Homework 3 Solutions

Econ 50 - Stanford University - Winter Quarter 2015/16

Exercise 1: Math Warmup: The Canonical Optimization Problems (Lecture 6)

For each of the following five "canonical" utility functions, find the point (x^*, y^*) that maximizes utility subject to the standard budget constraint $P_x x + P_y y = I$. In each case, indicate whether the solution is sometimes, always, or never found using the Lagrange method, and provide a brief, intuitive reason why. *Note: we already did some of these in lecture and section...*

(a) Cobb-Douglas: $u(x, y) = \alpha \ln x + (1 - \alpha) \ln y$

(b) Perfect Substitutes: $u(x, y) = \alpha x + (1 - \alpha)y$

(c) Perfect Complements: $u(x, y) = \min\{\frac{x}{\alpha}, \frac{y}{1-\alpha}\}$

(d) Quasilinear: $u(x, y) = \alpha \ln x + (1 - \alpha)y$

(e) CES: $u(x, y) = [\alpha x^r + (1 - \alpha)y^r]^{\frac{1}{r}}$ (optional; the math on this one can get hairy!)

Answer:

(It is okay on this problem to assume that α is strictly between 0 and 1.)

(a) $x^* = \frac{\alpha I}{p_x}, y^* = \frac{(1-\alpha)I}{p_y}$. The Lagrange method will always work in this case because Cobb-Douglas indifference curves behave nicely: they are continuously differentiable, strictly monotonic, strictly convex, and don't intersect the axes.

(b)

$$x^* = \begin{cases} 0 & \text{if } \frac{\alpha}{1-\alpha} < \frac{p_x}{p_y} \\ \frac{I}{p_x} & \text{if } \frac{\alpha}{1-\alpha} > \frac{p_x}{p_y} \end{cases}$$

$$y^* = \begin{cases} \frac{I}{p_y} & \text{if } \frac{\alpha}{1-\alpha} < \frac{p_x}{p_y} \\ 0 & \text{if } \frac{\alpha}{1-\alpha} > \frac{p_x}{p_y} \end{cases}$$

When the MRS is less than the slope of the budget line, utility is maximized by spending all income on y, and vice-versa. When $\frac{\alpha}{1-\alpha} = \frac{p_x}{p_y}$, anywhere on the budge line maximizes the utility. The Lagrange method doesn't work for perfect substitutes when $\frac{\alpha}{1-\alpha} \neq \frac{p_x}{p_y}$ because the indifference curves are linear and so won't have a point of tangency with the budget line. They are not strictly convex and they intersect the axes. When $\frac{\alpha}{1-\alpha} = \frac{p_x}{p_y}$, the Lagrange method works because the indifference curve is tangent to the budget line everywhere and everywhere is a solution.

(c) With perfect complements, the solution will be at a kink. The ray of kinks is given by $\frac{x}{\alpha} = \frac{y}{1-\alpha}$. Solving this and plugging into the budget constraint yields:

$$x^* = \frac{\alpha I}{\alpha p_x + (1-\alpha)p_y}$$

$$y^* = \frac{(1-\alpha)I}{\alpha p_x + (1-\alpha)p_y}$$

The Lagrange method will never work for perfect complements because the MRS is undefined at the kink point, which is where the maximum will be. The preferences are not continuously differentiable.

(d) Setting the MRS equal to the price ratio and then using the budget constraint gives

$$x = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_y}{p_x}\right)$$
$$y = \frac{I}{p_y} - \frac{\alpha}{1-\alpha}$$

It is possible for the expression of y to go negative so we must account for this. If it is negative we move to the nearest corner which will be at $y = 0$, so in that case all income is spent on x. We arrive at the following solutions:

$$x^* = \begin{cases} \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_y}{p_x}\right) & \text{if } I \geq \left(\frac{\alpha}{1-\alpha}\right) p_y \\ \frac{I}{p_x} & \text{if } I < \left(\frac{\alpha}{1-\alpha}\right) p_y \end{cases}$$

$$y^* = \begin{cases} \frac{I}{p_y} - \frac{\alpha}{1-\alpha} & \text{if } I \geq \left(\frac{\alpha}{1-\alpha}\right) p_y \\ 0 & \text{if } I < \left(\frac{\alpha}{1-\alpha}\right) p_y \end{cases}$$

In this case, the Lagrange method sometimes works: specifically when $I \geq \left(\frac{\alpha}{1-\alpha}\right) p_y$. Otherwise, the Lagrange method will give a negative value for y, so we move to a corner solution in those cases. Intuitively, this is because the indifference curves cross the axes and so can be tangent to the budget line below the first quadrant.

