date: monday, february 12, 2024

Group Theory and Linear Algebra

Matrices and Groups:

Let $M_n(\mathbb{R}) = \{$all $n \times n$ matrices in $\mathbb{R}\}$. If we ignore multiplication, then $M_n(\mathbb{R})$ is a group under addition:

- identity $= [0]$ <- zero matrix
- if $A = [a_{ij}]$, then $-A = [-a_{ij}]$

Set $M_n(\mathbb{R})$ is not a group under multiplication because many matrices (eg. $[0]$) has no multiplicative inverse.

Recall: $A \in M_n(\mathbb{R})$ is invertible <=> $\det(A) \neq 0$.

Let $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$.

Def$^{\underline{n}}$: General linear group $GL_n(\mathbb{R})$ = set of all invertible matrices.

Fact: $GL_n(\mathbb{R})$ is a group under multiplication.
$|GL_n(\mathbb{R})| = \infty$
$GL_n(\mathbb{R})$ is not abelian
identity is I_n
given $A \in GL_n(\mathbb{R})$, the inverse is A^{-1}

Theorem: Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ (this is a group under multiplication). Then the map $\det: GL_n(\mathbb{R}) \to \mathbb{R}^*$ given by $A \mapsto \det(A)$ is a group homomorphism.

Proof
Let $A, B \in GL_n(\mathbb{R})$. Then $\det(AB) = \det(A)\det(B)$. So det is a group homomorphism.

$\square$

Can use kernel of det to find normal subgroups:
$\ker(\det) = \{A \in GL_n(\mathbb{R}) \mid \det(A) = 1\}$.

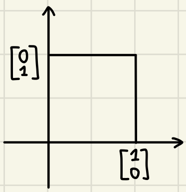
Def^{n}: Special linear group: $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid \det(A) = 1\}$.

Note: $SL_n(\mathbb{R})$ is normal in $GL_n(\mathbb{R})$

1st Isometry Theorem $GL_n(\mathbb{R}) / SL_n(\mathbb{R}) \simeq \mathbb{R}^*$

$\boxed{\text{Geometry}}$ A matrix A corresponds to a linear transformation. $T_A: \mathbb{R}^n \to \mathbb{R}^n$ given by $T_A(\vec{x}) = A\vec{x}$.

eg. $A = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$ What does T_A do to the unit square?

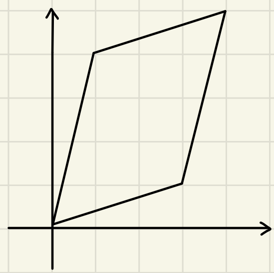


$T_A(\vec{x}) = A\vec{x}$

$T_A\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$T_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$T_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$



Recall: If S has area B in $\mathbb{R}^n$, the area of $T_A(S)$ is $|\det A| B$.

If $A \in SL_n(\mathbb{R})$, then the map $T_A: \mathbb{R}^2 \to \mathbb{R}^2$ maps the unit square to another shape of area 1.

$\boxed{\text{Orthogonal Group } O(n)}$

A matrix A is orthogonal if $A^T = A^{-1}$.

eg. $A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$, $A^T = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Alternative def^{n}: A orthogonal if:

- $\begin{bmatrix} a_{1,i} \\ a_{2,i} \\ \vdots \\ a_{n,i} \end{bmatrix}$ is the i^{th} column, then $\sqrt{a_{1,i}^2 + a_{2,i}^2 + \cdots + a_{n,i}^2} = 1$
- for columns $i \neq j$, $a_{1,i}a_{1,j} + a_{2,i}a_{2,j} + \cdots + a_{n,i}a_{n,j} = 0$ (saying the columns are orthogonal)

Def$^{\underline{n}}$: $O(n) = \{$set of all $n\times n$ orthogonal matrix$\}$

Claim: $O(n)$ is a group under multiplication.
Given $A, B \in O(n)$, want to show $AB \in O(n) \iff AB \in O(n) \iff (AB)^T = (AB)^{-1}$
$(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}$.

Fact: If $A \in O(n)$, then $\det(A) = \pm 1$.

Proof
$I_n = \det(I_n) = \det(A)\det(A^T) = \det(A)\det(A) = \det(A)^2$.
So $\det(A) = \pm 1$.

$\square$

Elements of $O(n)$ preserve distance.

Given $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, the distance between $\vec{v}$ and $\vec{w}$ is

$$\begin{aligned} \text{dist}(\vec{w}) &= \sqrt{(v_1 - w_1)^2 + \cdots + (v_n - w_n)^2} \\ &= \|\vec{v} - \vec{w}\| \end{aligned}$$

distance or norm

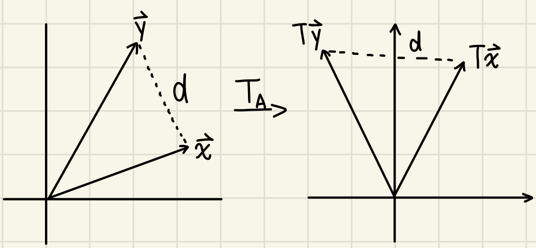
Theorem: $A \in O(n)$ iff

$$\|A\vec{x} - A\vec{y}\| = \|\vec{x} - \vec{y}\| \quad \text{for all } \vec{x}, \vec{y} \in \mathbb{R}.$$

distance between $A\vec{x}$ and $A\vec{y}$

distance between $\vec{x}$ and $\vec{y}$

Picture:



Def$^{\underline{n}}$: Special orthogonal group

$$\begin{aligned} SO(n) &= O(n) \cap SL_n(\mathbb{R}) \\ &= \{\text{all orthogonal matrices with } \det = 1\} \end{aligned}$$

$\boxed{\text{Special Case } O(2)}$ Elements in $O(2)$ come in two "flavours"

Note that $A \in O(2)$ is determined by where it takes $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

If $A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$, then $a^2+b^2=1$. Note this implies $A = \begin{bmatrix}a & *\\b & *\end{bmatrix}$.

Since columns orthogonal, then

① $A = \begin{bmatrix}a & -b\\b & a\end{bmatrix}$ or ② $A = \begin{bmatrix}a & b\\b & -a\end{bmatrix}$.

If ②,

$$A = \begin{bmatrix}a & b\\b & -a\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$$ for some θ (a rotation by θ).

If ①,

$$\begin{bmatrix}a & -b\\b & a\end{bmatrix} = \underbrace{\begin{bmatrix}a & b\\b & -a\end{bmatrix}}_{\text{rotation}}\underbrace{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}}_{\text{flip about } x\text{-axis}}$$