深蓝学院
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3D Point Clouds Lecture 1 – Introduction

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1、 Course Introduction

2、 Principle Component Analysis

3、 Downsampling

4、 Filtering

3D Point Cloud

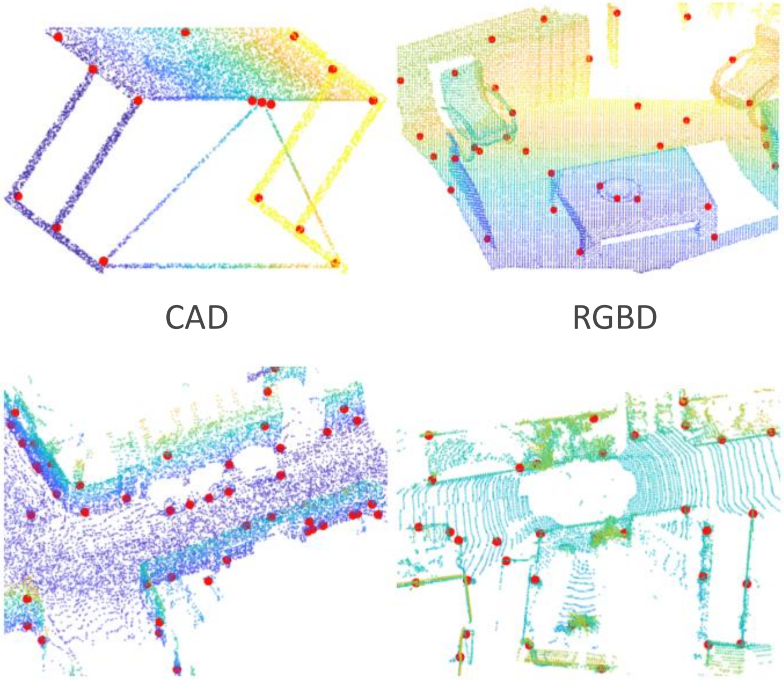
Representation

- $N \times 3$ matrix or $N \times (3 + D)$ matrix

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_N & y_N & z_N \end{bmatrix} \quad \begin{bmatrix} x_1 & y_1 & z_1 & d_{11} & \cdots & d_{1D} \\ x_2 & y_2 & z_2 & d_{21} & \cdots & d_{2D} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_N & y_N & z_N & d_{N1} & \cdots & d_{ND} \end{bmatrix}$$

Data sources:

- Lidar
- RGB-D
- CAD Models
- Structure-from-Motion (SfM)
- Depth from Images
- ……



CAD RGBD

2D Lidar Accumulation 64-beam Lidar

Applications of Point Clouds

Robotics, Autonomous driving

- Localization – SLAM, loop closure, registration
- Perception – object detection, classification
- Reconstruction – SfM, registration

Consumer Electronics

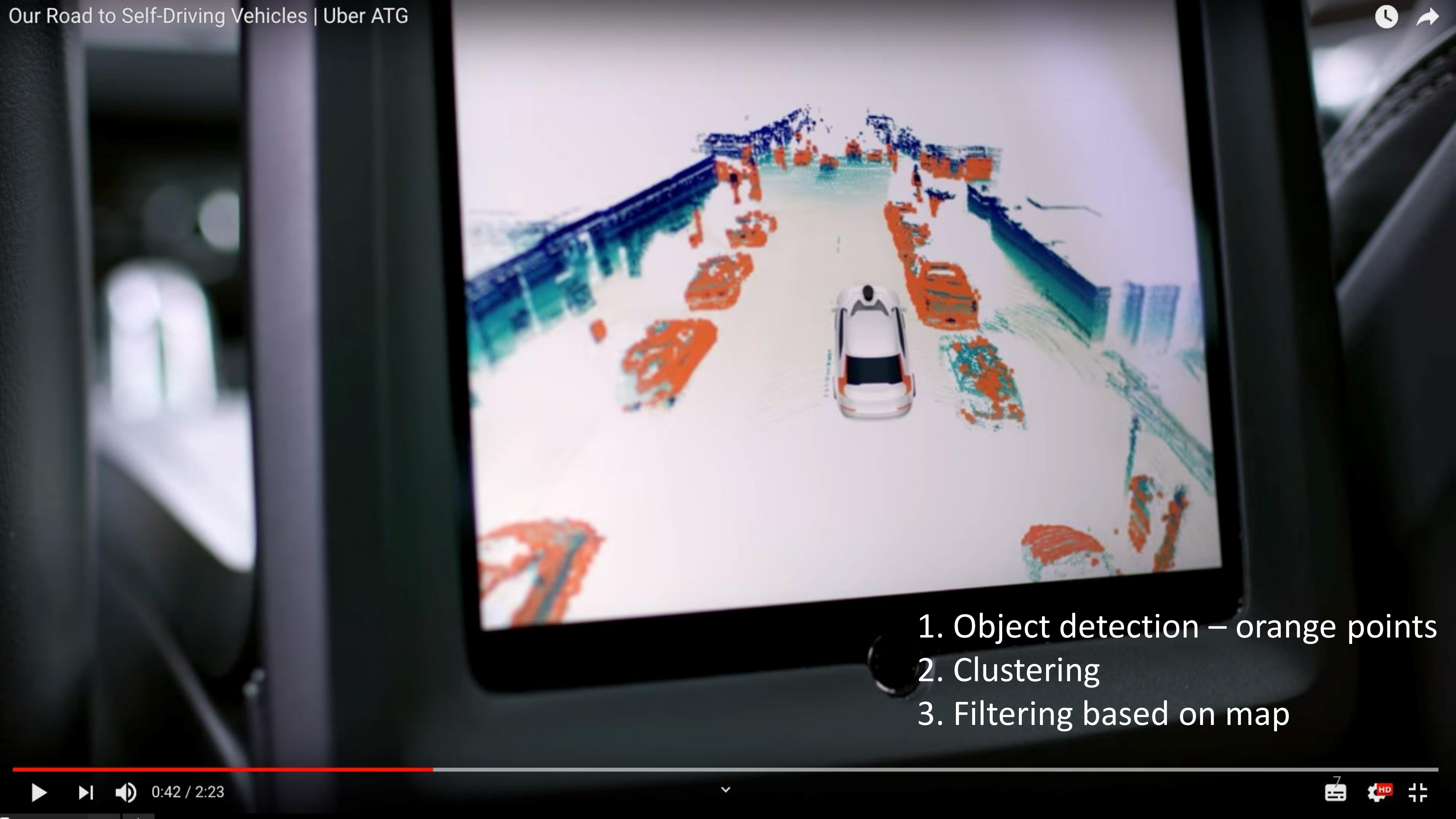
- Face detection / reconstruction – FaceID
- Hand pose – Hololens
- Human pose – Kinect

Applications

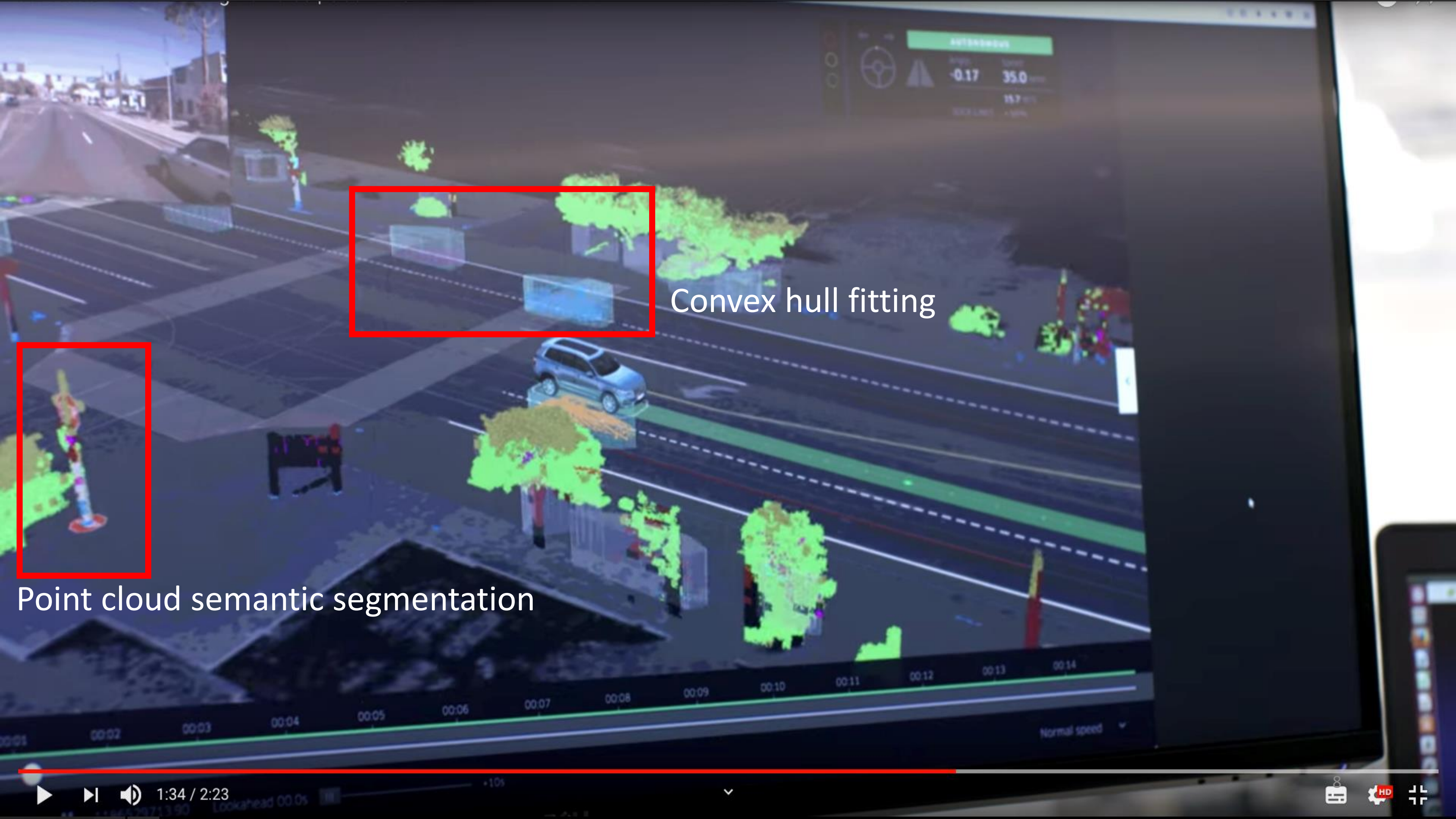
https://www.youtube.com/watch?v=27OuOCeZmwI

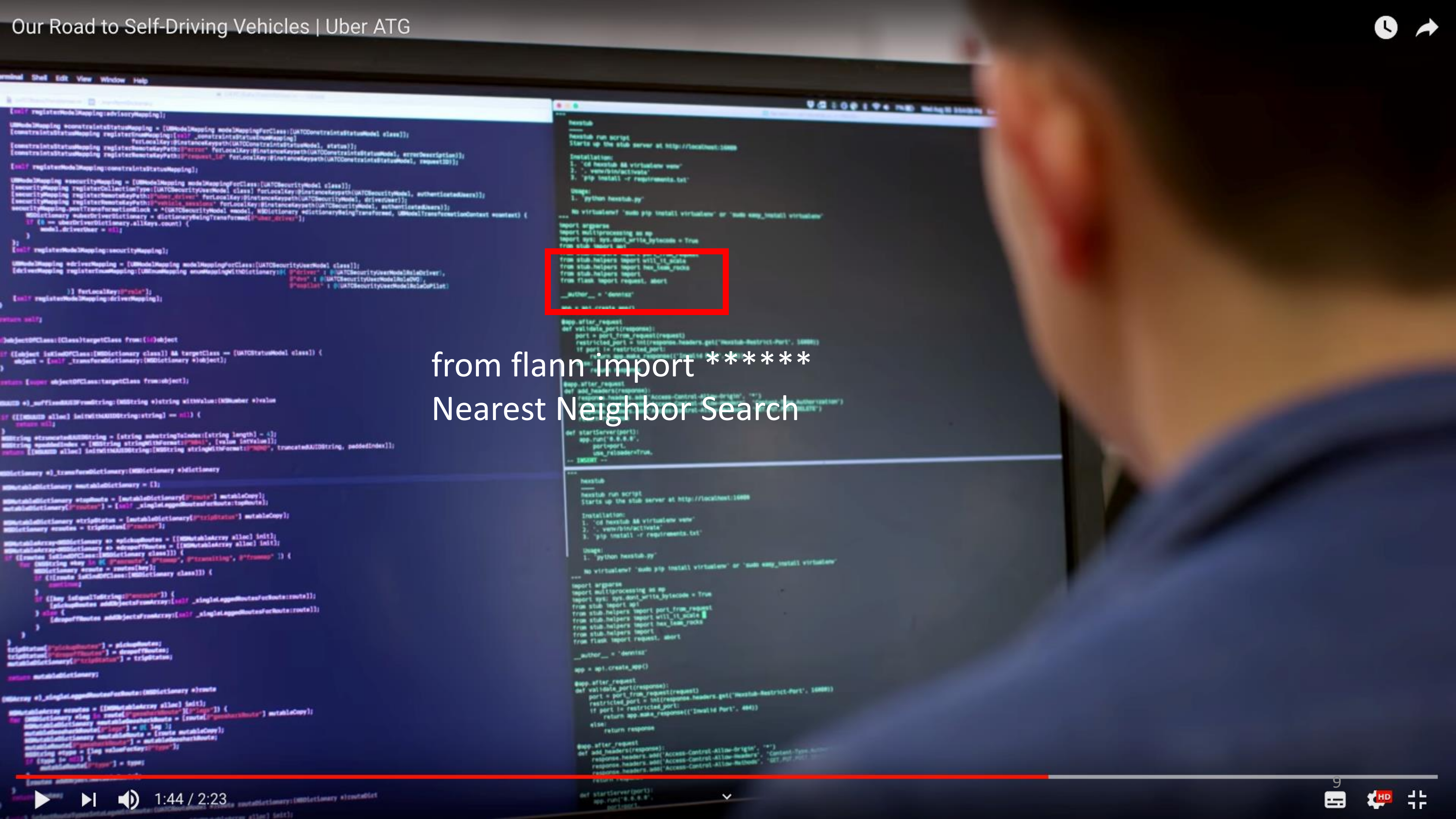
Our Road to Self-Driving Vehicles | Uber ATG