(e) The MRS is $\frac{\alpha x^{r-1}}{(1-\alpha)y^{r-1}}$. So setting this equal to the price ratio, plugging into the budget constraint, and then using symmetry yields the following solution:

$$x^* = \frac{I}{p_x\left[1 + \left(\frac{p_y}{p_x}\right)^{\frac{r}{r-1}} \left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{r-1}}\right]}$$

$$y^* = \frac{I}{p_y\left[1 + \left(\frac{p_x}{p_y}\right)^{\frac{r}{r-1}} \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{r-1}}\right]}$$

The Lagrange method can always be used in this case.

Exercise 2: Thinking on the Margin (Lecture 6)

This was a midterm question from last year.

(a) What does it mean if $MRS_{x,y} < \frac{P_x}{P_y}$ at a point along a consumer's budget constraint?

(b) If a consumer is in a position where that is true, can they always improve their utility by changing their consumption bundle? Why or why not? Illustrate your answer with one or two carefully drawn budget-line/indifference-curve diagrams.

(c) If they could improve their utility by changing their consumption bundle, would it involve consuming more X and less Y, more Y and less X, or would it depend upon the exact form of the utility function in question? Carefully state the assumptions underlying your answer.

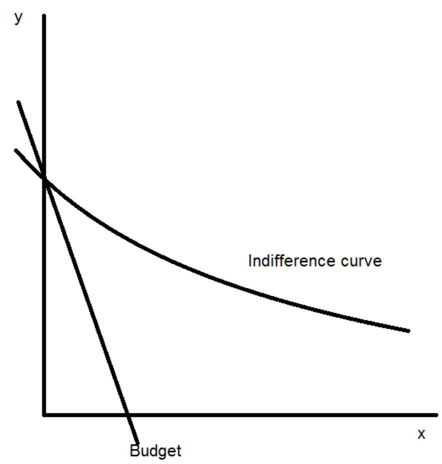
Answer:

(a) The relative cost of an additional unit of x (given by the price ratio $\frac{P_x}{P_y}$) is greater than the relative benefit (given by $MRS_{x,y}$). That means the consumer can increase utility by selling x (to buy more y).
Another way to see this is rearranging the equation into

$$MRS_{x,y} = \frac{MU_x}{MU_y} < \frac{P_x}{P_y} \implies \frac{MU_x}{P_x} < \frac{MU_y}{P_y}$$

The additional utility of every dollar spent on x is less than that spent on y. Therefore if the consumer sells a dollar worth of x to buy a dollar worth of y, the utility gained in y exceeds the utility lost in x.

(b) If the consumer is already consuming 0 units of x, then it's not possible to sell x to buy more y. In that case the consumer is at an optimal corner solution where equality of the above equation doesn't hold. In the graph below, the magnitude of the slope of the budget line $\frac{P_x}{P_y}$ is greater than the slope of the tangent to the indifference curve $MRS_{x,y}$. Seeking a higher indifference curve while staying on the budget line requires the consumer to consume negative x which is not possible.



(c) The argument in part (a) assumes that both MU_x and MU_y are positive. This means that the utility function is monotonic. Under this assumption, consuming more y and less x increases utility.

Exercise 3: Choosing a Budget Constraint (Lecture 6)

Suppose I have a budget of \$240 per year to spend on puppy food (good X) and puppy toys (good Y). Puppy toys cost $p_y = \$10$ each, and I always buy them from the local supermarket. At that store, I can buy puppy food for $p_x^S = \$20$ per bag; at Costco, it's only $p_x^C = \$10$ per bag for puppy food, but a Costco membership costs \$60 per year. Assume for simplicity that I'd only go to Costco to buy puppy food, and that my preferences are given by $u(x, y) = xy$.

(a) Draw my possible budget constraints with and without a Costco membership. (Make this a pretty big and very precise graph!)

(b) Solve my constrained optimization problem if I choose **not** to buy a Costco membership. How many bags of puppy food, and how many puppy toys, would I choose to buy? Carefully add this point, and the indifference curve passing through it, to the graph you drew in part (a).

(c) Solve my constrained optimization problem if I **do** choose to buy a Costco membership. How many bags of puppy food, and how many puppy toys, would I choose to buy? Again, add this point, and the indifference curve passing through it, to the graph you drew in part (a).

