Lecture 17 - Odds and Ends
Group Theory Review

I Structure of Groups - Finite Abelian Groups

A. Find all abelian gps of order 108
B. Show that there are two abelian gps of order 108 with an element of order 54
C. Suppose G is a finite abelian gp such that $10 \mid |G|$. Prove that G has a cyclic subgp of order 10
D. Give an example of ~~gp~~ abelian gp such that $4 \mid |G|$, but G has no cyclic subgp of order 4

A. $108 = 3^3 \cdot 2^2$. Can write 108 as

$$\begin{aligned}
3\times3\times3\times2\times2 &\longrightarrow \mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_2\times\mathbb{Z}_2\\
3^2\times3\times2\times2 &\longrightarrow \mathbb{Z}_9\times\mathbb{Z}_3\times\mathbb{Z}_2\times\mathbb{Z}_2\\
3^3\times2\times2 &\longrightarrow \mathbb{Z}_{27}\times\mathbb{Z}_2\times\mathbb{Z}_2\\
3\times3\times3\times2^2 &\longrightarrow \mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_4\\
3^2\times3\times2^2 &\longrightarrow \mathbb{Z}_9\times\mathbb{Z}_3\times\mathbb{Z}_4\\
3^3\times2^2 &\longrightarrow \mathbb{Z}_{27}\times\mathbb{Z}_4
\end{aligned}$$

B. The two gps $\mathbb{Z}_{27}\times\mathbb{Z}_2\times\mathbb{Z}_2$ and $\mathbb{Z}_{27}\times\mathbb{Z}_4$ have elements of order 54. In the first gp, look at $(1,1,0)$
In the second gp, consider $(1,2,0)$

C. Since $10 = 2\cdot5 \mid |G|$, we know $|G| = 2^a 5^b m$ with $a \geq 1, b \geq 1$ and $2 \nmid m$, $5 \nmid m$. So by Fundamental Thm of Finite Abelian gps
$$G \cong \mathbb{Z}_{2^{a_1}}\times\cdots\times\mathbb{Z}_{2^{a_t}}\times\mathbb{Z}_{5^{b_1}}\times\cdots\times\mathbb{Z}_{5^{b_s}}\times\cdots$$
with $a_1+\cdots+a_t = a$ and $b_1+\cdots+b_s = b$

The element 2^{a_1-1} has order 2 in $\mathbb{Z}_{2^{a_1}}$ and 5^{b_1-1} has order 5 in $\mathbb{Z}_{5^{b_1}}$
So the element
$$(2^{a_1-1}, 0, \ldots, 0, 5^{b_1-1}, 0, \ldots, 0)$$ has order 10 in G

D. Consider $\mathbb{Z}_2 \times \mathbb{Z}_2$. Then $4 \mid |\mathbb{Z}_2 \times \mathbb{Z}_2|$, but G has no cyclic subgp of order 4

II Structure of Finite Gps - Soluable gps

A. Find a composition series of $\mathbb{Z}_{20}$

B Suppose G has a series of subgps.

$$G = P_n \supset P_{n-1} \supset \dots \supset P_1 \supset P_0 = \{e\}$$

where P_i is normal in P_{i+1} and $|P_{i+1}/P_i|$ is a prime number. Why is G a soluable gp?

A. Hint: if $n = p_1^{a_1} \cdots p_r^{a_r}$, ~~then we can find a comp series~~ where we ~~when~~ find the comp series, we want $H_i/H_{i-1} \simeq \mathbb{Z}_{p_i}$ a_i times

If $n = 20$, $20 = 2^2 \cdot 5$ So, we want to find

$$\mathbb{Z}_{20} \supset H_1 \supset H_2 \supset \{0\}$$

where $H_2/\{0\} = \mathbb{Z}_5$, $H_1/H_2 \simeq \mathbb{Z}_2$, and $\mathbb{Z}_{20}/H_1 \simeq \mathbb{Z}_2$

Let $H_1 = \langle 2 \rangle = \{0, 2, 4, \dots, 18\}$

$H_2 = \langle 4 \rangle = \{0, 4, 8, 12, 16\}$

Then

$$\mathbb{Z}_{20} \supset H_1 \supset H_2 \supset \{0\} \text{ and } \mathbb{Z}_{20}/\langle 2 \rangle \simeq \mathbb{Z}_2, \ \langle 2 \rangle/\langle 4 \rangle \simeq \mathbb{Z}_2 \ \langle 4 \rangle/\langle 0 \rangle \simeq \mathbb{Z}_5$$

B. Recall $G \supset H_n \supset H_{n-1} \supset \dots \supset H_1 \supset \{0\}$ is a composition series if H_i is normal in H_{i+1} and H_{i+1}/H_i is ~~normal~~ simple for all i

In this problem, we are given that P_i is normal in P_{i+1}.

