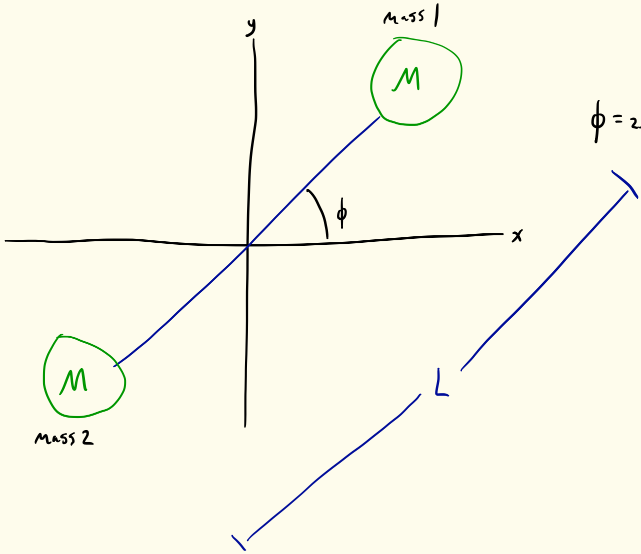
Problem 1

(a)



$$\phi = 2\pi f_{bar} t = \omega_b t$$

We compute the strain h_{ij} using the quadrupole formula $(c = G = 1)$

$$\bar{h}^{ij} = \frac{2}{r} \ddot{I}(t-r) \quad , \quad I = \int d^3x \, \underbrace{\mu(t,\vec{x})}_{\text{mass density}} x^i x^j$$

Here $\mu = M\left[\delta\left(\vec{x} - \vec{r}_1(t)\right) + \delta\left(\vec{x} - \vec{r}_2(t)\right)\right]$

where $\vec{r}_{1,2}$ are positions of masses 1 and 2.

$$\vec{r}_1 = \left\{ \frac{L}{2}\cos\phi , \frac{L}{2}\sin\phi , 0 \right\} \quad , \quad \vec{r}_2 = -\vec{r}_1$$

Evaluating the $\int d^3x$ with the δ functions

$$I^{ij}(t) = M\left(r_1^i(t) r_1^j(t) + r_2^i(t) r_2^j(t) \right) = 2M r_1^i(t) r_1^j(t)$$

$$I^{xx} = \frac{ML^2}{2}\cos^2\phi \quad , \quad I^{yy} = \frac{ML^2}{2}\sin^2\phi \quad , \quad I^{xy} = \frac{ML^2}{2}\cos\phi\sin\phi$$

$$\dot{I}^{xx} = -ML^2\cos\phi\sin\phi\,\dot\phi = -ML^2\omega_b\cos\phi\sin\phi \,,$$

$\cos 2\phi = \cos^2\phi - \sin^2\phi$

$$\ddot{I}^{xx} = -ML^2\omega_b\left[-\sin^2\phi\,\dot\phi + \cos^2\phi\,\dot\phi\right] = ML^2\omega_b^2\left[\sin^2\omega_b t - \cos^2\omega_b t\right] = -ML^2\omega_b^2\cos 2\phi$$

get $\ddot{I}^{yy}$ by taking $\phi \to \phi - \frac{\pi}{2}$ which sends $\sin \to -\cos$, $\cos \to \sin$

$$\boxed{\ddot{I}^{yy} = -\ddot{I}^{xx}}$$

use $\sin 2\phi = 2\cos\phi\sin\phi$

$$\dot{I}^{xy} = \frac{ML^2}{2}\left[-\sin^2\phi\,\dot\phi + \cos^2\phi\,\dot\phi\right] = \frac{ML^2\omega_b}{2}\left[\cos^2\phi - \sin^2\phi\right]$$

$$\ddot{I}^{xy} = \frac{ML^2\omega_b}{2}\left[-2\cos\phi\sin\phi\,\dot\phi - 2\sin\phi\cos\phi\,\dot\phi\right] = -2ML^2\omega_b^2\cos\phi\sin\phi = \boxed{-ML^2\omega_b^2\sin 2\phi}$$