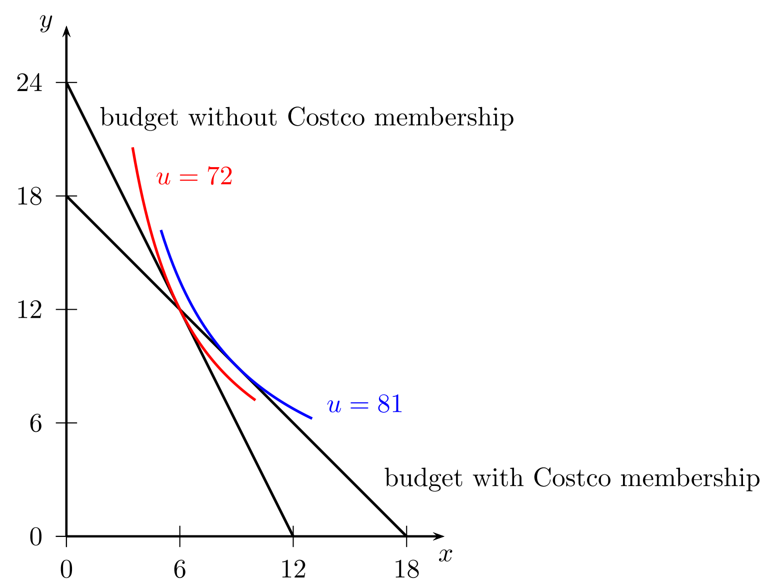
(d) Should I buy a Costco membership in order to take advantage of the discount on puppy food? Why or why not?

(e) My friend has faces the same choice, but she has a different sized dog, so her preferences are given by $u(x, y) = xy^2$. Solve her optimization problem with and without a Costco membership. Should she buy a Costco membership? Why or why not? Show your work!

(f) Based on these two utility functions (and your answers to the last two questions): who do you think has a bigger dog? Why?

Answer:

(a)



(b) Setting the MRS equal to the price ratio, we have $\frac{y}{x} = 2 \Rightarrow y = 2x$. Plugging this into the budget constraint gives: $20x + 20x = 240$. So $x = 6$ and $y = 12$. You would choose to buy 6 bags of puppy food and 12 puppy toys.

(c) When you buy the Costco membership, you only have \$180 left to spend on food and toys. Setting the MRS equal to the price ratio, we have $\frac{y}{x} = 1 \Rightarrow y = x$. Plugging this into the budget constraint gives: $10x + 10x = 180$. So $x = 9$ and $y = 9$. You would choose to buy 9 bags of puppy food and 9 puppy toys.

(d) Yes, you should buy the Costco membership because you can achieve a higher utility.

(e) Without the Costco membership, setting the MRS equal to the price ratio, we have $\frac{y}{2x} = 2 \Rightarrow y = 4x$. Plugging this into the budget constraint gives: $20x + 40x = 240$. So $x = 4$ and $y = 16$. This gives a utility of $u(4, 16) = 1024$.

With the Costco membership, setting the MRS equal to the price ratio, we have $\frac{y}{2x} = 1 \Rightarrow y = 2x$. Plugging this into the budget constraint gives: $10x + 20x = 180$. So $x = 6$ and $y = 12$. This gives a utility of $u(4, 16) = 864$.

Your friend should not purchase the Costco membership because she achieves a higher utility without it.

(f) You have a bigger dog than your friend because you get more utility out of a discount on dog food. Presumably the larger dog eats more.

Exercise 4: Predatory Lending and Borrowing (Lecture 7)

Sam and Gianna are two shark tour operators on Fisherman's Wharf, and they're each trying to make their consumption and savings decisions. Sam is very successful, and will earn \$100K this year. He plans on taking next year off to travel the world, living off savings from this year. Gianna will earn \$45K this year, but next year expects to take over over some of Sam's business, and so expects her earnings to increase to \$55K. They each have a utility function $u(c_1, c_2) = c_1 c_2$ over consumption this year (c_1) and consumption next year (c_2). Assume that each of them only plans two years in advance (so we can ignore decisions beyond next year, including savings decisions next year).

(a) Suppose both Sam and Gianna can borrow money interest-free, and also get no interest on savings. Write down their budget constraints and solve their optimization problem for c_1^* and c_2^*. How much will Sam save for next year? How much will Gianna borrow against her future earnings? Illustrate their consumption/savings decisions on a budget line-indifference curve diagram.

