## Lecture 13 - Supplement

The proof given in class for the Second Sylow theorem can be strengthened. (In fact, some textbooks use this as the statement of the second Sylow theorem)

(Strong Second Sylow Theorem) Let $G$ be a finite gp and $p$ a prime such that $p \mid |G|$. Suppose $H$ is a subgp of $G$ such that $|H| = p^l$ for some $r$. Then $H$ is contained in some Sylow $p$-subgp.

Proof The proof runs almost ~~all~~ the same as the one given in class we give all the details here

Note that since $|H| \mid |G|$ and $|H| = p^l$, this implies $|G| = p^r m$ with $r \geq l$ and $p \nmid m$. Let $P$ be a Sylow $p$-subgp with $|P| = p^r$ (such a gp exists by First Sylow Thm)

Let $S = \{gPg^{-1} \mid g \in G\}$ be all the conjugates of $P$

As shown in class, $p \nmid |S|$ (the proof in class is same to here)

(Here is where we change the proof). Let $H$ be our gp with $|H| = p^l$
For each $P_i \in S$, consider the $H$-conjugates of $P_i$, i.e
$\{hP_ih^{-1} \mid h \in H\}$. This is a subset of $S$

Also, $|\{hP_ih^{-1} \mid h \in H\}| = [H : N(P) \cap H]$ ← this is lemma 15.C

Since $|H| = |N(P) \cap H| [H : N(P) \cap H]$ and $|H| = p^l$, this forces $[H : N(P) \cap H]$ to be a multiple of $p$

The sets $A_1 = \{hP_1h^{-1} \mid h \in H\}$, $A_2 = \{hP_2h^{-1} \mid h \in H\}$, ..., $A_t = \{hP_th^{-1} \mid h \in H\}$

partition $S$, i.e.

$$S = A_1 \cup \cdots \cup A_t$$

(we are using fact that we are using a gp action $H \times S \to S$ to partition $S$)

If each $|A_i| \geq p$, then since $|A_i| = p^{a_i}$, this would imply $p \mid |S|$, which is not allowed. So some $|A_i| = p^0 = 1$.

This means $\{hP_ih^{-1} \mid h \in H\} = \{P_i\}$.

Note that ~~this~~ ~~then implies that~~ we have $hP_ih^{-1} = P_i$ for all $h \in H$. But since $h \in H$, $|h| = p^a$ for some $a$. So, by Lemma 15.5, we have $h \in P_i$ for all $h \in H$, i.e. $H \subseteq P_i$ ☐

(The proof is almost identical to the one given in class, but we use $H$ instead another Sylow $P$-subgp $Q$)