Our Road to Self-Driving Vehicles | Uber ATG
Velodyne 64
0:38 / 2:23



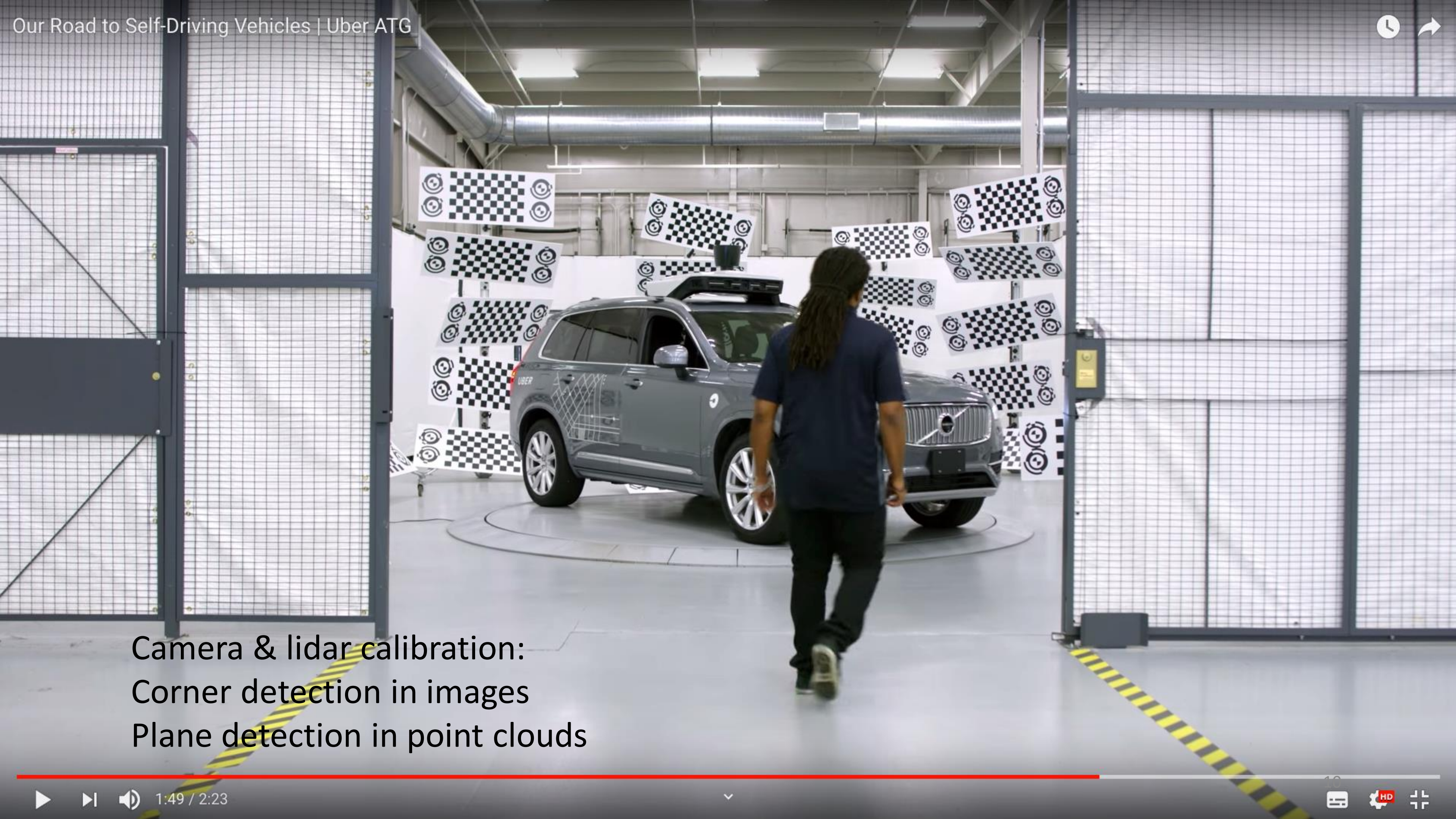
1. Object detection – orange points
2. Clustering
3. Filtering based on map


Convex hull fitting
Point cloud semantic segmentation
1:34 / 2:23



from flann import ******

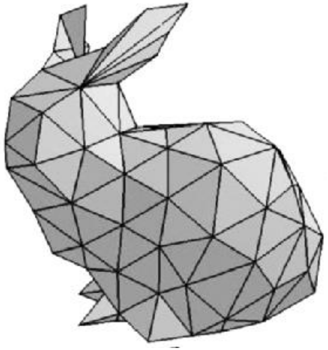
Nearest Neighbor Search


Our Road to Self-Driving Vehicles | Uber ATG
UBER
Camera & lidar calibration:
Corner detection in images
Plane detection in point clouds
1:49 / 2:23
HD

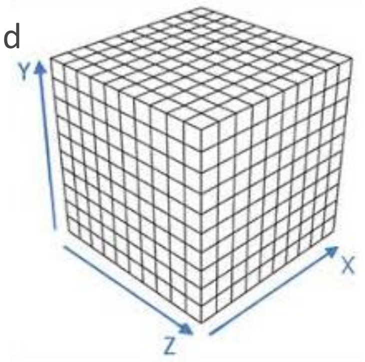
Advantages & Difficulties

What are the options for 3D?

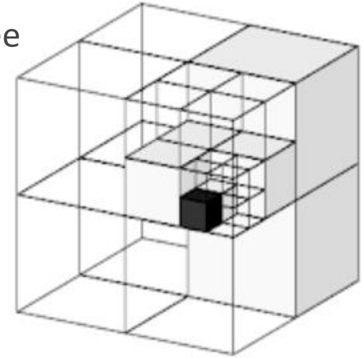
Mesh



Voxel Grid



Octree



Strength of Point Cloud

- 3D information
- Mathematically simple and concise

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_N & y_N & z_N \end{bmatrix}$$

Difficulties of Point Cloud Processing

- Sparsity
- Irregular – difficulty in neighbor searching
- Lack of texture information
- Un-ordered – difficulty in deep learning
- Rotation equivariance / invariance

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_N & y_N & z_N \end{bmatrix} = \begin{bmatrix} x_2 & y_2 & z_2 \\ x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_k & y_k & z_k \end{bmatrix}$$

Course Outline

Classical Methods – 60%

- Pros
  - Explainable – It follows physics and we know why it works/doesn’t work
  - Controllable – We know how to debug
- Cons
  - Hard to model semantics
  - User-unfriendly

Deep Learning Methods – 40%

- Pros
  - Simple!
  - High performance
  - Data driven
- Cons
  - Un-explainable – No one knows why / how.
  - Un-controllable – Black box
  - Requires special hardware – GPU / FPGA, etc.
  - Simple – The barrier is lower and lower means it will be more and more difficult to find a job.

Classical v.s. Deep Learning

Object Classification

- **Classical**
  - Keypoint detection
  - Keypoint description
  - Support Vector Machine
- **Deep learning**
  - Data collection
  - Data labeling
  - Train a network

Object Registration

- **Classical**
  - Nearest Neighbor Search
  - Iterative Closest Point
- **Deep learning**
  - Data collection
  - Data labeling
  - Train a network

Object Detection

- **Classical**
  - Background removal
  - Clustering
  - Classification
- **Deep learning**
  - Data collection
  - Data labeling
  - Train a network

Course Schedule

Week	Topic
1	Introduction to point clouds, PCA/kPCA, downsampling and filtering
2	Nearest neighbor algorithms
3	Clustering algorithms
4	Model fitting
5	Deep Learning with point clouds
6	Object detection
7	Keypoints and descriptors
8	Registration
9	2D and 3D SLAM with point clouds

Course Outline

- Theory = you know everything but nothing works.
- Practice = everything works but no one knows why.
- Theory + practice = nothing works and no one knows why.

- This module is around 50% theory and 50% practice
  - Theory – proof, derivation
  - Practice
    - Algorithms
    - Engineering practices
    - Assignments are mostly coding

Prerequisite

Basic Linear Algebra

- Eigen decomposition, SVD, PCA

Basic Probability Theory

- Bayes rule
- Marginalization
- Graph theory

Computer Vision

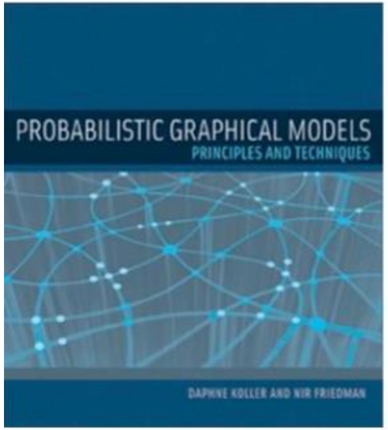
- Camera models

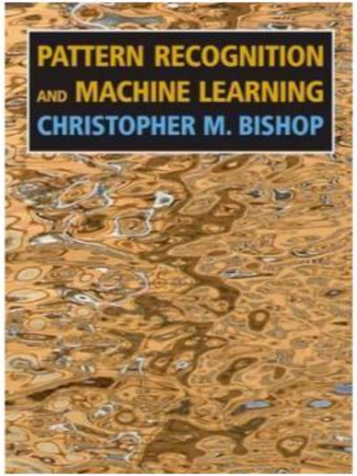
Simple Optimization

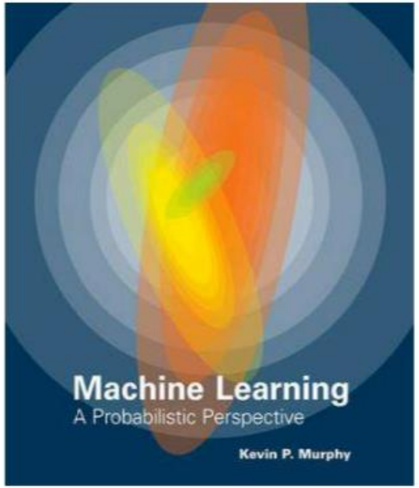
- Gradient Descent
- Least Square
  - Gauss-Newton
  - Levenberg Marquardt
- Lagrange Multiplier

Deep learning

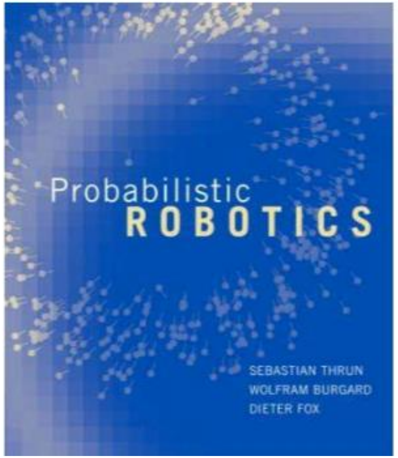
- Gradient descent optimizer
  - SGD, Adam, etc.
- Multi-Layer Perceptron (MLP)
- 2D / 3D Convolution
- Activation Function
  - The idea of non-linearity
- Normalization
  - Batch normalization

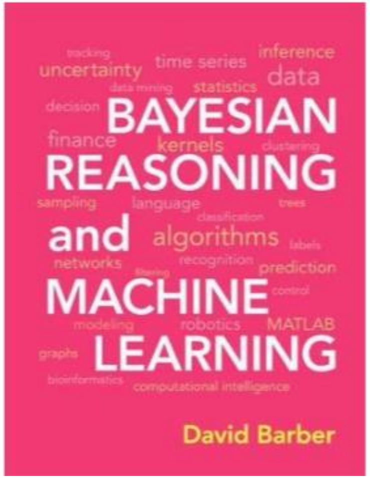

PROBABILISTIC GRAPHICAL MODELS
PRINCIPLES AND TECHNIQUES
DAPHNE KOLLER AND NIR FRIEDMAN


PATTERN RECOGNITION AND MACHINE LEARNING
CHRISTOPHER M. BISHOP


Machine Learning
A Probabilistic Perspective
Kevin P. Murphy


SIMON J. D. PRINCE
COMPUTER VISION
MODELS, LEARNING, AND INFERENCE

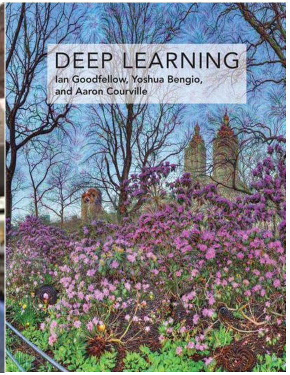

Probabilistic
ROBOTICS
SEBASTIAN THRUN
WOLFRAM BURGARD
DIETER FOX


BAYESIAN REASONING and MACHINE LEARNING
David Barber

Course Outline - Theory

Deep Learning

- Ian Goodfellow, Yoshua Bengio, Aaron Courville
- https://www.deeplearningbook.org/



Standford CS231n

- Li Fei-fei, Justin Johnson, Andrej Karpathy, etc
- http://cs231n.stanford.edu/

Common Tools for Practice

C++

- Point Cloud Library (PCL)
- Python Binding – pybind11
- Optimization Solver – g2o, Ceres
- Eigen

Python

- numpy
- scipy
- Open3D
- Pytorch
- Tensorflow

Assignments

Weekly Assignments

- Some assignments are compulsory
- Some are optional, for those who are interested.

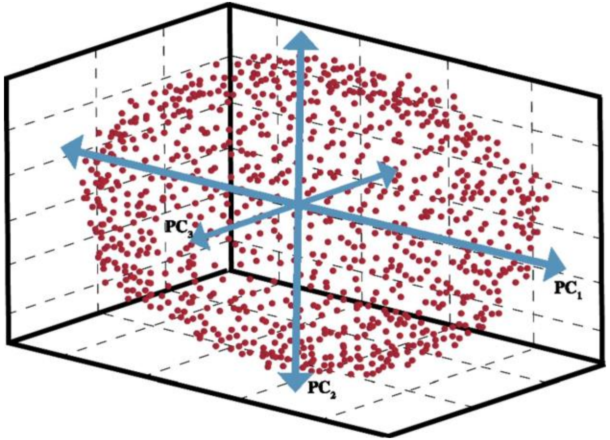
Big Projects

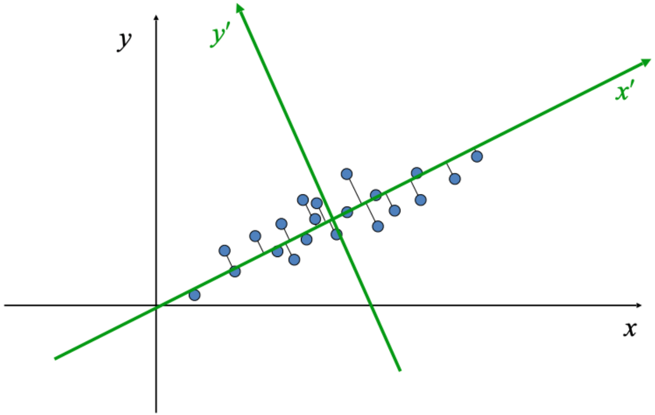
- Mid-term project
  - Ground estimation on KITTI dataset
- Final project
  - Object detection by Deep Learning and classical methods
    1. Use deep learning based object detector
    2. Use classical method (classification can be deep learning based)
    3. Try to combine both

Principle Component Analysis

- **PCA is to find the dominant directions of the point cloud**

- **Applications:**
  - Dimensionality reduction
  - Surface normal estimation
  - Canonical orientation
  - Keypoint detection
  - Feature description

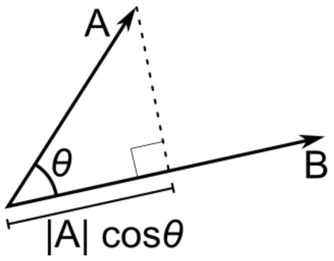




Principle Component Analysis

Let’s start with some physical intuitions.

