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Group Actions

Introduce group actions today: a group acting on a set.
Have seen something similar in linear algebra: If V is a vector space over a field F, then F acts on V by "scalar multiplication",
ie. $(F \times V) \to V$
$(c, v) \mapsto cv$

Group Actions and Examples

Def^n: Let X be a set and G a group. A (left) action of G on X is a map $G \times X \to X$ defined by

$$(g, x) \mapsto g \cdot x$$

may not be multiplication, $g \cdot x$ represents the action

where

$$(e, x) \mapsto e \cdot x = x$$
$$(g_1, (g_2, x)) \mapsto g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$$

Call X a G-set.

eg. ① Trivial action
$G \times X \to X$
$(g, x) \mapsto x$

② If $X = G$, then we can view the group operation as a group action:
$G \times G \to G$
$(g, x) \mapsto g * x$
group operation

③ Let $X = \mathbb{R}^2$ and $G = GL_2(\mathbb{R})$ ← all 2×2 invertible matrices
Define an action:
$G \times X \to X$
$\left(A, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \mapsto A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
2×2 matrix

This is a group action:

1) In G, $e=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and for any $\begin{bmatrix}x_1\\x_2\end{bmatrix}\in\mathbb{R}^2$,

$$\left(\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

2) Let $A,B\in G$,

$$\left(A,\left(B,\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\right)=\left(A,B\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=A\left(B\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)$$

$$\left(AB,\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=AB\begin{bmatrix}x_1\\x_2\end{bmatrix}\longleftarrow =$$

important

④ Let $X=G$ and let H be a subgroup of G. Define an H-action on G by

$(H\times G)\mapsto G$

$(h,g)\mapsto hgh^{-1}$ $\longleftarrow$ this is the action $h\cdot g$

This is a group action since

1) $e\in H$ and $(e,g)\mapsto ege^{-1}=g$

2) For any $h_1,h_2\in H$,

$$(h_1,(h_2,g))=(h_1,h_2gh_2^{-1})\mapsto h_1(h_2gh_2^{-1})h_1^{-1}=(h_1h_2)g(h_1h_2)^{-1}$$

also

$$(h_1h_2,g)=h_1h_2g(h_1h_2)^{-1}\longleftarrow =$$

⑤ Let $X=\{a_1,a_2,\dots,a_n\}$. Let $G=S_n$ $\leftarrow$ symmetric group on n-elements.

Then G acts on X by

$G\times X\to X$

$(\sigma,a_i)\mapsto a_{\sigma(i)}$

Example: $X=\{a_1,a_2,a_3\}$, S_3 and consider $\sigma=\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\in S_3$.

$(\sigma,a_1)\mapsto a_2$

$(\sigma,a_2)\mapsto a_1$

$(\sigma,a_3)\mapsto a_3$

eg. Do same for $\tau=\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$ for above example.

$(\tau,a_1)\mapsto a_3$

$(\tau,a_2)\mapsto a_2$

$(\tau,a_3)\mapsto a_1$

Group Actions and Equivalence Relations

Suppose G acts on X.

Def$^{\underline{n}}$: $x, y \in X$ are G-equivalent if there exists $g \in G$ such that $y = g \cdot x$. We write $x \sim y$ (or $x \sim_G y$).

Theorem: Let X be a G-set. Then G-equivalent is an equivalence relation on X.

Proof

(Reflexive) $x \sim x$ since $x = e \cdot x$

(Symmetric) Suppose $x \sim y \iff$ exists $g \in G$ such that $y = g \cdot x$. Then $g^{-1} \cdot y = g^{-1} \cdot (g \cdot x) = (g^{-1} g) \cdot x = x$. So $y \sim x$.

(Transitive) If $x \sim y$, then $y = g_1 \cdot x$ and if $y \sim z$, then $z = g_2 \cdot y$. So, $z = g_2 \cdot y = g_2 \cdot (g_1 \cdot x) = (g_2 g_1) \cdot x$. So $x \sim z$.

$\square$

Fix an $x \in X$. Define,

$$\mathcal{O}_x = \{g \cdot x \mid g \in G\} \leftarrow \text{the orbit of } x.$$

By previous result,

$$\mathcal{O}_x = \{y \mid x \sim y\} \leftarrow \text{equivalence class of } x \text{ under } \sim.$$

By general properties of equivalence relations,

Properties: ① $\mathcal{O}_{x_i} = \mathcal{O}_{x_j}$ or $\mathcal{O}_{x_i} \cap \mathcal{O}_{x_j} = \emptyset$ for all $x_i, x_j \in X$

② If $\mathcal{O}_{x_1}, \dots, \mathcal{O}_{x_j}$ are distinct orbits, then $\mathcal{O}_{x_1} \cup \mathcal{O}_{x_2} \cup \cdots \cup \mathcal{O}_{x_j}$ partitions X.

Moral: Group actions partition X.

eg. Let $X = \{1, 2, 3\}$ and $H = \{\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\}$.

Define action $H \times X \to X$

$$(\sigma_i, j) \mapsto \sigma_i(j)$$

$$\mathcal{O}_1 = \{\sigma_1(1), \sigma_2(1)\} = \{1, 2\}$$

$$\mathcal{O}_2 = \{\sigma_1(2), \sigma_2(2)\} = \{2, 1\}$$

$$\mathcal{O}_3 = \{\sigma_1(3), \sigma_2(3)\} = \{3\}$$

Thus, $X = \{1, 2, 3\} = \mathcal{O}_{x_1} \cup \mathcal{O}_{x_3}$

Going forward: Interested in the action $H \times G \to G$

$$(h,g) \mapsto hgh^{-1}$$

What are the orbits?
What are the sizes of orbits?
How many orbits?