Homework 4 Solutions

Econ 50 - Stanford University - Winter Quarter 2015/16

Tuesday, February 9

Exercise 1: Math warmup

(a) Solve the cost-minimization problem.

Answer: We set up the cost-minization problem:

$$\min_{x,y}\{P_x x + P_y y\} \quad \text{s.t. } \alpha \ln(x) + (1-\alpha)\ln(y) = U$$

And the associated Lagrangian:

$$\mathcal{L}(x,y,\lambda) = P_x x + P_y y + \lambda\big(U - \alpha\ln(x) - (1-\alpha)\ln(y)\big)$$

Partially differentiate with respect to x, y, λ and setting derivatives equal to 0 yields:

$$\begin{aligned}
\frac{\partial\,\mathcal{L}(x,y,\lambda)}{\partial x} &= P_x - \frac{\lambda\alpha}{x} = 0 && \Rightarrow \lambda = \frac{P_x x}{\alpha} \\
\frac{\partial\,\mathcal{L}(x,y,\lambda)}{\partial y} &= P_y - \frac{\lambda(1-\alpha)}{y} = 0 && \Rightarrow \lambda = \frac{P_y y}{1-\alpha} \\
\frac{\partial\,\mathcal{L}(x,y,\lambda)}{\partial \lambda} &= U - \alpha\ln(x) - (1-\alpha)\ln(y) = 0 && \Rightarrow \alpha\ln(x) + (1-\alpha)\ln(y) = U
\end{aligned}$$

Combining the first two equations to eliminate λ yields:

$$x = \frac{\alpha}{1-\alpha}\frac{P_y}{P_x}y$$

Substituting this expression for x into the constraint:

$$\begin{aligned}
U &= \alpha\ln\left(\frac{\alpha}{1-\alpha}\frac{P_y}{P_x}y\right) + (1-\alpha)\ln(y) \\
&= \alpha\ln\left(\frac{\alpha}{1-\alpha}\frac{P_y}{P_x}\right) + \ln(y)
\end{aligned}$$

That is:

$$\begin{aligned}
\ln(y) &= U - \ln\left[\left(\frac{\alpha}{1-\alpha}\frac{P_y}{P_x}\right)^{\alpha}\right] \\
\Rightarrow y^H(P_x, P_y, U) &= \left(\frac{\alpha}{1-\alpha}\frac{P_y}{P_x}\right)^{-\alpha} e^U \\
&= \left(\frac{1-\alpha}{\alpha}\frac{P_x}{P_y}\right)^{\alpha} e^U
\end{aligned}$$

And:

$$x^H(P_x,P_y,U) = \frac{\alpha}{1-\alpha}\frac{P_y}{P_x}y^H = \left(\frac{\alpha}{1-\alpha}\frac{P_y}{P_x}\right)^{1-\alpha} e^U$$

(b) Find the expenditure function.
Answer: Plugging $x^H(P_x,P_y,U), y^H(P_x,P_y,U)$ into the objective function yields:

$$\begin{aligned}
E(P_x,P_y,U) &= P_x x^H(P_x,P_y,U) + P_y y^H(P_x,P_y,U) \\
&= P_x\left(\frac{\alpha}{1-\alpha}\frac{P_y}{P_x}\right)^{1-\alpha} e^U + P_y\left(\frac{1-\alpha}{\alpha}\frac{P_x}{P_y}\right)^{\alpha} e^U \\
&= e^U P_x^{\alpha} P_y^{1-\alpha}\left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right]
\end{aligned}$$

(c) Find the indirect utility function.
Answer: Setting income equal to the expenditure function and rearrange for U:

$$\begin{aligned}
I &= e^U P_x^{\alpha} P_y^{1-\alpha}\left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right] \\
\Rightarrow e^U &= \frac{I}{P_x^{\alpha} P_y^{1-\alpha}\left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right]} \\
\Rightarrow V(P_x,P_y,I) &= \ln\left[\frac{I}{P_x^{\alpha} P_y^{1-\alpha}\left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right]}\right]
\end{aligned}$$

