Algorithms, 2025 Fall, Homework 1
(Due: Sep 24)

September 15, 2025

Readings: Our algorithm class assumes basic knowledge about basic algorithms knowledge (like what is a graph, BFS, DFS) and basic data structures (like array, link list, queue and stack), and college mathematics (basic calculus, linear algebra and probability). In case you have not learnt much before, you are required to read the following chapters.

1. Kleinberg-Tardos book Chapter 2.1-2.4 for big O notations, basic data structures. (You need to finish reading this chapter this week in order to do the homework)
2. Chapter 3 for BSF and DFS, also Chapter 22.1,22.2,22.3 in CLRS. (You need to finish reading this chapter this week in order to do the homework.)

You can discuss homeworks with your classmates or/and using ChatGPT (or other LLMs). But you will have to write down the solution on your own. TA will choose only 3 problems to grade.

Try to find the solution on your own first before consulting others or LLM. You need to acknowledge other students or LLMs who help you in the homework (specify which problem). If you read some other source that is helpful, you should list all of them. As I said in the class, if you use LLM for a problem but it fails (no matter how you prompt it), you can note this fact in your solution (we are collecting problems which LLM cannot answer correctly).

For those who have extensive experience of OI (show TA the proof), you do not need to solve the problems marked with @ (of course you can always choose 3 other problems to solve).

Problem 1 : @ Write down the pseudocode of BSF (using queues).

Problem 2 : @ KT book pp. 67 problem 4.

Problem 3 : @ KT book pp. 107 problem 3.

Problem 4 : @ KT book pp. 108 problem 5.

Problem 5 : @ KT book pp. 108 problem 6.

Problem 6 : @ KT book pp.110, problem 10) (Note: We only need the number, not the actually paths. Please learn BFS first.)

Problem 7 : We are given n balls and n bins. Each ball is thrown into a random bin (each bin is chosen with probability $1/n$). Prove that:

$$\lim_{n \to +\infty} \mathbb{E}\left[\frac{\text{the number of empty bins}}{n}\right] = \frac{1}{e}$$

where e is the base of the natural logarithm.

Problem 8 : For two women w and w', we write $w <_m w'$ to denote that w is worse than w' in the preference list of man m. Given two stable matchings f and f' (You can easily construct an example in which there are multiple stable matchings), define a mapping $g = f \vee f'$ as follows:

- for each man m, assign him more preferred partner

$$g(m) = f(m) \text{ if } f(m) \geq_m f'(m)$$

$$g(m) = f'(m) \text{ if } f'(m) >_m f(m)$$

- for each woman w, assign her less preferred partner

$$g(w) = f(w) \text{ if } f(w) \leq_w f'(w)$$

$$g(w) = f'(w) \text{ if } f'(w) <_w f(w)$$

Show that if both f and f' are stable matchings, so is g. (note: We can similarly define $f \wedge f'$. Then, all stable matchings form a distributive lattice, an abstract yet very popular object studied in combinatorics.)

Problem 9 : As we mentioned, there could be multiple stable matchings. The Gale-Shapley algorithm only finds one such stable matching. For any man m, let $best(m)$ be the best woman matched to m in all possible stable matchings. Show that Gale-Shapley algorithm is man-optimal, in the sense that it returns a stable matching where for any man m, m is matched to $best(m)$. (If we let women propose, the resulting matching is women-optimal).