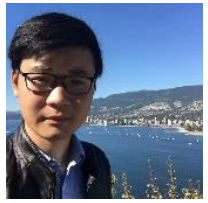
深蓝学院
shenlanxueyuan.com

Lecture 6

SOFT AND HARD CONSTRAINED TRAJECTORY OPTIMIZATION



主讲人 Fei Gao

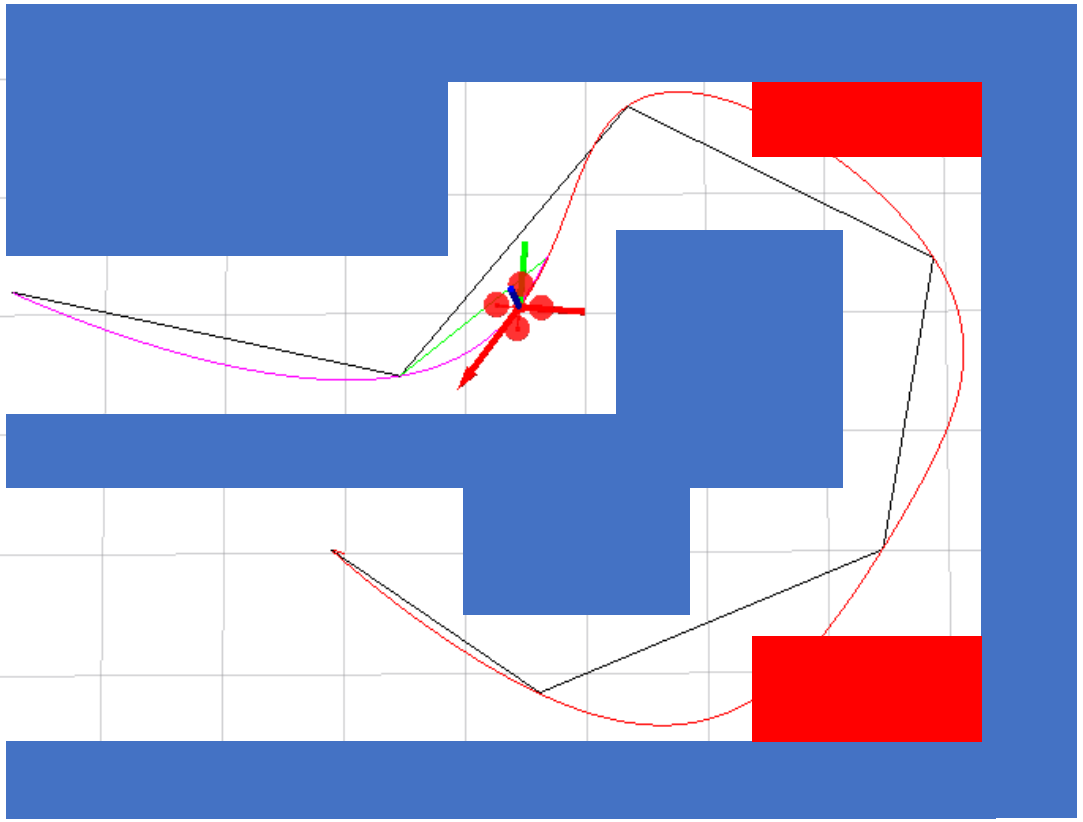
Ph.D. in Robotics
Hong Kong University of Science and Technology
Assistant Professor，Zhejiang University

Outline

1. Introduction
2. Soft-constrained Optimization
3. Hard-constrained Optimization
4. Case Study
5. Homework

Introduction

Minimum snap trajectory optimization



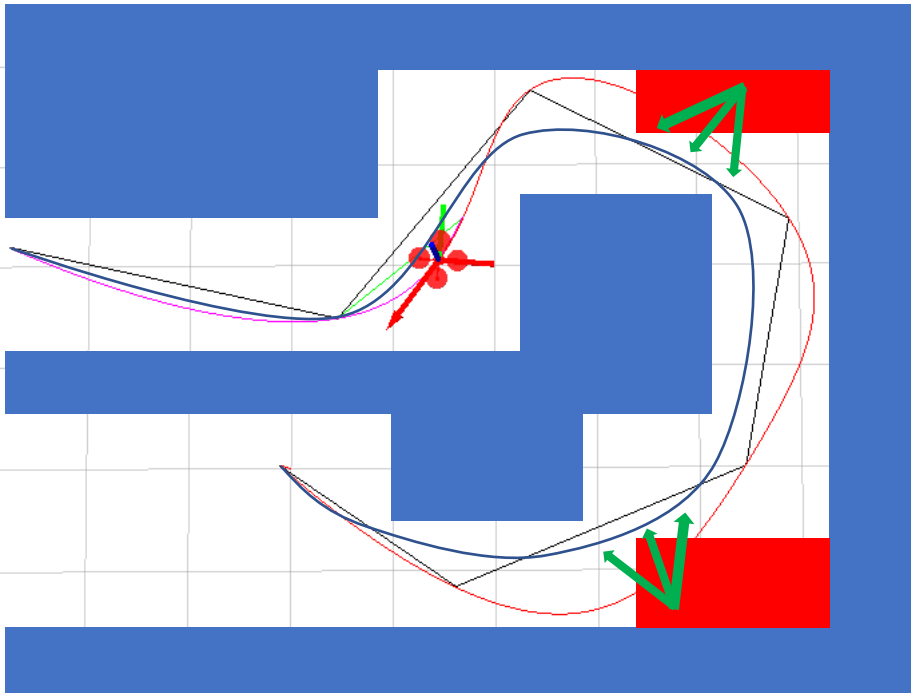
- We only constrain intermediate waypoints of the trajectory should pass.
- Computational cheap and easy to implement.
- No constraints on the trajectory itself.
- The “overshoot” of the trajectory unavoidable.

生成的轨迹不利于避障

The basic minimum snap framework is good for smooth curve generation, not for collision avoidance.

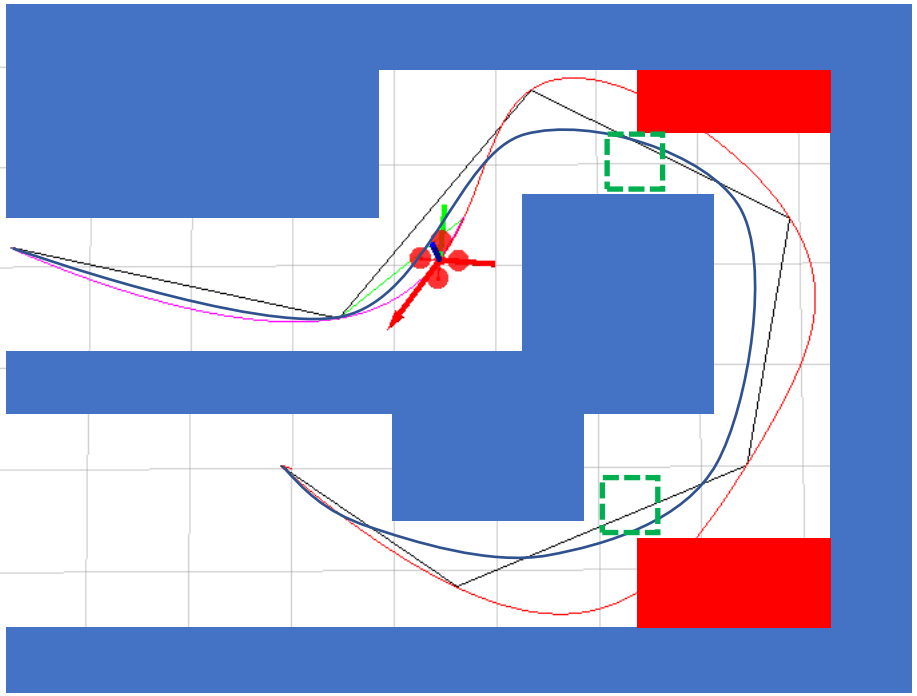
Minimum snap with safety constraints

Adding forces 增加推力



通过给轨迹施加推力，使得轨迹远离障碍物。

Adding bounds



增加约束使得轨迹在参数 m 在约束条件之内。

Hard/ Soft constraints

硬约束：要严格满足的条件

$$\min \quad f(x)$$

$$s.t. \quad g_i(x) = c_i, \quad i = 1, \cdots, n \quad \text{Equality constraints}$$

$$h_j(x) \geq d_j, \quad j = 1, \cdots, n \quad \text{Inequality constraints}$$

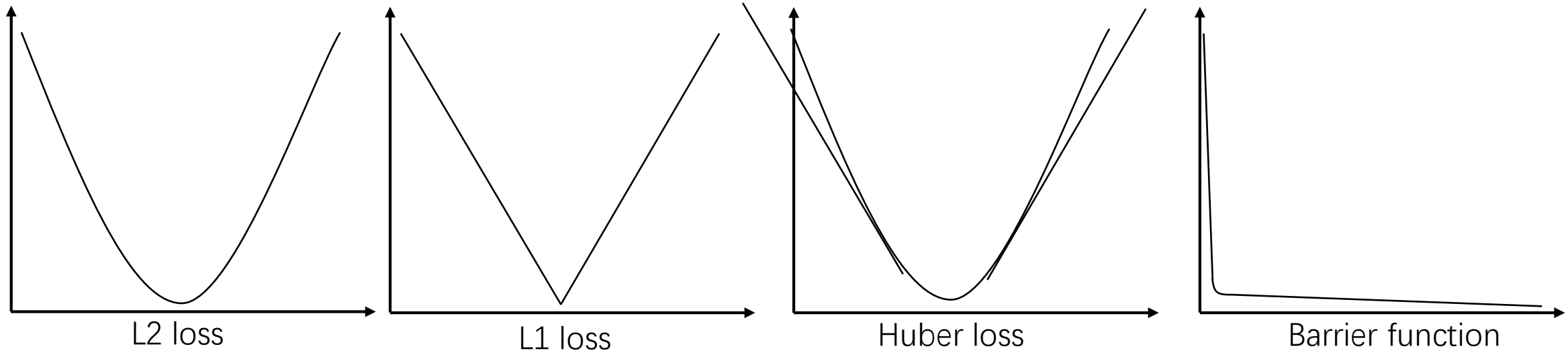
→ Required to be strictly satisfied.

由于单位不统一，所以需要使用系数 λ_1, λ_2

$$\min \quad f(x) + \lambda_1 \cdot g(x) + \lambda_2 \cdot h(x)$$

软约束：尽量满足的条件

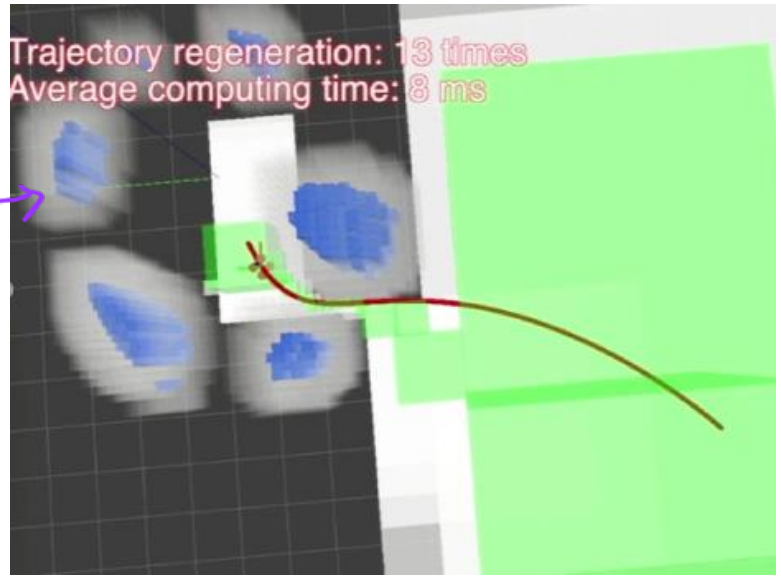
- Penalty terms / loss functions.
- Constraints which are preferred but not strictly required.
- Various kinds of loss functions.

