1 Problem 7

Statement: We are given n balls and n bins. Each ball is thrown into a random bin(each bin is chosen with probability $1/n$). Prove that:

$$\lim_{n\to+\infty} \mathbb{E}\left[\frac{the\ number\ of\ empty\ bins}{n}\right] = \frac{1}{e}$$

where e is the base of the natural logarithm.
Proof:
By the linearity of expectation:

$$\lim_{n\to+\infty} \mathbb{E}\left[\frac{the\ number\ of\ empty\ bins}{n}\right]$$

equals to

$$\lim_{n\to+\infty} \mathbb{E}\left[X\right]$$

X equals to 1 if a bin is empty after this procession and 0 otherwise.
This is equivalent to

$$\lim_{n\to+\infty} (1-\frac{1}{n})^n = \frac{1}{e}$$

2 Problem 8

Statement: For two women w and w', we write $w <_m w'$ to denote that w is worse than w' in the preference list of man m. Given two stable matchings f and f'(You can easily construct an example in with there are multiple stable matchings), define a mapping $g = f \vee f'$ as follows:

1. for each man m, assign him more preferred partner

$$g(m) = f(m) \text{ if } f(m) \geq_m f'(m)$$

$$g(m) = f'(m) \text{ if } f'(m) >_m f(m)$$

2. for each woman w, assign her less preferred partner

$$g(w) = f(w) \text{ if } f(w) \leq_w f'(w)$$

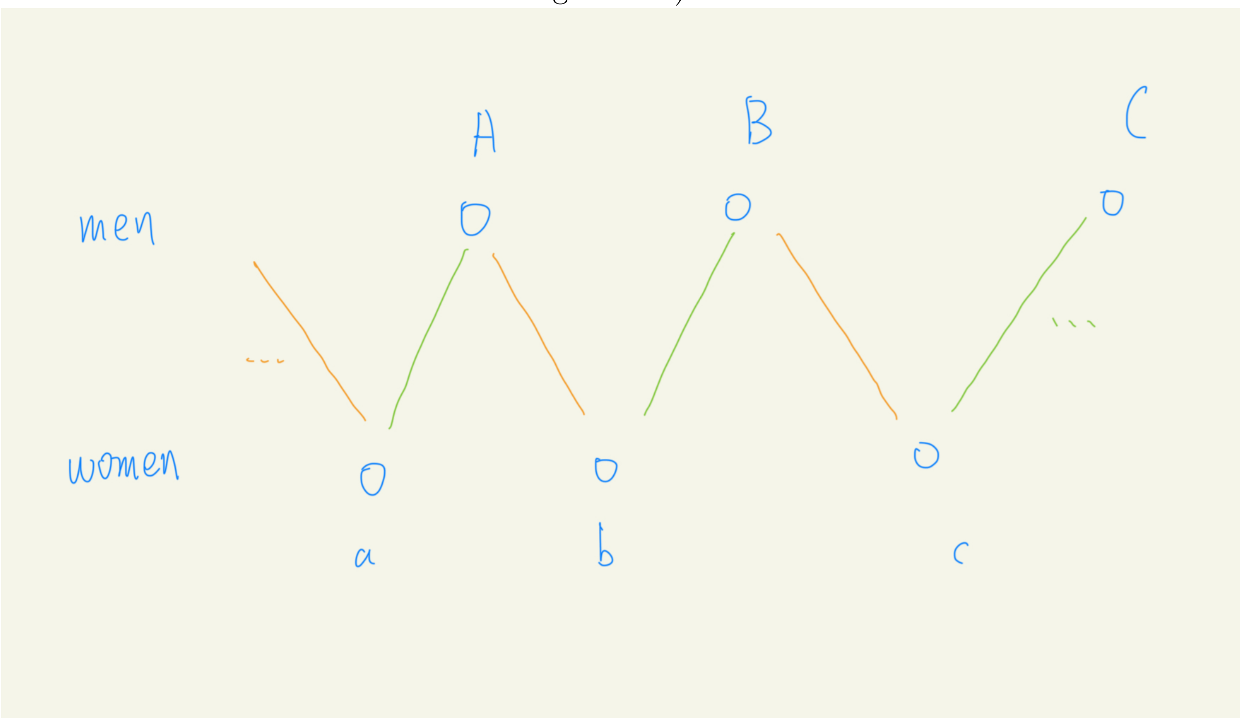
$$g(w) = f'(w) \text{ if } f'(w) <_w f(w)$$

Show that if both f and f' are stable matchings, so is g. (noto: We can similarly define $f \wedge f'$. Then all stable matchings form a distributive lattice, an abstract yet very popular object studied in combinatorics.)

