Asymptotic expansions of Whispering Gallery Modes in graded index optical micro-cavities

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Abstract

We are interested in resonance frequencies of two dimensional dielectric cavities—a component of optical micro-resonators—and more specifically in resonances corresponding to whispering gallery modes (WGM). WGM are optical waves with high polar mode index, circling around the cavity and almost perfectly guided by total internal reflection. For a cavity of general shape with a varying optical index n (graded index optical cavity), using a phase amplitude ansatz, we have obtained asymptotic expansions of resonances as the polar mode index becomes large. We have also found sufficient conditions linking the curvature of the cavity boundary and the optical index for such expansions to hold.

Keywords: Helmholtz transmission problem; Resonances; WKB method.

1 Problem setting

Resonant modes in an optical micro-cavity are particular time-harmonic solutions to the source free Maxwell equations inside and outside the dielectric cavity. We consider a 2D setting, a situation that arises as a simplification of the 3D resonance problem, e.g. by using the effective index approach. It is well know that the 2D Maxwell setting is the combination of two subsystems of equations referred to as transverse electric (TE) and transverse magnetic (TM). For brevity we report here only for the TM case.

We denote by Ω the bounded domain in $\mathbb{R}^2$ describing the dielectric cavity. The optical index n is 1 outside $\overline{\Omega}$ and coincides with a smooth function $n > 1$ in $\overline{\Omega}$. The resonance problem can be formulated as: Find $(k, u) \in \mathbb{C} \times \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^2)$ with $u \neq 0$ such that

$$- \Delta u - k^2 n^2 u = 0 \qquad \text{in } \mathbb{R}^2, \tag{1a}$$

$$u(r,\theta) = \sum_{m\in\mathbb{Z}} c_m \, \mathsf{H}^{(1)}_m(kr)\, \mathsf{e}^{im\theta} \qquad r > R_0. \tag{1b}$$

Here $\mathsf{H}^{(1)}_m$ refers to Hankel function of the first kind of order m. Equation (1b) expresses the radiation condition at infinity in polar coordinates for R_0 large enough. For real k, it corresponds to the outgoing Sommerfeld's condition. It is known that the solutions (k, u) to problem (1) are such that k has a negative imaginary part.

For a circular cavity with constant index, asymptotic expansions of the resonances can be obtained using expansions of Bessel functions [1]. When n is not constant in $\overline{\Omega}$, this approach is not applicable.

2 Disk cavity with radially varying index

Let R be the radius of the disk and assume n is a smooth radial function $r \mapsto n(r)$. Then problem (1) can be reduced to a family of 1D radial problems depending on an integer $m \in \mathbb{Z}$ referred as the *polar mode index*:

$$-\frac{1}{m^2}\,\mathcal{L}w + Vw = \lambda w \tag{4}$$

where $V(r) = \left[\frac{n(R)\,R}{n(r)\,r}\right]^2 - 1$ is an effective potential, $k = \frac{m}{Rn(R)}\sqrt{1+\lambda}$ and $\mathcal{L}w = \frac{n(R)^2\,R^2}{n(r)^2\,r}(rw')'$ is an elliptic operator. Since $V(R^-) = 0$ and $V(R^+) = n(R)^2 - 1 > 0$, we have a potential barrier at $r = R$. We note that $V'(R^-) = -2\breve{\kappa}$ where

$$\breve{\kappa} = \frac{1}{R} + \frac{n'(R)}{n(R)}. \tag{5}$$

We have identified three typical behaviors [2] for the solutions of (4), depending on the sign of $\breve{\kappa}$, relying on the spectral theory of Schrödinger operators.

a) Half-triangular potential well. If $\breve{\kappa} > 0$ then V is decreasing in a left neighborhood of R and has a local minimum at R. We have obtained an asymptotic expansion of the resonances in the form $k = m\,\mathcal{K}_{\mathrm{a}}(m^{-\frac{1}{3}})$ for a function $\mathcal{K}_{\mathrm{a}}$ in $\mathcal{C}^\infty([0,1])$ with determined Taylor expansion at 0. Moreover, the (quasi)mode u localizes at the boundary.

b) Half-quadratic potential well. If $\breve{\kappa} = 0$, under the additional condition $\frac{2}{R^2} - \frac{n''(R)}{n(R)} > 0$ the effective potential V has a local minimum at R. Our expansion is now $k = m\,\mathcal{K}_{\rm b}(m^{-\frac{1}{2}})$ for another function $\mathcal{K}_{\rm b} \in \mathcal{C}^\infty([0,1])$. Again, the (quasi)mode u localizes at the boundary.

c) Internal quadratic potential well. If $\breve{\kappa} < 0$, since $\lim_{r\searrow 0} V(r) = +\infty$, the effective potential V has at least a global minimum r_0 in $(0, R)$. Under the condition $\frac{2}{r_0^2} - \frac{n''(r_0)}{n(r_0)} > 0$ and that r_0 is the unique global minimum of V, our expansion has again the form $k = m\,\mathcal{K}_{\rm c}(m^{-\frac{1}{2}})$ with a different function $\mathcal{K}_{\rm c} \in \mathcal{C}^\infty([0,1])$, and now u localizes inside the cavity around $r = r_0$.