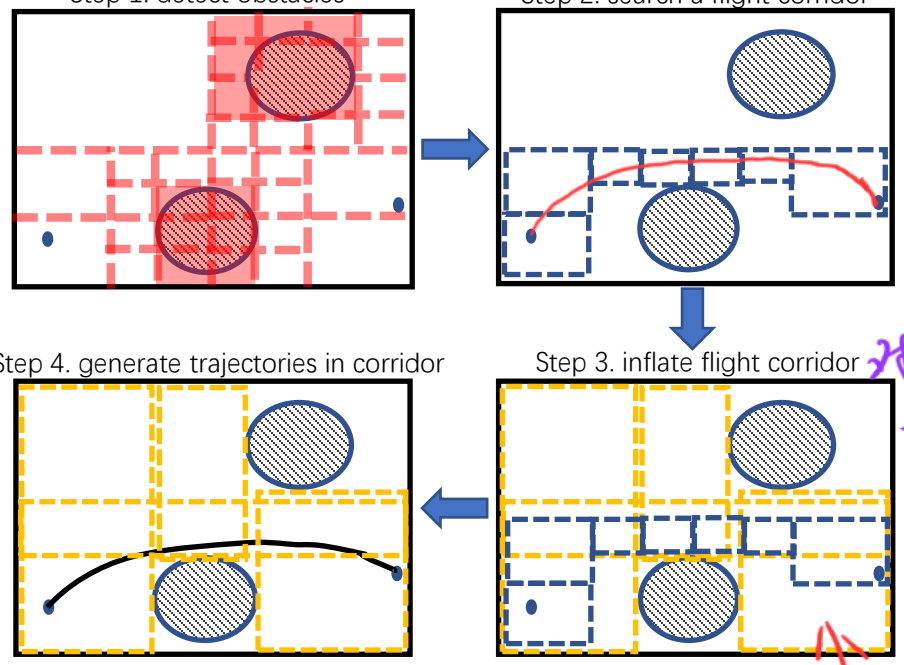


Hard-constrained Optimization

Corridor-based Trajectory Optimization

Corridor-based Smooth Trajectory Generation





- Differential flatness property

$$\{x, y, z, \dot{x}, \dot{y}, \dot{z}, \emptyset, \theta, \varphi, p, q, r\} \rightarrow \{x, y, z, \varphi\}$$

- Piecewise polynomial trajectory

$$f_\mu(t) = \begin{cases} \sum_{j=0}^{N} p_{1j}(t - T_0)^j & T_0 \le t \le T_1 \\ \sum_{j=0}^{N} p_{2j}(t - T_1)^j & T_0 \le t \le T_1 \\ \vdots & \vdots \\ \sum_{j=0}^{N} p_{Mj}(t - T_{M-1})^j & T_0 \le t \le T_1 \end{cases}$$

- Cost function (minimum jerk)

$$J = \sum_{\mu \in \{x,y,z\}} \int_0^T \left(\frac{d^k f_\mu(t)}{dt^k} \right)^2 dt$$

- Boundary constrains
- Continuity constrains
- Safety constrains

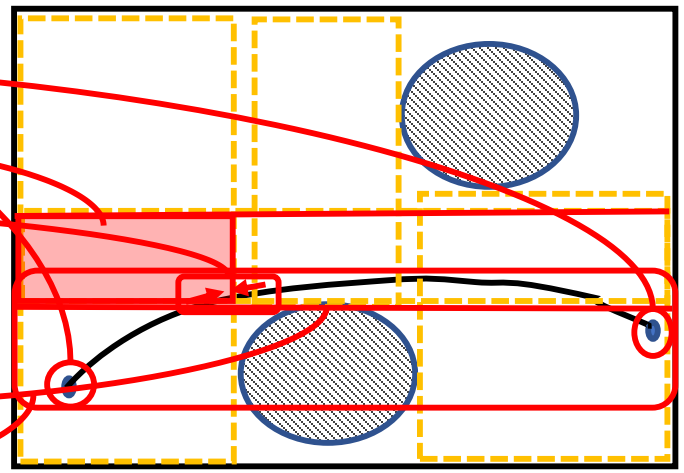
$$\begin{aligned} \min \ & \mathbf{p}^T \mathbf{H} \mathbf{p} \\ \text{s.t.} \ & \mathbf{A}_{eq}\mathbf{p} = \mathbf{b}_{eq} \\ & \mathbf{A}_{lq}\mathbf{p} \le \mathbf{b}_{lq} \end{aligned}$$

Quadratic Program

Online generation of collision-free trajectories for quadrotor flight in unknown cluttered environments, **Jing Chen** et al.

Problem formulation

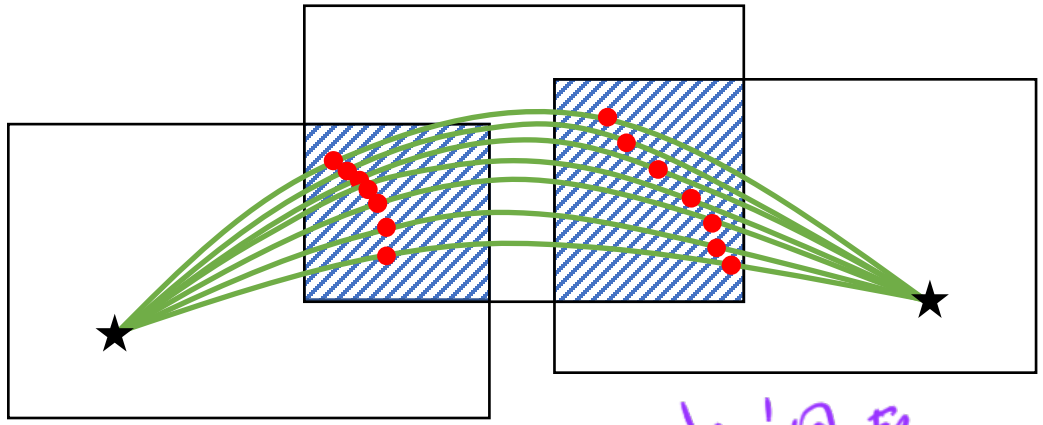
- Instant linear constraints:
  - Start, goal state constraint ($\mathbf{Ap} = \mathbf{b}$)
  - Transition point constraint ($\mathbf{Ap} = \mathbf{b}, \mathbf{Ap} \leq \mathbf{b}$)
  - Continuity constraint($\mathbf{Ap_i} = \mathbf{Ap_{i+1}}$)
- Interval linear constraints:
  - Boundary constraint ($\mathbf{A}(t)\mathbf{p} \leq \mathbf{b}, \forall t \in [t_l, t_r]$)
  - Dynamic constraint ($\mathbf{A}(t)\mathbf{p} \leq \mathbf{b}, \forall t \in [t_l, t_r]$)
    - Velocity constraints
    - Acceleration constraints



Advantages

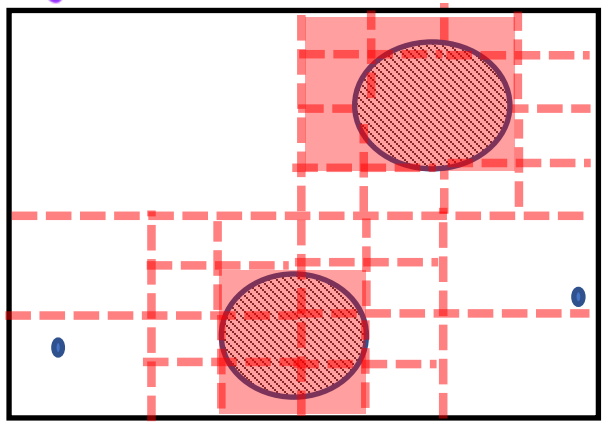
Many advantages:

- Efficiency: path search in the reduced graph, convex optimization in the corridor are efficient.
- High quality: corridor provides large optimization freedom.



具有附形加时间调整.

最优的轨迹一般都在边缘并且轨迹更少的弯曲.



$$\begin{aligned} \min \quad & \mathbf{p}^T\mathbf{H}\mathbf{p} \\ \text{s.t.} \quad & \mathbf{A}_{eq}\mathbf{p} = \mathbf{b}_{eq} \\ & \mathbf{A}_{lq}\mathbf{p} \leq \mathbf{b}_{lq} \end{aligned}$$

Quadratic Program

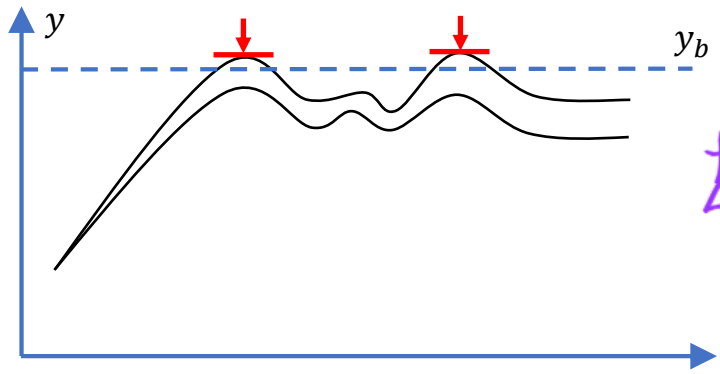
Disadvantages

硬约束加缺点：可能需要多次迭代求解，因为极值超过限制

Problem:

All constraints are enforced on piecewise joint points only, how to guarantee they are active along all the trajectory ?

- Iteratively check extremum and add extra constraints.



检查极值是否超过要求.

- Iterative solving is time consuming.
- If strictly no feasible solution meets all constraints. We have to run 10 iterations to determine the status of the solution ?

比如，位置y加极值超出了允许的区域，则此时给它添加一个极值的约束，同理速度加速度都可以这么做.

- Checking extremum is yet another polynomial root finding problem.
  - Up to quartic function, it's easy.
  - Higher order polynomials need numerical solutions.

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

Online generation of collision-free trajectories for quadrotor flight in unknown cluttered environments, **Jing Chen** et al.

Polynomial roots finding

Matlab function “roots”

For a general polynomial function: $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$,
It has the **companion matrix**: 伴随矩阵

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

A的特征值对应 P(x) 的零点

The characteristic polynomial $\det(xI - A)$ of A is the polynomial $p(x)$.

roots

Polynomial roots

Syntax

```
r = roots(p)
```

Description

`r = roots(p)` returns the roots of the polynomial represented by `p` as a column vector. Input `p` is a vector containing `n+1` polynomial coefficients, starting with the coefficient of x^n. A coefficient of `0` indicates an intermediate power that is not present in the equation. For example, `p = [3 2 -2]` represents the polynomial $3x^2 + 2x - 2$.

The `roots` function solves polynomial equations of the form $p_1 x^n + \ldots + p_n x + p_{n+1} = 0$. Polynomial equations contain a single variable with nonnegative exponents.

Algorithms

The `roots` function considers `p` to be a vector with `n+1` elements representing the `n`th degree characteristic polynomial of an `n`-by-`n` matrix, `A`. The roots of the polynomial are calculated by computing the eigenvalues of the companion matrix, `A`.

```
A = diag(ones(n-1,1),-1);
A(1,:) = -p(2:n+1)./p(1);
r = eig(A)
```

The results produced are the exact eigenvalues of a matrix within roundoff error of the companion matrix, `A`. However, this does not mean that they are the exact roots of a polynomial whose coefficients are within roundoff error of those in `p`.

Bezier Curve Optimization

贝塞尔曲线

Trajectory basis changing

- Use **Bernstein** polynomial basis.
- Change to basis of the trajectory from monomial polynomial to Bernstein polynomial

Bézier curve

$$P_j(t) = p_j^0 + p_j^1 t + p_j^2 t^2 + \cdots + p_j^n t^n$$

$$B_j(t) = c_j^0 b_n^0(t) + c_j^1 b_n^1(t) + \cdots + c_j^n b_n^n(t) = \sum_{i=0}^{n} c_j^i b_n^i(t)$$

$$b_n^i(t) = \binom{n}{i} \cdot t^i \cdot (1-t)^{n-i}$$

- Bézier curve is just a special polynomial, it can be mapped to monomial polynomial by: $p = M \cdot c$. And all previous derivations still hold.

$$\begin{bmatrix} P^0 \\ P^1 \\ P^2 \\ \vdots \end{bmatrix} \qquad \begin{bmatrix} C^0 \\ C^1 \\ \vdots \\ \vdots \end{bmatrix}$$

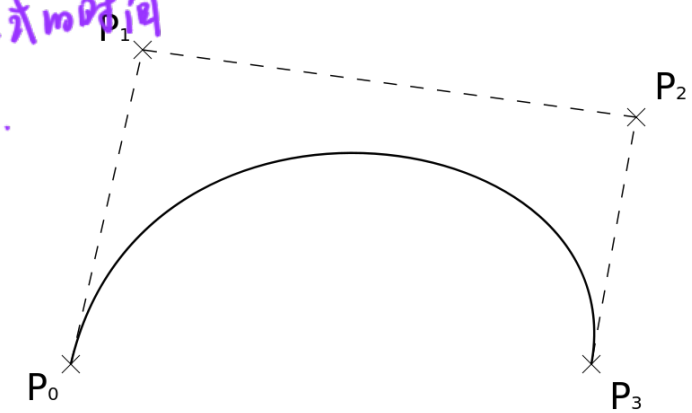
For 6 order:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 6 & 0 & 0 & 0 & 0 & 0 \\ 15 & -30 & 15 & 0 & 0 & 0 & 0 \\ -20 & 60 & -60 & 20 & 0 & 0 & 0 \\ 15 & -60 & 90 & -60 & 15 & 0 & 0 \\ -6 & 30 & -60 & 60 & -30 & 6 & 0 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{bmatrix}$$

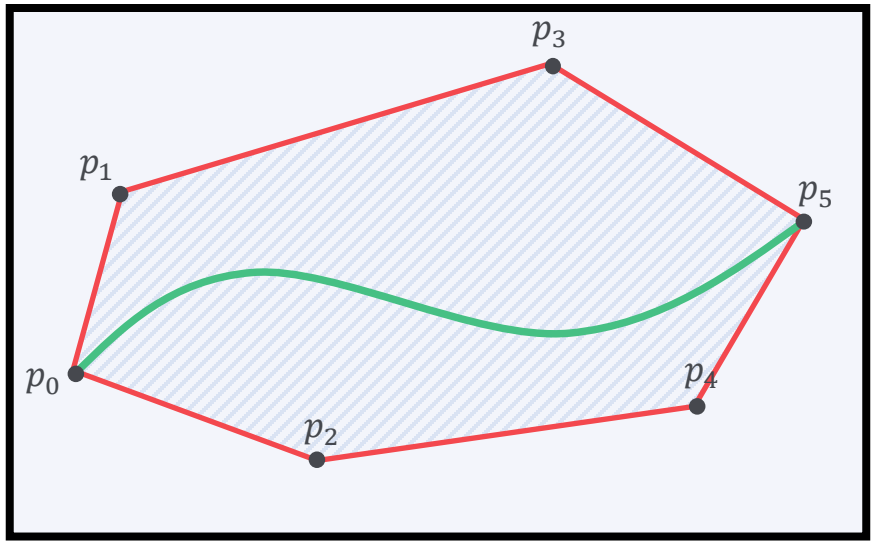
Trajectory basis changing

Properties:

- **Endpoint interpolation.** The Bezier curve always starts at the first control point, ends at the last control point, and never pass any other control points.
- **Convex hull.** The Bezier curve $B(t)$ consists of a set of control points ci are entirely confined within the convex hull defined by all these control points.
- **Hodograph.** The derivative curve $B'(t)$ of a Bezier curve $B(t)$ is called as hodograph, and it is also a Bezier curve with control points defined by $n \cdot (c_{i+1} - c_i)$, where n is the degree.
- **Fixed time interval**. A Bezier curve is always defined on [0,1]



