Math Review

Econ 50 - Winter Quarter 2015/2016

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1 Rules for Logs

The fundamental rule for natural logs is

$$\ln(a \times b) = \ln a + \ln B$$

Therefore logs transform a multiplicative relationship into an additive relationship. This is true for different numbers a and b, but also for the same number mutliple times; therefore it follows, for example, that

$$\ln a^3 = \ln(a \times a \times a) = \ln a + \ln a + \ln a = 3 \ln a$$

and more generally that

$$\ln a^b = b \ln a$$

Similarly, since we can rewrite $\ln a - \ln b$ as $\ln a + (-1 \times \ln b)$, and since $-1 \times \ln b = \ln b^{-1}$, it follows that

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

Finally, some special cases that will come up over and over again:

$$\ln(x^a y^b) = a \ln x + b \ln y$$

and in particular:

$$\ln(x^\alpha y^{1-\alpha}) = \alpha x + (1 - \alpha) y$$

2 Concavity and Convextiy

A **convex combination** of two quantities x_1 and x_2 is

$$\alpha x_1 + (1 - \alpha) x_2$$

for some α between 0 and 1. Note that when $\alpha = 0$ this expression is the same as x_2, and when $\alpha = 1$ this expression is the same as x_1. Therefore a convex combination can also be thought of as a weighted average of x_1 and x_2, where α is the weight placed on x_1 and $1 - \alpha$ is the weight placed on x_2.

For example, suppose $x_1 = 3$ and $x_2 = 12$. If $\alpha = \frac{1}{3}$ then

$$\alpha x_1 + (1 - \alpha) x_2 = \frac{1}{3} \times 3 + \frac{2}{3} \times 12 = 1 + 8 = 9$$

If you think about it, 9 lies two-thirds of the way between 3 and 12. If you repeat this for $\alpha = \frac{1}{2}$, you'll find the average value between the two numbers; and so forth.

The same logic applies to points or vectors. In this case the convex combination of two points, (x_1, y_1) and (x_2, y_2), is the set of points that lie along the line segment connecting those two points – i.e., $(\alpha x_1 + (1 - \alpha) x_2, \alpha y_1 + (1 - \alpha) y_2)$.

A function is said to be convex if

$$f(\alpha x_1 + (1 - \alpha) x_2) \leq \alpha f(x_1) + (1 - \alpha) f(x_2)$$

In other words, if you draw a segment between any two points of the function, that segment lies *above* the graph of the function itself.

Conversely, a function is said to be concave if

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2)$$

In other words, if you draw a segment between any two points of the function, that segment lies *below* the graph of the function itself.

Another way of validating that a function is convex is if the region above its graph is a **convex set**. A convex set is a set of points such that for any two points in that set, any convex combination of those points is also in the set. So for example, any interval on the real line (e.g., [0,1]) is a convex set, since for any two numbers in the interval, any weighted average of those two numbers is also in that interval. However, a set made up of the union of two disjoint intervals (e.g., [0,1] and [2,3]) is not a convex set, since the average of 0.5 and 2.5 is not in that set. For more on convexity and things that violate it, check out the wikipedia article on "Convex function" or look at the discussion in Varian, Chapter 3, section 3.5. You'll get some practice with convexity in Homework 2 as well.

3 The Chain Rule

See Math is Fun for a simple reminder of derivative rules such as the product rule, quotient rule, etc.

The most important rule for this course, though, is the chain rule; and it's especially important for the economic intuition behind what it means.

The mathematical formulation of the chain rule is this: if $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x))g'(x)$$

For example, suppose we have the function $h(x) = (3x+2)^2$. We can rewrite this as $f(g(x))$ if

$$f(g) = g^2$$
$$g(x) = 3x + 2$$

Therefore

$$\frac{df}{dg} = 2g$$
$$\frac{dg}{dx} = 3$$

so we have

$$\frac{dh}{dx} = \frac{df}{dg} \times \frac{dg}{dx} = 2(3x+2) \times 3 = 18x + 12$$

Note that if we had expanded $h(x)$ into $9x^2 + 12x + 4$, we would have gotten this same result by taking the simple derivative.

So, what's the intuition here? Basically, the question is how the whole expression changes due to a small change in the exogenous variable x. $f'(x)$ describes the rate at which the outer function changes per unit change in the inner function; and $g'(x)$ describes the rate at which the inner function changes per unit change in x. Put another way, the chain rule just says

$$\underbrace{\frac{df}{dx}}_{\text{change in } f \text{ per one-unit change in } x} = \underbrace{\frac{df}{dg}}_{\text{change in } f \text{ per unit change in } g} \times \underbrace{\frac{dg}{dx}}_{\#\text{ units change in } g \text{ per one-unit change in } x}$$

This "multiplication of rates" is familiar to us from algebra. For example, suppose a car gets 30 miles per gallon. We can therefore write the gallons required to drive m miles as

$$g(m) = \frac{m}{30}$$

If we want to figure out how many gallons we're using as a function of the number of hours driven, instead of miles, then this is further going to depend on the speed driven (i.e., how hours relate to miles). Therefore if we say that the number of miles driven in h hours is given by $m(h)$, then the gallons required to drive h hours becomes:

$$g(m(h)) = \frac{m(h)}{30}$$

Now suppose we're interested in the rate that we're using gasoline per hour; this is the derivative of $g(m(h))$ with respect to h. In order to find that, we can use the chain rule:

$$\frac{dg}{dh} = \frac{dg}{dm} \times \frac{dm}{dh}$$

which we can now interpret as

$$\frac{\text{gallons}}{\text{hour}} = \frac{\text{gallons}}{\text{mile}} \times \frac{\text{miles}}{\text{hour}}$$

Therefore, for example, if the car's fuel efficiency is given by $\frac{dg}{dm} = \frac{1}{30}$ gallons per mile, and is traveling at $\frac{dm}{dh} = 60$ miles per hour, it will use $\frac{1}{30} \times 60 = 2$ gallons per hour.