Vector Dot Product



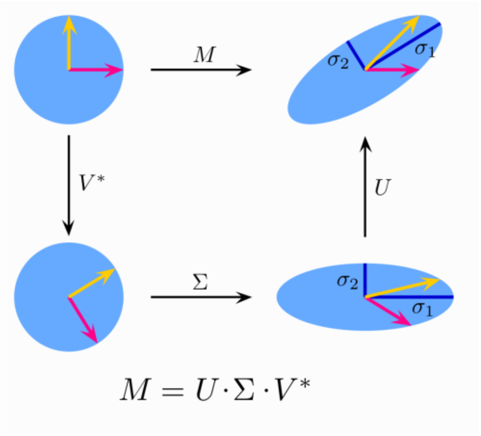
Matrix-Vector Multiplication

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ \vdots & \vdots & \vdots \\ x_{1N} & a_{2N} & a_{3N} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1\mathbf{a_1} + x_2\mathbf{a_2} + x_3\mathbf{a_3}$$

Linear combination

Singular Value Decomposition (SVD)



Principle Component Analysis

Spectral Theorem

Let $A \in R^{n,n}$ be symmetric, and $\lambda_i \in R, i = 1, 2, \cdots, n$ be the eigenvalues of A. There exists a set of orthonormal vectors $u_i \in R_n, i = 1, 2, \cdots, n$, such that $Au_i = \lambda_i u_i$. Equivalently, there exists an orthogonal matrix $U = [u_1, \cdots, u_n]$ (i.e., $UU^T = U^T U = I_n$), such that,

$$A = U \Lambda U^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T, \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$$

Principle Component Analysis

Rayleigh Quotients

Physical meaning of SVD!

Given a symmertric matrix $A \in S^n$,

$$\boxed{\lambda_{\text{min}}(A) \leq \frac{x^T A x}{x^T x} \leq \lambda_{\text{max}}(A), \forall x \neq 0}$$

$$\lambda_{max}(A) = \max_{x:\|x\|_2=1} x^T A x$$

$$\lambda_{min}(A) = \min_{x:\|x\|_2=1} x^T A x$$

The maximum and minimum are attained for $x = u_1$ and for $x = u_n$, respectively, where u_1 and u_n are the largest and smallest eigenvector of A, respectively.

Rayleigh Quotients – Proof:

- Apply the spectral theorem, U is orthogonal, Λ is diagonal

$$x^T A x = x^T U \Lambda U^T x = \bar{x}^T \Lambda \bar{x} = \sum_{i=1}^{n} \lambda_i \bar{x}_i^2$$

- Obviously,

$$\lambda_{\min} \sum_{i=1}^{n} \bar{x}_i^2 \le \sum_{i=1}^{n} \lambda_i \bar{x}_i^2 \le \lambda_{\max} \sum_{i=1}^{n} \bar{x}_i^2$$

- Also, orthogonal matrix U doesn't change the norm of any vector

$$\sum_{i=1}^{n} x_i^2 = x^T x = x^T U U^T x = (U^T x)^T (U^T x) = \bar{x}^T \bar{x} = \sum_{i=1}^{n} \bar{x}_i^2$$

- Combining the above 3 equations,

$$\lambda_{\min} x^T x \le x^T A x \le \lambda_{\max} x^T x$$

Principle Component Analysis

- Input: $x_i \in \mathbb{R}^n, i = 1, 2, \cdots, m$
- Output: principle vectors $z_1, z_2, \cdots, z_k \in \mathbb{R}^n, k \leq n$

Q: What is the most significant principle component?
A: A direction such that the variance of the projected data points on that direction is maximal.

Q: How to get the second significant one?
A: Deflation. Remove the most significant component from the data points, i.e., data point minus the projection. Find the most significant component for the deflated data.

Q: How to get the 3rd one?
A: Repeat the above steps.

Principle Component Analysis - Proof

- Normalize the data to be zero mean

$$\tilde{X} = [\tilde{x}_1, \cdots, \tilde{x}_m], \tilde{x}_i = x_i - \bar{x}, i = 1, \cdots, m \qquad \bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i$$

- PCA is to get largest variance when projected to a direction $z \in \mathbb{R}^n, \|z\|_2 = 1$

$$\alpha_i = \tilde{x}_i^T z, i = 1, \cdots, m$$

- The mean variance of the projections is

$$\frac{1}{m}\sum_{i=1}^{m}\alpha_i^2 = \frac{1}{m}\sum_{i=1}^{m} z^T \tilde{x}_i \tilde{x}_i^T z = \frac{1}{m} z^T \tilde{X}\tilde{X}^T z$$

- So, maximize it,

$$\max_{z \in R^n} z^T(\tilde{X}\tilde{X}^T)z, \text{s.t.:} \|z\|_2 = 1$$

Principle Component Analysis - Proof

- Now, maximize this

$$\max_{z \in R^n} z^T (\tilde{X}\tilde{X}^T) z, \text{ s.t.:} \|z\|_2 = 1$$

- Recall the Rayleigh Quotients $\lambda_{\min}(A) \le \frac{x^T A x}{x^T x} \le \lambda_{\max}(A), \forall x \neq 0$
- Recall our Spectral Theorem $A = U \Lambda U^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T, \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$
- Apply to PCA

$$H = \tilde{X}\tilde{X}^T = U_r \Sigma^2 U_r^T$$

- First principle vector $z_1 = u_1$, u_1 is the first column of U_r

Principle Component Analysis - Proof

- Let's take a look at $H = \tilde{X}\tilde{X}^T = U_r \Sigma^2 U_r^T$
- Perform SVD on $\tilde{X}$: $\tilde{X} = U_r \Sigma V_r^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T$

Spectral Theorem and SVD are closely related

- Find z_2 by deflation

$$\tilde{x}_i^{(1)} = \tilde{x}_i - u_1(u_1^T \tilde{x}_i), i = 1, \cdots, m$$

$$\tilde{X}^{(1)} = [\tilde{x}_1^{(1)}, \cdots, \tilde{x}_m^{(1)}] = (I_n - u_1 u_1^T)\tilde{X}$$

- Combine the above equations:

$$\begin{aligned}
\tilde{X}^{(1)} &= \sum_{i=1}^{r} \sigma_i u_i v_i^T - (u_1 u_1^T) \sum_{i=1}^{r} \sigma_i u_i v_i^T \\
&= \sum_{i=1}^{r} \sigma_i u_i v_i^T - \sum_{i=1}^{r} \sigma_i u_1 u_1^T u_i v_i^T \\
&= \sum_{i=1}^{r} \sigma_i u_i v_i^T - \sigma_1 u_1 v_1^T \quad \text{//U is orthogonal} \\
&= \sum_{i=2}^{r} \sigma_i u_i v_i^T
\end{aligned}$$

Principle Component Analysis - Proof

- We have removed the first components, finding z_2 is by

$$\max_{z \in R^n} z^T (\tilde{X}^{(1)} \tilde{X}^{(1)T}) z, \text{ s.t.:} \|z\|_2 = 1$$

$$\tilde{X}^{(1)} = \sum_{i=2}^{r} \sigma_i u_i v_i^T$$

- The result is simply $z_2 = u_2,\ u_2$ is the 2nd column of U_r

- $z_3, \cdots z_m$ can be found by similar deflation.

PCA - Summary

- **Given $x_i \in \mathbb{R}^n, i = 1,2,\cdots m$, perform PCA by:**

1. Normalized by the center

$$\tilde{X} = [\tilde{x}_1, \cdots, \tilde{x}_m], \tilde{x}_i = x_i - \bar{x}, i = 1, \cdots, m \qquad \bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i.$$

2. Compute SVD $H = \tilde{X}\tilde{X}^T = U_r \Sigma^2 U_r^T$
3. The principle vectors are the columns of U_r
   (Eigenvector of X = Eigenvector of H)

PCA – Dimensionality Reduction

Given $x_i \in \mathbb{R}^n, i = 1,2,\cdots m$**,** perform PCA to get l principle components $\{z_1, z_2, \cdots, z_l\}, z_j \in \mathbb{R}^n$

- Compress x_i from n dimension to l dimension, with $l \ll n$
- Encoder

$$\begin{bmatrix} a_{i1} \\ \vdots \\ a_{il} \end{bmatrix} = \begin{bmatrix} z_1^T \\ \vdots \\ z_l^T \end{bmatrix} x_i$$

- Reconstruct x_i from the principle components
- Decoder $\hat{x}_i = \sum_{j=1}^{l} a_j z_j = [z_1, \cdots, z_l] \begin{bmatrix} a_{i1} \\ \vdots \\ a_{il} \end{bmatrix}$

PCA – Dimensionality Reduction

Point cloud is projected
into two principle axis {1, 2}

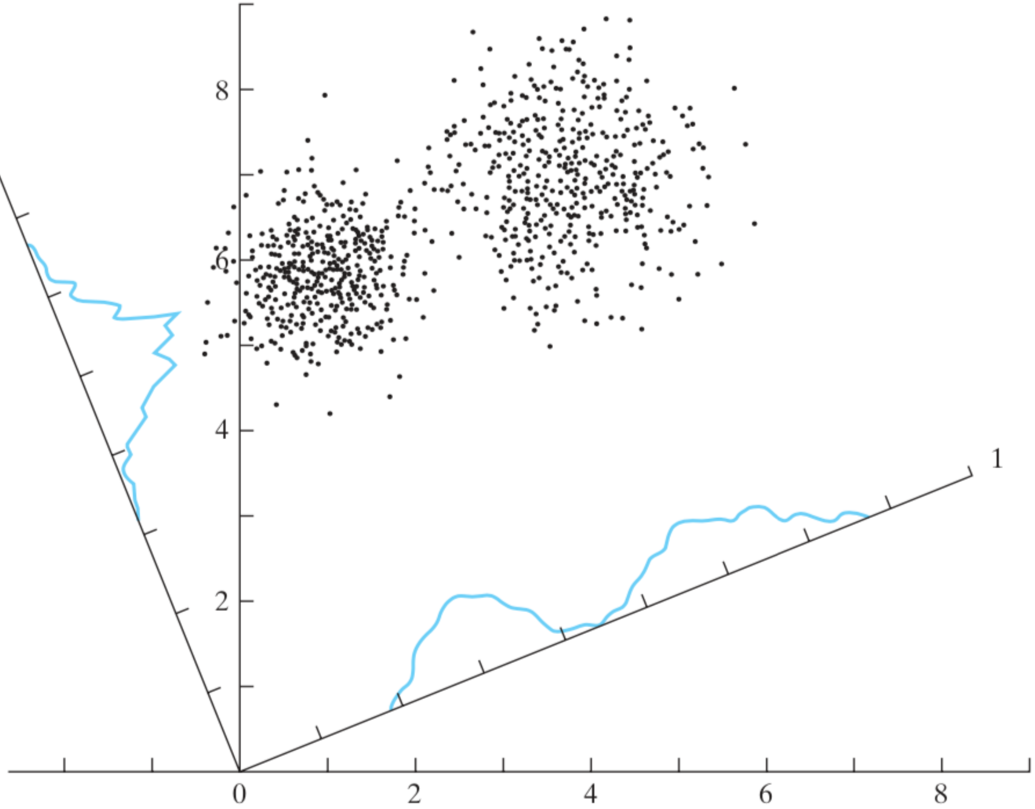


Image source: Neural Networks and Learning Machines, Simon Haykin

PCA – Dimensionality Reduction

- Represent a $H \times W$ binary/gray-scale image by a vector $x_i \in \mathbb{R}^n, n = HW$
- Get the principle vectors $\{z_1, \cdots, z_l\}, z_j \in \mathbb{R}^n$
- Digit recognition by clustering over the principle components $a_i = [a_1, \cdots a_l]^T \in \mathbb{R}^l$
- Similarly, face recognition by Eigenfaces

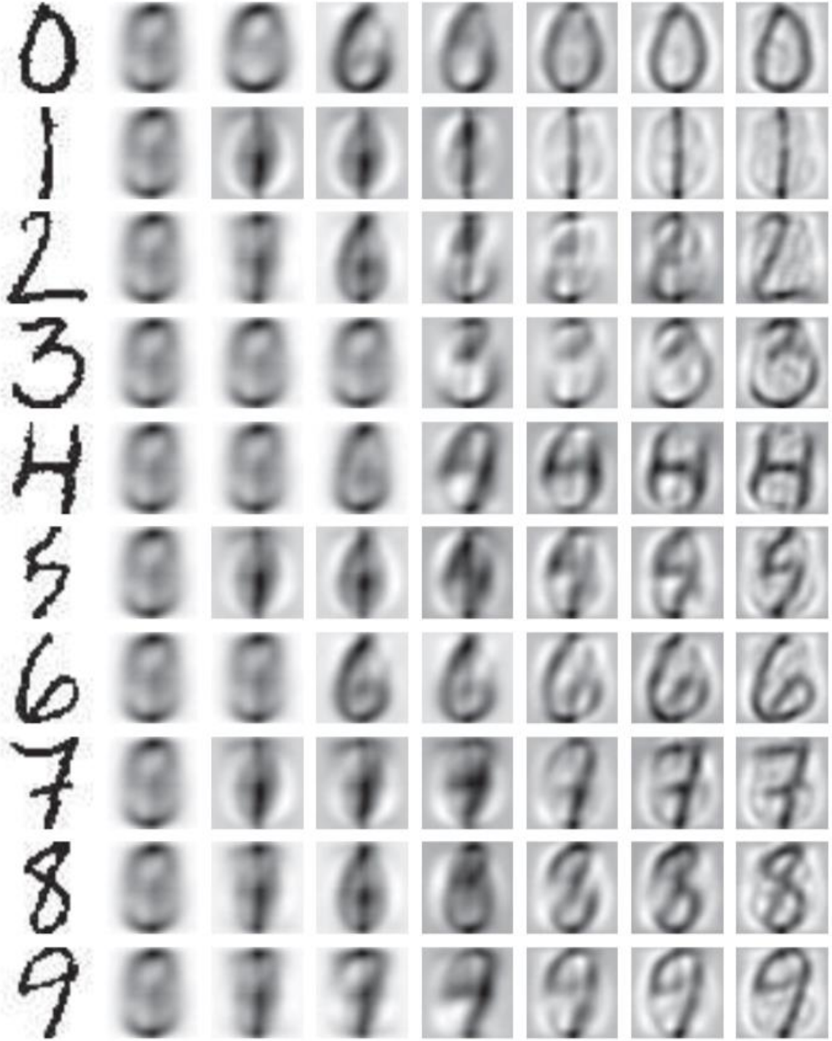
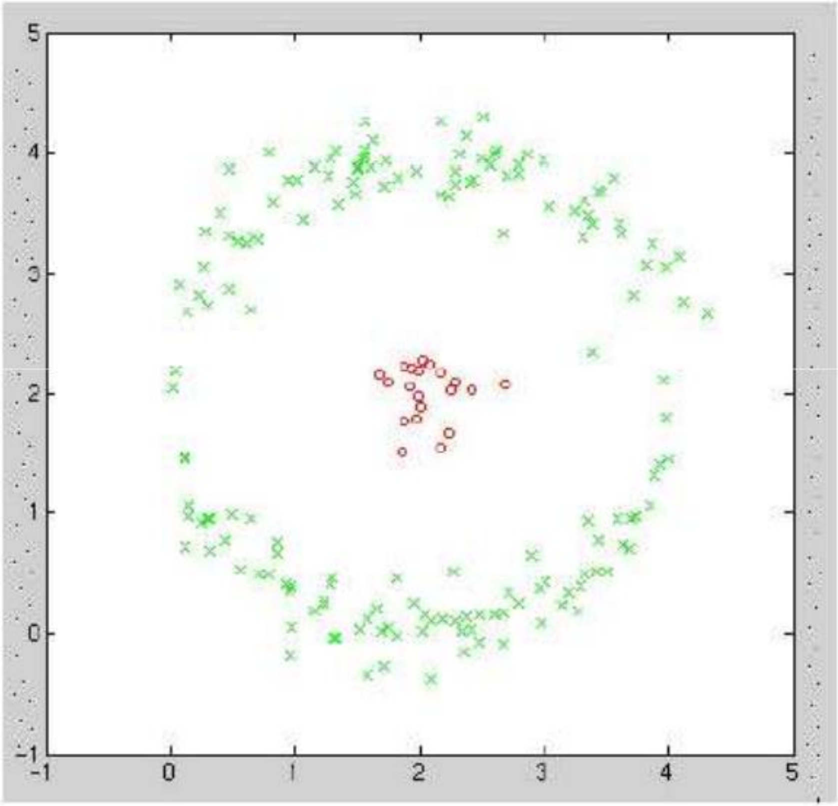