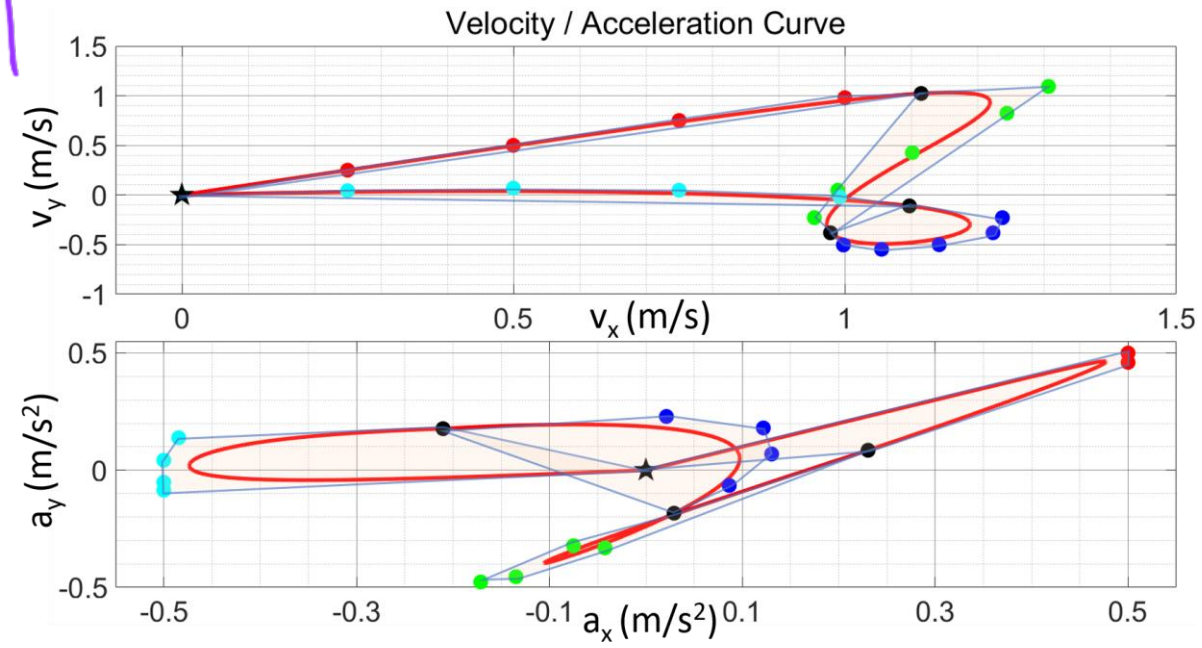
Convex hull property



- The flight corridor consists of convex polygons. 多边形
- Each cube corresponds to a piece of Bezier curve.
- Control points of this curve are enforced inside the polygon.
- The trajectory is entirely inside the convex hull of all points.

The trajectory is entirely inside the flight corridor.

某一段轨迹m控制点在方格内，则根据凸包性质，轨迹也一定在方格内



如果想使得v.a也处在限制范围内，只需要将Bezier求导之后m控制点限制范围。

Trajectory Generation Formulation

由于 Bezier 的起点，终点一定是第一个和最后一个控制点，所以对于起始位置只需限制其第一个控制点的位置，同理对起始速度只需要限制轨迹导数之后的控制点。

$C_{\mu j}^{i}$：第 j 段轨迹的第 i 个控制点的第 μ 个分量。

Define higher order (l^{th}) control points:

$$a_{\mu j}^{0,i} = c_{\mu j}^{i}, a_{\mu j}^{l,i} = \frac{n!}{(n-l)!} \cdot (a_{\mu j}^{l-1,i+1} - a_{\mu j}^{l-1,i}), \ \ l \geq 1$$

- Boundary Constraints:

$$a_{\mu j}^{l,0} \cdot s_j^{(1-l)} = d_{\mu j}^{(l)}$$

前一段轨迹的最后一个控制点等于后一段的第一个控制点

- Continuity Constraints:

$$a_{\mu j}^{\phi,n} \cdot s_j^{(1-\phi)} = a_{\mu,j+1}^{\phi,0} \cdot s_{j+1}^{(1-\phi)}, \ \ a_{\mu j}^{0,i} = c_{\mu j}^{i}.$$

- Safety Constraints:

$$\beta_{\mu j}^{-} \leq c_{\mu j}^{i} \leq \beta_{\mu j}^{+}, \ \ \mu \in \{x, y, z\}, \ \ i = 0, 1, 2, ..., n,$$

- Dynamical Feasibility Constraints:

$$v_m^{-} \leq n \cdot (c_{\mu j}^{i} - c_{\mu j}^{i-1}) \leq v_m^{+},$$
$$a_m^{-} \leq n \cdot (n-1) \cdot (c_{\mu j}^{i} - 2c_{\mu j}^{i-1} + c_{\mu j}^{i-2})/s_j \leq a_m^{+}$$

Stack all of these

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{Q}_o \mathbf{c} \\ \text{s.t.} \quad & \mathbf{A}_{eq}\mathbf{c} = \mathbf{b}_{eq}, \\ & \mathbf{A}_{ie}\mathbf{c} \leq \mathbf{b}_{ie}, \\ & \mathbf{c}_j \in \Omega_j, \quad j = 1, 2, ..., m, \end{aligned}$$

A typical convex QP formulation.

We only solve this program once to determine whether there is a qualified trajectory exists.

