\times T}(d)$, since $\mathsf{snd}(d) \lhd_T \Delta_T$ holds; thus $w\ H\ d$ allows to deduce $w\ H(\mathsf{fst}(c), \Delta_T)$ and $\exists$-elimination can be applied. In a similar way one can prove that also $w\ H(\Delta_S, \mathsf{snd}(c))$ holds and hence $w\ H\ c$ follows by conditions (i.2) and (iii) of Definition 3.3 of translation, since $(\mathsf{fst}(c), \Delta_T) \cdot_{S\times T} (\Delta_S, \mathsf{snd}(c)) \lhd_{S\times T} c$ holds; the other inclusion is trivial. $\square$

In the following, supposing $F \in \mathsf{Hom}(\mathscr{S}, \mathscr{W})$ and $G \in \mathsf{Hom}(\mathscr{T}, \mathscr{Z})$, we will write $F \times G$ to mean the translation $\langle F * \Pi_S, G * \Pi_T\rangle$ from $\mathscr{S}\times\mathscr{T}$ to $\mathscr{W}\times\mathscr{Z}$.

Even if the collection of points of an information base is never a set, since its elements are subsets, and hence we cannot define over it standard set operations like cartesian product, we can still show that there is a bijective correspondence between $\mathsf{Pt}(S\times T)$ and couple made by elements in $\mathsf{Pt}(S)$ and $\mathsf{Pt}(T)$. In fact, let γ be a point of $\mathscr{S}\times\mathscr{T}$; then we obtain a point of $\mathscr{S}$ and a point of $\mathscr{T}$ by setting

$$\alpha_\gamma \equiv \{a \in S \mid (a, \Delta_T)\varepsilon\gamma\},$$
$$\beta_\gamma \equiv \{b \in T \mid (\Delta_S, b)\varepsilon\gamma\}.$$

Moreover, supposing α is a point of $\mathscr{S}$ and β is a point of $\mathscr{T}$ we obtain a point of $\mathscr{S}\times\mathscr{T}$ by setting

$$\gamma_{\alpha,\beta} \equiv \{(a,b) \in S\times T \mid a\varepsilon\alpha \text{ and } b\varepsilon\beta\}.$$

Finally, the correspondence is clearly bijective; in fact, it is easy to see that $(a,b)\varepsilon\gamma_{\alpha_\gamma,\beta_\gamma}$ if and only if $(a,b)\varepsilon\gamma$; moreover, $a\varepsilon\alpha_{\gamma_{\alpha,\beta}}$ and $b\varepsilon\beta_{\gamma_{\alpha,\beta}}$ if and only if $a\varepsilon\alpha$ and $b\varepsilon\beta$ because $(a, \Delta_T) \cdot_{S\times T} (\Delta_S, b) =_{S\times T} (a,b)$.

4.3. Exponential of two information bases

The basic idea in constructing the exponential object of two information bases is to explain, by using only finite tokens of information, how a translation is defined. From a constructive point of view, this is not straightforward since a translation is just a propositional function, and we know it only intensionally. But, from a classical point of view, we can see it also extensionally, that is, like the collection of all the couples which satisfy such a propositional function. Hence a finite information on a translation is just a finite set of couples. The natural operation between two such finite sets of tokens of information is union, which collects the information on the translation contained in the two finite sets. Clearly, the unit for this operation is the empty-set which adds no information. If we want to remain within type theory, two problems arise in following this approach. First, the collection of finite subsets of a set is not a set because we cannot generate it by means of an inductive definition but some additional equations are needed (see for instance [1]). Moreover, a translation has to satisfy the positivity condition (iii) of Definition 3.3 of translation and hence any notion of *function space* has to take into account this fact. We solve these two problems by constructing the function space of two information bases $\mathscr{S}$ and $\mathscr{T}$ by using, instead of finite subsets, lists of couples whose first element is a *positive* element of S and second element is an element of T. Thus, we arrive at the following proposition where we use the set theoretic abbreviations that we introduced in Section 2.

Proposition 4.4 (Exponential of information bases). *Let $\mathscr{S}$ and $\mathscr{T}$ be information bases. Then the structure*

$$\mathscr{S} \Rightarrow \mathscr{T} \equiv \langle \mathscr{P}_{\omega}(\Sigma(S, \mathsf{Pos}_S) \times T), \sqcup, \emptyset, \mathsf{Pos}_{S\Rightarrow T}, \lhd_{S\Rightarrow T} \rangle,$$

where for any $l, m \in \mathscr{P}_{\omega}(\Sigma(S, \mathsf{Pos}_S) \times T)$:

$$\mathsf{Pos}_{S\Rightarrow T}(l) \equiv (\forall y \sqsubseteq l)\ \mathsf{Pos}_S(\odot_{\lambda x.\mathsf{fst}(\mathsf{fst}(x))} y) \rightarrow \mathsf{Pos}_T(\odot_{\lambda x.\mathsf{snd}(x)} y)$$

and

$$\begin{aligned} l \lhd_{S\Rightarrow T} m \equiv\ & \mathsf{Pos}_{S\Rightarrow T}(l) \rightarrow \\ & (\forall x \sqsubseteq m)(\exists y \sqsubseteq l)\ \ \odot_{\lambda x.\mathsf{fst}(\mathsf{fst}(x))}\, x\ \lhd_S \odot_{\lambda x.\mathsf{fst}(\mathsf{fst}(x))} y\ \ \& \\ & \qquad\qquad \odot_{\lambda x.\mathsf{snd}(x)}\, y \lhd_T \odot_{\lambda x.\mathsf{snd}(x)} x) \end{aligned}$$

is an information base.

The formal proof of this proposition is long and it is convenient to begin with some abbreviations and some lemmas. In the following we will abbreviate the set $\Sigma(S, \mathsf{Pos}_S) \times T$ by $\mathsf{Pos}_S \times T$ and, whenever it will be possible, we will indicate one of its elements by (s, t) instead that by $((s, \pi), t)$, where π is the proof that s is a positive element of S. Moreover, the set $\mathscr{P}_{\omega}(\Sigma(S, \mathsf{Pos}_S) \times T)$ will be abbreviated by $S \Rightarrow T$ and, for any $x \in \mathsf{Pos}_S \times T$, the element $\mathsf{fst}(\mathsf{fst}(x))$ of S will be abbreviated by x_S and the element $\mathsf{snd}(x)$ of T by x_T and hence the functions $\odot_{\lambda x.\mathsf{fst}(\mathsf{fst}(x))}$ and $\odot_{\lambda x.\mathsf{snd}(x)}$ will be abbreviated by $\odot_S$ and $\odot_T$, respectively. Thus, we will write $\mathsf{Pos}_{S\Rightarrow T}(l)$ as

$$(\forall y \sqsubseteq l)\ \mathsf{Pos}_S(\odot_S y) \rightarrow \mathsf{Pos}_T(\odot_T y)$$

and $l \lhd_{S\Rightarrow T} m$ as

$$\mathsf{Pos}_{S\Rightarrow T}(l) \to (\forall x \sqsubseteq m)(\exists y \sqsubseteq l)\ \odot_S x\ \lhd_S \odot_S y\ \&\ \odot_T y \lhd_T \odot_T x,$$

which look a bit more readable.

Let us now show that $\mathscr{S} \Rightarrow \mathscr{T}$ is indeed an information base.

Lemma 4.5. *Let $l, m \in S \Rightarrow T$. Then, if $l \sqsubseteq m$ then*

$$\odot_S m \lhd_S \odot_S l \quad \text{and} \quad \odot_T m \lhd_T \odot_T l.$$

Proof. The proof is by induction on the length of the list l and it is obtained by using $\cdot$-*left* on the information bases $\mathscr{S}$ and $\mathscr{T}$. □

Corollary 4.6. *Let $l \in S \Rightarrow T$. Then, if $l \sqsubseteq \emptyset$ then $\mathsf{Pos}_{S\Rightarrow T}(l)$.*

Proof. Suppose $y \sqsubseteq l$ and assume $l \sqsubseteq \emptyset$. Then $y \sqsubseteq \emptyset$. Hence, by the previous lemma, $\odot_T \emptyset \lhd_T \odot_T y$, but $\mathsf{Pos}_T(\odot_T \emptyset)$ holds since $\odot_T \emptyset = \Delta_T$, and hence by *monotonicity* in $\mathscr{T}$ we obtain $\mathsf{Pos}_T(\odot_T y)$ and thus the result follows immediately by logic. □

Lemma 4.7. *Let $l, m \in S \Rightarrow T$. Then, if $m \sqsubseteq l$ then $l \lhd_{S\Rightarrow T} m$.*

Proof. Let us suppose that $x \sqsubseteq m$; then the assumption $m \sqsubseteq l$ implies that $x \sqsubseteq l$ and hence we have found the “subset” of l we were looking for since obviously $\odot_S x \lhd_S \odot_S x$ and $\odot_T x \lhd_T \odot_T x$. □

We introduce now the new relation

$$l \lhd^1 m \equiv \mathrm{Pos}_{S\Rightarrow T}(l) \to (\forall x \varepsilon m)(\exists y \sqsubseteq l)\ x_S \lhd_S \odot_S y\ \&\ \odot_T y \lhd_T x_T.$$

We will prove that $\lhd^1$ is equivalent to the relation $\lhd_{S\Rightarrow T}$. We need to introduce $\lhd^1$ in order to show the validity of the $\cdot$-*right* condition for $\lhd_{S\Rightarrow T}$.

