Group Work Solutions - Quasilinear PCC, ICC, Demand and Engel Curves

Econ 50 - Winter Quarter 2015/2016

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1. **Initial optimization.**

   Since we didn't do a worked example of corner solutions last lecture, let's start by solving for one.

   (a) Find the values of (x^*, y^*) that the Lagrange method would give you for general P_x, P_y, and I.

   The marginal rate of substitution for this utility function is

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{12/x}{1} = \frac{12}{x}$$

   Setting this equal to the price ratio yields the relationship

$$\begin{aligned}
\frac{12}{x} &= \frac{P_x}{P_y} \\
P_x x &= 12P_y \\
x^* &= \frac{12P_y}{P_x}
\end{aligned}$$

   This gives us our optimal value of X, which we can see does not depend on income. Plugging the value for $P_x x$ (from the next-to-last line) into the budget constraint, we can solve for the optimal value of Y:

$$\begin{aligned}
P_x x + P_y y &= I \\
12P_y + P_y y &= I \\
P_y y &= I - 12P_y \\
y^* &= \frac{I}{P_y} - 12
\end{aligned}$$

   (b) For what values of P_x, P_y, and I does this approach produce an interior solution? For which values will they result in a corner solution?

   This produces an interior solutions as long as $x^* \geq 0$ and $y^* \geq 9$. Since $x^* \geq 0$ for any values of P_x and P_y, the possibility of a corner solution arises if $y^* < 0$. The condition for this occurring is:

$$\begin{aligned}
y^* = \frac{I}{P_y} - 12 &< 0 \\
\frac{I}{P_y} &< 12 \\
I &< 12P_y
\end{aligned}$$

(c) Write down the general formula (including corner solutions) for (x^*, y^*) as a function of P_x, P_y, and I.

$$x^*(P_x, P_y, I) = \begin{cases} \frac{12P_y}{P_x} \text{ if } I \geq 12P_y \\ \frac{I}{P_x} \text{ if } I < 12P_y \end{cases}$$

$$y^*(P_x, P_y, I) = \begin{cases} \frac{I}{P_y} - 12 \text{ if } I \geq 12P_y \\ 0 \text{ if } I < 12P_y \end{cases}$$

(d) Find the optimal consumption point if $P_x = 6$, $P_y = 6$, and $I = 120$.

In this case

$$\frac{I}{P_y} = \frac{120}{6} = 20$$

so $\frac{I}{P_y} - 12 > 0$ and we have an interior solution. In particular, we have

$$x^* = \frac{12P_y}{P_x} = \frac{12 \times 6}{6} = 12$$
$$y^* = \frac{I}{P_y} - 12 = \frac{120}{6} - 12 = 8$$

2. **Predict what the PCC and ICC curves will look like.**

   Based upon your intuition about this utility function (diminishing marginal utility for X, constant marginal utility for Y, sometimes corner solutions), what do you think the graphs of PCC_x, PCC_y, and ICC that pass through the optimal consumption point if $P_x = 6$, $P_y = 6$, and $I = 120$?

3. **Check your results.**

   Go to this site and click on "snap to optimal" to see plots of the PCC_x, PCC_y, and ICC. Do their shapes match your prediction? Do you understand why?

4. **Price-Consumption Curves**

   Derive the mathematical formulas for the price-consumption curves, and plot them precisely. Specifically:

(a) Holding $P_y = 6$ and $I = 120$ constant, derive and plot the price-consumption curve for good X (PCC_x) by finding the optimal points (x^*, y^*) as a function of P_x.

Since $\frac{I}{P_y} = \frac{120}{6} = 20 > 12$ no matter what P_x is, we'll always be at an interior solution for this problem.

$$x^* = \frac{12P_y}{P_x} = \frac{12 \times 6}{P_x} = \frac{72}{P_x}$$
$$y^* = \frac{I}{P_y} - 12 = \frac{120}{6} - 12 = 8$$

Since X varies while Y remains constant, this is a horizontal line at $y = 8$.

