# Derivations: LR and SR Profit Maximization

Econ 50 - Lecture 15

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Consider the production function

$$f(L,K) = L^{\frac{1}{4}} K^{\frac{1}{4}}$$

This firm can purchase labor and capital at prices $w$ and $r$ per unit; it can sell its output at the price $P$ per unit.

Let's derive the following for the short run and the long run:

- The total cost function
- The supply function
- The profit-maximizing demand for labor
- The firm's maximized profits
- The firm's producer surplus (and confirm Hotelling's lemma)

## 1 Short-Run Profit Maximization

In the short run, suppose this firm's capital is fixed at $\overline{K} = 4$.

### 1.1 Short run total cost

In the short run, $\overline{K} = 4$, so:

$$q = L^{\frac{1}{4}} \times 4^{\frac{1}{4}} \Rightarrow L = \frac{1}{4}q^4$$

Total cost is therefore

$$TC(q) = r\overline{K} + wL(q) = 4r + \frac{w}{4}q^4$$

and marginal cost is the derivative of this with respect to $q$:

$$MC(q) = wq^3$$

### 1.2 Short-run supply function

In the short run, the firm will set $P = MC(q)$. Using our expression for $MC(q)$ this means

$$P = wq^3$$
$$q^3 = \frac{P}{w}$$
$$q^*(P) = \left(\frac{P}{w}\right)^{\frac{1}{3}}$$

## 1.3 Short-run profit-maximizing demand for labor

We know from before that the firm's required labor, as a function of $q$, is

$$L(q) = \frac{1}{4}q^4$$

Plugging in our **supply function** $q^*(P) = \left(\frac{P}{w}\right)^{\frac{1}{3}}$ in for $q$, this becomes

$$L(q^*(P)) = \frac{1}{4}\left[\left(\frac{P}{w}\right)^{\frac{1}{3}}\right]^4 = \frac{1}{4}\left(\frac{P}{w}\right)^{\frac{4}{3}}$$

Notice that the conditional labor demand $L(q)$ was perfectly inelastic – it didn't depend at all on the wage rate $w$, since the amount of capital is fixed. However, the profit-maximizing quantity of labor demanded, $L(q^*(P))$, does depend on $w$!

## 1.4 Short-run maximized profits

Short-run profits are total revenues minus total costs:

$$\begin{aligned}
\Pi(P) &= TR(q^*(P)) - TC(q^*(P)) \\
&= P \times q^*(P) - \left[4r + \frac{w}{4}(q^*(P))^4\right] \\
&= P \times \left(\frac{P}{w}\right)^{\frac{1}{3}} - \left[4r + \frac{w}{4}\left(\frac{P}{w}\right)^{\frac{4}{3}}\right] \\
&= \frac{3}{4}P^{\frac{4}{3}}w^{-\frac{1}{3}} - 4r
\end{aligned}$$

## 1.5 Producer Surplus (and confirm Hotelling's Lemma)

The producer surplus at market price $P$ is the integral, from $p = 0$ to $p = P$, of the quantity supplied at price $p$:

$$\begin{aligned}
\int_{p=0}^{P} q^*(p)dp &= \int_{p=0}^{P} p^{\frac{1}{3}}w^{-\frac{1}{3}}dp \\
&= \left.\frac{3}{4}p^{\frac{4}{3}}w^{-\frac{1}{3}}\right|_{p=0}^{P} \\
&= \frac{3}{4}P^{\frac{4}{3}}w^{-\frac{1}{3}}
\end{aligned}$$

Note that adding fixed costs to producer surplus gives us profit.

The inverse of this procedure is, in effect the confirmation of Hotelling's lemma. Specifically, Hotelling's lemma states that the derivative of the profit function with respect to $P$, will give the optimal $q$:

$$\frac{d\Pi(P)}{dP} = P^{\frac{1}{3}}w^{-\frac{1}{3}}$$

So on the one hand, taking the integral of quantity gives us profit (if we then add fixed costs, which don't change with $P$). And on the other hand, taking the derivative of profit gives us quantity.

## 2 Profits and Losses

Will this firm be profitable?

The algebra here gets a little complex, so let's look at the case where $w = 8$ and $r = 24$. With these numbers, we have

$$
\begin{aligned}
TC(q) &= 96 + 2q^4 \\
MC(q) &= 8q^3 \\
q^*(P) &= \frac{1}{2}P^{\frac{1}{3}} \\
\Pi(P) &= \frac{3}{8}P^{\frac{4}{3}} - 96
\end{aligned}
$$

The firm's "break-even" price occurs when profit is equal to zero:

$$
\begin{aligned}
\frac{3}{8}P^{\frac{4}{3}} &= 96 \\
P^{\frac{4}{3}} &= 256 \\
P &= 64
\end{aligned}
$$

The quantity at that price is

$$q^*(P) = \frac{1}{2}P^{\frac{1}{3}} = \frac{1}{2} \times 4 = 2$$

At that quantity, the firm's costs are

$$
\begin{aligned}
TC(q) &= 96 + 2 \times 2^4 = 128 \\
MC(q) &= 8 \times 2^3 = 64
\end{aligned}
$$

Furthermore, notice that the average cost of producing $q = 2$ units is $AC(2) = TC(2)/2 = 64$. So, at a quantity of 2, $MC = AC = 64$.

(By the way, note that quantity might very well be measured in "millions of units per year," so $q = 2$ doesn't necessarily mean exactly two units of output...)

# 3 Long Run Profit Maximization

Now let's analyze this firm's profit-maximizing decision in the long run.

