Section 2 Solutions

Econ 50 - Stanford University - Winter Quarter 2015/16

January 22, 2016

Exercise 1: Quasilinear Utility Function

Solve the following utility maximization problem:

$$\max_{x,y}\{\sqrt{x}+y\} \quad \text{s.t. } p_x x + p_y y = I$$

Solution:
This problem can be solved using the standard Lagrange method. First we set up the Lagrangian to be:

$$\mathcal{L}(x,y,\lambda) = \sqrt{x} + y + \lambda(I - p_x x - p_y y)$$

Taking partial derivatives yields the following first-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial x} &= \frac{1}{2\sqrt{x}} - \lambda p_x = 0 && \Rightarrow x = \frac{1}{4\lambda^2 p_x^2}\\
\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial y} &= 1 - \lambda p_y = 0 && \Rightarrow \lambda = \frac{1}{p_y}\\
\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} &= I - p_x x - p_y y = 0 && \Rightarrow p_x x + p_y y = I
\end{aligned}$$

Substituting the expression for λ (equation 2) into the first equation gives us the consumer's demand for good X:

$$x(p_x, p_y, I) = \frac{1}{4}\frac{p_y^2}{p_x^2}$$

Substituting this back into the budget constraint gives us the consumer's demand for good Y:

$$y(p_x, p_y, I) = \frac{I}{p_y} - \frac{p_y}{4p_x}$$

Remember that the consumer can only purchase positive quantities of both goods. Since the expression for $y(p_x, p_y, I)$ involves subtraction, we must check for positivity here. Whenever this expression is negative, we need to enforce the condition $y = 0$ and arrive at a *corner solution.*

$$y(p_x, p_y, I) = \frac{I}{p_y} - \frac{p_y}{4p_x} \geq 0 \quad \Rightarrow \quad I \geq \frac{p_y^2}{4p_x}$$

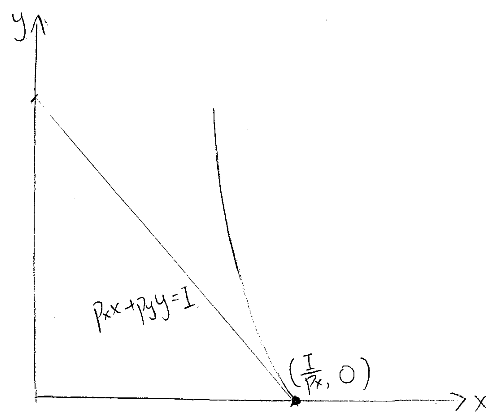
Hence the demand for good Y is:

$$y(p_x, p_y, I) = \begin{cases} 0, & \text{if } I < \frac{p_y^2}{4p_x} \\ \frac{I}{p_y} - \frac{p_y}{4p_x}, & \text{if } I \geq \frac{p_y^2}{4p_x} \end{cases}$$

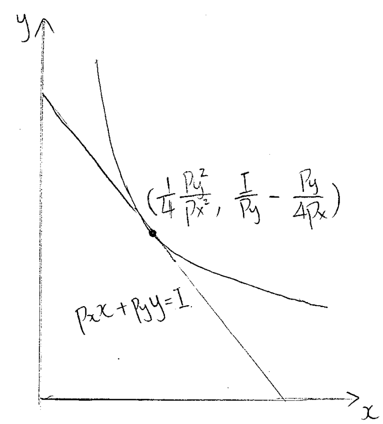
When $y = 0$, the consumer spends all of his income on good X. Thus the demand for good X is given by:

$$x(p_x, p_y, I) = \begin{cases} \frac{I}{p_x}, & \text{if } I < \frac{p_y^2}{4p_x} \\ \frac{1}{4}\frac{p_y^2}{p_x^2}, & \text{if } I \geq \frac{p_y^2}{4p_x} \end{cases}$$

The two types of solutions are illustrated by the following graph:



CORNER SOLUTION



INTERIOR SOLUTION.

<u>Interpretation of λ</u>

In this problem, the solution for λ is given by $\lambda = \frac{1}{p_y}$. Notice that it is independent of the quantities of goods X and Y and income I, and depends solely on p_y.

Recall that λ is the marginal change in the objective function when the constraint is relaxed by a little bit. In our problem, this translates into the marginal increase in utility when the consumer receives a little bit more income. Combined with the expression for λ, we can see that this consumer is always going to spend this extra income on purchasing good Y, which gives him a *marginal utility per dollar* of $\frac{1}{p_y}$.

Exercise 2: Utility Maximization with a Kinked Budget Constraint

Suppose that there are two types of goods in the economy: food F and composite goods C, which is a weighted mixture of all other goods one consumes apart from food. The consumer has a Cobb-Douglas utility function $u(C,F) = C^a F^{1-a}$, where a is between 0 and 1. His total budget comprises of \$100 worth of food stamps and \$100 of cash. The price of composite goods is normalized to 1, and the price of food is p_f. Solve his utility maximization problem.

