Homework 1 Solutions

Econ 50 - Stanford University - Winter Quarter 2014/15

January 16, 2015

Exercise 1: Constrained Optimization with One Variable

(a) For each function, write the "optimal" value(s) of x on the domain $0 \leq x \leq 10$; i.e., the value(s) that maximize(s) $f(x)$ on that domain. (This is what we did in group work in class.)

Answer:

i. $x = 2$

ii. $x = 2$

iii. $x = 10$

iv. $x = 0, 10$

v. $x = 0$

vi. Any $x \in [0, 10]$

(b) For each of the cases above, explain why taking the derivative $f'(x)$ and setting it equal to zero works, or does not work.

Answer:

i. This method works because the function is continuously differentiable everywhere, and the global maximum lies inside the domain.

ii. This method does not work because the derivative is undefined at the global maximum.

iii. This method does not work because the global maximum lies outside the domain.

iv. This method does not work because the function is convex and this gives us the global minimum.

v. Here $f'(x) = 0$ cannot be set to 0.

vi. This method technically works because the function is constant and the derivative is equal to zero everywhere.

(c) More generally, under what conditions will setting $f'(x) = 0$ get you to the solution to that problem?

Answer: The function needs to be continuously differentiable everywhere and concave, and the global maximum must lie inside the domain.

(d) Write down a series of steps (i.e., an algorithm or flowchart) which, if followed, would allow you to find the optimal value(s) of x for each of the functions above.

Answer: Below are just some model answers. Many answers, as long as they yield the correct outcome, are accepted.

S1: Is $f(x)$ continuously differentiable on the domain $0 \leq x \leq 10$?
If so, go to S2.
If not, go to S3.

S2: Take the derivative $f'(x)$. How many values of x are there that set $f'(x) = 0$?
If none, go to S4.
If one, go to S5.
If a finite number more than one, go to S6.
If an infinite number, then $f(x)$ is a horizontal line and all values of x are optimal. [END]

S3: Draw a graph of this function and find the maximum. [END]

S4: This function is linear. The maximum is $x = 0$ if it has a negative slope and $x = 10$ if it has a positive slope. [END]

S5: Denote x^* as the value of x that maximizes $f(x)$. Check if $f''(x^*)$ is negative.
If so, go to S7.
If not, then this is a minimum and we need to compare the values of the end points. The constrained maximum is given by $\max\{f(0), f(10)\}$. [END]

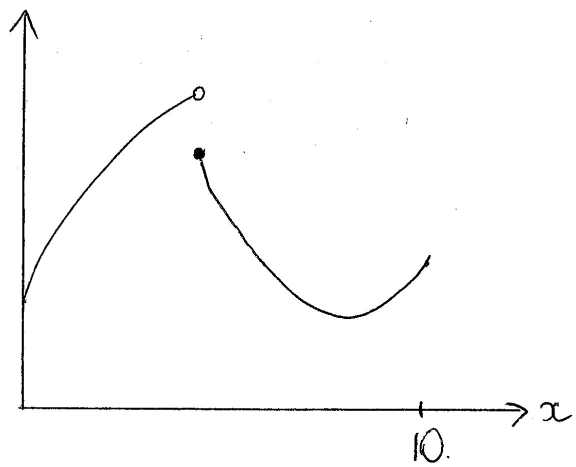
S6: Pick out all x_i^* such that $f'(x_i^*) = 0, f''(x_i^*) < 0$ and $x_i^* \in [0, 10]$, where $i = 1, 2, ..., N$. We need to compare the values of the local maxima and values of the end points. The constrained maximum is given by $\max\{f(0), f(10), f(x_1^*), ..., f(x_N^*)\}$. [END]

S7: Check if $x^* \in [0, 10]$.
If so, then x^* is the constrained maximum. [END]
If not, then we need to compare the values of the end points. The constrained maximum is given by $\max\{f(0), f(10)\}$. [END]

(e) Can you draw or describe a different kind of function that your algorithm would not work for?

Answer: Similar to part d), a wide range of answers will be accepted based on correctness.

The graph below illustrates a function that does not have a maximum.



