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Jordan-Hölder Theorem

Recall: A subnormal series $\{H_i\}$ is a composition series of G if

$$G = H_n \supset H_{n-1} \supset \cdots \supset H_1 \supset H_0 = \{e\}$$

if each H_i is normal in H_{i+1} and all H_{i+1}/H_i is simple.
Also, n = length = # of inclusions.

Jordan-Hölder: Suppose $\{H_i\}$ and $\{K_j\}$ are two composition series of G. Then $\{H_i\}$ is isomorphic to $\{K_j\}$, ie. there is a one-to-one correspondance between the sets $\{H_{i+1}/H_i\}$ and $\{K_{j+1}/K_j\}$.

Tools: (2nd Isomorphism Theorem) Let H be a subgroup of G (not necessarily normal) and let N be a normal subgroup of G. Then:
- $HN = \{hn \mid h \in H, n \in N\}$ is a subgroup of G
- $H \cap N$ is normal in H
- $H/H\cap N \simeq HN/N$

Lemma 1: Suppose H is normal in K. For any subgroup L of G, $H \cap L$ is normal in $K \cap L$.

Proof

Let $t \in H \cap L$ and $s \in K \cap L$. Since $t \in L$ and $s \in L$, $sts^{-1} \in L$ (L is a group). Since $t \in H$ and $s \in K$, $sts^{-1} \in sHs^{-1} \subseteq H$ since H is normal in K. So $sts^{-1} \in s(H\cap L)s^{-1} \in H \cap L$.
Let $s \in K \cap L$ and consider $sts^{-1} \in (H \cap L)s^{-1}$. So $t \in H$, so $sts^{-1} \in sHs^{-1} \subseteq H$.
So $t \in L$, so $sts^{-1} \in L$.

$\square$

Lemma 2: If A is normal in B and N is normal in A and B, then A/N is normal in B/N.

Proof

Want to show that $bN(A/N)(bN)^{-1} \subseteq A/N$ for all $bN \in B/N$.

Take $\ell \in bN(A/N)(bN)^{-1} \Rightarrow \ell = bNaNb^{-1}N$ for some $a \in A$. So $\ell = bab^{-1}N$. But A is normal in B, so $bab^{-1} \in bAb^{-1} \subseteq A$. So $bab^{-1}N = aN \in A/N$.

$\square$

Proof of Jordan-Hölder

Do induction on length of the smallest composition series of G. If $n=1$, $G \geq \{e\}$. So $G/\{e\} \simeq G$, and so G is simple. So this is the only possible composition series of G (note in composition series, H_{n-1} is normal in G, so this forces $H_{n-1} = H_0 = \{e\}$).

Suppose true for all $1 \leq k < n$. Consider two composition series:

$$G = H_n \supset H_{n-1} \supset H_{n-2} \supset \cdots \supset H_1 \supset H_0$$
$$G = K_m \supset K_{m-1} \supset K_{m-2} \supset \cdots \supset K_1 \supset K_0$$

By lemma 1) $H_i \cap K_{m-1}$ is normal in $H_{i+1} \cap K_{m-1}$ for $i = 0, \ldots, n-2$

2) $H_{n-1} \cap K_j$ is normal in $H_{n-1} \cap K_{j+1}$ for $j = 0, \ldots, m-2$

So we have two new subnormal series

$$G = H_n \supset H_{n-1} \supset \boxed{H_{n-1} \cap K_{m-1}} \supset H_{n-2} \cap K_{m-1} \supset H_{n-3} \cap K_{m-1} \supset \cdots H_0 \cap K_{m-1} = \{e\}$$
$$G = K_m \supset K_{m-1} \supset \boxed{H_{n-1} \cap K_{m-1}} \supset H_{n-1} \cap K_{m-2} \supset H_{n-1} \cap K_{m-3} \supset \cdots H_{n-1} \cap K_0 = \{e\}$$

same group

Want to show that these are new composition series, ie. each quotient is simple. Clear that H_n/H_{n-1} and K_m/K_{m-1} are simple by the initial set-up.