(b) Repeat part (a) if both Sam and Gianna can either borrow or save at 10% interest. Who is made better off, relative to the situation in part (a)? Who is made worse off?

(c) Suppose Gianna can borrow at 10% interest, but receives no interest on savings. Draw her new budget constraint, and again solve for her optimal consumption/savings decision. Will she still borrow against her future income?

(d) Suppose Gianna's credit isn't great, and the only person who will lend her money is a "loan shark" who charges 30% interest. Will she borrow any money at that rate? Why or why not? Illustrate this decision in a budget line-indifference curve diagram.

(e) What is the (approximate) highest interest rate Gianna will be willing to pay to borrow against her future earnings, if she receives no interest on savings? Give an intuitive explanation of your answer.

(f) Suppose Gianna's credit is so bad that she can't borrow money at all. What is the lowest interest rate that would make her save some of her current year's income?

Answer:

Let c_1 and c_2 be in thousands of dollars.

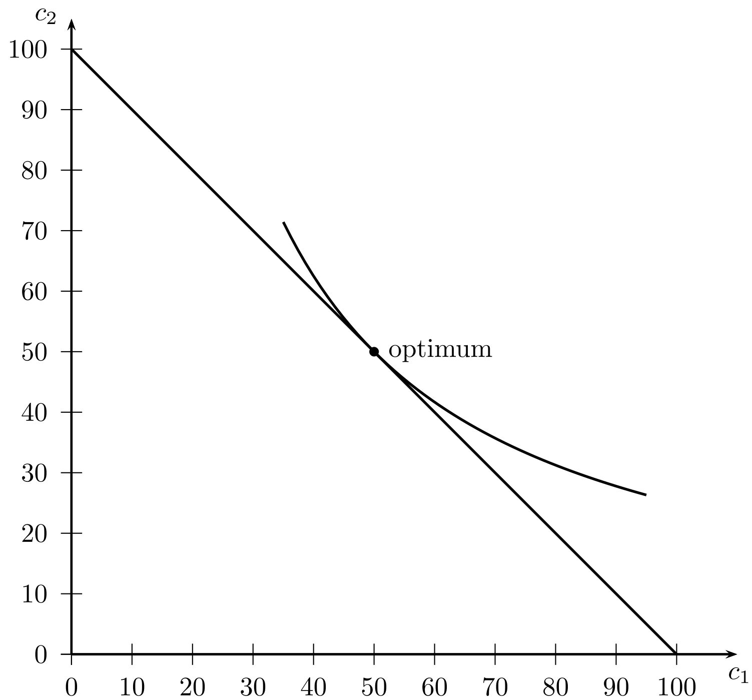
In general the solution will be given by using the budget constraint $(1 + r)c_1 + c_2 = (1 + r) + I_2$ and then setting the MRS equal to the "price ratio" just as before.

In this case we have $MRS_{c_1,c_2} = \frac{c_2}{c_1} = 1 + r$. From this we find that $c_2 = (1 + r)c_1$. Plugging into the budget constraint yields $c_1^* = \frac{(1+r)I_1+I_2}{2(1+r)}$ and $c_2^* = \frac{(1+r)I_1+I_2}{2}$. In words, the solutions show that c_1 is half the present value of income and c_2 is half the future value of income.

(a) With a zero interest rate, their budget constraints are the same: $c_1 + c_2 = 100$.

$MRS_{c_1,c_2} = \frac{c_2}{c_1}$. We set this equal to $1 + r = 1$. So we find that their optimal optimal choices are $c_1^* = c_2^* = 50$. Each of their utilities is 2500.

Sam will save \$50,000 for next year. Gianna borrows \$5,000 against her future income.