Image source: Neural Networks and Learning Machines, Simon Haykin

Kernel PCA

- PCA is linear
- How to handle data not linearly separable?
- Lift it to high dimension!



Kernel PCA

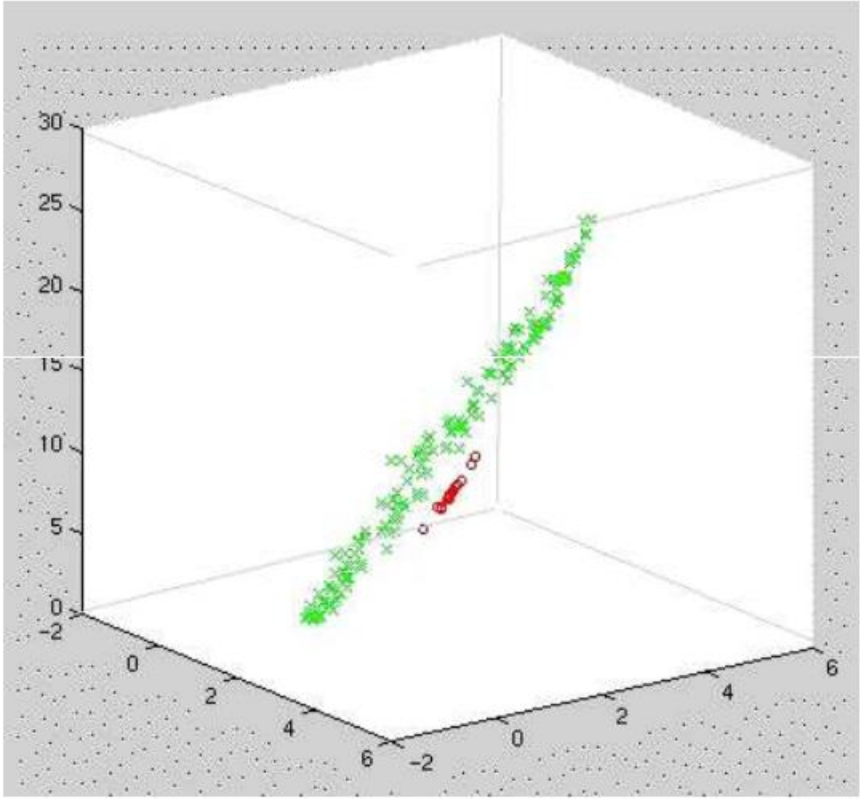
- Original data

$$x_i = [x_{i1}, x_{i2}] \in \mathbb{R}^2$$

- Lifted data

$$\phi(x_i) = [x_{i1}, x_{i2}, x_{i1}^2 + x_{i2}^2] \in \mathbb{R}^3$$

- They are separable now.
  - E.g., some principle component of $\phi(x_i)$ is able to tell the difference between the red and green



Kernel PCA

- Input data $x_i \in \mathbb{R}^{n_0}$, non-linear mapping $\phi: \mathbb{R}^{n_0} \to \mathbb{R}^{n_1}$
- Follow the standard Linear PCA on the lifted space $\mathbb{R}^{n_1}$
    1. Assume $\phi(x_i)$ is already zero-center
    $$\frac{1}{N}\sum_{i=1}^{N}\phi(x_i) = 0$$
    2. Compute correlation matrix
    $$\tilde{H} = \frac{1}{N}\sum_{i=1}^{N}\phi(x_i)\phi^T(x_i)$$
    3. Solve the eigenvectors/eigenvalues by $\tilde{H}\tilde{z} = \tilde{\lambda}\tilde{z}$
- Problem solved? No fully.
    - How to define ϕ?
    - Can we avoid working with the high dimension data?

Kernel PCA

- Note that eigenvectors can be expressed as linear combination of features

$$\tilde{z} = \sum_{j=1}^{N} \alpha_j \phi(x_j)$$

- Proof:

$$\tilde{H}\tilde{z} = \tilde{\lambda}\tilde{z}$$

$$\frac{1}{N}\sum_{i=1}^{N} \phi(x_i)\underbrace{\phi^T(x_i)\tilde{z}}_{\text{scalar}} = \tilde{\lambda}\tilde{z}$$

- Find the eigenvector $\tilde{z}$ = find the coefficient α_j

Kernel PCA

- Put that linear combination into $\tilde{H}\tilde{z} = \tilde{\lambda}\tilde{z}$

$$\frac{1}{N}\sum_{i=1}^{N}\phi(x_i)\phi^T(x_i)\left(\sum_{j=1}^{N}\alpha_j\phi(x_j)\right) = \tilde{\lambda}\sum_{j=1}^{N}\alpha_j\phi(x_j)$$

$$\frac{1}{N}\sum_{i=1}^{N}\phi(x_i)\left(\sum_{j=1}^{N}\alpha_j\phi^T(x_i)\phi(x_j)\right) = \tilde{\lambda}\sum_{j=1}^{N}\alpha_j\phi(x_j)$$

- Let's define kernel function $k(x_i, x_j) = \phi^T(x_i)\phi(x_j)$

$$\frac{1}{N}\sum_{i=1}^{N}\phi(x_i)\left(\sum_{j=1}^{N}\alpha_j k(x_i, x_j)\right) = \tilde{\lambda}\sum_{j=1}^{N}\alpha_j\phi(x_j)$$

- Multiply both sides by $\phi(x_k), k = 1, \cdots, N$,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_j k(x_k, x_i)k(x_i, x_j) = N\tilde{\lambda}\sum_{j=1}^{N}\alpha_j k(x_k, x_j), k = 1, \cdots, N$$

Kernel PCA

- Now define the Gram matrix $K \in \mathbb{R}^{N \times N}, K(i,j) = k(x_i, x_j)$
  - K is symmetric because $k\left(x_i, x_j\right) = k(x_j, x_i)$
- The above equation can be written as

$$K^2\alpha = N\tilde{\lambda}K\alpha$$

- Remove K on both sides

$$K\alpha = N\tilde{\lambda}\alpha$$
$$K\alpha = \lambda\alpha$$

- Again, get the eigenvectors α_r and eigenvalues λ_r, $r = 1, \cdots, l$
- However, we have to ensure that $\tilde{z}$ is unit vector
  Note that we are solving the linear PCA in the feature space

$$\tilde{H}\tilde{z} = \tilde{\lambda}\tilde{z} \qquad \tilde{z} = \sum_{j=1}^{N} \alpha_j \phi(x_j)$$

Kernel PCA

- The normalization of $\tilde{z}$ leads to

$$1 = \tilde{z}_r^T \tilde{z}_r$$

$$1 = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ri} \alpha_{rj} \phi^T(x_i) \phi(x_j)$$

$$1 = \alpha_r^T K \alpha_r$$

- Note that $K\alpha = \lambda\alpha$, we have $\alpha_r^T \lambda_r \alpha_r = 1, \forall r$
- That is, normalize α_r to be norm $1/\lambda_r$
- Now, the r^{th} principle vector in the lifted space is given below, which is unknown

$$\tilde{z}_r = \sum_{j=1}^{N} \alpha_{rj} \phi(x_j)$$

Kernel PCA

- Now, the r^{th} principle vector in the lifted space is given below

$$\tilde{z}_r = \sum_{j=1}^{N} \alpha_{rj}\phi(x_j)$$

- But we know the projection of data point x projected into principle component z_r

$$\phi^T(x)\tilde{z}_r = \sum_{j=1}^{N} \alpha_{rj}\phi^T(x)\phi(x_j) = \sum_{j=1}^{N} \alpha_{rj}k(x, x_j)$$

- One more thing, we assume $\phi(x_i)$ is of zero mean.

Kernel PCA

- Normalize $\phi(x_i)$ to be zero mean

$$\tilde{\phi}(x_i) = \phi(x_i) - \frac{1}{N}\sum_{j=1}^{N}\phi(x_j)$$

- The normalized kernel $\tilde{k}(x_i, x_j)$ is given by

$$\begin{aligned}
\tilde{k}(x_i, x_j) &= \tilde{\phi}^T(x_i)\tilde{\phi}(x_j) \\
&= \left(\phi(x_i) - \frac{1}{N}\sum_{k=1}^{N}\phi(x_k)\right)^T \left(\phi(x_j) - \frac{1}{N}\sum_{l=1}^{N}\phi(x_l)\right) \\
&= k(x_i, x_j) - \frac{1}{N}\sum_{k=1}^{N}k(x_i, x_k) - \frac{1}{N}\sum_{k=1}^{N}k(x_j, x_k) + \frac{1}{N^2}\sum_{k=1}^{N}\sum_{l=1}^{N}k(x_k, x_l)
\end{aligned}$$

- In the matrix form $\widetilde{K} = K - 2\mathbb{1}_{\frac{1}{N}}K + \mathbb{1}_{\frac{1}{N}}K\,\mathbb{1}_{\frac{1}{N}}$, where $\mathbb{1}_{\frac{1}{N}}(i,j) = \frac{1}{N}, \forall i, j$

Kernel PCA

- Kernel choices
  - Linear $k(x_i, x_j) = x_i^T x_j$
  - Polynomial $k(x_i, x_j) = (1 + x_i^T x_j)^p$
  - Gaussian $k(x_i, x_j) = e^{-\beta \|x_i - x_j\|_2}$
  - Laplacian $k(x_i, x_j) = e^{-\beta \|x_i - x_j\|_1}$

- Usually choose by experiments if there is no explicit knowledge what kernels best separate the data points.

Kernel PCA - Summary

- Select a kernel $k(x_i, x_j)$, compute the Gram matrix $K(i,j) = k(x_i, x_j)$
- Normalize K

$$\widetilde{K} = K - 2\mathbb{1}_{\frac{1}{N}}K + \mathbb{1}_{\frac{1}{N}}K\,\mathbb{1}_{\frac{1}{N}}$$

- Solve the eigenvector/eigenvalues of $\widetilde{K}$

$$\widetilde{K}\alpha_r = \lambda_r \alpha_r$$

- Normalize α_r to be $\alpha_r^T \alpha_r = \frac{1}{\lambda_r}$
- For any data point $x \in \mathbb{R}^n$, compute its projection onto r^{th} principle component $y_r \in \mathbb{R}$

