Derivations: Conditional Demand and Total Cost

Econ 50 - Lecture 13

February 18, 2016

1 Deriving Conditional Input Demand (and LR total cost)

For the function $f(L, K) = L^{\frac{1}{4}} K^{\frac{1}{4}}$:

1. Sketch an isoquant for $q = 10$.

   An isoquant for $q = 10$ will be various points such that

   $$L^{\frac{1}{4}} K^{\frac{1}{4}} = 10$$

   or

   $$LK = 10,000$$

   So for example, $(10, 1000)$ will work, as will $(20, 500)$, $(50, 200)$, and $(100, 100)$.

2. Calculate the $MRTS_{L,K}$.

   $$MP_L = \frac{1}{4} L^{-\frac{3}{4}} K^{\frac{1}{4}}$$
   $$MP_K = \frac{1}{4} L^{\frac{1}{4}} K^{-\frac{3}{4}}$$

   Therefore

   $$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{\frac{1}{4} L^{-\frac{3}{4}} K^{\frac{1}{4}}}{\frac{1}{4} L^{\frac{1}{4}} K^{-\frac{3}{4}}} = \frac{K}{L}$$

3. Derive the conditional demands for labor and capital – that is, $L^*(w, r, q)$ and $K^*(w, r, q)$.

   To do this we set $MRTS = \frac{w}{r}$:

   $$\frac{K}{L} = \frac{w}{r}$$

   or

   $$K = \frac{w}{r} L$$

We then substitute this back into the isoquant for q:

$$
\begin{aligned}
f(L,K) &= q \\
L^{\frac{1}{4}}K^{\frac{1}{4}} &= q \\
LK &= q^4 \\
L \times \frac{w}{r}L &= q^4 \\
L^2 &= \frac{r}{w}q^4 \\
L^* &= \sqrt{\frac{r}{w}}q^2 \\
K^* &= \frac{w}{r}L^* = \frac{w}{r}\sqrt{\frac{r}{w}}q^2 = \sqrt{\frac{w}{r}}q^2
\end{aligned}
$$

4. Find the total cost of producing q units, for general w and r.

$$
\begin{aligned}
TC(q) &= wL^*(w,r,q) + rK^*(w,r,q) \\
&= w\sqrt{\frac{r}{w}}q^2 + r\sqrt{\frac{w}{r}}q^2 \\
&= \sqrt{rw}q^2 + \sqrt{rw}q^2 \\
&= 2\sqrt{rw}q^2
\end{aligned}
$$

5. Confirm that when $w = 9$ and $r = 16$, we obtain $L^* = 133$,$K^* = 75$,$TC = \$2,400$ if we want to produce $q = 10$.

$$
\begin{aligned}
L^* &= \sqrt{\frac{r}{w}}q^2 = \sqrt{\frac{16}{9}} \times 10^2 = \frac{4}{3} \times 100 \approx 133 \\
K^* &= \sqrt{\frac{w}{r}}q^2 = \sqrt{\frac{9}{16}} \times 10^2 = \frac{3}{4} \times 100 = 75 \\
TC(10) &= 2\sqrt{rw}q^2 = 2\sqrt{9 \times 16} \times 10^2 = 2 \times 12 \times 100 = 2,400
\end{aligned}
$$

2 Short-run and long-run total cost curves

Let's look at the same production function ($f(L,K) = L^{\frac{1}{4}}K^{\frac{1}{4}}$) and now fix capital at some value $\overline{K}$.

In order to produce q units, therefore, the amount of labor required is

$$
\begin{aligned}
f(L,K) &= q \\
L^{\frac{1}{4}}\overline{K}^{\frac{1}{4}} &= q \\
L\overline{K} &= q^4 \\
L(q) &= \frac{q^4}{\overline{K}}
\end{aligned}
$$

Therefore the total cost of production is

$$
\begin{aligned}
TC(q) &= wL(q) + r\overline{K} \\
&= w\frac{q^4}{\overline{K}} + r\overline{K}
\end{aligned}
$$

Suppose $w = r = 10$. Then this becomes

$$TC(q) = \frac{10q^4}{\overline{K}} + 10\overline{K}$$

For different levels of capital, therefore, the cost function will be different. We can find the general cost function, as well as the specific costs of producing certain quantities

Capital ($\overline{K}$)	TC(q)	TC(3)	TC(4)	TC(5)	TC(6)
9	$\frac{10q^4}{9} + 90$	180	374	784	1530
16	$\frac{10q^4}{16} + 160$	211	320	551	970
25	$\frac{10q^4}{25} + 250$	282	352	500	768
36	$\frac{10q^4}{36} + 360$	382	431	534	720

From this we can see that the cost-minimizing quantity of capital depends on the amount of output we want to produce. Indeed, from our derivations above, we found that the cost-minimizing quantity of capital was $K^* = \sqrt{\frac{w}{r}}q^2$; in this case, since $w = r$, the optimal quantity of capital is q^2. So, the firm with $\overline{K} = 9$ has the lowest cost of producing $q = 3$ units of output, the firm with $\overline{K} = 16$ has the lowest cost of producing $q = 4$ units of output, and so on.

If we plot out the total cost curves of each of the firms, along with the long-run total cost curve when K is flexible, we can see that $TC(q)$ is actually the lower envelope of the "short-run" total cost curves for a fixed K:

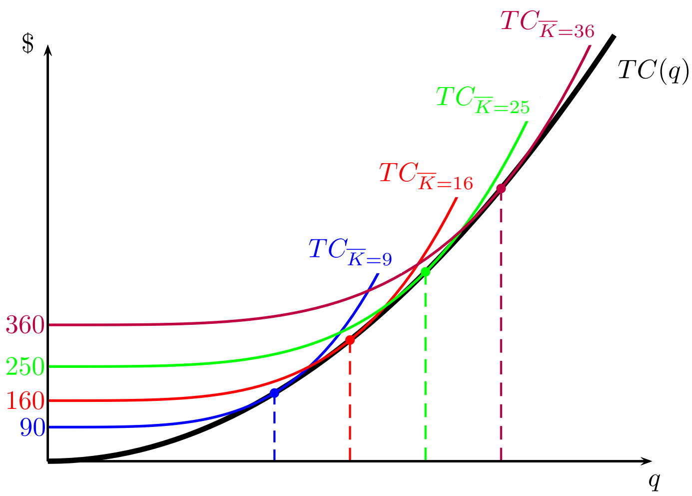

Figure 1: Short-Run and Long-Run Total Cost Curves