Exercise 2: Constrained Optimization with More Than One Variable

(a) What is the objective function for this problem?

Answer: The objective function is $P_E E + P_L L$, which the manager seeks to minimize.

(b) What is the constraint?

Answer: The constraint is that $Q = \sqrt{EL} = 200$, the requirement that the manager must produce 200 units of telephone service.

(c) Which of the variables $(Q, E, L, P_E$, and $P_L)$ are exogenous? Which are endogenous? Explain.

Answer: The exogenous variables are the ones that the manager doesn't get to choose: Q, P_E and P_L (we are told that the manager is a price taker when it comes to machine-hours and person-hours). The endogenous variables are the ones that the manager does get to choose: E and L.

(d) Write a statement of the constrained optimization problem.

Answer: The statement of the problem is: $\min_{E,L} P_E E + P_L L$ such that $\sqrt{EL} = 200$.

(e) Suppose $P_E = \$36$ and $P_L = \$9$. Use the method of Lagrange multipliers to find this firm's optimal choice of E and L. Show your work. Informally "check" your answer by showing two other combinations of E and L that would produce exactly 200 units of telephone service but that would be more costly than the one you identified in your solution.

Answer: First, we form the Lagrangian:

$$\mathcal{L} = P_E E + P_L L - \lambda(\sqrt{EL} - 200)$$

Then, we take the derivative of the Lagrangian with respect to E and set to 0:

$$\frac{\partial \mathcal{L}}{\partial E} = P_E - \lambda \frac{\sqrt{L}}{2\sqrt{E}} = 0$$

and we take the derivative of the Lagrangian with respect to L:

$$\frac{\partial \mathcal{L}}{\partial L} = P_L - \lambda \frac{\sqrt{E}}{2\sqrt{L}} = 0$$

Rearranging the two conditions, we have:

$$\frac{2P_E\sqrt{E}}{\sqrt{L}} = \lambda$$

$$\frac{2P_L\sqrt{L}}{\sqrt{E}} = \lambda$$

We can set these two equal and rearrange to get

$$P_E E = P_L L$$

We then make use of the constraint of $\sqrt{EL} = 200$, rearranged to $E = \frac{200^2}{L}$ to solve for L:

$$P_E \frac{200^2}{L} = P_L L \implies L = 200\sqrt{\frac{P_E}{P_L}} = 400$$

Then

$$E = \frac{200^2}{L} = 100$$

The cost is then

$$P_E E + P_L L = 7200$$

Compare this to the cost of setting $E = L = 200$:

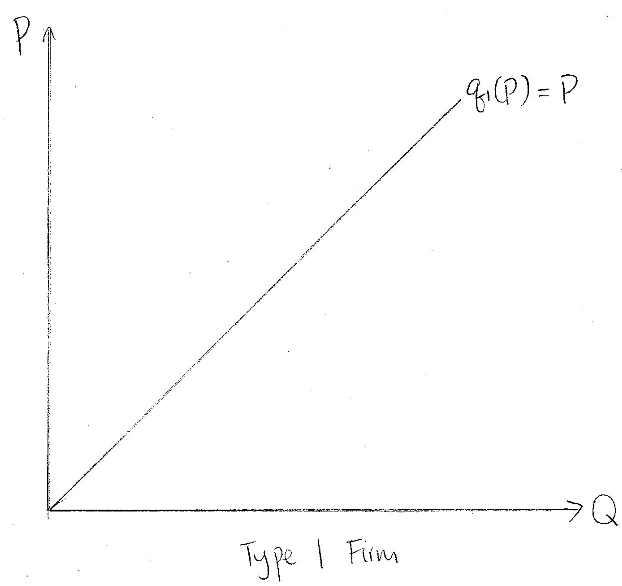
$$P_E 200 + P_L 200 = 9000$$

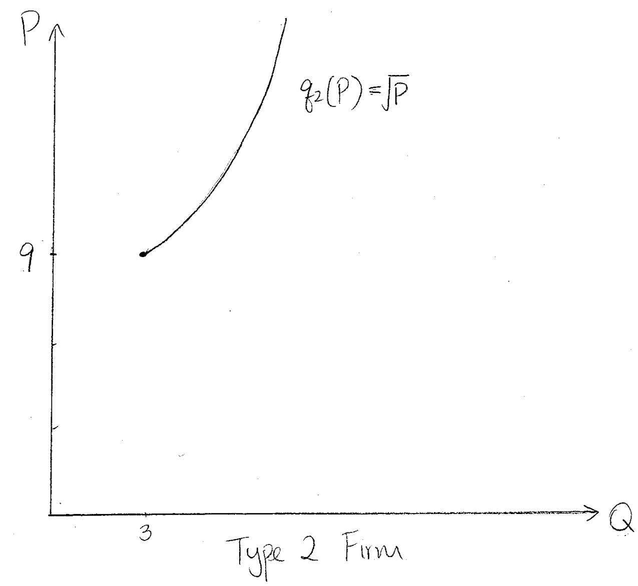
Or the cost of setting $E = 40000$ and $L = 1$:

$$P_E 40000 + P_L 1 = 1440009$$

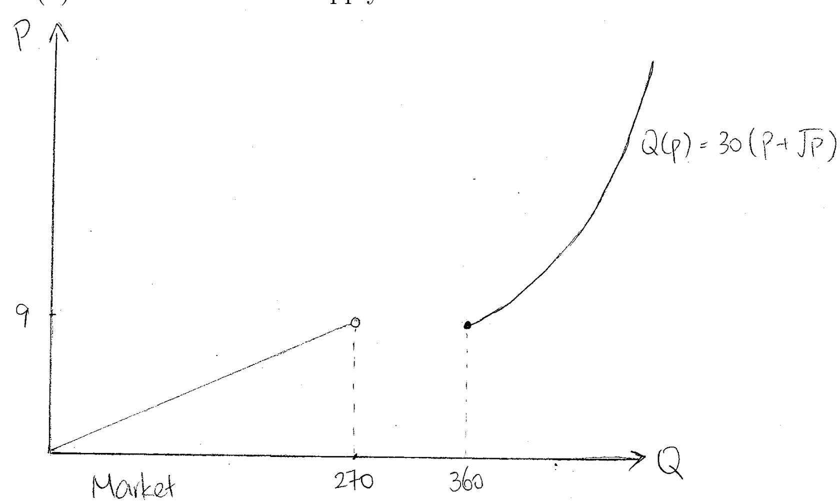
Exercise 3: Market Supply and Producer Surplus (Lecture 2)

(a) Sketch the individual supply curve for a representative firm of each type.





(b) Sketch the market supply curve.



(c) Suppose the price rises from \$4 to \$8. Illustrate the change in producer surplus and calculate its magnitude.

Answer: This price change only affects Type 1 suppliers. The total quantity produced by Type 1 suppliers is given by:

$$Q_1(P) = 30q_1(P) = 30P$$

The increase in producer surplus is thus:

$$\int_4^8 30P dP = \left[15P^2\right]_4^8 = 15(64 - 16) = 720.$$

(d) Suppose the price rises from \$8 to \$16. Illustrate the change in producer surplus and calculate its magnitude.

Answer: This price change affects both types of suppliers. Since Type 2 suppliers produce only when $P \geq 9$, to them this is essentially a price change from \$9 to \$16. For $P \geq 9$, the total quantity produced by Type 2 suppliers is given by:

$$Q_2(P) = 30q_2(P) = 30\sqrt{P}$$

The increase in producer surplus is given by:

$$\begin{aligned}\int_8^{16} 30P dP + \int_9^{16} 30\sqrt{P} dP &= \left[15P^2\right]_8^{16} + \left[20P^{\frac{3}{2}}\right]_9^{16} \\ &= 15(16^2 - 8^2) + 20(16^{\frac{3}{2}} - 9^{\frac{3}{2}}) \\ &= 3620\end{aligned}$$

Exercise 4: Parsing Demand Expressions (Lecture 2)

(a) For each expression, identify whether two goods are complements, substitutes, or neither; explain how you arrived at your conclusion.