Since $|P_{i+1}/P_i| = q_i$ for some prime q_i, we have $P_{i+1}/P_i \simeq \mathbb{Z}_{q_i}$

Since only one gp of order q_i. But $\mathbb{Z}_{q_i}$ is simple.

So G is solvable

III Characteristic Equation

A. Suppose $|G|=20$. Explain why this is not a valid class equation
$$20 = 1+2+3+6+10$$

B. Suppose G is an abelian gp with $|G|=2024$. What is its class equation?

Note that the class equation is
$$|G| = |Z(G)| + [G:C(x_i)] + \dots + [G:C(x_n)]$$
where x_i are the representatives for each nontrivial conjugacy classes.

Observe $Z(G)$ is a subgp of G, so $|Z(G)| \mid |G|$
Also $|G| = |C(x_i)|[G:C(x_i)]$. So $[G:C(x_i)] \mid |G|$.
Thus, each term in class equation must divide $|G|$.

A. Not a class equation since $3 \nmid 20$

B. If G is abelian, $G = Z(G) = \{g \mid gx = xg \text{ for all } x \in G\}$
So $|G| = |Z(G)| = 2024$.

IV Sylow Theorems

A. If $|G|=175$, prove that G is abelian.
B. Let P be a sylow p-subgp of G. Prove P is the only Sylow p-subgp contained in $N(P) = \{x \mid xPx^{-1} = P\}$
C. Suppose $|G| = p^k m$ and H is a normal subgp with $|H| = p^l$. Prove H is in all sylow p-subgps

A. We are given that $|G| = 175 = \cancel{5 \cdot 35}\ 5^2 \cdot 7$

By the 3^{rd} Sylow Thm, the # of Sylow $\cancel{5}$7-subgps is 1. Also the # of Sylow 5-subgps is 1

Let P be the Sylow 7-subgp. So $|P| = 7$ and P normal
Let Q " " " 5-subgp. So $|Q| = 25$ and Q normal.

Thus $G \simeq P \times Q$.
Since $|P| = 7$, $P \simeq \mathbb{Z}_7$ and since $|Q| = 5^2$. Then Q is abelian (Cor 14
So G is abelian

B. Suppose Q is a sylow p-subgp of G such that $Q \subseteq N(P)$
By second sylow theorem $Q = yPy^{-1}$ for some $y \in G$.
Now for any $g \in Q$, $gPg^{-1} = P$ since $g \in Q \subseteq N(P)$.

Recall lemma: If $|x| = p^l$ and $xPx^{-1} = P$ for a Sylow p-gp, then $x \in P$.
Since $g \in Q$ has $|g| = p^l$, $g \in P$. So $Q \subseteq P$. But both subgps have same order. So $Q = P$.

Alt proof Since $P \subseteq N(P)$, $|P| \mid |N(P)|$. Also, $|N(P)| \mid |G|$
So $|N(P)| = p^r n$ if $|G| = p^r m$ (i.e same power of p as $|G|$)
~~The~~ So P is a Sylow p-subgp of $N(P)$, and it is normal in $N(P)$
So, if Q is any other Sylow p-subgp in $N(P)$, it must be P
Since P is unique

C. By "strong" sylow, if $|H| = p^l$, $H \subseteq P$ when P is is some Sylow p-subgp. ~~For any $x \in G$,~~ ~~$xHx^{-1} \subseteq x$~~ For any other Sylow p-subgp Q, exists y such that $Q = yPy^{-1}$.
So $yHy^{-1} \subseteq yPy^{-1} = Q$. But H normal, so $H = yHy^{-1} \subseteq Q$.