

Kinematic Modeling in 2D

Course 1, Module 4, Lesson 1



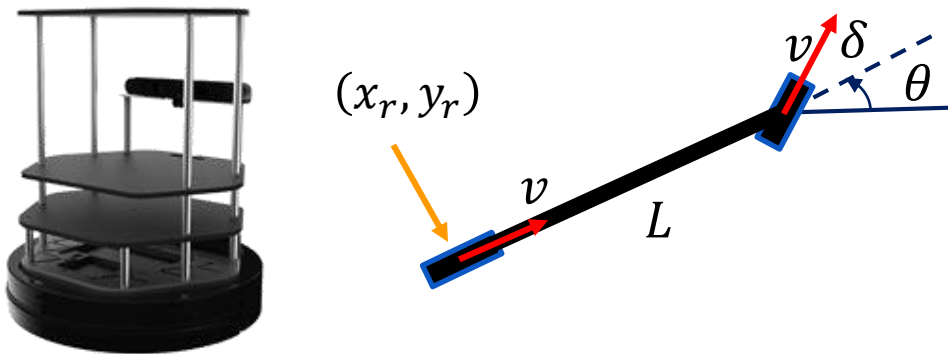
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FACULTY OF APPLIED SCIENCE & ENGINEERING

Overview of Module 4

- Basics of kinematic and coordinates
- Kinematic model development of a bicycle
- Basics of dynamic modeling
- Vehicle longitudinal dynamics and modeling
- Vehicle lateral dynamics and modeling
- Vehicle actuation system
- Tire slips and modeling

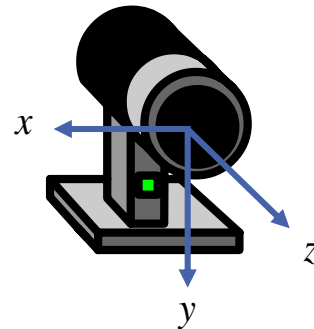
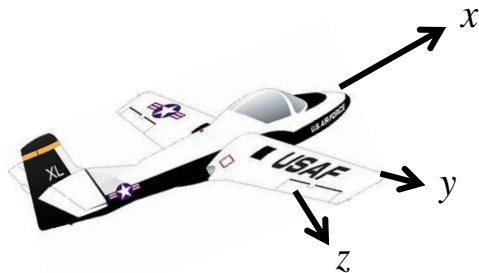
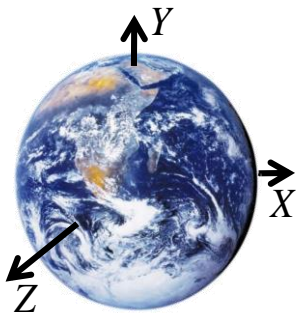
Kinematic Vs Dynamic Modeling

- At low speeds, it is often sufficient to look only at kinematic models of vehicles
 - Examples: Two wheeled robot, Bicycle model
- Dynamic modeling is more involved, but captures vehicle behavior more precisely over a wide operating range
 - Examples: Dynamic vehicle model

