The Fence Problem

Econ 50 - Lecture 1

January 14, 2015

1 The Basic Constrained Optimization Problem

You are given 40 linear feet of fence and need to make a rectangular enclosure. What is the length and width that maximize the area of the enclosure? In other words, find the maximum of the objective function $A(L, W) = LW$ subject to the constraint $2L + 2W \leq 40$. More formally, we can write

$$\max_{(L,W)} A(L, W) = LW$$

$$\text{s.t. } 2L + 2W \leq 40$$

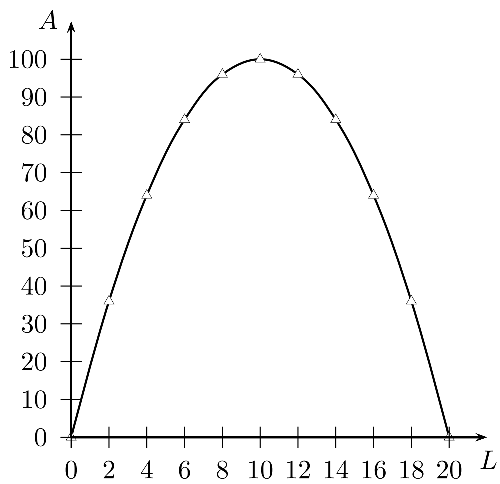
1.1 Brute Force Method

The "brute force method" is to just plug in different numbers and see if the solution pops out. For a simple problem like this one it works just fine to arrive at the solution, which, let's face it, you already know is a 10×10 square.

The only problem we have to overcome in this approach is the simultaneous choice of length and width; but given the assumptions of the problem, we can see an easy way to simplify things. First, we notice that there's never any reason to use less than the full 40 feet of fence, since "more area is always better." Given this, we can say that for any given length L, the amount left over for the width portions of the fence is $40 - 2L$, so each width portion of the fence will be half of that, or $W = \frac{1}{2}(40 - 2L)$. Now we can create a table and graph that plot a single choice variable, L, against the thing we're trying to maximize, A:

Length A	(Implied) Width $W = \frac{1}{2}(40 - 2L)$	Area $A = LW$
0	20	0
2	18	36
4	16	64
6	14	84
8	12	96
10	10	100
12	8	96
14	6	84
16	4	64
18	2	36
20	0	0

(a) Table of Length vs Area



(b) Graph of Length vs. Area

Figure 1: Brute Force Method

1.2 The Univariate Calculus Method

This optimization problem is fundamentally a two-variable choice problem: choose a length and a width to maximize an area. However, we noted above that the width may be written as a function of the length, assuming all the available fence is used. This means that the area, as a function only of length, may be written as:

$$\begin{aligned} A(L) &= L \times W(L) \\ &= L \times \frac{1}{2}(40 - 2L) \\ &= 20L - L^2 \end{aligned}$$

This is the equation for the smooth version of the curve plotted in Figure 1(b) above. Following the standard calculus rule for finding the maximum or minimum of a univariate function, we can first take the derivative of $A(L)$:

$$\frac{dA(L)}{dL} = 20 - 2L$$

At the optimal length, which we'll denote L^*, this will be equal to zero where the function is flat:

$$\begin{aligned} 20 - 2L^* &= 0 \\ 20 &= 2L^* \\ 10 &= L^* \end{aligned}$$

The corresponding optimal width, W^*, will then be

$$W^* = \frac{1}{2}(40 - 2L^*) = \frac{1}{2}(40 - 2 \times 10) = 10$$

and the optimized area, A^*, will be

$$A^* = A(L^*, W^*) = L^*W^* = 10 \times 10 = 100$$

1.3 The Analytical Proof Method

One non-calculus approach is to prove directly the hypothesis that the solution must be a square. In this approach, we start from the assumed solution, $L = W = 10$, and show that deviating from that solution would necessarily decrease the area.

In particular, let's suppose that we choose length $L(\Delta) = 10 + \Delta$ and width $W(\Delta) = 10 - \Delta$. It's easy to check that the resulting amount of fence is still 40 feet:

$$F = 2L + 2W = 2(10 + \Delta) + 2(10 - \Delta) = 20 + 2\Delta + 20 - 2\Delta = 40$$

The area, as a function of Δ, will be

$$A(\Delta) = L(\Delta)W(\Delta) = (10 + \Delta)(10 - \Delta) = 100 - \Delta^2$$

Note that just as before, we've reduced the problem to a single choice variable, Δ. Because this is equal to 100 when $\Delta = 0$ and is less than 100 whenever $\Delta \neq 0$, it is clear that area is optimized when $\Delta = 0$, or when the length and width are equal to one another.

