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Asymptotic modeling of linearly piezoelectric slender rods

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Abstract

The piezoelectric thin plate modeling already derived by the authors is extended to rod-like structures. Two models corresponding to sensor or actuator behavior are obtained. The conditions of existence of non-local terms in the limit models are discussed. ***To cite this article: T. Weller, C. Licht, C. R. Mecanique 336 (2008).***

Résumé

Modélisation asymptotique de poutres linéairement piézoélectriques. On étend aux structures de type poutre la modélisation de plaques minces linéairement piézoélectriques déjà obtenue par les auteurs. On met en évidence deux modèles correspondant à un comportement de type capteur ou actionneur. Les conditions d'apparition de termes non locaux dans les modèles limites sont discutées. ***Pour citer cet article : T. Weller, C. Licht, C. R. Mecanique 336 (2008).***

Keywords: Solids and structures; Piezoelectric rods; Asymptotic analysis; Non local effects

Mots-clés : Solides et structures ; Poutres piézoélectriques ; Analyse asymptotique ; Effets non locaux

1. Introduction

The mathematical modeling of elastic thin plates or slender rods by mean of asymptotic analysis is a classical topic (see [1–4] and the references therein): the thickness (in the case of plates) or the diameter (in the case of rods) is assigned to a role of *parameter* whose aim is to tend to zero. For plates, this method has rapidly been extended to linear piezoelectricity [5–9] and linear electromagneto-elasticity [10]. We have clearly shown in [9,10] how the electric (and possibly magnetic) boundary conditions lead to various models which correspond to the cases when the plate is used as a *sensor*, as an *actuator* or as a *mixed senso-actuator*. As pointed out in [11], because "beam *modeling* requires to condense on a line the properties of slender three-dimensional objects having one dimension prevailing on the others", it is more challenging than plate modeling. Here, we present *two* asymptotic models which involve a greater number of state variables than the couple (displacement/electrical potential) of the genuine three-dimensional

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physical problem. We therefore exhibit reduced formulations where the number of variables drops to one or two, one reduced problem being purely mechanical! We also discuss the conditions for which the elimination of additional variables leads to non-standard equations involving non-local terms.

2. Setting the problem

The reference configuration of a linearly piezoelectric slender rod is the closure in $\mathbb{R}^3$ of the set $\Omega^\varepsilon := \varepsilon\omega \times (0, L)$ where ω is a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\omega$, L is the length of the rod and ε a small positive number. We make no difference between $\mathbb{R}^3$ and the Euclidean physical space whose orthonormal basis is assumed to be the principal frame of inertia of the rod. Greek coordinate indices will run in $\{1, 2\}$ and Latin ones in $\{1, 2, 3\}$; for all $\xi = (\xi_1, \xi_2, \xi_3)$ of $\mathbb{R}^3$, $\hat{\xi}, \xi^R$ stand for $(\xi_1, \xi_2), (-\xi_2, \xi_1)$. Let $\Gamma^\varepsilon_{\mathrm{lat}} := \varepsilon\partial\omega \times (0, L)$, $\Gamma^\varepsilon_0 := \varepsilon\omega \times \{0\}$, $\Gamma^\varepsilon_L := \varepsilon\omega \times \{L\}$, $\Gamma_{0,L} := \Gamma_0 \cup \Gamma_L$, and two partitions of $\partial\Omega^\varepsilon$ $(\Gamma^\varepsilon_{mD}, \Gamma^\varepsilon_{mN})$, $(\Gamma^\varepsilon_{eD}, \Gamma^\varepsilon_{eN})$ with Γ^ε_{mD}, Γ^ε_{eD} of strictly positive surface measures and Γ_0 or Γ_L included in Γ^ε_{mD}. The rod is clamped along Γ^ε_{mD} and at an electrical potential φ^ε_0 on Γ^ε_{eD}. It is subjected to body forces f^ε in Ω^ε, surface forces g^ε on Γ^ε_{mN}, electrical loading w^ε on Γ^ε_{eN}. We note n^ε the outward unit normal to $\partial\Omega^\varepsilon$.The piezoelectric state $s^\varepsilon := (u^\varepsilon, \varphi^\varepsilon)$ at equilibrium satisfies:

$$\mathcal{P}(\Omega^\varepsilon) \quad \begin{cases} \operatorname{div}\sigma^\varepsilon + f^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \quad \sigma^\varepsilon n^\varepsilon = g^\varepsilon \quad \text{on } \Gamma^\varepsilon_{mN}, \quad u^\varepsilon = 0 \quad \text{on } \Gamma^\varepsilon_{mD} \\ \operatorname{div} D^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \quad D^\varepsilon \cdot n^\varepsilon = w^\varepsilon \quad \text{on } \Gamma^\varepsilon_{eN}, \quad \varphi^\varepsilon = \varphi^\varepsilon_0 \quad \text{on } \Gamma^\varepsilon_{eD} \\ (\sigma^\varepsilon, D^\varepsilon) = M^\varepsilon(x)\big(e(u^\varepsilon), \nabla\varphi^\varepsilon\big) \quad \text{in } \Omega^\varepsilon \end{cases}$$

where u^ε, φ^ε, σ^ε, $e(u^\varepsilon)$ and D^ε respectively stand for the displacement, the electric potential field, the stress tensor, the tensor of small strains and the electrical displacement. If we denote the set of all linear mappings from a space V into a space W by $\mathcal{L}(V, W)$, the set of all $N \times N$ symmetric matrices by S^N and define $\mathcal{H} := S^3 \times \mathbb{R}^3$, the operator M^ε is an element of $\mathcal{L}(\mathcal{H}, \mathcal{H})$ such that

$$\sigma^\varepsilon = M^\varepsilon_{mm} e(u^\varepsilon) - M^\varepsilon_{me}\nabla\varphi^\varepsilon, \qquad D^\varepsilon = M^{\varepsilon^T}_{me} e(u^\varepsilon) + M^\varepsilon_{ee}\nabla\varphi^\varepsilon \tag{1}$$

where M^ε_{mm}, M^ε_{me} and M^ε_{ee} are respectively the elastic, piezoelectric and dielectric tensors while the superscript T denotes the transpose operation. Of course, M^ε is not symmetric but under realistic assumption of boundedness of M^ε and of uniform ellipticity of M^ε_{mm} and M^ε_{ee}, the physical problem $\mathcal{P}(\Omega^\varepsilon)$ has a unique weak solution. Piezoelectric rod models are obtained by studying the limit behavior of s^ε when $\varepsilon \to 0$.

3. Convergence results

As in [9], we will show that two different limit behaviors, indexed by $p = 1, 2$, appear according to the type of electric boundary conditions and to the magnitude of the electrical external loading. In the sequel, any $h = (e, g) \in \mathcal{H}$ will be represented as $(\hat{e}, e_{\alpha 3}, e_{33}, \hat{g}, g_3)$ where $\hat{e}$ is the element of S^2 such that $\hat{e}_{\alpha\beta} = e_{\alpha\beta}$ while $h_{(m,3)}, h_{(e,3)}$ stand for e_{33}, g_3 respectively. For all $G \subset \mathbb{R}^N$, $H^1_g(G)$ denotes the subset of the Sobolev space $H^1(G)$ whose elements vanish on $g \subset \partial G$, except $H^1_m(\omega)$ which is the set of the elements of $H^1(\omega)$ with zero average on ω. Let us recall the main steps of the method.

First we come down to a fixed open set $\Omega := \omega \times (0, L)$ through the bijection $x = (x_1, x_2, x_3) \in \bar{\Omega} \mapsto x^\varepsilon = \pi^\varepsilon(x) = (\varepsilon x_1, \varepsilon x_2, x_3) \in \bar{\Omega}^\varepsilon$ and drop the index ε for the images by $(\pi^\varepsilon)^{-1}$ of the geometric sets defined at the beginning of Section 2. We assume that the electro-elastic coefficients and loading satisfy:

$$\begin{cases} M^\varepsilon(\pi^\varepsilon x) =: M(x), \quad M \in L^\infty\big(\Omega, \mathcal{L}(\mathcal{H}, \mathcal{H})\big), \quad \exists\kappa > 0\colon M(x)h \cdot h \geqslant \kappa |h|^2_{\mathcal{H}}, \quad \forall h \in \mathcal{H}, \text{ a.e. } x \in \Omega \\ \hat{f}^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 \hat{f}(x), \quad f^\varepsilon_3(\pi^\varepsilon x) = \varepsilon f_3(x), \quad \forall x \in \Omega \\ \hat{g}^\varepsilon(\pi^\varepsilon x) = \varepsilon^2 \hat{g}(x), \quad g^\varepsilon_3(\pi^\varepsilon x) = \varepsilon g_3(x), \quad \forall x \in \Gamma_{mN} \cap \Gamma_{0,L} \\ \hat{g}^\varepsilon(\pi^\varepsilon x) = \varepsilon^3 \hat{g}(x), \quad g^\varepsilon_3(\pi^\varepsilon x) = \varepsilon^2 g_3(x), \quad \forall x \in \Gamma_{mN} \cap \Gamma_{\mathrm{lat}} \\ w^\varepsilon(\pi^\varepsilon x) = \varepsilon^{2-p} w(x), \quad \forall x \in \Gamma_{eN} \cap \Gamma_{0,L}, \quad w^\varepsilon(\pi^\varepsilon x) = \varepsilon^{3-p} w(x), \quad \forall x \in \Gamma_{eN} \cap \Gamma_{\mathrm{lat}} \\ \varphi^\varepsilon_0(\pi^\varepsilon x) = \varepsilon^p \varphi_0(x), \quad \forall x \in \Gamma_{eD} \end{cases} \tag{2}$$