Answer:

i. The goods are complements because an increase in the price of good Y causes a decrease in the quantity demanded of good X:

$$\frac{\partial q_x^D(P_x, P_y, I)}{\partial P_y} = \frac{-I}{(P_x + P_y)^2} < 0$$

ii. The goods are neither complements nor substitutes because the demand for good X does not depend on P^y

$$\frac{\partial q_x^D(P_x, P_y, I)}{\partial P_y} = 0$$

iii. The goods are perfect substitutes because the consumer only buys the cheaper good.

iv. The goods are substitutes because an increase in the price of good Y causes an increase in the quantity demanded of good X:

$$\frac{\partial q_x^D(P_x, P_y, I)}{\partial P_y} = \frac{I}{(P_x + P_y)^2} > 0$$

(b) To the best of your ability, give a verbal description of the kind of consumer behavior each expression describes. (For example: one describes a consumer who always spends half their income on good X...which one?)

Answer:

i. The consumer buys the same quantity of each good. We can see this by solving for q_y^D:

$$q_y^D = \frac{I - P_x q_x^D}{P_y} = \frac{I - \frac{P_x}{P_x+P_y} I}{P_y} = \frac{I}{P_x + P_y} = q_x^D$$

ii. The consumer spends exactly half of his income on each good. We can see this by multiplying both sides of the equation by P_x, which yields $P_x q_x^D = \frac{I}{2}$.

iii. The consumer only buys the cheaper good.

iv. Multiplying both sides of the equation by $\frac{P_x}{I}$ gives us:

$$\frac{P_x q_x^D}{I} = \frac{P_y}{P_x + P_y} = \frac{1}{\frac{P_x}{P_y} + 1}$$

The ratio on the left is the proportion of total income spent on good X, and it increases as $\frac{P_x}{P_y}$, the relative price, decreases. In other words, the consumer spends a higher proportion of income on good X as good Y becomes relatively more expensive.

Exercise 5: How Elastic Are Those Sweatpants? [15 points]

Suppose the market demand for sweatpants (good X) is given by $Q_x = 20 + I - P_x - \frac{1}{2}P_y$, where I is the average income of consumers, P_x is the price of sweatpants, and P_y is the price of T-shirts.

(a) Compute the **own-price** elasticity of demand for sweatpants (ϵ_{Q_x,P_x}), the **cross-price** elasticity of demand for sweatpants with respect to

T-shirts (ϵ_{Q_x,P_y}), and the **income** elasticity of demand for sweatpants ($\epsilon_{Q_x,I}$). [6 points]

Answer:

$$\epsilon_{Q_x,P_x} = \frac{\partial Q_x}{\partial P_x}\frac{P_x}{Q_x} = -\frac{P_x}{Q_x} = -\frac{P_x}{20 + I - P_x - \frac{1}{2}P_y}$$

$$\epsilon_{Q_x,P_y} = \frac{\partial Q_x}{\partial P_y}\frac{P_y}{Q_x} = -\frac{1}{2}\frac{P_y}{Q_x} = -\frac{1}{2}\frac{P_y}{20 + I - P_x - \frac{1}{2}P_y}$$

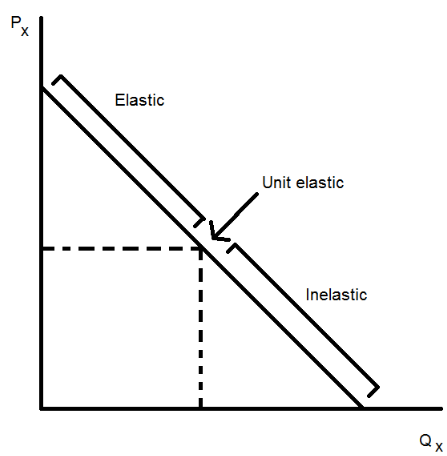
$$\epsilon_{Q_x,I} = \frac{\partial Q_x}{\partial I}\frac{I}{Q_x} = \frac{P_y}{Q_x} = \frac{I}{20 + I - P_x - \frac{1}{2}P_y}$$

(b) On a carefully drawn diagram of the demand for sweatpants, show where the demand for sweatpants is elastic, unit elastic, and inelastic. [6 points]

Answer: From part (a)

$$\epsilon_{Q_x,P_x} = -\frac{P_x}{Q_x}$$

Elastic: $\epsilon_{Q_x,P_x} < -1 \implies P_x > Q_x$
Unit elastic: $\epsilon_{Q_x,P_x} = -1 \implies P_x = Q_x$
Inelastic: $\epsilon_{Q_x,P_x} > -1 \implies P_x < Q_x$
Graphically, the demand function is a line with slope -1.



