

Lecture 2

SEARCH-BASED PATH FINDING



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Outline



1. Graph Search Basis



2. Dijkstra and A*



3. Jump Point Search



4. Homework

Graph Search Basis



Configuration Space



Configuration Space

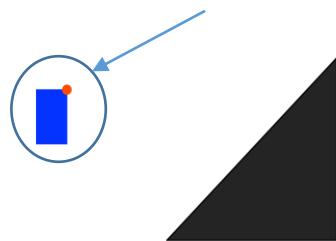
- **Robot configuration:** a specification of the positions of all points of the robot
- **Robot degree of freedom (DOF):** The minimum number n of real-valued coordinates needed to represent the robot configuration
- **Robot configuration space:** a n -dim space containing all possible robot configurations, denoted as **C-space**
- **Each robot pose is a point in the C-space**



Configuration Space Obstacle

- Planning in workspace
 - Robot has different shape and size
 - Collision detection requires knowing the robot geometry - time consuming and hard

Check collision for the robot geometry



(1) Rectangular mobile robot



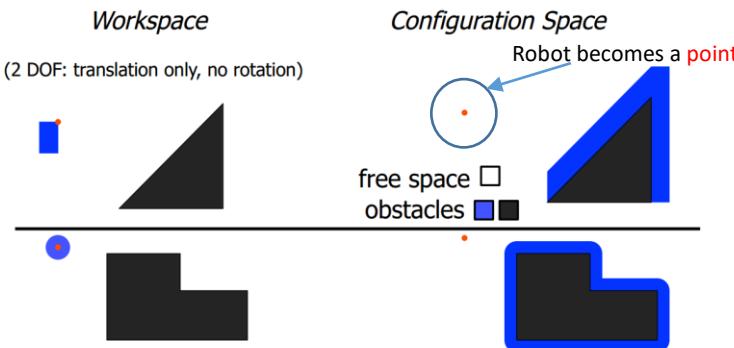
(2) Circular mobile robot



Configuration Space Obstacle

- Planning in configuration space

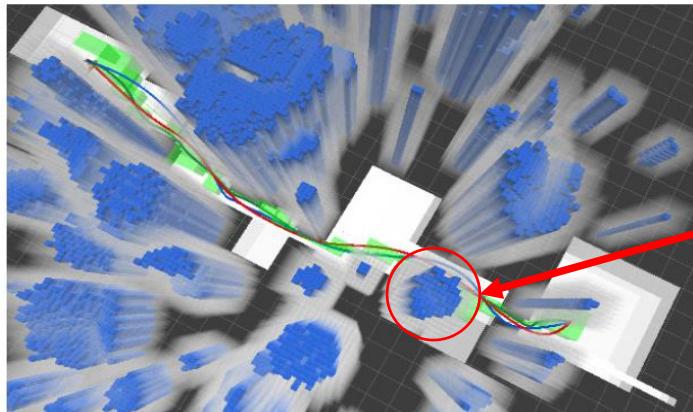
- Robot is represented by a point in C-space, e.g. position (a **point** in R^3), pose (a **point** in $SO(3)$),etc.
- Obstacles need to be represented in configuration space (one-time work prior to motion planning), called configuration space obstacle, or C-obstacle
- C-space = (C-obstacle) \cup (C-free)
- The path planning is finding a path between start **point** q_{start} and goal **point** q_{goal} within C-free





Workspace and Configuration Space Obstacle

- In workspace
 - Robot has shape and size (i.e. hard for motion planning)
- In configuration space: C-space
 - Robot is a **point** (i.e. easy for motion planning)
 - Obstacles are represented in C-space prior to motion planning
- Representing an obstacle in C-space can be extremely complicated. So approximated (but more conservative) representations are used in practice.



If we model the robot conservatively as a ball with radius δ_r , then the C-space can be constructed by inflating obstacle at all directions by δ_r .

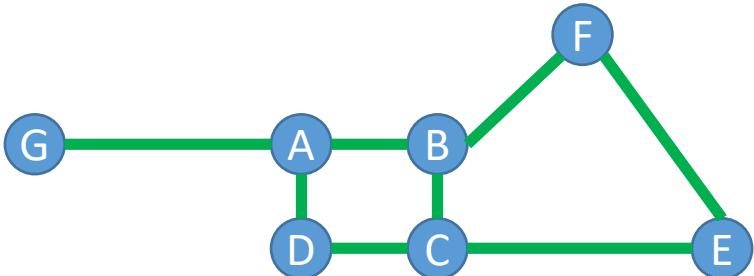
Graph and Search Method



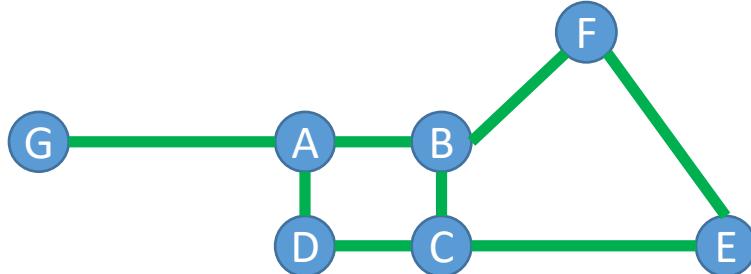
Search-based Method

Graphs

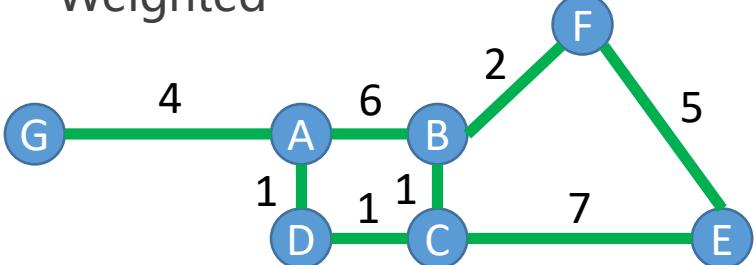
Graphs have **nodes** and **edges**



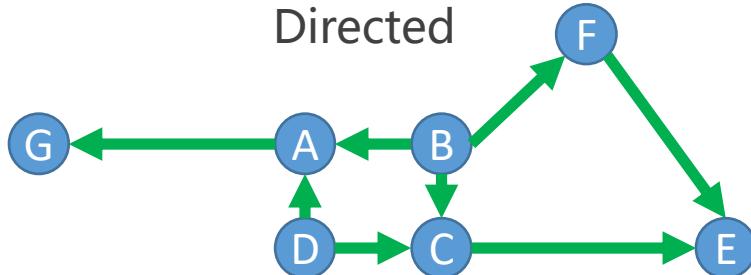
Undirected



Weighted



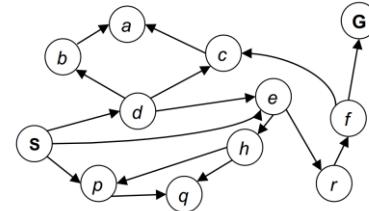
Directed



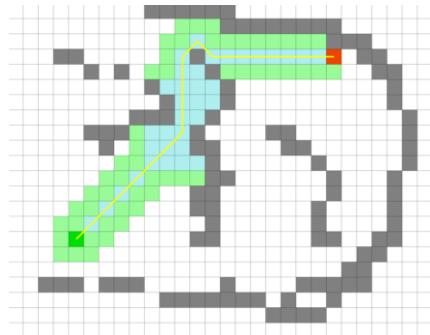


Search-based Method

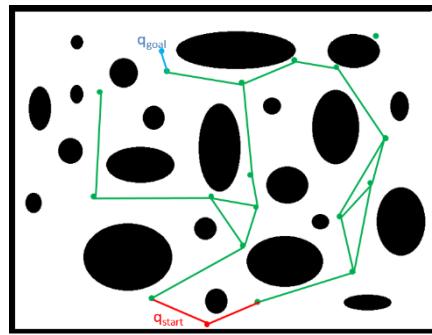
- State space graph: a mathematical representation of a **search algorithm**
 - For every search problem, there's a corresponding state space graph
 - Connectivity between nodes in the graph is represented by (directed or undirected) edges



*Ridiculously tiny search graph
for a tiny search problem*



Grid-based graph: use grid as vertices and grid connections as edges

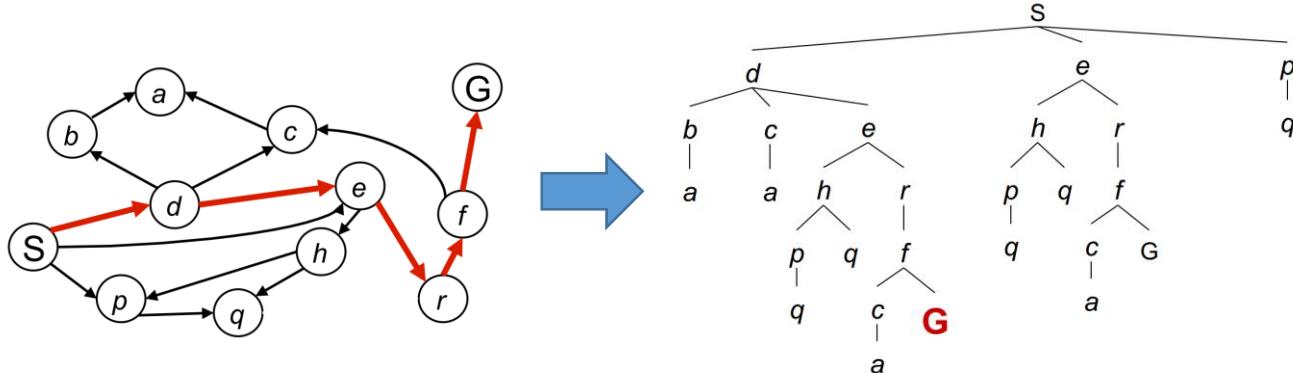


The graph generated by probabilistic roadmap (PRM)



Graph Search Overview

- The search always starts from start state X_s
 - Searching the graph produces a search tree
 - Back-tracing a node in the search tree gives us a path from the start state to that node
 - For many problems we can never actually build the whole tree, too large or inefficient – we only want to reach the goal node asap.





Graph Search Overview

- Maintain a **container** to store all the nodes **to be visited**
- The container is initialized with the start state X_S
- Loop
 - Remove a node from the container according to some pre-defined score function
 - Visit a node
 - Expansion: Obtain all **neighbors** of the node
 - Discover all its neighbors
 - Push them (**neighbors**) into the container
- End Loop



Graph Search Overview

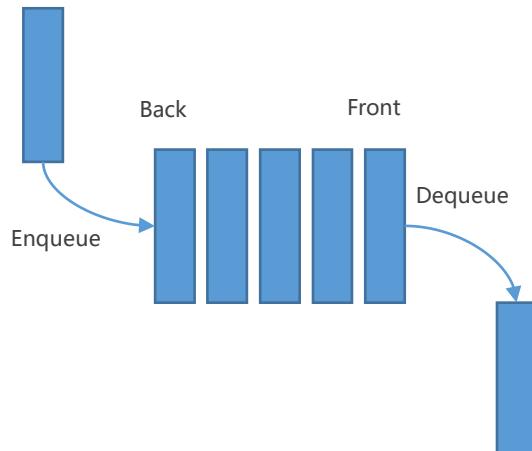
- Question 1: When to end the loop?
 - Possible option: End the loop when the container is empty
- Question 2: What if the graph is cyclic?
 - When a node is removed from the container (expanded / visited), it should never be added back to the container again
- Question 3: In what way to remove the right node such that the **goal state can be reached as soon as possible**, which results in less expansion of the graph node.