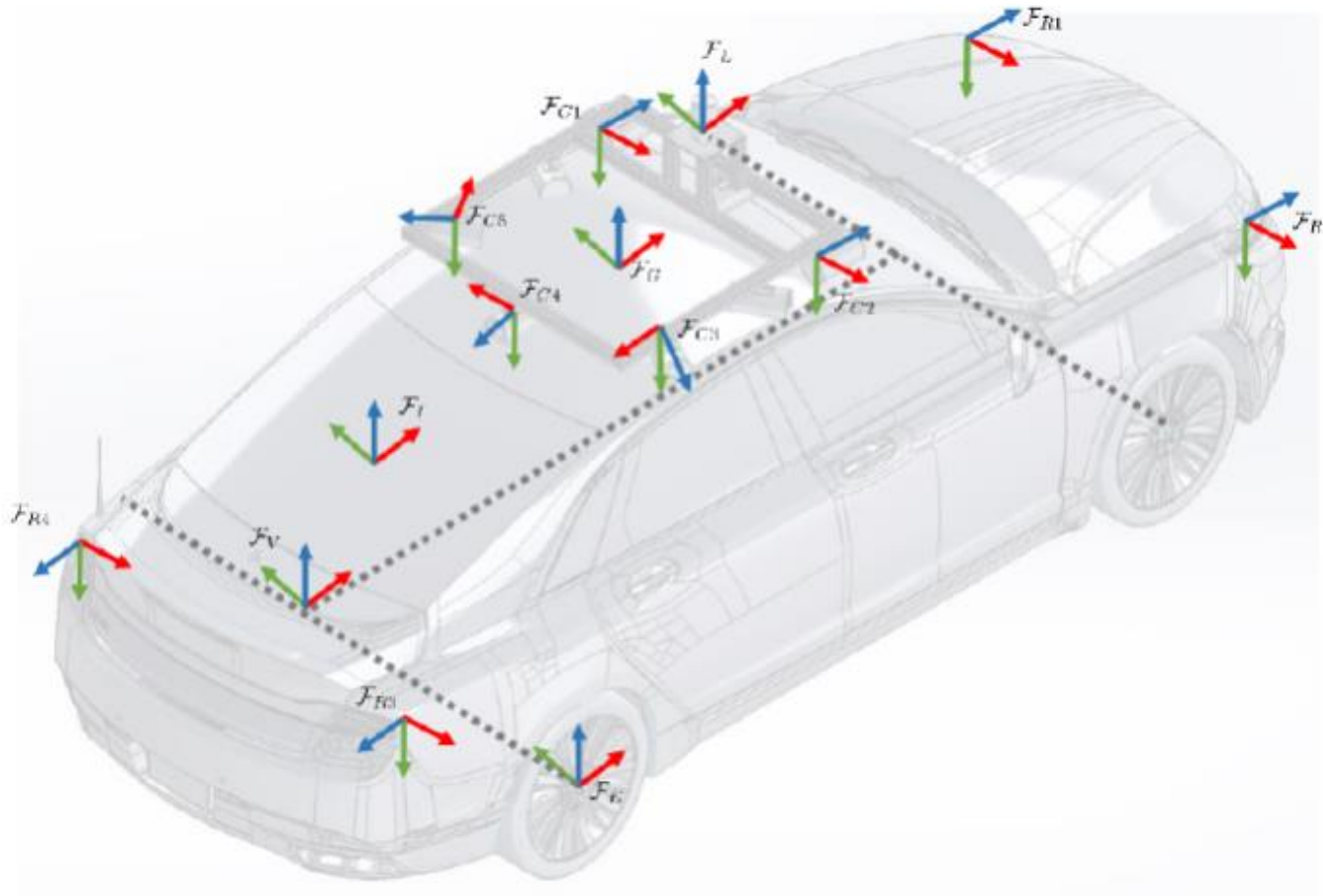


COORDINATE FRAMES

- Right handed by convention
- Inertial frame
 - Fixed, usually relative to earth
- Body frame
 - Attached to vehicle, origin at vehicle center of gravity, or center of rotation
- Sensor frame
 - Attached to sensor, convenient for expressing sensor measurements