1.4 The Multivariate Calculus (Lagrangian) Method

The Lagrangian method from multivariate calculus offers a solution to the problem that may at first seem unnecessarily cumbersome for just finding the solution, but captures the actual optimization problem most directly.

Instead of reducing the problem to a single choice variable, this method retains the double choice of L and W. To start out with, we consider the set of all combinations of L and W that satisfy the constraint. The shaded area in this graph represents all possible combinations of L and W such that $2L + 2W \leq 40$, with the downward-sloping line representing the set of points that meets the constraint exactly (i.e., $2L + 2W = 40$):

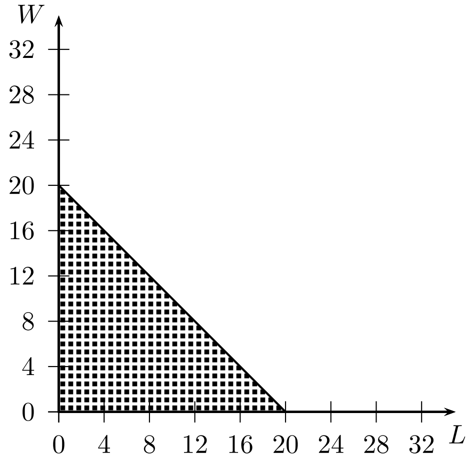


Figure 2: Feasible choices of L and W

Using this diagram, let's think about why the point $(16, 4)$ is suboptimal. A rectangle with width 16 and height 4 will have an area of $16 \times 4 = 64$. Other enclosures with that area (though not necessarily that same perimeter) include $(32, 2)$, $(8, 8)$, $(4, 16)$, and $(2, 32)$; indeed, any W and L such that $LW = 64$. If we superimpose $W = 64/L$ onto this graph, we can see that there are areas of the shaded "feasible set" that lie above this curve, and therefore have a larger area:

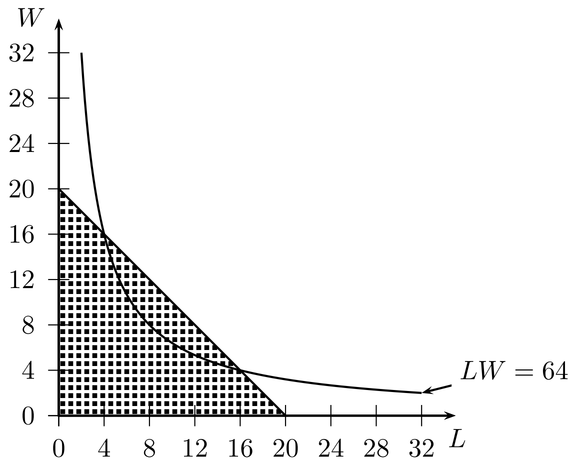


Figure 3: Feasible choices of L and W, showing combinations with an area of 64

We can repeat this process with different areas, again drawing a curve showing all the combinations that give us that area. Let's cheat a bit, and apply the fact that we know the solution is going to have an area of 100. If we add the curve $LW = 100$ to the diagram, we can see that this just touches the feasible set at the point $(10, 10)$, which is nice, because we know that's the optimal choice. Furthermore, if we add the curve $LW = 144$ to the diagram, we can see that this always lies above the feasible set, and is therefore unattainable with a 40-foot fence:

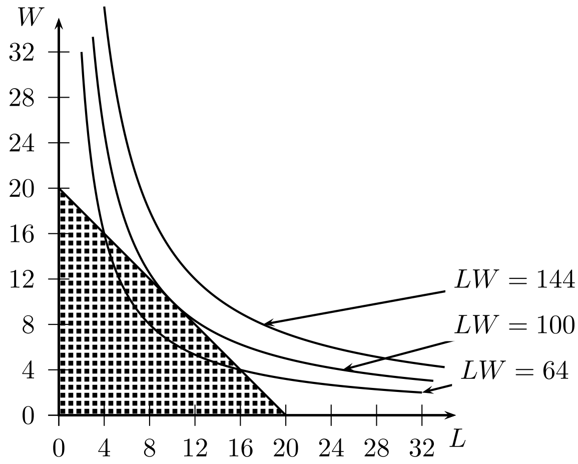


Figure 4: Feasible choices of L and W, showing combinations with areas of 64, 100, and 144