Simulation results

Blue curve : trajectory in execution horizon

Red curve : current trajectory

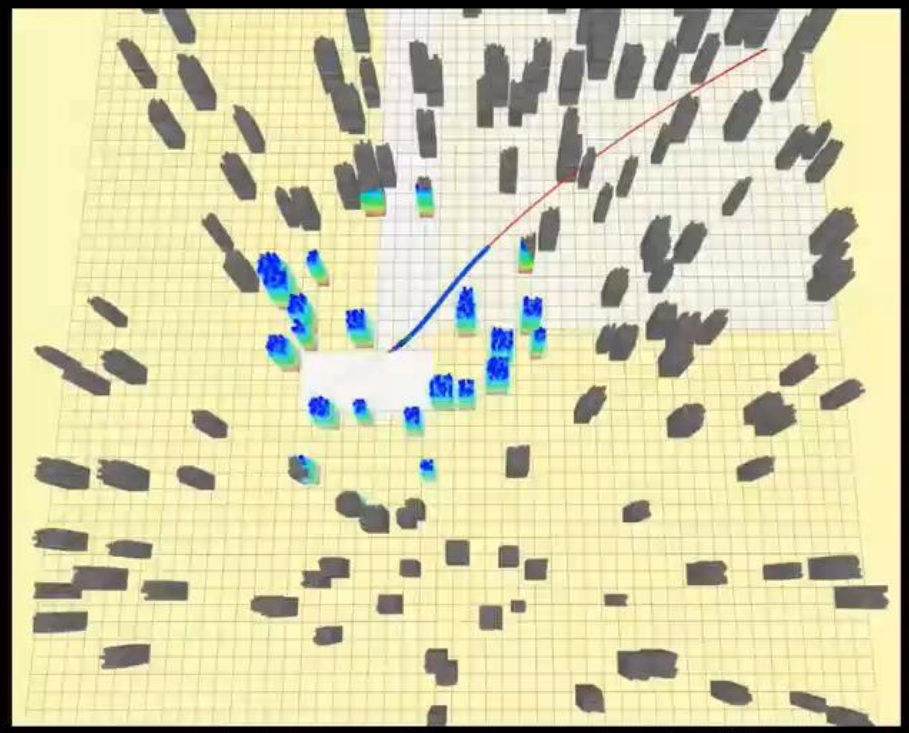
White cube : flight corridor

Colorful voxels : mapped obstacles

Grey voxels : un-mapped obstacles

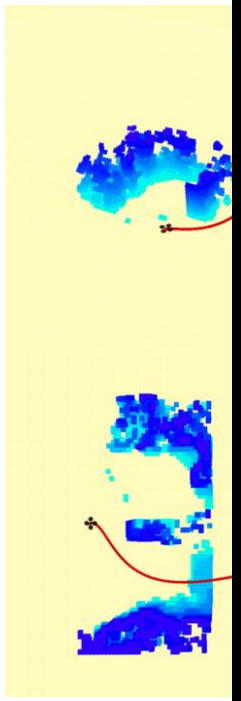
Greed arrow : velocity

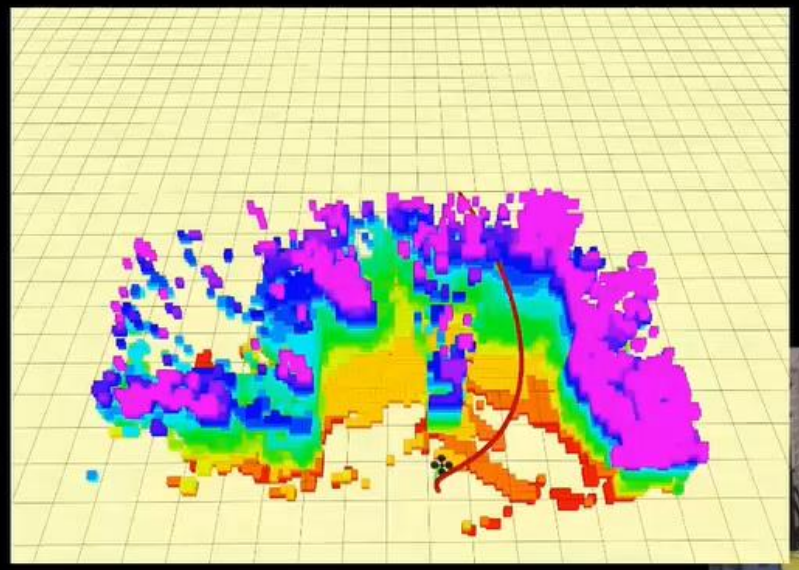
Yellow arrow : acceleration



Flight corridor is projected to x-y plane for visualization

Experimental results





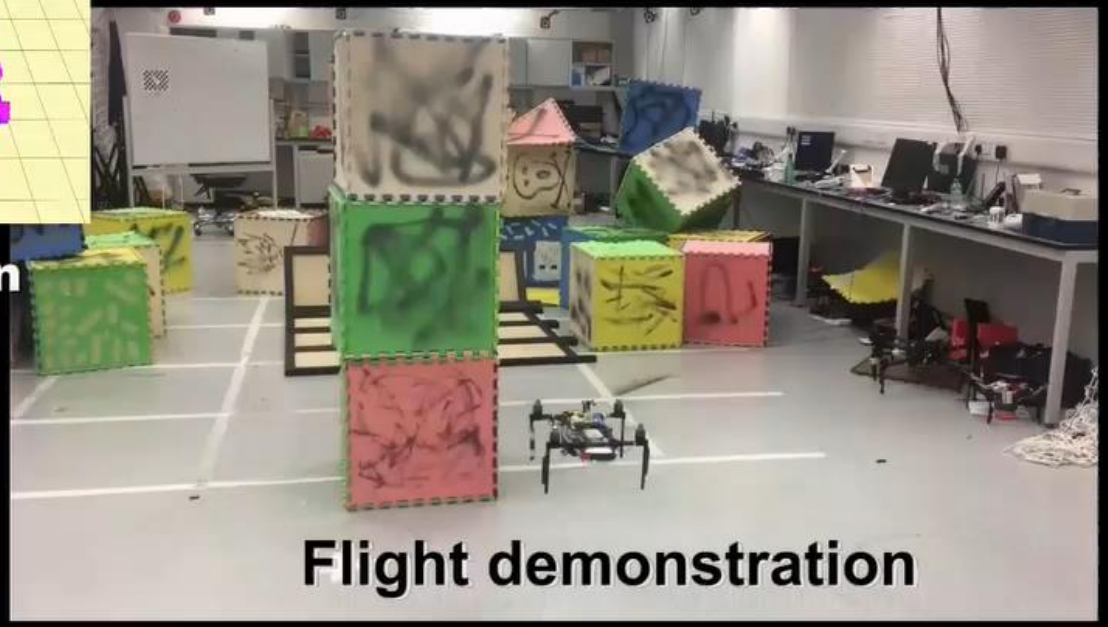
Trajectory visualization

Indoor autonomous flight 1

navigating among messy obstacles

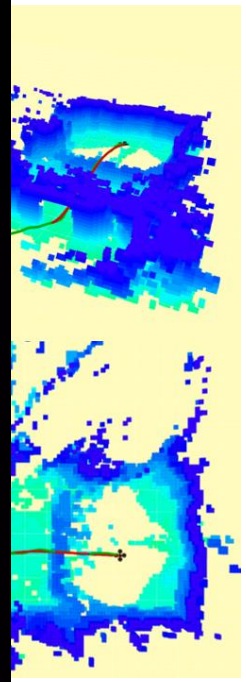
2 trajectories (re)generated

Average computing time : 41.2 ms



Flight demonstration

Speed : 1x



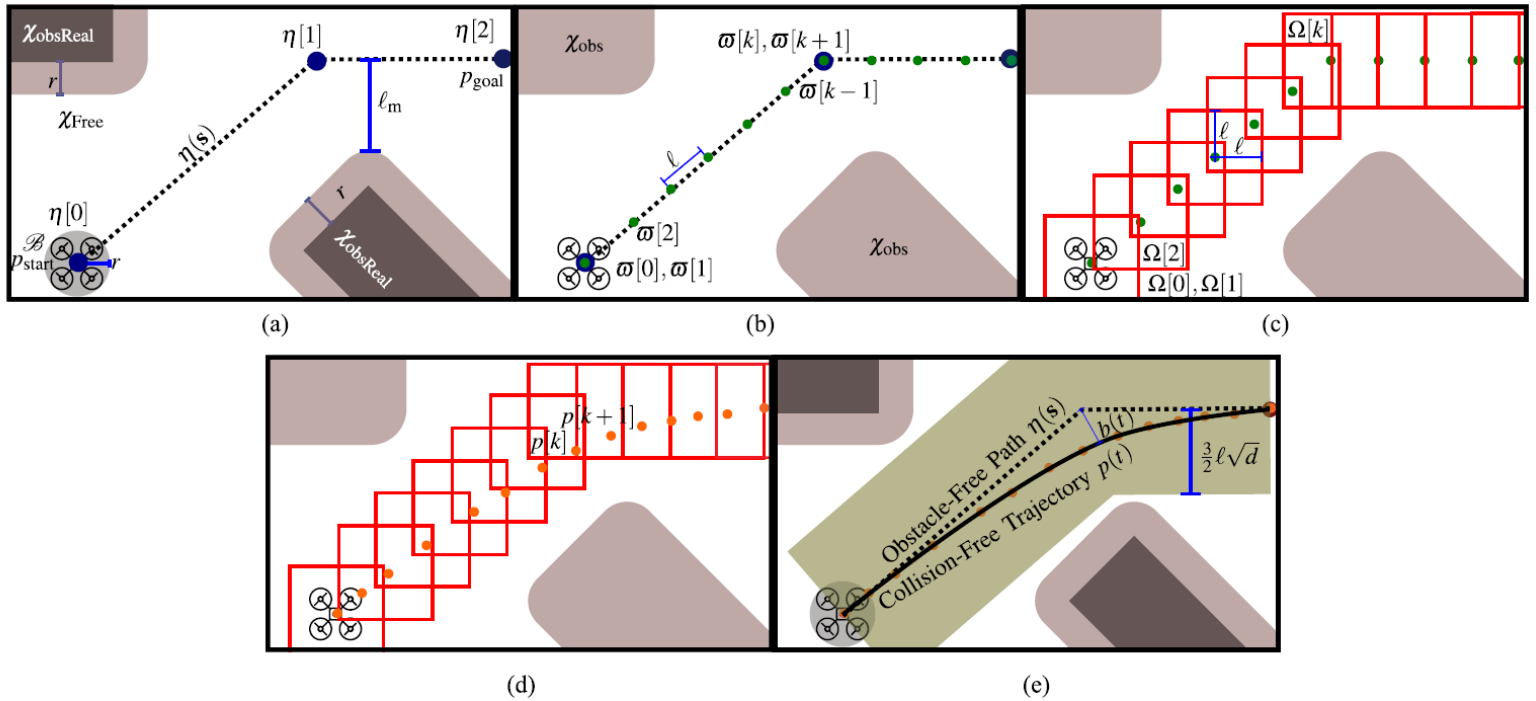
Online Safe Trajectory Generation For Quadrotors Using Fast Marching Method and Bernstein Basis Polynomial, **Fei Gao** et al.

Source code released at: https://github.com/HKUST-Aerial-Robotics/Btraj ← A complete UAV online motion planning framework

Other Options

Dense constraints

- Adding numerous constraints at discrete time ticks.
- Piecewise-constant accelerations at each tick.
- QP program solution.



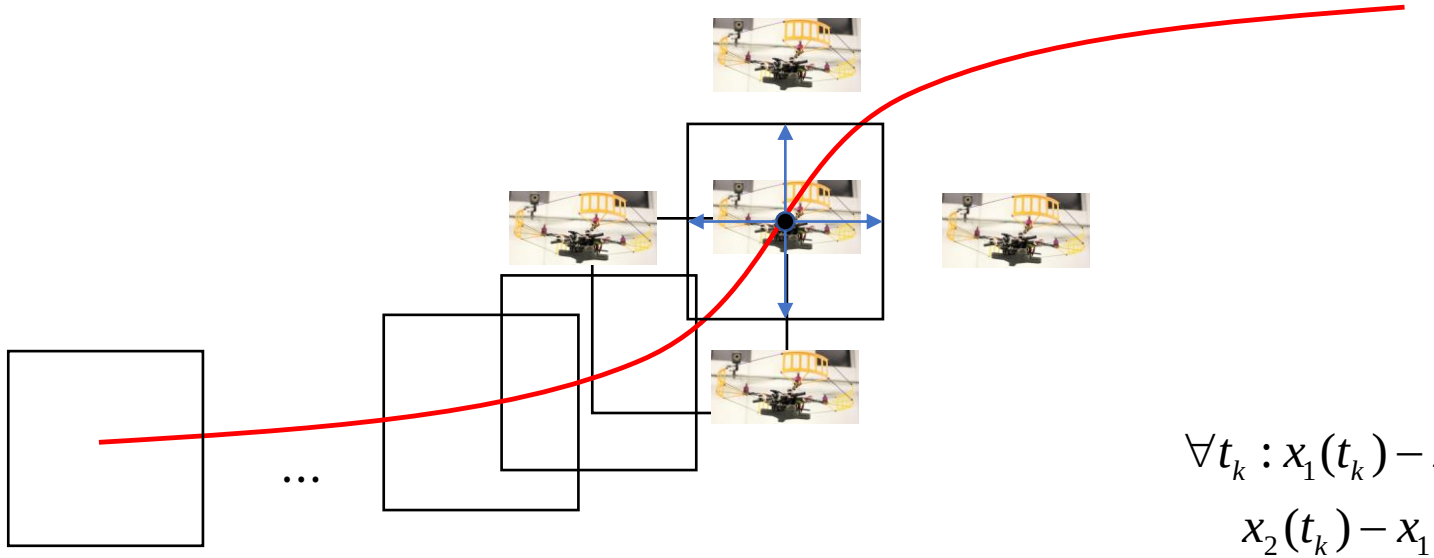
- Always generates over-conservative trajectories.
- Too many constraints, the computational burden is high.

A hybrid method for online trajectory planning of mobile robots in cluttered environments, **L Campos-Macías** et. al

Dense constraints

A hybrid method for online trajectory planning of mobile robots in cluttered environments, **L Campos-Macías** et. al

Mixed integer optimization



$$\forall t_k : x_1(t_k) - x_2(t_k) \le d_x$$
$$or \quad x_2(t_k) - x_1(t_k) \le d_x$$
$$or \quad y_1(t_k) - y_2(t_k) \le d_y$$
$$or \quad y_2(t_k) - y_1(t_k) \le d_y$$
$$or \quad z_1(t_k) - z_2(t_k) \le d_z$$
$$or \quad z_2(t_k) - z_1(t_k) \le d_z$$

- Mixed-integer QP.
- ‘Big M’ method.
- Super slow.

$$\forall t_k : x_1(t_k) - x_2(t_k) - c_0 \cdot M \le d_x$$
$$x_2(t_k) - x_1(t_k) - c_1 \cdot M \le d_x$$
$$y_1(t_k) - y_2(t_k) - c_2 \cdot M \le d_y$$
$$y_2(t_k) - y_1(t_k) - c_3 \cdot M \le d_y$$
$$z_1(t_k) - z_2(t_k) - c_4 \cdot M \le d_z$$
$$z_2(t_k) - z_1(t_k) - c_5 \cdot M \le d_z$$
$$\sum_{i=0}^{5} c_i = 5, \; M = 100000$$
$$c_i \in \{0,1\}$$

Mixed-integer quadratic program trajectory generation for heterogeneous quadrotor teams, **D. Mellinger** et al.

Dense constraints

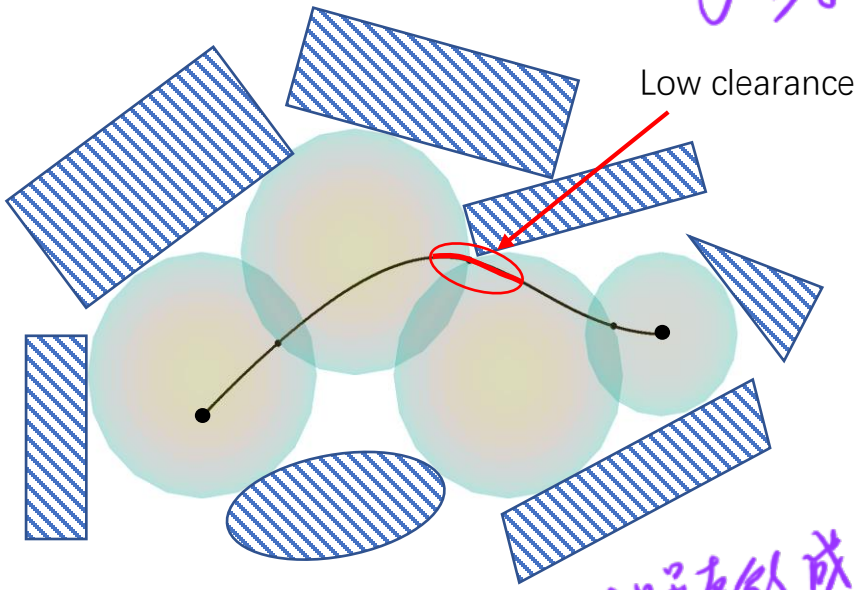
Mixed-integer quadratic program trajectory generation for heterogeneous quadrotor teams, **D. Mellinger** et al.

Soft-constrained Optimization

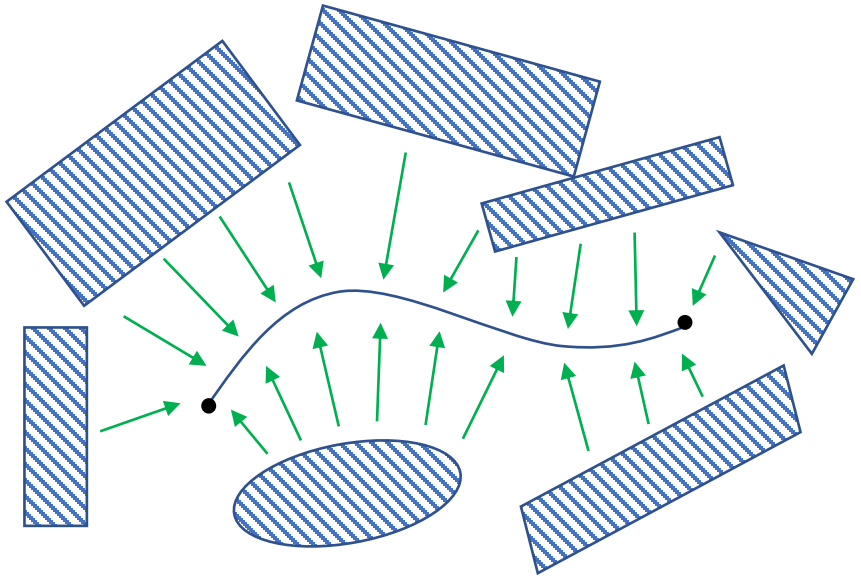
Distance-based Trajectory Optimization

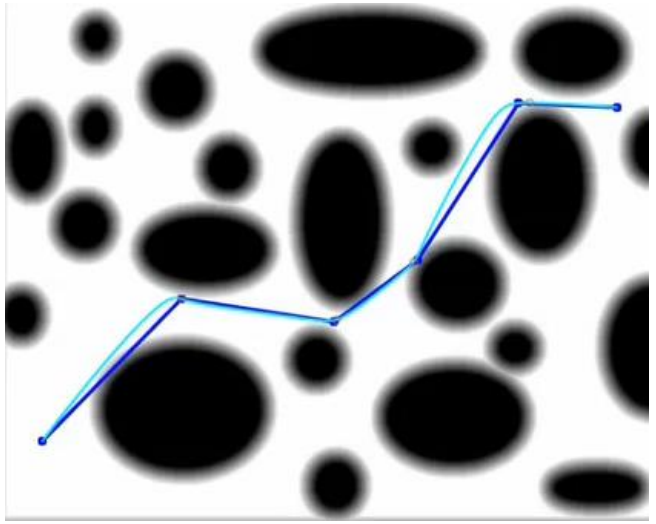
Motivation

Hard-constrained methods



Soft-constrained methods





Vison-based drone:

- Limited sensing range and quality
- Noisy depth estimation

Hard-constrained method:

- Treat all free space equally
- Solution space is sensitive to noise

Problem formulation

- Differential flatness property

$$\{x, y, z, \dot{x}, \dot{y}, \dot{z}, \emptyset, \theta, \varphi, p, q, r\} \rightarrow \{x, y, z, \varphi\}$$

- Piecewise polynomial trajectory 多项式轨迹.

$$f_\mu(t) = \begin{cases} \sum_{j=0}^{N} p_{1j}(t - T_0)^j & T_0 \le t \le T_1 \\ \sum_{j=0}^{N} p_{2j}(t - T_1)^j & T_0 \le t \le T_1 \\ \vdots & \vdots \\ \sum_{j=0}^{N} p_{Mj}(t - T_{M-1})^j & T_0 \le t \le T_1 \end{cases}$$

- Objective function

$$J = J_s + J_c + J_d$$

$$= \underbrace{\lambda_1 J_1}_{\text{Smoothness cost}} + \underbrace{\lambda_2 J_2}_{\text{Collision cost}} + \underbrace{\lambda_3 J_3}_{\text{Dynamical cost}}$$

- Smoothness cost: minimum snap formulation

$$J_s = \sum_{\mu \in \{x,y,z\}} \int_0^T \left(\frac{d^k f_\mu(t)}{dt^k} \right)^2 dt$$

$$= \begin{bmatrix} \boldsymbol{d}_F \\ \boldsymbol{d}_P \end{bmatrix}^T \boldsymbol{C}^T \boldsymbol{M}^{-T} \boldsymbol{Q} \boldsymbol{M}^{-1} \boldsymbol{C} \begin{bmatrix} \boldsymbol{d}_F \\ \boldsymbol{d}_P \end{bmatrix} = \begin{bmatrix} \boldsymbol{d}_F \\ \boldsymbol{d}_P \end{bmatrix}^T \begin{bmatrix} \boldsymbol{R}_{FF} & \boldsymbol{R}_{FP} \\ \boldsymbol{R}_{PF} & \boldsymbol{R}_{PP} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_F \\ \boldsymbol{d}_P \end{bmatrix}$$

- Collision cost: penalize on the distance to nearest obstacle

$$J_c = \int_{T_0}^{T_M} c(p(t)) ds \quad \text{对曲线积分}$$

$$= \sum_{k=0}^{T/\delta t} c(p(T_k)) \|v(t)\| \delta t, \quad T_k = T_0 + k\delta t$$

当前点在ESDF中的惩罚

- Dynamical Cost: penalize on the velocity and acceleration where exceeds limits (similar to collision term).