## 3.1 Long run total and marginal cost

In the long run, the firm can freely choose labor and capital. As we've found before, the **conditional** demands for labor and capital are:

$$\begin{aligned} L^*(q) &= r^{\frac{1}{2}} w^{-\frac{1}{2}} q^2 \\ K^*(q) &= r^{-\frac{1}{2}} w^{\frac{1}{2}} q^2 \\ TC^{LR}(q) &= 2r^{\frac{1}{2}} w^{\frac{1}{2}} q^2 \end{aligned}$$

The long-run marginal cost curve is the derivative of the long-run total cost curve:

$$MC^{LR}(q) = \frac{dTC(q)}{dq} = 4r^{\frac{1}{2}} w^{\frac{1}{2}} q$$

## 3.2 Long run supply function

In the long run, the firm will set $P = MC^{LR}(q)$:

$$\begin{aligned} P &= 4r^{\frac{1}{2}} w^{\frac{1}{2}} q \\ q^{LR}(P) &= \frac{1}{4} r^{-\frac{1}{2}} w^{-\frac{1}{2}} P \end{aligned}$$

## 3.3 Long run profit-maximizing demand for labor and capital

To find the long-run demand for labor, we plug the optimal quantity back into the conditional demand for labor:

$$\begin{aligned} L^{LR}(w, r, P) &= r^{\frac{1}{2}} w^{-\frac{1}{2}} [q^{LR}(P)]^2 \\ &= r^{\frac{1}{2}} w^{-\frac{1}{2}} \left[ \frac{1}{4} r^{-\frac{1}{2}} w^{-\frac{1}{2}} P \right]^2 \\ &= \frac{1}{16} r^{-\frac{1}{2}} w^{-\frac{3}{2}} P^2 \end{aligned}$$

We can do the same thing for capital; the algebra yields the symmetric result:

$$K^{LR}(w, r, P) = \frac{1}{16} r^{-\frac{3}{2}} w^{-\frac{1}{2}} P^2$$

## 3.4 Long-run maximized profits

Long-run profits are total revenues minus total costs:

$$\begin{aligned} \Pi^{LR}(P) &= TR(q^{LR}(P)) - TC(q^{LR}(P)) \\ &= P \times q^{LR}(P) - 2r^{\frac{1}{2}} w^{\frac{1}{2}} (q^{LR}(P))^2 \\ &= P \times \frac{1}{4} r^{-\frac{1}{2}} w^{-\frac{1}{2}} P - 2r^{\frac{1}{2}} w^{\frac{1}{2}} \left( \frac{1}{4} r^{-\frac{1}{2}} w^{-\frac{1}{2}} P \right)^2 \\ &= \frac{1}{4} r^{-\frac{1}{2}} w^{-\frac{1}{2}} P^2 - \frac{1}{8} r^{-\frac{1}{2}} w^{-\frac{1}{2}} P^2 \\ &= \frac{1}{8} r^{-\frac{1}{2}} w^{-\frac{1}{2}} P^2 \end{aligned}$$

### 3.5 Producer surplus and Hotelling's Lemma

As before, producer surplus at market price $P$ is the integral, from $p = 0$ to $p = P$, of the quantity supplied at price $p$:

$$\begin{aligned}
\int_{p=0}^{P} q^{LR}(p)dp &= \int_{p=0}^{P} \frac{1}{4}r^{-\frac{1}{2}}w^{-\frac{1}{2}}pdp \\
&= \frac{1}{8}r^{-\frac{1}{2}}w^{-\frac{1}{2}}p^2\bigg|_{p=0}^{P} \\
&= \frac{1}{8}r^{-\frac{1}{2}}w^{-\frac{1}{2}}P^2
\end{aligned}$$

In this case, because there are no fixed costs, this is exactly equal to the maximized profit.

We can again confirm Hotelling's Lemma holds:

$$\frac{d\Pi(P)}{dP} = \frac{1}{4}r^{-\frac{1}{2}}w^{-\frac{1}{2}}P = q^{LR}(P)$$

# 4 Comparison of SR and LR demand for labor

Let's look at the wage elasticity of labor in the long run and the short run. Taking the natural log of the short-run demand for labor and long-run demand for labor, we see:

$$\begin{aligned}
\ln(L^{SR}) &= \ln\frac{1}{4} - \frac{4}{3}\ln w + \frac{4}{3}\ln P \\
\ln(L^{LR}) &= \ln\frac{1}{16} - \frac{1}{2}\ln r - \frac{3}{2}\ln w - 2\ln P
\end{aligned}$$

We can now easily find the **wage elasticity of labor demand**:

$$\begin{aligned}
\epsilon_{L^{SR},w} &= \frac{\partial \ln L^{SR}(w,r,P)}{\partial \ln w} = -\frac{4}{3} \\
\epsilon_{L^{LR},w} &= \frac{\partial \ln L^{LR}(w,r,P)}{\partial \ln w} = -\frac{3}{2}
\end{aligned}$$

As expected, the demand for labor in the long run is **more elastic**, because in the long run the firm can substitute capital for labor.

We can also find the **cross-price elasticity of labor demand** (i.e., the elasticity of labor demand with respect to the price of capital, $r$):

$$\begin{aligned}
\epsilon_{L^{SR},r} &= \frac{\partial \ln L^{SR}(w,r,P)}{\partial \ln r} = 0 \\
\epsilon_{L^{LR},r} &= \frac{\partial \ln L^{LR}(w,r,P)}{\partial \ln r} = -\frac{1}{2}
\end{aligned}$$

Explaining what's going on here is an excellent potential final exam question...