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Class Equations: Applications and Worked Out Examples

Last time: G is a G-set by conjugation

$$G \times G \to G$$

$$(g,h) \mapsto ghg^{-1} \leftarrow \text{conjugate of } h$$

$C(h)=\{g\in G \mid ghg^{-1}=h \iff gh=hg\}$

$\mathcal{O}_h=\{ghg^{-1} \mid g\in G\}$

$|\mathcal{O}_h|=1 \iff \mathcal{O}_h=\{h\} \iff ghg^{-1}=h$ for all $g\in G$

$\iff C(h)=G$

$Z(G)=\{g\in G \mid gh=hg \text{ for all } h\in G\}$

$=\mathcal{O}_{h_1}\cup\mathcal{O}_{h_2}\cup\cdots\cup\mathcal{O}_{h_s}$ with $|\mathcal{O}_{h_i}|=1$

(Class Equation) Let $h_1,\dots,h_s$ be a representatives of conjugacy classes with $|\mathcal{O}_{h_i}|\geq 2$. Then,

$$|G|=|Z(G)|+|\mathcal{O}_{h_1}|+\cdots+|\mathcal{O}_{h_s}|$$

$$=\underbrace{|Z(G)|+[G:C(h_1)]+\cdots+[G:C(h_s)]}_{\text{class equation}}$$

Theorem: If $|G|=p^r$, then $|Z(G)|\geq p$.

Proof

By the class equation,

$$|Z(G)|=|G|-[G:C(h_1)]-\cdots-[G:C(h_s)]$$

Since $[G:C(h_i)]=\frac{|G|}{|C(h_i)|}\geq 2$. Since $|G|=p^r$, this means $|C(h_i)|=p^{r-n_i}$. So $[G:C(h_i)]=p^{n_i}$ with $n_i\geq 1$. (Note, we must have $n_i\geq 1$, since $[G:C(h_i)]\geq 2\geq p^0$). Thus, every term on RHS is divisible by p. So $p \mid |Z(G)|$.

$\square$

Theorem: If $|G|=p^2$, then G is abelian ($\Leftrightarrow G \simeq \mathbb{Z}_{p^2}$ or $\simeq \mathbb{Z}_p \times \mathbb{Z}_p$)

Proof

It is enough to show that $|Z(G)|=p^2 \Rightarrow Z(G)=G$. By previous result, $|Z(G)|=p^2$ or p. So, suppose $|Z(G)|=p$. Note $Z(G)$ is normal in G. Take $g\in G$ and $a\in Z(G)$. Then $gag^{-1}=gg^{-1}a=a\in Z(G)$. So, $G/Z(G)$ is defined and $|G/Z(G)|=p$. So $G/Z(G)$ is cyclic, ie. exists $h\in G$ such that $\langle hZ(G)\rangle = G/Z(G)$.

For any $g\in G$, $gZ(G)\in\langle hZ(G)\rangle$. Thus, exists m such that

$$gZ(G)=(hZ(G))^m=h^mZ(G).$$

Since $g\in gZ(G)$, there exists $x\in Z(G)$ such that $g=h^mx$. Take $g_1,g_2\in G$. So there exists m_1,m_2 and $x_1,x_2\in Z(G)$ such that $g_1=h^{m_1}x_1$ and $g_2=h^{m_2}x_2$. Then,

$$\begin{aligned} g_1g_2 &= h^{m_1}x_1h^{m_2}x_2 \quad x_1\in Z(G)\\ &= h^{m_1}h^{m_2}x_1x_2\\ &= h^{m_1+m_2}x_1x_2\\ &= h^{m_2}h^{m_1}x_1x_2 \quad x_2\in Z(G)\\ &= h^{m_2}x_2h^{m_1}x_1\\ &= g_2g_1. \end{aligned}$$

So, G is abelian.

$\square$

(Class Project) Determine the class equation Q_8 (quaternions)

$$1=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ I=\begin{bmatrix}0&1\\-1&0\end{bmatrix},\ J=\begin{bmatrix}0&i\\i&0\end{bmatrix},\ K=\begin{bmatrix}i&0\\0&-i\end{bmatrix} \quad (\text{here, } i^2=-1)$$

So $Q_8=\{\pm1,\pm I,\pm J,\pm K\}$.

Set has the following relations:

$$I^2=J^2=K^2=-1$$
$$IJ=K,\ JK=I,\ KI=J$$
$$JI=-K,\ KJ=-I,\ IK=-J$$

Cayley Table:

	1	-1	I	-I	J	-J	K	-K
1	1	-1	I	-I	J	-J	K	-K
-1	-1	1	-I	I	-J	J	-K	K
I	I	-I	-1	1	K	-K	-J	J
-I	-I	I	1	-1	-K	K	J	-J
J	J	-J	-K	K	-1	1	-I	I
-J	-J	J	K	-K	1	-1	I	-I
K	K	-K	J	-J	I	-I	-1	1
-K	-K	K	-J	J	-I	I	1	-1

$1^{-1}=1$

$-1^{-1}=-1$

$J^{-1}=-J$

$-J^{-1}=J$

$K^{-1}=-K$

$-K^{-1}=K$

$I^{-1}=-I$

$-I^{-1}=I$

Orbits:

$$\left.\begin{aligned} O_1 &= \{1\} = \{g1g^{-1} \mid g\in Q_8\} \\ O_{-1} &= \{-1\} = \{g(-1)g^{-1} \mid g\in Q_8\} \end{aligned}\right\} Z(G)=\{1,-1\}$$

$O_I = \{I,-I\} = \{g(I)g^{-1} \mid g\in Q_8\}$ (*)

$O_{-I} = \{I,-I\} = \{g(-I)g^{-1} \mid g\in Q_8\}$

$O_J = \{J,-J\} = \{g(J)g^{-1} \mid g\in Q_8\}$ (*)

$O_{-J} = \{J,-J\} = \{g(-J)g^{-1} \mid g\in Q_8\}$

$O_K = \{K,-K\} = \{g(K)g^{-1} \mid g\in Q_8\}$ (*)

$O_{-K} = \{K,-K\} = \{g(-K)g^{-1} \mid g\in Q_8\}$

Class equation: $8 = |Q_8| = |Z(G)| + |O_I| + |O_J| + |O_K|$

$= 2+2+2+2$

Shows $|G|=p^3$, but not abelian.