6.1

a) By definition, $Y = XW + b$

$\Rightarrow$ net $h1 = x_1 w_1 + x_2 w_2 + x_3 w_3 + x_4 w_4$

[given no bias]

$= 3(-0.2) + (1)(-0.1) + (-1)(0.2) + (2)(0.2)$

$= -0.5$

Similarly,

net $h2 = 3(0.9) + (1)(0.3) + (-1)0.5 + 2(-0.5)$

$= 1.5$

net $h3 = 3(0.4) + (1)(0.4) + (-1)(-0.7) + 2(0.5)$

$= 3.3$

Applyin leaky Relu $\Rightarrow f(x) = \begin{cases} 0.01x; & x<0 \\ x & ; x>0 \end{cases}$

$z_1' = -0.005$

$z_2' = 1.5$

$z_3' = 3.3$

Calculating output layer with softmax function:

$$O1 = (-0.005)(0.6) + (1.5)(-0.1) + (3.3)(-0.5)$$

$$= -1.803$$

$$O2 = (-0.005)(-0.2) + (1.5)(0.8) + (3.3)(-0.3)$$

$$= 0.211$$

Passing through softmax function

$$S(x_i) = \frac{e^{x_i}}{\sum_j e^{x_j}}$$

$$z_1^2 = \frac{e^{-1.803}}{e^{-1.803} + e^{0.211}}$$

$$= 0.117741$$

$$z_2^2 = 0.882259$$

Solution:

$h1 = -0.5$
$h2 = 1.5$
$h3 = 3.3$

$O_1 = -1.803$

$O_2 = 0.211$

$z_1^1 = -0.005$
$z_2^1 = 1.5$
$z_3^1 = 3.3$

$z_1^2 = 0.117741$

$z_2^2 = 0.882259$

b) Binary cross entropy loss:

$$L(w) = -\frac{1}{N}\sum_i y_i \cdot \log(P(y_i)) + (1-y_i)\cdot\log(1-P(y_i))$$

Given, $Y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$L(w) = -\frac{1}{2}\Big[((0)(\log(0.117741)) + (1-0)\cdot\log(1-0.117741)) + (1)(\log(0.882259)) + (1-1)\cdot\log(1-0.882259))\Big]$$

$$= 0.125269$$

$$L(w)_1 = -\frac{1}{2}(\log(0.882259)) = 0.062635$$

$$L(w)_2 = -\frac{1}{2}(-\log(0.882259)) = -0.062635$$

Given,

$$W^{(1)} = \begin{bmatrix} -0.2 & 0.9 & 0.4 \\ -0.1 & 0.3 & 0.4 \\ 0.2 & 0.5 & -0.7 \\ 0.2 & -0.5 & 0.5 \end{bmatrix}$$

$$W^{(2)} = \begin{bmatrix} 0.6 & -0.2 \\ -0.1 & 0.8 \\ -0.5 & -0.3 \end{bmatrix}$$

Calculating derivatives: = Layer 2

$$\frac{\partial L(w)}{\partial w_{11}^{(2)}} = \frac{\partial L(w)}{\partial z_1^2} * \frac{\partial z_1^2}{\partial o_1} * \frac{\partial o_1}{\partial w_{11}^{(2)}}$$

$$\frac{\partial L(w)}{\partial z_1^2} = \frac{\partial}{\partial z_1^2}\left(-\frac{1}{2}\left[y_{11}\cdot \log(z_1^2)) + (1-y_{11})\cdot \text{Log}(1-z_1^2) + y_{12}\cdot \log(z_2^2) + (1-y_{12})\cdot \log(1-z_2^2)\right]\right)$$

$$= -\frac{1}{2}\left(y_{11}\cdot \frac{1}{z_1^2} + (1-y_{11})\cdot \frac{-1}{1-z_1^2} + 0\right)$$

$$= -\frac{1}{2}\left(0 + \frac{-1}{1-0.117741}\right)$$

$$= 0.566727$$

$$\frac{\partial z_1^2}{\partial o_1} = \frac{\partial}{\partial o_1}\left(\frac{e^{o_1}}{e^{o_1}+e^{o_2}}\right)$$

$$= \frac{e^{o_1+o_2}}{(e^{o_1} + \quad)^2}$$

$$= \frac{e^{-1.803+0.211}}{(e^{-1.803}+e^{0.211})^2}$$

$$= 0.103878$$

$$\frac{\partial o_1}{\partial w_{11}^{(2)}} = \frac{\partial}{\partial w_{11}^{(2)}} \left(w_{11}^{(2)} z_1' + w_{12}^{(2)} z_2' + w_{13}^{(2)} z_3' \right)$$