对动力学的约束，可以离散 v. a. 当某一个 v. a 特别大时给以很大的惩罚

Objective/Jacobian evaluation

- Smoothness cost: minimum snap formulation

$$J_s = \sum_{\mu\in\{x,y,z\}} \int_0^T \left(\frac{d^k f_\mu(t)}{dt^k}\right)^2 dt$$

$$= \begin{bmatrix} \boldsymbol{d}_F \\ \boldsymbol{d}_P \end{bmatrix}^T \boldsymbol{C}^T \boldsymbol{M}^{-T} \boldsymbol{Q} \boldsymbol{M}^{-1} \boldsymbol{C} \begin{bmatrix} \boldsymbol{d}_F \\ \boldsymbol{d}_P \end{bmatrix} = \begin{bmatrix} \boldsymbol{d}_F \\ \boldsymbol{d}_P \end{bmatrix}^T \begin{bmatrix} \boldsymbol{R}_{FF} & \boldsymbol{R}_{FP} \\ \boldsymbol{R}_{PF} & \boldsymbol{R}_{PP} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_F \\ \boldsymbol{d}_P \end{bmatrix}$$

- The Jacobian with respect to free derivatives $\boldsymbol{d}_{p\mu}$ is:

$$\frac{\alpha J_s}{\alpha \boldsymbol{d}_{p\mu}} = 2\boldsymbol{d}_F^T \boldsymbol{R}_{FP} + 2\boldsymbol{d}_P^T \boldsymbol{R}_{PP}$$

- Collision cost: penalize on the distance to nearest obstacle

$$J_c = \int_{T_0}^{T_M} c(p(t))\, ds$$

Distance penalty at a point along the trajectory

$$= \sum_{k=0}^{T/\delta t} c(p(T_k))\|v(t)\|\delta t,\ T_k = T_0 + k\delta t$$

- The Jacobian with respect to free derivatives $\boldsymbol{d}_{p\mu}$ is:

$$\frac{\alpha J_c}{\alpha \boldsymbol{d}_{p\mu}} = \sum_{k=0}^{T/\delta t} \left\{ \forall_\mu c(p(T_k))\|v\|\boldsymbol{F} + c(p(T_k))\frac{v_\mu}{\|v\|}\boldsymbol{G} \right\} \delta t, \mu \in \{x, y, z\}$$

L_{dp} is the right block of matrix $M^{-1}C$ which corresponds to the free derivatives on the μ axis $d_{p\mu}$.

$F = TL_{dp}$, $G = TV_mL_{dp}$.

$\forall_\mu c(\cdot)$ is the gradient in μ axis of the collision cost.

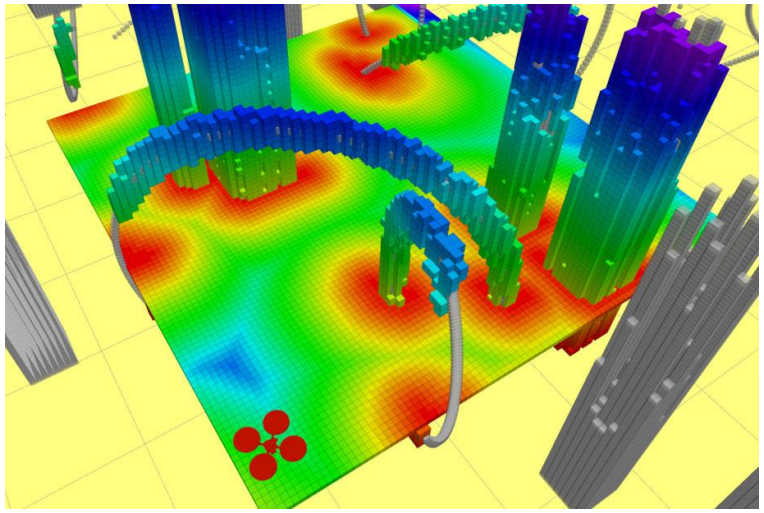
V_m maps the coefficients of the position to the coefficients of the velocity. $\boldsymbol{T} = [T_k^0, T_k^1, \cdots, T_k^n]$

$$\mathbf{H}_o = \left[\frac{\partial^2 f_o}{\partial \mathbf{d}_{P_x}^2}, \frac{\partial^2 f_o}{\partial \mathbf{d}_{P_y}^2}, \frac{\partial^2 f_o}{\partial \mathbf{d}_{P_z}^2}\right],$$

$$\frac{\partial^2 f_o}{\partial \mathbf{d}_{P\mu}^2} = \sum_{k=0}^{\tau/\delta t} \Big\{ \mathbf{F}^T \nabla_\mu c(p(\mathcal{T}_k) \frac{v_\mu}{\|v\|} \mathbf{G} + \mathbf{F}^T \nabla_\mu^2 c(p(\mathcal{T}_k)) \|v\| \mathbf{F}$$

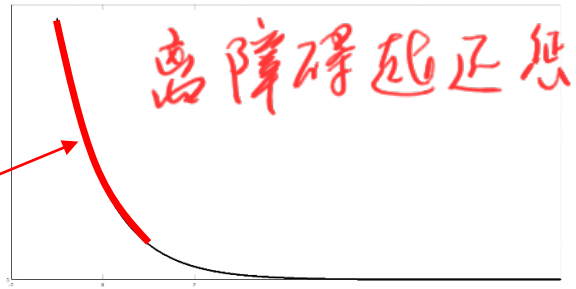
$$+ \mathbf{G}^T \nabla_\mu c(p(\mathcal{T}_k)) \frac{v_\mu}{\|v\|} \mathbf{F} + \mathbf{G}^T c(p(\mathcal{T}_k)) \frac{v_\mu^2}{\|v\|^3} \mathbf{G} \Big\} \delta t, \qquad (10)$$

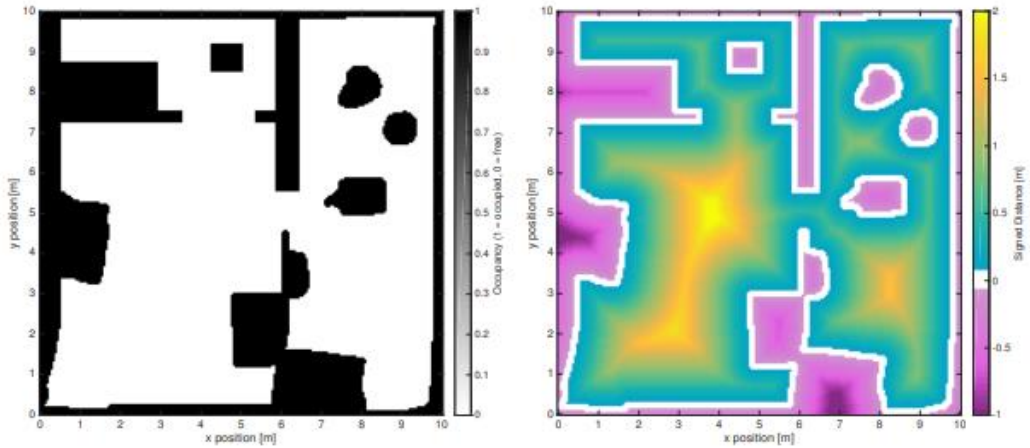
Euclidean signed distance field (ESDF)



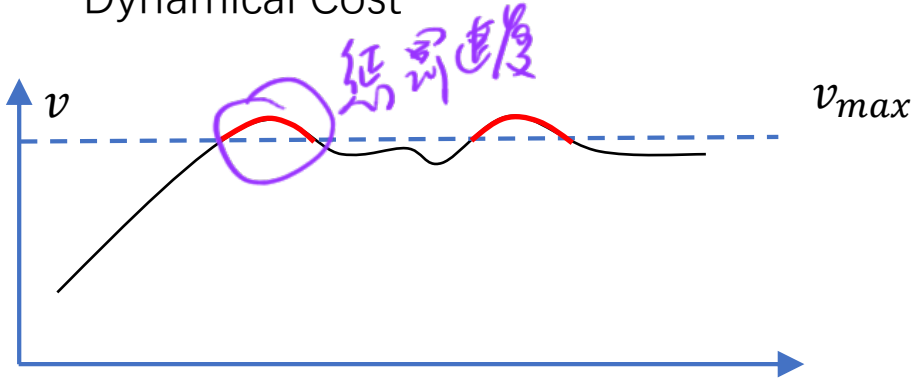
Use **exponential** function as cost function c, to prevent trajectory from be near to obstacle

Penalty explosion near obstacles





- Dynamical Cost



Numerical optimization

$$\text{minimize } f(x)$$

- Produce sequence of points $x^{(k)} \in dom\ f, k = 0,1, \ldots$with

$$f\left(x^{(k)}\right) \to p^{\star}$$

- Can be interpreted as iterative methods for solving optimality condition

$$\nabla f(x^{\star}) = 0$$

Descent method

$$x^{k+1} = x^k + t^k \Delta x^k \quad \text{with } f\left(x^{k+1}\right) < f(x^k)$$

- Other notations: $x^+ = x + t\Delta x,\ \ x := x + t\Delta x$
- Δx is the step, or search direction; t is the step size, or step length

---

General descent method
given a starting point $x \in dom\ f$.
repeat

1. Determine a descent direction Δx.
2. Line search. Choose a step size $t > 0$.
3. Update. $x := x + t\Delta x$.

until stopping criterion is satisfied.

Line search types

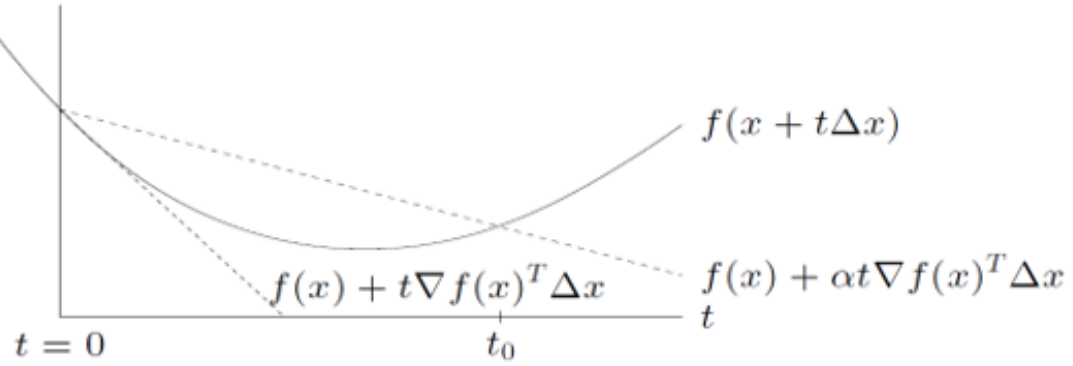
exact line search: $t = argmin_{t>0} f(x + t\Delta x)$

backtracking line search (with parameters $\alpha \in (0, 1/2), \beta \in (0,1)$)

- starting at $t = 1$, repeat $t := \beta t$ until

$$f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$$

- graphical interpretation: backtrack until $t \leq t_0$



First-order method

Gradient descent method

General descent method with $\Delta x = -\nabla f(x)$

---

given a starting point $x \in dom\ f$.

Repeat

1. $\Delta x := -\nabla f(x)$.
2. Line search. Choose step size t via exact or backtracking line search.
3. Update. $x := x + t\Delta x$.

until stopping criterion is satisfied.

- Stopping criterion usually of the form $\left|\left|\nabla f(x)\right|\right|_2 \leq \epsilon$

Second-order method

Newton method

$Taylor\ expansion\ at\ x^{(k)}$, and $second-$ order similarity $\Delta x = -\nabla^2 f(x)^{-1}\nabla f(x)$

---

given a starting point $x \in dom\ f$.

Repeat

1. $\Delta x := -\nabla^2 f(x)^{-1}\nabla f(x)$.
2. Line search. $t = argmin_{t>0} f(x^{(k)} - t\nabla^2 f(x)^{-1}\nabla f(x))$..
3. Update. $x := x + t\Delta x$.

until stopping criterion is satisfied.

- Stopping criterion usually of the form $\left|\left|\nabla f(x)\right|\right|_2 \leq \epsilon$
- $\nabla^2 f(x)$ is the Hessian matrix of the $f(x)$ at x
- For Gauss-newton: $\nabla^2 f(x) \approx J_f^T J_f$, J_f is Jacobi matrix, $\Delta x = -(J_f^T J_f)^{-1} J_f^T$

Second-order method

Levenberg-Marquardt method

Improvement of Gauss-newton method: $\Delta x = -(J_f^T J_f + \lambda I)^{-1} J_f^T$

---

given a starting point $x \in dom\, f, start\, \lambda_0 > 0$.

Repeat

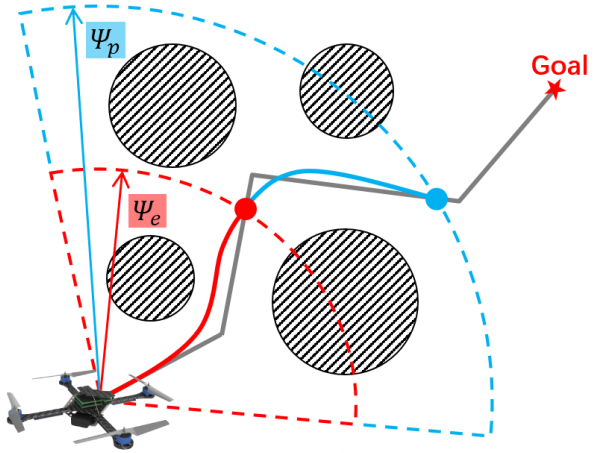
1. $\Delta x = -(J_f^T J_f + \lambda I)^{-1} J_f^T$.
2. Update. λ , the updating is controlled by the gain ratio
3. Update. $x := x + \Delta x$.

until stopping criterion is satisfied.