(b) Holding $P_x = 6$ and $I = 120$ constant, derive and plot the price-consumption curve for good Y (PCC_y) by finding the optimal points (x^*, y^*) as a function of P_y. (Hint: you might derive a general formula as a group, and then have individual team members find and evaluate (x^*, y^*) for $P_y = 1, 2, 3, 4, 8, 10, 12$.)

As long as $\frac{120}{P_y} > 12$, or $P_y < 10$, this will result in an interior solution, with x^* and y^* given by

$$x^* = \frac{12P_y}{P_x} = \frac{12P_y}{6} = 2P_y$$
$$y^* = \frac{I}{P_y} - 12 = \frac{120}{P_y} - 12$$

When $P_y \geq 10$, we will simply have $x^* = 20$ and $y^* = 0$.

Following the hint, we can make the following table:

P_y	x^*	y^*
1	2	108
2	4	48
3	6	28
4	8	18
6	12	8
8	16	3
10	20	0
12	20	0

This is a downward-sloping curve which ends at the point (10,0) along the X-axis.

5. **Income-Consumption Curve**

   Derive the mathematical formulas for the income-consumption curve, and plot it precisely. Specifically, holding $P_x = 6$ and $P_y = 6$ constant, derive and plot the income-consumption curve (ICC) by finding the optimal points (x^*, y^*) as a function of I.

   If $P_x = 6$ and $P_y = 6$, then the condition $\frac{I}{P_y} > 12$ becomes $I > 72$.

   We start at the origin when $I = 0$, since with no money the consumer can afford none of either good.

   As we increase I, we have corner solutions where $x^* = \frac{I}{6}$ and $y^* = 0$, which traces the X-axis, until we get to the point (12,0) when $I = 72$.

   From then on, we have $x^* = 12$ and $y^* = \frac{I}{6} - 12$, which is a vertical line at $x = 12$.

6. **Demand Curves**

   Derive the mathematical formulas for the demand curves for X and Y, and plot them precisely. Specifically:

   (a) Holding $P_y = 6$ and $I = 120$ constant, derive and plot the demand curve for good X $q_x^D(P_x)$. Using just the value for x^* from the PCC_x calculations, this is $q_x^D(P_x|P_y = 6, I = 120) = \frac{72}{P_x}$.

   (b) Holding $P_x = 6$ and $I = 120$ constant, derive and plot the price-consumption curve for good Y $q_y^D(P_y)$. (Hint: you can use the points you found while calculating the price-consumption curves...)

   We could use the table, or just the formula:

$$q_y^D(P_y|P_x = 6, I = 120) = \begin{cases} 0 \text{ if } P_y \geq 10 \\ \frac{120}{P_y} - 12 \text{ if } P_y < 10 \end{cases}$$

7. **Engel Curves**

   Derive the mathematical formulas for the Engel curves for X and Y, and plot them precisely. Specifically, holding $P_x = 6$ and $P_y = 6$ constant, derive and plot the Engel curves for X and Y. (Hint: you can use the points you found while calculating the income-consumption curves...)

   This is a little tricky. Remember that I is the independent variable, and (like P in a demand diagram) is plotted on the vertical axis. As we would expect from the logic of the ICC, there will be two segments of this curve: one when $I < 72$, and the other when $I \geq 72$.

   When $I < 72$, we have $y^* = 0$ and $x^* = \frac{I}{6}$; so the Engel curve for Y is a vertical line along the Y-axis, and the Engel curve for X slopes upward from the origin until the point (12, 72).

   When $I \geq 72$, we have $y^* = \frac{I}{6} - 12$ and $x^* = 12$; so the Engel curve for X is a vertical line at $x = 12$, and the Engel curve for Y slopes upward from the point (0,72) at a slope of 6 (since the consumer is spending all their additional money on Y, and each unit of Y (shown on the X-axis...confusing I know!) costs $P_x = 6$.