Meteo 515 – Assignment 3 – Linear Regression

In [1]:

```
from __future__ import division, print_function
#from collections import OrderedDict
from itertools import chain
#import datetime as dt

#import matplotlib.dates as mdates
import matplotlib.pyplot as plt
import numpy as np
import pandas as pd
import scipy.optimize as so
import scipy.stats as ss
#import sklearn as skl
import statsmodels.api as sm
```

In [2]:

```
plt.style.use('seaborn-darkgrid')
%matplotlib notebook
```

Load the data

We are using this dataset, the not-detrended and unsmoothed AMO index from NOAA ESRL. More info here: https://esrl.noaa.gov/psd/data/timeseries/AMO/

> **Note:** Currently (27-Sep-18 10:00 EST) the data on the site is messed up for 1980 onwards, though it was fine when I downloaded it the day before.

In [3]:

```
amo_fpath = './data/amon.us.long.mean.data'

amo_raw = np.genfromtxt(amo_fpath, skip_header=1, skip_footer=7)[:,1:]

with open(amo_fpath, 'r') as f: yr_range = f.readline().split()
t_amo = pd.date_range(start='{}/01/01'.format(yr_range[0]), freq='MS', periods=amo_raw.size)
amo_ndt_us = amo_raw.reshape((amo_raw.size,))

amo = pd.DataFrame({'amo': amo_ndt_us}, index=t_amo)
amo['julian_date'] = amo.index.to_julian_date()               # Julian Date (decimal days)
amo['t_elapsed'] = amo.julian_date - amo.julian_date.iloc[0]  # elapsed time (decimal days)
amo['year'] = amo.index.year                                  # integer year for each data point
amo['decyear'] = amo.index.year + (amo.index.month-1)/12      # decimal year

amo[amo == -99.99] = np.nan
amo.dropna(inplace=True)

#> don't include 2018 in the annual means, since 4 months are missing
grouped = amo.loc[amo.year<=2017, :].groupby(pd.Grouper(freq='A'))
amo_annual_mean = grouped.mean()
amo_annual_mean['sem_annual'] = grouped.amo.std().values / np.sqrt(12)  # stdev of the mean; the initial d
ata is monthly
# ^ note that this gives indices that are the last day of the year
#   and {year}.46 for decimal year, since initially the datetimes are first day of the year
```

In [4]:

```
#> plotting functions!
figsize_lin_reg = (9, 4.0)
figsize_res = (8, 3)
#degC = u'\u00B0C'
degC = u' (\u00B0C)'
```

```
def res_plot(y, y_hat, df_res=2, figid='ts_res'):
    """Plot the residuals
    could pass the OLS result in, but not doing that currently...
    """
    f, a = plt.subplots(figsize=figsize_res, num=figid)

    y_bar = y.mean()
    SSE = np.sum((y-y_hat)**2)  # sum of squared error
    SSR = np.sum((y_hat-y_bar)**2)  # residual sum of squares
    SST = np.sum((y-y_bar)**2)
    assert( np.isclose(SST, SSE+SSR) )
    #MSE = SSE/(res.df_resid)  # MSE == sample variance of the residuals
    MSE = SSE/df_res
    #assert( np.isclose(MSE, res.mse_resid) )  # res.mse_model == MSR; F = MSR/MSE
    s = '''
    SSE = {:.3g}
    SSR = {:.3g}
    MSE = {:.4g}
    '''.format(SSE, SSR, MSE)

    a.plot(t_plot, y-y_hat, '.-', c='purple', ms=6, lw=1, label=s)
    #a.set_ylabel('residual AMO index {:s}'.format(degC))
    a.set_ylabel('residual AMO index' + degC)
    a.text(1.01, 0.5, s,
           va='center', ha='left', transform=a.transAxes)
    #a.legend(loc='center left', bbox_to_anchor=(0.99, 0.5), labelspacing=1.5)

    f.tight_layout(rect=(0, 0, 0.85, 1.0))

    return f


def lin_reg_plot(t_plot, y, y_hat, s_model, figid='ts'):
    """Plot the data and least-squares regression result"""
    f, a = plt.subplots(figsize=figsize_lin_reg, num=figid)

    #xbar, se = y, y.std()
    #a.fill_between(amo_annual_mean.index, xbar-1.*se, xbar+1.*se, alpha=0.3, label='SEM')
    a.plot(t_plot, y, 'b.-', alpha=0.6, ms=6, lw=1, label='annual means')
    a.plot(t_plot, y_hat, '-', c='r', lw=2, label=s)

    #a.text(0.02, 0.98, s,
    #       va='top', ha='left', transform=a.transAxes)

    a.set_ylabel('AMO index' + degC)
    a.legend(loc='center left', bbox_to_anchor=(0.99, 0.5), labelspacing=1.5)

    f.tight_layout()

    return f
```