(d) Find the Marshallian demand functions.
Answer: Substituting the indirect utility function into the Hicksian demand $x^H(P_x,P_y,U)$ yields:

$$\begin{aligned}
x(P_x,P_y,I) &= x^H(P_x,P_y,V(P_x,P_y,I)) \\
&= \left(\frac{\alpha}{1-\alpha}\frac{P_y}{P_x}\right)^{1-\alpha} \frac{I}{P_x^{\alpha} P_y^{1-\alpha}\left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right]} \\
&= \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \frac{I}{P_x\left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\right]} \\
&= \frac{I}{P_x\left[1+\frac{1-\alpha}{\alpha}\right]} \\
&= \frac{\alpha I}{P_x}
\end{aligned}$$

Similarly:

$$
\begin{aligned}
y(P_x, P_y, I) &= y^H(P_x, P_y, V(P_x, P_y, I)) \\
&= \left(\frac{1-\alpha}{\alpha}\frac{P_x}{P_y}\right)^{\alpha} \frac{I}{P_x^{\alpha} P_y^{1-\alpha}\left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right]} \\
&= \left(\frac{1-\alpha}{\alpha}\right)^{\alpha} \frac{I}{P_y\left[\left(\frac{1-\alpha}{\alpha}\right)^{-(1-\alpha)} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right]} \\
&= \frac{I}{P_y\left[\frac{\alpha}{1-\alpha} + 1\right]} \\
&= \frac{(1-\alpha)I}{P_y}
\end{aligned}
$$

Exercise 2: It's all the same to me

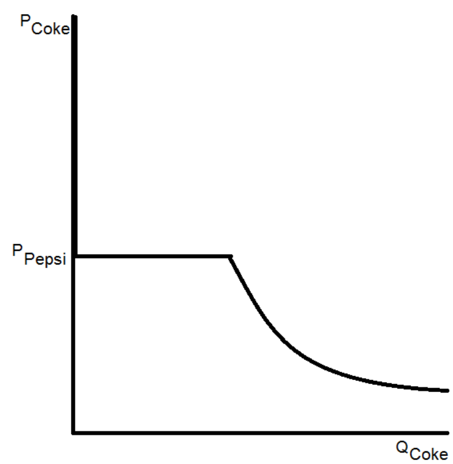
Nick cannot for the life of him tell the difference between Coke and Pepsi; he views them as perfect substitutes for one another.

(a) Clearly draw his demand curve for Coke on a carefully labeled diagram. [5 points]

Answer: If they are perfect substitutes, then Nick buys only Coke if it's cheaper than Pepsi. Otherwise he buys no Coke. Mathematically this is

$$Q_{Coke} = \frac{I}{P_{Coke}} \; if \; P_{Coke} < P_{Pepsi}$$

$$Q_{Coke} = 0 \; if \; P_{Coke} > P_{Pepsi}$$

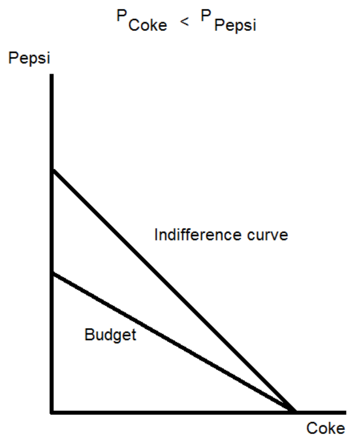


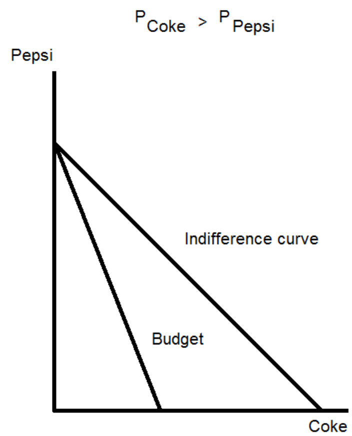
(b) Pick any point on the demand curve above, and label it point A. Use a budget-line/indifference curve diagram to explain what's going on at that point. [8 points]

Answer: The indifference curve is linear with a slope of -1 because consuming one less can of Coke and one more can of Pepsi leaves Nick indifference. The slope of the budget line is determined by the relative prices.

