Chapter 1

Linear Systems Review

1.1 Basic Properties

As mentioned in the introduction, it is assumed that you have a basic introduction to electronics. However, the topic of linear systems is so incredibly important as a foundation for power electronics that this chapter will provide a brief review of linear systems. A dedicated textbook for linear systems should be consulted for a comprehensive education on the topic.

Think of a linear system in the abstract. Here's a box, representing a linear system. Stuff gets shoved into the box, and comes out the other end looking different.

Figure 1.1: A box. I mean, a linear system.

This box has only one input and one output, but other systems can have multiple inputs and outputs. Linear systems have a couple properties that make them nice and predictable. If they're predictable, you can use some mathematical manipulation and analysis on them:

- *Gain Scaling* - If you take a given input / output combination and scale the gain of the input up, the same gain should show up on the output. If you stick a "1" in the front and get a "3" back out, then sticking in a "2" will produce a "6."

- *Time Delay* - For a given signal-in / signal-out pair, if you instead delay the input signal by 2 seconds, then the output will be delayed by 2 seconds as well.
- *Sine Wave In, Sine Wave Out* - Stick a sine wave into the system, and you'll get a sine wave out. It may have different amplitude and phase relative to input, but it'll be a sine wave with the same frequency. This is actually one of the key points about linear systems that will be heavily exploited later: gain and phase shift. (See Figure 1.2. The blue trace is the input signal to a transfer function, and the red trace is the resulting output.)[1]

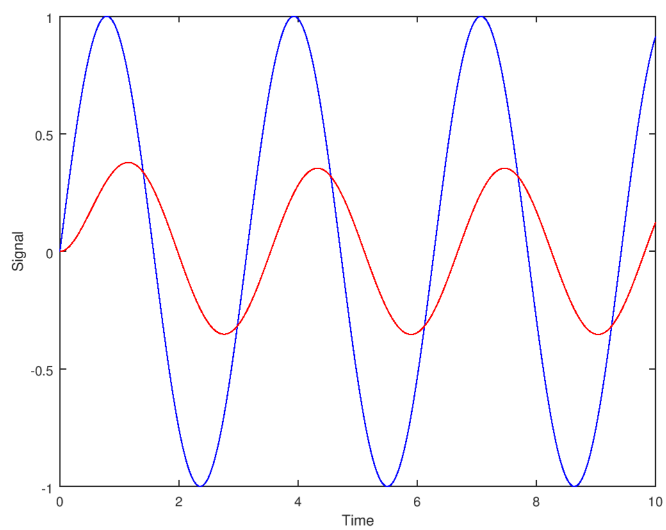

Figure 1.2: Sine Wave Goes In, Sine Wave Comes Out

Now, to be fair, many aspects of power converters are not linear. There's no getting around that. But *within a certain band of operating conditions* they do act pretty linear. That lets us set up linear equations to describe how the system works in said operating region and make predictions about how it will behave.

Linear systems are traditionally represented by transfer functions. One simple example of a low-pass filter with zero DC gain and a corner frequency

[1] The specific transfer function is $H(s) = \frac{1}{s+2}$, and the input is a simple sine wave $sin(t)$.

of about 3.18Hz is found in 1.1.

$$G(s) = \frac{20}{s+20} \tag{1.1}$$

The transfer function describes, mathematically speaking, what happens to a signal that's supplied to the function.[2] Because everything can be represented by a series of sinusoids[3], it is useful to think of how a given sinusoid is shifted in both magnitude and frequency. This is commonly referred to as "gain" and "phase." Sparing a lot of details, for a sinusoid of a given frequency ω, the transfer function imparts a specific gain shift and phase shift on the input sinusoid to create the output sinusoid.

$$\big|\big|G(s)\big|\big| \quad and \quad \angle G(s) \tag{1.2}$$

The gain is the complex magnitude of the transfer function, and the phase shift is the complex angle[4]. For convenience's sake, gain is almost always mathematically transformed into decibels, and phase is referred to in degrees. Plugging a few numbers into the transfer function given previously in Equation 1.1 results in the gain/phase values in Table 1.1.

ω	Equation	Gain (dB)	Phase (deg)
0	$\frac{20}{j0+20}$	0	0
1	$\frac{20}{j+20}$	-0.01	-2.86
5	$\frac{20}{j5+20}$	-0.26	-14.04
10	$\frac{20}{j10+20}$	-0.97	-26.57
20	$\frac{20}{j20+20}$	-3.01	-45.00
30	$\frac{20}{j30+20}$	-5.12	-56.31
50	$\frac{20}{j50+20}$	-8.60	-68.20
100	$\frac{20}{j100+20}$	-14.15	-78.69
150	$\frac{20}{j150+20}$	-17.58	-82.40
200	$\frac{20}{j200+20}$	-20.04	-84.29

Table 1.1: Manual Phase/Gain Calculations

[2] With transfer functions, remember that $s = j\omega$, where $j = \sqrt{-1}$ and ω is frequency in rad/sec.

