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Composition Series and Solvable Groups

The FTFAG classifies all finite abelian groups.

What about non-abelian groups? In the 20th century, big program was to classify non-abelian groups.

Main Idea: Reduce to understanding solvable and simple groups.

Defn: A subnormal series of a group G is a finite sequence of subgroups

$$G = H_n > H_{n-1} \supset H_{n-2} \supset \cdots \supset H_1 > H_0 = \{e\}$$

where H_i is normal in H_{i+1} for $i=0,\dots,n-1$.
If each H_i is normal in G, this is a normal series.
↳ length = # inclusions
↳ denoted $\{H_i\}$

eg. In an abelian group G, every subnormal series is also a normal series

$$\mathbb{Z} \geq 11\mathbb{Z} \geq 253\mathbb{Z} \geq 2024\mathbb{Z} \geq \{0\}$$

eg. A subnormal series in D_4.
$D_4 \supset \{(1),(12)(34),(13)(24),(14)(23)\} \geq \{(1),(12)(34)\} \geq \{(1)\}$
This is not a normal series since $\{(1),(12)(34)\}$ not normal in D_4.

Defn: A subnormal series $\{K_j\}$ is a refinement of a (sub)normal series $\{H_i\}$ if $\{H_i\} \subset \{K_j\}$. Ie. if the H_i's appear among the K_j's.

eg. $\mathbb{Z} \geq 11\mathbb{Z} \geq 253\mathbb{Z} \geq 2506\mathbb{Z} \geq 1012\mathbb{Z} \geq 2024\mathbb{Z} \geq \{0\}$ (this is a refinement of previous example)

Defn: Two (sub)normal series $\{H_i\}$ and $\{K_j\}$ are isomorphic if there is a one-to-one correspondance between the sets $\{H_i/H_{i-1}\}$ and $\{K_j/K_{j-1}\}$

Remark: Suppose $\langle a\rangle \subseteq \langle b\rangle \subseteq \mathbb{Z}_n$. Recall $|\langle a\rangle| = n/a$ and $\left|\langle b\rangle / \langle a\rangle\right| = \frac{n/b}{n/a} = \frac{a}{b}$.

eg. $\{H_i\}: \mathbb{Z}_{2024} \geq \langle 11\rangle \geq \langle 22\rangle \geq \langle 506\rangle \geq \langle 0\rangle$
$\{K_j\}: \mathbb{Z}_{2024} \geq \langle 23\rangle \geq \langle 46\rangle \geq \langle 506\rangle \geq \langle 0\rangle$

$\left\{ H_i / H_{i-1} \right\}$: $\mathbb{Z}_{2024}/\langle 11\rangle \simeq \langle 1\rangle / \langle 11\rangle = \mathbb{Z}_{11}$, $\left\{ K_j / K_{j-1} \right\}$: $\mathbb{Z}_{2024}/\langle 23\rangle \simeq \mathbb{Z}_{23}$

$\langle 11\rangle / \langle 22\rangle \simeq \mathbb{Z}_2$ $\qquad$ $\langle 23\rangle / \langle 46\rangle \simeq \mathbb{Z}_2$

$\langle 22\rangle / \langle 506\rangle \simeq \mathbb{Z}_{23}$ $\qquad$ $\langle 46\rangle / \langle 506\rangle \simeq \mathbb{Z}_{11}$

$\langle 506\rangle / \langle 0\rangle \simeq \mathbb{Z}_4$ $\qquad$ $\langle 506\rangle / \langle 0\rangle \simeq \mathbb{Z}_4$

Def$^{\underline{n}}$: If G has no subgroups, then G is simple.

eg. $\mathbb{Z}_p$ with p prime

Def$^{\underline{n}}$: A subnormal series $\{H_i\}$ of G is a composition series if all H_{i+1}/H_i are simple.
A normal series $\{H_i\}$ is a principal series if all H_{i+1}/H_i are simple.

eg. Previous example is not a composition series since $\mathbb{Z}_4$ is not simple.
However, $\mathbb{Z}_{2024} \geq \langle 23\rangle \geq \langle 46\rangle \geq \langle 506\rangle \geq \langle 1012\rangle \geq \langle 0\rangle$ has quotients $\mathbb{Z}_{23}, \mathbb{Z}_2, \mathbb{Z}_{11}, \mathbb{Z}_2, \mathbb{Z}_2$.
Note $2024 = 23 \cdot 11 \cdot 2^3$.