- Stopping criterion usually of the form $\left|\left|\nabla f(x)\right|\right|_2 \leq \epsilon$
- Whenλ->0, LM method -> Gauss-newton method
- Whenλ->∞, LM method -> Gradient descent method

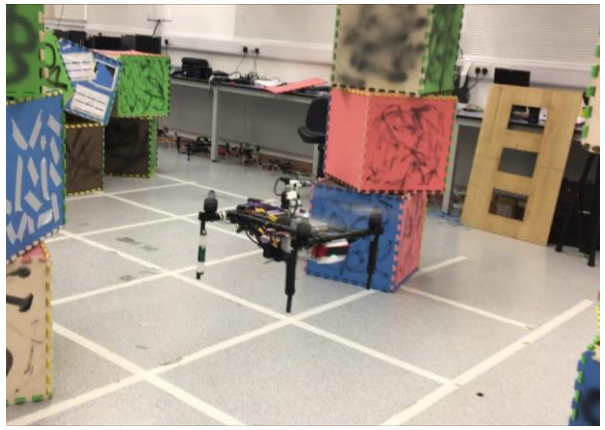
Planning strategy

- **Receding horizon re-planning** $\psi_p : planning\ horizon\ \ \psi_e : execution\ horizon$



- Get a global plan, search at the beginning of the navigation, or initialize as a straight line.
- Find a local path a local target by using the front-end.
- Generate a local trajectory by using the back-end.
- Track the trajectory within the execution horizon

- **Exploration strategy**

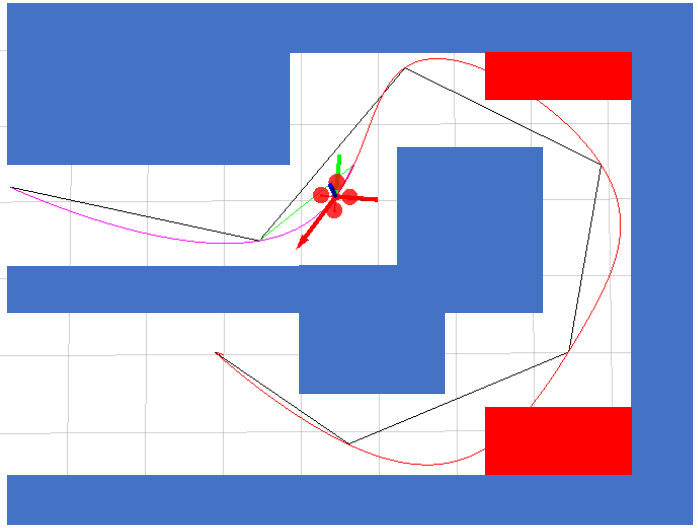


- Activated when outlier in mapping module blocks the way.
- Generate safe but short trajectories in nearby regions.
- Hover and observe.

Planning strategy

用右图红色轨迹做为初始值

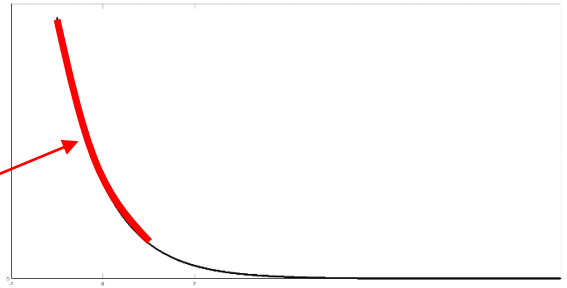
- Initialize as minimum snap trajectory
  - Good smoothness.
  - Collision may cost by overshoot. Unsafe initial value.



用右图直线轨迹做为初始值

- Initialize as straight-line trajectory following the path
  - Poor smoothness.
  - Collision free. Safe initial value.
  - Given collision-free initial trajectory, safety is achieved by using a **exponential penalty function** on collision cost.

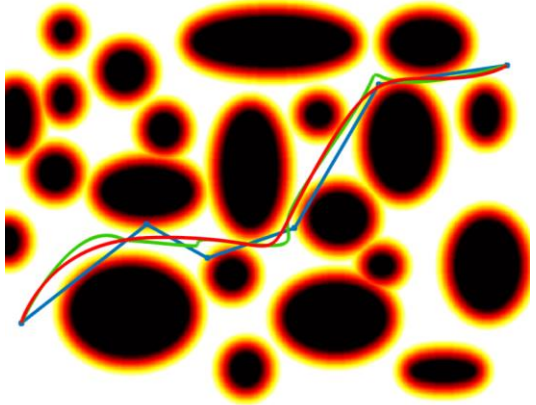
Penalty explosion near obstacles



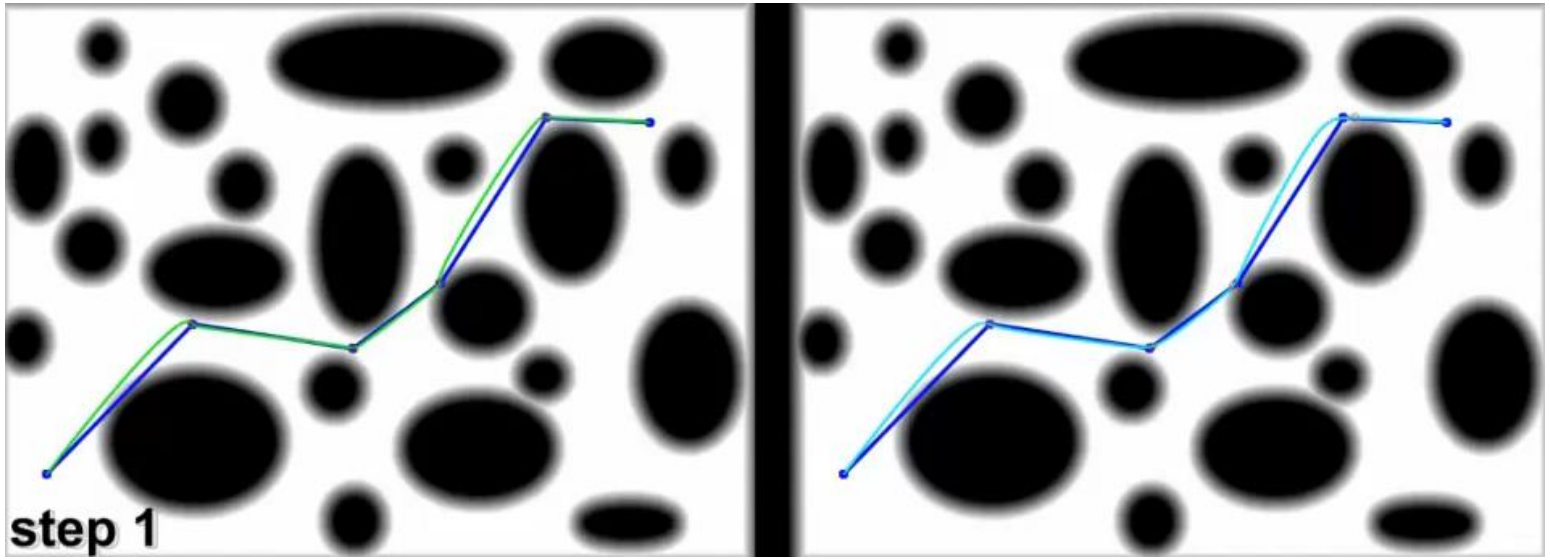
Optimization strategy

- two-step optimization

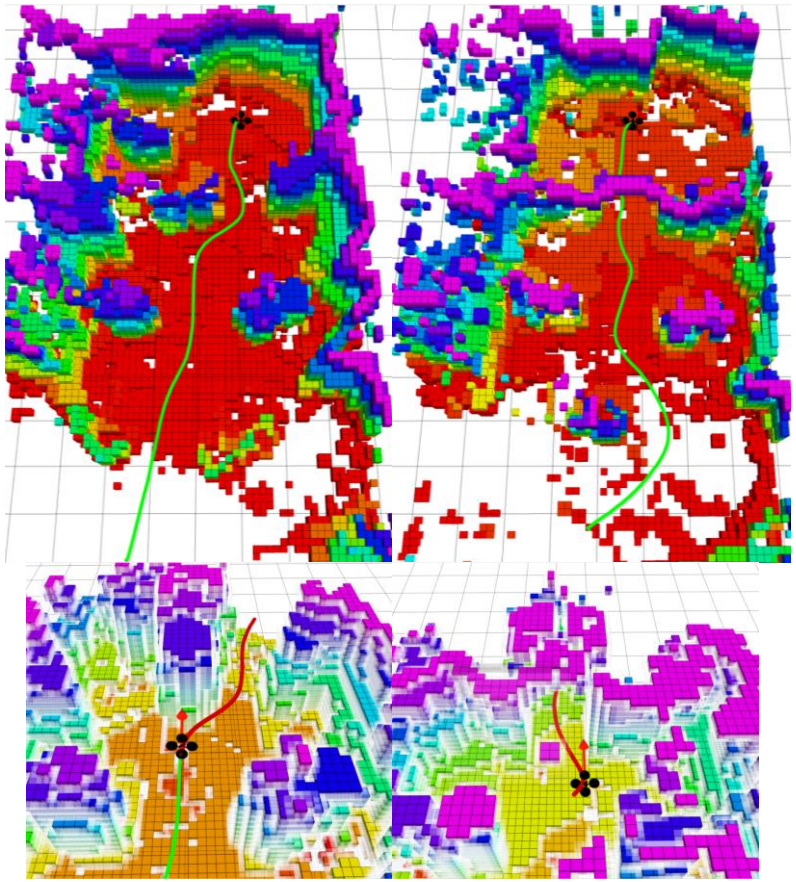
先优化轨迹使得距离远离障碍物，然后再执行

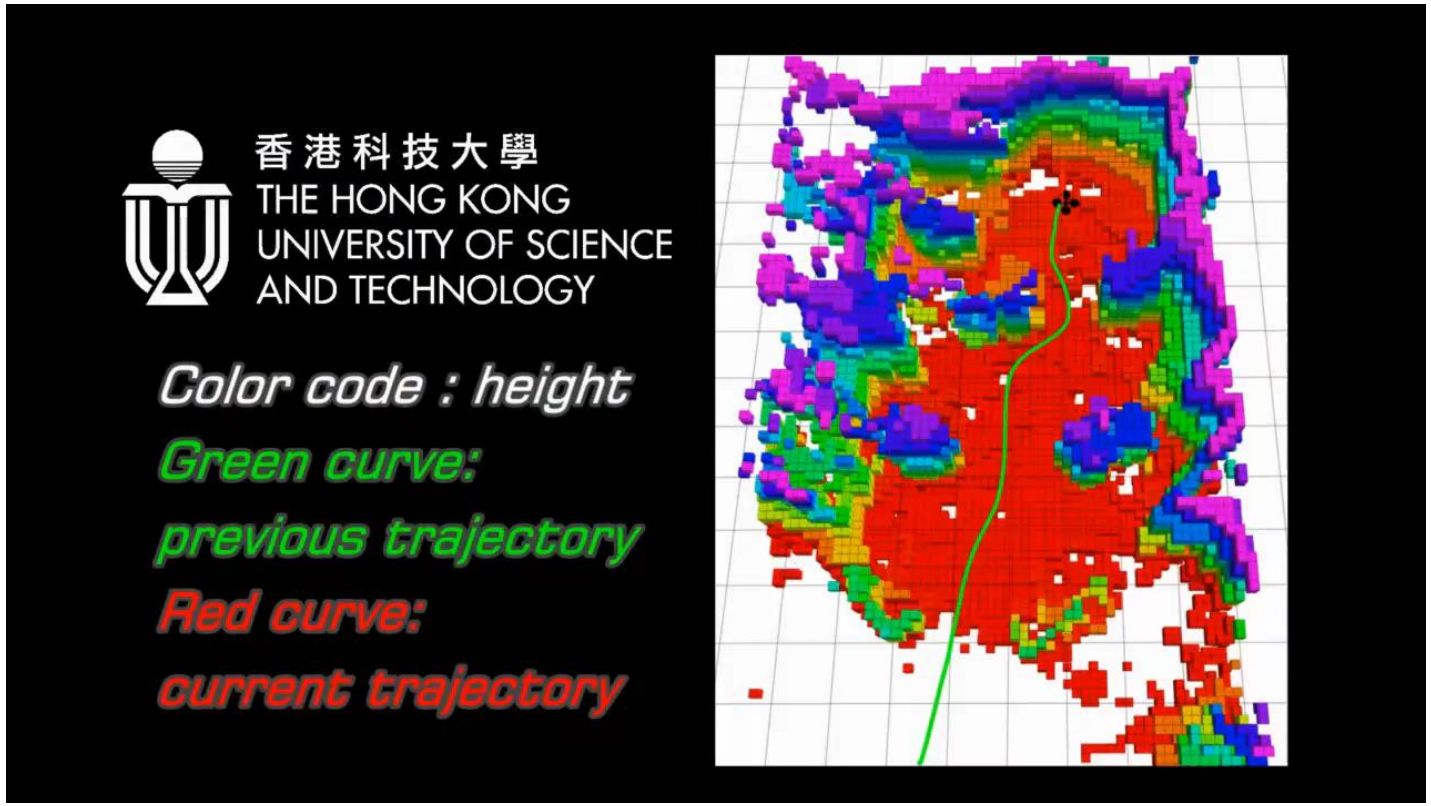


- Optimize the trajectory with collision cost only. 先initial优化
- Re-allocate time and re-parametrize the trajectory.
- Optimize the objective with a smoothness term and dynamical penalty term added.