$$y_r = \phi^T(x)\tilde{z}_r = \sum_{j=1}^{N} \alpha_{rj} k(x, x_j)$$

Kernel PCA - Example

Input data is not separable by linear PCA

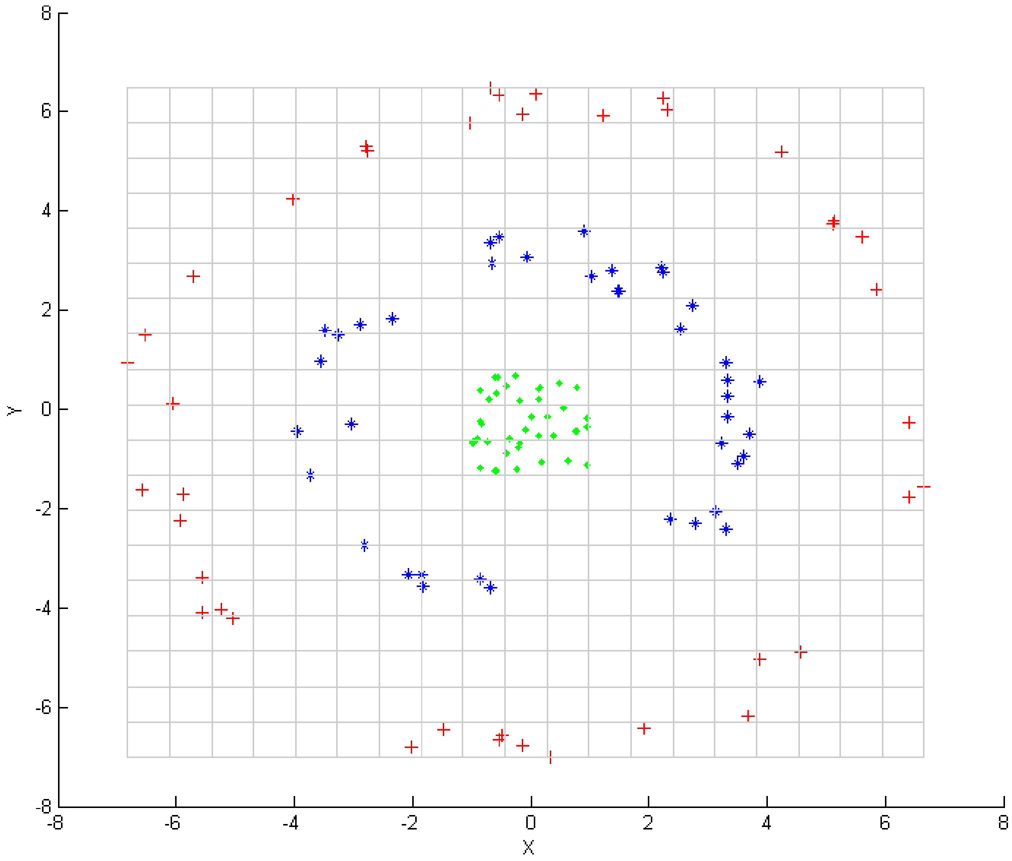


Image source: https://en.wikipedia.org/wiki/Kernel_principal_component_analysis

Kernel PCA - Example

- Projection into 1st and 2nd principle components

- kPCA polynomial kernel
$$k(x_i, x_j) = (1 + x_i^T x_j)^2$$

- Points can be separated by the first projection y_0

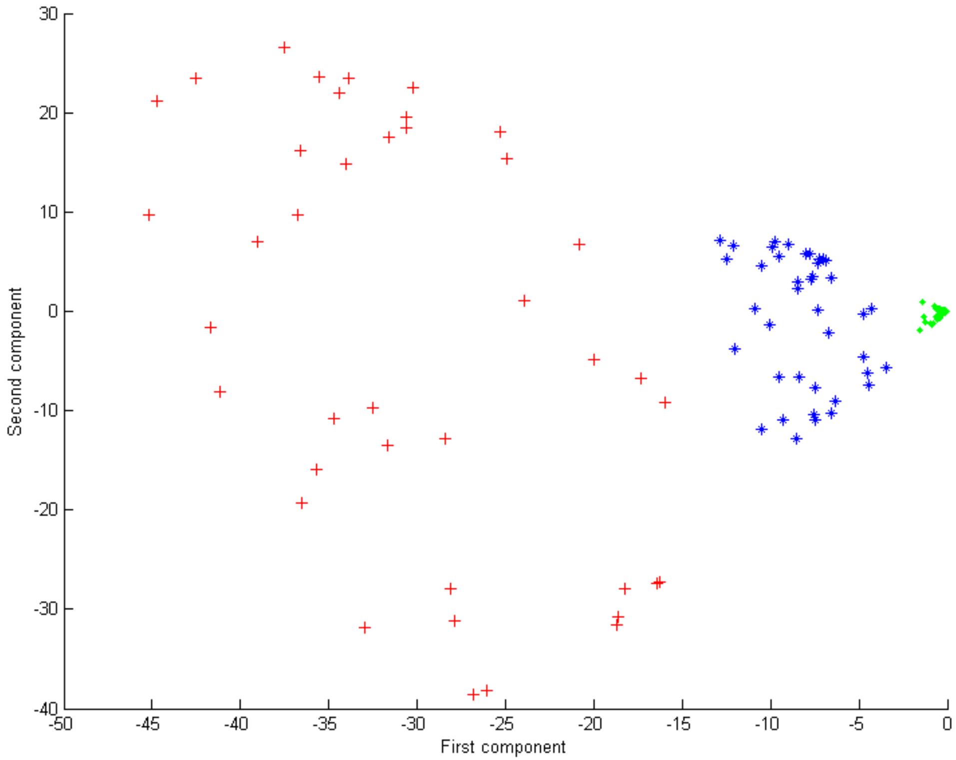


Image source: https://en.wikipedia.org/wiki/Kernel_principal_component_analysis

Kernel PCA - Example

- Projection into 1st and 2nd principle components

- kPCA Gaussian kernel
  $$k(x_i, x_j) = e^{-\beta\|x_i - x_j\|_2}$$

- Points can be separated by the first projection y_0

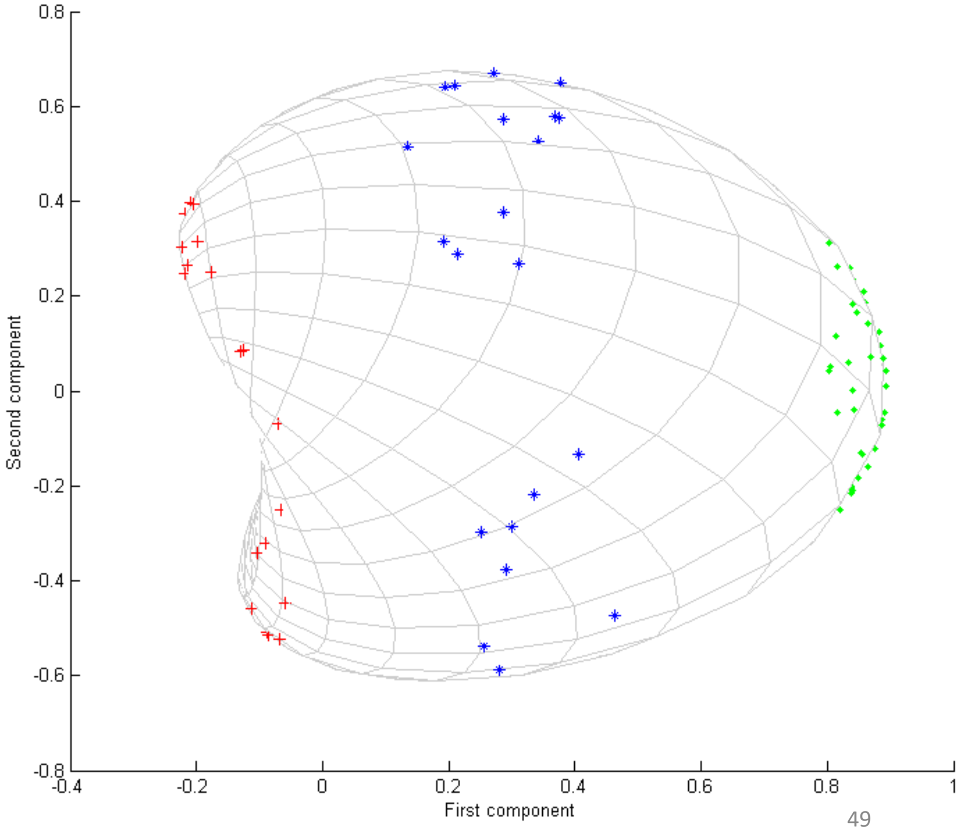
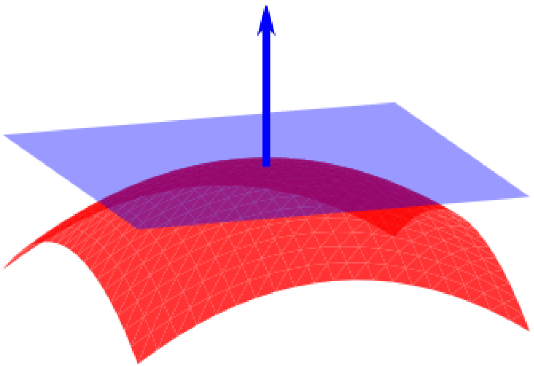


Image source: https://en.wikipedia.org/wiki/Kernel_principal_component_analysis

Surface Normal

Surface normal on surface

- The vector perpendicular to the tangent plane of the surface at a point P



Applications

- Segmentation / Clustering
- Plane detection
- Point cloud feature for applications like Deep Learning

Surface Normal – How to compute

Surface normal on 3D point cloud

1. Select a point P
2. Find the neighborhood that defines the surface
3. PCA
4. Normal -> the least significant vector
5. Curvature -> ratio between eigen values $\lambda_3/(\lambda_1 + \lambda_2 + \lambda_3)$

Intuitively it is obvious, can we prove it formally?

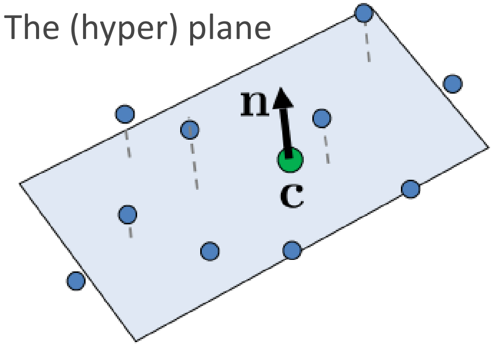
Surface Normal Estimation – Definition

Problem Definition

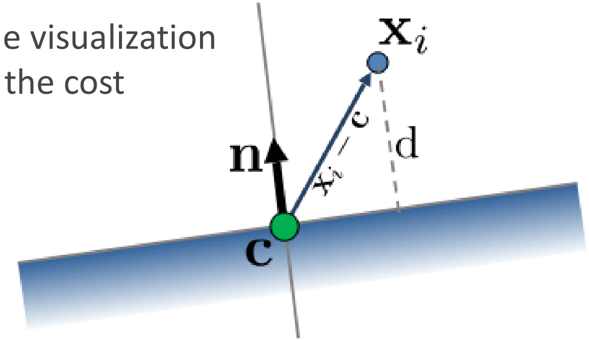
Denote data points as $x_i \in R^n, i = 1, 2, \cdots, m$, find a (hyper) plane, that passes through **a point c** with **normal vector n**, s.t.

$$\min_{\mathbf{c}, \mathbf{n}, \|\mathbf{n}\|=1} \sum_{i=1}^{m} ((\mathbf{x}_i - \mathbf{c})^T \mathbf{n})^2$$

The (hyper) plane



The visualization of the cost



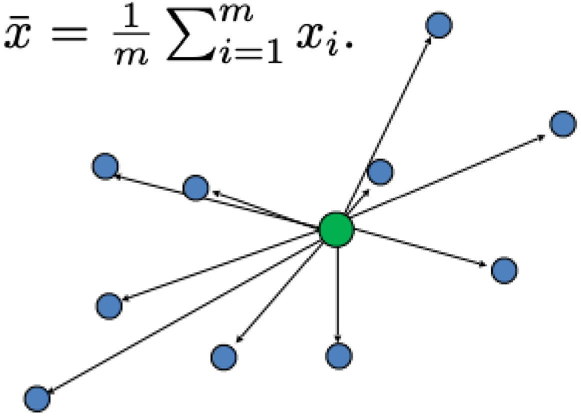
Surface Normal Estimation – Proof

- Since c and n are independent variables, let's look at c first

$$
\begin{aligned}
\mathbf{c}^* &= \arg\min_{\mathbf{c}} \sum_{i=1}^{m} ((\mathbf{x}_i - \mathbf{c})^T \mathbf{n})^2 \\
&= \arg\min_{\{c_j\}} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - c_j)^2 n_j^2 \\
&= \arg\min_{\{c_j\}} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij} - c_j)^2 \\
&= \arg\min_{\mathbf{c}} \sum_{i=1}^{m} \|\mathbf{x_i} - \mathbf{c}\|^2
\end{aligned}
$$

- That means c* is the center of the data points

$$\bar{x} = \frac{1}{m} \sum_{i=1}^{m} x_i .$$



Surface Normal Estimation – Proof

- So we normalize the data points by its center, similar to what we did in PCA proof.

$$\tilde{X} = [\tilde{x}_1, \cdots, \tilde{x}_m], \tilde{x}_i = x_i - \bar{x}, i = 1, \cdots, m$$

- Now the problem becomes,

$$\min_{n \in R^n} \sum_{i=1}^{m} (\tilde{x}_i^T n)^2, \text{s.t.: } \|n\|_2 = 1$$

$$\min_{n \in R^n} \sum_{i=1}^{m} n^T \tilde{x}_i \tilde{x}_i^T n, \text{s.t.: } \|n\|_2 = 1$$

$$\min_{n \in R^n} n^T \left(\sum_{i=1}^{m} \tilde{x}_i \tilde{x}_i^T \right) n, \text{s.t.: } \|n\|_2 = 1$$

$$\min_{n \in R^n} n^T \tilde{X} \tilde{X}^T n, \text{s.t.: } \|n\|_2 = 1$$

PCA:

$$\boxed{\max_{z \in R^n} z^T (\tilde{X} \tilde{X}^T) z, \text{s.t.:} \|z\|_2 = 1}$$

Surface Normal Estimation

What shall we do when there are noise?

1. Select neighbors according to problem
   E.g. Radius based neighbors
   a. Radius larger -> normal estimation is smoother, but affected by irrelevant objects
   b. Radius smaller -> normal estimation is sharper, but noisy

2. Weighted based on other features
   a. Lidar intensity
   b. RGB values

3. RANSAC
   a. Lecture 4

4. Deep Learning!

Surface Normal Estimation

Weighted normal estimation

$$\min_{n \in R^n} \sum_{i=1}^{m} w_i (\tilde{x}_i^T n)^2, \text{ s.t.: } \|n\|_2 = 1$$

$$\min_{n \in R^n} \sum_{i=1}^{m} w_i n^T \tilde{x}_i \tilde{x}_i^T n, \text{ s.t.: } \|n\|_2 = 1$$

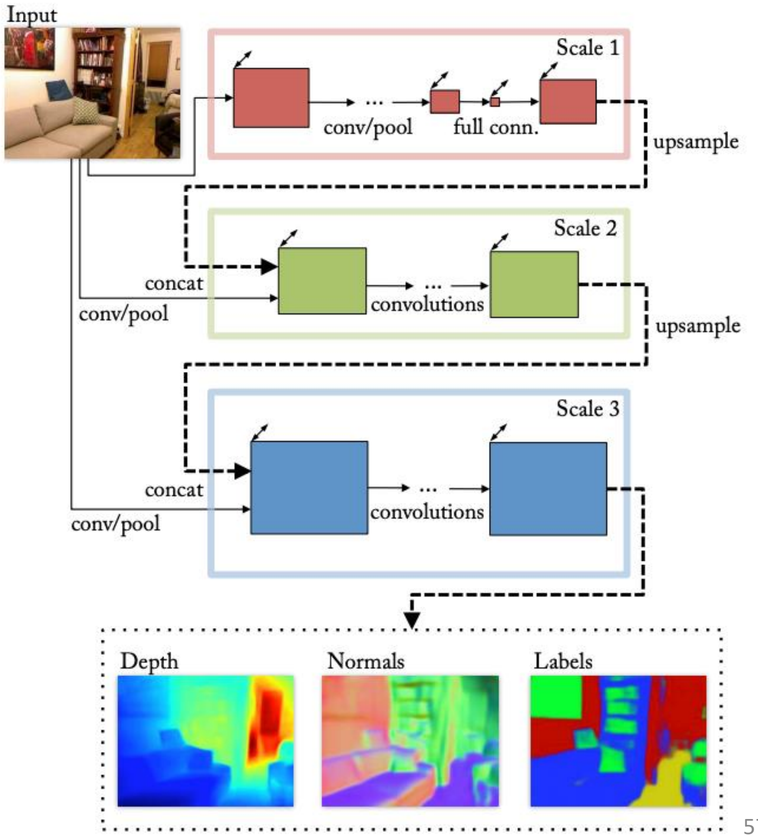
$$\min_{n \in R^n} n^T \left(\sum_{i=1}^{m} w_i \tilde{x}_i \tilde{x}_i^T \right) n, \text{ s.t.: } \|n\|_2 = 1$$