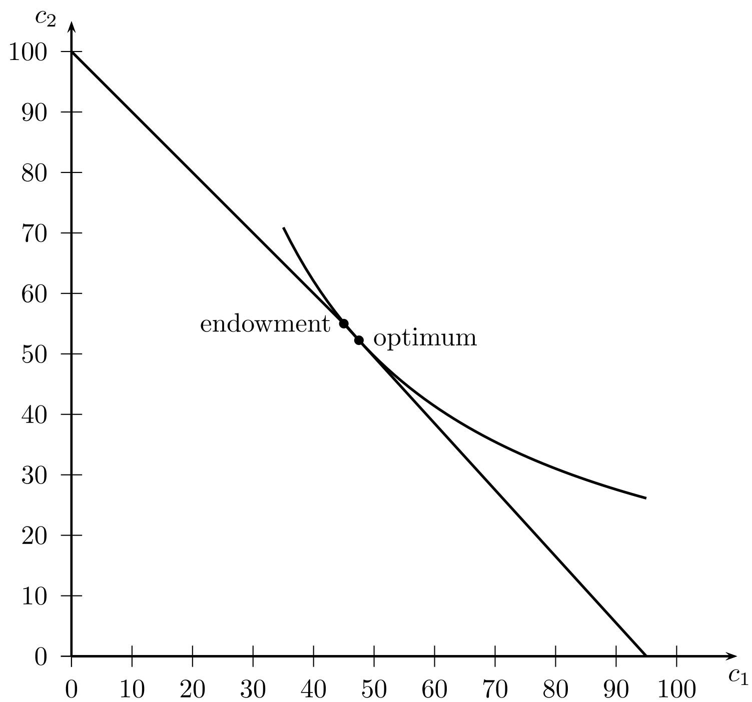
(b) Now we have that $c_2 = 1.1c_1$ (from setting the MRS equal to $1+r = 1.1$). Sam's budget constraint is now: $1.1c_1 + c_2 = 110$ and Gianna's is: $1.1c_1 + c_2 = 104.5$.

Sam will choose $c_1^* = 50, c_2^* = 55$. His utility is 2750.

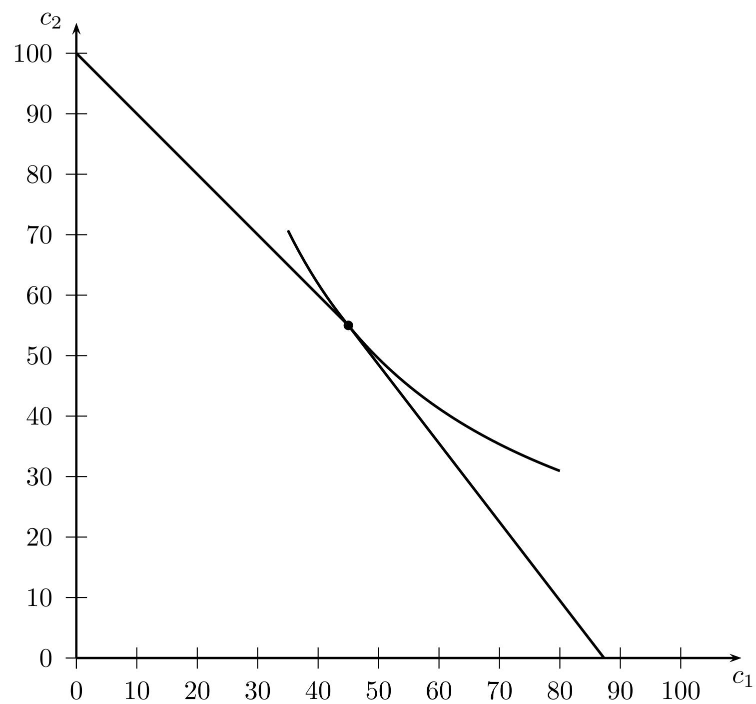
Gianna will choose $c_1^* = 47.5, c_2^* = 52.25$. Her utility is 2481.875.

We can tell from the utilities that Sam is made better off and Gianna is made worse off. Intuitively, we know that savers are better off when the interest rate goes up (because their savings accrue more interest) and borrowers are worse off (because it becomes more expensive to borrow).

(c) In this case, Gianna's consumption decision is the same as in (b) because she chose to borrow at 10% in (b) and with no savings interest rate, her utility at any point to the left of her endowment is lower now. So the optimum from (b) must still be the optimum.



(d) At her endowment of $(c_1, c_2) = (45, 55)$, Gianna's $MRS_{c_1,c_2} = \frac{c_2}{c_1} = \frac{55}{45} = 1.\overline{22}$. So if the interest rate were 30%, $1 + r$ would be 1.3. This is greater than Gianna's MRS, meaning she would like to move to the left, i.e. save. So she will not borrow at this rate. Since she cannot save at this rate and we know from part (a) she would not save at 0%, she must stay at her endowment (the kink).



(e) Gianna's MRS at her endowment is $1.\overline{22}$. If the interest rate (plus one) is less than this, she will want to move to the right and hence borrow. Intuitively, think of her MRS as how much more valuable additional consumption of c_1 is to her over c_2. If she can trade in some c_2 for c_1 at this "price" or less, she will. So she will borrow at any rate less than approximately 22.2%.

