Feynman Integrals from Positivity Constraints

February 2025

1 Proposal

We consider the problem of computing a family of Feynman integrals with L loops and n propagators,

$$I^d_{a_1,\ldots,a_n} = \left(\prod_{j=1}^{L} \int \frac{d^d \ell_j}{i\pi^{d/2}}\right) \frac{1}{D_1^{a_1} \cdots D_n^{a_n}}. \tag{1}$$

We already know that for a general family (1) of integrals, there exist **integral-by-parts (IBP)** relations among them, which are linear recurrence relations. Given such relations, every integral $I^d_{a_1,\ldots,a_n}$ of family (1) can be reduced to a linear combination of finitely many **master integrals (MI)**, namely

$$I^d_{a_1,\ldots,a_n} = \sum_p c_p(a_1,\ldots,a_n;d) I_p, \tag{2}$$

in which $I_p = I^d_{a_1^p,\ldots,a_n^p}$ denotes the p-th master integral, and each c_p is a rational function of the kinematic invariants as well as the dimension d. There are public programs (such as `FIRE`, `Reduze` or `Kira`) to perform such reduction. Once the reduction is done, the computation of an arbitrary $I^d_{a_1,\ldots,a_n}$ is reduced to computing all master integrals $\{I_p\}$.

In this work, we restrict ourselves to considering the computation of MIs under the following assumptions.

Assumption 1. *The kinematics lies in the Euclidean region.*

Assumption 2. *Each member I_p of the master integrals is finite.*

To formally define what the **Euclidean region** is, we introduce the Feynman parametrization of the integral $I^d_{a_1,\dots,a_n}$,

$$I^d_{a_1,\dots,a_n} = \frac{\Gamma(a-Ld/2)}{\Gamma(a_1)\cdots\Gamma(a_n)} \int_{\boldsymbol{x}\geqslant 0} d^n\boldsymbol{x}\,\delta\left(1-\sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} x_i^{a_i-1} \frac{\mathcal{U}(\boldsymbol{x})^{a-(L+1)d/2}}{\mathcal{F}(\boldsymbol{x})^{a-Ld/2}}, \tag{3}$$

in which $\mathcal{U}(\boldsymbol{x})$ and $\mathcal{F}(\boldsymbol{x})$ are known as Symanzik polynomials, and a is a shorthand for $\sum_{i=1}^{n} a_i$. The Euclidean region refers to any kinematic point that keeps both $\mathcal{U}(\boldsymbol{x})$ and $\mathcal{F}(\boldsymbol{x})$ non-negative throughout the integration range of $\boldsymbol{x}$. When this is the case, the integrand in (3) would be analytic to $\boldsymbol{x}$ throughout its integration range.

We remark that Assumption 2 can be easily realized by choosing a proper basis (i.e., a collection of finite, yet linearly independent MIs), and does not hurt the generality of our method. Although Assumption 1 seems to pose a severe limitation for our method, one may still treat the result in Euclidean regions as a starting point for other methods to evaluate master integrals, e.g., as a boundary condition for methods based on differential equations (DEs).

The basic idea of our method is to leverage *inequality constraints* to bound the values of master integrals. In the following, we would assume that t is the lower bound of $a = \sum_{i=1}^{n} a_i$ of all considered integrals $I^d_{a_1,\dots,a_n}$, including the MIs $\{I^d_{a_1^p,\dots,a_n^p}\}$. To simplify notations let us define

$$\hat{I}^d_{a_1,\dots,a_n} = \frac{\Gamma(a_1)\cdots\Gamma(a_n)}{\Gamma(a-Ld/2)} I^d_{a_1,\dots,a_n}, \tag{4}$$

and

$$\hat{I}_p = \frac{I_p}{\Gamma(t-Ld/2)}. \tag{5}$$

Notice that the expansion coefficients of any $\hat{I}^d_{a_1,\dots,a_n}$ into $\{\hat{I}_p\}$ are still rational functions of kinematic invariants and the dimension d. Moreover, as long as the considered integral $I^d_{a_1,\dots,a_n}$ is finite, and satisfies $a_i \geqslant 1$ $(i = 1,\dots,n)$, the corresponding $\hat{I}^d_{a_1,\dots,a_n}$ would be finite. Similarly, all "hatted" master integrals $\hat{I}_p$ are finite, given Assumption 2.