It is obvious that if $l \lhd_{S\Rightarrow T} m$ then $l \lhd^1 m$. In fact, supposing $x \varepsilon m$, by Lemma 2.1, we obtain $\{x\} \sqsubseteq m$ and hence the result is an immediate consequence of $l \lhd_{S\Rightarrow T} m$.

To prove the other implication we need to use one of the strongest property of constructive type theory, namely, the (extended) axiom of choice.

Lemma 4.8. *Let $l, m \in S \Rightarrow T$. Then, $l \lhd^1 m$ if and only if there exists a function f from $\Sigma(\mathsf{Pos}_S \times T, (x : \mathsf{Pos}_S \times T)\ x\varepsilon m)$ into $(S \Rightarrow T)$ such that, for all $x \in \Sigma(\mathsf{Pos}_S \times T, (x : \mathsf{Pos}_S \times T)\ x\varepsilon m)$,*

$$(f(x) \sqsubseteq l)\ \&\ (x_S \lhd_S \odot_S f(x))\ \&\ (\odot_T f(x) \lhd_T x_T).$$

Proof. After all the definitions are eliminated, the result is an immediate consequence of the application of the (extended) axiom of choice which asserts that

$$(\forall x : A)\ B(x) \to ((\exists y : C)\ D(x, y))$$

holds if and only if

$$(\exists f : \Sigma(A,B) \to C)(\forall z : \Sigma(A,B))\ D(\mathsf{fst}(z), f(z))$$

holds. Its proof in constructive type theory is a slight modification of the standard proof of the axiom of choice in [4]. In fact, supposing

$$h : (\forall x : A)\ B(x) \to ((\exists y : C)\ D(x,y)),$$

the choice function that we are looking for is

$$f \equiv \lambda z : \Sigma(A,B).\ h(\mathsf{fst}(z))(\mathsf{snd}(z)) \qquad \square$$

We can now finish the proof of the equivalence between the two relations $\lhd^1$ and $\lhd_{S\Rightarrow T}$.

Lemma 4.9. *Let $l,m \in S \Rightarrow T$. Then, if $l \lhd^1 m$ then $l \lhd_{S\Rightarrow T} m$.*

Proof. After Lemma 4.8, given any $z \sqsubseteq m$ we can use the choice function f to construct the "subset" $\bigsqcup_{x\varepsilon z} f(x)$ of l which satisfies the required conditions. $\square$

We can now verify that $\mathscr{S} \Rightarrow \mathscr{T}$ is an information base. Most of the necessary check are straightforward. Here, we only show the non-obvious cases.

- (*Monotonicity*) We have to show that if $\mathsf{Pos}_{S\Rightarrow T}(l)$ and $l \lhd_{S\Rightarrow T} m$ then $\mathsf{Pos}_{S\Rightarrow T}(m)$. Thus, let us suppose that $z \in S \Rightarrow T$, $z \sqsubseteq m$ and $\mathsf{Pos}_S(\odot_S z)$, then there exists $y \sqsubseteq l$ such that $\odot_S z \lhd_S \odot_S y$ and $\odot_T y \lhd_T \odot_T z$ since $l \lhd_{S\Rightarrow T} m$; hence $\mathsf{Pos}_S(\odot_S y)$ holds by *monotonicity* in $\mathscr{S}$; but $\mathsf{Pos}_{S\Rightarrow T}(l)$ implies $\mathsf{Pos}_S(\odot_S y) \to \mathsf{Pos}_T(\odot_T y)$, since $y \sqsubseteq l$, and so, by *monotonicity* in $\mathscr{T}$, $\mathsf{Pos}_T(\odot_T z)$.
- ($\cdot$-*right*) We have to show that if $l \lhd_{S\Rightarrow T} m$ and $l \lhd_{S\Rightarrow T} n$ hold then $l \lhd_{S\Rightarrow T} m \sqcup n$. The assumptions yield that $l \lhd^1 m$ and $l \lhd^1 n$. Suppose now that $x\varepsilon m \sqcup n$, then $x\varepsilon m$ or $x\varepsilon n$ and in both cases we can obtain the "subset" of l required to state $l \lhd^1 m \sqcup n$ by using the suitable assumption. But then $l \lhd_{S\Rightarrow T} m \sqcup n$ follows by Lemma 4.9.

It is worth noting that, in order to prove the validity of the $\cdot$-*right* condition for the exponential information base, we needed to consider the relation $\lhd^1$ instead of $\lhd_{S\Rightarrow T}$. In fact, if $x\varepsilon m \sqcup n$ we can prove that $x\varepsilon m$ or $x\varepsilon n$ but if we know that $y \sqsubseteq m \sqcup n$ we are not able to construct two "subsets" y_1 and y_2 such that $y_1 \sqsubseteq m$, $y_2 \sqsubseteq n$ and $y \cong y_1 \sqcup y_2$, unless the equality relation on $\mathsf{Pos}_S \times T$ is decidable.

After the previous results we can propose simple and intuitive explanations of the definitions we used for the positivity predicate and the cover relation for the exponential information base. Let us give first a definition.

Definition 4.10. Let R be a translation between the information bases $\mathscr{S}$ and $\mathscr{T}$ and l be a token of information in $S \Rightarrow T$. Then, we say that the translation R *contains* l if and only if $(\forall x\varepsilon l)\ x_S R x_T$.

We can prove the following theorems.

Lemma 4.11. *Let R be a translation between $\mathscr{S}$ and $\mathscr{T}$ and $l \in S \Rightarrow T$. Then, R contains l if and only if* $(\forall y \sqsubseteq l) \odot_S y R \odot_T y$.

Proof. Let us assume that R contains l and that $y \sqsubseteq l$. Then, for any $x \varepsilon y$, $x_S R x_T$ and hence $\odot_S y R \odot_T y$. On the other hand, for any $x \varepsilon l$, $\{x\} \sqsubseteq l$ and hence $(\forall y \sqsubseteq l) \odot_S y R \odot_T y$ yields $x_S =_S \odot_S \{x\} R \odot_T \{x\} =_T x_T$. $\square$

Theorem 4.12. *For any $l \in S \Rightarrow T$, $\mathsf{Pos}_{S \Rightarrow T}(l)$ holds if and only if there exists a translation R between $\mathscr{S}$ and $\mathscr{T}$ which contains l.*

Proof. Let us suppose that $\mathsf{Pos}_{S \Rightarrow T}(l)$ and define

$$sR_l t \equiv \mathsf{Pos}_S(s) \to (\exists y \sqsubseteq l)\, s \lhd_S \odot_S y \;\&\; \odot_T y \lhd_S t$$

Then, it is immediate to see that R_l is a translation. In fact, most of the cases that one has to check are straightforward; we will show here the only one which requires l to be a positive element of $\mathscr{S} \Rightarrow \mathscr{T}$, namely (i.4).

- If $\mathsf{Pos}_S(s)$ and $sR_l t$ then $\mathsf{Pos}_T(t)$. In fact, let us suppose $\mathsf{Pos}_{S \Rightarrow T}(l)$. Then $(\forall y \sqsubseteq l)$ $\mathsf{Pos}_S(\odot_S y) \to \mathsf{Pos}_T(\odot_T y)$, and hence, supposing, $y \sqsubseteq l$, $s \lhd_S \odot_S y$ and $\odot_T y \lhd_T t$, and $\mathsf{Pos}_S(s)$ we obtain first $\mathsf{Pos}_S(\odot_S y)$, by *monotonicity* in $\mathscr{S}$, and hence $\mathsf{Pos}_T(\odot_T y)$, by positivity of l, and finally $\mathsf{Pos}_T(t)$, by *monotonicity* in $\mathscr{T}$.

Moreover, Lemma 4.11 immediately yields that R_l contains l because, for any $y \sqsubseteq l$, $\odot_S y R_l \odot_T y$.

The other implication, that is, if there exists a translation R which contains l then l is positive, is immediate since supposing $y \sqsubseteq l$ we obtain $\odot_S y R \odot_T y$ by Lemma 4.11 and hence $\mathsf{Pos}_S(\odot_S y)$ yields $\mathsf{Pos}_T(\odot_T y)$ by condition (i.4) for the translation R. $\square$

It is interesting to note that the translation R_l that we defined in the proof of the previous theorem is the minimal translation which contains l, that is, for any translation R which contains l, if $sR_l t$ then sRt. In fact, we can prove the following lemma.

Lemma 4.13. *Let l be a positive element in $\mathscr{S} \Rightarrow \mathscr{T}$ and define R_l like in the proof of the previous theorem. Then R_l is contained in any translation which contains l.*

Proof. Let R be any translation which contains l and suppose $sR_l t$. Then, there exists $y \sqsubseteq l$ such that $s \lhd_S \odot_S y$ and $\odot_T y \lhd_T t$; then $\odot_S y R \odot_T y$ by Lemma 4.11 and hence sRt by conditions (i.3) and (ii) for the translation R. $\square$

We can exhibit an alternative characterization for the cover relation too.