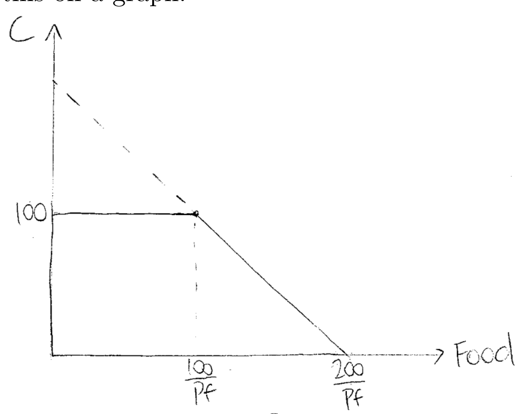
Solution:

a) Graph out this consumer's budget constraint.

Answer: Let's examine the consumer's budget set before solving this problem. Since food stamps can only be used to purchase food, the consumer can spend up to \$200 on food, but only up to \$100 on composite goods. Moreover, there is no trade-off between food and composite goods in the consumer's budget when he buys less than \$100 worth of food. Letting C be on the vertical axis and F on the horizontal axis, this budget constraint is characterized by the following equation:

$$C = \begin{cases} 100, & \text{if } F < 100 \\ 200 - p_f F, & \text{if } F \geq 100 \end{cases}$$

And it looks like this on a graph:



Budget Set

b) Solve for the consumer's demand for C and F using the equation that describes the part of the budget constraint that lies the below the kink.

Answer: To solve this problem, let's use the second part of the budget constraint, $C = \frac{200}{p_c} - \frac{p_f}{p_c}F$ to set up the Lagrangian:

$$\mathcal{L}(C, F, \lambda) = C^a F^{1-a} + \lambda(200 - p_c C - p_f F)$$

Taking partial derivatives yields the following first-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}(C, F, \lambda)}{\partial C} &= aC^{a-1}F^{1-a} - \lambda = 0 &\qquad \Rightarrow& \lambda = aC^{a-1}F^{1-a} \\
\frac{\partial \mathcal{L}(C, F, \lambda)}{\partial F} &= (1-a)C^a F^{-a} - \lambda p_f = 0 &\qquad \Rightarrow& \lambda = \frac{1-a}{p_f}C^a F^{-a} \\
\frac{\partial \mathcal{L}(C, F, \lambda)}{\partial \lambda} &= 200 - C - p_f F = 0 &\qquad \Rightarrow& C + p_f F = 200
\end{aligned}$$

Using the first two equations to eliminate λ, we arrive at:

$$\frac{C}{F} = \frac{a}{1-a}p_f \quad \Rightarrow \quad C = \frac{a}{1-a}p_f F$$

The consumer's demands for food and the composite good are, respectively:

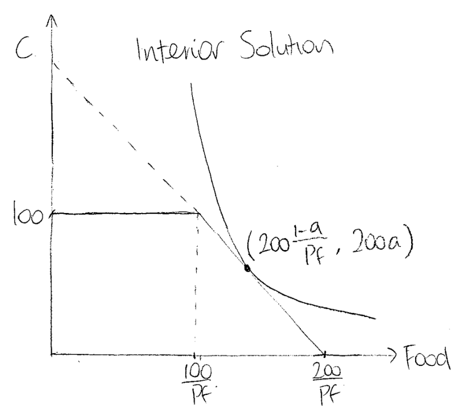
$$F = 200\frac{1-a}{p_f}, C = 200a$$

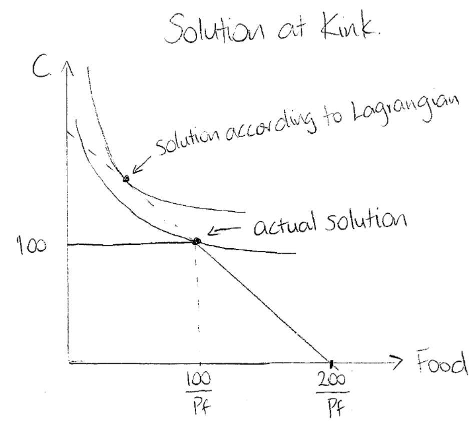
c) For what values of a do we get a solution that lies on the kink?

Answer: It is clear that the solution lies on the budget constraint if and only if $a \leq \frac{1}{2}$. For $a > \frac{1}{2}$, we get a solution at the kink. Note that the solution cannot lie on the horizontal part of the budget constraint.

Remember that a is a preference parameter, and roughly represents how much the consumer likes the composite good relative to food. Thus the intuitive understanding to why we get a solution at the kink when a is large ($a > \frac{1}{2}$) is that, the consumer likes to have a quantity of C that is too large to be supported by \$100.

On graphs, the two types of solutions look like this:


Interior Solution
C
100
$(200\frac{1-a}{P_F}, 200a)$
$\frac{100}{P_F}$
$\frac{200}{P_F}$
Food


Solution at Kink.
C
Solution according to Lagrangian
actual solution
100
$\frac{100}{P_F}$
$\frac{200}{P_F}$
Food