Section 1 Solutions

Econ 50 - Stanford University - Winter Quarter 2015/16

Friday, January 15, 2016

Problem 1: Constrained Optimization

You might be tempted to use the Lagrangian method here:

$$\mathcal{L}(x,y,\lambda) = x^2 + y^2 + \lambda(4 - x - y)$$

Taking the partial derivatives with respect to the three arguments (x, y, and λ) and setting them equal to zero yields the following 3 first order conditions:

$$\frac{\partial\,\mathcal{L}(x,y,\lambda)}{\partial x} = 2x - \lambda = 0 \qquad \Rightarrow x = \lambda/2$$
$$\frac{\partial\,\mathcal{L}(x,y,\lambda)}{\partial y} = 2y - \lambda = 0 \qquad \Rightarrow y = \lambda/2$$
$$\frac{\partial\,\mathcal{L}(x,y,\lambda)}{\partial \lambda} = 4 - x - y = 0 \qquad \Rightarrow x + y = 4$$

Together these yield the solution $x = y = 2$.

However it is not the right solution. You can verify this by plugging in values $x = 4, y = 0$ or $x = 0, y = 4$, both of which satisfy the constraint and yields $x^2 + y^2 = 16 > 8 = 2^2 + 2^2$.

We can see this on a graph:

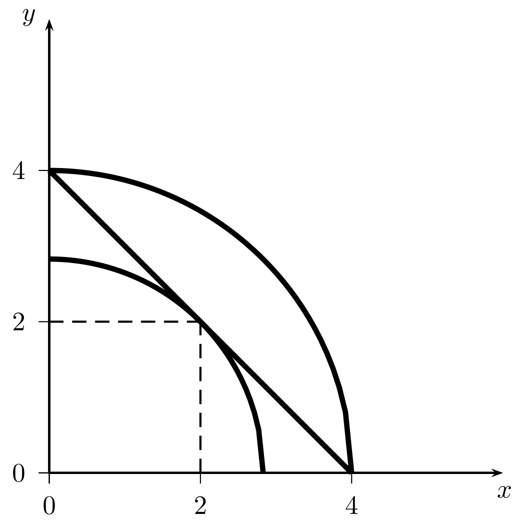

Figure 1

At point $(2,2)$, the line representing the constraint $x + y = 4$ is tangent to the smaller circle, which is a contour of the objective function: $x^2 + y^2 = 8$. Here the 3 first order condition are satisfied. However the actual optimum is find at points $(0,4)$ and $(4,0)$.

Why does taking the first order conditions of the Lagrangian not work here? It is helpful to plot the value of the objective function $x^2 + y^2$ as we move along the line that represents the constraint $x + y = 4$. Here I will plot it against the value of x. Notice that plotting $x^2 + y^2$ against x while satisfying $x + y = 4$ is equivalent to plotting $x^2 + (4 - x)^2$ against x.

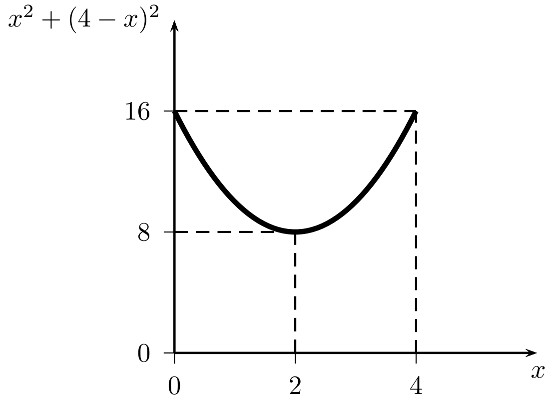

Figure 2

In this case, because of the shape of the objective function, taking the first order conditions would give you the *minimum* instead of the maximum.

(Notice that the objective function will go even higher if we look to the left of 0 or to the right of 4, which is why we had the additional constraint that $x \geq 0$ and $y \geq 0$. This is a minor point.)

Contrast this with the objective function of the fence problem, LW, subject to the constraint $2L + 2W \leq 40$. When you impose the equality constraint $L + W = 20$ and plot the objective function $L(20 - L)$ against L, it looks like this:

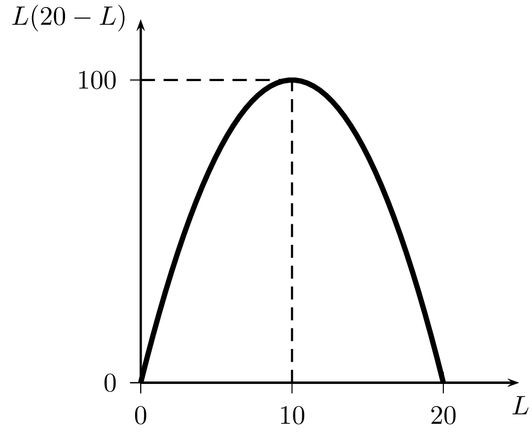

Figure 3

In this case, because of the shape of the objective function, taking the first order conditions *does* give you the maximum.