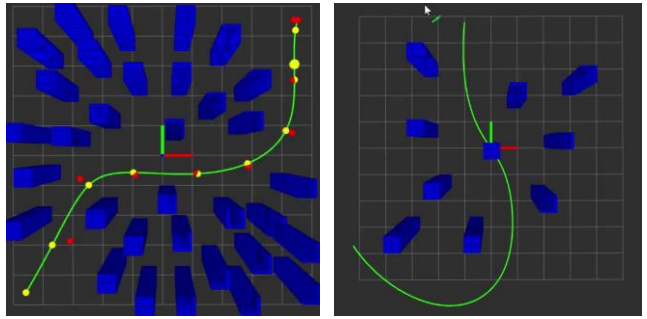
Results





Source code released at:
https://github.com/HKUST-Aerial-Robotics/grad_traj_optimization

A tool for local UAV trajectory optimization



Gradient-Based Online Safe Trajectory Generation for Quadrotor Flight in Complex Environments, **Fei Gao** et al.

Case Study

Fast Planner

Framework

Kinodynamic path searching

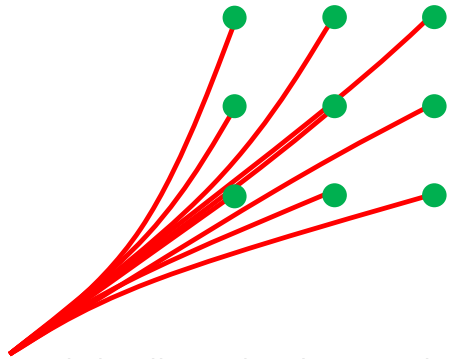
+ B-Spline trajectory optimization

+ time adjustment

B-Spline trajectory optimization

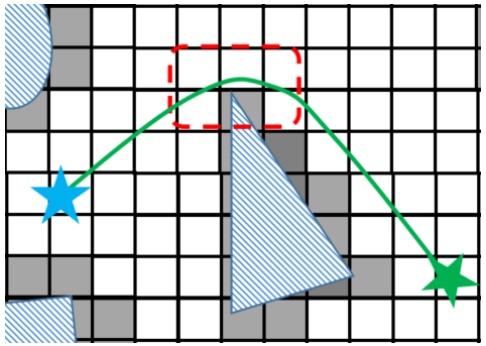
- Limitations of initial path

a) Suboptimality



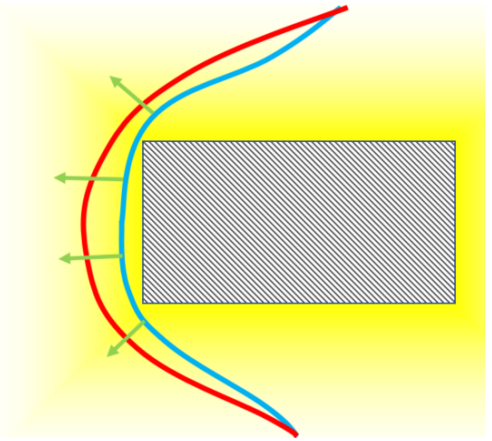
only search in discretized control space

b) Low clearance / high collision risk



some segments may be very close to obstacles

- Gradient-based trajectory optimization
  - Improve smoothness
  - Use Euclidean distance field (EDF) to improve clearance



B-Spline trajectory optimization

- Drawbacks of previous gradient-based methods:
  - Expensive computation of collision cost

$$f_{collision} = \int_{T_1}^{T_2} c(p(t))ds = \int_{T_1}^{T_2} c(p(t))||v(t)||dt = \sum_{k=0}^{(T_2-T_1)/\delta t} c(p(T_k))||v(t)||\delta t$$

small δt for accuracy → large $(T_2-T_1)/\delta t$ → high complexity

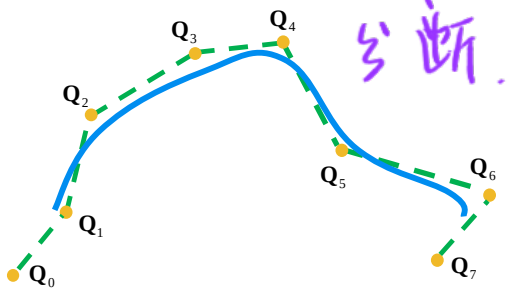
  - Also so for dynamic feasibility costs

$$f_{vel} = \sum_{\mu\in\{x,y,z\}} \int_{T_1}^{T_2} c_v\left(v_\mu(t)\right) ds = \sum_{\mu\in\{x,y,z\}} c_v\left(v_\mu(t)\right) \left\|a_\mu(t)\right\| dt$$

$$f_{acc} = \sum_{\mu\in\{x,y,z\}} \int_{T_1}^{T_2} c_a\left(a_\mu(t)\right) ds = \sum_{\mu\in\{x,y,z\}} c_a\left(a_\mu(t)\right) \left\|j_\mu(t)\right\| dt$$

B-Spline trajectory optimization

- Solution: B-Spline trajectory representation
  - Uniform B-Spline 不需要对轨迹分段.



Determined by its degree p_b, time interval Δt, and a set of $\{Q_0, Q_1, \ldots, Q_N\}$.

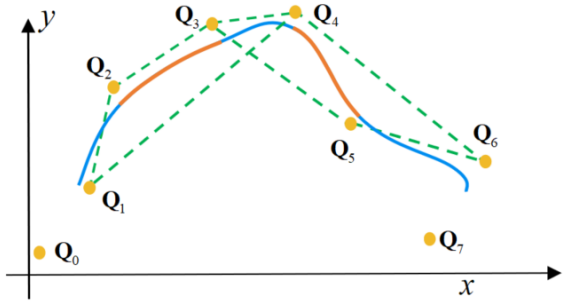
$$t \in [t_i, t_{i+1}) \xrightarrow{\text{normalize}} s(t) = (t - t_i)/\Delta t$$

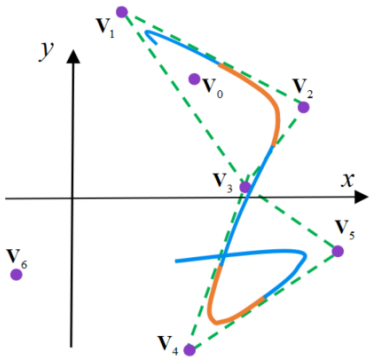
$$p(s(t)) = \begin{pmatrix} 1 \\ s \\ s^2 \\ . \\ . \\ . \\ s^{p_b} \end{pmatrix}^T M_{p_b+1} \begin{pmatrix} Q_{i-p_b} \\ Q_{i-p_b+1} \\ Q_{i-p_b+2} \\ . \\ . \\ . \\ Q_i \end{pmatrix}$$

  - Advantages

a) Convex hull property: simplify safety & dynamic constraints
b) Continuity: no need of constraints at segments joints

→ Speed up optimization

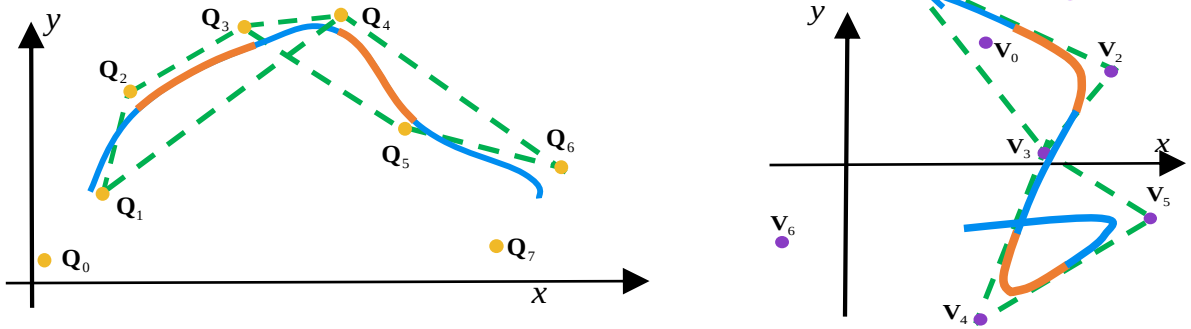




B-Spline trajectory optimization

- Convex hull property

使用样条曲线 点,需要优化控制点.
而且轨迹不需要分段.



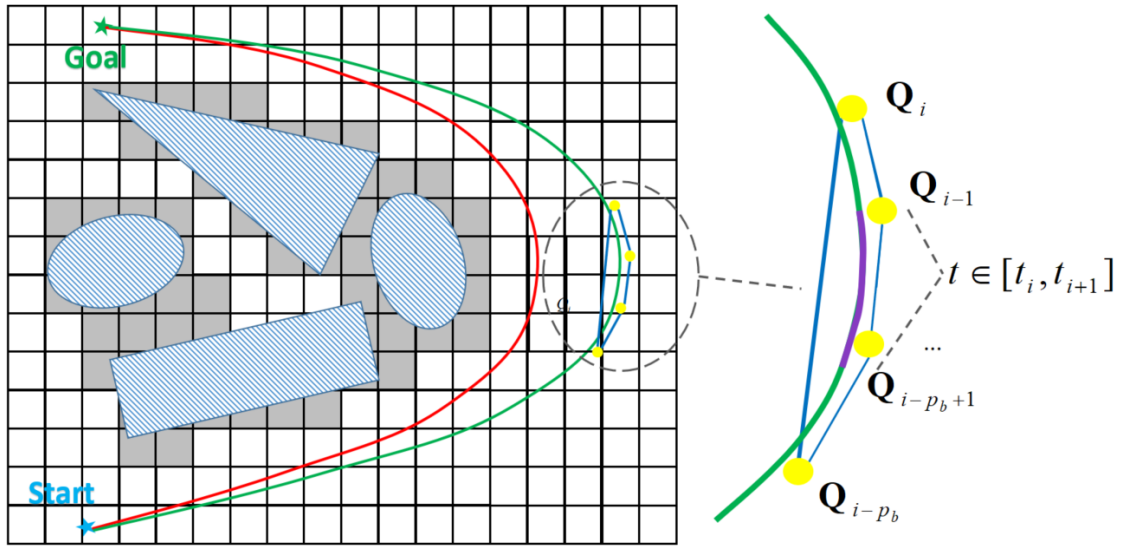
Each segment of a B-spline is **bounded** by the corresponding convex hull of control points (left), and so is the derivative (right).

- Simplify safety & dynamic constraints

Convex hulls are within feasible space —convex hull property→ The whole trajectory is feasible!

B-Spline trajectory optimization

- Ensure safety by convex hull



Push control points away from obstacles → Convex hulls are pushed to safe space → The whole trajectory is bounded in safe region

B-Spline trajectory optimization

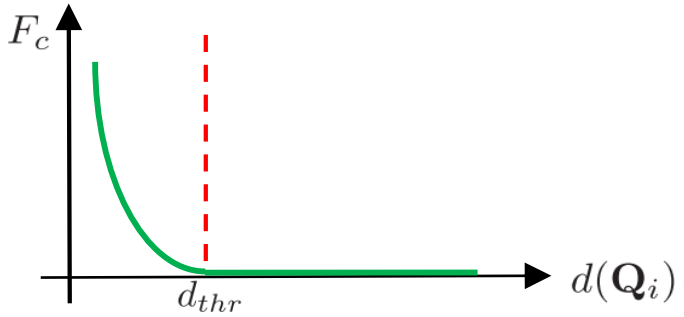
- Problem formulation $f_{total} = \lambda_1 f_s + \lambda_2 f_c + \lambda_3 (f_v + f_a)$
  - Smoothness - elastic band cost function

$$f_s = \sum_{i=p_b-1}^{N-p_b+1} \left\| \underbrace{(\mathbf{Q}_{i+1} - \mathbf{Q}_i)}_{F_{i+1,i}} - \underbrace{(\mathbf{Q}_{i-1} - \mathbf{Q}_i)}_{F_{i-1,i}} \right\|^2$$

  - Safety - potential function in EDF

$$f_c = \sum_{i=p_b}^{N-p_b} F_c(d(\mathbf{Q}_i))$$

$$F_c(d(\mathbf{Q}_i)) = \begin{cases} (d(\mathbf{Q}_i) - d_{thr})^2 & d(\mathbf{Q}_i) \le d_{thr} \\ 0 & d(\mathbf{Q}_i) > d_{thr} \end{cases}$$



Solve this non-linear optimization by the L-BFGS algorithm

  - Dynamic feasibility

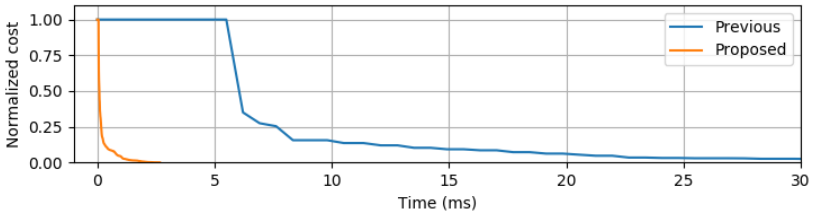
$$f_v = \sum_{\mu \in \{x,y,z\}} \sum_{i=p_b}^{N-p_b} F_v(V_{i\mu}), \quad f_a = \sum_{\mu \in \{x,y,z\}} \sum_{i=p_b-2}^{N-p_b} F_a(A_{i\mu})$$

$$F_v(v_\mu) = \begin{cases} (v_\mu^2 - v_{\max}^2)^2 & v_\mu^2 > v_{\max}^2 \\ 0 & v_\mu^2 \le v_{\max}^2 \end{cases}$$