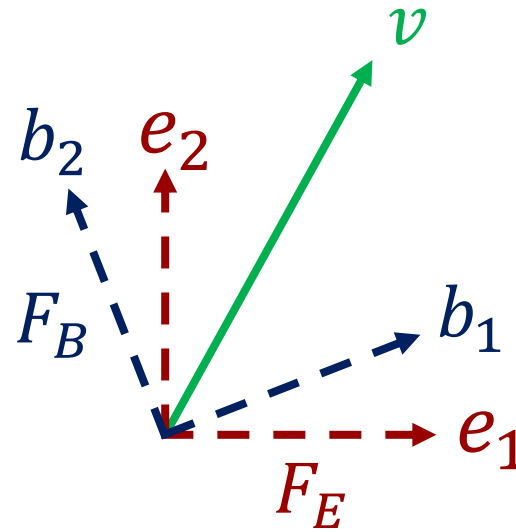


Why We Need Coordinate Transformation



Vectors

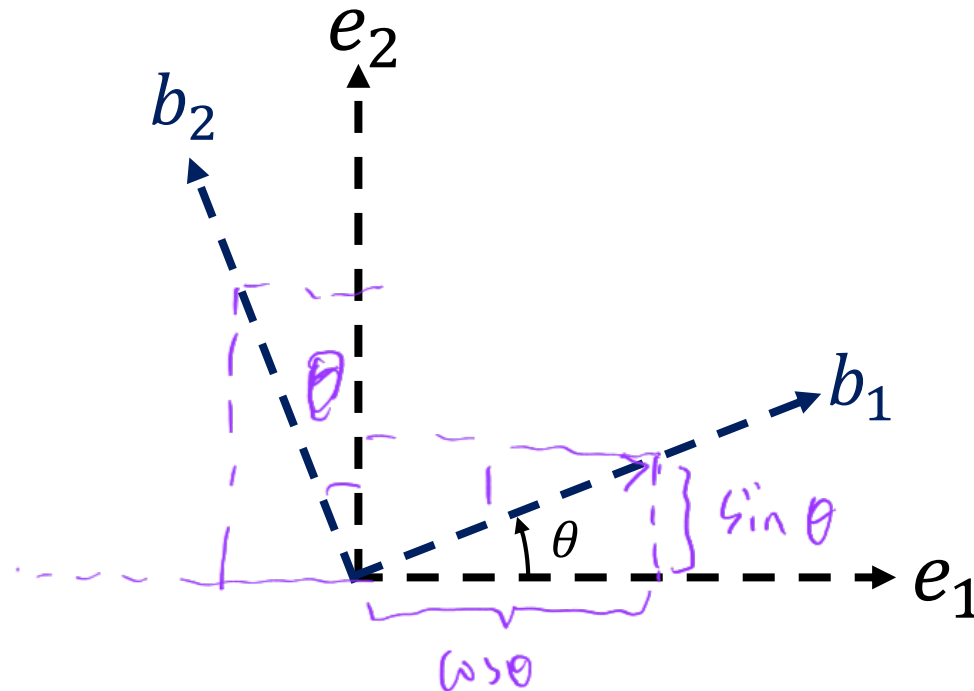
- Vectors are variables with both magnitude and direction
- In this figure, v is a vector
- The vectors $\{b_1, b_2\}$, $\{e_1, e_2\}$ define two different coordinate frames, F_B and F_E



Rotation Matrices in 2D

$$C_{EB} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$C_{BE} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



Coordinate Transformation

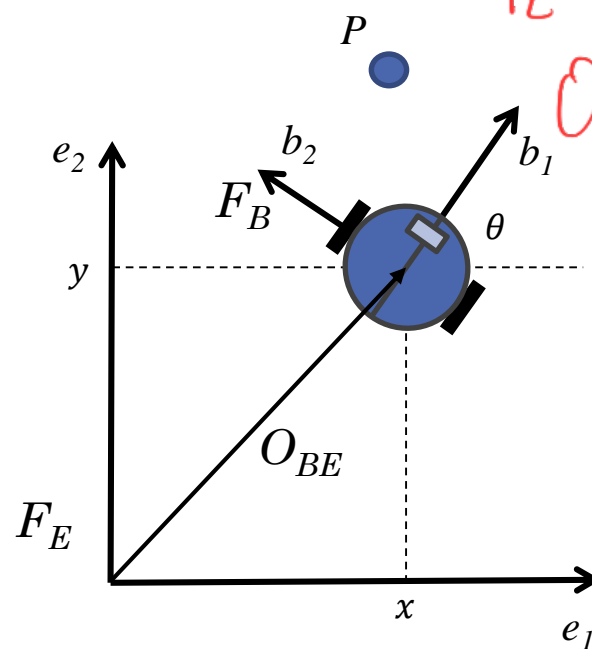
- Conversion between Inertial frame and Body coordinates is done with a translation vector and a rotation matrix

- Location of point (P) in Body Frame (B)

★ $P_B = C_{EB}(\theta) P_E + O_{EB}$ ← Translation term, expressed in body frame

- Location of point (P) in Inertial Frame (E)

