Application of an OIFS Method to MHD

Zachary M. Miksis
M.S. Candidate, Dept. of Applied Mathematics, IIT

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Abstract

Nek5000 is a highly-scalable spectral element solver for the unsteady incompressible Navier-Stokes equations, heat transfer problem, unsteady Stokes flow, and incompressible magnetohydrodynamics. In this project, an extension of a temporal discretization method, namely the OIFS method, used for the Navier-Stokes equations to the MHD equations is outlined, allowing for larger time-stepping and greater stability of solutions.

1 Introduction

As the capabilities of high-performance computing progress, so does the expanse of problems that can be solved and the accuracy at which this is done. Since 1986, Nek5000 has been a leading highly-scalable spectral element code for simulating unsteady incompressible Navier-Stokes, heat transfer, and many other applications [1]. Specifically, this project will focus on the application of Nek5000 to magnetohydrodynamics (MHD) problems, and further contributing to the ability to accurately simulate these flows.

We will consider the viscous incompressible MHD equations,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \tilde{p} + \frac{1}{Re} \nabla^2 \mathbf{u} + \mathbf{f}_L, \qquad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \frac{1}{Re_M} \nabla^2 \mathbf{B} \tag{2}$$

where $\mathbf{u}$ is the flow velocity field, $\mathbf{B}$ is the magnetic field, Re and Re_M are the Reynolds number and magnetic Reynolds number, respectively, $\tilde{p}$ is the pressure, and $\mathbf{f}_L$ is the Lorentz force, defined as

$$\mathbf{f}_L = (\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{2}\nabla|\mathbf{B}|^2 + \mathbf{B} \cdot \nabla\mathbf{B} \tag{3}$$

Nek5000 will discretize these equations and solve them using user defined boundary and initial conditions. The goal of this project is to improve the discretization methods used, and improve the simulations and computation time.

2 Current Status

Currently, Nek5000 reformulates the MHD equations using an Elsässer formulation as convenient, and then constructs the temporal discretization by employing a kth-order backward difference formula for the diffusive/solenoidal terms and a $k-1$ extrapolation for the nonlinear terms. For $k = 2$, this becomes

$$\frac{\beta_0\mathbf{z}_{\pm}^{n} - \beta_1\mathbf{z}_{\pm}^{n-1} - \beta_2\mathbf{z}_{\pm}^{n-1}}{\Delta t} - \nu_{+}\nabla^2\mathbf{z}_{\pm}^{n} - \nu_{-}\nabla^2\mathbf{z}_{\pm}^{n} + \nabla p_{\pm}^{n} =< \mathbf{z}_{\mp} \cdot \nabla\mathbf{z}_{\pm} >^{n} \tag{4}$$

where $< \mathbf{z}_{\mp} \cdot \nabla\mathbf{z}_{\pm} >^{n}$ is obtained using the $(k-1)$th-order extrapolation

$$< \mathbf{z}_{\mp} \cdot \nabla\mathbf{z}_{\pm} >^{n} := 2\bar{\mathbf{z}}_{\mp} \cdot \nabla\mathbf{z}_{\pm}^{n-1} - \bar{\mathbf{z}}_{\mp} \cdot \nabla\mathbf{z}_{\pm}^{n-2} \tag{5}$$

Here, we have β_i's are defined constants, $\nu_{\pm} := \frac{1}{2}\left(\frac{1}{Re} \pm \frac{1}{Re_M}\right)$, $p_{\pm} := p \pm p_M$, and $\mathbf{z}_{\pm} := \mathbf{u} \pm \mathbf{B}$. This can readily be extended to third-order in time.

Nek5000 currently uses the BDFk/EXT$k-1$ method when solving the MHD equations, but not when solving the general hydro Navier-Stokes equations. These are solved using the operator-integration-factor scheme (OIFS) method [2, 3], which will be described in the next section.

3 Proposed Work

The proposed work is to extend the use of the OIFS method from the hydro solver to the MHD solver in Nek5000. This method allows for larger time-stepping by circumventing the standard Courant stability constraints.

The idea of this method is instead to apply BDFk to the material derivative

$$\frac{D_{\mp}\mathbf{z}_{\pm}}{Dt} := \frac{\partial \mathbf{z}_{\pm}}{\partial t} + \mathbf{z}_{\mp} \cdot \nabla \mathbf{z}_{\pm} \approx \frac{\beta_0 \mathbf{z}_{\pm}^n - \beta_1 \bar{\mathbf{z}}_{\pm}^{n-1} - \beta_2 \bar{\mathbf{z}}_{\pm}^{n-2}}{\Delta t} \tag{6}$$

This then gives the fully implicit formulation

$$\frac{\beta_0}{\Delta t}\mathbf{z}_{\pm}^n - \nu_+ \nabla^2 \mathbf{z}_{\pm}^n - \nu_- \nabla^2 \mathbf{z}_{\pm}^n + \nabla p_{\pm}^n = \frac{\beta_1}{\Delta t}\bar{\mathbf{z}}_{\pm}^{n-1} - \frac{\beta_2}{\Delta t}\bar{\mathbf{z}}_{\pm}^{n-2}, \qquad \nabla \cdot \mathbf{z}_{\pm}^n = 0, \tag{7}$$

which is stable for all Δt if $k \leq 2$. Here, $\bar{\mathbf{z}}_{\pm}^{n-j}$ is an approximation of $\mathbf{z}_{\pm}$ at the foot of the $\mathbf{z}_{\mp}$ characteristic incident to each computational gridpoint. The OIFS scheme avoids off-gridpoint interpolation by solving the following pure convection problem for $\bar{\mathbf{z}}_{\pm}^{n-j}$.

$$\begin{aligned} \frac{\partial \tilde{\mathbf{u}}_l}{\partial t} + \mathbf{u} \cdot \nabla \tilde{\mathbf{u}} &= 0 \\ \tilde{\mathbf{u}}_l(\mathbf{x}, t^{n-l}) &= \mathbf{u}(\mathbf{x}, t^{n-l}) \end{aligned} \tag{8}$$

The initial value problem (8) is solved using an explicit fourth-order Runge-Kutta scheme with step size $\Delta s \leq \Delta t$ which satisfies appropriate CFL criteria. The values of $\mathbf{u}$ in (8) are interpolated/extrapolated from the previous velocity fields, $(\mathbf{u}^{n-1}, \ldots, \mathbf{u}^{n-l})$.

4 Expected Accomplishments

In the coming months, the goal is to begin the coding of the OIFS method in Nek5000, and to apply this to the MHD example in the current repository.

A large part of this will require a deeper understanding of the inner workings of the Nek5000 code, which will be the first task. The goal will be to have a full understanding of exactly how the OIFS method is applied to the Navier-Stokes equations. This will allow for a more standardized coding of the method for the MHD equations, and a full utilization of the pre-built mathematical methods in Nek.

The next step will be to compile a similar code for the MHD equations. This will take the most time, and is expected to take the majority of the semester.

The final step of the work will be to apply this to the current MHD example in the public repository, to debug and analyze its effectiveness over the BDFk/EXT$k-1$ method that is currently employed. This portion of the project may not be completed within the semester, though will be the final aim of this project.

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