Results

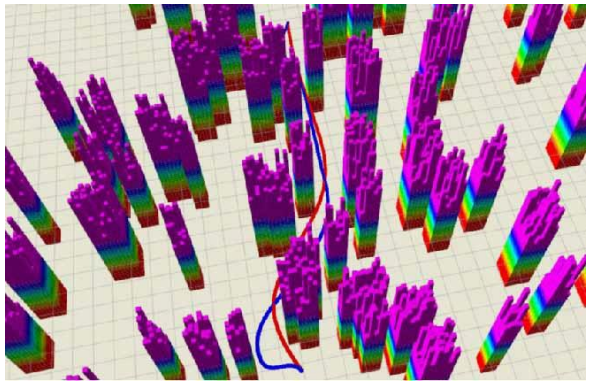
- Comparisons

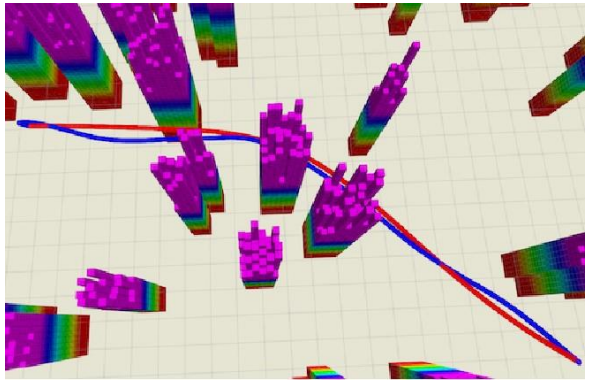
a) Faster convergence



b) Higher trajectory quality

	Integral of Jerk^2 (m^2/s^5)			Comp. Time(s)
	Mean	Max	Std	
Previous [1]	43.913	181.495	18.394	0.010
Proposed	**35.932**	**131.913**	**13.118**	0.001

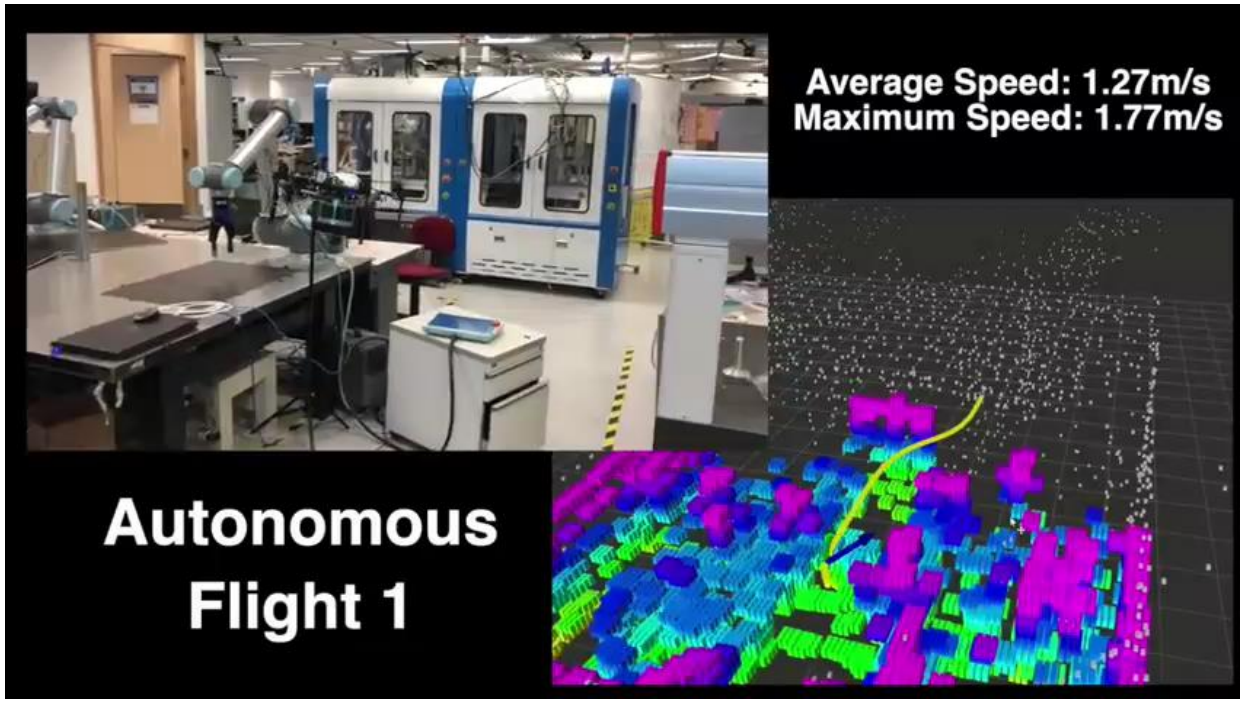


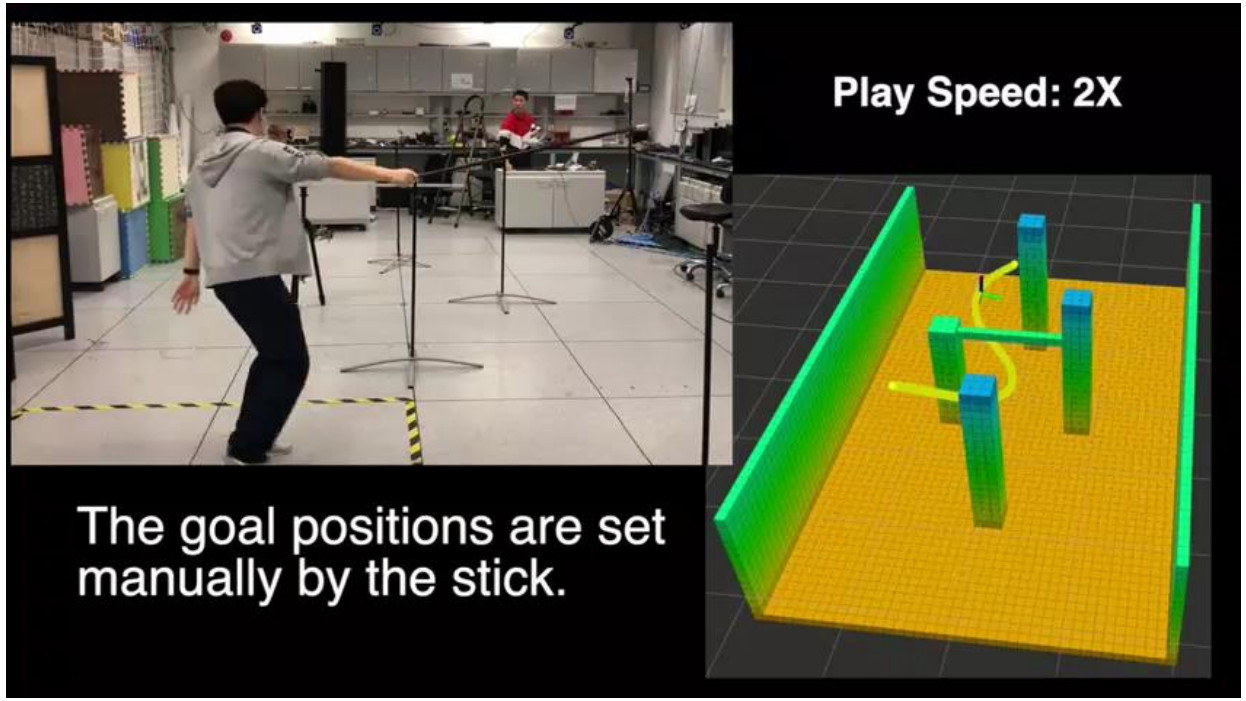


Trajectories generated by the proposed (red) and previous [1] (blue) method. Computation time for proposed method is 10 times shorter!

Gradient-Based Online Safe Trajectory Generation for Quadrotor Flight in Complex Environments, **Fei Gao** et al.

Results


Average Speed: 1.27m/s
Maximum Speed: 1.77m/s
Autonomous
Flight 1

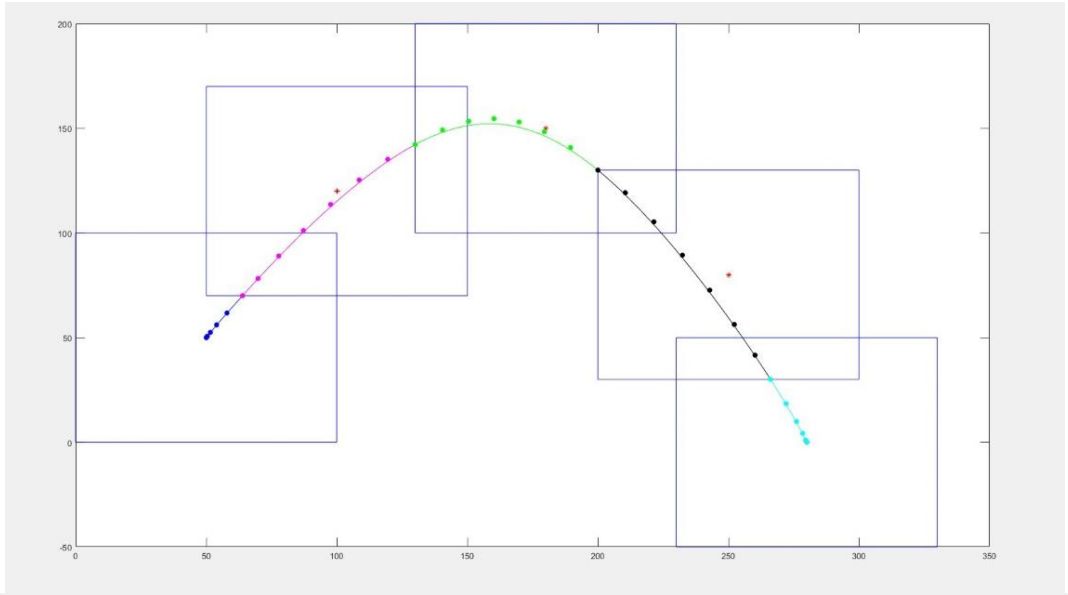

Play Speed: 2X
The goal positions are set
manually by the stick.

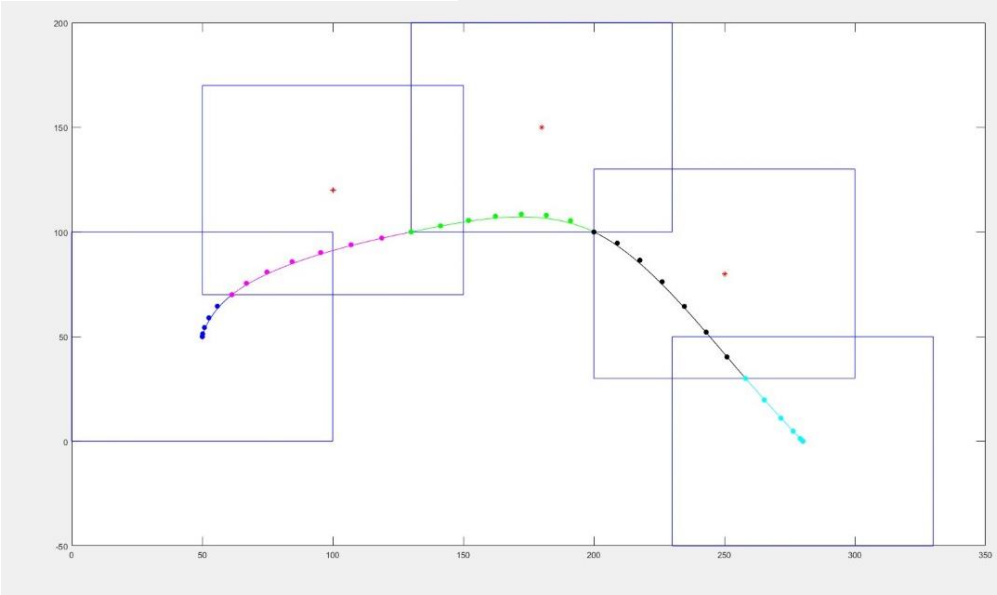
Homework

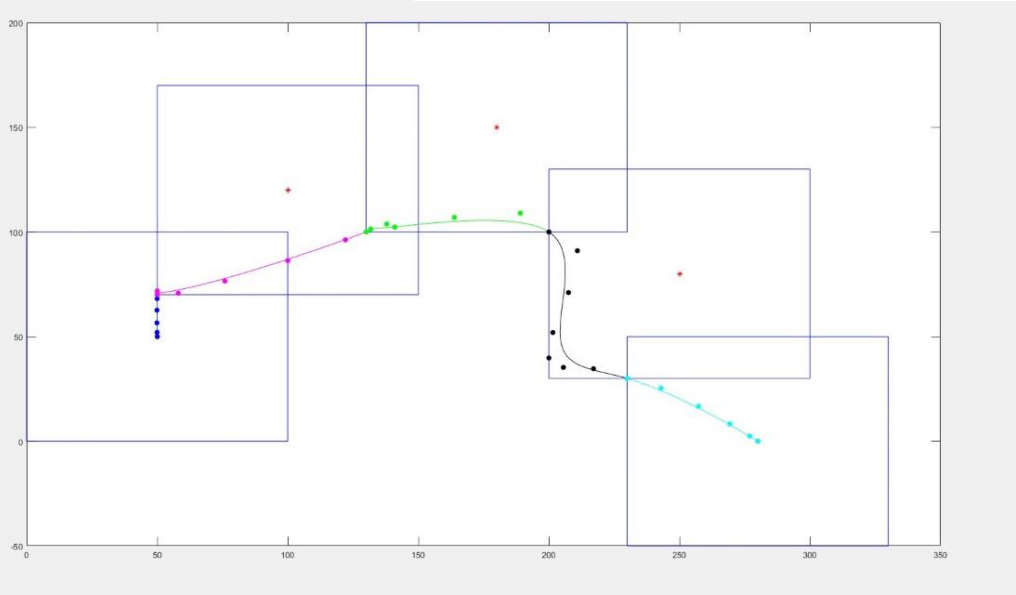
Homework

- In matlab, write a corridor-constrained piecewise Bezier curve generation.
- The conversion between Bezier to monomial polynomial is given.
- The corridor is pre-defined.
- Only position needs to be constrained.
- TA provides a video tutorial.

Homework


200
150
100
50
0
-50
0
50
100
150
200
250
300
350


200
150
100
50
0
-50
0
50
100
150
200
250
300
350


200
150
100
50
0
-50
0
50
100
150
200
250
300
350

Thanks for Listening!