Graph Traversal

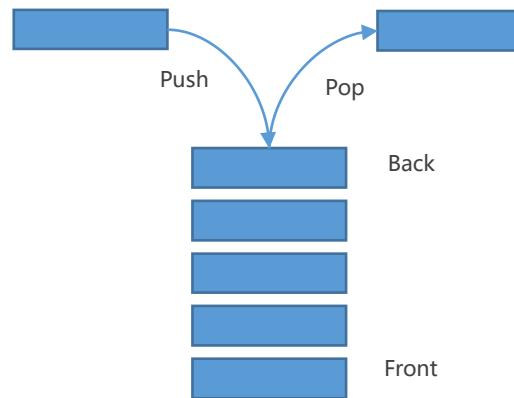
- Breadth First Search (BFS) vs. Depth First Search (DFS)

BFS uses "first in first out"



This is a **queue**

DFS uses "last in first out"



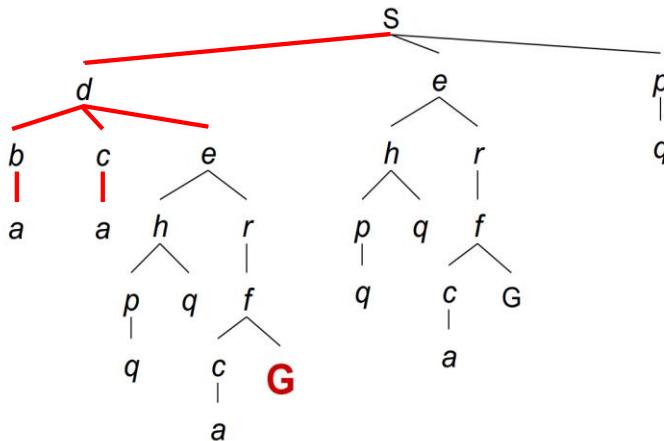
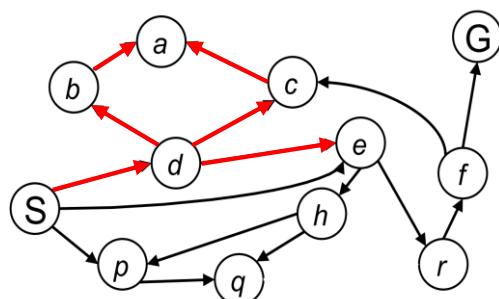
This is a **stack**



Depth First Search (DFS)

找最有可為 極限之路

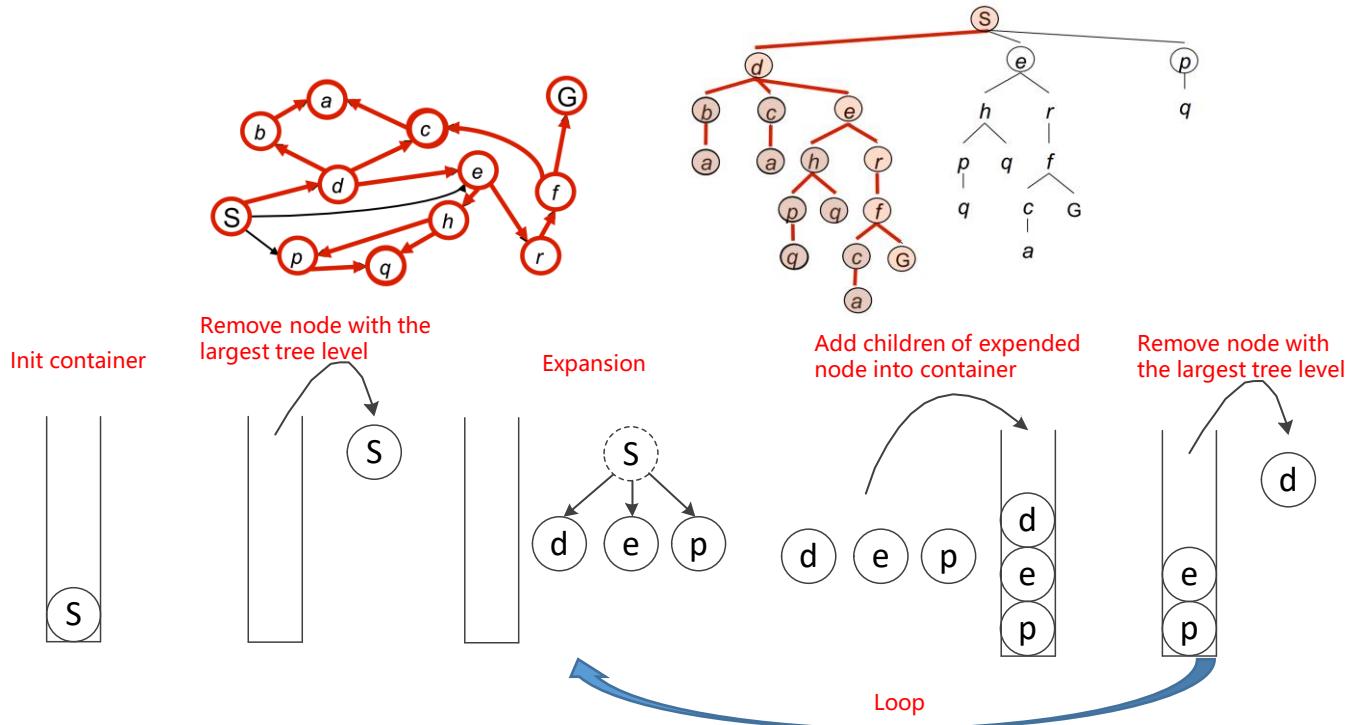
- Strategy: remove / expand the deepest node in the container





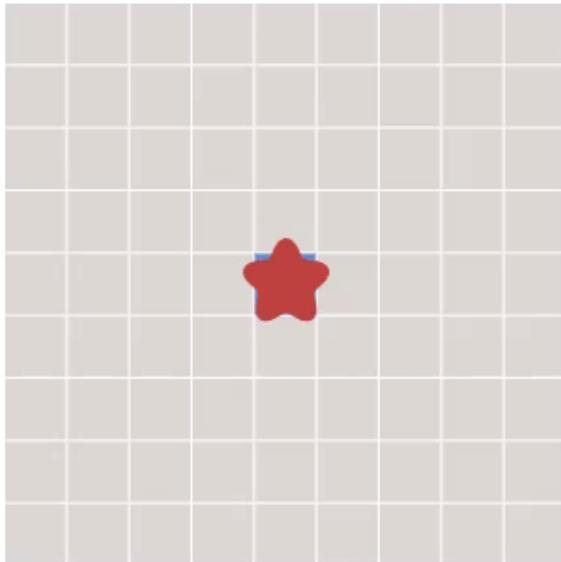
Depth First Search (DFS)

- Implementation: maintain a last in first out (LIFO) container (i.e. stack)





Depth First Search (DFS)

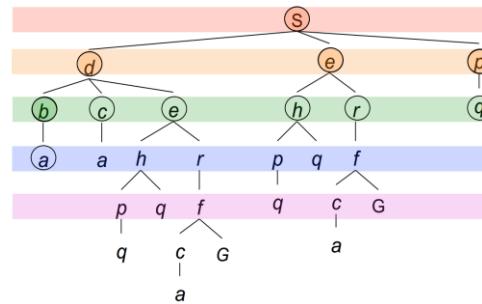
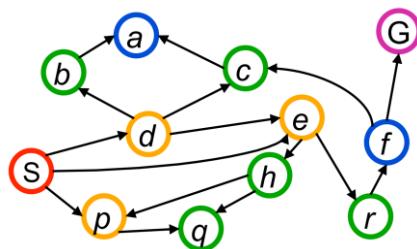


Courtesy: Amit Patel's Introduction to A*, Stanford



Breadth First Search (BFS)

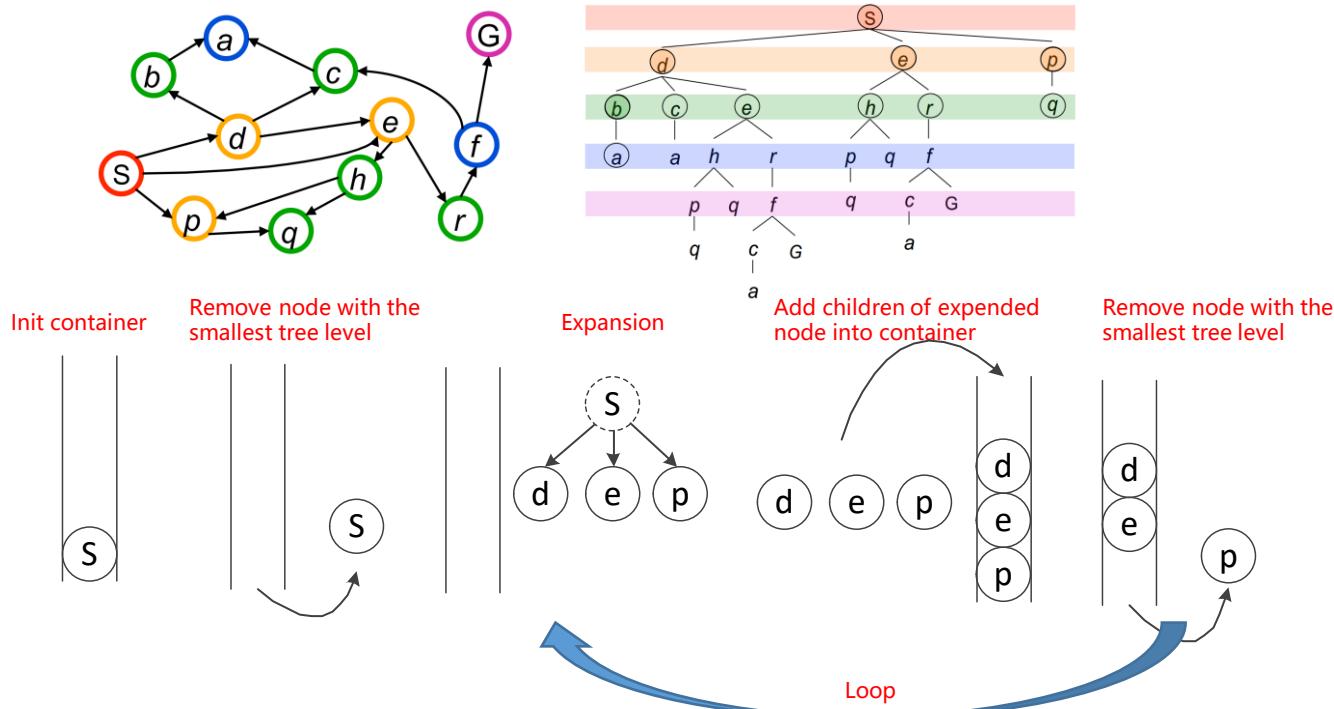
- Strategy: remove / expand the shallowest node in the container





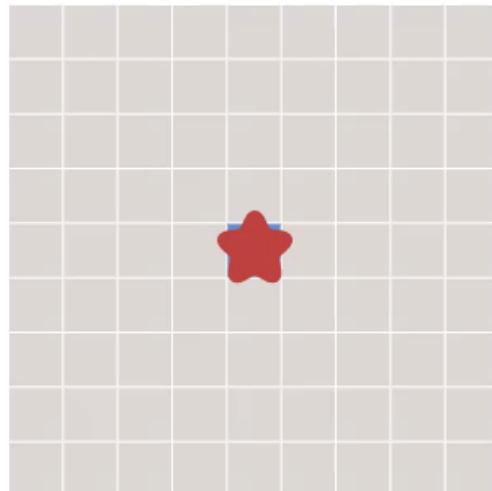
Breadth First Search (BFS)

- Implementation: maintain a first in first out (FIFO) container (i.e. queue)





Breadth First Search (BFS)

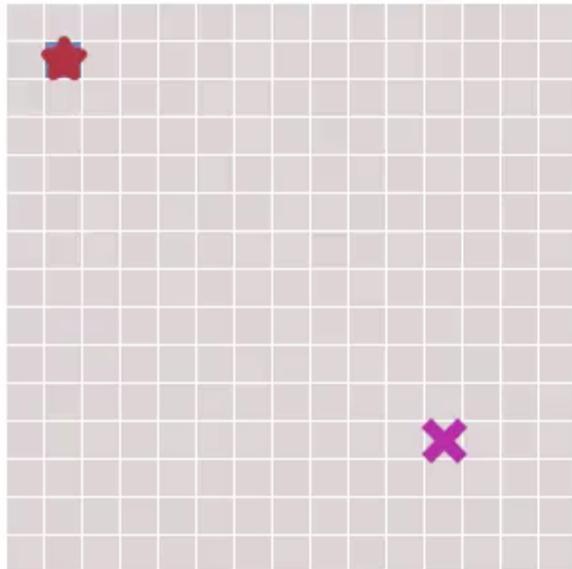


Courtesy: Amit Patel's Introduction to A*, Stanford

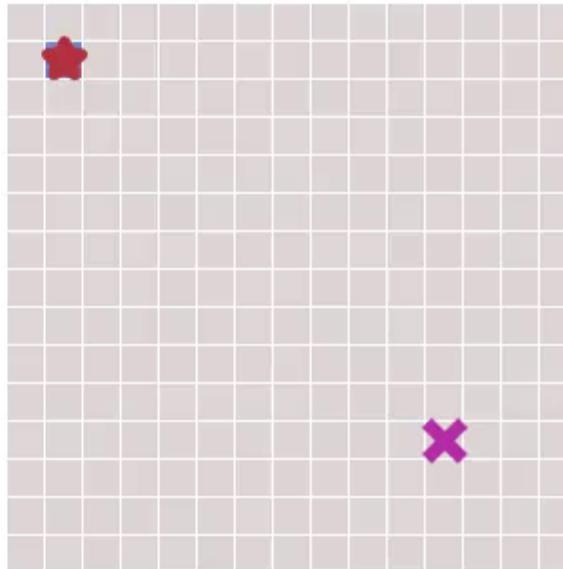


BFS vs. DFS: which one is useful?