$$= z_1' = -0.005$$

$$\frac{\partial L(w)}{\partial w_{11}^{(2)}} = 0.566727 * 0.103878 * (-0.005)$$

$$= -0.000294$$

Similarly for $w_{12}^{(2)}$:

$$\frac{\partial L(w)}{\partial w_{12}^{(2)}} = \frac{\partial L(w)}{\partial z_1^2} * \frac{\partial z_1^2}{\partial o_1} * \frac{\partial o_1}{\partial w_{12}^{(2)}}$$

$$\frac{\partial o_1}{\partial w_{12}^{(2)}} = \frac{\partial}{\partial w_{12}^{(2)}} \left(w_{11}^{(2)} z_1' + w_{12}^{(2)} z_2' + w_{13}^{(2)} z_3' \right)$$

$$= z_2' = 1.5$$

$$\frac{\partial L(w)}{\partial w_{12}^{(2)}} = 0.566727 * 0.103878 * 1.5$$

$$= 0.088306$$

$$\frac{\partial L(w)}{\partial w_{13}^{(2)}} = 0.566727 * 0.103878 * \overbrace{\frac{\partial o_1}{\partial w_{13}^{(2)}}}^{z_3' = 3.3}$$

$$= 0.194273$$

Again,

$$\frac{\partial L(w)}{\partial w_{21}^{(2)}} = \frac{\partial L(w)}{\partial z_2^2} * \frac{\partial z_2^2}{\partial o_2} * \frac{\partial o_2}{\partial w_{21}^{(2)}}$$

$$\frac{\partial L(w)}{\partial z_2^2} = \frac{\partial}{\partial z_2^2}\left(-\frac{1}{2}\left[y_{11}\cdot \log(z_1^2)) + (1-y_{11})\cdot \text{Log}(1-z_1^2) + y_{12}\cdot \log(z_2^2) + (1-y_{12})\cdot \log(1-z_2^2)\right]\right)$$

$$= -\frac{1}{2}\left(y_{12}\cdot \frac{1}{z_2^2} + (1-y_{12})\cdot \frac{-1}{z_2^2}\right)$$

$$= -\frac{1}{2}\left(1\cdot \frac{1}{0.882259} + 0\right)$$

$$= -0.566727$$

$$\frac{\partial z_2^2}{\partial o_2} = \frac{\partial}{\partial o_2}\left(\frac{e^{o_2}}{e^{o_1} + e^{o_2}}\right) = 0.103878$$

$$\frac{\partial o_2}{\partial w_{21}^{(2)}} = \frac{\partial}{\partial w^{(2)}}\left(w_{21}^{(2)} z_1^1 + w_{22}^{(2)} z_2^1 + w_{23}^{(2)} z_3^1\right)$$

$$= z_1^1 = -0.005$$

$$\frac{\partial L(w)}{\partial w_{21}^{(2)}} = -0.566727 * 0.103878 * (-0.005)$$

$$= -0.000294$$

$$\frac{\partial L(w)}{\partial w_{22}^{(2)}} = -0.566727 * 0.103878 * \underbrace{\frac{\partial o_2}{\partial w_{22}^{(2)}}}_{z_2^1 = 1.5}$$

$$= -0.088306$$

$$\frac{\partial L(w)}{\partial w_{23}^{(2)}} = -0.566727 * 0.103878 * \underbrace{\frac{\partial o_2}{\partial w_{23}}}_{z_3^1 = 3.3}$$

$$= -0.194273$$

Derivatives and values of layer 2:

$$\frac{\partial L(w)}{\partial w_{11}^{(2)}} = 0.000294 \qquad \frac{\partial L(w)}{\partial w_{21}^{(2)}} = -0.000294$$

$$\frac{\partial L(w)}{\partial w_{12}^{(2)}} = 0.088306 \qquad \frac{\partial L(w)}{\partial w_{22}^{(2)}} = -0.088306$$

$$\frac{\partial L(w)}{\partial w_{13}^{(2)}} = 0.194273 \qquad \frac{\partial L(w)}{\partial w_{23}^{(2)}} = -0.194273$$

Calculating derivatives for hidden layer:

For weight column 1 :

$$\frac{\partial L(w)}{\partial w_{11}^{(1)}} = \frac{\partial L(w)}{\partial z_1^1} * \frac{\partial z_1^1}{\partial h1} * \frac{\partial h1}{\partial w_{11}^{(1)}}$$

$$\frac{\partial L(w)}{\partial z_1^1} = \frac{\partial L_1}{\partial z_1^1} + \frac{\partial L_2}{\partial z_1^1}$$

$$= \frac{\partial L_1}{\partial o_1} * \frac{\partial o_1}{\partial z_1^1} + \frac{\partial L_2}{\partial o_2} * \frac{\partial o_2}{\partial z_1^1}$$

$$= \frac{\partial L_1}{\partial z_1^2} * \frac{\partial z_1^2}{\partial o_1} * \frac{\partial o_1}{\partial z_1^1} + \frac{\partial L_2}{\partial z_2^2} * \frac{\partial z_2^2}{\partial o_2} * \frac{\partial o_2}{\partial z_2^2}$$

$$\frac{\partial L_1}{\partial z_1^2} = \frac{\partial}{\partial z_1^2}\left(-\frac{1}{2}\left[y_{11} \cdot \log(z_1^2)\right) + (1-y_{11}) \cdot \mathrm{Log}(1-z_1^2)\right)$$

$$= -\frac{1}{2} \cdot \left(0 + 1 \cdot \frac{-1}{1-z_1^2}\right)$$

$$= 0.566727$$

$$\frac{\partial L_2}{\partial z_1^2} = \frac{\partial}{\partial z_1^2}\left(-\frac{1}{2}\left[y_{12} \cdot \log(z_1^2) + (1-y_{12}) \cdot \log(1-z_1^2)\right]\right)$$

$$= -\frac{1}{2}\left(1 \cdot \frac{1}{z_1^2}\right)$$

$$= -0.566727$$

Using the values calculated previously:

$$\frac{\partial L(w)}{\partial z_1^1} = 0.566727 * 0.103874 * 0.6 + (-0.566727) * 0.103874 * (-0.2)$$

$$= 0.0470946$$

Next,

$$\frac{\partial z_1'}{\partial h_1} = \frac{\partial}{\partial h_1}\left(f(h1) = \begin{cases}(0.01)h1 & ; h1<0 \\ h1 & ; h1>0\end{cases}\right)$$

$$= \begin{cases}0.01 & ; h1<0 \\ 1 & ; h1>0\end{cases}$$

$$\Rightarrow \frac{\partial L(w)}{\partial w_{11}^{(1)}} = \frac{\partial L(w)}{\partial z_1'} * \frac{\partial z_1'}{\partial h_1} * \overbrace{\frac{\partial h_1}{\partial w_{11}^{(1)}}}^{x_1}$$

$$= 0.0470946 * 0.01 * 3$$

$$= 0.001413$$

Similarly,

$$\underbrace{\frac{\partial h_1}{\partial w_{12}}}\ = x_2$$

$$\frac{\partial L(w)}{\partial w_{12}^{(1)}} = 0.0470946 * 0.01 * 1 = 0.000471$$

$$\underbrace{\frac{\partial h_1}{\partial w_{13}}}\ = x_3$$

$$\frac{\partial L(w)}{\partial w_{13}^{(1)}} = 0.0470946 * 0.01 * -1 = -0.000471$$

$$\underbrace{\frac{\partial h_1}{\partial w_{14}}}\ = x_4$$

$$\frac{\partial L(w)}{\partial w_{14}^{(1)}} = 0.0470946 * 0.01 * 2 = 0.000942$$

For weights column 2 :

$$\frac{\partial L(w)}{\partial w_{21}^{(1)}} = \frac{\partial L(w)}{\partial z_2'} * \frac{\partial z_2'}{\partial h2} * \frac{\partial h2}{\partial w_{21}^{(1)}}$$

$$\frac{\partial L(w)}{\partial z_2'} = \frac{\partial L_1}{\partial z_2'} + \frac{\partial L_2}{\partial z_2'}$$

$$= \frac{\partial L_1}{\partial o_1} * \frac{\partial o_1}{\partial z_2'} + \frac{\partial L_2}{\partial o_2} * \frac{\partial o_2}{\partial z_2'}$$

$$= \frac{\partial L_1}{\partial z_1^2} * \frac{\partial z_1^2}{\partial o_1} * \overbrace{\frac{\partial o_1}{\partial z_2^1}}^{w_{12}^{(2)}} + \frac{\partial L_2}{\partial z_2^2} * \frac{\partial z_2^2}{\partial o_2} * \overbrace{\frac{\partial o_2}{\partial z_2^1}}^{w_{22}^{(2)}}$$