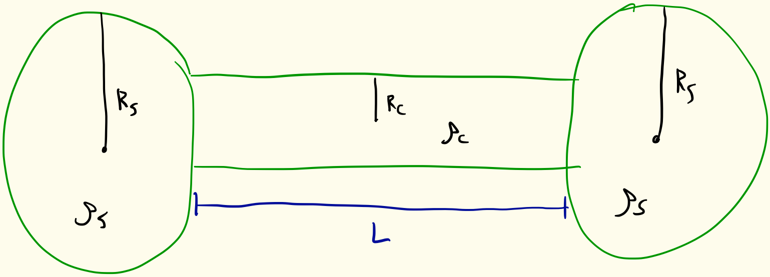
Hence, substituting into the quadrupole formula

restoring G and c

$$\bar{h}^{xx} = -\bar{h}^{yy} = \frac{-2ML^2\omega_b^2}{r}\cos\left[2\omega_b(t-r)\right] = \frac{-2GL^2\omega_b^2}{c^4 r}\cos\left[2\omega_b(t-r)\right]$$

$$\bar{h}^{xy} = \frac{-2ML^2\omega_b^2}{r}\sin\left[2\omega_b(t-r)\right] = \frac{-2GML^2\omega_b^2}{c^4 r}\sin\left[2\omega_b(t-r)\right]$$

(b) Let us imagine the dumbell consist of cylinder with cross-sectional radius R_c and two spheres with radius R_s. The spheres are made of a material with density ρ_s and the cylinders are made of a material with density ρ_c



We take the spheres to be made of copper and be as large as is practically possible

$\rho_s = 8960\,Kg\,m^{-3}$, $R_s = 10\,m$ $\rightarrow M = 3.8 \times 10^7\,Kg$

We design the dumbell to produces waves in LIGO band $\omega_b \approx 2\pi\, 10^2\,Hz$

We take the cylinder to be made of steel and have a radius of 1 m

$\rightarrow \rho_c = 7800\,Kg\,m^{-3}$, $R_c = 1\,m$

The length L is now determined by the condition that the cylinder not deform under the stress of accelerating the masses

The force on one face of the cylinder due to the sphere is

$$F = M \frac{V_{spher}^2}{(L/2)} = M \frac{(L/2\,\omega_b)^2}{(L/2)} = \frac{M}{2} L \omega_b^2$$

The stress on the cylinder is $T = \frac{F}{A} = \frac{M L \omega_b^2}{2\pi R_c^2}$

The corresponding strain is $S = \frac{T}{E}$, where E is the Young's modulus of the cylinder.

The strain $S \leq S_Y$, the yield strain of the material. Hence

$$S_Y E = T = \frac{M L \omega_b^2}{2\pi R_c^2} \rightarrow L = \frac{2\pi R_c^2 S_Y E}{M \omega_b^2} = 2.7 \times 10^2\,m$$

The strain calculated in part (a) has amplitude $A = \frac{2GML^2\omega_b^2}{c^4\, 3\lambda}$ when observed at $r = 3\lambda = \frac{c\,2\pi}{2\omega_b} = \frac{c\pi}{\omega_b}$. Substituting the above numbers gives

$$\boxed{A = 4.0 \times 10^{-33}}$$

, although note the dumbell designed here may be to difficult to construct experimentally.

Problem 2

(a) The binary system is identical to the spinning dumbell if we take

$M = 1M_\odot$, $f_b = \frac{1}{6.5h}$, $r = 11.7\,pc$

Kepler's law $w_b^2 = \frac{2MG}{L^3} \to L = \left(\frac{2MG}{w_b^2}\right)^{1/3} = 1.5\times10^9\,m$

For the face on inclination $\bar{h}^{ij}$ is already transverse and traceless.

Thus $\bar{h}^{ij} = \bar{h}^{ij}_{TT} = h^{ij}_{TT}$ since $\bar{h}^{ij} = h^{ij}$ in the TT gauge.