Breadth First Search



Depth First Search



Remember BFS.



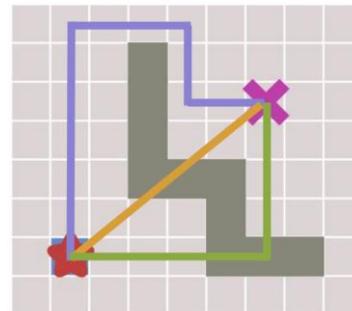
Heuristic search



Greedy Best First Search

- BFS and DFS pick the next node off the frontiers based on which was "first in" or "last in".
- Greedy Best First picks the "best" node according to some rule, called a **heuristic**.
- **Definition:** A heuristic is a **guess** of how close you are to the target.

- A heuristic guides you in the right direction.
- A heuristic should be easy to compute.



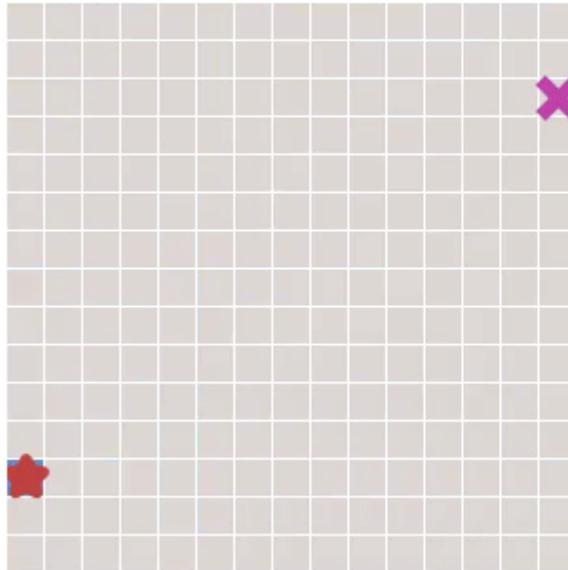
- Euclidean Distance
- Manhattan Distance

Both are approximations for the actual **shortest path**.

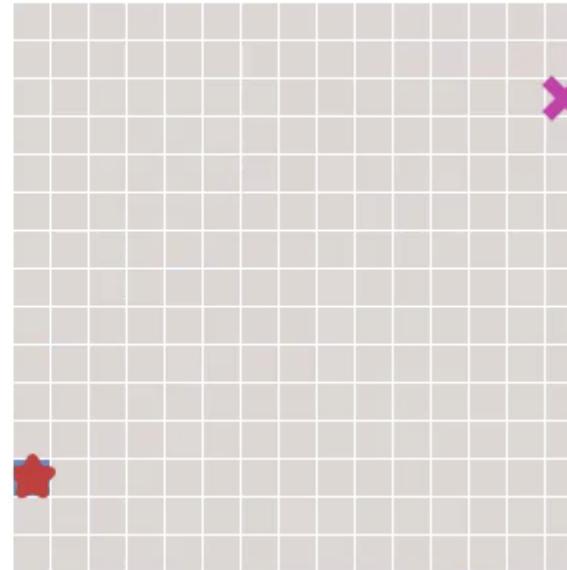


Greedy Best First Search

Breadth First Search



Greedy Best-First Search



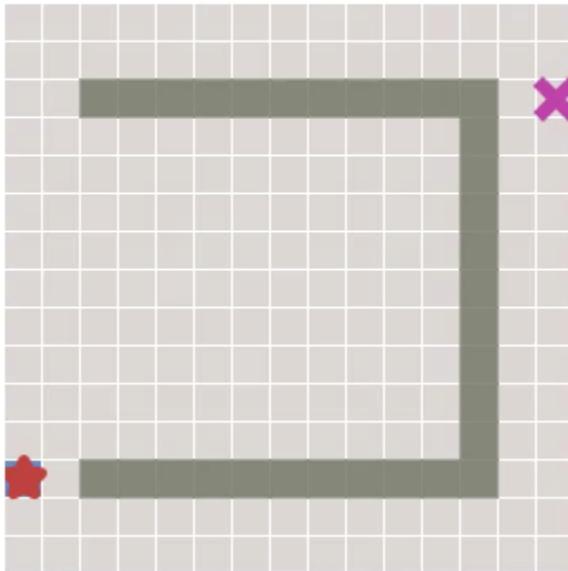
Looks pretty good.

Courtesy: Amit Patel's Introduction to A*, Stanford

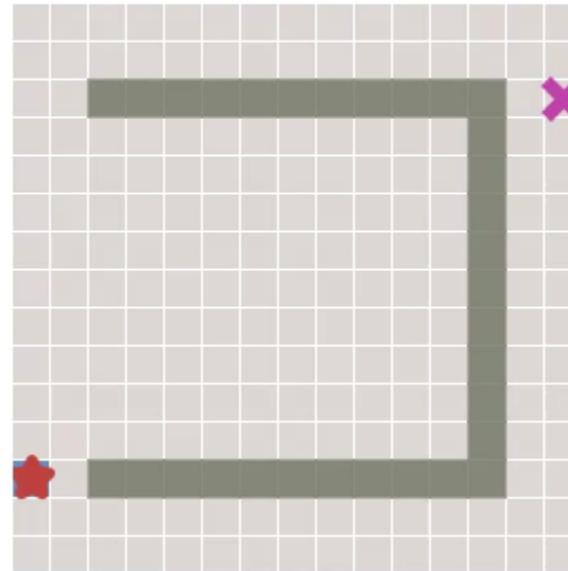


Greedy Best First Search

Breadth First Search



Greedy Best-First Search

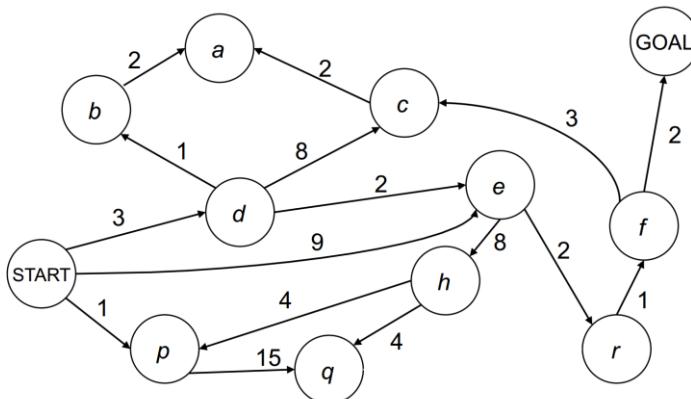


But with obstacles ...



Costs on Actions

- A practical search problem has a **cost “C”** from a node to its neighbor
 - Length, time, energy, etc.
- When all weight are 1, BFS finds the optimal solution
- For general cases, how to find the **least-cost path** as soon as possible?



Dijkstra and A*

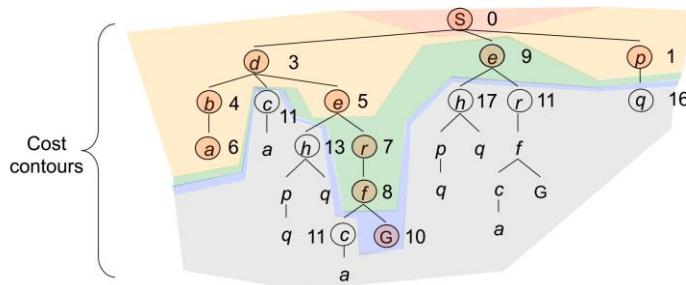
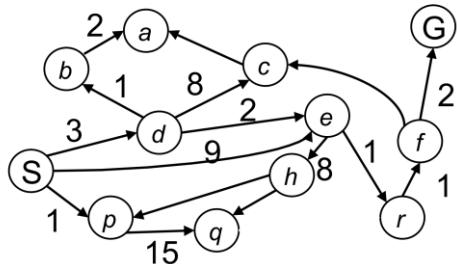


Algorithm Workflow



Dijkstra's Algorithm

- Strategy: expand/visit the node with **cheapest accumulated cost $g(n)$**
- $g(n)$: The current best estimates of the accumulated cost from the start state to node "n"
- Update the accumulated costs $g(m)$ for all unexpanded neighbors "m" of node "n"
- A node that has been expanded/visited is guaranteed to have the smallest cost from the start state





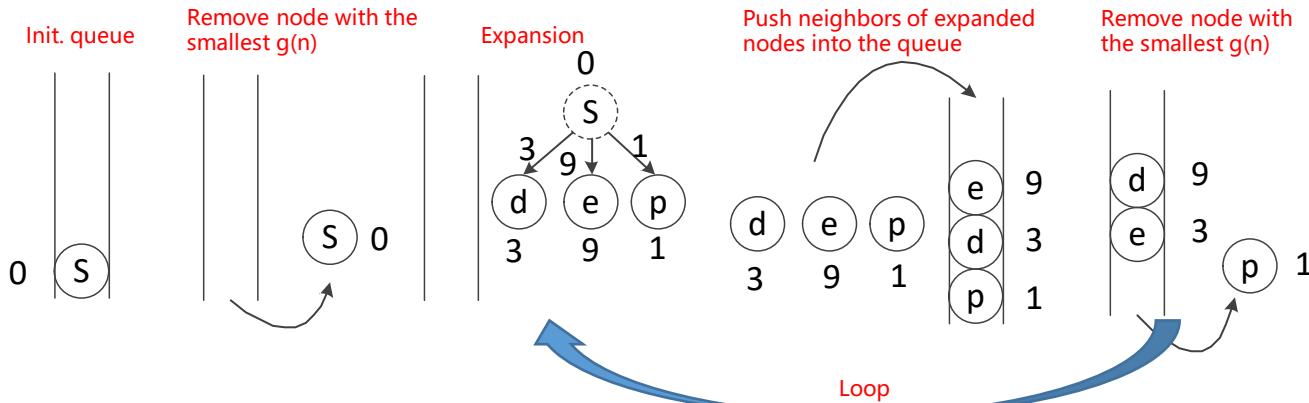
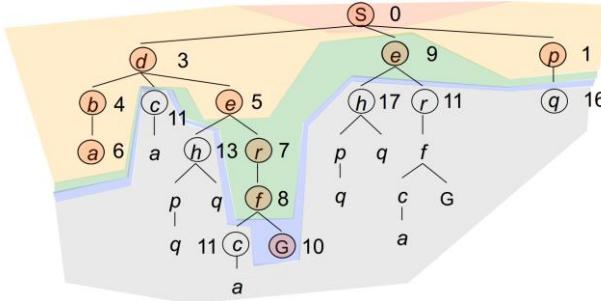
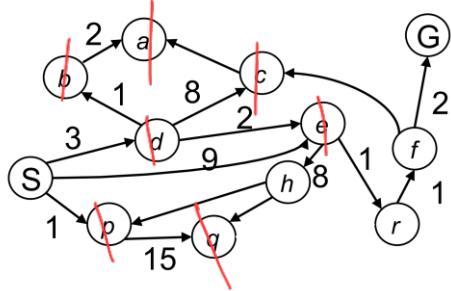
Dijkstra's Algorithm

