date: wednesday, march 13, 2024

Fields from Integral Domains

Goal of Ch. 18: Look at properties of integral domains.

Main example: $\mathbb{Z}$

Today: Given an integral domain D, we will construct a field F_D.

Main example: making $\mathbb{Q}$ from $\mathbb{Z}$

Recall: D an integral domain => D is commutative, has 1_D, and no zero divisors.

Let $S=\{(a,b) \mid a,b \in D, \text{ and } b \neq 0\}$. Define an equivalence relation $\sim$ on S: $(a,b)\sim(c,d) \iff ad=bc$.

Lemma: $\sim$ is an equivalence relation.

Proof

(transitive) Suppose $(a,b)\sim(c,d)$ and $(c,d)\sim(e,f)$. So $ad=bc$ and $cf=de$. So $adf=bcf$ and $bcf=bde$ (note $b\neq0$, $f\neq0$). So $adf=bde \iff (af-be)d=0$. Since $d\neq0$ and D an integral domain, $af-be=0$. Thus $af=be$ so $(a,b)\sim(e,f)$.

(reflexive) $(a,b)\sim(a,b)$ since $ab=ba$ due to D commutative.

(symmetric) Suppose $(a,b)\sim(c,d)$. So $ad=bc$. By commutativity, $cb=da$. So $(c,d)\sim(a,b)$.

$\square$

Def^{n}: $[a,b]=\{(c,d)\in S \mid (a,b)\sim(c,d)\}$ equivalence class of (a,b).

Def^{n}: $F_D=\{[a,b] \mid (a,b)\in S\}$

set of all equivalence classes

eg. When $D=\mathbb{Z}$, $S=\{(a,b) \mid a,b\in\mathbb{Z}, b\neq0\}$.

Consider $(2,7)\in S$.
$[2,7]=\{(c,d)\mid(2,7)\sim(c,d)\}=\{(c,d)\mid 2d=7c\}=\{\frac{c}{d}\in\mathbb{Q}\mid\frac{2}{7}=\frac{c}{d}\}$.
ie. When we write $\frac{2}{7}\in\mathbb{Q}$, we mean "all ways" to write $\frac{2}{7}$ (eg. $\frac{2}{7}=\frac{-2}{-7}$)

We put an addition and multiplication on F_D.

$[a,b]+[c,d]=[ad+bc,bd]$
$[a,b][c,d]=[ac,bd]$

Lemma: Both operations well defined.

Proof (addition in text)
Suppose $[a,b]=[a',b']$ and $[c,d]=[c',d']$. Want to show

$$[a,b][c,d]=[a',b'][c',d']$$

ie. $[ac,bd]=[a'c',b'd']$.
Given $ab'=a'b$ and $cd'=dc'$. So

$$ab'cd'=a'bc'd$$
$$\Rightarrow (ac)(b'd')=(a'c')(bd)$$
$$\Rightarrow [ac,bd]=[a'c',b'd']$$

$\square$

eg. $\frac{1}{2}=\frac{2}{4}$ and $\frac{2}{3}=\frac{-2}{-3}$
then $\frac{1}{2}\cdot\frac{2}{3}=\frac{2}{6}=\frac{-4}{-12}=\frac{2\cdot(-2)}{4\cdot(-3)}$

Theorem: F_D is a field.

eg. If $D=\mathbb{Z}$,
$F_D=\{[a,b]\mid a,b\in\mathbb{Z}, b\neq 0\}=\mathbb{Q}=\{\frac{a}{b}\mid a,b\in\mathbb{Z}, b\neq 0\}$

Proof
Verify all the properties to be a field.
Additive identity is $[0,1]=[0,2024]$,

$$\begin{aligned}[a,b]+[0,2024]&=[a\cdot 2024+b\cdot 0,\ 2024b]\\&=[a\cdot 2024,\ 2024b]\\&=[a,b]\end{aligned}$$

Multiplicative identity is $[1,1]=[2024,2024]$,

$$[a,b][1,1]=[a,b].$$

Suppose $[a,b]\in F_D$ and $a\neq 0$. Then $[b,a]\in F_D$ and this is the inverse since $[a,b][b,a]=[ab,ba]=[2024,2024]=[1,1]$.

Exercise: Show $[a,b]+([c,d]+[e,f])=([a,b]+[c,d])+[e,f]$.

$\square$

Def^{n}: The field F_D is called the field of fractions of D.

Theorem: Let D be an integral domain. Then D can be embedded into F_D ($\Leftrightarrow$ there exists an injective homomorphism $f: D \to F_D$).
(ie. F_D has a subring isomorphic to D)

<u>Proof</u>

Let $D'=\{[d,1] \mid d\in D\}\subseteq F_D$ $\leftarrow$ show D' is a subring.

Define a map $\varphi: D\to D'\subseteq F_D$ given by $\varphi(d)=[d,1]$. It is a ring homomorphism since

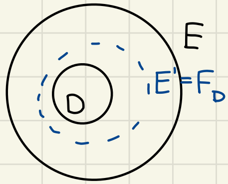
$$\begin{aligned}\varphi(d_1+d_2)&=[d_1+d_2,1]\\&=[d_1,1]+[d_2,1]\\&=\varphi(d_1)+\varphi(d_2)\\ \varphi(d_1d_2)&=[d_1d_2,1]\\&=[d_1,1][d_2,1]\\&=\varphi(d_1)\varphi(d_2)\end{aligned}$$

The map is injective and surjective on D'.

Note: Technically, $\mathbb{Z}$ is not a subring of $\mathbb{Q}$. But $\mathbb{Q}$ has a subring isomorphic to $\mathbb{Z}$.
We are sloppy and write $\mathbb{Z}\subseteq\mathbb{Q}$.

$$\mathbb{Z}\sim\mathbb{Z}'=\{\tfrac{2a}{2} \mid a\in\mathbb{Z}\}\subseteq\mathbb{Q}=F_{\mathbb{Z}}$$

Theorem: Suppose E is a field that contains an integral domain D. Then, there exists a subfield $E'\subseteq E$ such that $F_D\simeq E'\subseteq E$.



Corollary: If E is a field of characteristic 0, then E has a subfield isomorphic to $\mathbb{Q}$.

Proof

Let 1_E be the identity.

Set $D = \{n \cdot 1_E \mid n \in \mathbb{Z}\} \simeq \mathbb{Z}$. So $F_{\mathbb{Z}} \simeq \mathbb{Q} \simeq E' \subseteq E$.