Proof:

First, $f(m) = f'(m), f(w) = f'(w)$ is trivial. Let's skip this case.

Then we can observe that these two matchings form an undirected graph with circles(Nodes denote men and women. Each node's degree is 2.)



Assume that A prefers b than a.

b must prefer B to A, otherwise $(A, a), (B, b)$ are not stable matchings because of (A, b).

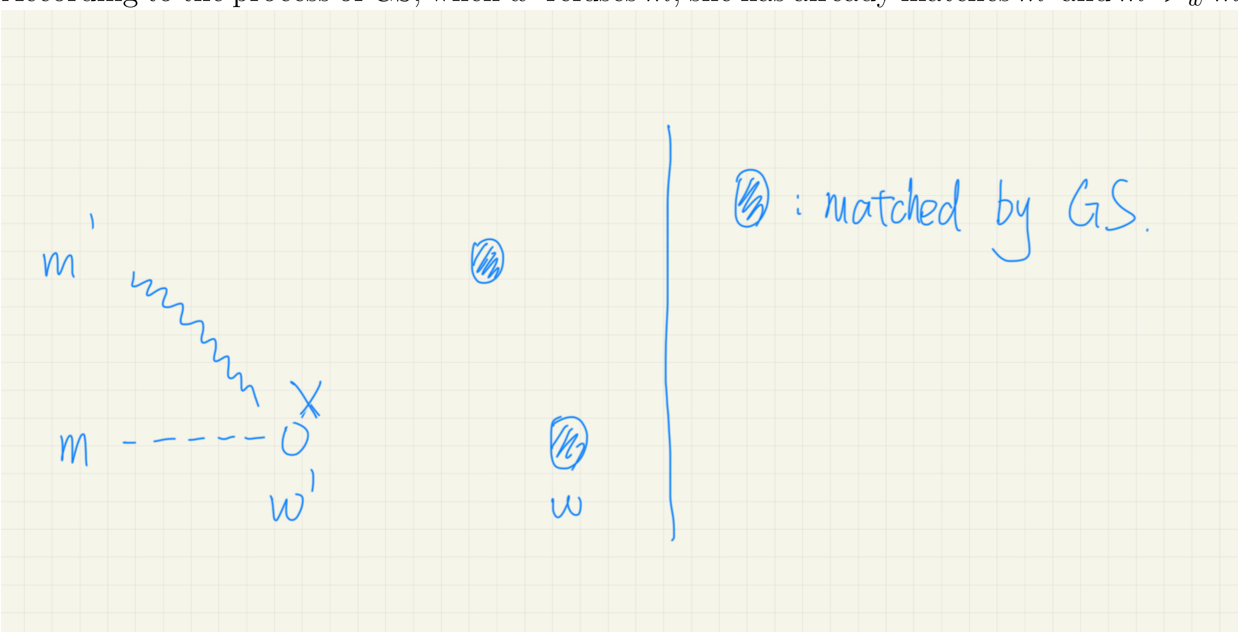
Also, B must prefer c to b, otherwise $(A, b), (B, c)$ are not stable matchings because of (B, b).

Thus, according to the statement, g is a stable matching which equals to the orange one.

This is because we have assumed that A prefers b than a. Likewise, if we consider A prefers a than b, we can get silimar conclusion. This shows that an undirected circle infers two directions(ways) of matchings.

3 Problem 9

Statement: As we mentioned, there could be multiple stable matchings. The Gale-Shapley algorithm only finds one such stable matching. For any man m, let $best(m)$ be the best woman matched to m in all possible stable matchings. Show that Gale-Shapley algorithm is man-optimal, in the sense that it returns a stable matching where for any m, m is matched to $best(m)$.(If we let women propose, the resulting matching is women-optimal)
Proof:
Consider a man m matches w by Gale-Shapley algorithm, and m matches w' in some stable matching S.($w' >_m w$)
According to the process of GS, when w' refuses m, she has already matches m' and $m' >_w m$.



No matter who m' matches at last, we can infer that (m', w') may cause unstable matching if m' matches a woman worse than w' in S.
Also, we have $w' \geq_{m'} f(m')$(let $f(m)$ denote the woman matched by GS).
So we can simply mark w' visited and recursively consider the same problem for m'(That we means w' can't be matched to others because we need (m, w')). But we can't recur forever. So it contradicts.