从小到大排序，队首的 cost 最小

- Maintain a **priority queue** to store all the nodes to be expanded
- The priority queue is initialized with the start state X_s
- Assign $g(X_s)=0$, and $g(n)=\text{infinite}$ for all other nodes in the graph
- Loop
 - If the queue is empty, return FALSE; break;
 - Remove the node "n" with the lowest $g(n)$ from the priority queue = **open list**
 - Mark node "n" as expanded
 - If the node "n" is the goal state, return TRUE; break;
 - For all unexpanded neighbors "m" of node "n"
 - If $g(m) = \text{infinite}$ **m 的 cost 是 infinite, 说明 m 没有被遍历过.**
 • $g(m) = g(n) + C_{nm}$
 • Push node "m" into the queue
 - If $g(m) > g(n) + C_{nm}$ **说明新路径的 cost 较低, 则更新成 new value.**
 • $g(m) = g(n) + C_{nm}$
 - end
- End Loop



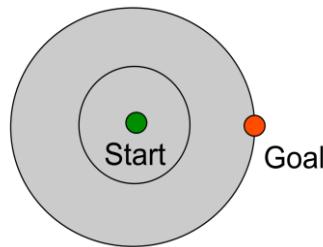
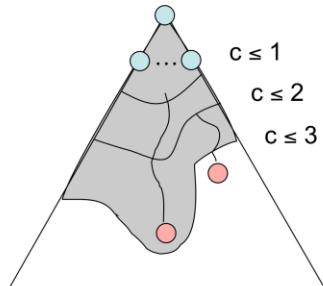
Dijkstra' s Algorithm





Pros and Cons of Dijkstra's Algorithm

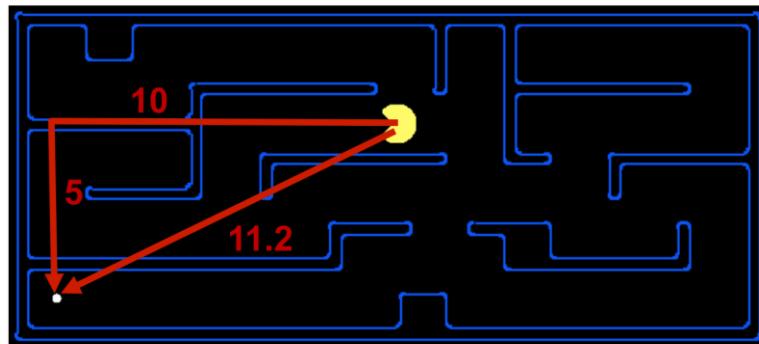
- The good:
 - Complete and optimal
- The bad:
 - Can only see the cost accumulated so far (i.e. the uniform cost), thus exploring next state in every "direction"
 - No information about goal location





Search Heuristics

- Recall the heuristic introduced in **Greedy Best First Search**
- Overcome the shortcomings of uniform cost search by **inferring the least cost to goal (i.e. goal cost)**
- Designed for particular search problem
- Examples: Manhattan distance VS. Euclidean distance





A*: Dijkstra with a Heuristic

- Accumulated cost
 - $g(n)$: The current best estimates of the accumulated cost from the start state to node “n”
- Heuristic
 - $h(n)$: The **estimated least cost** from node n to goal state (i.e. goal cost)
- The least estimated cost from start state to goal state passing through node “n” is $f(n) = g(n) + h(n)$
- Strategy: expand the node with **cheapest $f(n) = g(n) + h(n)$**
 - Update the accumulated costs $g(m)$ for all unexpanded neighbors “m” of node “n”
 - A node that has been expanded is guaranteed to have the smallest cost from the start state



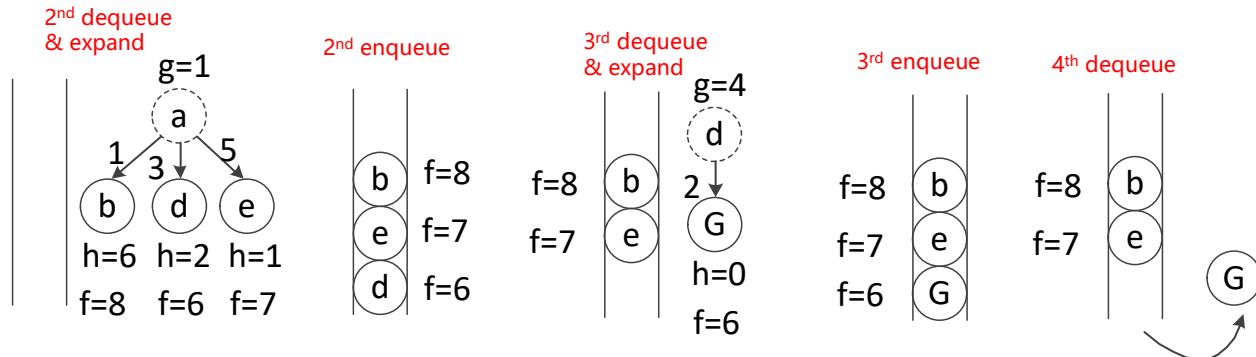
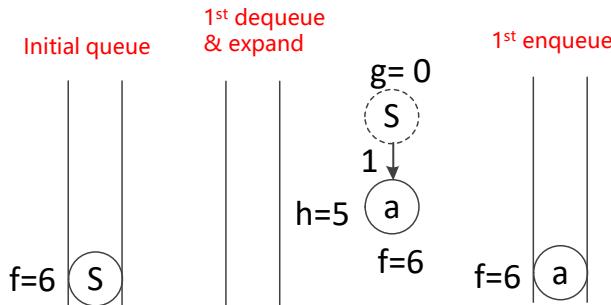
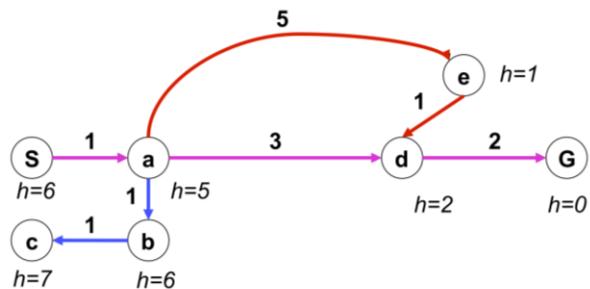
A* Algorithm

- Maintain a **priority queue** to store all the nodes to be expanded
- The heuristic function $h(n)$ for all nodes are pre-defined
- The priority queue is initialized with the start state X_S
- Assign $g(X_S)=0$, and $g(n)=\infty$ for all other nodes in the graph
- Loop
 - If the queue is empty, return FALSE; break;
 - Remove the node “n” with the lowest $f(n)=g(n)+h(n)$ from the priority queue
 - Mark node “n” as **expanded**
 - If the node “n” is the goal state, return TRUE; break;
 - For all **unexpanded** neighbors “m” of node “n”
 - If $g(m) = \infty$
 - $g(m)= g(n) + C_{nm}$
 - Push node “m” into the queue
 - If $g(m) > g(n) + C_{nm}$
 - $g(m)= g(n) + C_{nm}$
 - end
- End Loop

Only difference comparing to
Dijkstra’s algorithm

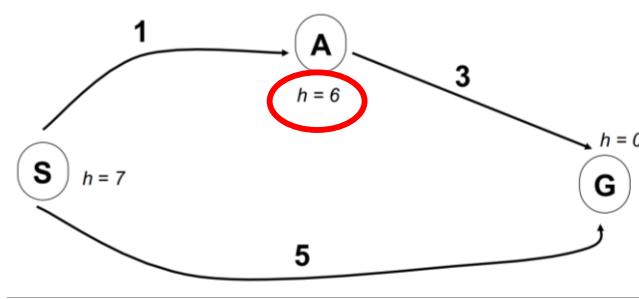


A* Example





A* Optimality

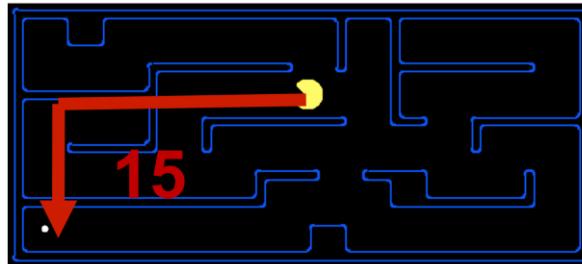
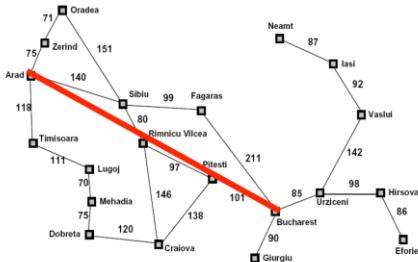


- What went wrong?
- For node A: actual least cost to goal (i.e. goal cost) < estimated least cost to goal (i.e. heuristic)
- We need the estimate to be **less than** actual least cost to goal (i.e. goal cost) **for all nodes!**



Admissible Heuristics

- A Heuristic h is **admissible** (optimistic) if:
 - $h(n) \leq h^*(n)$ for all node "n", where $h^*(n)$ is the true least cost to goal from node "n"
- If the heuristic is admissible, the A* search is optimal
满足 $h(n) < h^(n)$ 的时候*
函数才会寻到 A 最优*
- Coming up with admissible heuristics is most of what's involved in using A* in practice.
- Example:





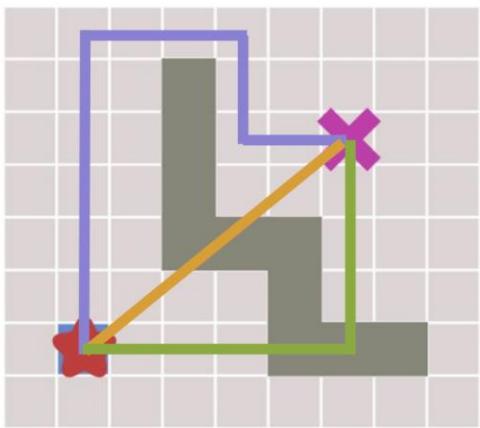
Heuristic Design

An admissible heuristic function has to be designed case by case.

- Euclidean Distance
- Manhattan Distance

Is Euclidean distance (L2 norm) admissible? **Always**

Is Manhattan distance (L1 norm) admissible? **Depends**



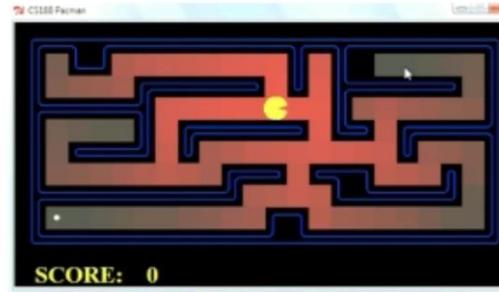
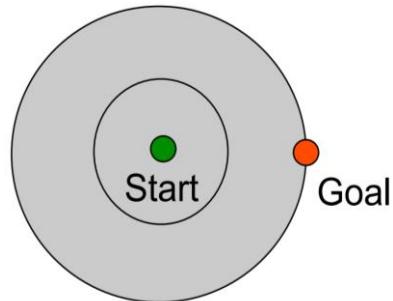
Is L^∞ norm distance admissible? **Always**

Is 0 distance admissible? **Always**

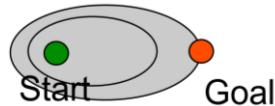


Dijkstra' s VS A*

- Dijkstra' s algorithm expanded in all directions



- A* expands mainly towards the goal, but does not hedge its bets to ensure optimality





Sub-optimal Solution

What if we intend to use an over-estimate heuristic?

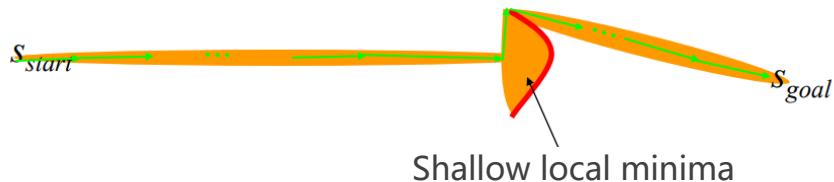


- Suboptimal path
- Faster



Weighted A*

Expands states based on
 $f = g + \epsilon h, \epsilon > 1$ = bias towards states that
are closer to goal.



