date: wednesday, january 31, 2024

Counting and Burnside's Equation

Applying group actions to solve counting problems.

Problem: We have a flag with six equal stripes. I can colour them red, blue, or green.

b
r
b
r
r
g

Want to count number of possible flags.
Wrong answer since this flag is the same if we flipped it upside down.
Note: A flag can be represented as a six-tuple $(c_1, c_2, c_3, c_4, c_5, c_6)$ with $c_i \in \{b, r, g\}$.

Let $X = \{\text{all such 6-tuples}\}$, we have $|X| = 3^6$.
Let τ be the permutation that corresponds to "flipping" the flag

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = (1\,6)(3\,4)(2\,5).$$

Let $G = \{(1), \tau\}$. Make X into a G-set by

$$G \times X \to X$$

$$(\sigma, (c_1, \ldots, c_6)) = \begin{cases} (c_1, \ldots, c_6) & \text{if } \sigma = (1) \\ (c_6, \ldots, c_1) & \text{if } \sigma = \tau \end{cases}$$

For any $x \in X$,

$$\begin{aligned} \mathcal{O}_x &= \{\sigma \cdot x \mid \sigma \in G\} \\ &= \{(c_1, \ldots, c_6), (c_6, \ldots, c_1) \mid \text{if } x = (c_1, \ldots, c_6)\} \end{aligned}$$

So

$$|\mathcal{O}_x| = \begin{cases} 1 & \text{if } x = (c_1, c_2, c_3, c_3, c_2, c_1) \\ 2 & \text{if } x \neq (c_1, c_2, c_3, c_3, c_2, c_1) \leftarrow \text{not symmetric} \end{cases}$$

Recall, orbits partition X,

$$X = \mathcal{O}_{x_1} \cup \mathcal{O}_{x_2} \cup \cdots \cup \mathcal{O}_{x_k}.$$

Solution to the problem = number of distinct orbits (each orbit consists of distinct flags)

Recall: Stabilizer of x, $G_x = \{g \in G \mid g \cdot x = x\}$

Lemma: Suppose X is a G set and $x \sim y$, ie. $y = g \cdot x$ for some G. Then

$$G_x \simeq G_y \Rightarrow |G_x| = |G_y|.$$

Proof

Let $g \in G$ such that $y = g \cdot x \Leftrightarrow g^{-1} \cdot y = x$. Define a map $\Phi : G_x \to G_y$
$a \mapsto gag^{-1}$

Note $gag^{-1} \in G_y$ since

$$\begin{aligned} gag^{-1}y &= ga(g^{-1} \cdot y) \\ &= ga \cdot x \\ &= g \cdot (a \cdot x) \quad a \in G_x \\ &= g \cdot x \\ &= y \end{aligned}$$

This is a homomorphism since

$$\Phi(ab) = gabg^{-1} = (gag^{-1})(gbg^{-1}) = \Phi(a)\Phi(b).$$

It is one-to-one since if

$$\Phi(a) = gag^{-1} = gbg^{-1} = \Phi(b)$$

we have $a = b$ via cancellation.

It is onto since if $h \in G_y$, then $g^{-1}hg \in G_x$ since

$$\begin{aligned} (g^{-1}hg) \cdot x &= g^{-1}h \cdot (g \cdot x) \quad y = g \cdot x \\ &= g^{-1}h \cdot y \\ &= g^{-1} \cdot (h \cdot y) \quad h \in G_y \\ &= g^{-1} \cdot y \\ &= x. \end{aligned}$$

Then $\Phi(g^{-1}hg) = g(g^{-1}hg)g^{-1} = h$.

$\square$

Theorem: (Burnside) Let G be a finite group acting on a set X. If K is the number of distinct orbits of X, then

$$K = \frac{1}{|G|} \sum_{g \in G} |X_g| \text{ where } X_g = \{x \mid g \cdot x = x\}.$$

Proof

We want to count all solutions to $g \cdot x = x$. Count in 2 ways.

Method 1: Fix g and count all $x \in X$ such that

$$g \cdot x = x \iff |X_g|$$

So, if we sum over all $g \in G$, the number of solutions is

$$\sum_{g \in G} |X_g|.$$

Method 2: Fix an x and count all $g \in G$ such that $g \cdot x = x$.

$$G_x = \{g \mid g \cdot x = x\}.$$

Summing over all x,

$$\sum_{x \in X} |G_x|.$$

So,

$$\sum_{g \in G} |X_g| = \sum_{x \in X} |G_x|.$$

refine the index of summation

Recall $X = \mathcal{O}_{x_1} \cup \mathcal{O}_{x_2} \cup \cdots \cup \mathcal{O}_{x_k}$. So,

$$\sum_{x \in X} |G_x| = \sum_{x \in \mathcal{O}_{x_1}} |G_x| + \sum_{x \in \mathcal{O}_{x_2}} |G_x| + \cdots + \sum_{x \in \mathcal{O}_{x_k}} |G_x|.$$

By the lemma, $|G_x| = |G_y|$ for all $x, y \in \mathcal{O}_{x_i}$.

So,

$$\sum_{x \in \mathcal{O}_{x_i}} |G_x| = |G_x||\mathcal{O}_{x_i}|.$$

So,

$$\sum_{x \in X} |G_x| = |G_{x_1}||\mathcal{O}_{x_1}| + \cdots + |G_{x_k}||\mathcal{O}_{x_k}|.$$

But

$$|\mathcal{O}_{x_i}| = \frac{|G|}{|G_{x_i}|} = [G : G_{x_i}].$$

Thus,

$$\sum_{x \in X} |G_x| = |G_{x_1}| \frac{|G|}{|G_{x_1}|} + \cdots + |G_{x_k}| \frac{|G|}{|G_{x_k}|} = k|G|.$$

Hence,

$$\sum_{g \in G} |X_g| = k|G| \iff k = \frac{1}{|G|} \sum_{g \in G} |X_g|. \qquad \square$$

Return to the flag problem: $G = \{(1), \tau\}$ so $|G| = 2$.

We need to compute

$$X_{(1)} = \{x \in X \mid (1) \cdot x = x\} = X \implies |X_{(1)}| = 3^6$$

$$X_{(\tau)} = \{x \in X \mid \tau \cdot x = x\} = \{(c_1, c_2, c_3, c_4, c_5, c_6) \in X \mid c_1 = c_6,\ c_2 = c_5,\ c_3 = c_4\} \implies |X_\tau| = 3^3$$

So,

$$\#\text{ of flags} = \#\text{ of orbits} \overset{\text{Burnside}}{=} \frac{1}{2}(|X_{(1)}| + |X_\tau|) = \frac{1}{2}(3^6 + 3^3) = 378.$$