CH3 Gaussian Filter
Exercise 3 Solution

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1 Problem 1

Let X_t denote the state of the car at time t.

1.1 Question (a)

The state vector should be:
$x_t = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$

1.2 Question (b)

Goal: State Transition model of $P(x_t|u_t, x_{t-1})$
Solution:

Let u_t denote operation at time t, then
$u_t \sim N(0, \sigma^2)$

Suppose during Δt, the acceleration is constand, thus
$\ddot{x_t} = u_t$
$\dot{x_t} = \dot{x_{t-1}} + u_{t-1}\Delta t$
$x_t = x_{t-1} + \dot{x_{t-1}}\Delta t + u_{t-1}\Delta t^2/2$

$$X_t = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} X_{t-1} + \begin{pmatrix} 0.5\Delta t^2 \\ \Delta t \end{pmatrix} \ddot{x}_{t-1}$$

since $u_{t-1} \sim N(0, \sigma^2)$, we have $\ddot{x}_{t-1} \sim N(0, \sigma^2)$

$$A = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \epsilon = \begin{pmatrix} 0.5\Delta t^2 u \\ \Delta t u \end{pmatrix}$$

where $u \sim N(0, \sigma^2)$
as $\Delta t = 1$

$$\epsilon = \begin{pmatrix} 0.5u \\ u \end{pmatrix}$$

so,

$$R = \begin{pmatrix} 1/4 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

1.3 Question (c)

Suppose

$$\Sigma_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

For $t = 1, 2, 3, 4, 5$

$$\bar{\Sigma}_1 = A\Sigma_0 A^T + R = \begin{pmatrix} 0.25 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$$

$$\bar{\Sigma}_2 = A\Sigma_1 A^T + R = \begin{pmatrix} 2.5 & 2.0 \\ 2.0 & 2.0 \end{pmatrix}$$

$$\bar{\Sigma}_3 = A\Sigma_2 A^T + R = \begin{pmatrix} 8.75 & 4.5 \\ 4.5 & 3.0 \end{pmatrix}$$

$$\bar{\Sigma}_4 = A\Sigma_3 A^T + R = \begin{pmatrix} 21.0 & 8.0 \\ 8.0 & 4.0 \end{pmatrix}$$

$$\bar{\Sigma}_5 = A\Sigma_4 A^T + R = \begin{pmatrix} 41.25 & 12.5 \\ 12.5 & 5.0 \end{pmatrix}$$

1.4 Question (d)

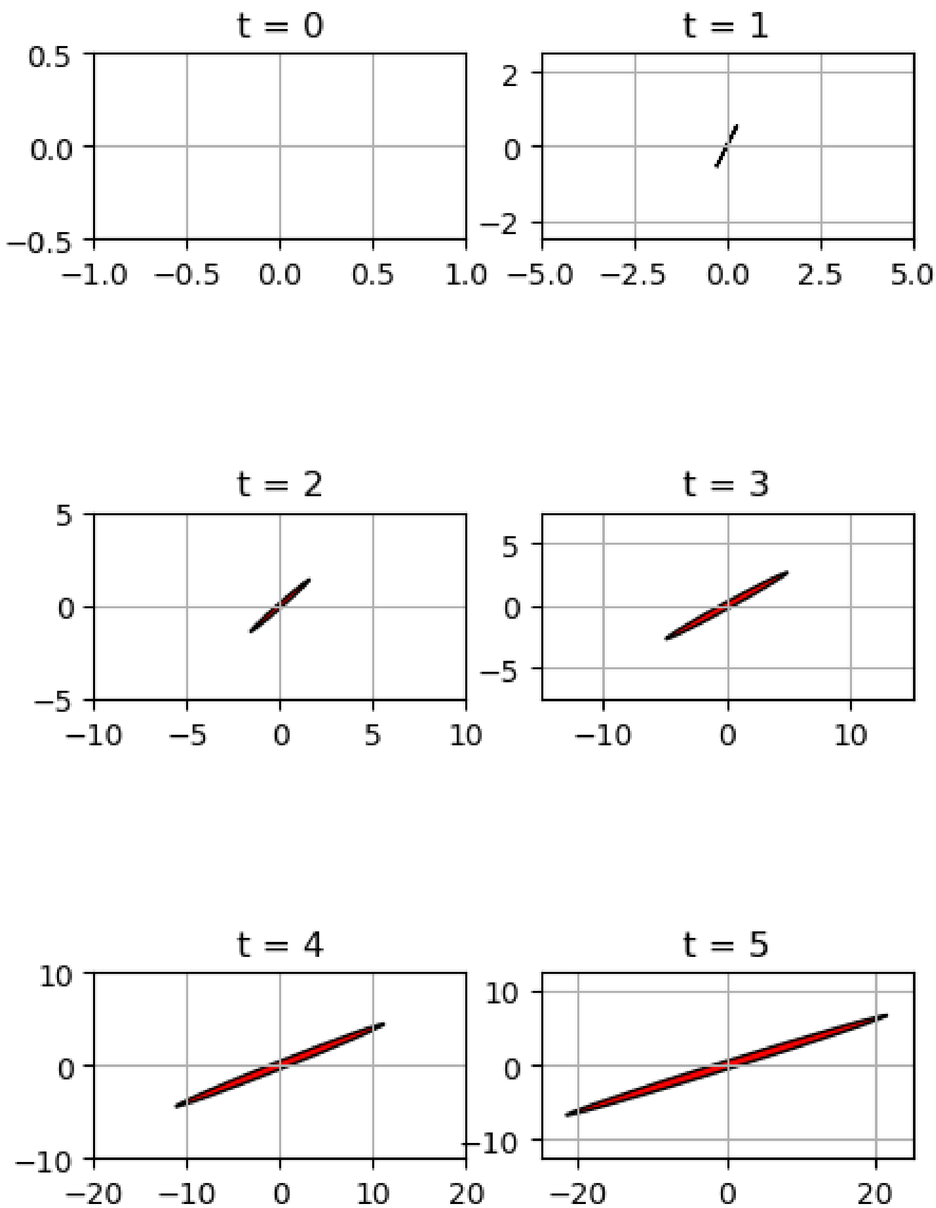

Figure 1: Uncertainty Ellipse

1.5 Question (e)

As $t \to \infty$, the uncertainty ellipse will keep growing.