$$\min_{n \in R^n} n^T \tilde{X} W \tilde{X}^T n, \text{ s.t.: } \|n\|_2 = 1$$

$$W = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_m \end{bmatrix}$$

Deep Learning about Surface Normal

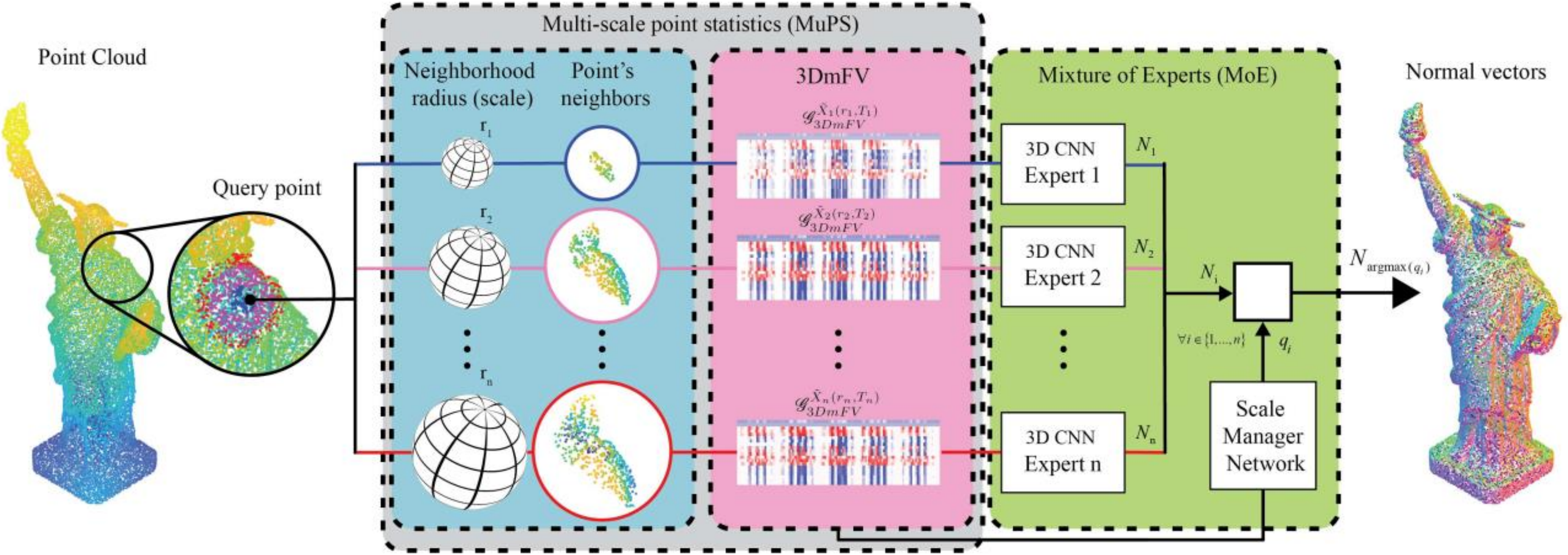
- **Predicting Depth, Surface Normals and Semantic Labels with a Common Multi-Scale Convolutional Architecture**
  - ICCV 2015
  - Joint estimation of depth and surface normal improves the depth result



Deep Learning about Surface Normal

Nesti-Net: Normal Estimation for Unstructured 3D Point Clouds using Convolutional Neural Networks

- CVPR 2019



Filters

Noise removal
- Radius Outlier Removal
- Statistical Outlier Removal

Downsampling
- Voxel Grid Downsampling
  - Exact / Approximated
  - Centroid / Random Selection
- Farthest Point Sampling
- Normal Space Sampling

Upsampling / Smoothing / Noise Removal
- Bilateral Filter

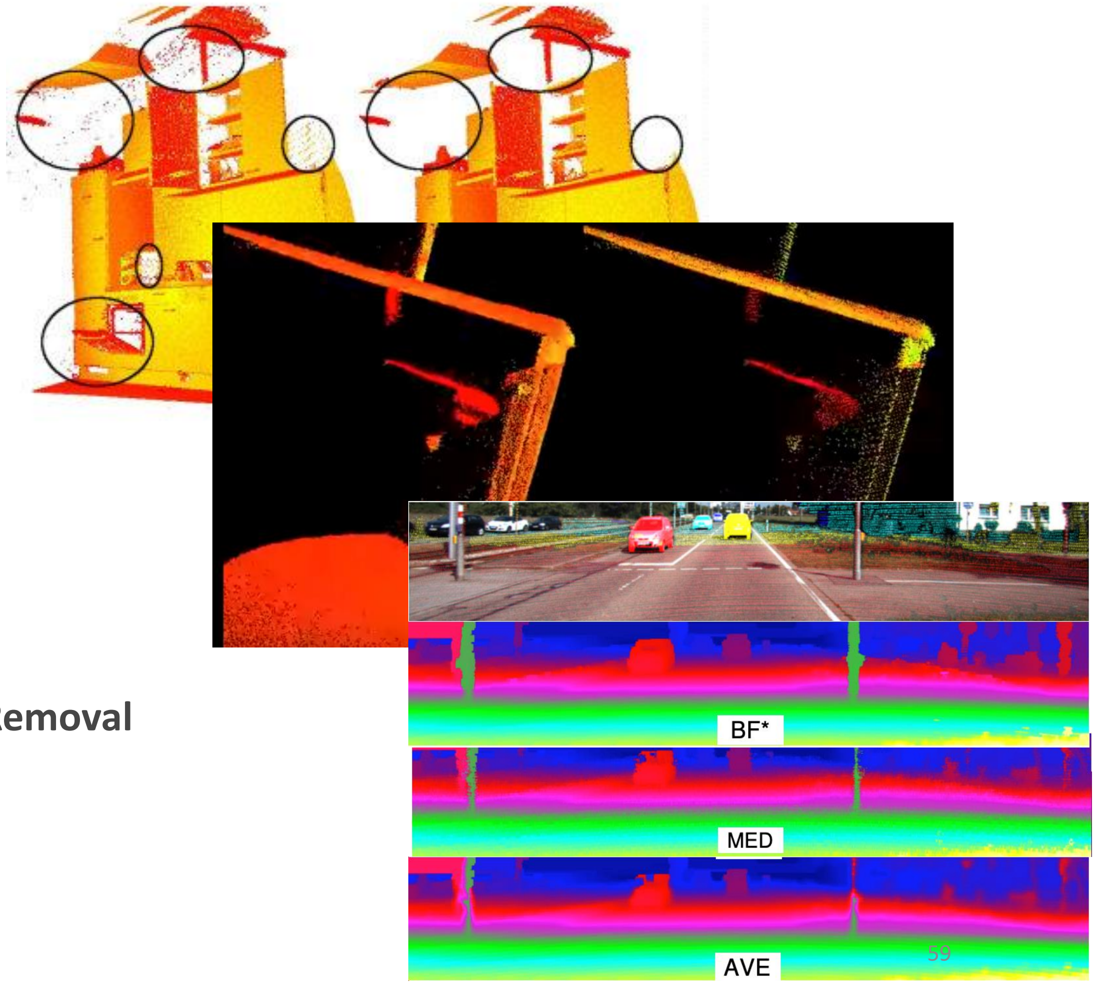


Image source: http://pointclouds.org/

Radius Outlier Removal

1. For each point, find a radius = r neighborhood
2. If number of neighbor k < k*, remove the point

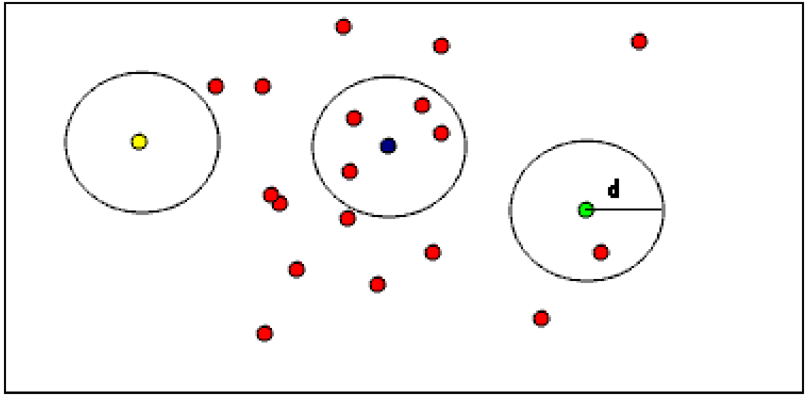


Image source: http://pointclouds.org/

Statistical Outlier Removal

1. For each point, find a neighborhood
2. Compute its distance to its neighbors $d_{ij}, i = [1, \dots, m], j = [1, \dots, k]$
3. Model the distances by Gaussian distribution $d \sim N(\mu, \sigma)$

$$\mu = \frac{1}{nk} \sum_{i=1}^{m} \sum_{j=1}^{k} d_{ij}, \ \sigma = \sqrt{\frac{1}{nk} \sum_{i=1}^{m} \sum_{j=1}^{k} (d_{ij} - \mu)^2}$$

4. For each point, compute its mean distance to its neighbors
5. Remove the point, if the mean distance is outside some confidence according to the Gaussian distribution
   E.g. Remove if

$$\sum_{j=1}^{k} d_{ij} > \mu + 3\sigma \text{ or } \sum_{j=1}^{k} d_{ij} < \mu - 3\sigma$$

Statistical Outlier Removal

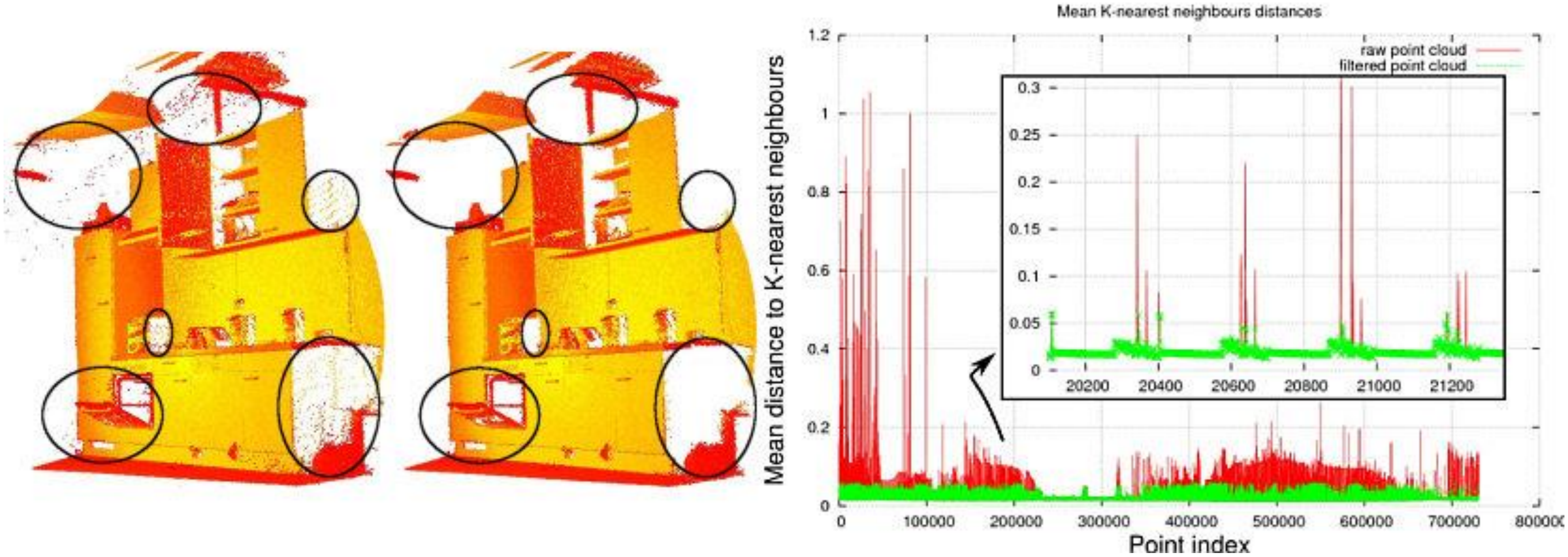


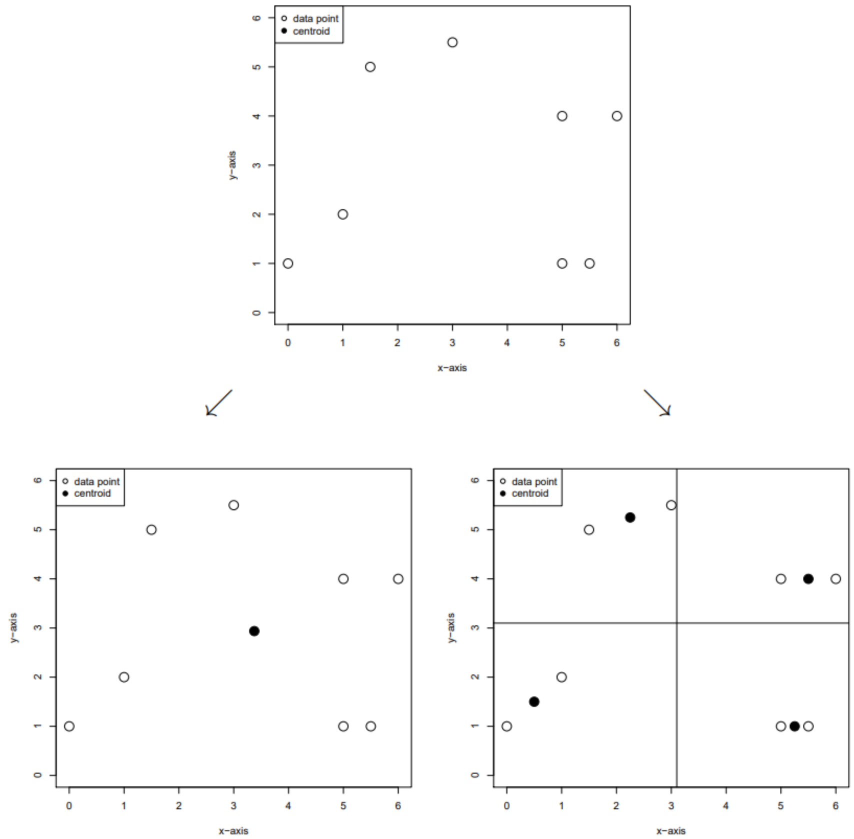
Image source: http://pointclouds.org/

Voxel Grid Downsampling

1. Build a voxel grid that contains the point cloud
2. Take one point in each cell

Q1, how to “take one point”?

Q2, how to make it efficient?



Voxel Grid Downsampling

How to “take one point” from a cell in the grid?