[3] See the works of Joseph Fourier.

[4] Or "argument," if you will

Graphing the gain and phase of a transfer function is called making a "Bode plot." Doing all that by hand is not fun. Fortunately, we have computers to do tedious math and make pretty pictures in the process. With two simple commands in GNU Octave, you can get a nice graph of the transfer function from Equation 1.1 in the form of Figure 1.3.

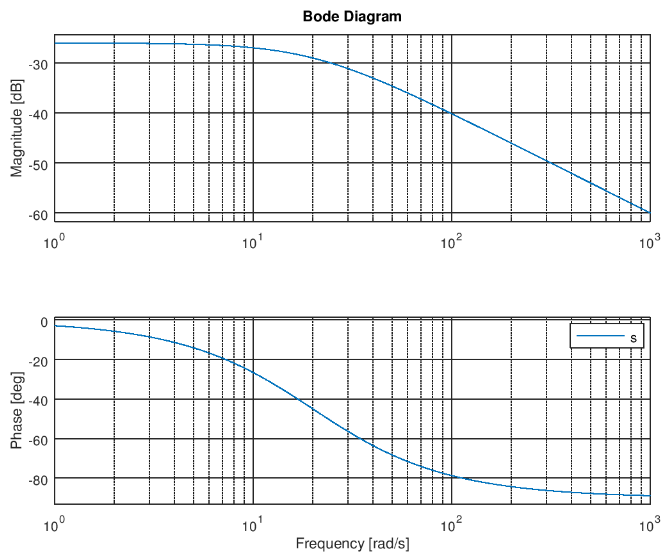

Figure 1.3: Simple Low-Pass Filter

Here you can see the behavior of the system in much more detail. A few things become apparent:

- There's a specific frequency around which "things happen." This is often called the corner frequency, or bandwidth, or frequency location[5]. For this system, the corner frequency is 20 rad/sec, or approximately 3.18Hz.
- At the corner frequency, the gain starts shifting. In this case, it starts decreasing at about -20dB / decade.
- Starting one decade lower than the corner frequency, the phase starts shifting noticeably. At the corner frequency, it is 45° off from the

[5]Example of the latter: when referencing the location of poles and zeros in a transfer function, you could say "the double pole at 37kHz and the zero at 5kHz." This is how you'll most often use the term in designing power converters and their control systems.

original (in this case, $-45°$). At one decade higher than the corner frequency, it has mostly stabilized around $90°$ off from original.

All this is because of that pesky s term. The complex portion of the transfer function changes with frequency. At low frequency, all the s terms basically go away. At middling frequencies, there's a sort of balance between real and complex portions of the transfer function. At high frequencies, the complex terms dominate.

1.2 Poles and Zeros

With transfer functions introduced, I want to spend a little more time describing poles and zeros. "What do you mean 'poles and zeros'?" you may ask. What most people mean with poles and zeros are the polynomial roots of the denominator and numerator (respectively) of a transfer function.

$$G(s) = \frac{s+10}{s(s^2+5s+6)} \tag{1.3}$$

Take the transfer function represented by Equation 1.3. There are several values of s that result in the numerator or denominator equalling zero. For the numerator, when $s = -10$ the transfer function will equal zero (due to the numerator being zero). This value of -10, by no coincidence, is a "zero" of the transfer function $G(s)$.

Additionally, there are multiple values of s that would result in the denominator being zero. Those values are 0, -2, and -3. Now, if we call the top roots "zeros" because the transfer function goes to zero, we *don't* call the roots that cause it to go to infinity "infinities" or "divides by zero." Instead, imagine a 3D surface, with the two "flat" axes being the real and imaginary axes, and the "height" being the magnitude. The general shape around one of the denominator roots kind of resembles a tent with a big center pole going upward to infinity (because the function is some number divided by zero). So the thing that causes the tent shape is a "pole." I didn't make this up; just roll with it and call the denominator roots "poles" and everything will be fine.

On a side note, there's no reason we cannot make a Bode plot of the transfer function in Equation 1.3. That's given below in Figure 1.4.