(f) If the interest rate goes above Gianna's MRS at her endowment, she will want to move left and hence save. Similar logic to (e) applies. If the rate is greater than her relative value of c_1, she will trade in some c_1 for c_2 because c_2 is cheaper in the market than her value for it. She will save at any rate greater than approximately 22.2%.

Exercise 5: Endowment Budget Constraint (Lecture 7)

Suppose that instead of having a fixed income I, you have an endowment of $y^E = 12$ units of good Y. You can sell each of these units of good Y at price P_y and use the proceeds to buy good X, which has a price of P_x.

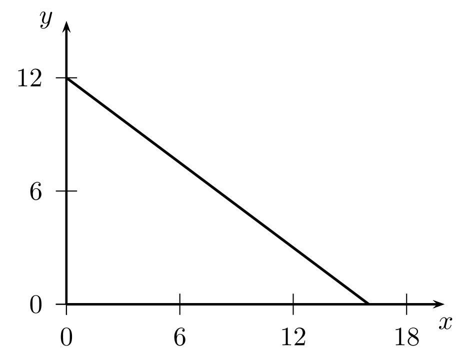
(a) Draw your budget line if $P_x = 3$ and $P_y = 4$. Draw what happens to your budget line if P_x increases from 3 to 4, or if P_y decreases from 4 to 3. How is this different from the effect of price changes if you had a fixed dollar income?

(b) Suppose you have the simple Cobb-Douglas utility function $u(x, y) = xy$. Solve for your optimal consumption (x^*, y^*) as a function of the exogenous variables y^E, P_x, and P_y.

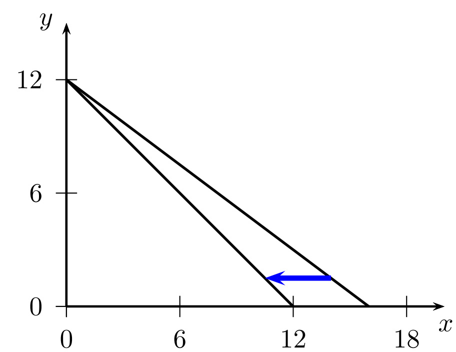
(c) In the canonical Cobb-Douglas case, X and Y are neither complements nor substitutes, because the quantity demanded of each depends only on its own price. Is that still the case with an endowment budget line? Why or why not? (What is particularly weird about your demand for good Y in this scenario...?)

Answer:

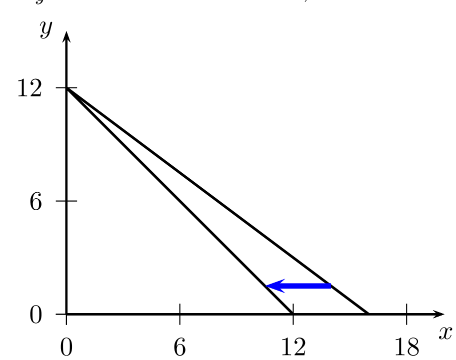
(a)



If P_x increases from 3 to 4, the line shifts according to the blue arrow below:



If P_y decreases from 4 to 3, the line shifts according to the blue arrow below:



Note that P_x increasing from 3 to 4 has the exact same effect as P_y decreasing from 4 to 3 in this problem. The first change is the same as with a fixed dollar income: the amount of y that can be purchased is unchanged but the amount of x that can be purchased decreases.

The change in P_y is different from a fixed dollar income. Because there is an endowment of all y, the amount of y that can be afforded does not depend on the price. Instead, the decrease in price of y makes one less able to purchase x because selling y yields less income.

(b) MRS $= \frac{y}{x} = \frac{P_x}{P_y} \Rightarrow y = \frac{P_x}{P_y} \cdot x$. The budget is given by $P_y y^E = P_y y + P_x x$. Solving, we find:

$$x^* = \frac{P_y y^E}{2P_x}$$

$$y^* = \frac{y^E}{2}$$

(c) Clearly x^* depends on P_y. On the other hand, y^* doesn't depend on either price.

No, the quantity demanded of each good no longer depends only on its own price.

Notice, however, that each demand function still represents the classic Cobb-Douglas behavior of spending a fixed fraction of income on each good. The difference is that income in this scenario depends on P_y: the income is $P_y y^E$.