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FYS 3150 Project 1

Amund Bremer
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Github browser-link to this repository: https://github.com/zmbdr/FYS3150/tree/main/project1

Problem 1

Given

$$u(x) = 1 - (1 - e^{-10})x - e^{-10x},$$

we find that

$$u'(x) = (1 - e^{-10}) - (-10)e^{-10x},$$

and

$$u''(x) = -100e^{-10x}.$$

With the supplied source term $f(c)$ given by

$$f(x) = 100e^{-10x},$$

we get

$$u''(x) = -f(x).$$

The above is valid for $x \in [0, 1]$, and the boundary conditions $u(0) = u(1) = 0$ are satisfied. Hence, $u(x)$ is an exact solution to the given problem.

Problem 2

A simple c++ script to dump $u(x_i)$ for $x = ih$ is supplied in the file `main_2.cpp`. The plot in Figure **??** is generated in the file `plot_2.py`

Problem 3

It is well established that an approximation to $u''(x)$ is given as follows:

$$u''(x) \approx= \frac{u(x+h) - 2u(x) + u(x-h)}{h^2},$$

where h is an appropriately chosen small step-size. For the problem at hand, we are interested in approximating u'' over the unit interval. We define $x_i = i/N$, discretizing the unit interval into N non-overlapping intervals, and let $i = 0 \dots N$. Furthermore, we introduce $u(x_i) = u_i$, and $f(x_i) = f_i$.

Thus, from the original Poisson equation, we can write

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \text{higher order terms} \;=\; f_i, \quad i = 1, \dots N-1.$$

This leads to the simple discrete approxmation v_i:

$$-(v_{i+1} - 2v_i + v_{i-1}) = h^2 f_i, \quad i = 1, \dots N-1.$$

Here, $v_0 = u_0 = u(0) = 0$, $v_N = u_N = u(1) = 0$, and $h = 1/N$. This is the discrete version of the original Poisson equation, and if this is solvable (it is!), we have found approximation to $u(x)$ on the discrete grid x_i.

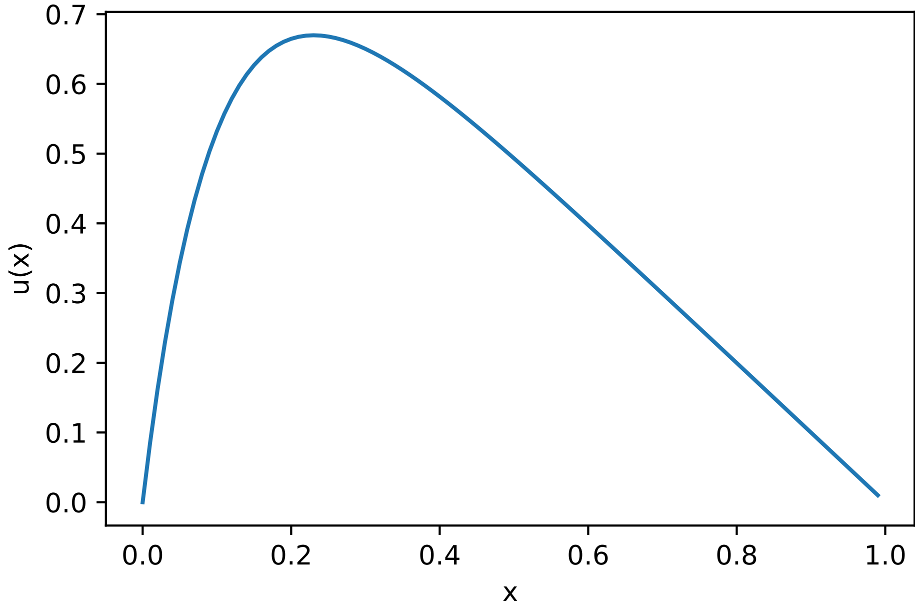

FIG. 1: The exact solution $u(x)$ to the Poisson equation with the given source term over its domain.

Problem 4

The discrete equation above can be written as follows in matrix form:

$$-\begin{bmatrix}1 & -2 & 1\end{bmatrix}\begin{bmatrix}v_{i-1}\\ v_i\\ v_{i+1}\end{bmatrix} = h^2 f_i, \quad i = 2, \ldots N-2;$$

$$-\left(\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}v_i\\ v_{i+1}\end{bmatrix} + v_{i-1}\right) = h^2 f_i, \quad i = 1, \text{ and}$$

$$-\left(\begin{bmatrix}1 & -2\end{bmatrix}\begin{bmatrix}v_{i-1}\\ v_i\end{bmatrix} + v_{i+1}\right) = h^2 f_i, \quad i = N.$$

Combining these three equations we get

$$\begin{bmatrix}2 & -1 & 0 & \cdots & 0\\ -1 & 2 & 1 & \cdots & 0\\ \vdots & \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & -1 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_{N-1}\end{bmatrix} - \begin{bmatrix}v_0\\ 0\\ \vdots\\ v_N\end{bmatrix} = h^2\begin{bmatrix}f_1\\ \vdots\\ f_{N-1}\end{bmatrix}$$

This is what we want;

$$\mathbf{A}\vec{v} = \vec{g},$$

where $g_1 = h^2 f_1 + v_0$, $g_{N-1} = h^2 f_{N-1} + v_N$, and $g_i = h^2 f_i, i = 2, \ldots N-2$.