Problem 2: Constrained Optimization

This is a straightforward application of the Lagrangian method:

$$\mathcal{L}(x, y, \lambda) = a \ln x + b \ln y + \lambda(I - p_x x - p_y y)$$

Taking the partial derivatives with respect to the three arguments (x, y, and λ) and setting them equal to zero yields the following 3 first order conditions:

$$
\begin{aligned}
\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} &= \frac{a}{x} - \lambda p_x = 0 && \Rightarrow x = a/\lambda p_x \\
\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} &= \frac{b}{y} - \lambda p_y = 0 && \Rightarrow y = b/\lambda p_y \\
\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} &= I - p_x x - p_y y = 0 && \Rightarrow p_x x + p_y y = I
\end{aligned}
$$

These yield the solution

$$
\begin{aligned}
x &= \frac{a}{a+b}\frac{I}{p_x} \\
y &= \frac{b}{a+b}\frac{I}{p_y}
\end{aligned}
$$

Note that, if you see I as the total budget of the consumer, p_x and p_y as prices of 2 goods X and Y, and x and y as the quantities demanded of the 2 goods, then the solution implies that

$$
\begin{aligned}
p_x x &= \frac{a}{a+b} I \\
p_y y &= \frac{b}{a+b} I
\end{aligned}
$$

That is, the consumer always spends fraction $\frac{a}{a+b}$ of her income on good X and $\frac{b}{a+b}$ on good Y.

Problem 3: Marginal Cost; Interpretation of Lagrangian Multiplier λ

If we fix input prices P_E and P_L, the *total cost* of the firm for output level Q is how much it costs to produce Q units of output when inputs are chosen at the *optimal* levels:

$$TC(Q) = P_E E^*(Q) + P_L L^*(Q)$$

(We are omitting the input prices P_E and P_L from the optimal input levels $E^*(Q, P_E, P_L)$ and $L^*(Q, P_E, P_L)$, since input prices are assumed to be fixed throughout the problem.)

The *marginal cost* is the derivative of the total cost with respect to output level Q:

$$MC(Q) = TC'(Q) = P_E E^{*'}(Q) + P_L L^{*'}(Q)$$

The original problem is to maximize $-P_E E - P_L L$ (i.e. minimize cost $P_E E + P_L L$) subject to $\sqrt{EL} = Q$.

$$\mathcal{L}(E, L, \lambda) = -P_E E - P_L L + \lambda(Q - \sqrt{EL})$$

First order conditions are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}(E,L,\lambda)}{\partial E} &= -P_E - \lambda \frac{1}{2}\sqrt{\frac{L}{E}} = 0 \\
\frac{\partial \mathcal{L}(E,L,\lambda)}{\partial L} &= -P_L - \lambda \frac{1}{2}\sqrt{\frac{E}{L}} = 0 \\
\frac{\partial \mathcal{L}(E,L,\lambda)}{\partial \lambda} &= Q - \sqrt{EL} = 0
\end{aligned}
$$

These yield solution $E^*(Q) = Q\sqrt{\frac{P_L}{P_E}}$ and $L^*(Q) = Q\sqrt{\frac{P_E}{P_L}}$. Hence

$$TC'(Q) = P_E E^{*'}(Q) + P_L L^{*'}(Q) = P_E\sqrt{\frac{P_L}{P_E}} + P_L\sqrt{\frac{P_E}{P_L}} = 2\sqrt{P_E P_L}$$

Note that if you plug $E^*(Q) = Q\sqrt{\frac{P_L}{P_E}}$ and $L^*(Q) = Q\sqrt{\frac{P_E}{P_L}}$ into the first 2 first order conditions, you will find that $\lambda = -2\sqrt{P_E P_L}$. This is not a coincidence. There is a general result that in this type of constrained optimization problems, the Lagrangian multiplier λ reflects the marginal change in the optimized objective function with respect to relaxing the constraint. In our example this marginal change in the objective function is really the marginal cost. However in our case the cost function is the negative of the objective function (of the maximization problem, since we are doing cost minimization), so we carry a negative sign.