1) Simple linear regression with year as predictor

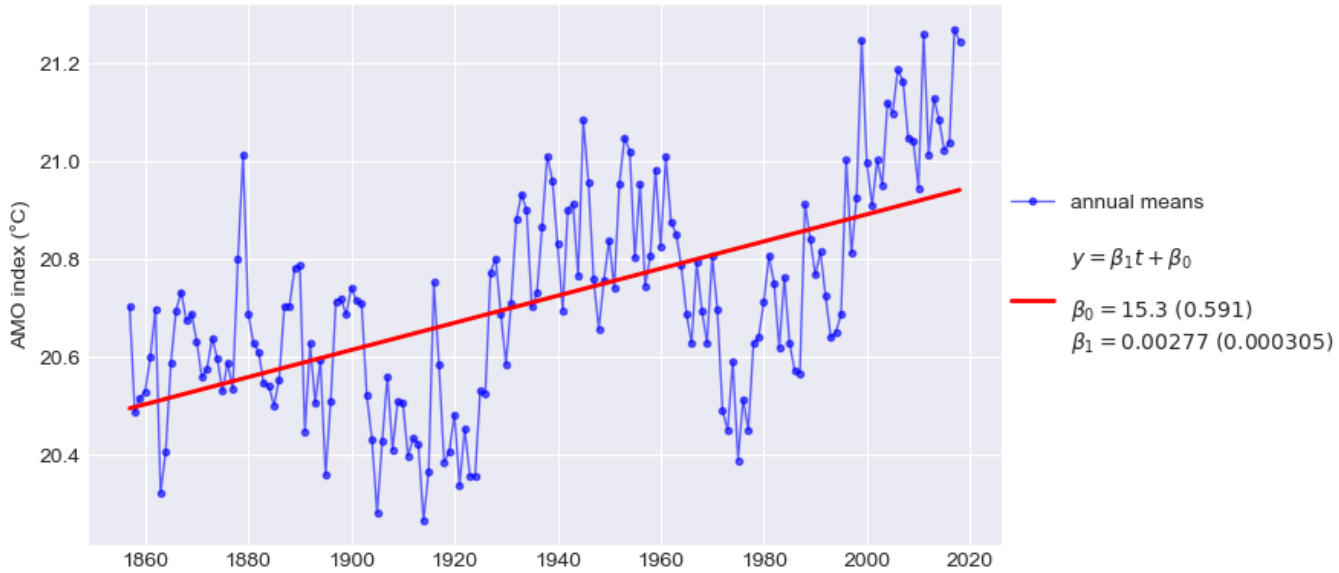
Note that using the monthly data vs the annual means gives slightly different answers, mainly for the error. Change `df_reg` to see.

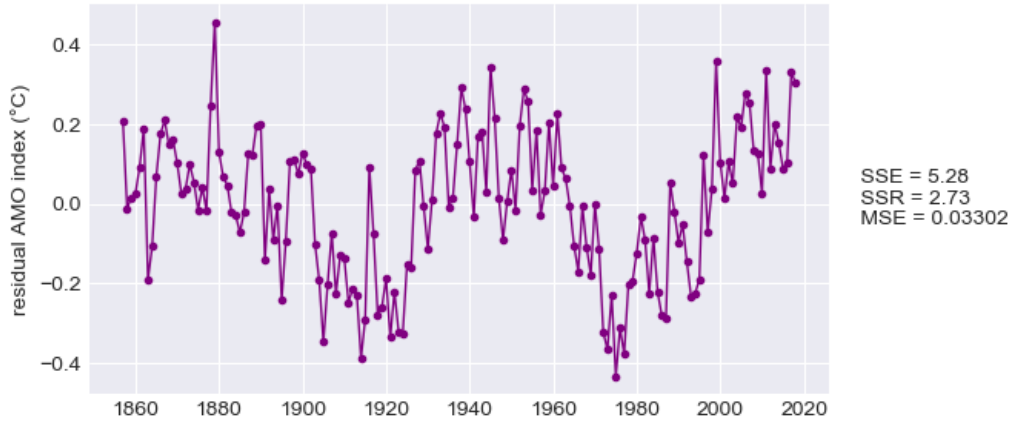
In [5]:

```
df_reg = amo_annual_mean  # {amo_annual_mean, amo}
t_reg = df_reg.year
t_plot = df_reg.index

res = sm.OLS(df_reg['amo'], sm.add_constant(t_reg), ).fit()

print(res.summary())  # note: can plot res using `sm.graphics.abline_plot(model_results=res)`

param_lines = '\n'.join([r'$\beta_{:d} = {{:.3g}} \ ({{:.3g}})$'.format(i) for i in range(len(res.params))
])
s = r'$y=\beta_1 t + \beta_0$' + '\n\n' + param_lines
s = s.format(*chain.from_iterable(zip(res.params, res.bse)))

y = df_reg['amo'].values
y_hat = t_reg*res.params[1]+res.params[0]
```

```
f1 = lin_reg_plot(t_plot, y, y_hat, s, 'ts_ols1')

f1b = res_plot(y, y_hat, df_res=res.df_resid, figid='ts_ols1_res');
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    amo   R-squared:                       0.340
Model:                            OLS   Adj. R-squared:                  0.336
Method:                 Least Squares   F-statistic:                     82.56
Date:                Wed, 03 Oct 2018   Prob (F-statistic):           3.73e-16
Time:                        21:33:48   Log-Likelihood:                 47.402
No. Observations:                 162   AIC:                            -90.80
Df Residuals:                     160   BIC:                            -84.63
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         15.3469      0.591     25.952      0.000      14.179      16.515
year           0.0028      0.000      9.086      0.000       0.002       0.003
==============================================================================
Omnibus:                        3.980   Durbin-Watson:                   0.612
Prob(Omnibus):                  0.137   Jarque-Bera (JB):                2.915
Skew:                          -0.182   Prob(JB):                        0.233
Kurtosis:                       2.454   Cond. No.                     8.02e+04
==============================================================================

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
[2] The condition number is large, 8.02e+04. This might indicate that there are
strong multicollinearity or other numerical problems.
```





