date: monday, april 1, 2024

Compass and Straight-Edge Constructions

Next two classes: classical (~2500 years) old problems in math

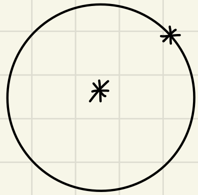
Straight edge: No markings on this straight edge (so not a ruler). Give two points, can draw a straight line in between the two points.

Compass: A tool to draw circles. Assume our compass can draw circles of radius 0 to ∞. The compass is "collapsable" (when you lift the compass, you cannot keep distance)

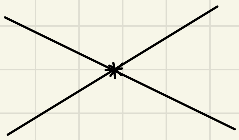
Basic Constructions:

① Create a line through two points:

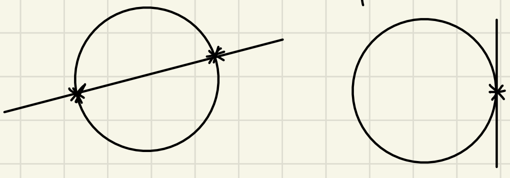
② Create a circle using two points where one point is the center and the other is on the perimeter.



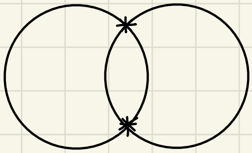
③ Intersection of two lines gives us a point



④ Create one or two points by intersecting a line and circle



⑤ Create one or two points via intersecting two circles



Using these rules, what can we make?

eg. How to find the midpoint between any two points.

① Given two points

② Draw a line between them

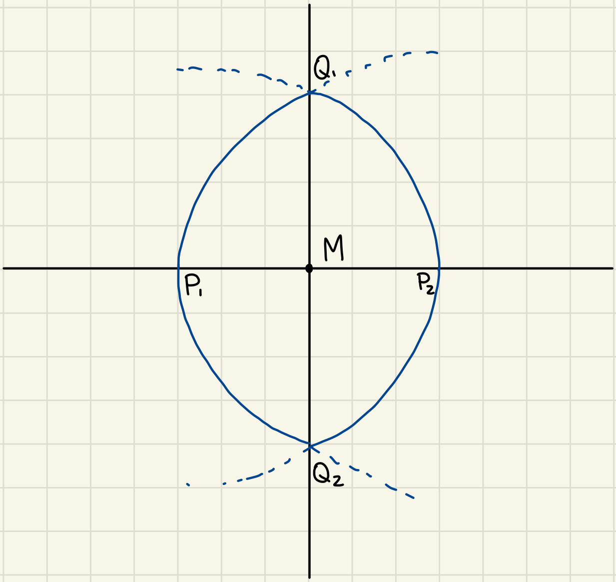
③ Create a circle C_1 with center at P_1 and perimeter with P_2

- create a circle C_2 with center at P_2 and perimeter point P_1

④ Label two new points Q_1, Q_2

⑤ Draw edge between Q_1 to Q_2

⑥ The intersection of two lines M is the midpoint.



Other constructions:

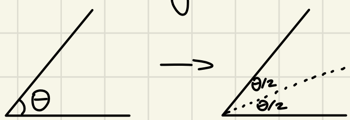
① The above example actually creates the perpendicular bisector.

② Given a line segment A ——— B, can create a square whose sides have length AB.

③ Given a line L and a point P off of L $*P$ ——— L, can create a line L' through P parallel to L.

④ Given a rectangle with area A, can construct a square with area A.

⑤ Bisect an angle:



Classical Problems:

① Trisecting an angle

② Squaring the circle: given a circle, create a square with equal area

③ Duplication of the cube: given a cube, construct a cube whose volume is double the original.

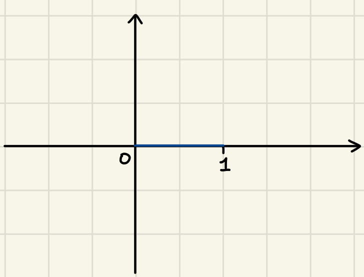
Answer: NOT POSSIBLE! Need field theory to prove this:

① and ③ proved by Wentzel in 1837.

② follows from Lindemann's 1882 proof that π is transcendental.

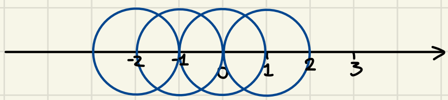
Constructible Numbers

Starting with a unit length, a straightedge, and a compass, what points can you construct?



Def^{n}: A real number α is constructible if we can create a line segment of length $|\alpha|$ in a finite number of steps using only a compass and straightedge.

eg. Every element of $\mathbb{Z}$ is constructible.



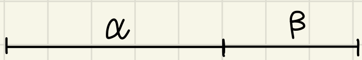
Note, we can make $\frac{1}{2}$ by taking the midpoint.

Theorem: The set $F = \{\text{all constructible numbers}\}$ is a subfield of $\mathbb{R}$.

<u>Proof</u>

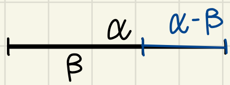
Given $\alpha, \beta \in F$, need to show that $\alpha \pm \beta \in F$, $\alpha\beta \in F$, and $\frac{\alpha}{\beta} \in F$.
Assume $\alpha > \beta$.

Then,

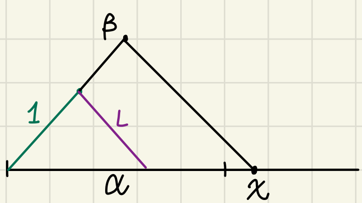


<– length $\alpha + \beta$

And,



To make $\alpha\beta$, consider the case $\beta > 1$ and make a triangle with sides of length 1 and α



Extend side length of 1 to β. Draw a line parallel to L through β. Extend the line of length α until it intersects this line to get x. By similar triangles $\frac{1}{\alpha} = \frac{\beta}{x} \Rightarrow x = \alpha\beta$.
See the textbook for α/β.