$P_E = C_{BE}(\theta) P_B + O_{BE}$ ← Translation term, expressed in inertial frame



$$P_B = C_{EB} \cdot P_E + O_{EB}$$

P_B : B系下的一个点

C_{EB} : B系到E系的旋转矩阵

P_E : E系下的一个点

O_{EB} : 在B系下B→E的向量

Homogeneous Coordinate Form

- A 2D vector in homogeneous form

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \bar{P} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Transforming a point from body to inertial coordinates with homogeneous coordinates

$$\overline{P}_E = [C_{EB}(\theta) \mid O_{EB}] \overline{P}_B$$

2D Kinematic Modeling

- The kinematic constraint is nonholonomic
 - A constraint on rate of change of degrees of freedom
 - Vehicle velocity always tangent to current path

$$\frac{dy}{dx} = \tan \theta = \frac{\sin \theta}{\cos \theta}$$

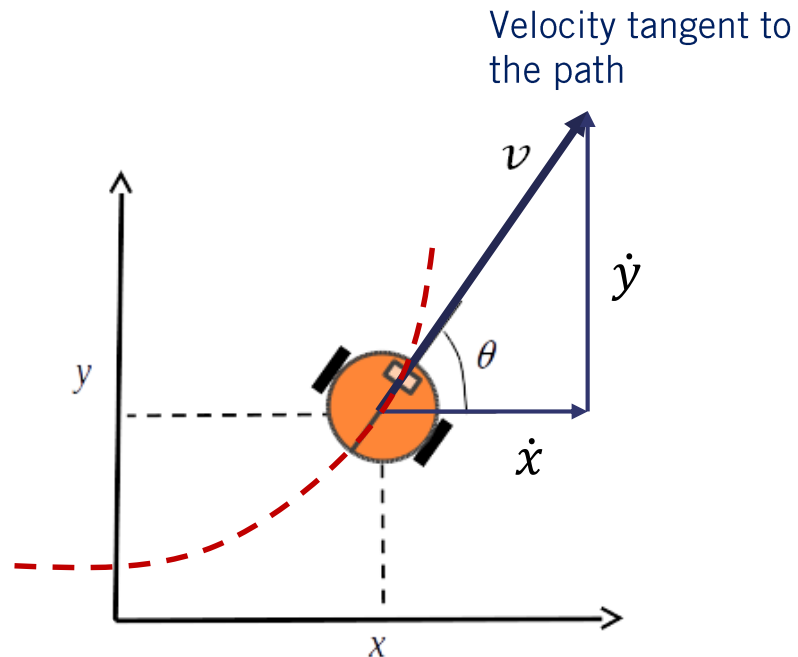
- Nonholonomic constraint

$$\dot{y} \cos \theta - \dot{x} \sin \theta = 0$$

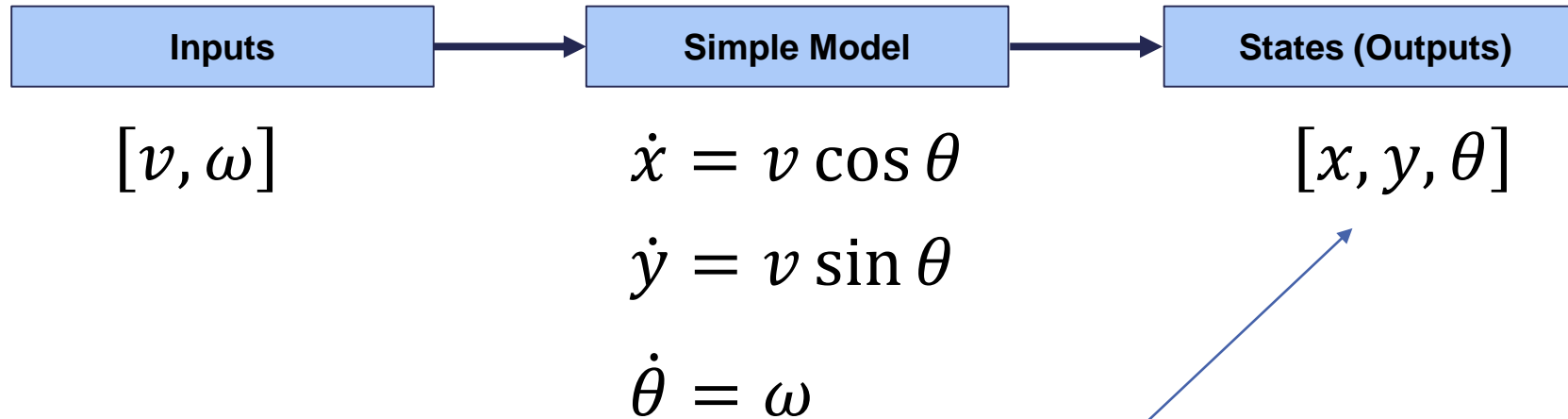
- Velocity components

$$\dot{x} = v \cos \theta$$

$$\dot{y} = v \sin \theta$$



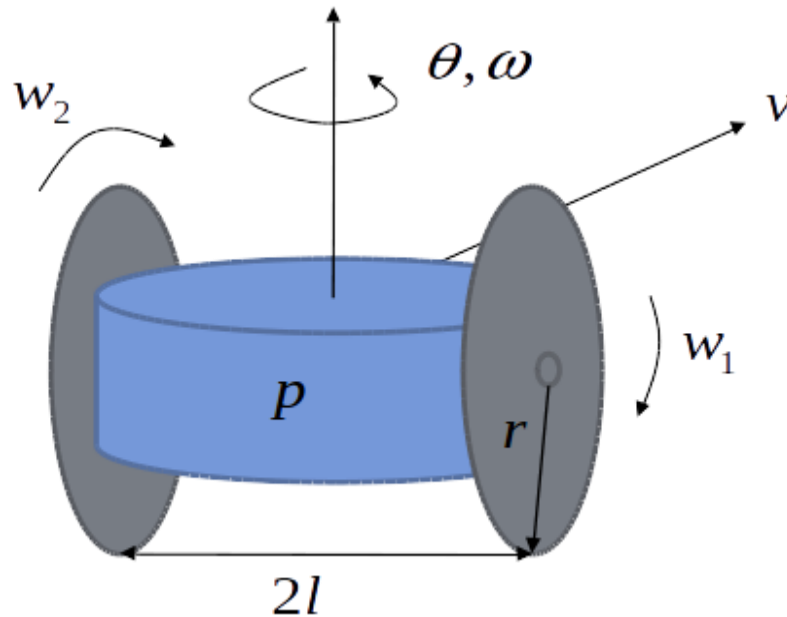
Simple Robot Motion Kinematics



The **state** is a set of variables (often arranged in the form of a vector) that fully describe the system at the current time.

