Derivations: Returns to Scale and Elasticity of Substitution

Econ 50 - Winter 2015/2016 - Lecture 12

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1 Definitions

1.1 Returns to Scale

A production function exhibits:

- Increasing returns to scale if $f(tL, tK) > tf(L, K)$
- Constant returns to scale if $f(tL, tK) = tf(L, K)$
- Decreasing returns to scale if $f(tL, tK) < tf(L, K)$

for some "scale factor" $t > 1$.

1.2 Marginal Rate of Technical Substitution

The $MRTS_{L,K}$ is the analog of the $MRS_{x,y}$ for a utility function. It is the (negative of the) slope of an isoquant at a particular point, and represents the rate at which a firm can substitute labor for capital: specifically, the amount by which the quantity of capital can be reduced per one-unit increase in labor, while keeping output constant.

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{\partial f(L,K)/\partial L}{\partial f(L,K)/\partial K}$$

1.3 Elasticity of Substitution

The elasticity of substitution (σ) is the ratio of a percentage change in the capital-labor ratio K/L to a percentage change in the $MRTS$:

$$\sigma = \frac{\%\Delta \frac{K}{L}}{\%\Delta MRTS}$$

Since it's more natural to think of the capital-labor ratio as "exogenous" and the $MRTS$ as "endogenous," it's probably better to think about this as the inverse of the elasticity of the $MRTS$ with respect to K/L: that is, think of $MRTS$ as a function of K/L, and take its elasticity.

$$\sigma = \frac{1}{\%\Delta MRTS / \%\Delta \frac{K}{L}}$$

2 Cobb-Douglas production function: $q = f(L, K) = AL^{\alpha}K^{\beta}$

2.1 Returns to Scale for Cobb-Douglas

Multiplying each of the inputs by $t > 1$ yields

$$\begin{aligned} f(tL, tK) &= A(tL)^{\alpha}(tK)^{\beta} \\ &= t^{\alpha+\beta}AL^{\alpha}K^{\beta} \\ &= t^{\alpha+\beta}f(L, K) \end{aligned}$$

This is greater than $tf(L, K)$, and therefore exhibits increasing returns to scale, if $\alpha + \beta > 1$.
This is equal to $tf(L, K)$, and therefore exhibits constant returns to scale, if $\alpha + \beta = 1$.
This is less than $tf(L, K)$, and therefore exhibits decreasing returns to scale, if $\alpha + \beta < 1$.
For example, suppose $t = 2$, so we are considering doubling inputs. For simplicity, let's suppose $A = 1$.

- If $\alpha = \beta = 1$, then $f(L, K) = LK$. Therefore

  $$f(2L, 2K) = (2L)(2K) = 4LK$$

  $$2f(L, K) = 2LK$$

  Since $f(2L, 2K) > 2f(L, K)$, we have increasing returns to scale.

- If $\alpha = \beta = \frac{1}{2}$, then $f(L, K) = L^{\frac{1}{2}}K^{\frac{1}{2}}$. Therefore

  $$f(2L, 2K) = (2L)^{\frac{1}{2}}(2K)^{\frac{1}{2}} = 2L^{\frac{1}{2}}K^{\frac{1}{2}}$$

  $$2f(L, K) = 2L^{\frac{1}{2}}K^{\frac{1}{2}}$$

  Since $f(2L, 2K) = 2f(L, K)$, we have constant returns to scale.

- If $\alpha = \beta = \frac{1}{4}$, then $f(L, K) = L^{\frac{1}{4}}K^{\frac{1}{4}}$. Therefore

  $$f(2L, 2K) = (2L)^{\frac{1}{4}}(2K)^{\frac{1}{4}} = 2^{\frac{1}{2}}L^{\frac{1}{4}}K^{\frac{1}{4}}$$

  $$2f(L, K) = 2L^{\frac{1}{4}}K^{\frac{1}{4}}$$

  Since $f(2L, 2K) < 2f(L, K)$, we have decreasing returns to scale.

