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Applications of Sylow Theorems

Two main applications:

① For some n, can classify all groups with $|G|=n$

② For some n, we can show that all groups with $|G|=n$ are not simple <=> G must have a nontrivial subgroup

eg of ① (last class) If $|G|=pq$ with $p<q$ and $q\not\equiv 1 \bmod p$, then $G\simeq \mathbb{Z}_{pq}$.

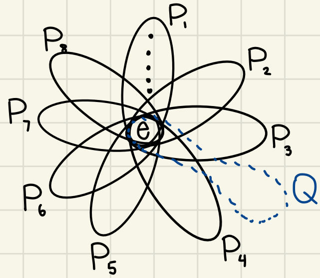
eg. If $|G|=15=3\cdot 5 \Rightarrow G\simeq\mathbb{Z}_{15}$

For ②, rely on the fact:

Theorem: Suppose $p \,\|\, |G|$. Then G has a unique Sylow p-subgroup iff the Sylow p-subgroup is normal in G.

eg. Show that if $|G|=20$, then G is not simple.
$20=2^2\cdot 5$
n_5 = number of Sylow 5-subgroups satisfies $n_5\equiv 1 \bmod 5$ and $n_5 \mid 20 \iff \{1,2,4,5,10,20\}$
So $n_5=1$

eg. Show that any group G with $|G|=56=2^3\cdot 7$ is not simple.
By 3rd Sylow theorem,
n_7 = number of Sylow 7-subgroups satisfies $n_7\in\{1,8,15,22,29,36,43,50\}$
$n_7\in\{1,2,4,7,8,14,28,56\}$.
Show $n_7=1$ or 8.
If $n_7=1$, happy 🙂, only 1 Sylow 7-subgroup.
What happens if $n_7=8$?
Let $P_1,P_2,\ldots,P_8$ be these 8 Sylow 7-subgroups.
Note $|P_i|=7$ for all i and $|P_i\cap P_j|=1$ (since $|P_i\cap P_j| \mid |P_i|$, the number is 1 or 7. But can't be 7 since this would imply $P_i\cap P_j=P_i$).



In total, we have $6\times 8+1=49$ elements of G.
By the first Sylow theorem, there is a Sylow 2-subgroup Q with $|Q|=8$.
Also $|Q\cap P_i|=1$, for all i. So Q has identity plus the other 7 elements. Can have at exactly <u>one</u> Sylow 2-subgroup. But then this subgroup is normal.

$\square$

eg. Suppose $|G|=p^n k$ with $p>k$. Prove that G is not simple.
$|G|=p^n\cdot k$ Sylow p group of order p^n.
$n_p\equiv 1 \bmod p$
$n_p \mid p^n\cdot k$
$n_p=1+ap$, $a\in\mathbb{Z}$, $a\geq 0$
So $1+ap \mid p^n\cdot k$. But $\gcd(1+ap,p^n)=1$. So $1+ap\mid k$. So $k\geq 1+ap$. If $a\geq 1$, this means $k\geq 1+ap\geq 1+p>k$. A contradiction. So $a=0$, ie. $n_p=1$.

eg. Any group of order 33 is not simple since $33=11\cdot 3$.

Theorem: Let $G'=\langle aba^{-1}b^{-1} \mid a,b\in G\rangle$. Subgroup generated by the product of this terms. Then,
- G' is not normal in G
- G/G' is abelian
- N is normal and G/N is abelian $\iff G'\subseteq N$

G' is called the commutator

Problem: Show all groups with $|G|=255=3\cdot 5\cdot 17$ are cyclic

<u>Proof</u>
Since $(3\cdot 5)<17$, our previous result implies G has one Sylow 17 subgroup and it's normal.
Let H be this group. Then

$$G/H \text{ is a group with order } \frac{255}{17}=15.$$

By previous fact, $G/H\simeq\mathbb{Z}_{15}$ <- abelian
Let G' be the commutator subgroup. By commutator result, $G'\subseteq H$.

So $|G'|=1$ or $|G'|=17$. If $|G'|=1$, then $G/G' \simeq G$ is abelian.
So $G \simeq \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \simeq \mathbb{Z}_{255}$ via the Fundamental Theorem of Finite Abelian Groups. So suppose $|G'|=17$. Count number of Sylow 3-subgroups and 5-subgroups.

Divisors of 255	mod 3	mod 5
1	1	1
3	0	3
5	2	0
17	2	2
$3 \cdot 5$	0	0
$3 \cdot 17$	0	1
$5 \cdot 17$	1	0
$3 \cdot 5 \cdot 17$	0	0

So number of Sylow 3-subgroups $= 1$ or 85 and number of Sylow 5-subgroups $= 1$ or 51.

Can't have both 85 Sylow 3-groups and 51 Sylow 5-groups. If Q is a Sylow 3-group and P a Sylow 5-group, and $|Q \cap P| = 1$.
The 85 Sylow 5-subgroup consists of $2 \times 85 + 1 = 170$ elements + identity
The 51 Sylow 5-subgroup consists of $4 \times 51 + 1 = 204$ elements + identity
But those are $374 + 1$ distinct elements. But $|G| = 255$.
If only one Sylow 3-subgroup Q, then Q is normal and $|G/Q| = 5 \cdot 17$.
But by Theorem, implies $G/Q \simeq \mathbb{Z}_{5 \cdot 17}$. So $G' \subseteq Q$. So $|G'|=17$, $|Q|=3$ $\Rightarrow\Leftarrow$.
If only one Sylow 5-subgroup P, then P normal and $|G/P| = 3 \cdot 17$.
By same theorem, $G/P \simeq \mathbb{Z}_{3 \cdot 17}$ so $G' \subseteq P \Rightarrow 17 = |G'| \mid |P| = 5$ $\Rightarrow\Leftarrow$.
Thus, must have $|G'| = 1 \Rightarrow G/G' = G$ is abelian.

$\square$