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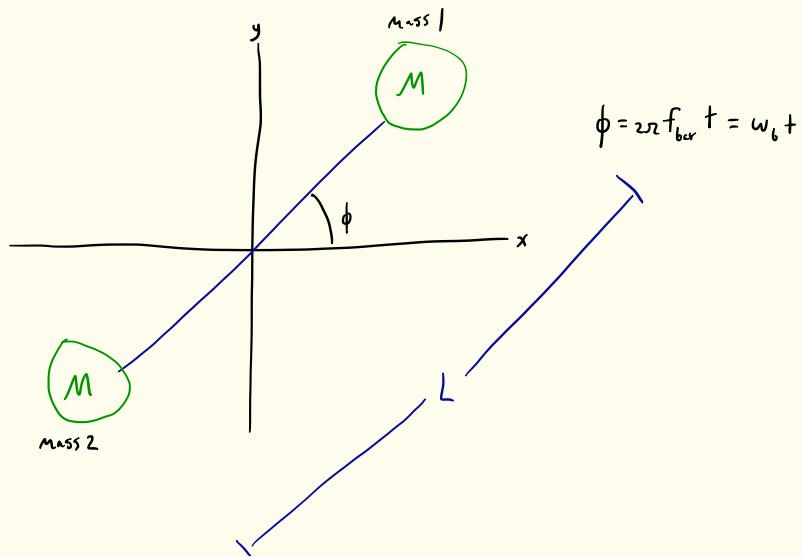
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Problem 1

(a)



$$\phi = 2\pi f_{bar} t = \omega_b t$$

We compute the strain  $h_{ij}$  using the quadrupole formula ( $c=G=1$ )

$$\vec{h}^{ij} = \frac{2}{r} \vec{I}(t-r) , \quad \vec{I} = \int \delta^3 x \underbrace{u(t, \vec{x})}_{\text{mass density}} \vec{x}^i \vec{x}^j$$

$$\text{Here } u = M \left[ \delta(\vec{x} - \vec{r}_1(t)) + \delta(\vec{x} - \vec{r}_2(t)) \right]$$

where  $\vec{r}_{1,2}$  are positions of masses 1 and 2.

$$\vec{r}_1 = \left\{ \frac{L}{2} \cos \phi, \frac{L}{2} \sin \phi, 0 \right\} , \quad \vec{r}_2 = -\vec{r}_1$$

Evaluating the  $\int \delta^3 x$  with the 8 functions

$$\vec{I}^{ii}(t) = M \left( r_1^i(t) r_1^i(t) + r_2^i(t) r_2^i(t) \right) = 2M r_1^i(t) r_1^i(t)$$

$$\vec{I}^{xx} = \frac{ML^2}{2} \cos^2 \phi , \quad \vec{I}^{yy} = \frac{ML^2}{2} \sin^2 \phi , \quad \vec{I}^{xy} = \frac{ML^2}{2} \cos \phi \sin \phi$$

$$\dot{I}^{xx} = -ML^2 \cos\phi \sin\phi \dot{\phi} = -ML^2 w_b \cos\phi \sin\phi ,$$

$$\ddot{I}^{xx} = -ML^2 w_b \left[ -\sin^2\phi \dot{\phi} + \cos^2\phi \ddot{\phi} \right] = ML^2 w_b^2 \left[ \sin^2 w_b t - \cos^2 w_b t \right] \stackrel{\downarrow}{=} -ML^2 w_b^2 \cos 2\phi$$

get  $\ddot{I}^{yy}$  by taking  $\phi \rightarrow \phi - \frac{\pi}{2}$  which sends  $\sin \rightarrow -\cos$ ,  $\cos \rightarrow \sin$

$$\boxed{\ddot{I}^{yy} = -\ddot{I}^{xx}}$$

$$\dot{I}^{xy} = \frac{ML^2}{2} \left[ -\sin^2\phi \dot{\phi} + \cos^2\phi \ddot{\phi} \right] = \frac{ML^2 w_b}{2} \left[ \cos^2\phi - \sin^2\phi \right]$$

$$\ddot{I}^{xy} = \frac{ML^2 w_b}{2} \left[ 2\cos\phi \sin\phi \dot{\phi} - 2\sin\phi \cos\phi \ddot{\phi} \right] = -2ML^2 w_b^2 \cos\phi \sin\phi \dot{\phi} = \boxed{-ML^2 w_b^2 \sin 2\phi}$$