Two scenarios are given by the equations in part (a). In the first scenario, the slope of the budget line is shallower than the slope of the indifference curve, causing the optimal bundle to be at a corner solution on the Coke axis. Nick consumes all Coke and no Pepsi and this corresponds to the "curved" section in part (a).

In the second scenario, the slope of the budget line is steeper than the indifference curve, causing the optimal bundle to be at a corner solution on the Pepsi axis. Nick consumes no Coke, which corresponds to the vertical section in part (a).





There is a third scenario when the prices are equal. This corresponds to the horizontal segment in part (a). Nick is indifferent between Coke and Pepsi and can consume any amount of Coke up to the point that his budget is exhausted. The rest of the budget will be spent on Pepsi.

(c) Explain what would happen if the price of Coke dropped a little bit (say, a penny) from the point you chose in part (b). [2 points]

Answer: In the first scenario, Nick still consumes all Coke because it's still cheaper. He now consumes a bit more. In the second scenario, Pepsi is still cheaper so Nick still consumes 0 Coke. In the third scenario, he could have started with any amount of Coke. After the price change, he consumes all Coke.

Exercise 3: Define and Quantify

Suppose Buster's preferences over bats (x) and mitts (y) are summarized by the utility function $u(x, y) = x + 2y$.

(a) $x* = \begin{cases} \frac{I}{Px} & 2P_x < P_y \\ 0 & 2P_x > P_y \\ [0, \frac{I}{P_x}] & 2P_x = P_y \end{cases}$

$$y* = \begin{cases} 0 & 2P_x < P_y \\ \frac{I}{P_y} & 2P_x > P_y \\ [0, \frac{I}{P_y}] & 2P_x = P_y \end{cases}$$

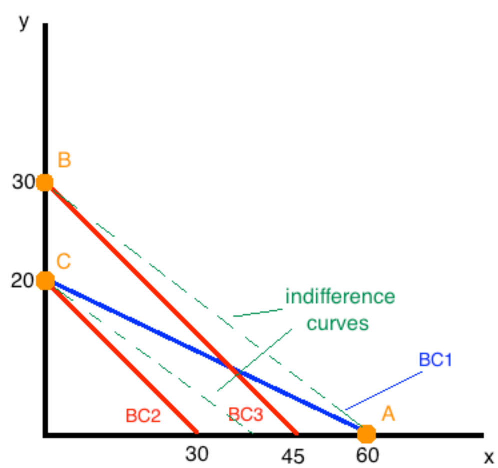
$$V = \begin{cases} \frac{I}{Px} & 2P_x < P_y \\ \frac{2I}{P_y} & 2P_x > P_y \\ \frac{I}{P_x} = \frac{2I}{P_y} & 2P_x = P_y \end{cases}$$

$$E = \begin{cases} P_x U & 2P_x < P_y \\ P_y(\frac{1}{2}U) & 2P_x > P_y \\ P_x U = P_y(\frac{1}{2}U) & 2P_x = P_y \end{cases}$$

$$x^H = \begin{cases} U & 2P_x < P_y \\ 0 & 2P_x > P_y \\ [0, U] & 2P_x = P_y \end{cases}$$

$$y^H = \begin{cases} 0 & 2P_x < P_y \\ \frac{1}{2}U & 2P_x > P_y \\ [0, \frac{1}{2}U] & 2P_x = P_y \end{cases}$$

(b) Originally, $P_x = 1$ and $P_y = 3$. Therefore, $\frac{MU_x}{P_x} = \frac{1}{1}$ and $\frac{MU_y}{P_y} = \frac{2}{3}$, so Buster will spend all of his income on x (as $1 > \frac{2}{3}$). Point A = (60, 0). When P_x changes to 2, $\frac{MU_x}{P_x} = \frac{1}{2}$ now, so he will spend his entire income on y (as $\frac{1}{2} < \frac{2}{3}$). Point C = (0, 20). Buster wants to find a point that will make him exactly as happy as he was at point A but under new prices. Under the new prices, we just mentioned he will spend his entire income on y, so that means we must find the point on his original indifference curve where he's spending his entire income on y. This is at point B = (0, 30).
The substitution effect (A $\rightarrow$ B) = (-60, 30). The income effect (B $\rightarrow$ C) = (0, -10). The total effect then (A $\rightarrow$ C) = (-60, 20). Note that because the goods are perfect substitutes, the substitution effect dominates the income effect (as we evaluate their impacts on y).