Thus $h_+ = \frac{-2GML^2w_b^2}{rc^4}\cos\left[2w_b(t-r)\right]$ and $h_\times = \frac{-2GML^2w_b^2}{rc^4}\sin\left[2w_b(t-r)\right]$

Both h_+ and $h_\times$ have frequency $f = 2f_b = 8.5\times10^{-5}\,Hz$

Both h_+ and $h_\times$ have amplitudes

$$A = \frac{2GML^2w_b^2}{rc^4} = 1.79\times10^{-20}$$

although they are out of phase by $\frac{\pi}{2}$.

(b) The GW luminosity is the time averaged radiated power. It can also be expressed in terms of derivatives of the quadrupole moment (see Hartle, table 23.)

$$\mathcal{L}_{GW} = \frac{1}{5}\left(\dddot{I\!\!-}_{ij}\,\dddot{I\!\!-}^{ij}\right) \quad \text{with} \quad I\!\!-_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I^{k}{}_{k}$$

Recall from 1(a): $\ddot{I}^{xx} = -\frac{M_{tot}}{2}L^2\Omega^2\cos 2\Omega t = -\ddot{I}^{yy}$, $\ddot{I}^{xy} = -\frac{M_{tot}}{2}L^2\Omega^2\sin 2\Omega t$

where we replace $M \to \frac{1}{2}M_{tot}$ where M_{tot} is the total mass, and $\omega_b = \Omega = \sqrt{M_{tot}/L^3}$ is the orbital frequency.

Calculating the third derivative

$$\dddot{I}^{xx} = -\dddot{I}^{yy} = M_{tot}L^2\Omega^3\sin\Omega t, \quad \dddot{I}^{xy} = -M_{tot}L^2\Omega^3\cos\Omega t$$

Note $\dddot{I}^{k}{}_{k} = 0$ so $\dddot{I}_{ij} = \dddot{I\!\!-}_{ij}$

Hence

$$\mathcal{L}_{GW} = \frac{1}{5}\left\langle \dddot{I\!\!-}_{ij}\,\dddot{I\!\!-}^{ij}\right\rangle = \frac{1}{5}\left\langle \dddot{I}_{xx}^{\,2} + \dddot{I}_{yy}^{\,2} + 2\dddot{I}_{xy}^{\,2}\right\rangle = \frac{2}{5}\left\langle \dddot{I}_{xx}^{\,2} + \dddot{I}_{xy}^{\,2}\right\rangle$$

$$= \frac{2}{5}\left\langle (M_{tot}L^2\Omega^3)^2\sin^2 2\Omega t + (M_{tot}L^2\Omega^3)^2\cos^2\Omega t\right\rangle$$

$$= \frac{2}{5}M_{tot}^2L^4\Omega^6 = \boxed{\frac{2}{5}\left(\frac{M_{tot}}{L}\right)^5} = \frac{2}{5}M_{tot}^{10/3}\Omega^{10/3} \checkmark$$

We now deduce the slow evolution of L (with respect to the orbital period) using energy Balance

$$\frac{dE}{dt} = -\mathcal{L}_{GW}$$

The Newtonian energy of the Binary is

$$E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{m_1 m_2}{L}$$

Kepler

In the COM frame for equal masses $m_1 = m_2 = \frac{M_{tot}}{2}$, $v_1 = v_2 = \frac{L}{2}\Omega = \frac{1}{2}\sqrt{M/L}$

$$E = \frac{M_{tot}}{2} \frac{1}{4} \frac{M_{tot}}{L} - \frac{M_{tot}^2}{4L} = - \frac{M_{tot}^2}{8L}$$

Hence

$$\frac{dE}{dt} = \frac{M_{tot}^2}{8L^2} \frac{dL}{dt} = - \mathcal{L}_{GW} = \frac{-2}{5} \left(\frac{M_{tot}}{L} \right)^5$$