3 General cavity with variable index

In the general case, the phase function is not known and we have used a phase amplitude ansatz, the famous WKB method, to find asymptotic expressions of the resonances in tubular coordinates along the boundary of Ω [2]. This leads to an eikonal equation coupled with a Schrödinger equation. A small parameter h appears naturally, and has to be quantized so that the phase is well-defined, giving rise to a generalized notion of polar mode index. We have constructed quasi-resonances (k, u) in the sense of [3] in a similar, but more general, form than in case a) above, when the condition $\kappa + \frac{\partial_\nu n}{n} > 0$ is fulfilled all along $\partial\Omega$ where κ is the curvature of $\partial\Omega$. The asymptotic expansions are computed using a computer algebra system.

The advantage of the asymptotic formulas we have obtained is twofold. They provide accurate approximations of resonances at high frequencies when the use of standard numerical approximation is difficult. For lower frequencies, combined with finite element (FE) computations, they provide information on the localization of the resonances in the FE matrix spectrum.

4 Numerical Experiments

For numerical illustration we consider an elliptic cavity with perimeter L and constant index n. Our 4-term asymptotic expansion of k reads

$$k_j^{[4]}(m) = \frac{2\pi m}{Ln}\Bigg[1 + \frac{a_j}{2}\langle\kappa^{\frac{2}{3}}\rangle h_m^{\frac{2}{3}} - \frac{n\langle\kappa\rangle}{2\sqrt{n^2-1}}h_m + \frac{a_j^2}{12}\left(\langle\kappa^{\frac{2}{3}}\rangle^2 - \frac{\langle\kappa^{\frac{4}{3}}\rangle}{10} - \frac{4}{45}\langle\kappa'^2\kappa^{-\frac{8}{3}}\rangle\right)h_m^{\frac{4}{3}} - \frac{a_j n_0}{12\sqrt{n^2-1}}\left(\langle\kappa^{\frac{2}{3}}\rangle\langle\kappa\rangle + \frac{\langle\kappa^{\frac{5}{3}}\rangle}{n^2-1}\right)h_m^{\frac{5}{3}}\Bigg]$$

as $m \to +\infty$, where $h_m = \frac{L}{\pi m}$ and $\langle\cdot\rangle$ is the mean value along $\partial\Omega$. Here j is a natural integer (radial mode index) and a_j is the j-th zero of the reverse Airy function $x \mapsto \mathsf{Ai}(-x)$.

On Figure 1, we have compared our N-terms asymptotic expansion to FE computation for an ellipse of semi-major axis 1 and eccentricity 0.5 with constant index $n = 5$. We have used a *Perfectly Matched Layer* with a structured quadrangular mesh of geometric degree 3 and a FE space of degree 7 with 64156 dofs. For $N = 0$, $k_0^{[0]}(m)$ is the principal term $\frac{2\pi m}{Ln}$, for $N = 1$, $k_0^{[1]}(m)$ is $k_0^{[0]}(m)$ plus one corrective term, etc. Note that, problem (1) has two solutions for each m.

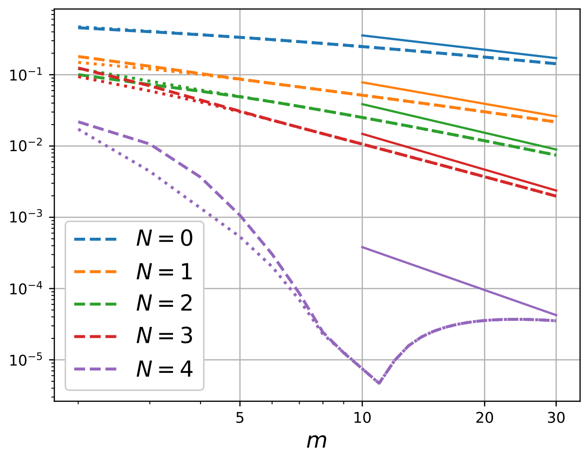

Figure 1: Relative difference between FE resonances and N-terms asymptotic expansions, for $j = 0$. The slopes of the solid lines are $-\frac{N+2}{3}$.

References

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This work has been performed within the framework of the CNRS INFINITI project SONATINE in collaboration with FOTON Laboratory. The authors also acknowledge support of the Centre Henri Lebesgue ANR-11-LABX-0020-01.