- Weighted A* Search:

- Optimality vs. speed
- ϵ -suboptimal:
$$\text{cost(solution)} \leq \epsilon \text{cost(optimal solution)}$$
- It can be orders of magnitude faster than A*

Weighted A* → Anytime A* → ARA* → D*

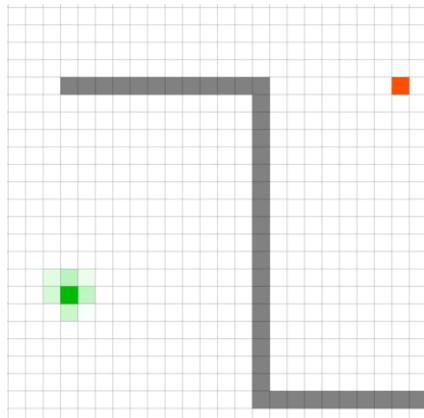


Beyond the scope of this course



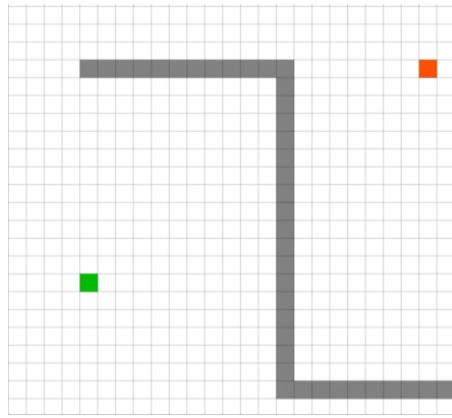
Greedy Best First Search vs. Weighted A* vs. A*

$$f = \textcolor{red}{a} \cdot g + \textcolor{red}{b} \cdot h$$



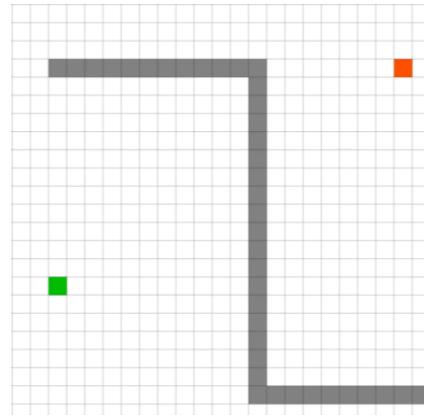
Most Greedy

$$a = 0, b = 1$$



Tunable Greediness

$$a = 1, b = \varepsilon > 1$$



Optimal

$$a = 1, b = 1$$



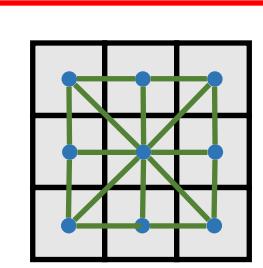
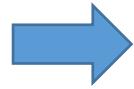
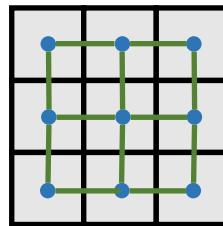
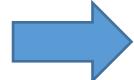
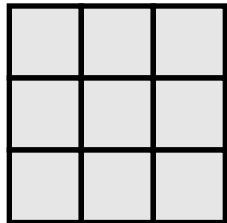
Engineering Considerations



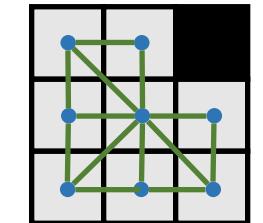
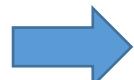
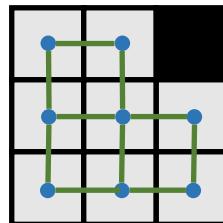
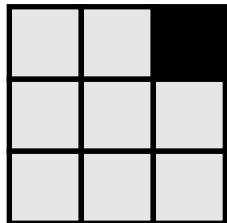
Example: Grid-based Path Search

How to represent grids as graphs?

Each cell is a node. Edges connect adjacent cells.



Common Choice!



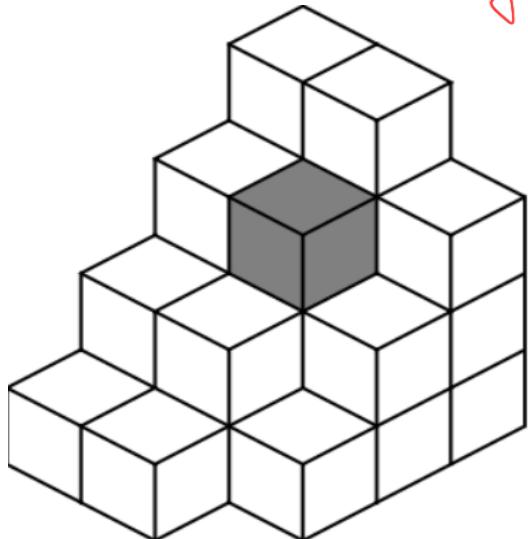
4 connection

8 connection



Grid-based Path Search: Implementation

进入入门使用 grid map



- Create a dense graph.
- Link the occupancy status stored in the grid map.
- Neighbors discovered by grid index.
- Perform A* search.

• Priority queue in C++

- `std::priority_queue`
- `std::make_heap`
- `std::multimap`



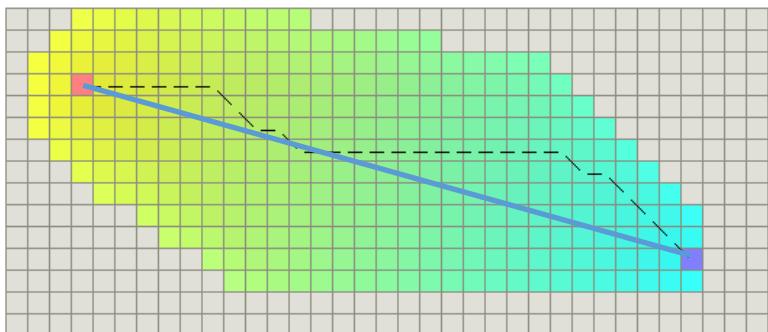
The Best Heuristic

- Recall:

- Is Euclidean distance (L2 norm) admissible?
- Is Manhattan distance (L1 norm) admissible?
- Is L^∞ norm distance admissible?
- Is 0 distance admissible?



Euclidean Heuristic



They are useful, but none of them is the best choice, why?

Because none of them is **tight**.

Tight means who close they measure the true shortest distance.

Why so many nodes expanded?

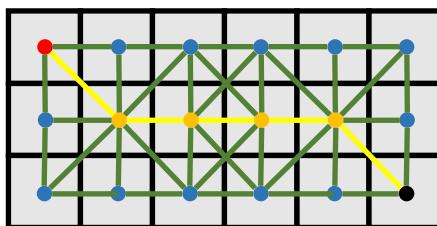
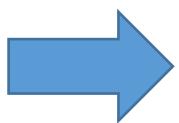
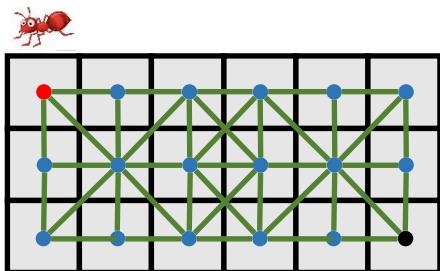
Because Euclidean distance is far from the **truly theoretical optimal solution**.



The Best Heuristic

How to get the **truly theoretical optimal solution?**

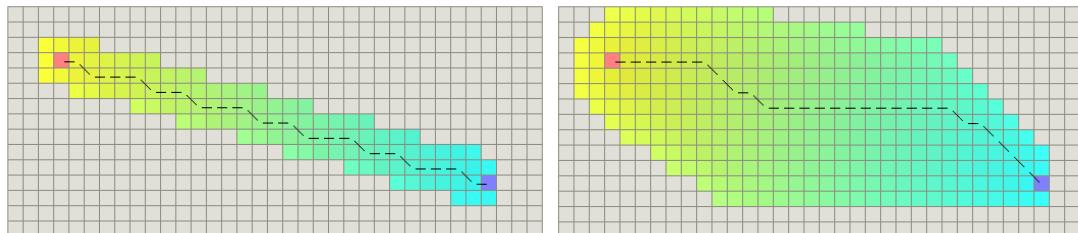
Fortunately, the grid map is highly structural.



$$h = h^*$$

Compare

Diagonal Heuristic



对于 grid map 有方法叫做
h_{grid} 使用

- You don't need to search the path.
- It has the **closed-form solution!**

```
dx=abs(node.x -goal.x)  
dy=abs(node.y -goal.y)  
h=(dx+dy)+(\sqrt{2}-2)*min(dx,dy)
```

For 3D case, we also have a similar version of this.



Tie Breaker 打破平衡, 抵抗对称性

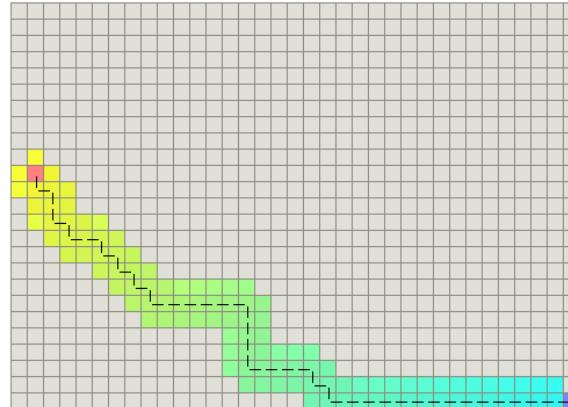
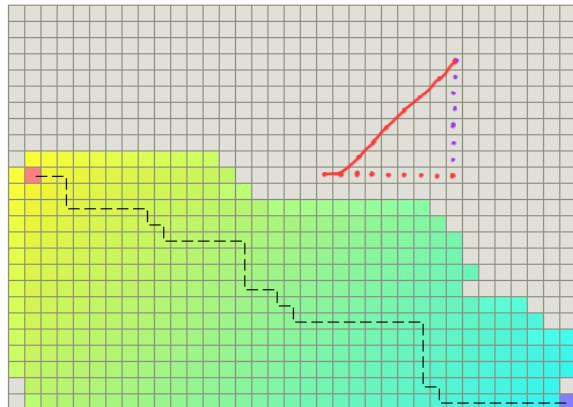
- Many paths have the same f value.
- No differences among them making them explored by A* equally.

- Manipulate the f value breaks the tie.
- Make same f values differ.
- Interfere h slightly.

$$h = h \times (1.0 + p)$$

放大 h

$$p < \frac{\text{minimum cost of one step}}{\text{expected maximum path cost}}$$



Slightly breaks the admissibility of h , does it matter?

