date: wednesday, april 3, 2024

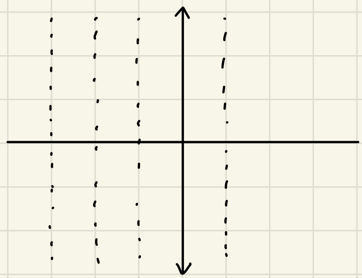
Geometric Constructions III

Recall: $\alpha \in \mathbb{R}$ is constructible if we can construct a line segment of length $|\alpha|$ using a straightedge and compass.

$$F = \{\alpha \in \mathbb{R} \mid \alpha \text{ is constructible}\} \subseteq \mathbb{R}$$

this is a subfield. In fact, F is an extension of $\mathbb{Q}$

Def^{n}: A point $P=(a,b)$ is constructible if a,b are constructible.



Lemma: Let F be a subfield of $\mathbb{R}$.

Ⓐ If a line L contains two points P_1, P_2 in F, then its equation has form

$$ax+by+c=0 \quad \text{with} \quad a,b,c \in F.$$

Ⓑ If a circle has a center P in F and a radius r (also in F), then equation has the form

$$x^2+y^2+dx+ey+f=0 \quad \text{with} \quad d,e,f \in F.$$

Proof

Ⓑ Let $P=(a,b)$ and radius r equation is $(x-a)^2+(y-b)^2=r^2$.

$\Rightarrow x^2-2ax+a^2+y^2-2by+b^2-r^2=0.$

$\Rightarrow x^2+y^2+\underbrace{(-2a)}_{\in F}x+\underbrace{(-2b)}_{\in F}y+\underbrace{(a^2+b^2-r^2)}_{\in F}=0.$

$\square$

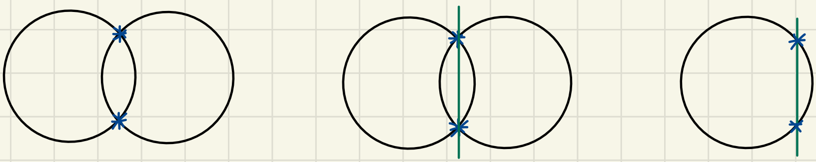
Starting with a field F of constructible numbers, recall how we add "new" points.

① Intersect two lines, each of which passes through two points with coordinates in F.

② Intersecting a line and a circle (center in F and radius in F)

③ Intersect two circles where centers and radii are in F.

Note: Case ③ reduces to case ②



In case ①, since the two equations have form $ax+by+c=0$ with coefficient in F, the intersection will have coordinates in F.

In case ②, we want to solve

$$ax+by+c=0$$
$$x^2+y^2+dx+ey+f=0$$

Set $x=-\frac{a}{b}x-\frac{c}{b}$ and sub into second equation

$$x^2+\left(-\frac{a}{b}x-\frac{c}{b}\right)^2+dx+e\left(-\frac{a}{b}x-\frac{c}{b}\right)+f=0.$$

After expanding and collecting, you get

$$Ax^2+Bx+C=0 \quad \text{with } A,B,C\in F.$$

So,

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A} \quad \text{and} \quad y=-\frac{a}{b}\left(\frac{-B\pm\sqrt{B^2-4AC}}{2A}\right)-\frac{c}{b}.$$

So $x,y\in F(\sqrt{\alpha})$ with $\alpha=B^2-4AC$. Note this means that $\sqrt{\alpha}$ is constructible.

To recap: when creating points by intersecting a circle and a line, get points with coordinates in $F(\sqrt{\alpha})$ for some α.

Observation: $[F(\sqrt{\alpha}):F]=1$ or 2, $F(\sqrt{\alpha})$ is a quadratic extension.

Theorem: $\alpha \in \mathbb{R}$ is constructible iff there is a sequence of fields

$$\mathbb{Q} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k$$

such that $F_i = F_{i-1}(\sqrt{\alpha_i})$ for some $\alpha_i \in F_{i-1}$. In particular,

$$[\mathbb{Q}(\alpha):\mathbb{Q}] = 2^k.$$

Idea

Note all points of $\mathbb{Q}$ are constructible. Suppose we construct $\alpha \in \mathbb{R}$, we "build" it by adding $\sqrt{\alpha_1}$ to $\mathbb{Q}$ then $\sqrt{\alpha_2}$ to $\mathbb{Q}(\sqrt{\alpha_1})$ until we get $\alpha \in \mathbb{Q}(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \ldots, \sqrt{\alpha_n})$.

So,

$$\begin{aligned}[\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})] &= [\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}) : \mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_{n-1}})] \\ &\quad [\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_{n-1}}) : \mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_{n-2}})] \\ &\quad \vdots \\ &\quad [\mathbb{Q}(\sqrt{\alpha_1}) : \mathbb{Q}] = 2^n.\end{aligned}$$

So $\mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$.

So $\underbrace{[\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}) : \mathbb{Q}]}_{= 2^n} = [\mathbb{Q}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}) : \mathbb{Q}(\alpha)]$.

So $[\mathbb{Q}(\alpha):\mathbb{Q}] = 2^k$ for some k.

$\square$

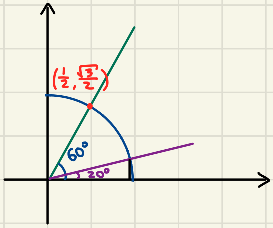
Proofs of "Impossible" Results:

1. Squaring a circle.
   Given a circle of radius 1 and area π. Want to make an square of area π. So need $\alpha = \sqrt{\pi}$ to be constructible. But $[\mathbb{Q}(\sqrt{\pi}):\mathbb{Q}] = \infty$ since π (and $\sqrt{\pi}$) is transcendental.

2. Doubling volume of a cube
   Given a unit cube, we need to make a cube with dimensions $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2}$. But $\sqrt[3]{2}$ is not constructible since

$$[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}] = 3 \neq 2^k.$$

3. Trisect an angle
   Note that you can construct $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. This point is on the unit circle and 60° above the x-axis.



If we construct a point R that trisects 60°, would construct cos 20°.

By trig identities,

$$\begin{aligned}\cos 3\theta &= \cos(2\theta+\theta)\\ &= \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta\\ &= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta\\ &= (2\cos^2\theta-1)\cos\theta - 2(1-\cos^2\theta)\cos\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

So, $\cos 2\theta$ satisfies

$$4(\cos 20°)^3 - 3(\cos 20°) - \cos(60°) = 0$$
$$\iff 4(\cos 20°)^3 - 3(\cos 20°) - \tfrac{1}{2} = 0$$

So $\cos 20°$ is a root of $4x^3 - 3x - \frac{1}{2} = 0$ <- irreducible are $\mathbb{Q}$.
So $[\mathbb{Q}(\cos 20°) : \mathbb{Q}] = 3 \neq 2^k$.