(c) According to this demand function, are sweatpants and T-shirts complements, substitutes, or neither? How do you know? [3 points]
Answer: The change in quantity demanded for x in response to a change in price of y is $\frac{\partial Q_x}{\partial P_y} = -\frac{1}{2} < 0$. They are complements.

Exercise 6: Elasticity and Logs (Lecture 3)

(a) For each of these four functions, write them in log-log form. That is, write $\ln Q^S$ as a function of $\ln P$, $\ln w$, and $\ln N_F$; then do the same for the other three. (Once you do this, the following calculations should take approximately 10 seconds each!)
Answer:
i. $\ln(Q^S) = \ln(N_F) + \ln(P) - \ln(w)$
ii. $\ln(Q^D) = \ln(N_C) + \ln(\frac{1}{4}) + \ln(I) - \ln(P)$
iii. $\ln(P^E) = \ln(\frac{1}{2}) + \frac{1}{2}\big(\ln(I) + \ln(w) + \ln(N_C) - \ln(N_F)\big)$
iv. $\ln(Q^E) = \ln(\frac{1}{2}) + \frac{1}{2}\big(\ln(I) + \ln(N_F) + \ln(N_C) - \ln(w)\big)$

(b) Calculate the income elasticity of demand.
Answer: This is given by $\frac{\partial \ln(Q^D)}{\partial \ln(I)} = 1$.

(c) Calculate the wage elasticity of supply.
Answer: This is given by $\frac{\partial \ln(Q^S)}{\partial \ln(w)} = -1$.

(d) Calculate the elasticity of the equilibrium price with respect to the number of consumers.
Answer: This is given by: $\frac{\partial \ln(P^E)}{\partial \ln(N_C)} = \frac{1}{2}$.

(e) Calculate the elasticity of the equilibrium quantity with respect to consumer income.
Answer: This is given by $\frac{\partial \ln(Q^E)}{\partial \ln(I)} = \frac{1}{2}$.

Optimal Extension

The market arrives at at equilibrium when quantity demanded is equal to quantity supplied, that is, when $Q^S = Q^D$:

$$N_F \frac{P}{w} = N_C \frac{\frac{1}{4} I}{\tau P}$$

Rearrange the above equation to get an expression for the equilibrium price P_S^E received by suppliers:

$$P_S^E = \sqrt{\frac{N_C}{N_F} \frac{\frac{1}{4} w I}{\tau}}$$

And the price paid by consumers is given by $P_C^E = \tau P_S^E = \sqrt{\tau \frac{N_C}{N_F} \frac{1}{4} w I}$

Substitute this back into the supply equation to get the equilibrium quantity Q^E:

$$Q^E = \sqrt{N_C N_F \frac{\frac{1}{4} I}{w\tau}}$$

The elasticity of P_S^E and P_C^E with respect to τ can be computed using the log-log method. Applying logs onto both sides of the expression for P_S^E and P_C^E yields

$$\ln P_S^E = \frac{1}{2}\Big(\ln N_C - \ln N_F + \ln\frac{1}{4} + \ln w + \ln I - \ln\tau\Big)$$

$$\ln P_C^E = \frac{1}{2}\Big(\ln N_C - \ln N_F + \ln\frac{1}{4} + \ln w + \ln I + \ln\tau\Big)$$

And the elasticities are

$$\frac{\partial P_S^E}{\partial\tau} = -\frac{1}{2}$$

$$\frac{\partial P_C^E}{\partial\tau} = \frac{1}{2}$$

This means that, for every 1% rise in taxes, the price received by suppliers decreases by 0.5% and the price paid by consumers increases by 0.5%. The tax burden is shared equally between consumers and suppliers.