在有障礙物的情況下 這種做法沒



Tie Breaker

不会带来任何的不利影响。

Core idea of tie breaker:

Find a preference among same cost paths

- When nodes having same f , compare their h .
- Add deterministic random numbers to the heuristic or edge costs (A hash of the coordinates).
- Prefer paths that are along the straight line from the starting point to the goal.

$$dx1 = \text{abs}(\text{node}.x - \text{goal}.x)$$

$$dy1 = \text{abs}(\text{node}.y - \text{goal}.y)$$

$$dx2 = \text{abs}(\text{start}.x - \text{goal}.x)$$

$$dy2 = \text{abs}(\text{start}.y - \text{goal}.y)$$

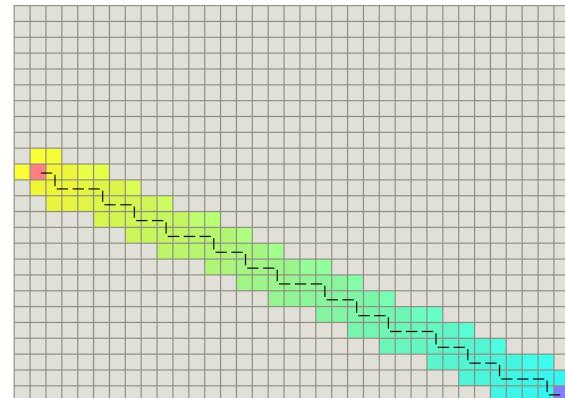
$$\text{cross} = \text{abs}(dx1 \times dy2 - dx2 \times dy1)$$

$$h = h + \text{cross} \times 0.001$$

- ... Many customized ways

f 一样的话就比较 h
cross 做 path 的好坏
越低越好。

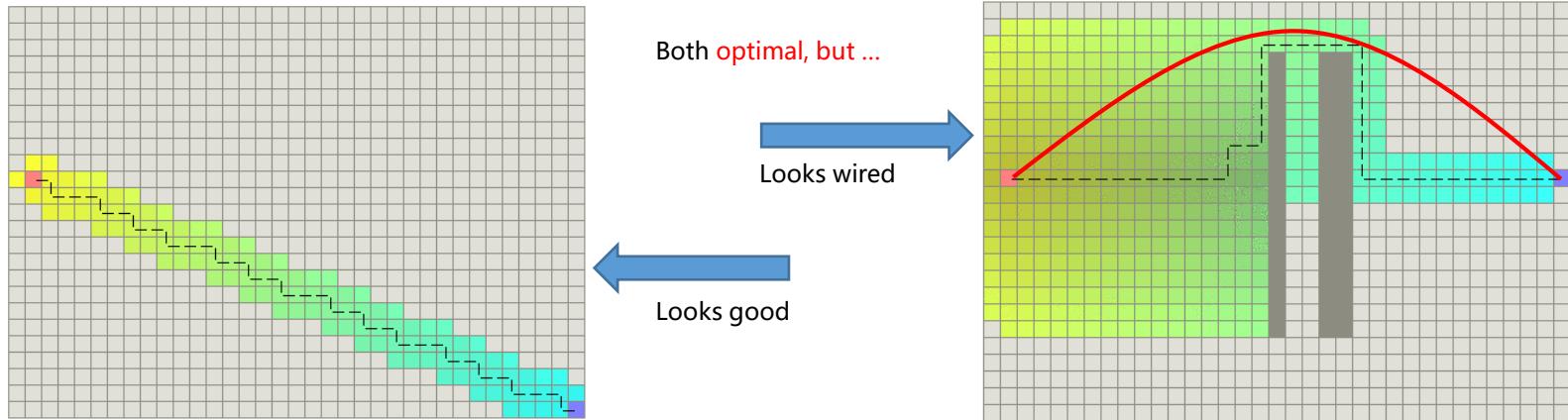
Better/other tie breaker?





Tie Breaker

- Prefer paths that are along the straight line from the starting point to the goal.



It's the shortest path, but harm for trajectory generation (smoothing).

Or a systematic approach: **Jump Point Search (JPS)**

Jump Point Search



Algorithm Workflow

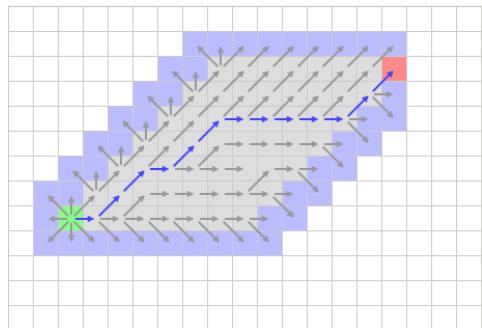


Jump Point Search

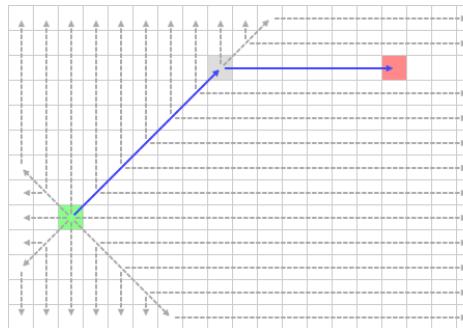
Core idea of JPS:

Find symmetry and break them.

A* explore all symmetric path.



JPS choose one path.



Grey node: Added in the Open List.

JPS explores intelligently, because it always looks ahead based on a rule.



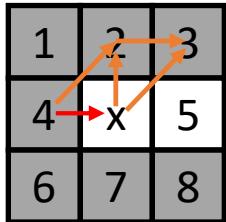
Look Ahead Rule

如果 x 的父节点到达某个节点，如不经过

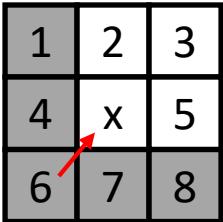
x 更近，则就没有经过 x 而及更佳

对角线的情况要排除等于！

straight

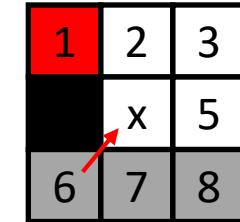
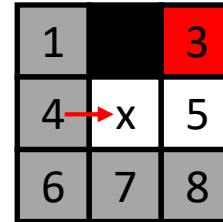


diagonal



Consider:

- current node x
- x 's expanded direction



Neighbor Pruning

劣性节点，规划时不需要考虑。

- Gray nodes: inferior neighbors, when going to them, the path without x is cheaper. Discard.
- White nodes: natural neighbors. 自然邻居
- We only need to consider natural neighbors when expand the search.

Forced Neighbors

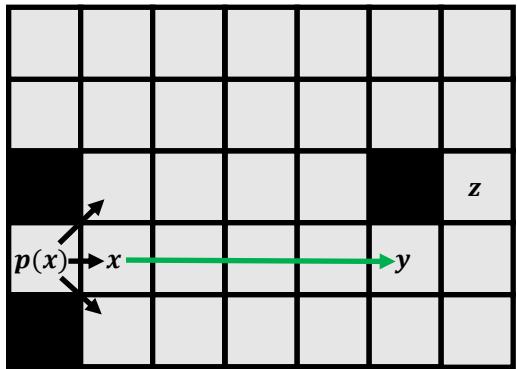
- There is obstacle adjacent to x
- Red nodes are forced neighbors. ↗
- A cheaper path from x 's parent to them is blocked by obstacle.

→ result: ① 红点或有终点



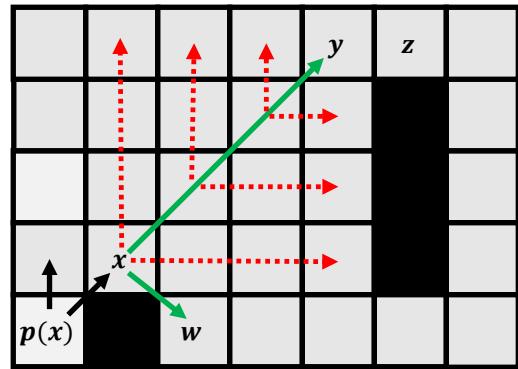
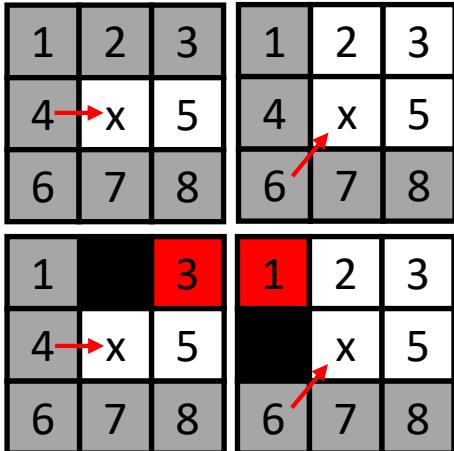
Jumping Rules

jump point : ① \exists forced neighbor
 ② 当前点有 forced neighbor.
 ③ 如果 parent 到 y 是对角线移动且 y 经过其他节点
 垂直移动可以到达 y，则 y 是跳点。



Jumping Straight

Look Ahead Rule

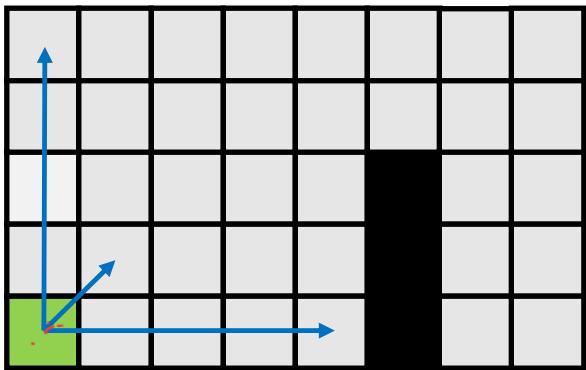


Jumping Diagonally

- Recursively apply **straight** pruning rule and identify y as a **jump point successor** of x. This node is interesting because it has a neighbor z that cannot be reached optimally except by a path that visits x then y.
- Recursively apply the **diagonal pruning** rule and identify y as a **jump point successor** of x.
- Before each diagonal step we first recurse straight. Only if both straight recursions fail to identify a jump point do we step diagonally again.
- Node w, a forced neighbor of x, is expanded as normal. (also push into the open list, the **priority queue**)



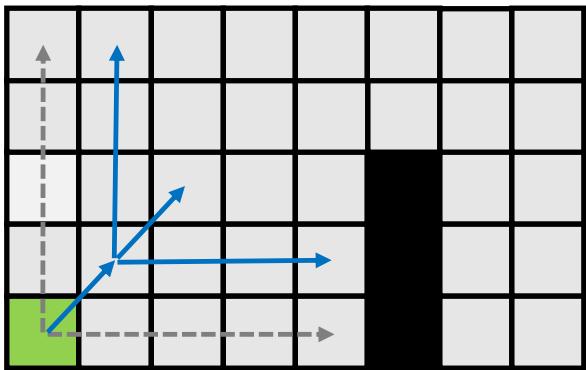
Jump Point Search



- Expand horizontally and vertically.
- Both jumps end in obstacles.
- Move diagonally.



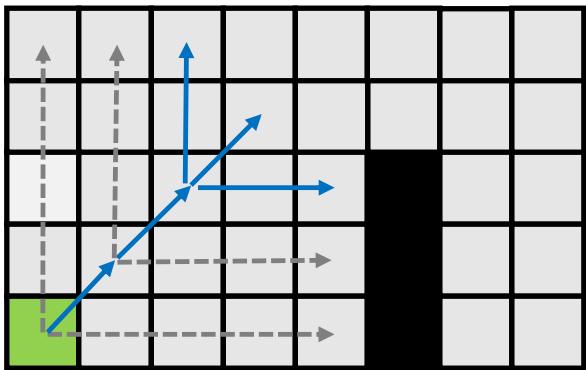
Jump Point Search



- Expand horizontally and vertically.
- Both jumps end in obstacles.
- Move diagonally.



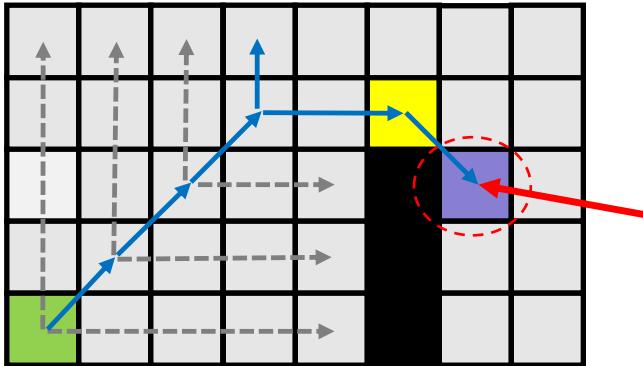
Jump Point Search



- Expand horizontally and vertically.
- Both expansions end in obstacles.
- Move diagonally.



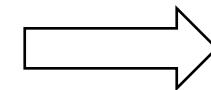
Jump Point Search



- **Remember:** you can only jump straight or diagonally; never piecewise jump
- Vertically expansion end in obstacle.
- Right-ward expansion finds a node with a **forced neighbor**.

Recall the rule

1		3
4	x	5
6	7	8

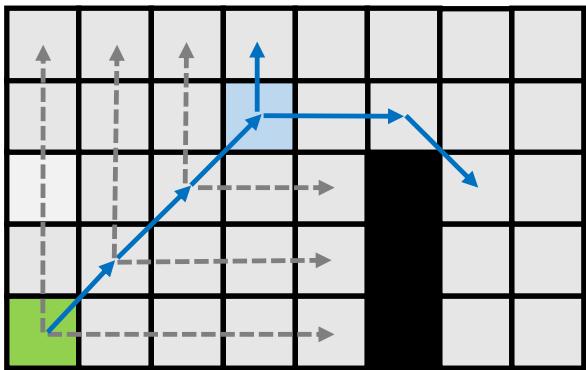


So we have

1	2	3
4	x	5
6		8



Jump Point Search



- Now this node is of interested.
- Put it to open list.

