电磁场与电磁波笔记

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目录

1. 矢量分析

笔者注: 严格的来讲, 矢量符号应该都是粗体, 但为了方便, 本文中部分矢量符号没有加粗

1.1. 矢量表达

1.1.1. 三维直角坐标系

此坐标系中任一矢量 $\boldsymbol{A}$ 可表示为:

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{e}_x A_x + \boldsymbol{e}_y A_y + \boldsymbol{e}_z A_z \\ &= \sum_i A_i \boldsymbol{e}_i \quad (i = 1, 2, 3 \text{ 表示 } x, y, z) \\ &= A_i \boldsymbol{e}_i \quad (\text{爱因斯坦求和约定})\end{aligned} \tag{1}$$

单位矢量 $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$ 满足:

$$e_i \cdot e_j = \delta_{ij} \tag{2}$$

$$e_i \times e_j = \varepsilon_{ijk} e_k \tag{3}$$

其中 δ_{ij} 叫做克罗内克 δ 函数

$$\delta_{ij} = \begin{cases} 1\ ,\ i = j \\ 0\ ,\ i \neq j \end{cases} \tag{4}$$

ε_{ijk} 为列维-奇维塔符号

$$\varepsilon_{ijk} = \begin{cases} +1 & (\text{当} i, j, k \text{ 是偶排列, 即} (1,2,3) \text{ 的循环排列}) \\ -1 & (\text{当} i, j, k \text{ 是奇排列, 即} (1,2,3) \text{ 的反循环排列}) \\ 0 & (\text{当} i = j, \text{或} i = k, \text{或} j = k) \end{cases} \tag{5}$$

例: $e_z \times e_y = \varepsilon_{zyx} e_x + \varepsilon_{zyy} e_y + \varepsilon_{zyz} e_z = -e_x$

- 矢量点乘

$$\boldsymbol{A}\cdot\boldsymbol{B}=(A_ie_i)\cdot\left(B_je_j\right)=A_iB_j\left(e_i\cdot e_j\right)=A_iB_i\delta_{ij}=A_iB_i \tag{6}$$

- 矢量叉乘

$$\boldsymbol{A}\times\boldsymbol{B}=(A_ie_i)\times\left(B_je_j\right)=A_iB_j\left(e_i\times e_j\right)=A_iB_j\varepsilon_{ijk}e_k \tag{7}$$

例: 当 k 为 x 时, 上式为 $A_yB_z\varepsilon_{yzx}e_x+A_zB_y\varepsilon_{zyx}e_x=e_x\left(A_yB_z-A_zB_y\right)$, 此即为叉乘结果的 x 分量

叉乘结果也可写为

$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix} e_x & e_y & e_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{8}$$

一个列维-奇维塔符号的性质:

$$\varepsilon_{ijk}\varepsilon_{mnk}=\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm} \tag{9}$$

由此可证明 $A\cdot(B\times C)=C\cdot(A\times B)=B\cdot(C\times A)$, 过程如下

$$\begin{aligned}
&\underbrace{A}_{k}\cdot\left(\underbrace{B}_{i}\times\underbrace{C}_{j}\right)\\
&=A_ke_k\cdot\left(e_mB_iC_j\varepsilon_{ijm}\right)\\
&=\delta_{km}\varepsilon_{ijm}A_kB_iC_j=\varepsilon_{ijk}A_kB_iC_j\\
&=\varepsilon_{jki}A_kB_iC_j\quad(B\cdot(C\times A))\\
&=\varepsilon_{kij}A_kB_iC_j\quad(C\cdot(A\times B))
\end{aligned} \tag{10}$$

1.2. ∇ 算符运算

∇算符 读作 “nabla” 或 “del”, 其既有矢量性, 也有微分性

$$\begin{aligned}\nabla &= \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z} \\ &= \boldsymbol{e}_i \partial_i\end{aligned} \tag{11}$$

- 标量场的梯度

$$\nabla U(\boldsymbol{r}) = e_i \partial_i U(\boldsymbol{r}) \tag{12}$$

- 矢量场的散度

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = e_i \partial_i \cdot A_j e_j = \delta_{ij} \partial_i A_j = \partial_i A_i \tag{13}$$

- 矢量场的旋度

$$\begin{aligned}\nabla \times \boldsymbol{A}(\boldsymbol{r}) &= e_i \partial_i \times e_j A_j = \varepsilon_{ijk} e_k \partial_i A_j \\ &= \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} & \frac{\partial}{\partial_z} \\ A_x & A_y & A_z \end{vmatrix}\end{aligned} \tag{14}$$

几个典型性质:

1. 假设f, g都是标量, 则

$$\nabla(fg) = g(\nabla f) + f(\nabla g) \tag{15}$$

证明:

$$\nabla(fg) = e_i\partial_i(fg) = e_i(g(\partial_i f) + f(\partial_i g)) = g(\nabla f) + f(\nabla g) \tag{16}$$

2. 假设$\boldsymbol{A}$是矢量, 则

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \tag{17}$$

证明:

$$\begin{aligned}
\nabla \times (\nabla \times A) &= \nabla \times (e_k \varepsilon_{mnk} \partial_m A_n) \\
&= \left(e_j \partial_j\right) \times (e_k \varepsilon_{mnk} \partial_m A_n) \\
&= e_i \varepsilon_{ijk} \varepsilon_{mnk} \partial_j \partial_m A_n \\
&= e_i \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) \partial_j \partial_m A_n \\
&= e_i \partial_j \partial_i A_j - e_i \partial_j \partial_j A_i \\
&= e_i \partial_i \left(\partial_j A_j\right) - \partial_j^2 e_i A_i \\
&= \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}
\end{aligned} \tag{18}$$