1 Minimum number of terms (alternating test)

The error in a decreasing alternating series is bounded by the next term in the series. For a series

$$S = \sum_{k=0}^{\infty} (-1)^k a_k \ \ ,$$

the minimum number of terms (N) needed to be certain of accuracy within a chosen tolerance ϵ (positive) is

$$|a_{N+1}| \leq \epsilon \ \ . \tag{1.1}$$

Here, we have

$$S = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k + 1 + \ln(k^2 + 1)}$$

so we need to find N such that

$$|a_{N+1}| = \left| \frac{1}{2N + 1 + \ln(N^2 + 1)} \right| \leq \epsilon \quad . \tag{1.2}$$

Since $N > 0$ then $\ln(N^2 + 1)$ is positive, and the denominator is positive. So we can say

$$\begin{aligned} \frac{1}{2N + 1 + \ln(N^2 + 1)} &\leq \epsilon \\ \implies 2N + 1 + \ln(N^2 + 1) &\geq \frac{1}{\epsilon} \end{aligned} \tag{1.3}$$

Now, $x \geq \ln(x + 1)$ when $x > -1$, so $N^2 \geq \ln(N^2 + 1)$ for $N^2 > 0$. Therefore,

$$\begin{aligned} 2N + 1 + N^2 &\geq \frac{1}{\epsilon} \\ (N + 1)^2 &\geq \frac{1}{\epsilon} \\ N &\geq \sqrt{\frac{1}{\epsilon}} - 1 \end{aligned} \tag{1.4}$$