Notice that

$$\begin{aligned} \hat{I}^d_{a_1,\dots,a_n} &= \int_{\boldsymbol{x}\geqslant 0} d^n\boldsymbol{x}\,\delta\left(1-\sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} x_i^{a_i-1} \frac{\mathcal{U}(\boldsymbol{x})^{a-(L+1)d/2}}{\mathcal{F}(\boldsymbol{x})^{a-Ld/2}} \\ &= \int_{\boldsymbol{x}\geqslant 0} d^n\boldsymbol{x}\,\delta\left(1-\sum_{i=1}^{n} x_i\right) \prod_{i=1}^{n} \tilde{x}_i^{a_i-1} \frac{\mathcal{U}(\boldsymbol{x})^{n-(L+1)d/2}}{\mathcal{F}(\boldsymbol{x})^{n-Ld/2}}, \end{aligned} \tag{6}$$

where the last step introduces $\tilde{x}_i = (\mathcal{U}(\boldsymbol{x})/\mathcal{F}(\boldsymbol{x}))x_i$. Now, let $Q(\tilde{\boldsymbol{x}})$ be a polynomial of $\tilde{\boldsymbol{x}}$ that stays non-negative throughout the range of $\boldsymbol{x}$,

$$Q(\tilde{\boldsymbol{x}}) = \sum_q b_q \left(\prod_{i=1}^n \tilde{x}_i^{a_i^q - 1} \right) \geqslant 0, \quad \forall \boldsymbol{x} \text{ such that } \boldsymbol{x} \geqslant 0 \text{ and } \sum_{i=1}^n x_i \leqslant 1. \tag{7}$$

Then the following inequality

$$\begin{aligned} 0 &\leqslant \int_{\boldsymbol{x}\geqslant 0} d^n\boldsymbol{x}\, \delta\left(1 - \sum_{i=1}^n x_i\right) Q(\tilde{\boldsymbol{x}}) \frac{\mathcal{U}(\boldsymbol{x})^{n-(L+1)d/2}}{\mathcal{F}(\boldsymbol{x})^{n-Ld/2}} \\ &= \sum_q b_q \hat{I}^d_{a_1^q,\dots,a_n^q} \end{aligned} \tag{8}$$

must hold, as long as the involved integrals are all finite. Since every $\hat{I}^d_{a_1^q,\dots,a_n^q}$ can be expanded as a linear combination of $\{\hat{I}_p\}$, the inequality (8) serves as a constraint on the values of $\{\hat{I}_p\}$. The number of such constraints is in principle infinite. As more constraints are taken into account, we presumably obtain better estimates of the values of $\{\hat{I}_p\}$.

Such an approach can be slightly adapted to compute the ϵ-*expansion* of the master integrals $\{\hat{I}_p\}$. To this end, let us define $d = d_0 - 2\epsilon$, and denote the ϵ-expansion of an arbitrary $\hat{I}^d_{a_1,\dots,a_n}$ by

$$\hat{I}^d_{a_1,\dots,a_n} = \hat{I}_{a_1,\dots,a_n}\Big|_{\epsilon^0} + \epsilon \cdot \hat{I}_{a_1,\dots,a_n}\Big|_{\epsilon^1} + \epsilon^2 \cdot \hat{I}_{a_1,\dots,a_n}\Big|_{\epsilon^2} + \cdots. \tag{9}$$

Due to (6), we have

$$\hat{I}_{a_1,\dots,a_n}\Big|_{\epsilon^k} = \int_{\boldsymbol{x}\geqslant 0} d^n\boldsymbol{x}\, \delta\left(1 - \sum_{i=1}^n x_i\right) \prod_{i=1}^n \tilde{x}_i^{a_i - 1} \frac{\mathcal{U}^{n-(L+1)d_0/2}}{\mathcal{F}^{n-Ld_0/2}} \frac{1}{k!} \log^k \frac{\mathcal{U}^{L+1}}{\mathcal{F}^L}. \tag{10}$$