Theorem 4.14. *For any $l, m \in S \Rightarrow T$, $l \lhd_{S \Rightarrow T} m$ if and only if any translation which contains l contains m too.*

Proof. Let us assume that $l \lhd_{S \Rightarrow T} m$ and that R is a translation between $\mathscr{S}$ and $\mathscr{T}$ which contains l. Then l is positive by Theorem 4.12. Then, for any $x \sqsubseteq m$, there exists $y \sqsubseteq l$ such that $\odot_S yR \odot_T y$, by Lemma 4.11, and $\odot_S x \lhd_S \odot_S y$ and $\odot_T y \lhd_T \odot_T x$, by definition of the cover relation in $\mathscr{S} \Rightarrow \mathscr{T}$. Then $\odot_S xR \odot_T x$ by conditions (i.3) and (ii) for the translation R and hence R contains m by Lemma 4.11.

On the other hand, if all translations contain m whenever they contain l then we can prove that $l \lhd^1 m$ holds, and hence $l \lhd_{S \Rightarrow T} m$ follows by Lemma 4.9. In fact, let us suppose that $x \varepsilon m$ and assume that $\mathsf{Pos}_{S \Rightarrow T}(l)$. Then a translation R_l can be defined as in the proof of Theorem 4.12 and it contains l; hence, by the assumption, it also contains m, so that $x_S\, R_l x_T$ which yields $\mathsf{Pos}_S(x_S) \to (\exists y \sqsubseteq l)\ x_S \lhd_S \odot_S y\ \&\ \odot_T y \lhd_T x_T$. But $\mathsf{Pos}_S(x_S)$ holds because x is an element in $\mathsf{Pos}_S \times T$. $\square$

Thus, our definitions of the positivity predicate and the cover relation are just a fully constructive way to express the more perspicous conditions that in Theorems 4.12 and 4.14 we proved to be equivalent to them. The reason we could not use these conditions directly in the definitions of the positivity predicate and the cover relation is that they cannot be expressed in a constructive way since they would require an existential quantification (in the case of the positivity predicate) or an universal quantification (in the case of the cover relation) over the *collection* of all the translations and such quantifications are meaningless since only quantification over the elements of a set can be given a constructive meaning.

The following theorem completely characterizes $\mathscr{S} \Rightarrow \mathscr{T}$ as the categorical exponential object of the information bases $\mathscr{S}$ and $\mathscr{T}$.

Theorem 4.15. *Let* $G \in \mathsf{Hom}(W \times S, T)$ *and* $H \in \mathsf{Hom}(W, S \Rightarrow T)$; *then there exist a unique translation* $\Lambda(G) \in \mathsf{Hom}(W, S \Rightarrow T)$ *and a translation* $\mathsf{Ap} \in \mathsf{Hom}((S \Rightarrow T) \times S, T)$ *such that the following equations hold*:

$$\mathsf{Ap} * (\Lambda(G) \times \mathsf{Id}_S) = G,$$

$$\Lambda(\mathsf{Ap} * (H \times \mathsf{Id}_S)) = H.$$

Proof. For any $w \in W$ and $l \in S \Rightarrow T$, put

$$w\Lambda(G)l \equiv \mathsf{Pos}_W(w) \to (\forall c \varepsilon l)(w, c_S)\, G\, c_T$$

and, for any $l \in S \Rightarrow T$, $s \in S$ and $t \in T$, put

$$(l,s)\ \mathsf{Ap}\ t \equiv (\forall y \in \mathsf{Pos}_{(\mathscr{S} \Rightarrow \mathscr{T}) \times \mathscr{S}}((l,s)))\ l \lhd_{\mathscr{S} \Rightarrow \mathscr{T}} \{((s, \mathsf{snd}(y)), t)\}.$$

It is easy to check that $\Lambda(G)$ and Ap are indeed translations.

A bit more complex is to show that the two equations hold. We will first prove that, for any $G \in \mathsf{Hom}(W \times S, T)$, $\mathsf{Ap} * (\Lambda(G) \times \mathsf{Id}_S) = G$. Let us suppose that $(w,s) \in W \times S$ and $t \in T$ and assume that $\mathsf{Pos}_{W \times S}((w,s))$; then, if $(w,s)\ \mathsf{Ap} * (\Lambda(G) \times \mathsf{Id}_S)\ t$ then there exists $(l,u) \in (S \Rightarrow T) \times S$ such that $(w,s)\ \Lambda(G) \times \mathsf{Id}_S\ (l,u)$, that is, $w\,\Lambda(G) l$

and $s \lhd_S u$, and $(l,u)\,\mathsf{Ap}\,t$. Then $\mathsf{Pos}_{S\Rightarrow T}(l)$ and $\mathsf{Pos}_S(u)$ and hence $(l,u)\,\mathsf{Ap}\,t$ yields $l \lhd_{S\Rightarrow T} \{((u,\pi),t)\}$, where π is the proof that u is a positive element of S. Thus there exists $y \sqsubseteq l$ such that $u \lhd_S \odot_S y$, and hence both $s \lhd_S \odot_S y$, because $s \lhd_S u$, and $\odot_T y \lhd_T t$ hold. Now, observe that $w\,\Lambda(G)\,l$ means that $(\forall c\varepsilon l)\ (w,c_S)\,G\,c_T$ and hence $(w,\odot_S y)\,G\odot_T y$ which yields $(w,s)\ G\,t$ since $(w,s)\lhd_{W\times S}(w,\odot_S y)$ and $\odot_T y \lhd_T t$.

To prove the other inclusion let us suppose that $(w,s)\ G\ t$ holds; then we immediately obtain that $w\ \Lambda(G)\ \{((s,\pi),t\}$, where π is the proof that s is positive. But we also have that $s \lhd_S s$, i.e. $s\ \mathsf{Id}_S\,s$, and $(\{((s,\pi),t)\},s)\ \mathsf{Ap}\,t$ since $(\forall y \in \mathsf{Pos}_{(S\Rightarrow T)\times S}((\{((s,\pi),t)\},s))\ \{((s,\pi),t)\} \lhd_{S\Rightarrow T} \{((s,\mathsf{snd}(y)),t)\}$.

Let us suppose now that $H \in \mathsf{Hom}(W, S\Rightarrow T)$. Then $\Lambda(\mathsf{Ap} * (H\times \mathsf{Id}_S)) = H$. In fact, supposing $w\in W$, $l \in S\Rightarrow T$ and $\mathsf{Pos}_W(w)$, $w\ \ \Lambda(\mathsf{Ap} * (H\times\mathsf{Id}_S))l$ yields $(\forall c\varepsilon l)\ (w,c_S)\ \mathsf{Ap} * (H\times \mathsf{Id}_S)c_T$ and hence there exist $m\in S\Rightarrow T$ and $u \in S$ such that $w\,H\,m$, $c_S \lhd_S u$ and $(m,u)\ \mathsf{Ap}\,c_T$. But, $(m,u)\ \mathsf{Ap}\,c_T$ yields $m \lhd_{S\Rightarrow T} \{((u,\pi),c_T\}$, where π is the proof that u is a positive element of S obtained by *monotonicity* from $c_S \lhd_S u$ since c_S is a positive element of S because $c\varepsilon l$ and $l \in \mathsf{Pos}_S \times T$. Moreover, $c_S \lhd_S u$ yields $\{((u,\pi),c_T\} \lhd_{S\Rightarrow} \{c\}$, and hence $m \lhd_{S\Rightarrow T} \{c\}$. Thus, for all $c\varepsilon l$, $w\,H\,\{c\}$ and hence $w\,H\,l$.

On the other hand, if $w\,H\,l$, then for any $c\varepsilon l$, $w\,H\,\{c\}$ since $l \lhd_{S\Rightarrow_T} \{c\}$ holds. Then $(w,c_S)\ H\times \mathsf{Id}_S\,(\{c\},c_S)$. Moreover, $(\{c\},c_S)\,\mathsf{Ap}\,c_T$ and hence $(w,c_S)\,\mathsf{Ap} * (H\times\mathsf{Id}_S)\ c_T$, that is, we proved that $w\,\Lambda(\mathsf{Ap} * (H\times \mathsf{Id}_S))l$. $\square$

After the categorical characterization of the exponential object it can be useful to see more directly the relation between the information base $\mathscr{S}\Rightarrow\mathscr{T}$ and the translations between $\mathscr{S}$ and $\mathscr{T}$. In fact, a full information in $\mathscr{S}\Rightarrow\mathscr{T}$ is not a token but a point. And indeed we can prove that there is a bijective correspondence between points of $\mathscr{S}\Rightarrow\mathscr{T}$ and translations between $\mathscr{S}$ and $\mathscr{T}$. We need first a technical lemma.

Lemma 4.16. *Let $\mathscr{S}$ and $\mathscr{T}$ be two information bases, l be an element in $S\Rightarrow T$ and Φ be a point of $\mathscr{S}\Rightarrow\mathscr{T}$. Then, $l\varepsilon\Phi$ if and only if $(\forall x\varepsilon l)\ \{x\}\varepsilon\Phi$.*

Proof. Let us suppose that $x\varepsilon l$; then $\{x\}\sqsubseteq l$ and hence $l \lhd_{S\Rightarrow T} \{x\}$, by Lemma 4.7, and thus $l\varepsilon\Phi$ yields $\{x\}\varepsilon\Phi$.

