date: wednesday, february 14, 2024

Review

I. Finite Abelian Groups

A. Find all abelian groups of order 108

$108 = 2^2 \cdot 3^3 \Rightarrow \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^3}$

$\Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^3}$

$\Rightarrow \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_{3^2}$

$\Rightarrow \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

$\Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^2}$

$\Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

B. Show that two groups in A have an element of order 54.

$\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^3}$: $(2,1)$ since $|2| = 2$ and $|1| = 27$ and $\text{lcm}(2,27) = 54$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^3}$: $(1,1,1)$ since $|1| = 2$ and $|1| = 27$ and $\text{lcm}(2,27) = 54$

C. Suppose G is a finite abelian group and $10 \mid |G|$. Show G has an element of order 10.

$10 \mid |G|$ implies $2 \mid |G|$ and $5 \mid |G|$. $\gcd(2,5) = 1$. $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_5$ which has an element $(1,1)$ of order 10.

D. Give an example of a group G where $4 \mid |G|$, but no element of order 4.

$\mathbb{Z}_2 \times \mathbb{Z}_2$

II. Solvable Groups

A. Find a composition series of $\mathbb{Z}_{20}$.

$$\mathbb{Z}_{20} \supset \langle 5 \rangle \supset \langle 10 \rangle \supset \{e\}$$

$$\mathbb{Z}_{20}/\langle 5 \rangle \simeq \mathbb{Z}_5 \quad \langle 5 \rangle / \langle 10 \rangle \simeq \mathbb{Z}_2$$

B. Suppose G has a subnormal series

$$G = P_n \supset P_{n-1} \supset \cdots \supset P_1 \supset P_0 = \{e\}$$

where P_i is normal in P_{i+1}. If $|P_{i+1}/P_i|$ is prime for all i, why is G a solvable group?

$|P_{i+1}/P_i|$ prime $\Rightarrow \simeq \mathbb{Z}_p$ which is abelian $\Rightarrow G$ solvable

III. Characteristic Equation

A. Suppose $|G| = 20$. Explain why

$$20 = 1 + 2 + 3 + 4 + 10$$

is not a valid class equation.

$3 \nmid 20$

B. G is an abelian group with $|G| = 2024$. What is the class equation?

G abelian $\Rightarrow |G| = |Z(G)| = 2024$

IV. Sylow Theorems

A. If $|G| = 175$, prove G is abelian.

$175 = 5^2 \cdot 7$

3^{rd} Sylow: $n_5 \in \{1, 6, 11, 16, \ldots$

$n_5 \in \{1, 5, 7, 25, 35, 175\}$

$n_7 \in \{1, 8, 15, 22, 29, \ldots$

$n_7 \in \{1, 5, 7, 25, 35, 175\}$

One $\underbrace{\text{Sylow 5-subgroup}}_{H}$ and $\underbrace{\text{Sylow 7-subgroup}}_{K}$ which are normal.

$H \cap K = \{e\}$. $G \simeq H \times K$

$K \simeq \mathbb{Z}_7 \leftarrow$ abelian

$|H| = 5^2 \leftarrow \simeq \mathbb{Z}_{25}$ or $\simeq \mathbb{Z}_5 \times \mathbb{Z}_5$ $\leftarrow$ abelian

$\Rightarrow G$ abelian

B. Let P be a Sylow p-subgroup of G. Prove P is the only Sylow subgroup contained in $N(P) = \{x \mid xPx^{-1} = P\}$.

$N(P)$ subgroup of G. Let $|G| = p^r m$. Then $|P| = p^r$ and so $|P| \mid |N(P)|$ $\Rightarrow p^r \mid |N(P)| \Rightarrow |N(p)| = p^r \ell$ and $|N(P)| \mid |G|$. So P is a Sylow p-subgroup of $N(P)$. $N(P)$ is by construction the largest subgroup so P is normal in $N(P)$. So P is the only Sylow subgroup contained in $N(P)$.