Discussion:

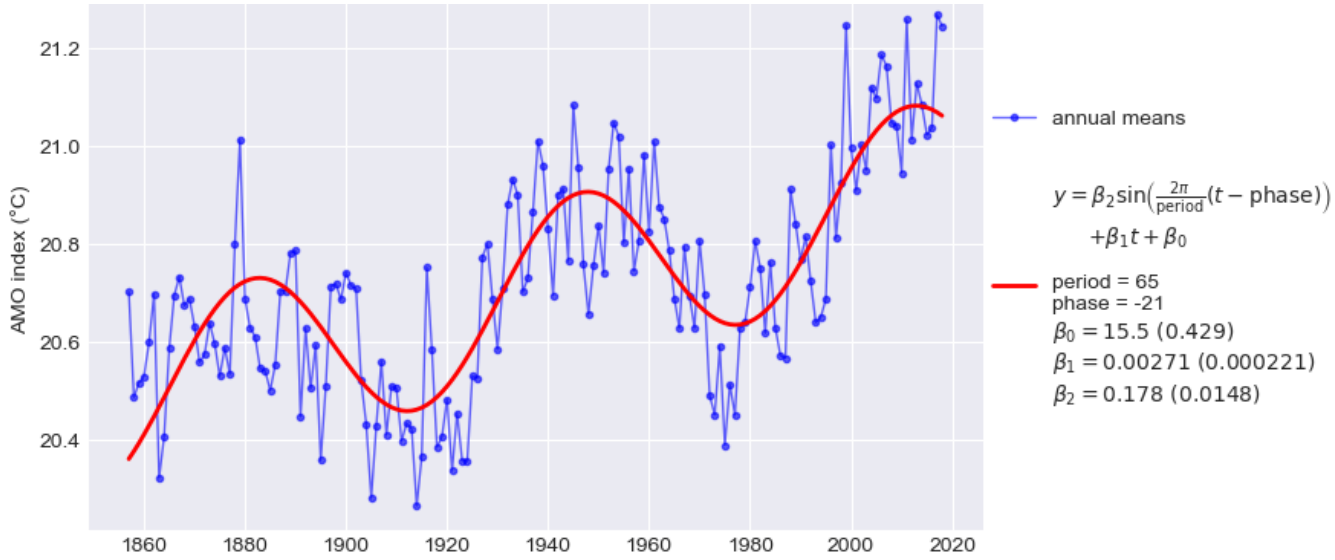
Our R^2 here is pretty low. The value (0.34) indicates that our model explains only 34% of the variation in AMO index. In the plot of the residuals, we see a clear oscillatory pattern, which next we add to our model to see how that improves things...