where (f, g, w, φ_0) is an element (independent of ε) of $L^2(\Omega)^3 \times L^2(\Gamma_{mN})^3 \times L^2(\Gamma_{eN}) \times H^1(\Omega)$ and:

$$\begin{cases} \text{if } p=1\text{: } \varphi_0 \text{ does not depend on } \hat{x} \text{ and } \Gamma_{eD} \subset \Gamma_{0,L} \\ \text{if } p=2\text{: } \exists \gamma_{eD} \subset \gamma := \partial\omega \text{ with positive length such that } (\gamma \setminus \gamma_{eD}) \times (0,L) \subset \Gamma_{eN} \text{ and} \\ \quad \text{either } \Gamma_{eN} \cap \Gamma_{0,L} = \emptyset \text{ or } w = 0 \text{ on } \Gamma_{eN} \cap \Gamma_{0,L} \end{cases} \tag{3}$$

These assumptions, similar to those of [9,10], make possible to control the magnitude of the electromechanical loading with respect to the slenderness of the beam. In the purely mechanical case, they supply a rational justification of the Bernoulli–Navier theory (cf. [2–4]).

Next, with the true physical state $s^\varepsilon = (u^\varepsilon, \varphi^\varepsilon)$ defined on Ω^ε, we associate a *scaled* piezoelectric state $s_p(\varepsilon) := (u_p(\varepsilon), \varphi_p(\varepsilon))$ defined by:

$$\hat{u}^\varepsilon(x^\varepsilon) = \big(\hat{u}_p(\varepsilon)\big)(x), \quad u_3^\varepsilon\big(x^\varepsilon\big) = \varepsilon\big(u_p(\varepsilon)\big)_3(x), \quad \varphi^\varepsilon(x^\varepsilon) = \varepsilon^p \varphi_p(\varepsilon)(x), \quad \forall x = \pi^\varepsilon(x) \in \Omega \tag{4}$$

so that $s_p(\varepsilon)$ is the unique solution of the following mathematical problem:

$$\mathcal{P}(\varepsilon, \Omega)_p \quad \begin{cases} \text{Find } s_p(\varepsilon) \in (0, \varphi_0) + V \text{ such that } m_p(\varepsilon)\big(s_p(\varepsilon), r\big) = L(r), \quad \forall r \in V, \\ V := \big\{ r = (v, \psi) \in H^1_{\Gamma_{mD}}(\Omega)^3 \times H^1_{\Gamma_{eD}}(\Omega) \big\} \end{cases}$$

equivalent to the genuine physical one, with

$$\begin{cases} m_p(\varepsilon)(s, r) := \displaystyle\int_\Omega M(x) k_p(\varepsilon, s) \cdot k_p(\varepsilon, r)\, \mathrm{d}x, \quad k_p(\varepsilon, r) := k_p\big(\varepsilon, (v, \psi)\big) = \big(e(\varepsilon, v), \nabla_p(\varepsilon, \psi)\big) \\ e_{\alpha\beta}(\varepsilon, v) := \varepsilon^{-2} e_{\alpha\beta}(v), \quad e_{\alpha 3}(\varepsilon, v) := \varepsilon^{-1} e_{\alpha 3}(v), \quad e_{33}(\varepsilon, v) := e_{33}(v) \\ 2e_{ij}(v) := \partial_i v_j + \partial_j v_i, \quad \nabla_p(\varepsilon, \phi)_\alpha := \varepsilon^{p-2} \partial_\alpha \varphi, \quad \nabla_p(\varepsilon, \varphi)_3 := \varepsilon^{p-1} \partial_3 \varphi \\ L(r) := L(v, \psi) = \displaystyle\int_\Omega f \cdot v\, \mathrm{d}x + \int_{\Gamma_{mN}} g \cdot v\, \mathrm{d}x + \int_{\Gamma_{eN}} w\psi\, \mathrm{d}x \end{cases} \tag{5}$$