Note: Not every group has a composition or principal series.
$\mathbb{Z} \geq 11\mathbb{Z} \geq 253\mathbb{Z} \geq 2024\mathbb{Z} \geq \langle 0\rangle$
$2024\mathbb{Z}/\langle 0\rangle$ is not simple.

In fact, if $\mathbb{Z} \geq H_1 \geq H_2 \geq \cdots \geq H_k \geq \{0\}$ is a composition of $\mathbb{Z}$. Then, $H_k \simeq t\mathbb{Z}$ for some t. And then $H_k/\langle 0\rangle \simeq t\mathbb{Z}$ which is not simple.

eg. Composition series, if they exist, may not be unique

$$\left.\begin{array}{l}\mathbb{Z}_{2024} \supset \langle 11 \rangle \supset \langle 22 \rangle \supset \langle 506 \rangle \supset \langle 1012 \rangle \supset \langle 0 \rangle \\ \mathbb{Z}_{2024} \supset \langle 23 \rangle \supset \langle 46 \rangle \supset \langle 92 \rangle \supset \langle 1012 \rangle \supset \langle 0 \rangle\end{array}\right\} \text{both composition series}$$

Theorem: (Jordan-Hölder) Any two composition series of G are isomorphic.

Def^{n}: G is a solvable group if it has a subnormal series $\{H_i\}$ such that all $\{H_{i+1}/H_i\}$ are abelian.

eg. Big theorem: A_n with $n \geq 5$ is not solvable.

Application of the Jordan-Hölder Theorem: Fundamental Theorem of Arithmetic

We know that every integer can be factored into primes (existence).
We use Jordan-Hölder for uniqueness. Suppose

$$n = p_1 p_2 \cdots p_r = q_1 \cdots q_s \quad \leftarrow \text{not necessarily distinct}$$

So, we make a composition series

$$\mathbb{Z}_n \supset \langle p_1 \rangle \supset \langle p_1 p_2 \rangle \supset \cdots \supset \langle p_1 \cdots p_r \rangle \supset \langle 0 \rangle$$

with $\langle p_1 \cdots p_{i-1} \rangle / \langle p_1 \cdots p_i \rangle \simeq \mathbb{Z}_{p_i}$ $\leftarrow$ simple.

Also, $\mathbb{Z}_n \supset \langle q_1 \rangle \supset \langle q_1 q_2 \rangle \supset \cdots \supset \langle q_1 \cdots q_s \rangle \supset \langle 0 \rangle$ with $\langle q_1 \cdots q_{j-1} \rangle / \langle q_1 \cdots q_j \rangle \simeq \mathbb{Z}_{q_j}$.

By Jordon-Hölder, series are isomorphic, so $r=s$ and after relabelling $\mathbb{Z}_{p_i} \simeq \mathbb{Z}_{q_i} \iff p_i = q_i$.

$\square$

Can use proof to find composition series of $\mathbb{Z}_n$.
$n = 2024 = 23 \cdot 2 \cdot 11 \cdot 2 \cdot 2$
$\mathbb{Z}_{2024} \supset \langle 23 \rangle \supset \langle 23 \cdot 2 \rangle \supset \langle 23 \cdot 2 \cdot 11 \rangle \supset \langle 23 \cdot 2 \cdot 11 \cdot 2 \rangle \supset \langle 0 \rangle$.