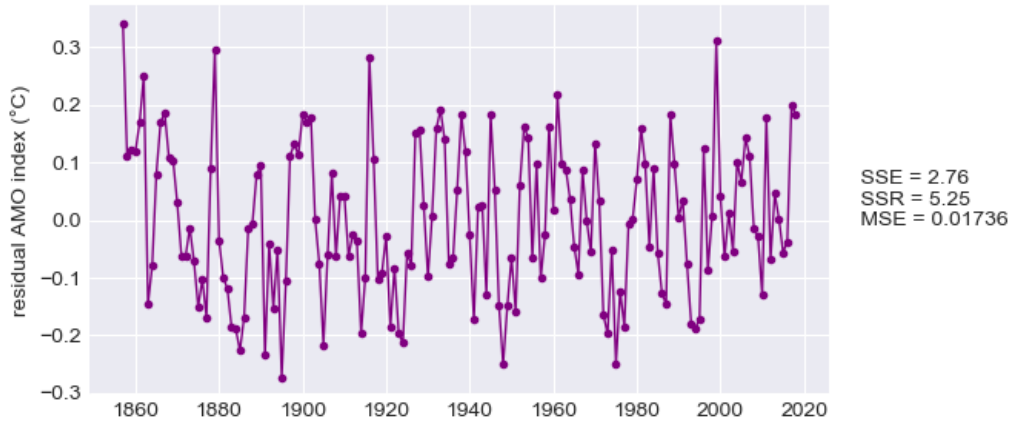
2) Adding a prescribed oscillation

In [6]:

```
phase = -21  # years
period = 65  # in years

y = df_reg['amo']  # AMO index

x0 = np.ones(df_reg['amo'].shape)  # intercept
x1 = df_reg['year'].values  # year
x2 = np.sin(2*np.pi/period*(x1-phase))  # oscillation with prescribed period and phase

#X = np.vstack((x2, x1, x0)).T
X = pd.DataFrame(data={'0-const': x0, '1-year': x1, '2-osc_amp': x2}, index=df_reg.index)

res = sm.OLS(y, X).fit()

y_hat = np.dot(X, res.params)

print(res.summary())

param_lines = '\n'.join([r'$\beta_{:d} = {{:.3g}} \ ({{:.3g}})$'.format(i) for i in range(len(res.params))
])
s = r'''
$y = \beta_2 \sin\left( \frac{{2 \pi}}{{\mathrm{{period}}}} (t - \mathrm{{phase}}) \right)$
      $+ \beta_1 t + \beta_0$

period = {:d}
phase = {:d}
''' + param_lines
s = s.format(period, phase, *chain.from_iterable(zip(res.params, res.bse)))

f2 = lin_reg_plot(t_plot, y, y_hat, s, 'ts_ols2')

f2b = res_plot(y, y_hat, df_res=res.df_resid, figid='ts_ols2_res');
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    amo   R-squared:                       0.655
Model:                            OLS   Adj. R-squared:                  0.651
Method:                 Least Squares   F-statistic:                     151.2
Date:                Wed, 03 Oct 2018   Prob (F-statistic):           1.65e-37
Time:                        21:33:48   Log-Likelihood:                 99.994
No. Observations:                 162   AIC:                            -194.0
Df Residuals:                     159   BIC:                            -184.7
Df Model:                           2
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
0-const       15.4580      0.429     36.044      0.000      14.611      16.305
1-year         0.0027      0.000     12.234      0.000       0.002       0.003
2-osc_amp      0.1779      0.015     12.056      0.000       0.149       0.207
==============================================================================
Omnibus:                        5.696   Durbin-Watson:                   1.167
Prob(Omnibus):                  0.058   Jarque-Bera (JB):                3.405
Skew:                           0.153   Prob(JB):                        0.182
Kurtosis:                       2.359   Cond. No.                     8.03e+04
==============================================================================

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
[2] The condition number is large, 8.03e+04. This might indicate that there are
strong multicollinearity or other numerical problems.
```





Discussion:

Adding a prescribed oscillation to our model has greatly improved the fit: our R^2 has increased by about a factor of 2, and MSE correspondingly decreased by about the same. The plot of the residuals is now more like what we would like to see (looking just noisy). However, the values in the first few decades seem to be noticeably and consistently a bit higher than later years, suggesting that maybe the increasing trend didn't start until sometime after 1857. In (3) we will add that to the model and see how things improve...