(c) Buster's CV is equal to the extra income he needs to achieve his initial utility at the new prices. The formula can be written as $CV = E(P_{x2}, P_y, U_1) - I$. The original utility was 60, and we showed at the new prices that $2P_x > P_y$, so $E = P_y(\frac{1}{2}U) = (3)(\frac{1}{2})(60) = \$90$. Therefore, the $CV = 90 - 60 = \$30$.
Equivalent variation is the change in income required to achieve his new utility at the initial

prices. The expenditure formula for EV is $E(P_{x1}, P_y, U_2)$. His new utility = 40 under the new prices (because at point C he is buying 0 x and 20 y). At the original prices, $2P_x < P_y$, so $E = P_x U = (1)(40) = 40$. Therefore, the change in income is \$20.

Exercise 4: A calculation and drawing exercise

Before we do anything, it shall be convenient to derive the Marshallian demand functions. This time I will take a shortcut - I will derive the optimality condition by equating MRS to the price ratio, and use the optimality condition along with the budget constraint to solve for the demand functions.

The optimality condition is given by:

$$\text{MRS} = \frac{MU_x}{MU_y} = \frac{y^2}{2xy} = \frac{y}{2x} = \frac{P_x}{P_y}$$

Rearranging yields:

$$P_y y = 2P_x x$$

Substuting this expression into the budget constraint gives us:

$$P_x x + 2P_x x = 3P_x x = I$$

Thus:

$$x(P_x, P_y, I) = \frac{I}{3P_x}$$
$$y(P_x, P_y, I) = \frac{2I}{3P_y}$$

(a) Calculate Fred's optimal consumption bundle at $P_x = 1, P_y = 1, I = 120$

Answer: Plugging in relevant numbers into the demand functions gives us:

$$x(1, 1, 120) = \frac{120}{3} = 40$$
$$y(1, 1, 120) = \frac{240}{3} = 80$$

His budget line, BL_A, is given by the linear equation $x + y = 120$.

(b) Calculate Fred's optimal consumption bundle at $P_x = 8, P_y = 1, I = 120$

Answer: Plugging in relevant numbers into the demand functions gives us:

$$x(8, 1, 120) = \frac{120}{24} = 5$$
$$y(8, 1, 120) = \frac{240}{3} = 80$$

His budget line, BL_C, is given by the linear equation $8x + y = 120$. This is obtained by rotating BL_A inwards.

(c) Derive the equation for Fred's price-consumption curve for good X, holding $P_y = 1$ and $I = 120$ constant.

Answer: Fred's demand for good X is independent of the price of good Y. Thus, it is a horizontal line at $y = 80$.

(d) Derive the equation for Fred's income-consumption curve for good X when $P_x = 1$ and when $P_x = 8$, holding $P_y = 1$ and $I = 120$ constant.

Answer: the income-consumption curve is given by the optimality condition $y = 2\frac{P_x}{P_y}x$. Plugging in prices yields:

$$ICC_1 : y = 2x$$
$$ICC_8 : y = 16x$$

(e) Derive the equation for Fred's Marshallian demand curve for good X, holding $P_y = 1$ and $I = 120$ constant.

Answer: We can plug $P_y = 1$ and $I = 120$ into the demand curve for good X obtained at the start of the question:

$$x(P_x, 1, 120) = \frac{120}{3P_x} = \frac{40}{P_x}$$

(f) Derive Fred's indirect utility function. Determine his utility at points A and C. Write down the equations for the indifference curves passing through those two points.