Two-Wheeled Robot Kinematic Model

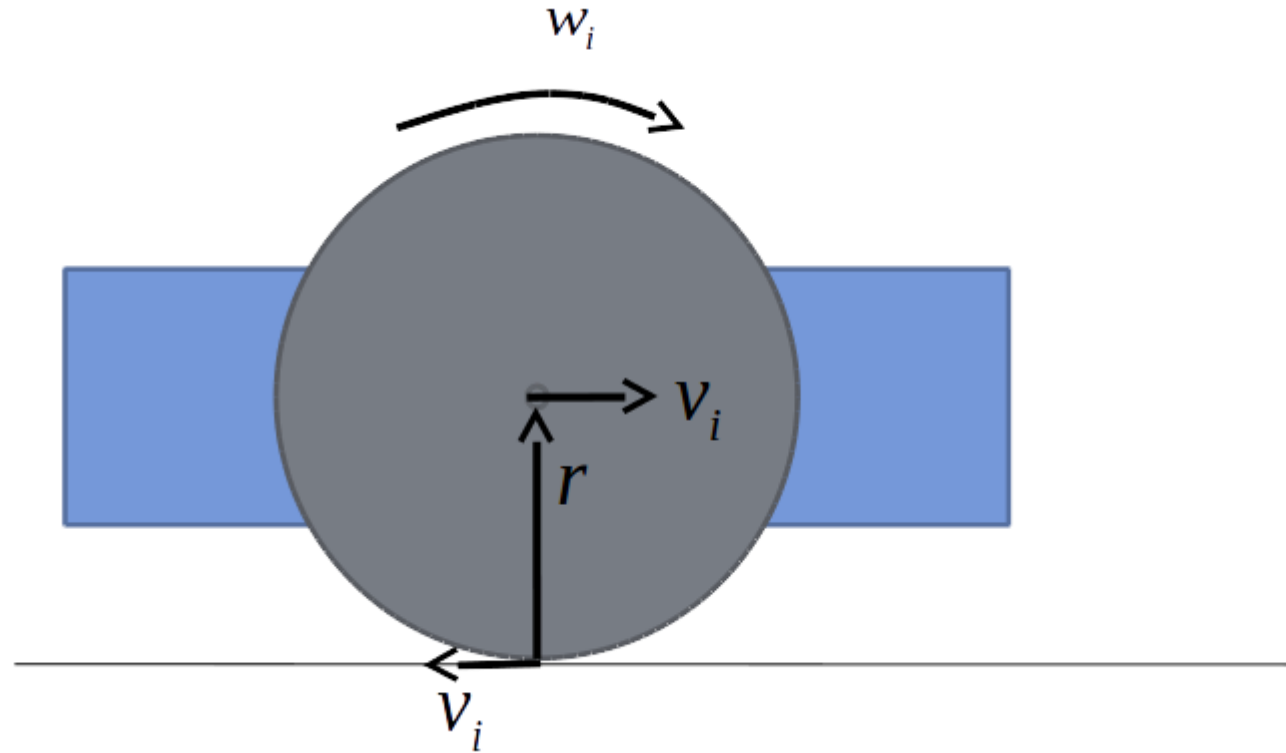
- Assume control inputs are wheel speeds
 - Center: p
 - Wheel to center: l
 - Wheel radius: r
 - Wheel rotation rates: w_1, w_2



