date: wednesday, april 10, 2024

Exam Review

Format: Part A: 20 points
9 short answer questions
Part B: 30 points
8 proof questions, choose 6
Bonus: 2 points

1. Give examples of:
(a) a domain that is not a PID $\mathbb{R}[x,y]$

(b) a p-group that is not cyclic $\mathbb{Z}_2 \times \mathbb{Z}_2$

(c) field extension that is not algebraic $\mathbb{Q}(\pi)$

(d) a simple group $\mathbb{Z}_p$ p prime, A_n for $n \geq 5$

(e) irreducible polynomial $x^2+1 \in \mathbb{R}[x]$, $x+1 \in \mathbb{R}[x]$

(f) prime ideal that is not maximal $\{0\}$ in $\mathbb{Z}$ as $\mathbb{Z}/\{0\}$ is a domain but not a field. $\mathbb{R}[x,y]$ where $P=\langle x \rangle$ is prime but not maximal

(g) a field that is not algebraically closed $\mathbb{R}, \mathbb{Q}, \mathbb{Z}_3$

(h) Euclidean domain $\mathbb{R}, \mathbb{Z}[i], \mathbb{Z}$

(i) non-commutative ring $M_2(\mathbb{R})$

(j) non-solvable group The Monster, A_5

2. Find all finite abelian groups of order $42 = 2 \times 3 \times 7$

Only one: $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7$

3. Show all groups with $|G| = 42$ are not simple.

G must have number of Sylow 7-subgroups $\equiv 1 \pmod 7$ so only one => normal => not simple.

4. Give an example of a non-abelian group of order 42.

D_{21}

5. Give an example and non-example of a class equation for a group of order 2024.

For an abelian group:

$2024 = |Z(G)| = 2024$

For a nonabelian group.

$2024 = 2000 + 24$ as $2000 \nmid 2024$.

6. Find two different factorizations of $x^2 + x + 8$ in $\mathbb{Z}_{10}[x]$.

$$\begin{array}{r l} & x + 2 \\ x - 1 \;\big) & x^2 + x + 8 \\ & -(x^2 - x) \\ \hline & 2x + 8 \\ & -(2x - 2) \\ \hline & 10 \end{array}$$

$(x-1)(x+2)$
$\equiv (x+9)(x+2)$

$(x+a)(x+b) = x^2 + (a+b)x + ab$

$a + b \equiv 1 \pmod{10}$
$ab \equiv 8 \pmod{10}$

$a = 4$, $b = 7$ $\quad$ $a = 9$, $b = 2$ $\quad$ $(x+9)(x+2) \equiv (x+4)(x+7)$

7. Find a basis for $\mathbb{Q}(\sqrt{8})$ over $\mathbb{Q}(\sqrt{2})$.

$\mathbb{Q}(\sqrt{8}) = \mathbb{Q}(2\sqrt{2})$

$a + b\sqrt{2}$

$(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})(2\sqrt{2})$
$= a_1 + b_1\sqrt{2} + 2\sqrt{2}\,a_2 + b_2 \cdot 2 \cdot 2$

$\mathbb{Q}(\sqrt{8}) = \mathbb{Q}(\sqrt{2})$

$\sqrt{8}$ is a root of $x - 2\sqrt{2} \in \mathbb{Q}(\sqrt{2})(x)$

=> Basis is $\{1\}$

Office Hours: Friday, April 19th