2 Problem 2

As in Problem 1

$x_t = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$

2.1 Question (a)

Let z_t denote a measurement, since only the displacement is measured

$$z_t = \begin{pmatrix} 1 & 0 \end{pmatrix} x_t + \delta_t$$

where $\delta_t \sim N(0, 10)$ and $Q = [10]$ (a single element matrix).

2.2 Question (b)

From problem 1,

$$\bar{\mu}_5 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \bar{\Sigma}_5 = \begin{pmatrix} 41.25 & 12.5 \\ 12.5 & 5.0 \end{pmatrix}$$

$$K_5 = \bar{\Sigma}_5 C_5^T (C_5 \bar{\Sigma}_5 C_5^T + Q_5)^{-1}$$

where

$$Q_5 = [10], C_5 = [1, 0]^T$$

so

$$K_5 = \begin{pmatrix} 0.80 \\ 0.24 \end{pmatrix}$$

$$\mu_5 = \bar{\mu_5} + K_5(z_5 - C_t \bar{\mu_5}) = \begin{pmatrix} 4.02 \\ 1.22 \end{pmatrix}$$

$$\Sigma_5 = (I - K_t C_5)\bar{\Sigma}_5 = \begin{pmatrix} 8.05 & 2.44 \\ 2.44 & 1.95 \end{pmatrix}$$

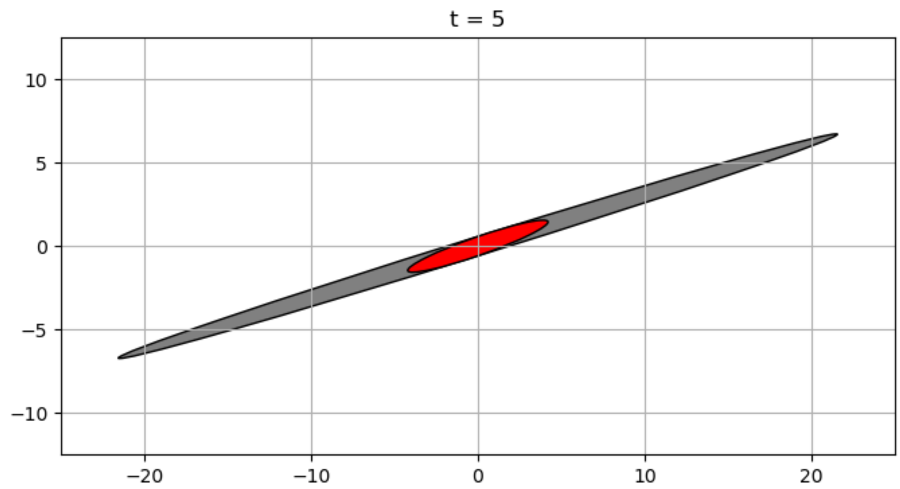

Figure 2: Uncertainty Ellipse: Before(Gray) and After(Red) Observation

3 Problem 3

As know, if $X_t \sim N(\mu, \sigma^2)$ the characteristic function of it should be

$$C_X(t) = exp(iut - \frac{\sigma^2 t^2}{2})$$

and

$$Y = aX + b \Rightarrow C_Y(t) = e^{ibt} C_X(at)$$
$$Z = X + Y \Rightarrow C_Z(t) = C_X(t) C_Y(t)$$

For prediction step of KF

$$x_t = Ax_{t-1} + Bu_t + \epsilon_t$$

where

$$x_{t-1} \sim N(\mu_{t-1}, \Sigma_{t-1})$$
$$\epsilon_t \sim N(0, R_t)$$

so $(Ax_{t-1} + Bu_t) \sim N(A\mu_{t-1} + Bu_t, A\Sigma_{t-1}A^T)$
and $x_t = (Ax_{t-1} + Bu_t) + \epsilon_t$

thus

$$\begin{aligned}
C_{x_t}(t) &= C_{(Ax_{t-1}+Bu_t)}(t)C_{\epsilon_t}(t) \\
&= exp < iA\mu_{t-1} + iBu_t, t > exp[-1/2 < A\Sigma_{t-1}A^T t, t >] \\
&\times exp < 0, t > exp[-1/2 < R_t t, t >] \\
&= exp < i(A\mu_{t-1} + Bu_t), t > exp[-1/2 < (A\Sigma_{t-1}A^T + R_t)t, t >]
\end{aligned}$$

transform back to probability space

$$x_t \sim N(A\mu_{t-1} + Bu_t, A\Sigma_{t-1}A^T + R_t)$$

this is

$$\begin{aligned}
b\bar{e}l(x_t) &= A_t\mu_{t-1} + Bu_t \\
\bar{\Sigma}_t &= A\Sigma_{t-1}A^T + R_t
\end{aligned}$$

4 Problem 4