$$= 0.566727 * 0.103874 * (-0.1) + (-0.566727) * 0.103874 * (0.8)$$

$$= -0.052981$$

$$\frac{\partial L(w)}{\partial w_{21}^{(1)}} = -0.052981 * \underbrace{1}^{\text{since } h2 > 0} * 3 = -0.158943$$

$$\frac{\partial L(w)}{\partial w_{22}^{(1)}} = -0.052981 * 1 * 1 = -0.052981$$

$$\frac{\partial L(w)}{\partial w_{23}^{(1)}} = -0.052981 * 1 * (-1) = 0.052981$$

$$\frac{\partial L(w)}{\partial w_{24}^{(1)}} = -0.052981 * 1 * 2 = -0.105962$$

For weights column 3:

$$\frac{\partial L(w)}{\partial w_{31}^{(1)}} = \frac{\partial L(w)}{\partial z_3^1} * \frac{\partial z_3^1}{\partial h3} * \frac{\partial h3}{\partial w_{31}^{(1)}}$$

$$\frac{\partial L(w)}{\partial z_3^1} = \frac{\partial L_1}{\partial z_3^1} + \frac{\partial L_2}{\partial z_3^1}$$

$$= \frac{\partial L_1}{\partial o_1} * \frac{\partial o_1}{\partial z_3^1} + \frac{\partial L_2}{\partial o_2} * \frac{\partial o_2}{\partial z_3^1}$$

$$= \frac{\partial L_1}{\partial z_1^2} * \frac{\partial z_1^2}{\partial o_1} * \overbrace{\frac{\partial o_1}{\partial z_3^1}}^{w_{12}^{(2)}} + \frac{\partial L_2}{\partial z_2^2} * \frac{\partial z_2^2}{\partial o_2} * \overbrace{\frac{\partial o_2}{\partial z_3^1}}^{w_{22}^{(2)}}$$

$$= 0.566727 * 0.103874 * (-0.5) + (-0.566727) * 0.103874 * (-0.3)$$

$$= -0.011789$$

$$\frac{\partial L(w)}{\partial w_{31}^{(1)}} = -0.011789 * \overbrace{1}^{\text{since } h3 > 0} * 3 = -0.035367$$

$$\frac{\partial L(w)}{\partial w_{32}^{(1)}} = -0.011789 * 1 * 1 = -0.011789$$

$$\frac{\partial L(w)}{\partial w_{33}^{(1)}} = -0.011789 * 1 * (-1) = 0.011789$$

$$\frac{\partial L(w)}{\partial w_{34}^{(1)}} = -0.011789 * 1 * 2 = -0.023578$$

Derivatives and values for layer 1:

$\frac{\partial L(w)}{\partial w_{11}^{(1)}}$	0.001413	$\frac{\partial L(w)}{\partial w_{21}^{(1)}}$	-0.158943
$\frac{\partial L(w)}{\partial w_{12}^{(1)}}$	0.000471	$\frac{\partial L(w)}{\partial w_{22}^{(1)}}$	-0.052981
$\frac{\partial L(w)}{\partial w_{13}^{(1)}}$	-0.000471	$\frac{\partial L(w)}{\partial w_{23}^{(1)}}$	0.052981
$\frac{\partial L(w)}{\partial w_{14}^{(1)}}$	0.000942	$\frac{\partial L(w)}{\partial w_{24}^{(1)}}$	-0.105962
$\frac{\partial L(w)}{\partial w_{31}^{(1)}}$	-0.035367	$\frac{\partial L(w)}{\partial w_{32}^{(1)}}$	-0.011789
$\frac{\partial L(w)}{\partial w_{33}^{(1)}}$	0.011789	$\frac{\partial L(w)}{\partial w_{34}^{(1)}}$	-0.023578

c) Given learning rate = 0.1,

$$w_{11}^{(1)} = w_{11}^{(1)} - 0.1 * \frac{\partial L(w)}{\partial w_{11}^{(1)}}$$

$$= -0.2 - 0.1 * 0.001413$$

$$= -0.2001413$$

Similarly updating all the weights

$$W^{(1)} = \begin{bmatrix} -0.2001413 & 0.9158943 & 0.4035367 \\ -0.1000471 & 0.3052981 & 0.4011789 \\ 0.2000471 & 0.4947019 & -0.7011789 \\ 0.1999058 & -0.4894038 & 0.5023578 \end{bmatrix}$$

$$W^{(2)} = \begin{bmatrix} 0.5999706 & -0.1999706 \\ -0.1088306 & 0.8088306 \\ -0.5194273 & -0.2805727 \end{bmatrix}$$