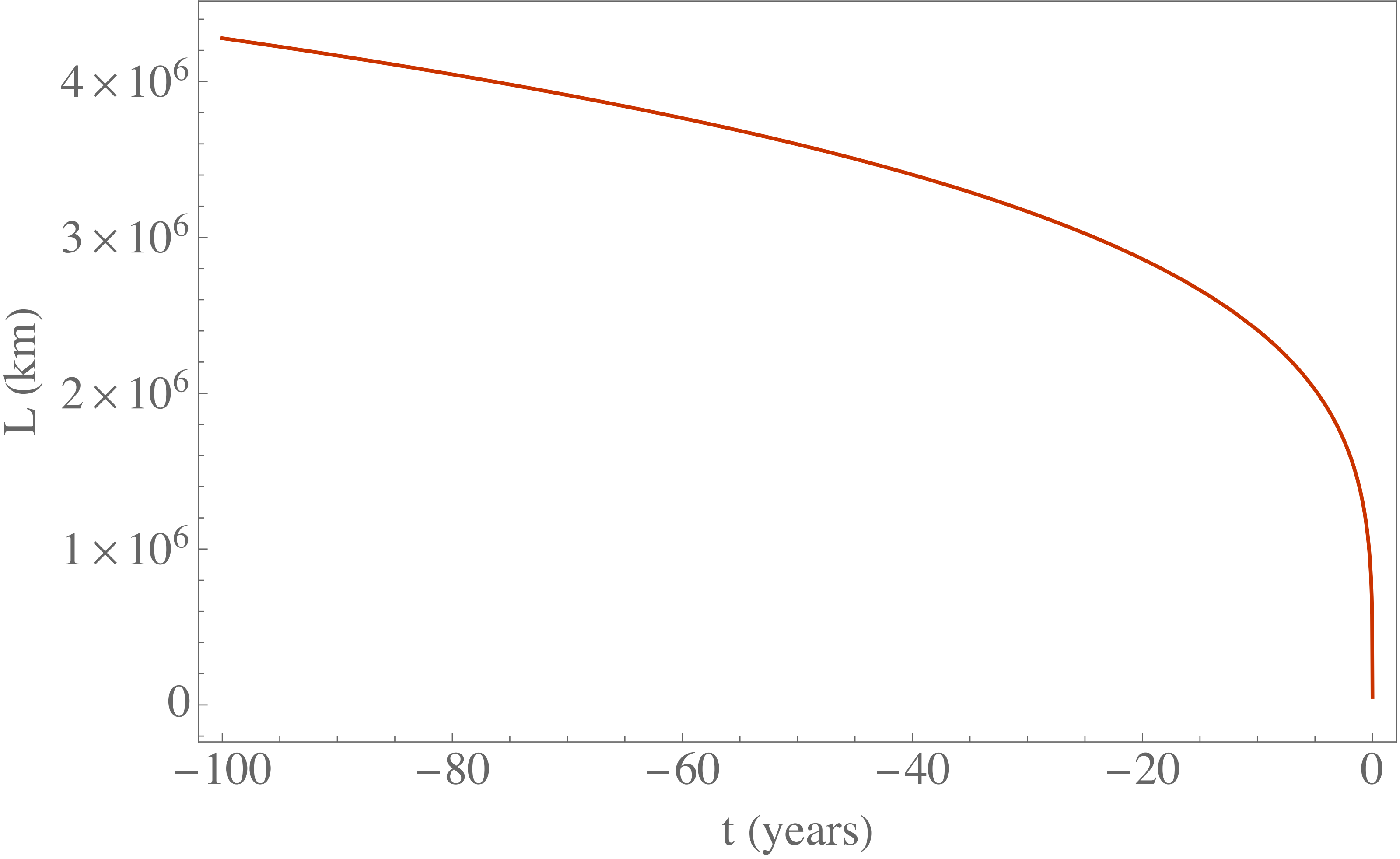
$$\rightarrow \frac{dL}{dt} = -\frac{16}{5} \left(\frac{M_{tot}}{L} \right)^3 \rightarrow \int dL\, L^3 = -\frac{16}{5} M_{tot}^3 t \rightarrow \frac{1}{4} L^4 = \frac{16}{5} M_{tot}^3 (t - t_c)$$

where t_c is the coalescence time

$$\rightarrow \boxed{L = \left(\frac{64}{5} M_{tot}^3 (t - t_c) \right)^{1/4}}$$

restoring powers of G and c

$$L = \left(\frac{64}{5} \frac{G^3 M_{tot}^3}{c^5} (t - t_c) \right)^{1/4}$$



4×10^6
3×10^6
2×10^6
1×10^6
0
L (km)
−100
−80
−60
−40
−20
0
t (years)