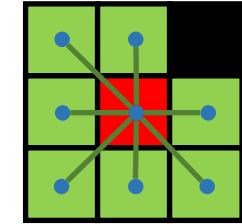


Jump Point Search

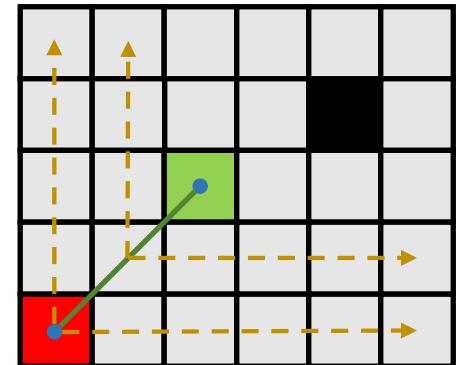
Recall A*'s pseudo-code, JPS' s is all the same!

- Maintain a priority queue to store all the nodes to be expanded
- The heuristic function $h(n)$ for all nodes are pre-defined
- The priority queue is initialized with the start state X_S
- Assign $g(X_S)=0$, and $g(n)=\infty$ for all other nodes in the graph
- Loop
 - If the queue is empty, return FALSE; break;
 - Remove the node "n" with the lowest $f(n)=g(n)+h(n)$ from the priority queue
 - Mark node "n" as expanded
 - If the node "n" is the goal state, return TRUE, break;
 - For all unexpanded neighbors "m" of node "n"
 - If $g(m) = \infty$
 - $g(m) = g(n) + C_{nm}$
 - Push node "m" into the queue
 - If $g(m) > g(n) + C_{nm}$
 - $g(m) = g(n) + C_{nm}$
 - end
- End Loop

A*: "Geometric" neighbors



JPS: "Jumping" neighbors



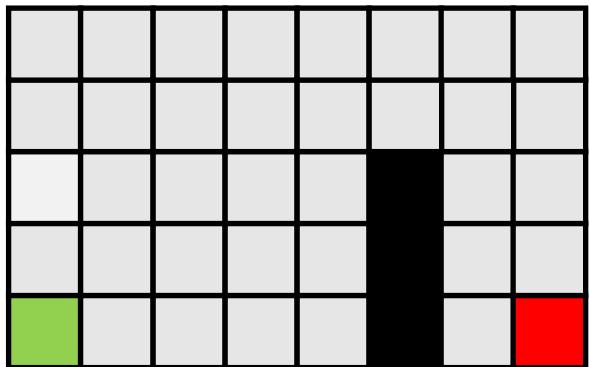


Example



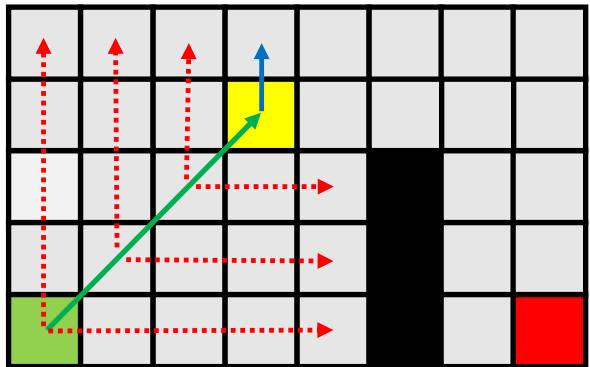
Jumping Example

Planning Case





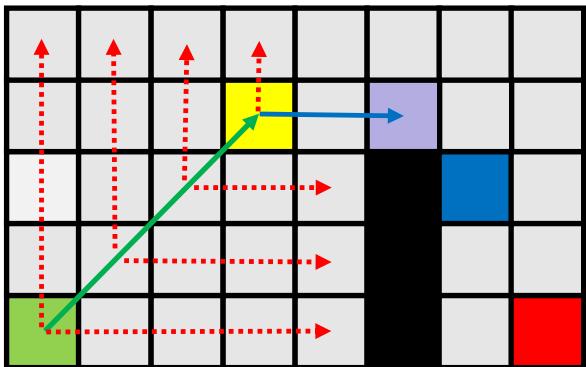
Jumping Example



- Expand—>move diagonally
- Find a critical node finally, add it into open list.
- Pop it (the only one) from the open list.
- Expand vertically, end at obstacles.



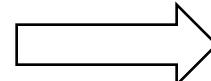
Jumping Example



- Expand horizontally, meets a node with a forced neighbor.
- Add it to open list

Recall the rule

1		3
4 → x		5
6	7	8

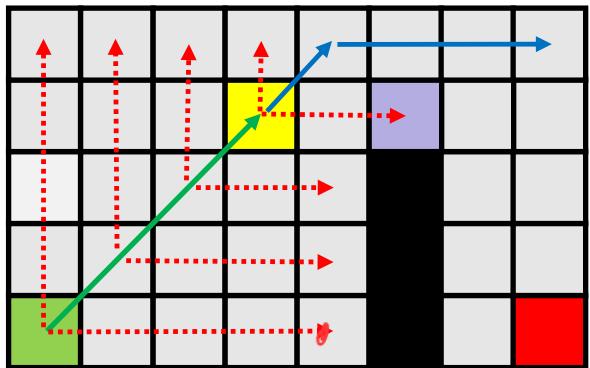


So we have

1	2	3
4 → x		5
6		8



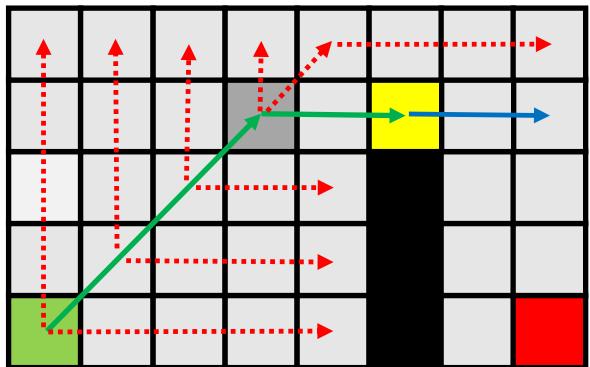
Jumping Example



- Expand diagonally, expand, find nothing.
- Finish the expansion of the current node.



Jumping Example



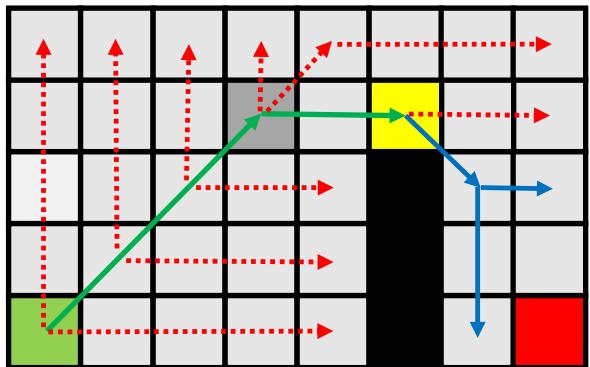
- Examine the “new best” node in the open list.
- Expand horizontally.
- Finds nothing.

Remember the rule

1	2	3
4	x	5
6		8



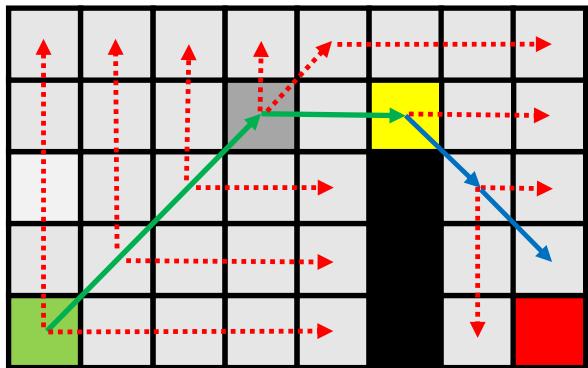
Jumping Example



- Move diagonally.
- Expand along vertical and horizontal first.



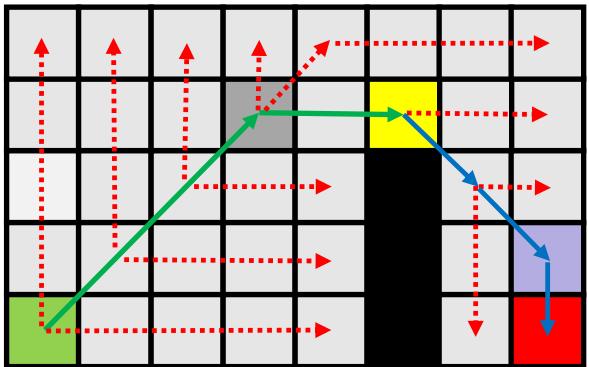
Jumping Example



- Finds nothing.
- Move diagonally.



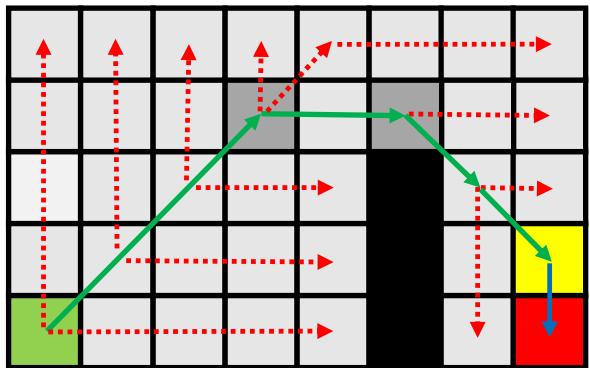
Jumping Example



- Expand horizontally and vertically.
- Finds the goal. Equally interested as finding a node with a forced neighbor.
- Add this node to open list.
- Finish the expand of the current node (No naturally neighbors left).
- Pop it out of the open list.



Jumping Example

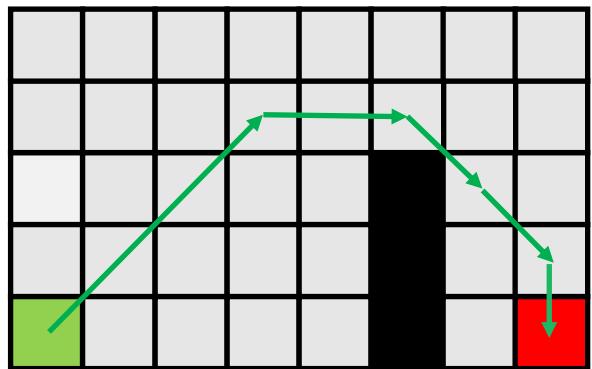


- Examine the “new best” node in the open list.
- Expand horizontally (nowhere), and vertically (finds the goal).
- The end.



