Midterm - Part II - Questions 5 and 6 - 45 Points

Econ 50 - Stanford University - Winter Quarter 2014/15

February 9, 2015

Write your name and your TA's name (**Rui** Xu, **Michael** Zhang, or **Connor** Scherer), and sign the statement on the cover of the exam (below).

"The answers written on these pages are entirely my own. I attest that in taking this exam, I am fully complying with all provisions of Stanford's Fundamental Standard and Honor Code."

Signature:

Printed Name:

TA's Name:

Please do not open this exam until it is time to begin. Good luck!

For Questions 5 and 6, suppose Wilson's preferences over X and Y are summarized by the utility function

$$u(x,y) = \frac{xy}{x+y}$$

which has an associated marginal rate of substitution of

$$MRS_{x,y} = \frac{y^2}{x^2}$$

As usual, he has a total of \$$I$ available to spend on X and Y at prices P_x and P_y per unit, respectively.

Question 5: Utility function deep dive: Demand derivations [20 points]

(a) Derive Wilson's **Marshallian demand functions**, $x^*(P_x, P_y, I)$ and $y^*(P_x, P_y, I)$. [10 points]
Answer: The indifference curve for any given u can be written as

$$y = \frac{ux}{x-u}$$

We don't consider $x < u$ because then $y < 0$. The indifference curves never cross the axes. Drawing the indifference curves also show that utility is convex and monotonic. The optimal is given by the tangent condition

$$MRS_{x,y} = \frac{y^2}{x^2} = \frac{P_x}{P_y} \implies y = \sqrt{\frac{P_x}{P_y}}x$$

Substitute this equation into the budget constraint

$$I = P_x x + P_y y = P_x x + P_y\sqrt{\frac{P_x}{P_y}}x \implies x^* = \frac{I}{P_x + \sqrt{P_x P_y}}$$

Substitute this back into either the MRS or budget equation (or by symmetry) gives

$$y^* = \frac{I}{P_y + \sqrt{P_x P_y}}$$

(b) Write down expressions for Wilson's **indirect utility function** $V(P_x, P_y, I)$, and his **expenditure function** $E(P_x, P_y, U)$. [5 points]

Answer: Substitute x^* and y^* into the utility function:

$$V = \frac{\frac{I}{P_x+\sqrt{P_xP_y}}\frac{I}{P_y+\sqrt{P_xP_y}}}{\frac{I}{P_x+\sqrt{P_xP_y}} + \frac{I}{P_y+\sqrt{P_xP_y}}} = \frac{\frac{I^2}{(P_x+\sqrt{P_xP_y})(P_y+\sqrt{P_xP_y})}}{\frac{I(P_y+\sqrt{P_xP_y})+I(P_x+\sqrt{P_xP_y})}{(P_x+\sqrt{P_xP_y})(P_y+\sqrt{P_xP_y})}} = \frac{I}{P_x + P_y + 2\sqrt{P_xP_y}}$$

The expenditure function is obtained by inverting V

$$I = V(P_x + P_y + 2\sqrt{P_xP_y}) \implies E = U(P_x + P_y + 2\sqrt{P_xP_y})$$

(c) Write down expressions for Wilson's **Hicksian demand functions**, $x^H(P_x, P_y, U)$ and $y^H(P_x, P_y, U)$. [5 points]

Answer: Take the Marshallian demands x^* and y^* and substitute out the I for U using the expenditure function in part (b)

$$x^H = \frac{I}{P_x + \sqrt{P_xP_y}} = \frac{U(P_x + P_y + 2\sqrt{P_xP_y})}{P_x + \sqrt{P_xP_y}} = \frac{U(\sqrt{P_x} + \sqrt{P_y})^2}{\sqrt{P_x}(\sqrt{P_x} + \sqrt{P_y})} = \frac{U(\sqrt{P_x} + \sqrt{P_y})}{\sqrt{P_x}}$$

$$y^H = \frac{I}{P_y + \sqrt{P_xP_y}} = \frac{U(P_x + P_y + 2\sqrt{P_xP_y})}{P_y + \sqrt{P_xP_y}} = \frac{U(\sqrt{P_x} + \sqrt{P_y})^2}{\sqrt{P_y}(\sqrt{P_x} + \sqrt{P_y})} = \frac{U(\sqrt{P_x} + \sqrt{P_y})}{\sqrt{P_y}}$$

Question 6: Utility function deep dive: Comparative statics analysis (25 points)

Now assume Wilson's income is $I = \$288$ and the price of good Y is $P_y = \$1$ per unit. This question will ask you to carefully analyze the effect of a price decrease in P_x from \$9 to \$4 per unit.