The above analysis combines the objective function and the constraint on a single graph. To use this method mathematically, we first set up the Lagrange function, which also combines the objective function and the constraint. To do so, we follow these two steps:

- Rewrite the constraint $2L + 2W \leq 40$ as the non-negativity constraint $40 - 2L - 2W \geq 0$
- "Punish" the objective function by adding this constraint to the objective function: that is,

$$\mathcal{L}(L, W, \lambda) = LW + \lambda(40 - 2L - 2W)$$

In other words, if we violate the constraint by choosing too high values L and W, such that $40-2L-2W < 0$, the value of the Lagrangian decreases.[1]

To find the optimal choice of L and W, we take the partial derivatives with respect to the three arguments (L, W, and λ) and set them equal to zero:

$$\begin{aligned}
\frac{\partial\,\mathcal{L}(L, W, \lambda)}{\partial L} &= W - 2\lambda = 0 && \Rightarrow W = 2\lambda \\
\frac{\partial\,\mathcal{L}(L, W, \lambda)}{\partial W} &= L - 2\lambda = 0 && \Rightarrow L = 2\lambda \\
\frac{\partial\,\mathcal{L}(L, W, \lambda)}{\partial \lambda} &= 40 - 2L - 2W\lambda = 0 && \Rightarrow 2L + 2W = 40
\end{aligned}$$

Right away here, we can see that $W = L$ since they are both equal to 2λ. Solving for L, W, and λ, we find $W = L = 10$, as before, and also that $\lambda = 5$.

But what is the interpretation of λ? For that, we look to the more general problem.

[1] Note that it's completely equivalent to write

$$\mathcal{L}(L, W) = LW - \lambda(2L + 2W - 40),$$

which is the way some people prefer to do it.

2 The General Constrained Optimization Problem

Suppose that instead of 40 feet of fence material, we have material of an unknown length F. How do the optimal length and width, as well as the optimized area, change as F changes?

To answer this question, it's easy to see that the different approaches we tried before will yield varying levels of insight. Let's look at each one in turn.

2.1 The Brute Force Method

The brute force method, simple for a specific value of F, is now pretty useless. If you want to be mean to yourself, try writing down the table and graph from figure with an F floating around. Blech.

2.2 The Univariate Calculus Method

It's actually pretty simple to write down the new simplified version of the problem. Instead of the width being $\frac{1}{2}(40-2L)$, it's now $\frac{1}{2}(F-2L)$, so the area as a function of length only is

$$\begin{aligned} A(L) &= L \times W(L) \\ &= L \times \frac{1}{2}(F-2L) \\ &= \frac{1}{2}FL - L^2 \end{aligned}$$

This is the equation for the smooth version of the curve plotted in Figure 1(b) above. Following the standard calculus rule for finding the maximum or minimum of a univariate function, we can first take the derivative of $A(L)$:

$$\frac{dA(L)}{dL} = \frac{1}{2}F - 2L$$

At the optimal length, which we'll denote L^*, this will be equal to zero where the function is flat:

$$\begin{aligned} \frac{1}{2}F - 2L^* &= 0 \\ \frac{1}{2}F &= 2L^* \\ \frac{F}{4} &= L^* \end{aligned}$$

The corresponding optimal width, W^*, will then be

$$\begin{aligned} W^* &= \frac{1}{2}(F-2L^*) \\ &= \frac{1}{2}\left(F - 2 \times \frac{F}{4}\right) \\ &= \frac{1}{2}\left(\frac{F}{2}\right) \\ &= \frac{F}{4} \end{aligned}$$

and the optimized area, A^*, will be

$$\begin{aligned} A^* &= A(L^*, W^*) \\ &= L^* W^* \\ &= \frac{F}{4} \times \frac{F}{4} \\ &= \frac{F^2}{16} \end{aligned}$$

We can see that for $F = 40$, $L^* = W^* = \frac{F}{4} = 10$ and $A^* = \frac{F^2}{16} = \frac{1600}{16} = 100$, as before.