1. Centroid
   a. For coordinates, compute the average in the cell
   b. For other attributes, voting / average
   c. More accurate but slower

2. Random select
   a. Randomly select a point in the cell
   b. Less accurate but faster

Voxel Grid Downsampling - Exact

1. Compute the min or max of the point set $\{p_1, p_2, \cdots p_N\}$

$$x_{max} = \max(x_1, x_2, \cdots, x_N), x_{min} = \min(x_1, x_2, \cdots, x_N), y_{max} = \cdots\cdots$$

2. Determine the voxel grid size r
3. Compute the dimension of the voxel grid

$$D_x = (x_{max} - x_{min})/r$$
$$D_y = (y_{max} - y_{min})/r$$
$$D_z = (z_{max} - z_{min})/r$$

4. Compute voxel index for each point

$$h_x = \lfloor (x - x_{min})/r \rfloor$$
$$h_y = \lfloor (y - y_{min})/r \rfloor$$
$$h_z = \lfloor (z - z_{min})/r \rfloor$$
$$h = h_x + h_y * D_x + h_z * D_x * D_y$$

5. Sort the points according to the index in Step 4
6. Iterate the sorted points, select points according to Centroid / Random method
   0, 0, 0, 0, 3, 3, 3, 8, 8, 8, 8, 8, 8, 8, 8, ……

Voxel Grid Downsampling - Exact

Int32 overflow!

- Example, 3D lidar in autonomous driving. Detection range 200m, voxel grid resolution r=0.05m, assume we crop z to be [-10, 10]
- Dimension of the voxel grid: (20/0.05) * (400/0.05) * (400/0.05)=2.56×10^{10}
- 2^32 = $4.3 \times 10^{9} < 2.56 \times 10^{10}$

Voxel Grid Downsampling - Exact

Strict Weak Ordering!

- In cpp, the sort function in <algorithm> supports customized comparator
- However, the comparator should follow the strict weak ordering:

Expression	Return type	Requirements
`comp(a, b)`	implicitly convertible to `bool`	Establishes strict weak ordering relation with the following properties • For all a, `comp(a,a)==false` • If `comp(a,b)==true` then `comp(b,a)==false` • if `comp(a,b)==true` and `comp(b,c)==true` then `comp(a,c)==true`

- In the voxel grid downsampling setting, the sorting comparator should be *a.index < b.index*, instead of *a.index <= b.index*
- Otherwise this is undefined behavior that may lead to segmentation fault

Voxel Grid Downsampling – Approximated

- Exact voxel grid downsampling requires sorting O(N*log(N))
- However, in most cases, the voxel is SPARSE
- Image we have N=10000 points, we know after downsamling the number M < 100. (E.g, 95)
- Can we have a magic function, that maps the 10000 points into the 100 containers?
- Finally we just extract one point from the 100 containers. Ideally there will be 95 non-empty containers, and 5 empty.

Hash Table!

Voxel Grid Downsampling – Approximated

1. Compute the min / max of each coordinate
2. Determine the voxel grid size r
3. Compute the dimension of the voxel grid
4. Compute voxel index for each point
5. Use a *hash* function to map each point to a container G_i in $\{G_1, G_2, \ldots, G_M\}$
6. Iterate $\{G_1, G_2, \ldots, G_M\}$ and get M point!

- That *hash* function is

$$\text{hash}(h_x, h_y, h_z) : \mathbb{R}^3 \rightarrow \mathbb{R}$$

E.g., $hash(h_x, h_y, h_z) = (h_x + h_y * D_x + h_z * D_x * D_y)\ \%\ container\_size$

Voxel Grid Downsampling – Approximated

The *hash* function is not magic, not perfect!

- Different voxel will map into the same value

$$\text{hash}(h_x, h_y, h_z) = \text{hash}(h'_x, h'_y, h'_z), h_x \neq h'_x \ or \ h_y \neq h'_y \ or \ h_z \neq h'_z$$

- Consequence: The 10000 points should fill in 95 containers, but in fact fill only 80. You are missing 15 points!

This is called *conflict* in hash table

Voxel Grid Downsampling – Approximated

- How to solve *conflict* in hash table?

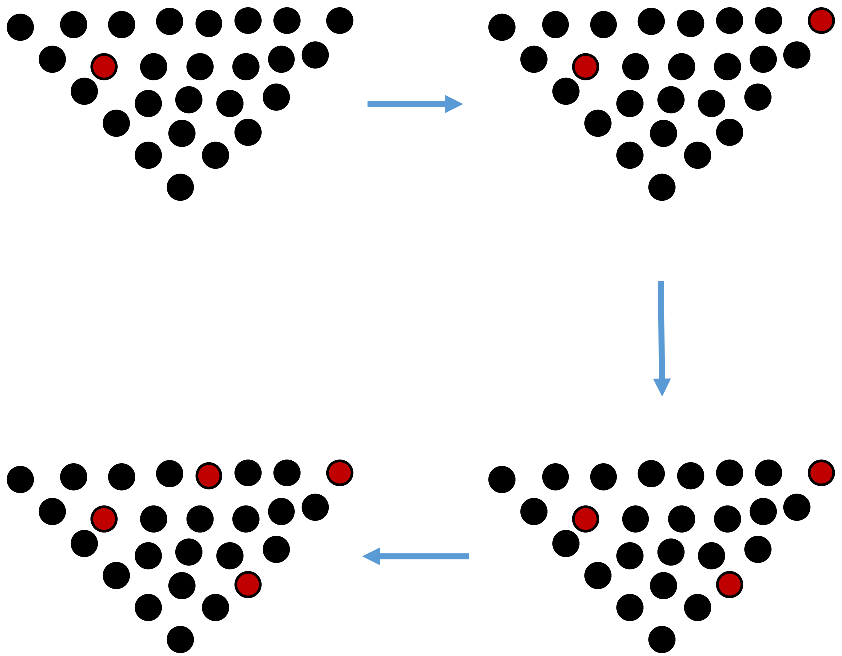
- Detect it!

$$\mathrm{hash}(h_x, h_y, h_z) = \mathrm{hash}(h'_x, h'_y, h'_z), \boxed{h_x \neq h'_x \ or \ h_y \neq h'_y \ or \ h_z \neq h'_z}$$

- If you find a *conflict*, what do you do?
  1. Select a point from the container
  2. Empty that container

Farthest Point Sampling (FPS)

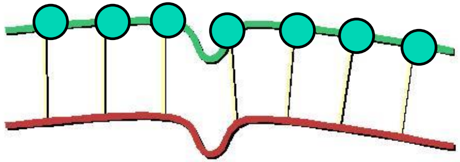
1. Randomly choose a point to be the first FPS point
2. Iterate until we get the desired number of points
   a. For each point in the original point cloud, compute its distance to the nearest FPS point
   b. Choose the point with the largest value, add to FPS set



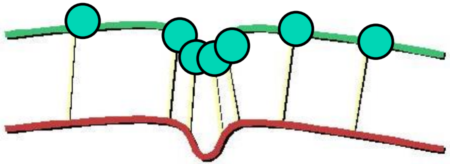
Normal Space Sampling (NSS)

Used in Iterative Closest Point

1. Construct a set of buckets in the normal space
2. Put all points into bucket according to the surface normals
3. Uniformly pick points from all buckets until we have the desired number of points



Uniform Sampling



Normal Space Sampling

Learning to Sample

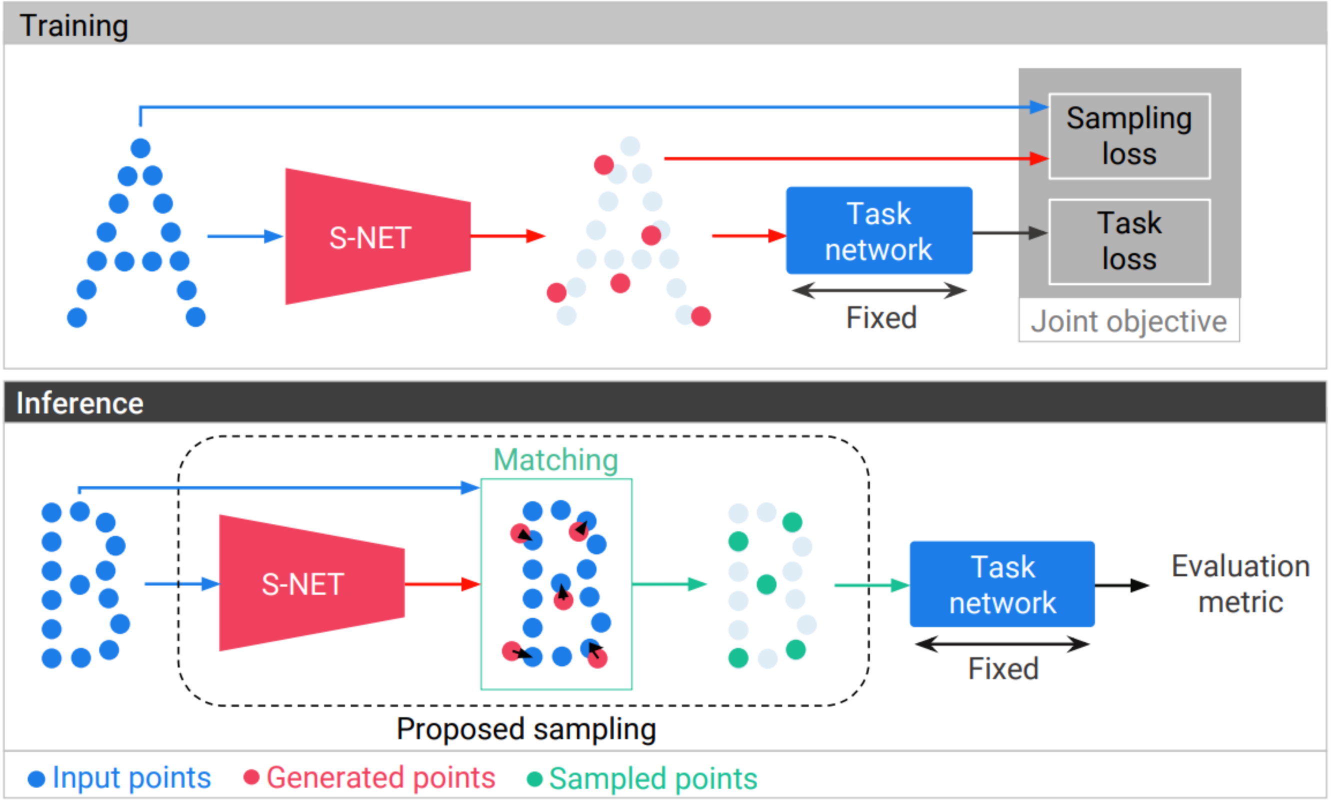


Image source: Learning to Sample, Oren Dovrat, et. al.

Learning to Sample

Problem statement Given a point set $P = \{p_i \in \mathbb{R}^3, i = 1, \ldots, n\}$, a sample size $k \leq n$ and a task network T, find a subset S^* of k points that minimizes the task network's objective function f:

$$S^* = \underset{S}{\operatorname{argmin}}\, f(T(S)), \quad S \subset P, \quad |S| = k \leq n. \quad (1)$$

Sampling based on Semantics

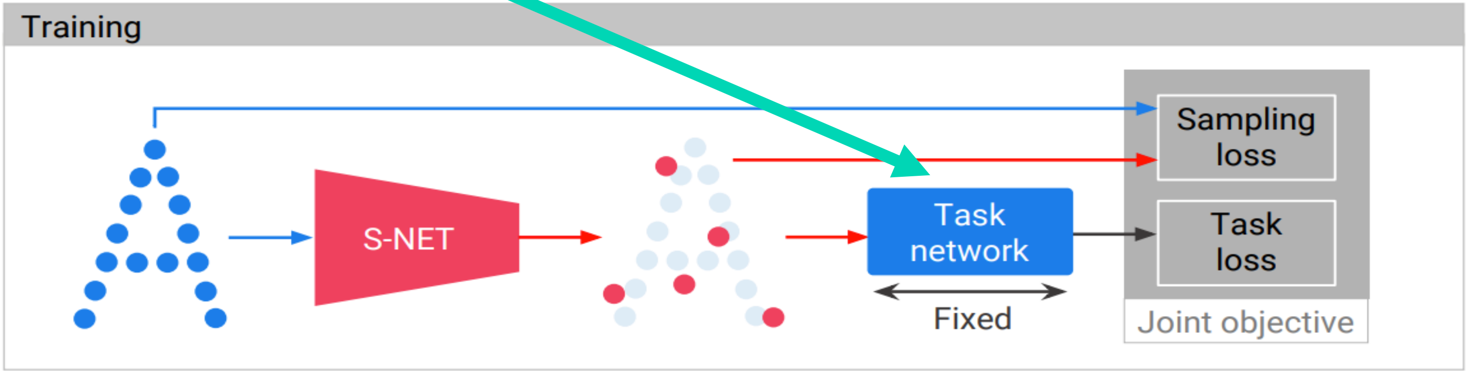


Image source: Learning to Sample, Oren Dovrat, et. al.

Learning to Sample

Sampling with Geometric Constraints

Let us denote the generated point set as G and the input point set as P. We construct a sampling regularization loss, composed out of three terms:

$$L_f(G,P) = \frac{1}{|G|}\sum_{g\in G}\min_{p\in P}||g-p||_2^2 \tag{2}$$

$$L_m(G,P) = \max_{g\in G}\min_{p\in P}||g-p||_2^2 \tag{3}$$

$$L_b(G,P) = \frac{1}{|P|}\sum_{p\in P}\min_{g\in G}||p-g||_2^2. \tag{4}$$

(2) + (4) = Chamfer loss

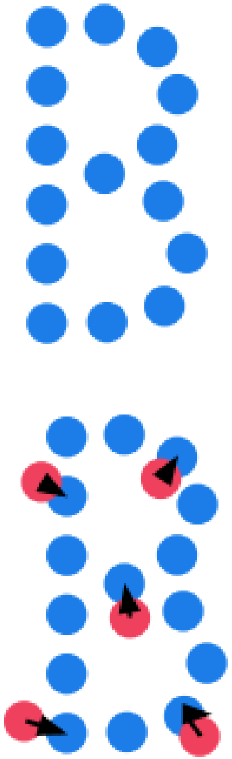
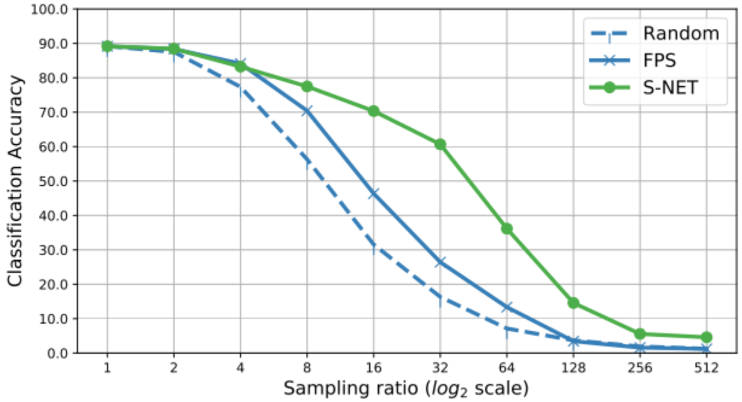


Image source: Learning to Sample, Oren Dovrat, et. al.