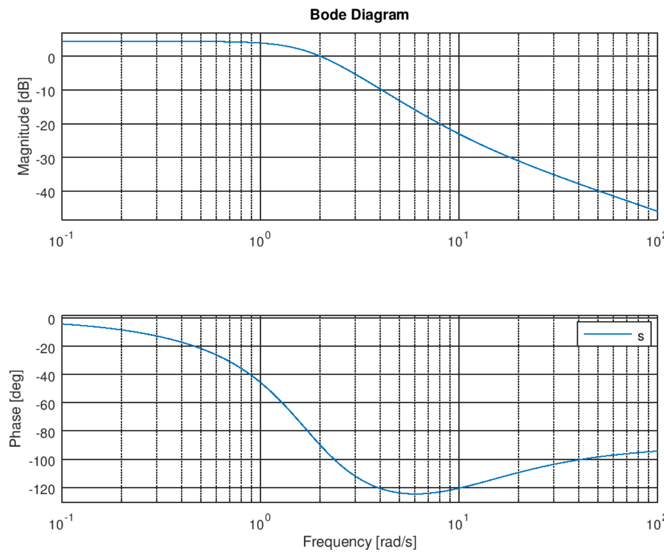

Figure 1.4: Bode Plot for TF in Equation 1.3

Go back to Equation 1.3 and think about the numerator. As s changes, how does the gain and phase of just that term change? When s is small, the fixed "10" dominates the gain, with no phase shift. As s increases, both the gain and the phase shift increase. A similar thought process can be done with the denominator. Remember that all the terms are "acting" on the TF's total gain and phase shift at all times.

Take the generic form of a pole or zero (pulled out of the transfer function and set up on its own):

$$\left(\frac{s}{\omega_0} + 1\right) \tag{1.4}$$

The frequency ω_0 is called the "corner frequency," "3dB frequency," or just "frequency" of the respective pole or zero. (This might not be a real number for a root with complex solutions.) There are two key points to keep in mind with respect to this frequency (mentioned in passing previously, but reiterated here because they're important):

- At the corner frequency, the gain of that specific pole/zero starts shifting by 20dB / decade. (At the corner frequency, it has already shifted by 3dB, but it starts majorly shifting here.)

- At one tenth the corner frequency, the phase shift contribution of that specific pole/zero starts becoming noticeable. At the corner frequency, it has shifted 45°. At ten times the corner frequency, the phase shift has mostly stabilize at 90°.

The direction of the phase or gain shifts depend on the form of the pole or zero. Typically, poles cause a negative shift in phase and gain, and zeros cause a positive shift in phase and gain. (Looking backwards, Figure 1.3 is the graph of a pole with a corner frequency of 3.18Hz.) There are a few exceptions to this, such as a "right-hand-plane zero," but that'll be covered later.

1.3 Convolution And Laplace Transforms

If you are wondering where all these "s" terms are coming from and why they're used, then here is a brief digression on Laplace Transforms.

In the time domain, Equation 1.5 gives the mathematical expression for what you get as an output $y(t)$ for a particular input $x(t)$ and system $h(t)$.

$$x(t) * h(t) = y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \tag{1.5}$$

(Note that the star operator $*$ as used here does not mean multiplication, but convolution.) On a conceptual level, it is what happens when you "sweep" the input signal function through the system function and integrate the intersection of areas for each time step $d\tau$. Doing that integral by hand is a mess, though you will come out in the end with a time-domain function for what the output signal will be. This process is highy-simplified by moving everything from the time domain into the frequency domain.

The Laplace Transform, named after Pierre Simon Laplace, takes a given time-domain function $f(t)$ and converts it into a frequency-domain function $F(s)$ [6]:

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \tag{1.6}$$

[6] $s = j\omega$, where ω is frequency in radians and $j = \sqrt{-1}$, since electrical engineers use i for current and not imaginary numbers.

Again, these integrals are not the easiest to do by hand, and require some manipulation based on complex number identities. Fortunately again, tables of common signal transformations are widely available. Not only can the Laplace transform be used to convert signals from time domain to frequency domain, you can exploit the properties of frequency-domain functions to simplify common mathematical operations like integration, derivation, and especially convolution.

A few identities and properties most relevant to power electronics are given below in Table 1.2. Full Laplace Transform tables and properties can be found elsewhere.

Time Domain	Frequency Domain	Description
$\delta(t)$	1	Impulse Function
e^{at}	$\frac{1}{s-a}$	Exponential
$sin(kt)$	$\frac{k}{s^2+k^2}$	Sine Wave
te^{at}	$\frac{1}{(s-a)^2}$	Time-Ramped Exponential
$e^{at}f(t)$	$F(s-a)$	Multiply By Exponential
$\delta(t-t_0)$	e^{-st_0}	Time-Shift
$f'(t)$	$sF(s)-f(0)$	Derivation
$\int f(t)dt$	$\frac{1}{s}F(s)$	Integration
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	Convolution

Table 1.2: Selected Laplace Transforms and Identities

Especially look at the last item in the list: convolution changes into simple multiplication. Also, gain scaling is preserved across the Laplace conversion. Regarding derivation, you can often ignore the constant expression in practical applications.