On the other hand, if $(\forall x\varepsilon l)\ \{x\}\varepsilon\Phi$ then $l\varepsilon\Phi$ can be proved by induction on the length of l by using condition (i.2) in the definition of point. $\square$

Theorem 4.17. *Let $\mathscr{S}$ and $\mathscr{T}$ be two information bases. Then there is a bijective correspondence between the collection of the points of $\mathscr{S}\Rightarrow\mathscr{T}$ and the collection of the translations between $\mathscr{S}$ and $\mathscr{T}$.*

Proof. Let Φ be a point of $\mathscr{S}\Rightarrow\mathscr{T}$ and put

$$sR_\Phi t \equiv \mathsf{Pos}_S(s) \to \{(s,t)\}\varepsilon\Phi.$$

Then, it is straightforward to prove that R_Φ is a translation between $\mathscr{S}$ and $\mathscr{T}$. Suppose now that R is a translation between $\mathscr{S}$ and $\mathscr{T}$ and put

$$\Phi_R \equiv \{l \in S \Rightarrow T \mid R \text{ contains } l\}.$$

Then, Φ_R is a point of $\mathscr{S} \Rightarrow \mathscr{T}$.

Moreover, the correspondence is bijective. In fact, $R_{\Phi_R} = R$, because

$$\begin{array}{lll} sR_{\Phi_R}t & \text{iff } \mathsf{Pos}_S(s) \rightarrow \{(s,t)\}\varepsilon\Phi_R & \text{by definition of } R_{\Phi_R} \\ & \text{iff } \mathsf{Pos}_S(s) \rightarrow sRt & \text{by definition of } \Phi_R \\ & \text{iff } sRt & \text{by condition (iii) on } R \end{array}$$

and $\Phi_{R_\Phi} = \Phi$, because

$$\begin{array}{lll} l\varepsilon\Phi_{R_\Phi} & \text{iff } R_\Phi \text{ contains } l & \text{by definition of } \Phi_{R_\Phi} \\ & \text{iff } (\forall x\varepsilon l)\ x_S R_\Phi x_T & \text{by definition of “contains”} \\ & \text{iff } (\forall x\varepsilon l)\ \mathsf{Pos}_S(x_S) \rightarrow \{(x_S, x_T)\}\varepsilon\Phi & \text{by definition of } R_\Phi \\ & \text{iff } (\forall x\varepsilon l)\ \{(x_S, x_T)\}\varepsilon\Phi & \text{since } \mathsf{Pos}_S(x_S) \text{ holds} \\ & & \text{because } l \in S \Rightarrow T \\ & \text{iff } l\varepsilon\Phi & \text{by lemma 4.16} \end{array}$$

5. The generic information base

In this section we want to show how to construct, within intuitionistic type theory, the information base which corresponds to a generic *set-based* Scott domain. We will not propose here a new construction but we simply show how to formalize the one in [8].

A Scott domain $\mathscr{D} \equiv (D, \leqslant)$, where D is a collection and $\leqslant$ a order relation over D, is called *set-based* if the subcollection of its compact elements can be indexed by means of a set, that we will call K_D. From now on, in order to keep the notation clearer, we will confuse the set of indexes K_D with the subcollection of the compact elements of $\mathscr{D}$. We can use K_D to define the information base that we are looking for. The hint to find the correct definition comes from the topological intuition. To this aim, let us recall the definition of Scott topology on a CPO.

Definition 5.1. In any CPO $\mathscr{D}$, a sub-collection $\mathscr{O}$ is called (Scott) *open* if it is *upward closed*, that is, if $x \in \mathscr{O}$ and $x \leqslant y$ then $y \in \mathscr{O}$, and *smooth*, that is, for each directed subset U, if $\bigvee U \in \mathscr{O}$ then $(\exists u \in U)\ u \in \mathscr{O}$.

It is well known (see for instance [2]) that Scott opens form a topology on $\mathscr{D}$, which is usually called the *Scott topology*.

If $\mathscr{D}$ is not only a CPO but a Scott domain, then it is completely determined by its Scott topology. In fact, given a base $\mathscr{B}$ for the Scott topology on $\mathscr{D}$, $x \leqslant y$ if and only if $(\forall \mathscr{O} \in \mathscr{B})\ (x \in \mathscr{O}) \rightarrow (y \in \mathscr{O})$. This remark suggests that we need a base, in the usual topological sense, in order to find the information base that we are looking for. A base for the Scott topology on $\mathcal{D}$ is usually obtained by considering all the subcollections $\uparrow a \equiv \{x \in D \mid a \leqslant x\}$ for $a \in K_D$ and possibly by adding the empty set. Here, this must be refined a little to avoid any definition or proof based on the distinction between the cases $\uparrow a \cap \uparrow b = \uparrow(a \vee b)$ and $\uparrow a \cap \uparrow b = \emptyset$, that is, between $\{a, b\}$ bounded or not. Then, the idea is to move from elements to finite subsets of K_D and consider, for any $U \in \mathscr{P}_\omega(K_D)$, the subcollection of its upper bounds $\mathcal{O}_U \equiv \{x \in D \mid U \leqslant x\}$, where $U \leqslant x$ is an abbreviation for $a \leqslant x$ for any $a \varepsilon U$. It is easy to check that $\{\mathcal{O}_U \mid U \in \mathscr{P}_\omega(K_D)\}$ is a base for the Scott topology on $\mathcal{D}$.

So, apart from foundational matters, the information base is now disclosed; the foundational problem is that $\{\mathcal{O}_U \mid U \in \mathscr{P}_\omega(K_D)\}$ is not a set, but a set-indexed family of subcollections of D and hence it cannot be used to define an information base. The standard way out in formal topology is to build up an information base $\mathscr{S}_\mathcal{D}$ by pulling the structure of the base $\{\mathcal{O}_U \mid U \in \mathscr{P}_\omega(K_D)\}$ back to the index set $\mathscr{P}_\omega(K_D)$. In detail, we provide $\mathscr{P}_\omega(K_D)$ with an operation of combination $\cdot_{\mathscr{S}_\mathcal{D}}$ such that $\mathcal{O}_{U \cdot_{\mathscr{S}_\mathcal{D}} V} = \mathcal{O}_U \cap \mathcal{O}_V$, that is, we put

$$U \cdot_{\mathscr{S}_\mathcal{D}} V \equiv U \sqcup V.$$

Then, the unit element of $\mathscr{S}_\mathcal{D}$ is $\emptyset \in \mathscr{P}_\omega(K_D)$, which can also be seen by observing that $\mathcal{O}_\emptyset = D$ and hence $\mathcal{O}_\emptyset \cap \mathcal{O}_U = \mathcal{O}_U$ for any U.

We say that U is positive when $\mathcal{O}_U$ is inhabited; so we put

$$\mathsf{Pos}_{\mathscr{S}_\mathcal{D}}(U) \equiv (\exists a \in K_D)\ (U \leqslant a)$$

and in this way $\mathsf{Pos}_{\mathscr{S}_\mathcal{D}}$ is a subset of $\mathscr{P}_\omega(K_D)$. Note that U is positive if and only if $\bigvee U \equiv \bigvee\{a \in K_D \mid a \varepsilon U\}$ exists.

Finally, we want U to be covered by W when $\mathcal{O}_U \subseteq \mathcal{O}_W$, which is clearly equivalent to: if $\bigvee U$ exists, then $W \leqslant \bigvee U$. Thus, we put

$$U \lhd_{\mathscr{S}_\mathcal{D}} W \equiv \mathsf{Pos}_{\mathscr{S}_\mathcal{D}}(U) \to W \leqslant \bigvee U.$$

It is obvious now that

$$\mathscr{S}_\mathcal{D} \equiv \langle \mathscr{P}_\omega(K_D), \cdot_{\mathscr{S}_\mathcal{D}}, \emptyset, \mathsf{Pos}_{\mathscr{S}_\mathcal{D}}, \lhd_{\mathscr{S}_\mathcal{D}} \rangle$$

is an information base.

Moreover, $\mathscr{S}_\mathcal{D}$ is the information base that we were looking for. In fact, the domains $\mathcal{D}$ and $\mathsf{Pt}(\mathscr{S}_\mathcal{D})$ are isomorphic. The easiest way to find out an isomorphism, is to specialize to the base $\{\mathcal{O}_U \mid U \in \mathscr{P}_\omega(K_D)\}$ the fact that a domain is completely determined by a base for its Scott topology. In fact, in this way we obtain that $x \leqslant y$ if and only if $(\forall \mathcal{O}_U)(x \in \mathcal{O}_U \to y \in \mathcal{O}_U)$, which can equivalently be expressed in our framework as $(\forall U \in \mathscr{P}_\omega(K_D))\ (U \leqslant x \rightarrow U \leqslant y)$, that is, $\{U \in \mathscr{P}_\omega(K_D) \mid U \leqslant x\} \subseteq \{U \in \mathscr{P}_\omega(K_D) \mid U \leqslant y\}$. It is easy to check that, for any $x \in D$, the subset[3] $\{U \in \mathscr{P}_\omega(K_D) \mid U \leqslant x\}$ is a point of $\mathscr{S}_\mathscr{D}$. Hence putting

$$f : x \mapsto \{U \in \mathscr{P}_\omega(K_D) \mid U \leqslant x\}$$

defines a map from $\mathscr{D}$ into $\mathsf{Pt}(\mathscr{S}_\mathscr{D})$, which is monotonic and one–one; to conclude we must only show that f is onto and hence an isomorphism since any bijective monotonic function respects all suprema. To this aim, observe that if α is a point of $\mathscr{S}_\mathscr{D}$ then, for any $W \in \mathscr{P}_\omega(K_D)$, $W \varepsilon \alpha$ if and only if $(\forall a \varepsilon W)\ \{a\} \varepsilon \alpha$, that is, α is determined by the singletons it contains; hence the element of D whose image under f is α must be $\bigvee\{a \in K_D \mid \{a\} \varepsilon \alpha\}$, which exists since $\{a \in K_D \mid \{a\} \varepsilon \alpha\}$ is directed. So we have proved:

Theorem 5.2. *Any set-based Scott domain $\mathscr{D}$ is isomorphic to the points of a suitable information base $\mathscr{S}_\mathscr{D}$.*

6. Some properties of the category InfBas

In this section we will present some useful categorical constructions which are possible in InfBas.