Learning to Sample – for classification

- The Learning to Sample is targeted to some specific task, e.g., Classification
- Semantics based downsampling instead of pure geometric based.



#Sampled points	Random	FPS	S-NET
1024	89.2	89.2	89.2
512	88.2	**88.3**	87.8
256	86.6	88.1	**88.3**
128	86.2	87.9	**88.6**
64	81.5	86.1	**87.7**
32	77.0	82.2	**87.3**
16	65.8	76.7	**85.6**
8	45.8	61.6	**83.6**
4	26.9	35.2	**73.4**
2	16.6	18.3	**53.0**

Image source: Learning to Sample, Oren Dovrat, et. al.

Learning to Sample – for reconstruction

- **NRE:**
  Normalized Reconstructed Error

- **The output of S-Net is visually similar to that of FPS**
  - This is expected because of Chamfer loss

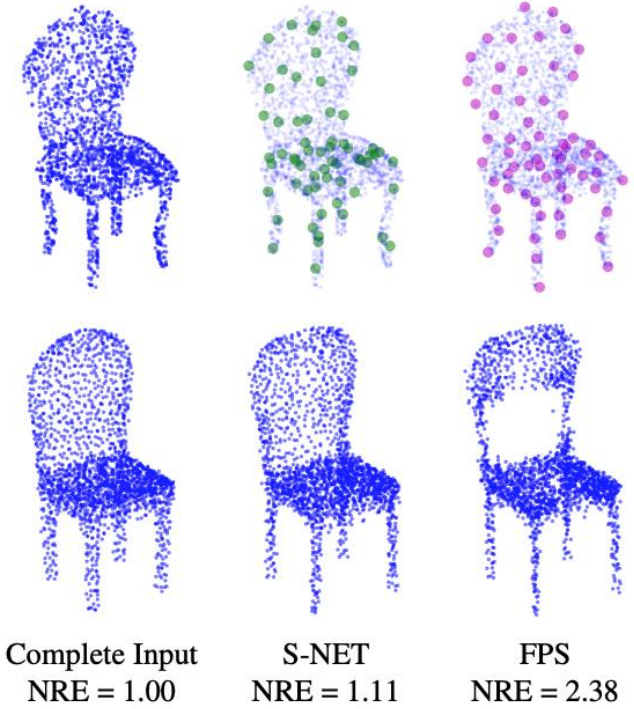
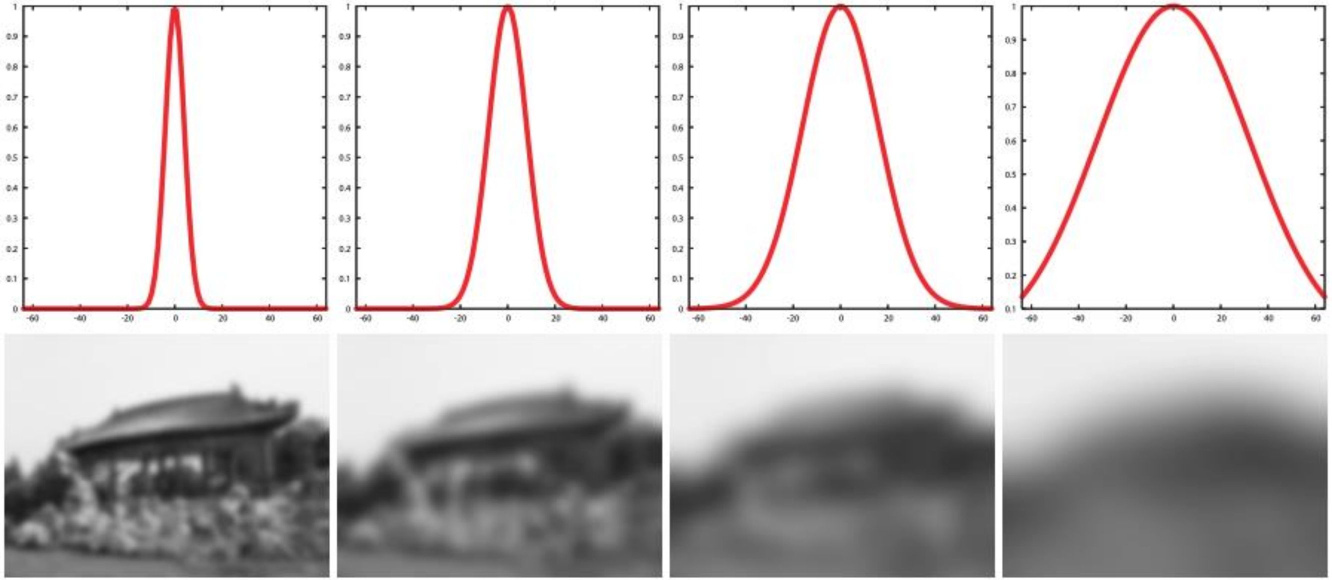


Image source: Learning to Sample, Oren Dovrat, et. al.

Bilateral Filter – Gaussian Filter

$$GB[I]_{\mathbf{p}} = \sum_{\mathbf{q} \in \mathcal{S}} G_\sigma(\|\mathbf{p} - \mathbf{q}\|)\, I_{\mathbf{q}}, \qquad G_\sigma(x) = \frac{1}{2\pi\,\sigma^2} \exp\left(-\frac{x^2}{2\,\sigma^2}\right)$$



$\sigma = 4$ $\qquad$ $\sigma = 8$ $\qquad$ $\sigma = 16$ $\qquad$ $\sigma = 32$

Image source: A Gentle Introduction to Bilateral Filtering and its Applications, Sylvain Paris, et. al.

Edge Preserving Blurring

1 iteration 2 iteration 4 iteration

Image source: A Gentle Introduction to Bilateral Filtering and its Applications, Sylvain Paris, et. al.

Bilateral Filter

- **Given image I, for each pixel p, find its neighbor S.**
- **each pair (p, q)**
  - Compute distance weight G_{σ_s} intensity weight G_{σ_r}

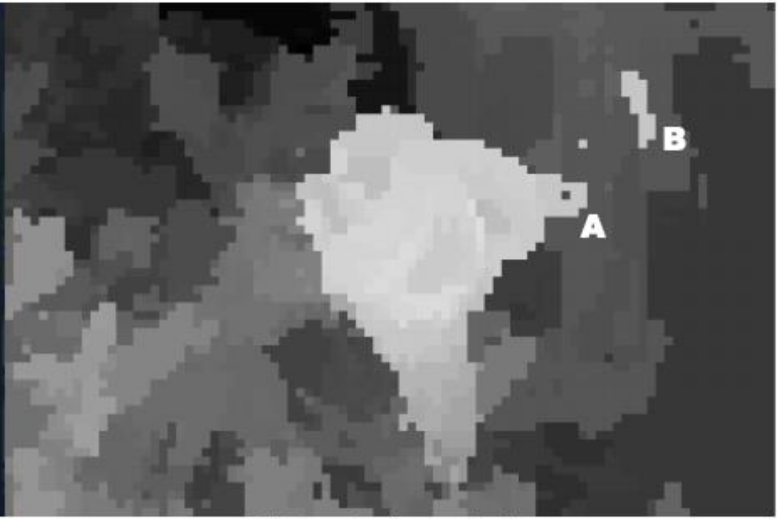
$$G_\sigma(x) = \frac{1}{2\pi\,\sigma^2}\ \exp\left(-\frac{x^2}{2\,\sigma^2}\right)$$

  - Apply Bilateral Filter to get intensity of pixel p

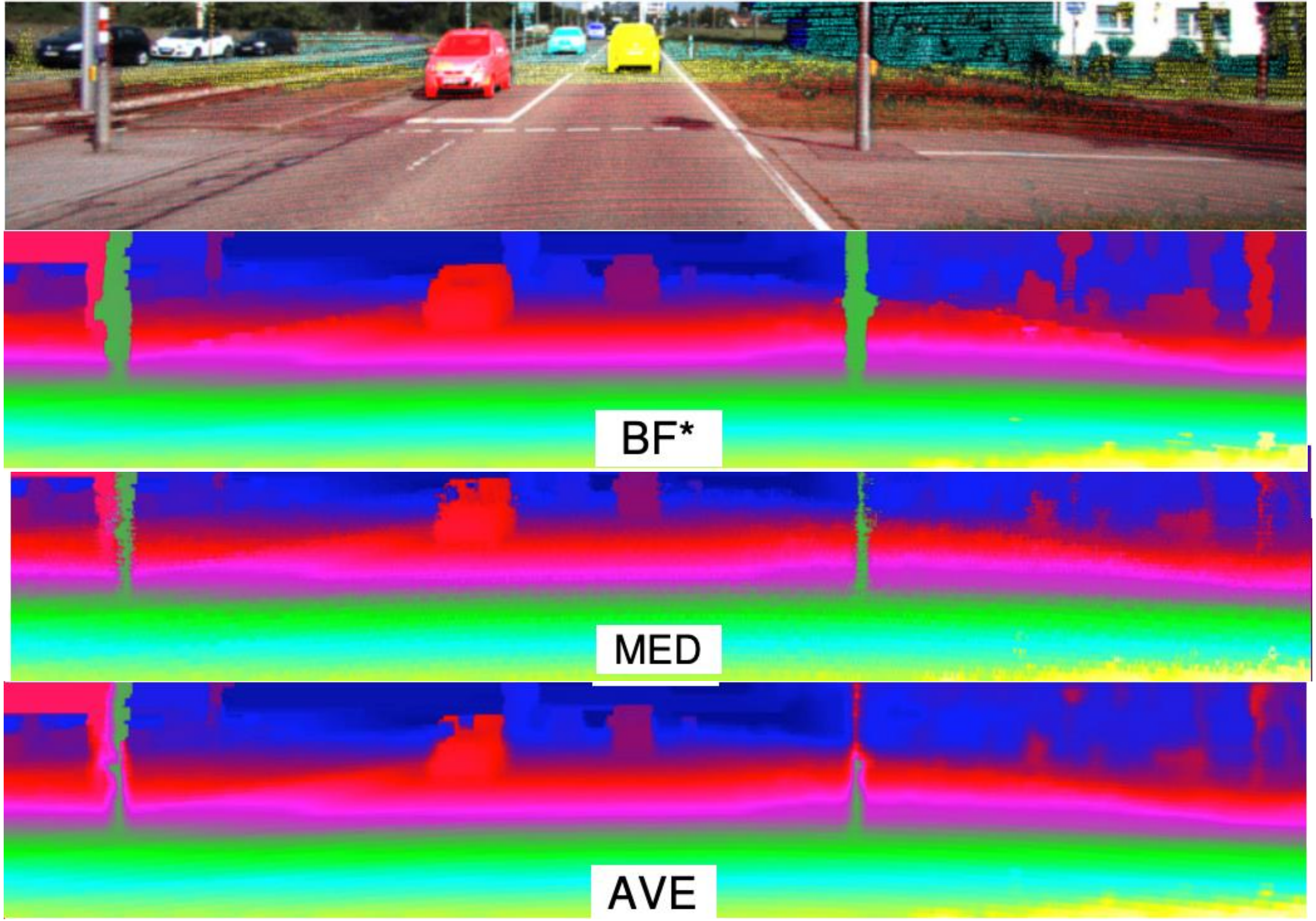
$$BF[I]_{\mathbf{p}} = \frac{1}{W_{\mathbf{p}}} \sum_{\mathbf{q}\in\mathcal{S}} G_{\sigma_s}(\|\mathbf{p}-\mathbf{q}\|)\, G_{\sigma_r}(I_{\mathbf{p}} - I_{\mathbf{q}})\, I_{\mathbf{q}}$$

$$W_{\mathbf{p}} = \sum_{\mathbf{q}\in\mathcal{S}} G_{\sigma_s}(\|\mathbf{p}-\mathbf{q}\|)\, G_{\sigma_r}(I_{\mathbf{p}} - I_{\mathbf{q}})$$

a) Input image Flower



b) Input depth map (8×8)



Bilateral for Point Cloud Upsampling

Homework - Compulsory

1. Build dataset for Lecture 1
   a. Download ModelNet40 dataset
   b. Select one point cloud from each category
2. Perform PCA for the 40 objects, visualize it.
3. Perform surface normal estimation for each point of each object, visualize it.
4. Downsample each object using voxel grid downsampling (exact, both centroid & random). Visualize the results.
5. Write your own code, DO NOT call apis (PCL, open3d, etc.) except for visualization.

Homework – Optional

KITTI depth dataset

- http://www.cvlibs.net/datasets/kitti/eval_depth_all.php
- Download and get familiar KITTI dataset

Perform depth upsampling / completion for the validation dataset

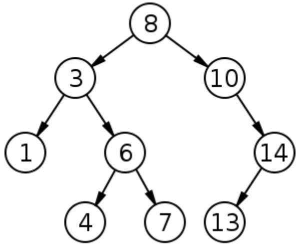
- Use whatever method you want, except Deep Learning method.

Evaluate the result using the evaluation code provided in the kitti-depth development kit.

Lecture Outline

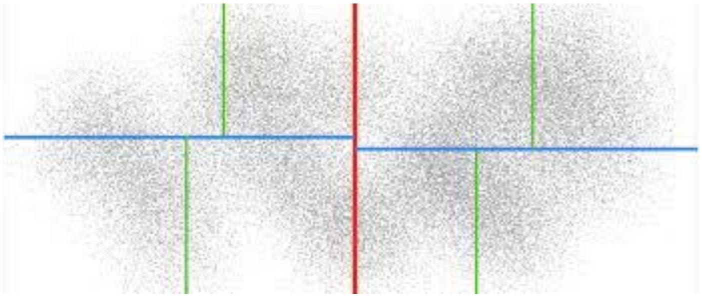
Binary Search Tree

- Basic knowledge about trees
- 1D NN problem
- With Python codes



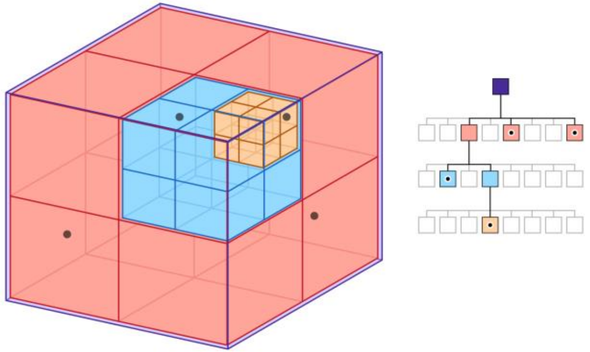
Kd-tree

- Works for data of any dimension
- Illustrated in 2D
- With Python codes



Octree

- Specifically designed for 3D data
- Illustrated in 2D/3D
- With Python codes



Thanks for Listening!