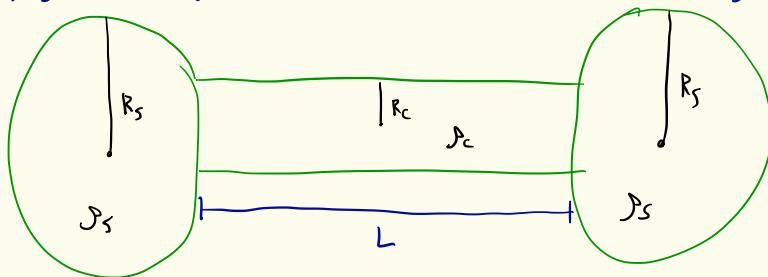
$$\text{use } \sin 2\phi = 2 \cos\phi \sin\phi$$

Hence, substituting into the quadrupole formula

restoring G and c

$$\begin{aligned} \bar{h}^{xx} &= -\bar{h}^{yy} = \frac{-2ML^2 w_b^2}{r} \cos [2w_b(t-r)] \\ &\quad \stackrel{\leftarrow}{=} \frac{-2GM^2 w_b^2}{c^4 r} \cos [2w_b(t-r)] \\ \bar{h}^{xy} &= \frac{-2ML^2 w_b^2}{r} \sin [2w_b(t-r)] \\ &\quad \stackrel{\downarrow}{=} \frac{-2GM^2 w_b^2}{c^4 r} \sin [2w_b(t-r)] \end{aligned}$$

- (b) Let us imagine the dumbbell consist of cylinders with cross-sectional radius  $R_c$  and two spheres with radius  $R_s$ . The spheres are made of a material with density  $\rho_s$  and the cylinders are made of a material with density  $\rho_c$



We take the spheres to be made of copper and be as large as is practically possible

$$\rho_s = 8960 \text{ kg m}^{-3}, R_s = 10 \text{ m} \rightarrow M = 3.8 \times 10^7 \text{ Kg}$$

We design the dumbbell to produce waves in LIGO band  $\omega_b \approx 2\pi \times 10^2 \text{ Hz}$

We take the cylinder to be made of steel and have a radius of 1m

$$\rightarrow \rho_c = 7800 \text{ kg m}^{-3}, R_c = 1 \text{ m}$$

The length L is now determined by the condition that the cylinder not deform under the stress of accelerating the masses

The force on one face of the cylinder due to the sphere is

$$F = M \frac{\nu_{\text{spher}}^2}{(L/2)} = M \frac{(\nu_b \omega_b)^2}{(L/2)} = \frac{M}{2} L \omega_b^2$$

$$\text{The stress on the cylinder is } T = \frac{F}{A} = \frac{ML\omega_b^2}{2\pi R_c^2}$$

The corresponding strain is  $S = \frac{T}{E}$ , where E is the Young's modulus of the cylinder.

The strain  $S \leq S_y$ , the yield strain of the material. Hence

$$S_y E = T = \frac{ML\omega_b^2}{2\pi R_c^2} \rightarrow L = \frac{2\pi R_c^2 S_y E}{M \omega_b^2} = 2.7 \times 10^2 \text{ m}$$

The strain calculated in part (a) has amplitude  $A = \frac{26 ML^2 \omega_b^2}{c^4 3\lambda}$  when observed at  $r = 3\lambda = \frac{c\tau}{2\omega_b} = \frac{c\tau}{\omega_b}$ . Substituting the above numbers gives

$$A = 4.0 \times 10^{-33}$$
, although note the dumbbell designed here may be too difficult to construct experimentally.

## Problem 2

(a) The binary system is identical to the spinning dumbbell if we take

$$M = 1 M_{\odot}, P_b = \frac{1}{6.5 h}, r = 11.7 \text{ pc}$$

$$\text{Kepler's law } w_b^2 = \frac{2MG}{L^3} \rightarrow L = \left( \frac{2MG}{w_b^2} \right)^{1/3} = 1.5 \times 10^9 \text{ m}$$

For the face-on inclination  $\bar{h}^{ij}$  is already transverse and traceless.

Thus  $\bar{h}^{ij} = \bar{h}_{TT}^{ij} = h_{TT}^{ij}$  since  $\bar{h}^{ij} = h^{ij}$  in the TT gauge.

$$\text{Thus } h_t = \frac{-2GM L^2 w_b^2}{r c^4} \cos[2w_b(t-r)] \text{ and } h_x = \frac{-2GM L^2 w_b^2}{r c^4} \sin[2w_b(t-r)]$$

Both  $h_t$  and  $h_x$  have frequency  $f = 2f_b = 8.5 \times 10^{-5} \text{ Hz}$

Both  $h_t$  and  $h_x$  have amplitudes

$$A = \frac{2GM L^2 w_b^2}{r c^4} = 1.79 \times 10^{-20}$$

although they are out of phase by  $\frac{\pi}{2}$ .