6.1. The initial object

Since InfBas is a category equivalent to ScDom there is no initial object, but we can modify InfBas in a very simple way in order to have them. Indeed, it is sufficient to drop the condition that, for any information base $\mathscr{S}$, $\mathsf{Pos}_S(\Delta_S)$ holds and we will be able to prove the following theorem.

Theorem 6.1. *Let $\mathscr{E} \equiv (E, \cdot_E, \Delta_E, \mathsf{Pos}_E, \lhd_E)$ be any information base with no positive element. Then, for any information base $\mathscr{S}$, the total relation, which holds for any $e \in E$ and $s \in S$, is the unique translation between $\mathscr{E}$ and $\mathscr{S}$.*

Proof. It is obvious that the total relation is a translation. Moreover, if F is any translation between $\mathscr{E}$ and $\mathscr{S}$, then, since $\neg\mathsf{Pos}_E(e)$ holds for any $e \in E$, $\mathsf{Pos}_E(e) \rightarrow eFs$ holds by logic and thus eFs follows by the last condition on a translation. $\square$

[3] The fact that $\{U \in \mathscr{P}_\omega(K_D) \mid U \leqslant x\}$ is a subset, that is, a propositional function over $\mathscr{P}_\omega(K_D)$, is not so immediate. Given $x \in D$, consider the subset $\downarrow_K(x) \equiv \{a \in K_D \mid a \leqslant x\}$ of K_D; then $U \leqslant x$ means that $(\exists a \varepsilon \downarrow_K(x))\ (U \leqslant a)$ which is a propositional function with U free.

We can easily build a structure $\perp$ which is like an information base except for the fact that $\neg\mathsf{Pos}_\perp(e)$ holds for any $e \in \perp$. For instance, we can use the one element set $\top$, whose only element is $*$, and declare it not positive. We thus arrive at the following definitions:

$$\cdot_\perp \equiv (x : \top)(y : \top)\ *,$$

$$\Delta_\perp \equiv *,$$

$$\mathsf{Pos}_\perp \equiv (x : \top)\ \neg(x =^\top x),$$

$$\lhd_\perp \equiv (x : \top)(y : \top)\ x =^\top y.$$

It is obvious that the collection of points of any initial information base is empty because of the condition (ii) on points.

6.2. The separated sum

No co-product can be defined in InfBas, but still we can constructively define two kinds of sum of information bases, that is, the *separated* and the *coalesced* sum. We will show the former in this section and the latter in the next one.

Let $\mathscr{S}$ and $\mathscr{T}$ be two information bases. Then, from a topological point of view the information base $\mathscr{S} \oplus \mathscr{T}$ of the separated sum of $\mathscr{S}$ and $\mathscr{T}$ is just the disjoint union of $\mathscr{S}$ and $\mathscr{T}$. Hence, we obtain a base for such a topological space by putting together the elements in the base S and those in the base T and by adding a new element to mean the whole topological space. But we have to add also another element in order the operation $\cdot_{S\oplus T}$, which means the intersection between two elements of the disjoint union, be always defined, namely, also when an element in $\mathscr{S}$ is considered together with an element in $\mathscr{T}$. Thus, the new base can be defined by using the disjoint sum $S+T$ of S and T and by adding two new elements by using the type constructor Succ (see Section 2); thus the set that we are looking for is $\mathsf{Succ}(\mathsf{Succ}(S+T))$.

Let us use the following short-hands, for any $s \in S$ and $t \in T$:

$$\Delta_{S\oplus T} \equiv \mathsf{succ}(1_{\mathsf{Succ}(S+T)}),$$

$$\perp_{S\oplus T} \equiv 1_{\mathsf{Succ}(\mathsf{Succ}(S+T))},$$

$$(s)_S \equiv \mathsf{succ}(\mathsf{succ}(\mathsf{i}(s))),$$

$$(t)_T \equiv \mathsf{succ}(\mathsf{succ}(\mathsf{j}(t))).$$

Note that if $(s_1)_S =^{S\oplus T} (s_2)_S$, then $s_1 =^S s_2$ and if $(t_1)_T =^{S\oplus T} (t_2)_T$, then $t_1 =^T t_2$.

The composition operation $\cdot_{S\oplus T}$ works according to the following table:

$\cdot_{S\oplus T}$	$\bot_{S\oplus T}$	$(s_2)_S$	$(t_2)_T$	$\Delta_{S\oplus T}$
$\bot_{S\oplus T}$	$\bot_{S\oplus T}$	$\bot_{S\oplus T}$	$\bot_{S\oplus T}$	$\bot_{S\oplus T}$
$(s_1)_S$	$\bot_{S\oplus T}$	$(s_1 \cdot s_2)_S$	$\bot_{S\oplus T}$	$(s_1)_S$
$(t_1)_T$	$\bot_{S\oplus T}$	$\bot_{S\oplus T}$	$(t_1 \cdot t_2)_T$	$(t_1)_T$
$\Delta_{S\oplus T}$	$\bot_{S\oplus T}$	$(s_2)_S$	$(t_2)_T$	$\Delta_{S\oplus T}$

It is not difficult to formalize it within intuitionistic type theory.

Note that

- If $c \cdot_{S\oplus T} d =^{S\oplus T} \Delta_{S\oplus T}$, then $c =^{S\oplus T} \Delta_{S\oplus T}$ and $d =^{S\oplus T} \Delta_{S\oplus T}$.
- If $c \cdot_{S\oplus T} d =^{S\oplus T} (s)_S$, then

$$(c =^{S\oplus T} \Delta_{S\oplus T} \ \&\ d =^{S\oplus T} (s)_S) \ \vee\ (c =^{S\oplus T} (s)_S \ \&\ d =^{S\oplus T} \Delta_{S\oplus T}) \ \vee$$
$$((\exists s_1, s_2 \in S)\ c =^{S\oplus T} (s_1)_S \ \&\ d =^{S\oplus T} (s_2)_S \ \&\ s =^S s_1 \cdot_S s_2).$$

- If $c \cdot_{S\oplus T} d =^{S\oplus T} (t)_T$, then

$$(c =^{S\oplus T} \Delta_{S\oplus T} \ \&\ d =^{S\oplus T} (t)_T) \ \vee\ (c =^{S\oplus T} (t)_T \ \&\ d =^{S\oplus T} \Delta_{S\oplus T}) \ \vee$$
$$((\exists t_1, t_2 \in T)\ c =^{S\oplus T} (t_1)_T \ \&\ d =^{S\oplus T} (t_2)_T \ \&\ t =^T t_1 \cdot_T t_2).$$

A token of information in $\mathcal{S} \oplus \mathcal{T}$ is positive when it is positive in $\mathcal{S}$ or in $\mathcal{T}$, and hence, given any element $c \in S \oplus T$ we put

$$\begin{aligned}\mathsf{Pos}_{S\oplus T}(c) \equiv (c =^{S\oplus T} \Delta_{S\oplus T}) \ \vee \\ ((\exists s \in S)\ \mathsf{Pos}_S(s) \ \&\ c =^{S\oplus T} (s)_S) \ \vee \\ ((\exists t \in T)\ \mathsf{Pos}_T(t) \ \&\ c =^{S\oplus T} (t)_T)\end{aligned}$$

Note that to assume $\mathsf{Pos}_{S\oplus T}(\bot_{S\oplus T})$ means that

$$\begin{aligned}&(\bot_{S\oplus T} =^{S\oplus T} \Delta_{S\oplus T}) \ \vee \\ &((\exists s \in S)\ \mathsf{Pos}_S(s) \ \&\ \bot_{S\oplus T} =^{S\oplus T} (s)_S) \ \vee \\ &((\exists t \in T)\ \mathsf{Pos}_T(t) \ \&\ \bot_{S\oplus T} =^{S\oplus T} (t)_T)\end{aligned}$$

holds. Hence we get $\neg\mathsf{Pos}_{S\oplus T}(\bot_{S\oplus T})$ because, as we observed in the end of Section 2, all the disjoints lead to a contradiction.

It is worth noting also that $\mathsf{Pos}_{S\oplus T}((s)_S)$ yields $\mathsf{Pos}_S(s)$ and $\mathsf{Pos}_{S\oplus T}((t)_T)$ yields $\mathsf{Pos}_T(t)$.