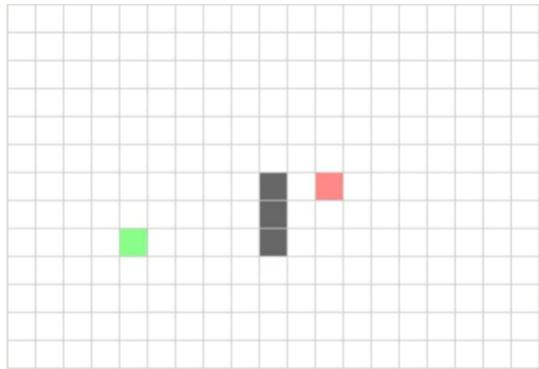
Jumping Example

Final Path

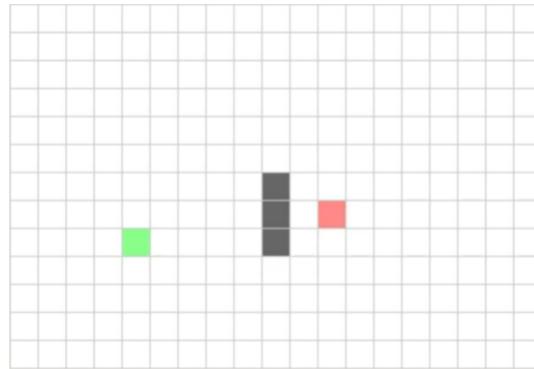




Example



(1)



(2)

Thanks:

<https://zerowidth.com/2013/a-visual-explanation-of-jump-point-search.html>



Extension



3D JPS

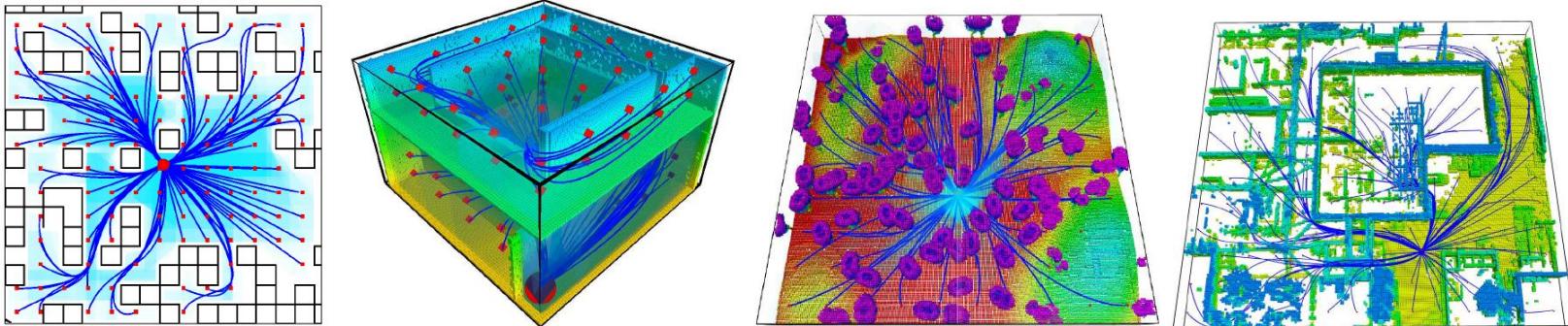


Table 1. Trajectory Generation Run Time (sec)

Map	Size	# of Cells	# of Trajs	Time (s)	Path Planning		Convex Decomp	Traj Opt	Replan (JPS)
					A*	JPS			
Random Blocks	$40 \times 40 \times 1$	1.4×10^6	130	Avg	0.57	0.034	0.0021	0.028	0.065
				Std	1.26	0.034		0.022	0.051
				Max	9.98	0.19			
Multiple Floors	$10 \times 10 \times 6$	5.9×10^5	147	Avg	6.12	0.039	0.0064	0.082	0.13
				Std	15.77	0.046		0.041	0.081
				Max	84.56	0.22			
The Forest	$50 \times 50 \times 6$	1.8×10^6	89	Avg	0.65	0.033	0.0039	0.055	0.094
				Std	1.57	0.044		0.024	0.068
				Max	7.78	0.20			
Outdoor Buildings	$100 \times 110 \times 7$	6.2×10^5	127	Avg	0.54	0.028	0.0066	0.099	0.14
				Std	1.46	0.045		0.064	0.10
				Max	10.96	0.27			

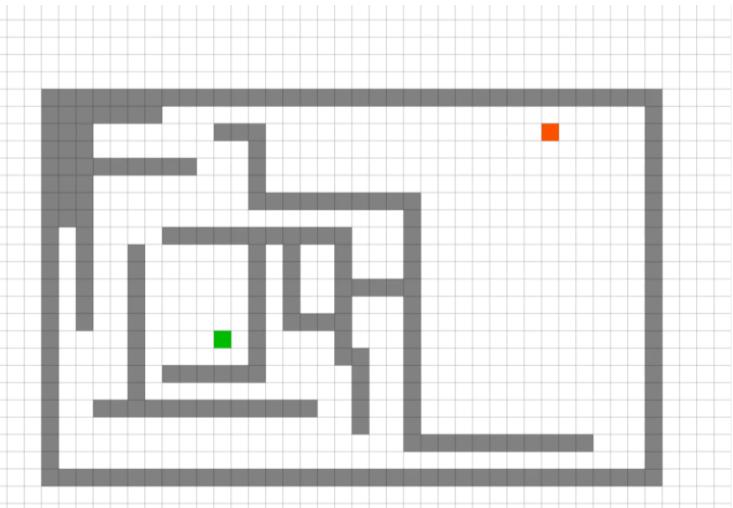
Planning Dynamically Feasible Trajectories for Quadrotors using Safe Flight Corridors in 3-D Complex Environments,
Sikang Liu, RAL 2017

<https://github.com/KumarRobotics/jps3d>

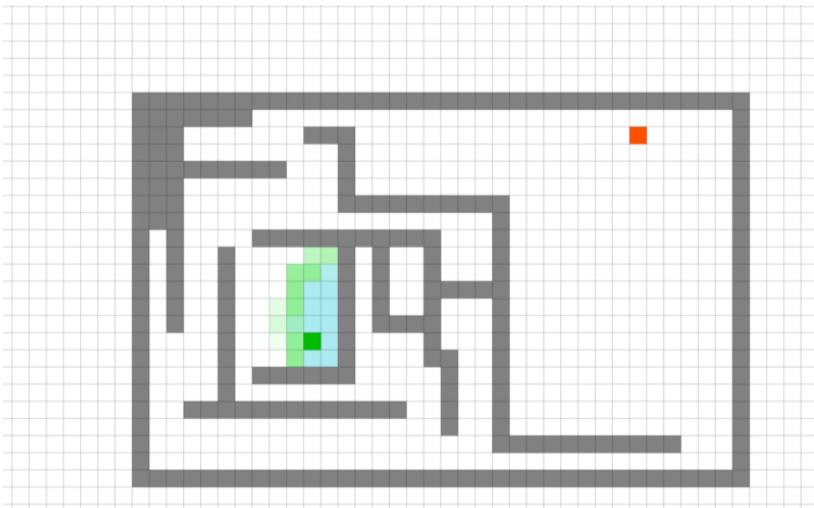


Is JPS always better?

Maze-like environments



JPS



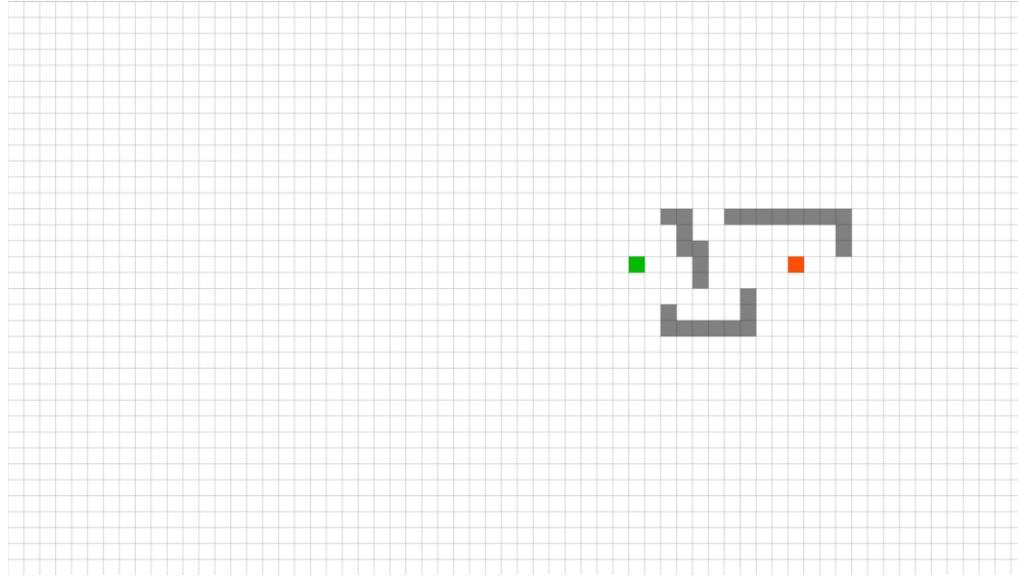
A*

Do more tests by yourself!

Thanks: <http://qiao.github.io/PathFinding.js/visual/>



Is JPS always better?



- This is a simple example saying "No."
- This case may commonly occur in robot navigation.
- Robot with limited FOV, but a global map/large local map.

Conclusion:

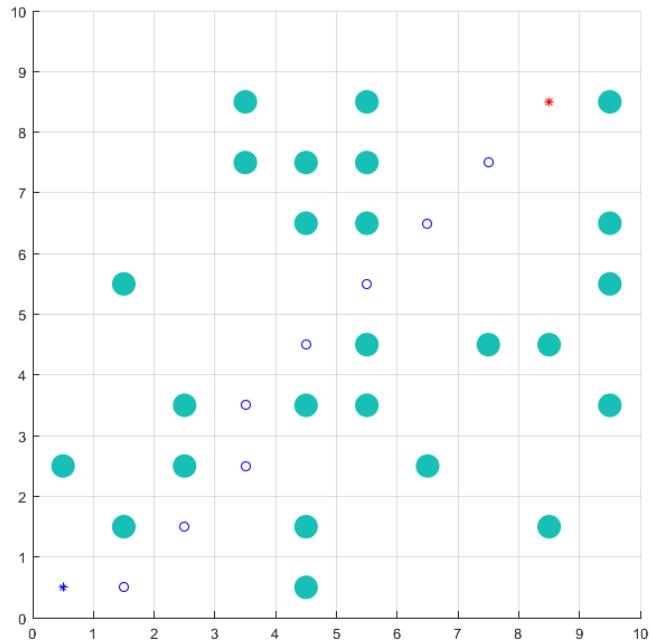
- Most time, especially in complex environments, JPS is better, but far away from "always". Why?
- JPS reduces the number of nodes in **Open List**, but increases the number of **status query**.
- You can try JPS in Homework 2.
- **JPS's limitation:** only applicable to uniform grid map. **but A* can !**

Homework



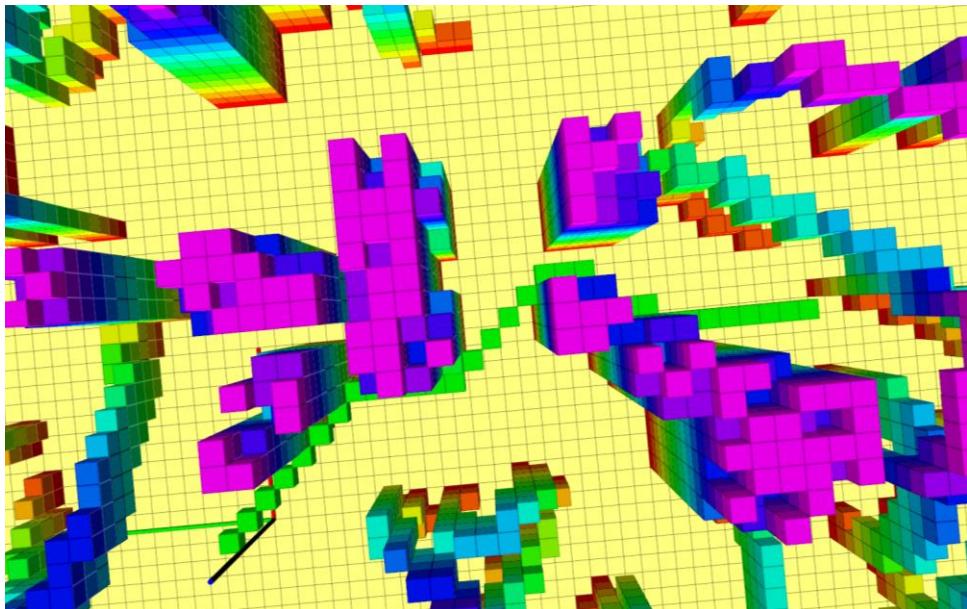
Testing Environment

Matlab



Basic

C++



Advance



Assignment: Basic

- This project work will focus on path finding and obstacle avoidance in a 2D grid map.
- A 2D grid map is generated randomly every time the Project is run, which contains the obstacles, start point and target point locations will be provided. You can also change the probability of obstacles in the map in `obstacle_map.m`
- You need to implement a 2D A* path search method to plan an optimal path with safety guarantee.



Assignment: Advance

- I highly suggest you implement Dijkstra/A* with C++/ROS.
- Complex 3d map can be generated randomly. The sparsity of obstacles in this map is tunable.
- An implementation of JPS is also provided. Comparisons can be made between A* and JPS in different map set-up.



Thanks for Listening!