(a) Find the quantity of X and Y that Wilson will choose to buy if $P_x = 9$ and if $P_x = 4$. Use these two points to sketch a reasonable price-consumption curve for X (i.e., PCC_X) and (Marshallian) demand curve in two carefully-drawn diagrams. *Be sure to label your axes!* [8 points]

Answer: Using the demand functions in part (a), for $P_x = 9$

$$x^* = \frac{288}{9+\sqrt{9}} = 24$$

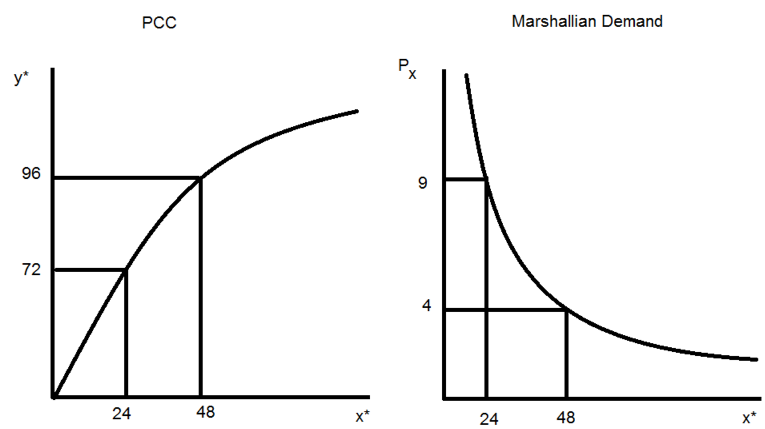
$$y^* = \frac{288}{1+\sqrt{9}} = 72$$

For $P_x = 4$

$$x^* = \frac{288}{4+\sqrt{4}} = 48$$

$$y^* = \frac{288}{1+\sqrt{4}} = 96$$

The functions x^* and y^* both increase when P_x decreases, implying that the PCC is upward sloping. This calculation shows that when x^* doubles, y^* less than doubles, meaning the curve is concave. For the demand curve, the function x^* shows that the curve asymptotes towards both axes. When $P_x \to 0$, $x^* \to \infty$. When $P_x \to \infty$, $x^* \to 0$.



(b) Does Wilson view these two goods as complements or substitutes? How do you know? [2 points]

Answer: Both part (a) and the function x^* show that

$$\frac{\partial y^*}{\partial P_x} < 0$$

They are complements. A decrease in price of X leads to an increase in the demand for Y.

(c) On a carefully drawn Slutsky diagram, show the effect of a price change from $P_x = 9$ to $P_x = 4$. Label your initial point A, the final point C, and the Slutsky decomposition point B. Clearly show the coordinates for those points, as well as the coordinates of the intercepts of all relevant budget lines. [10 points]

Answer: From part (a), $A(24, 72)$ and $C(48, 96)$. The utility at A is

$$u(A) = \frac{24 \cdot 72}{24 + 72} = 18$$

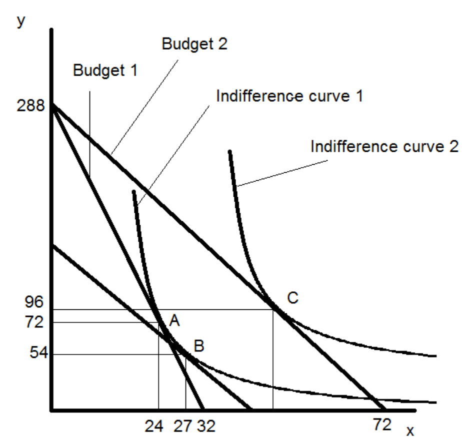
This is also $u(B)$. Using the Hicksians in Question 5, with new prices $P_x = 4$ and $P_y = 1$,

$$x^H = \frac{18(\sqrt{4} + \sqrt{1})}{\sqrt{4}} = 27, y^H = \frac{18(\sqrt{4} + \sqrt{1})}{\sqrt{1}} = 54$$

which are the coordinates for $B(27, 54)$. Note that even if you didn't compute the Hicksians in Question 5, you could get this directly using the tangency condition: we're looking for the point along the indifference curve $u(x, y) = 18$ where $MRS = 4$. Since $MRS = 4$ when $y = 2x$, you can plug $y = 2x$ into the utility function to obtain

$$u(x, 2x) = \frac{x \times 2x}{x + 2x} = \frac{2x^2}{3x} = \frac{2}{3}x.$$

Setting this equal to 18 gives us $x = 27$, so $y = 2x = 54$.



(d) Compute the **compensating variation** and **equivalent variation** for this price change. [5 points]

Answer: The utility at C is

$$u(C) = \frac{48 \cdot 96}{48 + 96} = 32$$

Using the expenditure function from Question 5, CV is the difference in expenditure evaluated at the old utility of 18 while EV is the difference in expenditure evaluated at the new utility of 32.

$$CV = E(9, 1, 18) - E(4, 1, 18) = 18(9 + 1 + 2\sqrt{9 \cdot 1}) - 18(4 + 1 + 2\sqrt{4 \cdot 1}) = 126$$

$$EV = E(9, 1, 32) - E(4, 1, 32) = 32(9 + 1 + 2\sqrt{9 \cdot 1}) - 32(4 + 1 + 2\sqrt{4 \cdot 1}) = 224$$

As above, if you hadn't been able to find a good expenditure function, you could also compute the coordinates for the decomposition points directly from the tangency condition, and evaluate the expenditure required to purchase them at the appropriate prices.

(e) Extra credit: On the back of this sheet, illustrate the compensating variation in a diagram showing the relevant Marshallian and Hicksian demand curves. [+5 points]

Answer: Graph the Hicksian for the old utility of 18. The CV is the shaded area.