This same principle of multiplied rates also true if the functions in question are multivariate. Harvey Mudd College has a good quick tutorial on this. To choose an example that's a harbinger of things to come, let's assume that there's a utility function $u(x, y)$ that assigns a total "utility" to consuming x units of good X and y units of good Y. Furthermore, let's assume that given the prices P_x and P_y and income I, we've figured out "demand equations" $q_x^D(P_x, P_y, I)$ and $q_y^D(P_x, P_y, I)$. We now want to figure out how much your utility would change due to a change in income. Well, we can write

$$V(P_x, P_y, I) = u(q_x^D(P_x, P_y, I), q_y^D(P_x, P_y, I))$$

for what we'll call the *indirect utility* of P_x, P_y, and I: that is, the utility you would get if you made the optimal purchase decision based on P_x, P_y, and I. The derivative of this with respect to income is

$$\frac{\partial V(P_x, P_y, I)}{\partial I} = \underbrace{\frac{\partial u(q_x^D, q_y^D)}{\partial q_x^D} \times \frac{\partial q_x^D}{\partial I}}_{\Delta V \text{ due to } \Delta q_x^D \text{ due to } \Delta I} + \underbrace{\frac{\partial u(q_x^D, q_y^D)}{\partial q_y^D} \times \frac{\partial q_y^D}{\partial I}}_{\Delta V \text{ due to } \Delta q_y^D \text{ due to } \Delta I}$$

This looks scary, but it's actually not so bad. What happens when income changes? You change your consumption of X and Y. What happens when you change your consumption of X and Y? Your utility changes. So the total change in your utility due to a change in income is composed of two parts:

1. the change in $u(q_x^D, q_y^D)$ per unit change in q_x^D, multiplied by

   the number of units q_x^D changed due to the change in I

2. the change in $u(q_x^D, q_y^D)$ per unit change in q_y^D, multiplied by

   the number of units q_y^D changed due to the change in I

These are represented by the two terms of the expression on the right.

4 Implicit differentiation

Consider the level curve of a multivariate function

$$f(x, y) = k$$

We can think about this level curve defining a relationship between y and x; that is, near any point (x, y) along this level curve, we can write the function $y(x)$ to describe the relationship of y to x, and $\frac{dy}{dx}$ corresponds to the slope of the level curve in that area.

To solve for $\frac{dy}{dx}$, we take the derivative of both sides with respect to x:

$$\frac{d}{dx}[f(x, y(x))] = \frac{d}{dx}[k]$$

The right-hand side is just equal to zero; applying the chain rule as we did in the previous section, therefore, we get

$$\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx} = 0$$

In short, the rate at which the left-hand side changes due to a change in x is the direct change $\left(\text{that is, } \frac{\partial f(x,y)}{\partial x}\right)$ plus the change due to the change in y, due to the change in x $\left(\text{that is, } \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx}\right)$.

Solving this for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = -\frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}}$$

See Khan Academy's videos on implicit differentiation for a good introduction and example.

5 Lagrange multiplier method

Suppose we want to maximize a function $f(x, y)$ subject to the constraint $g(x, y) = 0$. If lots of good conditions hold (and we'll get into these in class), the necessary "first-order conditions" for an optimum are obtained by writing down the function

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

and taking the derivatives of this function with respect to x, y, and λ, and setting those equal to zero:

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} = \frac{\partial f(x,y)}{\partial x} + \lambda\frac{\partial g(x,y)}{\partial x} = 0 \qquad \Rightarrow \lambda = -\frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial g(x,y)}{\partial x}} \tag{1}$$

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} = \frac{\partial f(x,y)}{\partial y} + \lambda\frac{\partial g(x,y)}{\partial y} = 0 \qquad \Rightarrow \lambda = -\frac{\frac{\partial f(x,y)}{\partial y}}{\frac{\partial g(x,y)}{\partial y}} \tag{2}$$

$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} = g(x, y) = 0 \tag{3}$$

Condition (3) just says that the optimal solution must lie along the constraint. Let's look at conditions (1) and (2), though. Setting the values of λ equal to each other, we have

$$-\frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial g(x,y)}{\partial x}} = -\frac{\frac{\partial f(x,y)}{\partial y}}{\frac{\partial g(x,y)}{\partial y}}$$

Cross-multiplying yields the "tangency condition"

$$\frac{\frac{\partial f(x,y)}{\partial x}}{\frac{\partial f(x,y)}{\partial y}} = \frac{\frac{\partial g(x,y)}{\partial x}}{\frac{\partial g(x,y)}{\partial y}}$$

The expressions on either side of this equation are familiar from our previous analysis of implicit differentiation: namely, they're the slopes of the level curves of $f(x, y)$ and $g(x, y)$. Now, the constraint is a specific level curve: in particular, it's the level curve where $g(x, y) = 0$. So the Lagrange method works by finding the point along the level curve $g(x, y) = 0$ at which the slope of the level curve of the objective function matches the slope of the constraint.

6 Summary

A proficiency with logs, convexity, taking derivatives, implicit differentiation, and the Lagrange method are not the only mathematical techniques we'll be using in this class. However, they represent the areas where most students who have taken the course in the past have found difficulty. This brief review is not a substitute for a deeper understanding of these techniques, and the purpose of the math review session for which these are notes was not to teach the material as new material. So please use this review sheet mostly as a list of things to know and review in greater detail if you're rusty.