Week 4: Basics of Control Theory and Laplace Transform and Its Applications

AE 315 - Systems and Control

Zeyad M. Manaa

Aerospace Engineering, King Fahd University of Petroleum and Minerals.
Dhahran, Saudi Arabia.

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Outline

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Open-loop control system

Definition (Open loop control system)

An open-loop control system utilizes an actuating device to control the process directly without using feedback.

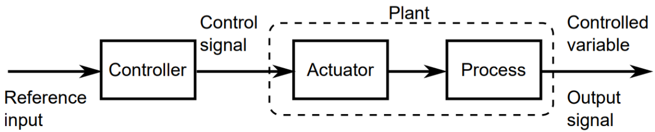

Figure: Open-loop control system

An open-loop system usually contains the following:

- process to be controlled that form a plant together with an actuator.
- control signal (manipulated variable) is varied by the controller.
- controlled variable – quantity that is measured or controlled.
- reference input, which dictates the desired value of the plant output.
- controller that acts upon the reference input signal.

Open-loop control system cont.

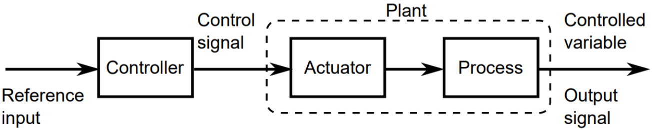

Figure: Open-loop control system

- In any open-loop control system the output is not compared with the reference input. Thus, such system works in fixed operating conditions.
- Open-loop control systems can be used in practice, only if the relationship between the input and output is known and if there are neither internal nor external disturbances.
- The accuracy of the system depends on calibration. Recalibration is necessary from time to time.
- Open-loop control systems less expensive and easier to construct than a corresponding closed-loop systems. No stability issues.

Closed-loop control system

Definition (Closed loop control system)

Closed-loop (or feedback) control system maintains a prescribed relationship between the output and the reference input (command) by comparing them and using the difference as a means of control.

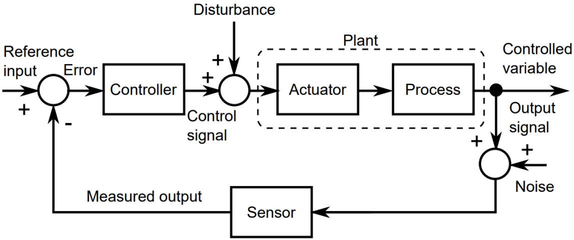

Figure: Closed-loop control system

Closed-loop control system Cont.

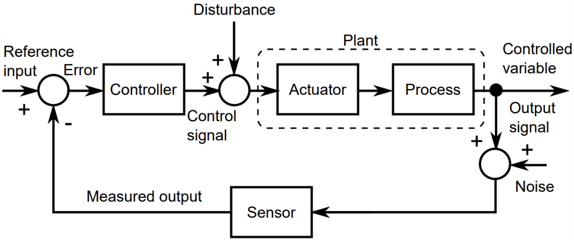

Figure: Closed-loop control system

- The feedback loop where the output signal is measured with a sensor. The measured signal is fed back to the summing junction.
- An error signal (actuating signal) is generated in the summing junction as a difference between the reference signal and the measured output.
- Disturbance and sensor noise adversely affect the value of the output of a system.

Properties of feedback control system

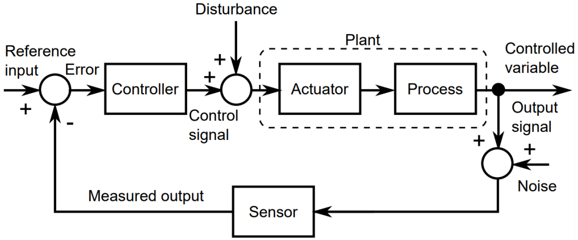

Figure: Closed-loop control system

- Stability. The system must be stable at all times (absolute requirement).
- Tracking The system output must track the command reference signal as closely as possible.
- Distrubance rejection. The system output must be as insensitive as possible to distrubance signals.
- Robustness. The control goals must be met even if the model used in control design is not completely accurate or if the dynamics of the system change over time.

Outline

Controller functionality

- **Controller Functionality**: The controller compares desired behavior (reference input) with actual behavior to generate control signals.
- In the context of motion control of a robot arm:
  $\theta_d(t)$ represents the desired behavior (desired motion or reference input)
  $\theta(t)$ represents the actual motion.
  $\theta_e(t)$ represent the error.
- **Control Objective**: Minimize or eliminate the error, making the actual system response closely follow the desired response.

Error Dynamics

Definition (Error Dynamics)

Let us define

- The error signal: the difference between the desired response (reference signal) and the actual response of the system being controlled.
- Error dynamics: refers to the analysis and study of how the error signal $e(t)$ behaves over time in a control system.

$$\theta_e(t) = \theta_d(t) - \theta(t) \tag{1}$$

- Error dynamics involves understanding system's properties, such as stability, convergence, overshoot, settling time, etc.

Performance metrices

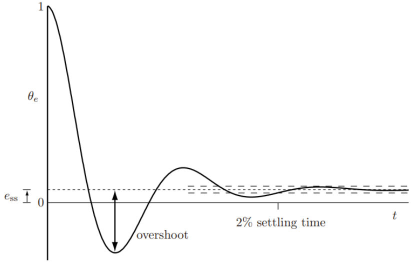

Figure: An example error response showing steady-state error ess, the overshoot, and the 2% settling time.