2.3 The Analytical Proof Method

There actually wasn't anything about the analytical proof method that relied on a particular perimeter, since what we proved was that the solution to this problem must always be a square. Therefore this method already showed that $L^* = W^* = \frac{F}{4}$ and $A^* = \frac{F^2}{16}$.

2.4 The Multivariate Calculus (Lagrangian) Method

Here, as in the univariate calculus method, we can replace the 40 with an F, so the Lagrangian becomes

$$\mathcal{L}(L, W, \lambda) = LW + \lambda(F - 2L - 2W)$$

and the first-order conditions (FOC's) become

$$\begin{aligned}
\frac{\partial \mathcal{L}(L, W, \lambda)}{\partial L} &= W - 2\lambda = 0 && \Rightarrow W = 2\lambda \\
\frac{\partial \mathcal{L}(L, W, \lambda)}{\partial W} &= L - 2\lambda = 0 && \Rightarrow L = 2\lambda \\
\frac{\partial \mathcal{L}(L, W, \lambda)}{\partial \lambda} &= F - 2L - 2W\lambda = 0 && \Rightarrow 2L + 2W = F
\end{aligned}$$

Note that the first two FOC's are unchanged; they still say that $L = W = 2\lambda$. Plugging $L = W$ into the third FOC yields the now-familiar answer, $L^* = W^* = \frac{F}{4}$; therefore, we also have $\lambda = \frac{1}{2}L = \frac{1}{2}W = \frac{F}{8}$. So what is the interpretation of this? To answer that, we turn to comparative static analysis.

3 Comparative Statics

Comparative statics analysis is useful for seeing how the solution to a problem changes when its underlying parameters change. In this case, we've solved for the optimal length and width, and found an expression for the optimized area of the enclosure, all as a function of a general length F:

$$\begin{aligned}
L^*(F) &= \frac{F}{4} \\
W^*(F) &= \frac{F}{4} \\
A^*(F) &= \frac{F^2}{16}
\end{aligned}$$

Comparative statics examines how these three endogenous variables (L, W, and A) change when the exogenous variable (F) changes. The simple answer is to take the derivatives of $L^*(F)$, $L^*(F)$, and $L^*(F)$ with respect to F:

$$\begin{aligned}
\frac{dL(F)}{dF} &= \frac{1}{4} \\
\frac{dW(F)}{dF} &= \frac{1}{4} \\
\frac{dA(F)}{dF} &= \frac{F}{8}
\end{aligned}$$

The first two results make sense: since we'll always make a square, the additional fence will be distributed evenly on all four sides. (That is, if F increases by 1, each side will increase by $\frac{1}{4}$.)

The third result should look familiar to us: it's the λ from the Lagrangian method! But how do we interpret this?

Recall that the Lagrangian function $\mathcal{L}(L, W, \lambda)$ represents the "adjusted" objective function. Suppose that we were to think of this function not only in terms of L, W, and λ, but also F:

$$\mathcal{L}(L, W, \lambda, F) = LW + \lambda(F - 2L - 2W)$$

Now it take its (partial) derivative with respect to F, we can begin to get a clue as to the interpretation of λ:

$$\frac{\partial \mathcal{L}(L, W, \lambda, F)}{\partial F} = \lambda$$

In short, λ represents the effect on the constrained optimum of the objective function (in this case, area) resulting from a loosening of the constraint.

To understand this visually, think of the plot of the function $A^*(F)$:

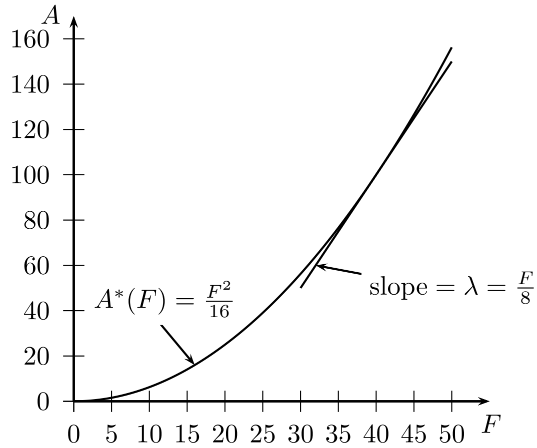


Figure 5: Optimized area (A^*) as a function of available fence (F)

The slope of that function, $\frac{dA^*(F)}{dF}$, is λ.