Two-wheeled kinematic model

- Kinematic constraint

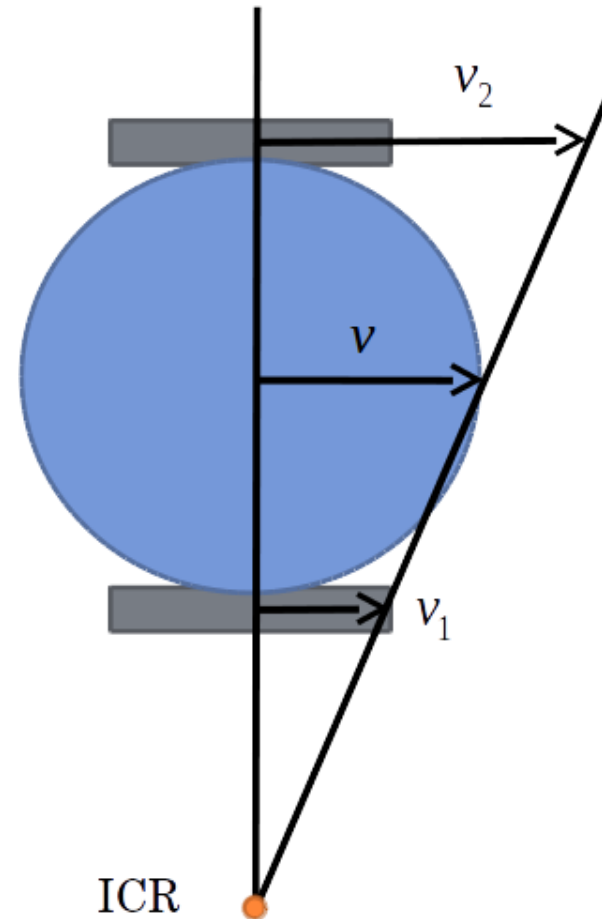
$$v_i = r w_i$$



Two-wheeled Kinematic Model

- Velocity is the average of the two wheel velocities

$$v = \frac{v_1 + v_2}{2} = \frac{rw_1 + rw_2}{2}$$

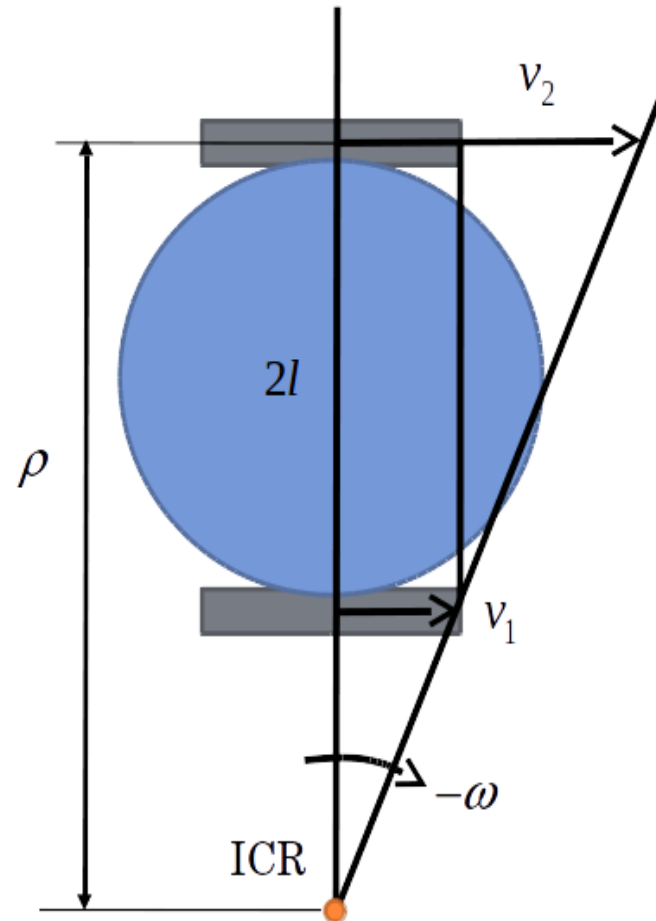


Two-wheeled Kinematic Model

- Use the instantaneous center of rotation (ICR)
- Equivalent triangles give the angular rate of rotation

$$\omega = \frac{-v_2}{\rho} = \frac{-(v_2 - v_1)}{2l}$$

$$\omega = \frac{rw_1 - rw_2}{2l}$$



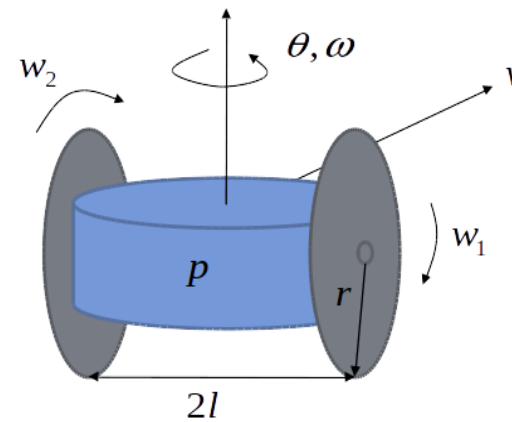
Kinematic Model of a Simple 2D Robot

- Continuous time model:

$$\dot{x} = \left[\left(\frac{rw_1 + rw_2}{2} \right) \cos \theta \right]$$

$$\dot{y} = \left[\left(\frac{rw_1 + rw_2}{2} \right) \sin \theta \right]$$

$$\dot{\theta} = \left(\frac{rw_1 - rw_2}{2l} \right)$$



- Discrete time model:

$$x_{k+1} = x_k + \left[\left(\frac{rw_{1,k} + rw_{2,k}}{2} \right) \cos \theta_k \right] \Delta t$$

$$y_{k+1} = y_k + \left[\left(\frac{rw_{1,k} + rw_{2,k}}{2} \right) \sin \theta_k \right] \Delta t$$

$$\theta_{k+1} = \theta_k + \left(\frac{rw_{1,k} - rw_{2,k}}{2l} \right) \Delta t$$

Summary

What we have learned from this lesson:

- Basics of 2D kinematics
- Coordinate frames and transformations
- Continuous and discrete kinematic model of a two wheeled robot

What is next?

- Going through the kinematics formulation of a bicycle model