- **Settling time**
  The time it takes for the system to converge to its steady state.
- **Steady state error** $e_s(t)$
  The difference between the steady-state output and the desired output.
- **Overshoot percentage**
  How much the peak level is higher than the steady state, normalized against the steady state.

Outline

Importance of Laplace Transform

1. The Laplace transform in one of the most important mathematical tools available for modeling and analyzing linear systems.
2. Operations such as differentiation and integration can be replaced by algebraic operations in the complex plane.
3. Linear differential equations can be transformed into an algebraic equations. Both transient and steady-state component of the solution can be obtained simultaneously.
4. The Laplace transform allows the use of various techniques for predicting the system performance and synthesis of controllers.

Laplace Transform

Definition (Laplace Transform)

Let us define

- $f(t)$: a function of time t such that $f(t) = 0$ for $t < 0$.
- $s = \sigma + j\omega$: a complex variable.
- $F(s)$: Laplace transform of $f(t)$.

Then the Laplace transform of $f(t)$ is given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \tag{2}$$

where $\mathcal{L}\{.\}$ – an operational symbol indicating that the quantity that if prefixes is to be transformed by the Laplace integral.

Example

Example (1)

Evaluate the Laplace transform of the constant function 1:

The constant function 1 is defined as $f(t) = 1$ for all $t \geq 0$. By the definition of the Laplace transform,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} dt = \left[\frac{-1}{s} e^{-st} \right]_0^\infty = \boxed{\frac{1}{s}}$$

Example (2)

Evaluate the Laplace transform of the function $f(t) = \sin(t)$:

The function $f(t) = \sin(t)$ for all $t \geq 0$. By the definition of the Laplace transform,

$$\mathcal{L}\{\sin(t)\} = \int_0^\infty e^{-st} \sin(t) dt = \left[\frac{-1}{s} e^{-st} \sin(t) + \frac{1}{s^2} e^{-st} \cos(t) \right]_0^\infty = \boxed{\frac{1}{s^2+1}}$$

Laplace Transform Tables

Table of Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8.	$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$

Differentiation Property

Differentiation

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0) \tag{3}$$

where $f(0)$ is the initial value of $f(t)$ at $t = 0$.
This can expand to **second order** derivatives by

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - sf(0) - f'(0) \tag{4}$$

Generalizing to **higher order** derivatives,

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) \cdots - f^{(n-1)}(0) \tag{5}$$

This will be the workhorse on solving ODEs.

Solving ODEs

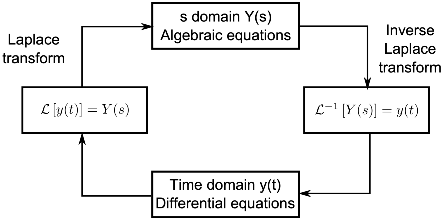

Figure: Laplace transform for solving ODE equations

1. Transform the **linear ODE** to the s–domain by the **Laplace transform**.
2. Manipulate the transformed **algebraic equation** and solve for the transform of the unknown function.
3. Perform **partial-fraction expansion** to the transform of the unknown function.
4. Obtain the **inverse Laplace transform** from the Laplace transform table.

Go to example 3

SISO

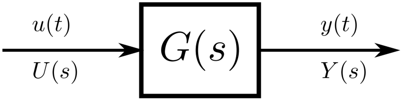

Figure: Block diagram of a single input single output (SISO) system

Consider the continuous, **linear time-invariant(LTI)** system defined by **linear constant coefficient ordinary differential equation**

$$\boxed{\begin{aligned} & a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y \\ & = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u \end{aligned}}$$

Transfer Function

Definition (Transfer Function)

The transfer function **G(s)** of a linear, time-invariant differential equation system is defined as the ratio of the Laplace transform of the output **Y(s)** (**response function**) to the Laplace transform of the input **U(s)** (**driving function**) under the assumption that all **initial conditions are zero.**

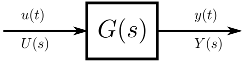

Figure: Block diagram of a single input single output (SISO) system

The Laplace of the system is given by,

$$\boxed{\begin{aligned} &a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \\ &= b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 \end{aligned}}$$

Then,

$$\boxed{\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = G(s)}$$

Transfer Function Cont.

Definition (Characteristic equation)

The characteristic equation of a system is defined as the equation obtained by setting the characteristic polynomial (The denominator) of a transfer function G(s) in Eq. to zero, i.e.:

$$N(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

Definition (Pole)

The roots of the characteristic equation $N(s) = 0$ are called poles of a transfer function G(s).

Definition (Zero)

The roots of the nominator of the transfer functions are called zeros of a transfer function G(s), i.e.:

$$M(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0$$

Outline

Solving ODEs Cont.

Example (3)

Consider the mechanical system depicted in the figure. The input signal is given by the external force $F(t) = 3$ N for $t \geq 0$ acting on the mass $m = 1$ kg. The displacement $x(t)$ of the mass is the output signal. The displacement is measured from the equilibrium position in the absence of the external force. Let $k = 5\ N/m$ be the spring constant, $c = 2\ Ns/m$ be the damping coefficient.

1. Write the equations of motion for the system.
2. Derive the transfer function
3. Examine all of
   - Impulse
   - Step

   responses.

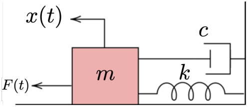

Figure: Mechanical system