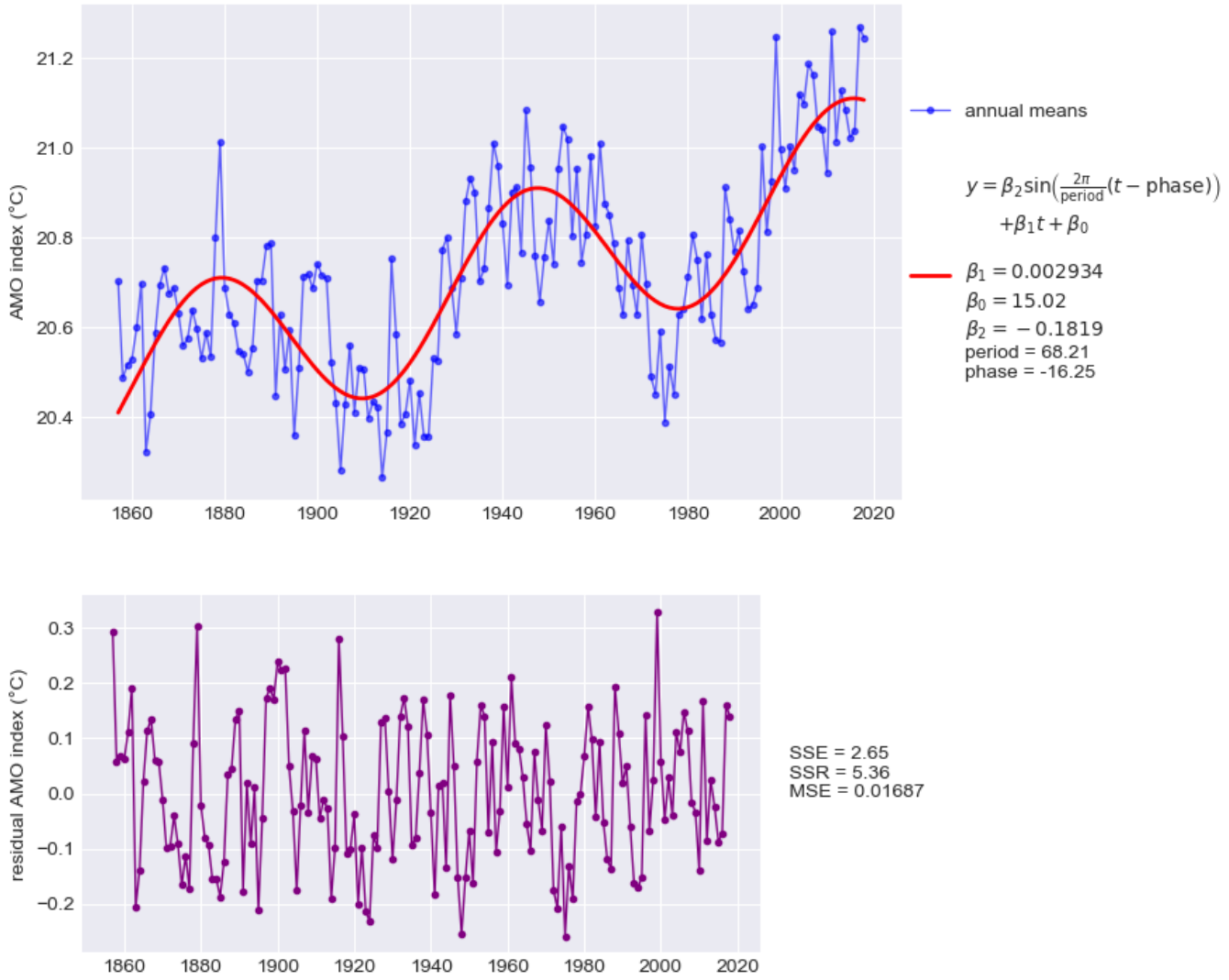
But first:

2.5) Find a better oscillation using optimization

In [7]:

```
def fit1(t, slope, const, amp, period, phase):
    return amp*np.sin(2*np.pi/period*(t-phase)) + slope*t + const

popt, pcov = so.curve_fit(fit1, df_reg.year, df_reg.amo,
                          p0=(0.0027, 15.5, 0.18, 65, 75))

popt[-1] = popt[-1] #% popt[-2]

y = df_reg.amo
y_hat = fit1(df_reg.year, *popt)

#print(res.summary())

s = r'''
$y = \beta_2 \sin\left( \frac{{2 \pi}}{{\mathrm{{period}}}} (t - \mathrm{{phase}}) \right)$
      $+ \beta_1 t + \beta_0$
```

```
$\beta_1 = {:.4g}$
$\beta_0 = {:.4g}$
$\beta_2 = {:.4g}$
period = {:.4g}
phase = {:.4g}
'''.format(*popt)

f2p5 = lin_reg_plot(t_plot, y, y_hat, s, 'ts_ols2.5')

n = df_reg.index.size
f2p5b = res_plot(y, y_hat, df_res=n-5, figid='ts_ols2.5_res');
```



Discussion:

The improvement in fit is more modest this time, and in fact pretty similar to what we will see below in (3).

