Derivations: Cost Curves and Shephard's Lemma

Econ 50 - Lecture 14

February 23, 2016

Let's return to the function $f(L, K) = L^{\frac{1}{4}} K^{\frac{1}{4}}$. Last time we derived the conditional demands

$$L^* = \sqrt{\frac{r}{w} q^2}$$
$$K^* = \sqrt{\frac{w}{r} q^2}$$

From there we found the long-run total cost of producing q units, for general w and r.

$$\begin{aligned} TC(w, r, q) &= wL^*(w, r, q) + rK^*(w, r, q) \\ &= w\sqrt{\frac{r}{w} q^2} + r\sqrt{\frac{w}{r} q^2} \\ &= \sqrt{rw} q^2 + \sqrt{rw} q^2 \\ &= 2\sqrt{rw} q^2 \end{aligned}$$

1 Cost Curves

If we fix capital at some value $\overline{K}$, then in order to produce q units, therefore, the amount of labor required is

$$\begin{aligned} f(L, K) &= q \\ L^{\frac{1}{4}} \overline{K}^{\frac{1}{4}} &= q \\ L\overline{K} &= q^4 \\ L(q) &= \frac{q^4}{\overline{K}} \end{aligned}$$

Therefore the short-run total cost of production is

$$\begin{aligned} STC(w, r, q, \overline{K}) &= wL(q) + r\overline{K} \\ &= w\frac{q^4}{\overline{K}} + r\overline{K} \end{aligned}$$

For simplicity let's assume $w = r = 1$ and fix $\overline{K} = 4$. Then our long-run and short-run total cost become

$$TC(q) = 2q^2$$
$$STC(q) = 4 + \frac{1}{4} q^4$$

From here let's derive the other cost curves.

The fixed cost is the portion of the short-run total cost that doesn't change in the short run, i.e., $r\overline{K}$. In this case since $r = 1$ and $\overline{K} = 4$, we have

$$FC = r\overline{K} = 4$$

The variable cost is the portion of the short-run total cost that does change in the short run:

$$VC(q) = \frac{1}{4}q^4$$

Marginal cost is the derivative of $TC(q)$:

$$\begin{aligned} MC(q) &= TC'(q) = \frac{d}{dq}(2q^2) = 4q \\ SMC(q) &= MTC'(q) = \frac{d}{dq}\left(4 + \frac{1}{4}q^4\right) = q^3 \end{aligned}$$

Average cost is the total cost divided by q; short-run average cost, average fixed cost, and average variable cost are likewise those functions divided by q:

$$\begin{aligned} AC(q) &= \frac{TC(q)}{q} = \frac{2q^2}{q} = 2q \\ SAC(q) &= \frac{STC(q)}{q} = \frac{\left(4 + \frac{1}{4}q^4\right)}{q} = \frac{4}{q} + \frac{q^3}{4} \\ AFC(q) &= \frac{FC}{q} = \frac{4}{q} \\ AVC(q) &= \frac{AVC(q)}{q} = \frac{q^3}{4} \end{aligned}$$

Note that just as $STC(q) = FC + VC(q)$, we have $SAC(q) = AFC(q) + AVC(q)$ (since each element of that equation is just divided through by q).

2 Shephard's Lemma and the Dual Method

If we have the total cost function, we can "back out" the conditional input demand functions and the production function as follows.

Shephard's Lemma states that the conditional demand curves may be found by taking the partial derivative of long-run total cost with respect to input prices. In this case we have

$$TC(w, r, q) = 2\sqrt{rw}q^2$$

Therefore we have

$$\begin{aligned} L^*(w, r, q) &= \frac{\partial TC(w, r, q)}{\partial w} = \sqrt{\frac{r}{w}}q^2 \\ K^*(w, r, q) &= \frac{\partial TC(w, r, q)}{\partial r} = \sqrt{\frac{w}{r}}q^2 \end{aligned}$$

which we see are the same as the conditional demands found above.

Since the conditional demand functions depend only on the ratio w/r, we can solve each of them for w/r:

$$\begin{aligned} L &= \sqrt{\frac{r}{w}}q^2 \\ \sqrt{\frac{w}{r}} &= \frac{q^2}{L} \\ \frac{w}{r} &= \frac{q^4}{L^2} \end{aligned}$$

$$\begin{aligned} K &= \sqrt{\frac{w}{r}}q^2 \\ \frac{K}{q^2} &= \sqrt{\frac{w}{r}} \\ \frac{K^2}{q^4} &= \frac{w}{r} \end{aligned}$$

Setting the two values of w/r equal to each other, we can solve for q as a function of L and K:

$$\begin{aligned} \frac{K^2}{q^4} &= \frac{q^4}{L^2} \\ L^2K^2 &= q^8 \\ q &= L^{\frac{1}{4}}K^{\frac{1}{4}} \end{aligned}$$

which is indeed the production function we started with.