Answer: We can derive Fred's indirect utility function by plugging in $x(P_x, P_y, I), y(P_x, P_y, I)$ into the utility function:

$$\begin{aligned} V(P_x, P_y, I) &= u(x(P_x, P_y, I), y(P_x, P_y, I)) \\ &= \frac{I}{3P_x}\left(\frac{2I}{3P_y}\right)^2 \\ &= \frac{4I^3}{27P_xP_y^2} \end{aligned}$$

Fred's utility at points A and C are, respectively:

$$u_A : V(1, 1, 120) = 40 * 80^2 = 256000$$
$$u_C : V(8, 1, 120) = 5 * 80^2 = 32000$$

And the indifference curves passing through A and C are described by:

$$IC_A : y = \sqrt{\frac{256000}{x}} = 80\sqrt{\frac{40}{x}}$$
$$IC_C : y = \sqrt{\frac{32000}{x}} = 80\sqrt{\frac{5}{x}}$$

(g) Calculate the point at which the indifference curve passing through point A intersects the ICC passing through point C.

Answer: This point is given by the intersection of $IC_A : y = 80\sqrt{\frac{40}{x}}$ and $ICC_8 : y = 16x$. Equating the two expressions to find the x-intercept:

$$16x = 80\sqrt{\frac{40}{x}}$$
$$\Rightarrow x^2 = \frac{1000}{x}$$
$$\Rightarrow x = 10, y = 160$$

(h) Calculate the point at which the indifference curve passing through point C intersects the ICC passing through point A.

Answer: This point is given by the intersection of $IC_C : y = 80\sqrt{\frac{5}{x}}$ and $ICC_1 : y = 2x$. Equating the two expressions to find the x-intercept:

$$2x = 80\sqrt{\frac{5}{x}}$$
$$\Rightarrow x^2 = \frac{8000}{x}$$
$$\Rightarrow x = 20, y = 40$$

(i) Derive Fred's expenditure function. Use it to determine the income $I_{A,8}$ required to afford the utility at point A if $P_x = 8$, and the income $I_{C,1}$ required to afford the utility at point C if $P_x = 1$.

Answer: We arrive at the expenditure function by rearranging the indirect utility function and expressing income as a function of everything else:

$$4I^3 = 27P_xP_y^2U$$
$$\Rightarrow E(P_x, P_y, U) = \left(\frac{27P_xP_y^2U}{4}\right)^{\frac{1}{3}}$$

The income $I_{A,8}$ required to afford the utility at point A if $P_x = 8$ is:

$$E(8, 1, 256000) = \left(\frac{27 * 8 * 256000}{4}\right)^{\frac{1}{3}} = 240$$

The income $I_{C,1}$ required to afford the utility at point C if $P_x = 1$ is:

$$E(1, 1, 32000) = \left(\frac{27 * 32000}{4}\right)^{\frac{1}{3}} = 60$$

(j) See illustration.

(k) Derive Fred's Hicksian demand function for good X.

Answer: We plug the expenditure function into the Marshallian demand function for good X to arrive at Fred's Hicksian demand function for good X:

$$x^H(P_x, P_y, U) = \frac{\left(\frac{27P_xP_y^2U}{4}\right)^{\frac{1}{3}}}{3P_x} = \left(\frac{P_y}{2P_x}\right)^{\frac{2}{3}} U^{\frac{1}{3}}$$

(l) Derive the change in Fred's consumer surplus, his compensating variation and his equivalent variation.

Answer: The change in consumer surplus is given by:

$$\int_1^8 x(P_x, 1, 120)dP_x = \int_1^8 \frac{40}{P_x} dP_x = 40\big[\ln(P_x)\big]_1^8 = 40\ln(80)$$