3) Same as (2), but linear trend predictor starts in 1900

Also, we have the option of using the period and phase from the optimized fit in (2.5), but this turned out to not be the best option, since that fit did not include the specification that year only starts as a predictor in 1900, so instead we use the same values as in (2)

Note that the following code for conf and prediction intervals is strictly valid only for **simple linear regression, not multiple.**

```
#> compute confidence and prediction intervals for the predicted
#  x: year
#  y: AMO
n = len(df_reg.index)
x = df_reg['year'].values
x_bar = x.mean()
```

```
    x_bar = x.mean()
    s_x = np.sqrt( np.sum((x-x_bar)**2) / (n-1) )  # x.std() does population version!!
    y_hat = np.dot(X, fit.params)  # should give same as fit.predict(X).values, or fit.fittedvalues.val
    ues
    resid = y - y_hat
    df_resid = n - 3  # 3 predictors (2 + const); or can use fit.df_resid from StatsModels results
    s_y = np.sqrt( np.sum(resid**2) / df_resid )  # standard error (deviation) of the residuals
    t_star = ss.t.ppf(1-0.05/2, df_resid)  # or ss.t.isf(0.05/2, df_resid)
    conf_pm = t_star * s_y * np.sqrt( 1/n + ((x-x_bar)**2)/((n-1)*s_x**2) )
    conf_int = (y_hat-conf_pm, y_hat+conf_pm)
    #pred_pm = t_star * s_y * np.sqrt( 1 + 1/n + ((x-x_bar)**2)/((n-1)*s_x**2) )  # same as below
    pred_pm = t_star * s_y * np.sqrt( 1 + 1/n + ((x-x_bar)**2)/(np.sum((x-x_bar)**2)) )
    pred_int = (y_hat-pred_pm, y_hat+pred_pm)
```

These formulas use equations 7.22 and 7.23 from Wilks. For the prediction variance, e.g., the formula for simple linear regression is

$$s^2_{y_0} = s^2_e\left[1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right] = s^2_e\left[1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{(n-1)s^2_x}\right]$$

at point x_0. However, in multiple linear regression (MLR), the situation is a little more complicated, since x_0 is now a vector $\mathbf{x_0}$ with dimensions ($k+1$,), where k is the number of non-constant predictors, and n the number of sets of predictors (note that this could be different from the number of observations used to fit the model, as long as they lie within the bounds of the data, but here we use the same x_{ij} values). In this problem, it is the set of predictor values at a particular time. $\hat{y}_0$ corresponds to the predicted value of y at $\mathbf{x_0}$ using the estimated model parameter values from the fit. For multiple linear regression, the prediction variance is

$$s^2_{y_0} = s^2_e\left(1 + \mathbf{x_0'}\left(\mathbf{X'X}\right)^{-1}\mathbf{x_0}\right)$$

Note that the estimate for the variance of the residuals s^2_e or $\hat{\sigma}^2$ is equal to the mean squared error (MSE). Note that in this case, using the MLR version, as opposed to the SLR version in the code above, the prediction interval is very slightly larger. The confidence interval for the regression is more noticeably larger. For both, the shapes are very similar.

s_e above is the sample standard deviation of the residuals $e_i = y_i - \hat{y}_i$

$$s_e = \sqrt{\frac{\sum_{i=1}^{n} e_i^2}{n-k-1}} = \frac{\text{SSE}}{n-k-1}$$

where SSE = $\mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y}$, where $\mathbf{I}$ is the identity matrix (order n) and $\mathbf{H} = \mathbf{X}\left(\mathbf{X'X}\right)^{-1}\mathbf{X}'$ is the "hat matrix" ($\hat{\mathbf{y}} = \mathbf{Hy}$).

Note that $\mathbf{C}$, the covariance matrix for the model parameter estimates, is equal to $s^2_e\left(\mathbf{X'X}\right)^{-1}$. Thus another formula for the prediction variance for MLR is

$$s^2_e + \sum_{j=1}^{k}\left[\mathbf{X} \circ (\mathbf{CX}')'\right]_j$$

where $\circ$ denotes elementwise multiplication (Hadamard product) and j denotes the column, corresponding to predictor j out of k. This is the formula that the function from StatsModels uses (source code). In the StatsModels fit result, $\mathbf{C}$ = `fit.cov_params()` and MSE = `fit.mse_resid`.