Finally, supposing c and d are two elements in $S \oplus T$, c is covered by d in $S \oplus T$ if, whenever c and d are obtained from two elements c' and d' of the same information base, c' is covered in such an information base by d'. Thus, we put

$$\begin{aligned}
c \lhd_{S\oplus T} d \equiv \mathsf{Pos}_{S\oplus T}(c) \rightarrow & \\
& (c =^{S\oplus T} \Delta_{S\oplus T} \rightarrow d =^{S\oplus T} \Delta_{S\oplus T}) \ \wedge \\
& ((\exists s_1 \in S)\ c =^{S\oplus T} (s_1)_S \rightarrow (d =^{S\oplus T} \Delta_{S\oplus T} \vee \\
& \qquad ((\exists s_2 \in S)\ d =^{S\oplus T} (s_2)_S\ \&\ s_1 \lhd_S s_2))) \ \wedge \\
& ((\exists t_1 \in T)\ c =^{S\oplus T} (t_1)_T \rightarrow (d =^{S\oplus T} \Delta_{S\oplus T} \vee \\
& \qquad ((\exists t_2 \in T)\ d =^{S\oplus T} (t_2)_T\ \&\ t_1 \lhd_T t_2))).
\end{aligned}$$

Observe that if $(s_1)_S \lhd_{S\oplus T} (s_2)_S$, then $s_1 \lhd_S s_2$ and if $(t_1)_T \lhd_{S\oplus T} (t_2)_T$, then $t_1 \lhd_T t_2$.
Then we arrive at the following result.

Theorem 6.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be two information bases and put*

$$\mathcal{S} \oplus \mathcal{T} \equiv \langle S \oplus T, \cdot_{S\oplus T}, \Delta_{S\oplus T}, \mathsf{Pos}_{S\oplus T}, \lhd_{S\oplus T}\rangle.$$

Then, $\mathcal{S} \oplus \mathcal{T}$ is an information base.

Proof. Many checks are required, but most of them are immediate; here we will show only the cases which are not straightforward.

- (*Monotonicity*) If $\mathsf{Pos}_{S\oplus T}(c)$ and $c \lhd_{S\oplus T} d$, then $\mathsf{Pos}_{S\oplus T}(d)$. In fact, supposing $\mathsf{Pos}_{S\oplus T}(c)$, $c \lhd_{S\oplus T} d$ yields
  (1) $c =^{S\oplus T} \Delta_{S\oplus T} \rightarrow d =^{S\oplus T} \Delta_{S\oplus T}$.
  (2) $(\exists s_1 \in S)\ c =^{S\oplus T} (s_1)_S \rightarrow$
  $(d =^{S\oplus T} \Delta_{S\oplus T} \vee ((\exists s_2 \in S)\ d =^{S\oplus T} (s_2)_S\ \&\ s_1 \lhd_S s_2))$.
  (3) $(\exists t_1 \in T)\ c =^{S\oplus T} (t_1)_T \rightarrow$
  $(d =^{S\oplus T} \Delta_{S\oplus T} \vee ((\exists t_2 \in T)\ d =^{S\oplus T} (t_2)_T\ \&\ t_1 \lhd_T t_2))$.
  Now, observe that there are three possibilities for c to be positive:
  - $c =^{S\oplus T} \Delta_{S\oplus T}$. In this case (1) yields $d =^{S\oplus T} \Delta_{S\oplus T}$ and hence d is positive.
  - $((\exists s_1 \in S)\ \mathsf{Pos}_S(s_1)\ \&\ c =^{S\oplus T} (s_1)_S)$. In this case (2) yields that $(d =^{S\oplus T} \Delta_{S\oplus T} \vee ((\exists s_2 \in S)\ d =^{S\oplus T} (s_2)_S\ \&\ s_1 \lhd_S s_2))$; if $d =^{S\oplus T} \Delta_{S\oplus T}$ then it is trivially positive otherwise, by *monotonicity* in $\mathcal{S}$ we obtain $\mathsf{Pos}_S(s_2)$ and thus also in this case d is positive.
  - $((\exists t_1 \in T)\ \mathsf{Pos}_T(t_1)\ \&\ c =^{S\oplus T} (t_1)_T)$. Completely analogous to the previous one.
- ($\cdot$-*left*) If $a \lhd_{S\oplus T} c$, then $a \cdot_{S\oplus T} b \lhd_{S\oplus T} c$. First note that, by *monotonicity*, if $\mathsf{Pos}_{S\oplus T}(a \cdot_{S\oplus T} b)$ then $\mathsf{Pos}_{S\oplus T}(a)$. Hence $a \lhd_{S\oplus T} c$ yields
  - $a =^{S\oplus T} \Delta_{S\oplus T} \rightarrow c =^{S\oplus T} \Delta_{S\oplus T}$.
  - $(\exists s_1 \in S)\ a =^{S\oplus T} (s_1)_S \rightarrow$
    $(c =^{S\oplus T} \Delta_{S\oplus T}\ \vee\ (\exists s_2 \in S)\ c =^{S\oplus T} (s_2)_S\ \&\ s_1 \lhd_S s_2)$.
  - $(\exists t_1 \in T)\ a =^{S\oplus T} (t_1)_T \rightarrow$
    $(c =^{S\oplus T} \Delta_{S\oplus T}\ \vee\ (\exists t_2 \in T)\ c =^{S\oplus T} (t_2)_T\ \&\ t_1 \lhd_T t_2)$.

Now the result follows by a case analysis on the shape of $a \cdot_{S\oplus T} b$.

- ($\cdot$-*right*) If $a \lhd_{S\oplus T} b$ and $a \lhd_{S\oplus T} c$ then $a \lhd_{S\oplus T} b \cdot_{S\oplus T} c$. Let us assume $\mathsf{Pos}_{S\oplus T}(a)$. Then we obtain the result by a case analysis on the possible shape for a.
  - $a =^{S\oplus T} \Delta_{S\oplus T}$. Then, from $a \lhd_{S\oplus T} b$, we obtain that $b =^{S\oplus T} \Delta_{S\oplus T}$ and, from $a \lhd_{S\oplus T} c$, we obtain that $c =^{S\oplus T} \Delta_{S\oplus T}$. Hence the result is immediate by definition of $\cdot_{S\oplus T}$.
  - $a =^{S\oplus T} (s)_S$ for some $s \in S$. Then, from $a \lhd_{S\oplus T} b$, we obtain that $b =^{S\oplus T} \Delta_{S\oplus T}$ or $b =^{S\oplus T} (s_1)$ and $s \lhd_S s_1$; in a similar way, by $a \lhd_{S\oplus T} c$, we obtain that $c =^{S\oplus T} \Delta_{S\oplus T}$ or $c =^{S\oplus T} (s_2)$ and $s \lhd_S s_2$. Now the result is straightforward by logic and $\cdot$-*right* in $\mathcal{S}$.
  - $a =^{S\oplus T} (t)_T$ for some $t \in T$. Completely analogous to the previous one.

The separated sum of information bases is not a categorical co-product since it is not possible to define the necessary translations. Anyhow, it is possible to prove the following theorem.

Theorem 6.3. *Let $\mathcal{S}$ and $\mathcal{T}$ be two information bases. Then the propositional functions defined by putting, for any $s \in S$, $t \in T$ and $w \in S \oplus T$*

$$s\mathsf{L}_{\mathrm{sep}}w \equiv \mathsf{Pos}_S(s) \to (s)_S \lhd_{S\oplus T} w,$$

$$t\mathsf{R}_{\mathrm{sep}}w \equiv \mathsf{Pos}_T(t) \to (t)_T \lhd_{S\oplus T} w$$

are translations between $\mathcal{S}$ and $\mathcal{S} \oplus \mathcal{T}$ and $\mathcal{T}$ and $\mathcal{S} \oplus \mathcal{T}$, respectively. Moreover, supposing $\mathcal{Z}$ is any information base, $F \in \mathsf{Hom}(\mathcal{S}, \mathcal{Z})$ and $G \in \mathsf{Hom}(\mathcal{T}, \mathcal{Z})$, the propositional function defined by putting, for any $w \in S \oplus T$ and $z \in Z$,

$$\begin{aligned} w\{F, G\}z \equiv \mathsf{Pos}_{S\oplus T}(w) \to & \\ & \Delta_Z \lhd_Z z \ \vee \\ & ((\exists s \in S)\ w =^{S\oplus T} (s)_S \ \&\ sFz) \ \vee \\ & ((\exists t \in T)\ w =^{S\oplus T} (t)_T \ \&\ tGz) \end{aligned}$$

is a translation and the following equations hold:

$$\{F, G\} * \mathsf{L}_{\mathrm{sep}} = F,$$

$$\{F, G\} * \mathsf{R}_{\mathrm{sep}} = G,$$

$$\{H * \mathsf{L}_{\mathrm{sep}}, H * \mathsf{R}_{\mathrm{sep}}\} = H \quad \textit{for any translation } H \in \mathsf{Hom}(\mathcal{S} \oplus \mathcal{T}, \mathcal{Z}) \textit{ such that } \Delta_{S\oplus T}Hz \textit{ if and only if } \Delta_Z \lhd_Z z.$$

6.3. The coalesced sum

The second kind of sum that we can define in InfBas is the coalesced sum. Also, in this case we will not obtain a categorical co-product. The main difference with respect to the previous kind of sum is that, supposing $\mathcal{S}$ and $\mathcal{T}$ are two information bases, the base for the coalesced sum $\mathscr{S} \uplus \mathscr{T}$ is obtained by identifying the two unit elements Δ_S of $\mathscr{S}$ and Δ_T of $\mathscr{T}$. Thus, most of the definitions are like in the previous section, that is, the basic opens are the elements of the set $\mathsf{Succ}(\mathsf{Succ}(S+T))$, and $\Delta_{S\uplus T}$, $\perp_{S\uplus T}$, the operation $\cdot_{S\uplus T}$ and the positivity predicate $\mathsf{Pos}_{S\uplus T}$ are defined exactly as the corresponding objects of $\mathscr{S} \oplus \mathscr{T}$.