Now, let

$$Q\left(\tilde{\boldsymbol{x}}, \log\frac{\mathcal{U}(\boldsymbol{x})^{L+1}}{\mathcal{F}(\boldsymbol{x})^L}\right) = \sum_q \frac{b_q}{k_q!} \left(\prod_{i=1}^n \tilde{x}_i^{a_i^q - 1} \right) \log^{k_q} \frac{\mathcal{U}(\boldsymbol{x})^{L+1}}{\mathcal{F}(\boldsymbol{x})^L} \tag{11}$$

be a polynomial of $\tilde{\boldsymbol{x}}$ and $\log(\mathcal{U}^{L+1}/\mathcal{F}^L)$ that stays non-negative throughout the integration range of $\boldsymbol{x}$. Then we have an inequality

$$\begin{aligned} 0 &\leqslant \int_{\boldsymbol{x}\geqslant 0} d^n\boldsymbol{x}\, \delta\left(1 - \sum_{i=1}^n x_i\right) Q\left(\tilde{\boldsymbol{x}}, \log\frac{\mathcal{U}(\boldsymbol{x})^{L+1}}{\mathcal{F}(\boldsymbol{x})^L}\right) \frac{\mathcal{U}(\boldsymbol{x})^{n-(L+1)d_0/2}}{\mathcal{F}(\boldsymbol{x})^{n-Ld_0/2}} \\ &= \sum_q b_q\, \hat{I}_{a_1^q,\dots,a_n^q}\Big|_{\epsilon^{k_q}} \end{aligned} \tag{12}$$

that constrains ϵ-expansion terms of different $\hat{I}_{a_1,\dots,a_n}$'s, and thus constrains ϵ-expansion terms of master integrals.

2 Implementation

To make our proposed method work, we need the following inputs:

- a set of (properly chosen) master integrals $\{\hat{I}_p\}$,
- an IBP table, keeping track of how an arbitrary $\hat{I}^d_{a_1,\ldots,a_n}$ (satisfying $a_i \geqslant 1$ and $a \geqslant t$) expands into $\{\hat{I}_p\}$,
- the exact or numerical value of at least one master integral,
- a non-negative polynomial $Q(\tilde{\boldsymbol{x}}, \log(\mathcal{U}^{L+1}/\mathcal{F}^L))$.

In our implementation, we adopt the following ansatz for $Q(\tilde{\boldsymbol{x}}, \log(\mathcal{U}^{L+1}/\mathcal{F}^L))$,

$$\begin{aligned} Q\left(\tilde{\boldsymbol{x}}, \log\frac{\mathcal{U}(\boldsymbol{x})^{L+1}}{\mathcal{F}(\boldsymbol{x})^L}; \boldsymbol{c}\right) &= Q_0\left(\tilde{\boldsymbol{x}}, \log\frac{\mathcal{U}(\boldsymbol{x})^{L+1}}{\mathcal{F}(\boldsymbol{x})^L}\right) \\ &\times \left(\sum_{0\leqslant j_1+\cdots+j_n\leqslant N_1} \sum_{0\leqslant k\leqslant N_2} c_{j_1,\ldots,j_n,k}\tilde{x}_1^{j_1}\cdots\tilde{x}_n^{j_n}\log^k\frac{\mathcal{U}(\boldsymbol{x})^{L+1}}{\mathcal{F}(\boldsymbol{x})^L}\right)^2, \end{aligned} \tag{13}$$

where $Q_0(\tilde{\boldsymbol{x}}, \log(\mathcal{U}^{L+1}/\mathcal{F}^L))$ is a non-negative *prefactor*, and $\boldsymbol{c} = (c_{j_1,\ldots,j_n,k})$ denotes arbitrary parameters. Plugging (13) into the RHS of (12) results in a quadratic form of $\boldsymbol{c}$, and the inequality (12) essentially requires that the quadratic form is **positive semi-definite**. Now the problem of evaluating master integrals $\{\hat{I}_p\}$ has turned into that of solving those semi-definite constraints, the latter falling under a well-known subfield of convex optimization—**semi-definite programming**, to which numerous efficient algorithms can apply.