2.2 Marginal Rate of Technical Substitution for Cobb-Douglas

The marginal product of labor and capital are given by

$$MP_L = \frac{\partial f(L, K)}{\partial L} = \alpha AL^{\alpha-1}K^{\beta}$$

$$MP_K = \frac{\partial f(L, K)}{\partial L} = \beta AL^{\alpha}K^{\beta-1}$$

Therefore

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{\alpha AL^{\alpha-1}K^{\beta}}{\beta AL^{\alpha}K^{\beta-1}} = \frac{\alpha K}{\beta L}$$

2.3 Elasticity of Substitution for Cobb-Douglas

Example: $f(L,K) = 2L^{\frac{1}{3}}K^{\frac{2}{3}}$, so $MRTS = \frac{K}{2L}$.

Percent change in $MRTS$ due to a 1% change in $\frac{K}{L}$ is

$$\begin{aligned}\frac{\%\Delta MRTS}{\%\Delta \frac{K}{L}} &= \frac{dMRTS}{d\frac{K}{L}} \times \frac{\frac{K}{L}}{MRTS} \\ &= \frac{1}{2} \times \frac{\frac{K}{L}}{\frac{K}{2L}} \\ &= 1\end{aligned}$$

Remember that σ is the inverse of this (which is also 1 in this case, which can be a little confusing!):

$$\sigma = \frac{1}{\%\Delta MRTS/\%\Delta \frac{K}{L}} = \frac{1}{1} = 1$$

In general, note that $MRTS$ is **linear in** $\frac{K}{L}$, regardless of the values of A, α and β. Therefore its elasticity with respect to $\frac{K}{L}$ is always 1:

$$\begin{aligned}MRTS &= \frac{\alpha}{\beta}\frac{K}{L} \\ \ln(MRTS) &= \ln(\alpha) - \ln(\beta) + \ln\left(\frac{K}{L}\right) \\ \frac{\%\Delta MRTS}{\%\Delta \frac{K}{L}} = \frac{d\ln(MRTS)}{d\ln\left(\frac{K}{L}\right)} &= 1\end{aligned}$$

3 Linear production function: $q = f(L,K) = aL + bK$

3.1 Returns to Scale for Linear

Multiplying each of the inputs by $t > 1$ yields

$$\begin{aligned}f(tL,tK) &= a(tL) + b(tK) \\ &= t(aL + bK) \\ &= tf(L,K)\end{aligned}$$

so this is always constant returns to scale (i.e., always equals $tf(L,K)$).

There is a “generalized” linear production function, $q = f(L,K) = (aL + bK)^{\gamma}$, which is increasing returns to scale if $\gamma > 1$, constant returns to scale if $\gamma = 1$, and decreasing returns to scale if $\gamma < 1$.

3.2 Marginal Rate of Technical Substitution for Linear

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{a}{b}$$

3.3 Elasticity of Substitution for Linear

Since the $MRTS$ doesn’t change with the capital/labor ratio, the elasticity of substitution is infinite.

4 Fixed-proportions production function: $f(L, K) = \min(aL, bK)$

4.1 Returns to Scale for Fixed Proportions

Multiplying each of the inputs by $t > 1$ yields

$$\begin{aligned} f(tL, tK) &= \min(atL, btK) \\ &= t \times \min(aL, bK) \\ &= tf(L, K) \end{aligned}$$

so this is always constant returns to scale (i.e., always equals $tf(L, K)$).

As with the linear case, there is a "generalized" fixed-proportions production function, $q = f(L, K) = [min(aL, bK)]^\gamma$, which is increasing returns to scale if $\gamma > 1$, constant returns to scale if $\gamma = 1$, and decreasing returns to scale if $\gamma < 1$.

4.2 Marginal Rate of Technical Substitution for Fixed Proportions

As with utility functions, this is either undefined (if $K/L \geq a/b$) or zero (if $K/L < a/b$)

4.3 Elasticity of Substitution for Fixed Proportions

This is zero, since you cannot substitute capital for labor at all.