Proof of Claim Skipped in Class (Lecture 7-Extra)

Claim (1) $H_{n-1} \cap K_{m-1}$ is normal in H_{n-1}

(2) $H_{n-1} / H_{n-1} \cap K_{m-1}$ is simple

Proof (1) Take $h \in H_{n-1} \cap K_{m-1}$ and $g \in H_{n-1}$. Want to show $ghg^{-1} \in H_{n-1} \cap K_{m-1}$. Since h, g, and $g^{-1} \in H_{n-1}$, we have $ghg^{-1} \in H_{n-1}$. Because K_{m-1} is normal in G and $h \in K_{m-1}$ and $g \in H_{n-1} \subseteq G$, we have $ghg^{-1} \in K_{m-1}$. Thus, $ghg^{-1} \in H_{n-1} \cap K_{m-1}$, as desired

(2) By the second isomorphism theorem

$$H_{n-1} / (H_{n-1} \cap K_{m-1}) \simeq H_{n-1}K_{m-1} / K_{m-1}$$

We first

Want to show $H_{n-1}K_{m-1}/K_{m-1}$ is normal in G/K_{m-1}

Let $hkK_{m-1} \in H_{n-1}K_{m-1}/K_{m-1}$ and $gK_{m-1} \in G$.

Since $h \in H_{n-1}$ and H_{n-1} normal in G, $ghg^{-1} \in H_{n-1}$

Since $k \in K_{m-1}$ and K_{m-1} normal in G, $gkg^{-1} \in K_{m-1}$

So $(gK_{m-1})(hkK_{m-1})(g^{-1}K_{m-1}) = g(hk)g^{-1}K_{m-1}$

But $(ghg^{-1})(gkg^{-1}) = ghkg^{-1} \in H_{n-1}K_{m-1}$. So

$$g(hk)g^{-1}K_{m-1} \in H_{n-1}K_{m-1}/K_{m-1}$$

So $H_{n-1}K_{m-1}/K_{m-1}$ is normal in $G/K_{m-1} = K_m/K_{m-1}$

But K_m/K_{m-1} is simple, so

$$H \cap K_{m-1}/_{K_{m-1}} = K_{m-1}/_{K_{m-1}} \quad \text{or} \quad H \cap K_{m-1}/_{K_{m-1}} = K_m/_{K_{m-1}}$$

the trivial gp

In both cases, the group is simple