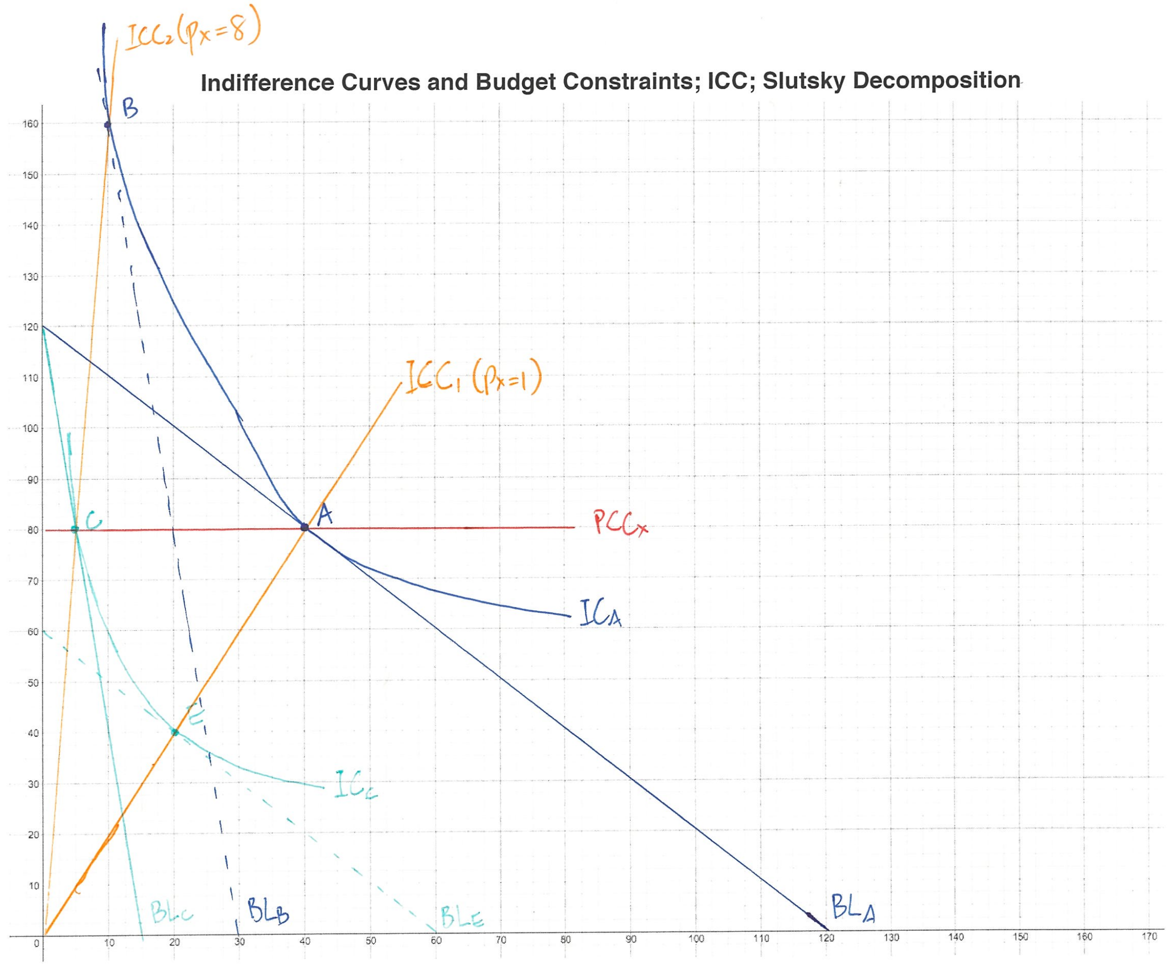
Fred's compensating variation is given by:

$$E(8, 1, 256000) - E(1, 1, 256000) = \left(\frac{27 * 8 * 256000}{4}\right)^{\frac{1}{3}} - \left(\frac{27 * 256000}{4}\right)^{\frac{1}{3}} = 240 - 120 = 120$$

Fred's equivalent variation is given by:

$$E(8, 1, 32000) - E(1, 1, 32000) = \left(\frac{27 * 8 * 32000}{4}\right)^{\frac{1}{3}} - \left(\frac{27 * 32000}{4}\right)^{\frac{1}{3}} = 120 - 60 = 60$$

Indifference Curves and Budget Constraints; ICC; Slutsky Decomposition



Marshallian and Hicksian Demands; Compensating and Equivalent Variation