The real novelty is the definition of the cover relation. In fact, let us suppose that c and d are two elements in $S \uplus T$; then, c is covered by d in $S \uplus T$ if, whenever c and d are obtained from two elements c' and d' of the same information base, c' is covered in such an information base by d', but we also have that $\Delta_{S\uplus T}$ is covered by $(s)_S$ for any element $s \in S$ which covers Δ_S and by $(t)_T$ for any element $t \in T$ which covers Δ_T. Thus, we put

$$
\begin{aligned}
c \lhd_{S\uplus T} d \equiv\ & \mathsf{Pos}_{S\uplus T}(c) \rightarrow \\
& (c =^{S\uplus T} \Delta_{S\uplus T} \rightarrow \\
& \quad (d =^{S\uplus T} \Delta_{S\uplus T} \vee \\
& \quad ((\exists s \in S)\ d =^{S\uplus T} (s)_S\ \&\ \Delta_S \lhd_S s) \vee \\
& \quad ((\exists t \in T)\ d =^{S\uplus T} (t)_T\ \&\ \Delta_T \lhd_T t)) \wedge \\
& ((\exists s_1 \in S)\ \mathsf{Pos}_S(s_1)\ \&\ c =^{S\uplus T} (s_1)_S \rightarrow \\
& \quad (d =^{S\uplus T} \Delta_{S\uplus T} \vee \\
& \quad ((\exists s \in S)\ d =^{S\uplus T} (s)_S\ \&\ \Delta_S \lhd_S s) \vee \\
& \quad ((\exists t \in T)\ d =^{S\uplus T} (t)_T\ \&\ \Delta_T \lhd_T t) \vee \\
& \quad ((\exists s_2 \in S)\ d =^{S\oplus T} (s_2)_S\ \&\ s_1 \lhd_S s_2))) \wedge \\
& ((\exists t_1 \in T)\ \mathsf{Pos}_T(t_1)\ \&\ c =^{S\oplus T} (t_1)_T \rightarrow \\
& \quad (d =^{S\uplus T} \Delta_{S\uplus T} \vee \\
& \quad ((\exists s \in S)\ d =^{S\uplus T} (s)_S\ \&\ \Delta_S \lhd_S s) \vee \\
& \quad ((\exists t \in T)\ d =^{S\uplus T} (t)_T\ \&\ \Delta_T \lhd_T t) \vee \\
& \quad ((\exists t_2 \in T)\ d =^{S\oplus T} (t_2)_T\ \&\ t_1 \lhd_T t_2))).
\end{aligned}
$$

Then, we arrive at the following result.

Theorem 6.4. *Let $\mathscr{S}$ and $\mathscr{T}$ be two information bases and put*

$$\mathscr{S} \uplus \mathscr{T} \equiv \langle S \uplus T, \cdot_{S\uplus T}, \Delta_{S\uplus T}, \mathsf{Pos}_{S\uplus T}, \lhd_{S\uplus T} \rangle.$$

Then, $\mathscr{S} \uplus \mathscr{T}$ is an information base.

Moreover the following theorem holds.

Theorem 6.5. *Let $\mathscr{S}$ and $\mathscr{T}$ be two information bases. Then the propositional functions defined by putting, for any $s \in S$, $t \in T$ and $w \in S \uplus T$*

$$s \mathsf{L}_{\mathrm{coal}} w \equiv \mathsf{Pos}_S(s) \rightarrow (s)_S \lhd_{S\uplus T} w,$$

$$t \mathsf{R}_{\mathrm{coal}} w \equiv \mathsf{Pos}_T(t) \rightarrow (t)_T \lhd_{S\uplus T} w$$

are translations between $\mathscr{S}$ and $\mathscr{S} \uplus \mathscr{T}$ and $\mathscr{T}$ and $\mathscr{S} \uplus \mathscr{T}$, respectively. Moreover, supposing $\mathscr{Z}$ is any information base, $F \in \mathsf{Hom}(\mathscr{S}, \mathscr{Z})$ and $G \in \mathsf{Hom}(\mathscr{T}, \mathscr{Z})$, the propositional function defined by putting, for any $w \in S \uplus T$ and $z \in Z$,

$$\begin{aligned} w\langle F, G\rangle z \equiv\ & \mathsf{Pos}_{S \uplus T}(w) \rightarrow \\ & \Delta_Z \lhd_Z z\ \vee \\ & ((\exists s \in S)\ w =^{S \uplus T} (s)_S\ \&\ sFz)\ \vee \\ & ((\exists t \in T)\ w =^{S \uplus T} (t)_T\ \&\ tGz). \end{aligned}$$

is a translation and the following equations hold:

$$\langle F, G\rangle * \mathsf{L}_{\mathrm{coal}} = F \ \textit{ iff } \ \Delta_S\, F\, z \Rightarrow \Delta_T\, G\, z,$$

$$\langle F, G\rangle * \mathsf{R}_{\mathrm{coal}} = G \ \textit{ iff } \ \Delta_T\, G\, z \Rightarrow \Delta_S\, F\, z,$$

$$\langle H * \mathsf{L}_{\mathrm{coal}}, H * \mathsf{R}_{\mathrm{coal}}\rangle = H \quad \textit{for any translation } H \in \mathsf{Hom}(\mathscr{S} \uplus \mathscr{T}, \mathscr{Z}).$$

6.4. Fixed-point property

One of the most interesting property of the category ScDom is the possibility to deal with fixed-points therein. In fact, supposing f is an approximable function from the Scott domain $\mathscr{D}$ into itself, there exists an element $d \in D$ such that $f(d) = d$. Moreover, such a fixed point can be found in a uniform way, that is, there exists a function fix from $\mathscr{D} \Rightarrow \mathscr{D}$ into $\mathscr{D}$ such that, when applied to any function f, gives the smallest, with respect to the order in $\mathscr{D}$, of its fixed points, that is, $f(\mathsf{fix}(f)) = \mathsf{fix}(f)$ and, for any $z \in D$, $f(z) = z$ yields $\mathsf{fix}(f) \leqslant z$. The technique to define the map fix is well known: provided the bottom element in $\mathscr{D}$ is denoted by $\perp_D$, put

$$\mathsf{fix}(f) \equiv \bigvee_{n \in \mathsf{Nat}} f^n(\perp_D).$$

In fact, the set-indexed collection $\{f^n(\perp_D) \mid n \in \mathsf{Nat}\}$ is directed and hence its supremum exists in $\mathscr{D}$ and it obviously satisfies the required conditions.

The main problem in looking for a constructive counterpart of this definition is the presence of the limit process, but this limit process is so much uniform that a solution can be found. Let us analyse it. Suppose $\mathscr{S}$ and $\mathscr{T}$ are two information bases and suppose that f is any approximable function from $\mathsf{Pt}(\mathscr{S})$ into $\mathsf{Pt}(\mathscr{T})$; as we noticed in the end of Section 3, we can define a translation F_f from $\mathscr{S}$ in $\mathscr{T}$ such that $f = \mathsf{Pt}(F_f)$ by putting

$$s\, F_f\, t \equiv \mathsf{Pos}_S(s) \rightarrow (t \varepsilon f(\uparrow s)),$$

where $\uparrow s \equiv \{u \in S \mid s \lhd_S u\}$ is the point of $\mathscr{S}$ which contains all the elements of S which cover s. If we would apply directly this technique to the case of the previous function

fix we would obtain the following propositional function between $\mathscr{S} \Rightarrow \mathscr{S}$ and $\mathscr{S}$:

$$l \ \mathsf{Fix}\ s \equiv \mathsf{Pos}_{S\Rightarrow S}(l) \rightarrow s\varepsilon\mathsf{fix}(\uparrow l).$$

The problem is that the point $\uparrow l$ of $\mathscr{S} \Rightarrow \mathscr{S}$ is *not* an approximable function from $\mathsf{Pt}(\mathscr{S})$ into itself and hence we *cannot* apply the function fix to it. But we already showed in Section 4.3 how a translation, and hence also an approximable function, is associated with any point of $\mathscr{S} \Rightarrow \mathscr{S}$: the approximable function $f_{\uparrow l}: \mathsf{Pt}(\mathscr{S}) \rightarrow \mathsf{Pt}(\mathscr{S})$ associated with the point $\uparrow l$ of $\mathscr{S} \Rightarrow \mathscr{S}$ is

$$f_{\uparrow l}(\alpha) \equiv \bigcup_{s\varepsilon\alpha} \{u \in S \mid sR_{\uparrow l}u\},$$

where, according to the notation that we used in the proof of Theorem 4.17, $R_{\uparrow l}$, defined by setting $s_1R_{\uparrow l}s_2$ if and only if $\mathsf{Pos}_S(s_1) \rightarrow \{(s_1,s_2)\}\varepsilon \uparrow l$, is the translation associated with the point $\uparrow l$.