Claim: $H_{n-1} \cap K_{m-1}$ is normal in H_{n-1} and $H_{n-1}/H_{n-1} \cap K_{m-1}$ is simple.

(We'll come back to this)

By the 2nd Isomorphism Theorem,

$$\frac{H_{i+1} \cap K_{m-1}}{H_i \cap K_{m-1}} = \frac{H_{i+1} \cap K_{m-1}}{H_i \cap (H_{i+1} \cap K_{m-1})} \simeq \frac{H_i(H_{i+1} \cap K_{m-1})}{H_i}$$

for $i = 0, \ldots, n-2$.

Claim: $H_i(H_{i+1} \cap K_{m-1})$ is normal in H_{i+1}.

Let $ab \in H_i(H_{i+1} \cap K_{m-1})$ with $a \in H_i$ and $b \in (H_{i+1} \cap K_{m-1})$. Let $\ell \in H_{i+1}$. Since H_i is normal in H_{i+1}, $\ell a \ell^{-1} \in H_i$. Also, $\ell b \ell^{-1} \in H_{i+1}$ since $b, \ell \in H_{i+1}$. Finally since K_{m-1} is normal in G, $\ell b \ell^{-1} \in K_{m-1}$. So

$$\ell ab \ell^{-1} = \ell a e b \ell^{-1} = (\ell a \ell^{-1})(\ell b \ell^{-1}) \in H_i(H_{i+1} \cap K_{m-1}).$$

Because H_i is normal in $H_i(H_{i+1} \cap K_{m-1})$ and H_{i+1} and $H_i(H_{i+1} \cap K_{m-1})$ is normal in H_{i+1} by lemma 2. $\frac{H_i(H_{i+1} \cap K_{m-1})}{H_i}$ is normal in H_{i+1}/H_i. But H_{i+1}/H_i is simple, so

$$\frac{H_i(H_{i+1}\cap K_{m-1})}{H_i} = H_i/H_i \text{ or } H_{i+1}/H_i \iff H_i(H_{i+1}\cap K_{m-1}) = H_i \text{ or } H_{i+1}.$$

So, by removing non-proper inclusions,

$$H_{n-1} \supset H_{n-1}\cap K_{m-1} \supset H_{n-2}\cap K_{m-1} \supset \cdots \supset H_0 \cap K_{m-1} \quad (*)$$

is a composition series.
By the same reasoning,

$$K_{m-1} \supset H_{n-1}\cap K_{m-1} \supset H_{n-1}\cap K_{m-2} \supset \cdots \supset H_{n-1}\cap K_0$$

is a composition series.
By the induction hypothesis, this composition series $(*)$ is the same as

$$H_{n-1} \supset H_{n-2} \supset \cdots \supset H_0.$$

If $G = H_n \supset H_{n-1} \supset \cdots$ and $G = K_m \supset K_{m-1} \supset \cdots$, then if $H_{n-1} = K_{m-1}$, then the first quotients are the same, ie. $H_n/H_{n-1} \simeq K_m/K_{m-1}$ and the rest are the same by induction.
If $H_{n-1} \neq K_{m-1}$, look at

$$\begin{aligned} G &= H_n \supset H_{n-1} \supset H_{n-1}\cap K_{m-1} \supset \cdots \\ &= K_m \supset K_{m-1} \supset H_{n-1}\cap K_{m-1} \supset \cdots \end{aligned}$$

Note $K_{m-1}/H_{n-1}\cap K_{m-1} \simeq \dfrac{H_{n-1}K_{m-1}}{H_{n-1}} = \dfrac{G}{H_{n-1}}$ by 2^{nd} Isomorphism Theorem.

Similarly, $H_{n-1}/H_{n-1}\cap K_{m-1} \simeq G/K_{n-1}$.