(1) The GW luminosity is the time averaged radiated power. It can also be expressed in terms of derivatives of the quadrupole moment (see Hartle, table 23.)

$$\mathcal{L}_{GW} = \frac{1}{5} \langle \ddot{\mathbb{I}}_{ij} \ddot{\mathbb{I}}^{ij} \rangle \text{ with } \ddot{\mathbb{I}}_{ij} = \ddot{\mathbb{I}}_{ij} - \frac{1}{3} \delta_{ij} \ddot{\mathbb{I}}^k_k$$

Recall from 1(a):  $\ddot{\mathbb{I}}^{xx} = -\frac{M_{tot} L^2}{2} \Delta^2 \cos 2\omega t = -\ddot{\mathbb{I}}^{yy}$ ,  $\ddot{\mathbb{I}}^{xy} = -\frac{M_{tot} L^2}{2} \Delta^2 \sin 2\omega t$   
where we replace  $M \rightarrow \frac{1}{2} M_{tot}$  where  $M_{tot}$  is the total mass, and  $\omega_1 = \omega = \sqrt{M_{tot}/L^3}$   
is the orbital frequency.

Calculating the third derivative

$$\dddot{\mathbb{I}}^{xx} = -\ddot{\mathbb{I}}^{yy} = M_{tot} L^2 \Delta^3 \sin \omega t, \quad \ddot{\mathbb{I}}^{xy} = -M_{tot} L^2 \Delta^3 \cos \omega t$$

Note  $\ddot{\mathbb{I}}^k_k = 0$  so  $\ddot{\mathbb{I}}_{ij} = \ddot{\mathbb{I}}_{ij}$

Hence

$$\begin{aligned} \mathcal{L}_{GW} &= \frac{1}{5} \langle \ddot{\mathbb{I}}_{ij} \ddot{\mathbb{I}}^{ij} \rangle = \frac{1}{5} \left\langle \ddot{\mathbb{I}}_{xx}^2 + \ddot{\mathbb{I}}_{yy}^2 + 2 \ddot{\mathbb{I}}_{xy}^2 \right\rangle = \frac{2}{5} \left\langle \ddot{\mathbb{I}}_{xx}^2 + \ddot{\mathbb{I}}_{xy}^2 \right\rangle \\ &= \frac{2}{5} \left\langle (M_{tot} L^2 \Delta^3)^2 \sin^2 \omega t + (M_{tot} L^2 \Delta^3)^2 \cos^2 \omega t \right\rangle \\ &= \frac{2}{5} M_{tot}^2 L^4 \Delta^6 = \boxed{\frac{2}{5} \left( \frac{M_{tot}}{L} \right)^5} = \frac{2}{5} M_{tot}^{10/3} \Delta^{10/3} \quad \checkmark \end{aligned}$$

We now deduce the slow evolution of  $L$  (with respect to the orbital period)  
using energy Balance

$$\frac{dE}{dt} = -\mathcal{L}_{GW}$$

The Newtonian energy of the binary is

$$E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{m_1 m_2}{L}$$

In the COM frame for equal masses  $m_1 = m_2 = \frac{M_{\text{tot}}}{2}$ ,  $v_1 = v_2 = \frac{L}{2}$ ,  $L = \frac{1}{2} \sqrt{M/L}$

$$E = \frac{M_{\text{tot}}}{2} \frac{1}{4} \frac{M_{\text{tot}}}{L} - \frac{M_{\text{tot}}^2}{4L} = - \frac{M_{\text{tot}}^2}{8L}$$

$\downarrow$   
Kepler

Hence

$$\frac{dE}{dt} = \frac{M_{\text{tot}}^2}{8L^2} \frac{dL}{dt} = - \mathcal{L}_{\text{GW}} = - \frac{2}{5} \left( \frac{M_{\text{tot}}}{L} \right)^5$$

$$\rightarrow \frac{dL}{dt} = - \frac{16}{5} \left( \frac{M_{\text{tot}}}{L} \right)^3 \rightarrow \int dL L^3 = - \frac{16}{5} M_{\text{tot}}^3 t \rightarrow \frac{1}{4} L^4 = \frac{16}{5} M_{\text{tot}}^3 (t - t_c)$$

where  $t_c$  is the coalescence time

$$\boxed{\rightarrow L = \left( \frac{64}{5} M_{\text{tot}}^3 (t - t_c) \right)^{1/4}}$$

restoring powers of  $G$  and  $c$

$$L = \left( \frac{64}{5} \frac{G M_{\text{tot}}^3}{c^5} (t - t_c) \right)^{1/4}$$