If we consider now the case $f_{\uparrow l}$ is applied to the bottom element of $\mathsf{Pt}(\mathscr{S})$, that is, the case $\alpha \equiv \{\Delta_S\}$, we obtain

$$f_{\uparrow l}(\{\Delta_S\}) \equiv \{u \in S \mid \Delta_S R_{\uparrow l} u\}$$

and hence

$$f^n_{\uparrow l}(\{\Delta_S\}) \equiv \{u \in S \mid \Delta_S R^n_{\uparrow l} u\}.$$

Thus,

$$\bigcup_{n\in\mathsf{Nat}} f^n_{\uparrow l}(\{\Delta_S\}) \equiv \{u \in S \mid (\exists n \in \mathsf{Nat})\ \Delta_S R^n_{\uparrow l} u\}$$

since $\Delta_S R_{\uparrow l} \Delta_S$ holds. We can simplify a bit the last equivalence if we note that the translation $R_{\uparrow l}$ coincides with the translation R_l that we introduced in the proof of Theorem 4.12. In this way we obtain that

$$l\,\mathsf{Fix}\,s \equiv \mathsf{Pos}_{S\Rightarrow S}(l) \rightarrow (\exists n \in \mathsf{Nat})\ \Delta_S R^n_l s,$$

which has a clear independent meaning. In fact, it states that, given any partial information l concerning a translation, in order to find a fixed point of such a translation we have to collect all the tokens of information into which the whole space, that is, Δ_S, is mapped at some moment.

It is now obvious the following theorem.

Theorem 6.6. *Let $\mathscr{S}$ be an information base and put, for any $l \in S \Rightarrow S$ and $s \in S$,*

$$l\mathsf{Fix}\,s \equiv \mathsf{Pos}_{S\Rightarrow S}(l) \rightarrow (\exists n \in \mathsf{Nat})\ \Delta_S R^n_l s.$$

Then Fix is a translation between $\mathscr{S} \Rightarrow \mathscr{S}$ and $\mathscr{S}$.

Proof. The proof is just a check. We will show the only not-obvious case. Suppose $l_1, l_2 \in S \Rightarrow S$ and $s \in S$, then if $l_1 \lhd_{S\Rightarrow S} l_2$ and $l_2\,\mathsf{Fix}\,s$ then $l_1\,\mathsf{Fix}\,s$. In fact, if $l_1 \lhd_{S\Rightarrow S} l_2$

then R_{l_2} is contained in R_{l_1} since, by Theorem 4.14, $l_1 \lhd_{S\Rightarrow S} l_2$ yields that any translation containing l_1 also contains l_2 and hence R_{l_1} contains l_2 since it contains l_1; but, by Lemma 4.13, R_{l_2} is the minimal translation which contains l_2. $\square$

Fix is the translation that we are looking for. In fact, for any translation $F \in \mathsf{Hom}(\mathscr{S}, \mathscr{S})$, we can define the following translation between a terminal information base $\top$ and the information base $\mathscr{S} \Rightarrow \mathscr{S}$, by putting, for any $t \in \top$ and $l \in S \Rightarrow S$,

$$t\, 1_F\, l \equiv \mathsf{Pos}_\top(t) \to (\forall c \varepsilon l)\ c_1\, F\, c_2,$$

where $c_1 \equiv \mathsf{fst}(\mathsf{fst}(c))$ and $c_2 \equiv \mathsf{snd}(c)$.

It is interesting to note that the translation 1_F can be used to "determine" the translation F "inside" the information base $\mathscr{S} \Rightarrow \mathscr{S}$. In fact, we can first define a point of $\mathscr{S} \Rightarrow \mathscr{S}$ by setting

$$\Phi_F \equiv \{l \in S \Rightarrow S \mid \Delta_\top 1_F\, l\}$$

and then, as we did in the proof of Theorem 4.17, such a point can be associated to the translation R_{Φ_F}, defined by putting

$$s_1\, R_{\Phi_F}\, s_2 \text{ iff } \mathsf{Pos}_S(s_1) \to \{(s_1, s_2)\} \varepsilon \Phi_F.$$

Now we can see that R_{Φ_F} and F coincides. In fact, supposing $s_1, s_2 \in S$, we have

$$\begin{aligned} s_1\ R_{\Phi_F}\ s_2 &\text{ iff } \mathsf{Pos}_S(s_1) \to \{(s_1, s_2)\} \varepsilon \Phi_F \\ &\text{ iff } \mathsf{Pos}_S(s_1) \to \Delta_\top\, 1_F\, \{(s_1, s_2)\} \\ &\text{ iff } \mathsf{Pos}_S(s_1) \to s_1\, F\, s_2 \\ &\text{ iff } s_1\, F\, s_2. \end{aligned}$$

Now, we can prove the following theorem.

Theorem 6.7 (Fixed point). *Let $\mathscr{S}$ be an information base. Then, for any translation F between $\mathscr{S}$ and $\mathscr{S}$,*

$$F * \mathsf{Fix} * 1_F = \mathsf{Fix} * 1_F.$$

Proof. Let us first observe that, if $t \in \top$, $s \in S$ and $\mathsf{Pos}_\top(t)$, then $t\, \mathsf{Fix} * 1_F\, s$ means that there exists $l \in S \Rightarrow S$ such that $t\, 1_F\, l$, that is, l is contained in F, and $(\exists n \in \mathsf{Nat})\, \Delta_S R_l^n\, s$; but the former yields that R_l is contained in F and hence the second yields $(\exists n \in \mathsf{Nat})\, \Delta_S\, F^n\, s$, that is, $\Delta_S\, F\, s_1 \ldots s_n\, F\, s$. Therefore, we can consider the list $l^* \equiv \{(\Delta_S, s_1), \ldots, (s_n, s)\}$: it satisfies both $t\, 1_F\, l^*$ and $(\exists n \in \mathsf{Nat})\, \Delta_S\, R_{l^*}^n\, s$. Hence, supposing $\mathsf{Pos}_\top(t)$, $t\, \mathsf{Fix} * 1_F\, s$ holds if and only if $(\exists n \in \mathsf{Nat})\, \Delta_S\, F^n\, s$.

Now the result is almost immediate. In fact, let us suppose $\mathsf{Pos}_{\top}(t)$; then,

$$
\begin{aligned}
t\,F * \mathsf{Fix} * 1_F\, s \text{ if and only if there exists } u \in S \text{ such that}\\
t\,\mathsf{Fix} * 1_F\, u \text{ and } u\,F\,s,\\
\text{if and only if there exists } u \in S \text{ such that}\\
(\exists n \in \mathsf{Nat})\, \Delta_S F^n u \text{ and } u\,F\,s,\\
\text{if and only if } (\exists k \in \mathsf{Nat})\, \Delta_S F^k s,
\end{aligned}
$$

where in the last step it can be necessary to use the fact $\Delta_S\ F\ \Delta_S$.

References

[1] R. Backhouse, P. Chisholm, G. Malcom, E. Saaman, Do-it-yourself Type Theory, Formal Aspects of Computing, vol. 1, 1989, pp. 19–84.

[2] H. Barendregt, The Lambda Calculus, its Syntax and Semantics, Studies in Logic and Foundations of Mathematics, North-Holland, Amsterdam, 1984.

[3] M.E. Maietti, About effective quotients in constructive type theory, in: T. Altenkirch, W. Naraschewski, B. Reus (Eds.), Types for Proofs and Programs, International Workshop "Types'98", Lecture Notes in Computer Science, vol. 1657, Springer, Berlin, 1999, pp. 164–178.

[4] P. Martin-Löf, Intuitionistic Type Theory, notes by G. Sambin of a series of lectures given in Padua, Bibliopolis, Naples, 1984.

[5] B. Nordström, K. Peterson, J. Smith, Programming in Martin-Löf's Type Theory, An introduction, Clarendon Press, Oxford, 1990.

[6] G. Sambin, Intuitionistic formal spaces—a first communication, in: D. Skordev (Ed.), Mathematical Logic and its Applications, Plenum Press, New York, 1987, pp. 187–204.

[7] G. Sambin, S. Valentini, Building up a tool-box for Martin-Löf intuitionistic type theory, in: G. Sambin, J. Smith (Eds.), Twenty-five Years of Constructive Type Theory, Oxford Logic Guides, vol. 36, Oxford University Press, New York, 1998, pp. 221–244.

[8] G. Sambin, S. Valentini, P. Virgili, Constructive domain theory as a branch of Intuitionistic Pointfree Topology, Theoret. Comput. Sci. 159 (1996) 319–341.

[9] D.S. Scott, Lectures on a mathematical theory of computation, Oxford University Computing Laboratory Technical Monograph PRG-19, 1981.

[10] D.S. Scott, Domains for denotational semantics, in: M. Nielsen, E.M. Schmidt (Eds.), Automata, Languages and Programming, Springer, Berlin, 1982, pp. 577–613.

[11] S. Valentini, Decidability in Intuitionistic Theory of Types is functionally decidable, Math. Logic Quart. 42 (1996) 300–304.

[12] S. Valentini, Extensionality versus Constructivity, Mathematical Logic Quart. 48(2) (2002) 179–187.