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Group Actions of the Class Equation

Last time: Let X be a G-set, ie. there is a group action

$$G \times G \to X$$
$$(g,x) \mapsto g \cdot x$$

- $x \sim y$ if there exists a $g \in G$ such that $y = g \cdot x$
- this is an equivalence relation on X
- orbit of $x \in X$: $\mathcal{O}_x = \{g \cdot x \mid g \in G\} = \{y \mid x \sim y\}$ <- equivalence class of x
- so if $\mathcal{O}_{x_1}, \dots, \mathcal{O}_{x_s}$ are distinct classes, $X = \mathcal{O}_{x_1} \cup \dots \cup \mathcal{O}_{x_s}$ is a partition

Def$^{\underline{n}}$: Fixed point of $g \in G$, $X_g = \{x \in X \mid g \cdot x = x\} \subseteq X$
Stabilizer subgroup of $x \in X$, $G_x = \{g \in G \mid g \cdot x = x\} \subseteq G$.

Lemma: G_x is a subgroup of G.

Proof

1) $G_x \neq \emptyset$ since $e \cdot x = x$, so $e \in G_x$.
2) Let $g \in G_x$. So $g \cdot x = x$. So $g^{-1} \cdot (g \cdot x) = g^{-1} \cdot x \iff (g^{-1}g) \cdot x = g^{-1}x \iff x = g^{-1}x$. So $g^{-1} \in G_x$.
3) $g, h \in G_x$. Then
$$\begin{aligned}(gh) \cdot x &= g \cdot (h \cdot x) & h \in G_x \\ &= g \cdot x & g \in G_x \\ &= x.\end{aligned}$$
So $gh \in G_x$.

$\square$

Q: How many elements in $\mathcal{O}_x$?

Theorem: $|\mathcal{O}_x| = \frac{|G|}{|G_x|} = \underbrace{[G : G_x]}_{\text{index of } G_x \text{ in } G} = \#$ distinct left cosets.

Proof

$[G : G_x] = \#$ distinct left cosets of G_x. Let $\mathcal{L}_{G_x}$ = set of distinct left cosets $= \{gG_x \mid g \in G\}$.

Note that if $y \in \mathcal{O}_x$, there exists a $g \in G$ such that $y = g \cdot x$. Define a map

$$\begin{aligned} \Phi : \mathcal{O}_x &\to \mathcal{L}_{G_x} \\ y &\mapsto gG_x \text{ where } y = g \cdot x. \end{aligned}$$

If we show this is a bijection, then $|\mathcal{O}_x| = |\mathcal{L}_{G_x}|$.

The map is surjective, because if $gG_x \in \mathcal{L}$, then $y = g \cdot x \in \mathcal{O}$ and $\Phi(y) = gG_x$.

The map is injective because if

$$\Phi(y_1) = g_1 G_x = g_2 G_x = \Phi(y_2)$$

with $y_1 = g_1 \cdot x$ and $y_2 = g_2 \cdot x$. So there exists $g \in G_x$ such that $g_2 = g_1 g$.

So

$$\begin{aligned} y_2 = g_2 \cdot x &= (g_1 g) \cdot x \quad \text{group action property} \\ &= g_1 \cdot (g \cdot x) \quad g \in G_x \\ &= g_1 \cdot x \\ &= y_1. \end{aligned}$$

So Φ is injective.

$\square$

eg. $X = \{1,2,3,4\}$, $G = \{\sigma_1, \sigma_2\} = \{(1), (1\,2)(3\,4)\}$ and $\begin{aligned} G \times X &\to X \\ (\sigma, i) &\mapsto \sigma(i) \end{aligned}$.

$\mathcal{O}_1 = \{\sigma_1(1), \sigma_2(1)\} = \{1, 2\}$.

$G_1 = \{\sigma \in G \mid \sigma(1) = 1\} = \{\sigma_1\}$.

So $|\mathcal{O}_1| = \frac{|G|}{|G_1|} = 2/1 = 2$.

Observation: If $|\mathcal{O}_x| = 1$, then $\{g \cdot x \mid g \in G\} = \{x\}$. So if X is a G-set, then the set of all fixed points

$$X_G = \{x \mid g \cdot x = x \text{ for all } g \in G\} = \mathcal{O}_x \cup \cdots \cup \mathcal{O}_{x_s}$$

where $|\mathcal{O}_{x_i}| = 1$.

Summary: Let X be a G-set, and let $x_1, \dots, x_n$ be the distinct cosets representatives. Then,

$$X = \underbrace{\mathcal{O}_{x_1} \cup \cdots \cup \mathcal{O}_{x_s}}_{|\mathcal{O}_i| > 1} \cup \underbrace{\mathcal{O}_{x_{s+1}} \cup \cdots \cup \mathcal{O}_{x_n}}_{|\mathcal{O}_{x_i}| = 1} = \mathcal{O}_{x_1} \cup \cdots \cup \mathcal{O}_{x_j} \cup X_G$$

Thus,

$$\boxed{|X| = |\mathcal{O}_{x_1}| + \cdots + |\mathcal{O}_{x_s}| + |X_G| = [G : G_{x_1}] + \cdots + [G : G_{x_s}] + |X_G|}$$

Class Equation

We specialize these results to the following case: $G \times G \to G$ (where $X = G$)
$(g \cdot x) \mapsto \underbrace{gxg^{-1}}_{\text{sometimes called conjugation}}$

Set of fixed points: $Z(G) = \{x \in G \mid gxg^{-1} = x \text{ for all } g \in G\}$
↳ This is the center of G (subgroup) $\Updownarrow$ $gx = xg$

The stabilizer subgroup of X: $C(x) = \{g \mid gxg^{-1} = x \iff gx = xg\}$ ← all things that commute with x
↳ This is called the centralizer subgroup

The orbits of $x \in G$ (called conjugacy classes of x): $\mathcal{O}_x = \{gxg^{-1} \mid g \in G\}$.

Theorem: Let G be a finite group and consider the group action of conjugation:

$$G \times G \to G$$
$$(g, x) \mapsto gxg^{-1}$$

If $x_1, \dots, x_n$ are the distinct coset representation of this action, then

$$G = \mathcal{O}_{x_1} \cup \cdots \cup \mathcal{O}_{x_n},$$

Furthermore, if $|\mathcal{O}_{x_1}|, \dots, |\mathcal{O}_{x_s}| > 1$ and $|\mathcal{O}_{s+1}| = \cdots = |\mathcal{O}_n| = 1$, then

$$|G| = |\mathcal{O}_{x_1}| + \cdots + |\mathcal{O}_{x_s}| + |Z(G)|$$
$$\boxed{(*) = [G : C(x_1)] + \cdots + [G : C(x_s)] + |Z(G)|.}$$

Def^n: (*) is the class equation of G.