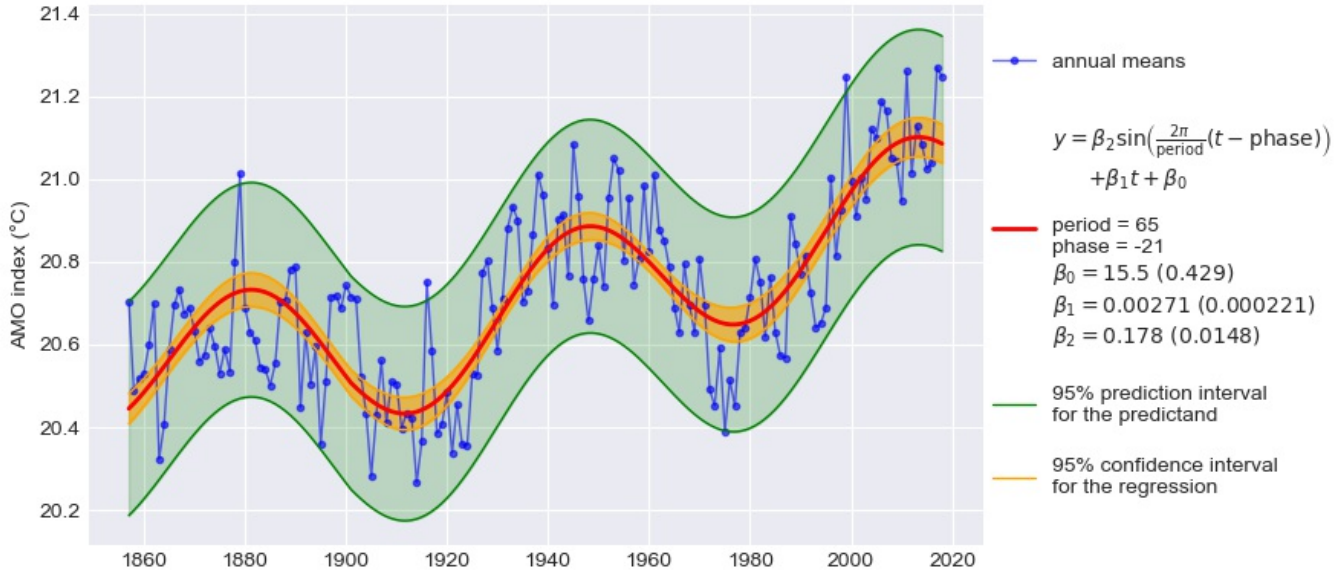
In [8]:

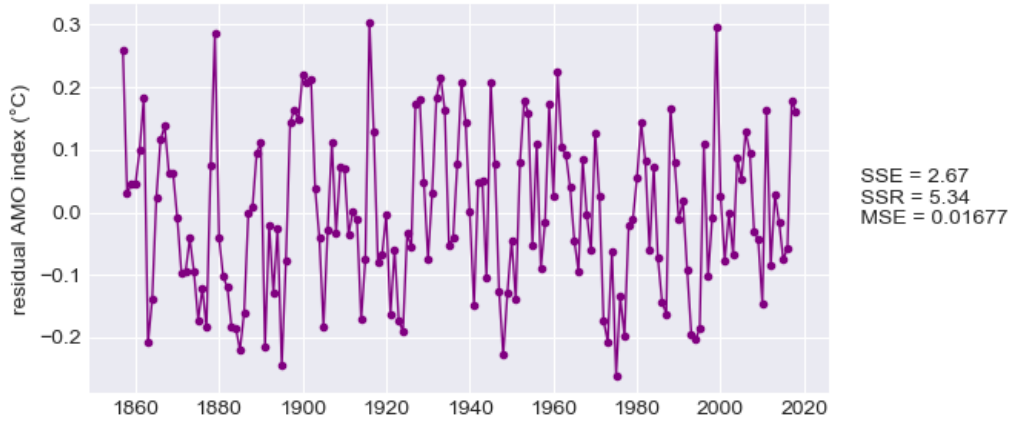
```
period = 65#popt[-2]  # in years
phase  = -21#popt[-1]#% period  # years

y = df_reg['amo']  # AMO index

x0 = np.ones(df_reg['amo'].shape)  # intercept
x1 = np.copy(df_reg['year'].values)  # year
x2 = np.sin(2*np.pi/period*(x1-phase))  # oscillation with prescribed period and phase
```

```
x2 = np.sin(2*np.pi/period*(x1-phase))  # oscillation with prescribed period and phase
x1[x1 < 1900] = 1900  # this is the only change!! in the linear model setup wrt (2)

#X = np.vstack((x2, x1, x0)).T
X = pd.DataFrame(data={'0-const': x0, '1-year': x1, '2-osc_amp': x2}, index=df_reg.index)

fit = sm.OLS(y, X).fit()  # model fit (coefficients etc.) for the least-squares soln

y_hat = np.dot(X, fit.params)  # should give same as fit.predict(X).values, or fit.fittedvalues.values

print(fit.summary())

#> compute confidence and prediction intervals
xis = np.copy(X.values)  # sets of predictor values at which to evaluate to compute the intervals
confvar = np.zeros(y.shape)
predvar = np.zeros_like(confvar)
for i in range(xis.shape[0]):
    xi = xis[i,:]
    confvar[i] = fit.mse_resid *      xi.T @ np.linalg.inv(X.values.T @ X.values) @ xi
    predvar[i] = fit.mse_resid * (1 + xi.T @ np.linalg.inv(X.values.T @ X.values) @ xi)
confstd = np.sqrt(confvar)
predstd = np.sqrt(predvar)

df_resid = n - 3  # 3 predictors; or can use res.df_resid from StatsModels results
t_star = ss.t.ppf(1-0.05/2, df_resid)  # or ss.t.isf(0.05/2, df_resid)

conf_pm = t_star * confstd
conf_int = (y_hat-conf_pm, y_hat+conf_pm)
pred_pm = t_star * predstd
pred_int = (y_hat-pred_pm, y_hat+pred_pm)

#> ask StatsModels to do the prediction interval calculation for us, and compare
#  this formula uses the covariance matrix of the model param estimates instead of just looking at x
from statsmodels.sandbox.regression.predstd import wls_prediction_std
_, sm_pred_l, sm_pred_u = wls_prediction_std(fit, alpha=0.05)
assert(np.allclose(pred_int[0], sm_pred_l))

param_lines = '\n'.join([r'$\beta_{:d} = {{:.3g}} \ ({{:.3g}})$'.format(i) for i in range(len(res.params))
])
s = r'''
$y = \beta_2 \sin\left( \frac{{2 \pi}}{{\mathrm{{period}}}} (t - \mathrm{{phase}}) \right)$
      $+ \beta_1 t + \beta_0$

period = {:.4g}
phase = {:.4g}
''' + param_lines
s = s.format(period, phase, *chain.from_iterable(zip(res.params, res.bse)))

f3 = lin_reg_plot(t_plot, y, y_hat, s, 'ts_ols4')

a = plt.gca()
a.fill_between(t_plot, *pred_int, color='green', alpha=0.2)
a.plot(t_plot, pred_int[0], 'green', lw=1.0, label='95% prediction interval\nfor the predictand')
a.plot(t_plot, pred_int[1], 'green', lw=1.0)
a.fill_between(t_plot, *conf_int, color='orange', alpha=0.6)
a.plot(t_plot, conf_int[0], 'orange', lw=1.0, label='95% confidence interval\nfor the regression')
a.plot(t_plot, conf_int[1], 'orange', lw=1.0)
a.legend(loc='center left', bbox_to_anchor=(0.99, 0.5), labelspacing=1.5)

f3b = res_plot(y, y_hat, df_res=res.df_resid, figid='ts_ols4_res');
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    amo   R-squared:                       0.667
Model:                            OLS   Adj. R-squared:                  0.663
Method:                 Least Squares   F-statistic:                     159.3
Date:                Wed, 03 Oct 2018   Prob (F-statistic):           1.07e-38
Time:                        21:33:50   Log-Likelihood:                102.78
No. Observations:                 162   AIC:                            -199.6
Df Residuals:                     159   BIC:                            -190.3
Df Model:                           2
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
0-const       14.2701      0.508     28.095      0.000      13.267      15.273
```

```
1-year         0.0033       0.000      12.668       0.000        0.003        0.004
2-osc_amp      0.1691       0.015      11.636       0.000        0.140        0.198
==============================================================================
Omnibus:                        8.271   Durbin-Watson:                   1.207
Prob(Omnibus):                  0.016   Jarque-Bera (JB):                4.140
Skew:                           0.145   Prob(JB):                        0.126
Kurtosis:                       2.272   Cond. No.                     9.70e+04
==============================================================================

Warnings:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
[2] The condition number is large, 9.7e+04. This might indicate that there are
strong multicollinearity or other numerical problems.
```





Discussion:

This model gives the best fit:

- highest $R^2 = 0.667$ compared to 0.655 in (2)
- slightly lower MSE (0.01677 vs 0.01736), but only very slightly lower than the MSE for the optimized fit in (2.5)

Our 95% CIs for the regression are pretty tight, indicating that if we were to resample from the true population a large number of times, and recompute the regression, we would get very similar results about 95% of time. The small width of the CI relative to the AMO index values tells us that there is little uncertainty in the *expected value* of AMO index at any year within our range. Note that it is smallest around the mean year (about 1930). In SLR, the width would increase outward from the central x value, but our case here is a little more complex.

The 95% PIs, however, are much wider. This is because the PI needs to account for variability in AMO index with respect to the expected value. We see that the PI envelope roughly encloses the fluctuations in the annual AMO index time series that we see in the blue color.

To save figures locally if you wish...

Probably a silly thing for a Jupyter Notebook to do but whatever

Put the code in a cell and run it to save the figs to local dir 'figs'.

```
#> save figs; requires that the 'figs' dir exists..
for n in plt.get_fignums():
    f = plt.figure(n)
    f.savefig('./figs/hw3_{:s}.pdf'.format(f.get_label()),
        transparent=False,
        bbox_inches='tight', pad_inches=0.05,
        )
```