Finding the limit problems is slightly more difficult than in the case of plates because they involve a greater number of state variables: $\tilde{s}_1 = (v, w, \psi)$ and $\tilde{s}_2 = (v, w)$ are added to the initial state variable $s = (u, \phi)$ and we let $\mathsf{s}_p = (s, \tilde{s}_p)$; they belong to spaces defined as follows:

$$D_m := \big\{ x_3 \in \{0, L\};\ \omega \times \{x_3\} \supset \Gamma_{mD} \cap \Gamma_{0,L} \big\}, \qquad D_e := \big\{ x_3 \in \{0, L\};\ \omega \times \{x_3\} \supset \Gamma_{eD} \cap \Gamma_{0,L} \big\}$$

$$\begin{aligned} V_{\mathrm{BN}}(\Omega) &:= \big\{ u \in H^1_{\Gamma_{mD}}(\Omega)^3;\ e_{\alpha\beta}(u) = e_{\alpha 3}(u) = 0 \big\} \\ &= \left\{ u;\ \exists (u^b, u^s) \in H^2_{D_m}(0, L)^2 \times H^1_{D_m}(0, L)\text{: } \hat{u}(x) = u^b(x_3),\ u_3(x) = u^s(x_3) - x_\alpha \frac{\mathrm{d}u^b_\alpha}{\mathrm{d}x_3} \right\} \end{aligned}$$

$$R(\Omega) := \big\{ v\text{: } \exists c \in H^1_{D_m}(0, L),\ \hat{v}(x) = c(x_3) x^R,\ v_3 \in L^2\big(0, L; H^1_m(\omega)\big) \big\}$$

$$RD_2^\perp(\Omega) := \left\{ w\text{: } \hat{w} \in L^2\big(0, L; H^1_m(\omega)\big),\ w_3(x) = 0 \text{ and } \int_\omega x^R \cdot \hat{w}(x)\, \mathrm{d}\hat{x} = 0,\ \text{a.e. } x_3 \in (0, L) \right\}$$

$$\Phi_1 := \big\{ \phi;\ \exists \varphi \in H^1_{D_e}(0, L)\text{: } \phi(x) := \varphi(x_3) \big\}, \quad \Phi_2 := L^2\big(0, L; H^1_{\gamma_e}(\omega)\big), \quad \Psi_1 := L^2\big(0, L; H^1_m(\omega)\big)$$

$$V_1 := V_{\mathrm{BN}}(\Omega) \times \Phi_1 \times R(\Omega) \times RD_2^\perp(\Omega) \times \Psi_1, \quad V_2 := V_{\mathrm{BN}}(\Omega) \times \Phi_2 \times R(\Omega) \times RD_2^\perp(\Omega)$$

If $\mathsf{s}_{1_0} := (0, \varphi_0, 0, 0, 0)$, $\mathsf{s}_{2_0} := (0, \varphi_0, 0, 0)$ the limit problems read as

$$\bar{\mathsf{P}}(\Omega)_p\text{: Find } \bar{\mathsf{s}}_p \in \mathsf{s}_{p_0} + V_p \text{ such that for all } \mathsf{s}'_p \in V_p \int_\Omega M(x) \mathsf{k}_p(\bar{\mathsf{s}}_p) \cdot \mathsf{k}_p(\mathsf{s}'_p)\, \mathrm{d}x = L(\mathsf{s}'_p)$$

$\mathsf{k}_1(\mathsf{s}), \mathsf{k}_2(\mathsf{s}) \in L^2(\Omega; \mathcal{H})$ being represented by $(\hat{e}(w), e_{\alpha 3}(v), e_{33}(u), \hat{\nabla}\psi, \frac{\mathrm{d}\phi}{\mathrm{d}x_3})$, $(\hat{e}(w), e_{\alpha 3}(v), e_{33}(u), \hat{\nabla}\phi, 0)$. In the formulae defined *supra*, $V_{\mathrm{BN}}(\Omega)$ stands for the Bernoulli–Navier displacements space while $R_b(\Omega)$ and $RD_2^\perp(\Omega)$ respectively describe the rotations and the displacements orthogonal to rigid displacements in cross sections (see [2]). We have the following convergence result:

Theorem 1. *When $\varepsilon \to 0$, the family $(s_p(\varepsilon))_{\varepsilon>0}$ of the unique solutions of $\mathcal{P}(\varepsilon,\Omega)_p$ is such that $(s_p(\varepsilon), k(\varepsilon, s_p(\varepsilon))$ converges strongly in $V_{\mathrm{BN}}(\Omega) \times \Phi_p \times L^2(\Omega;\mathcal{H})$ to $(\bar{s}_p, \mathsf{k}_p(\bar{\mathsf{s}}_p))$, where $\bar{\mathsf{s}}_p = (\bar{s}_p, \bar{\tilde{s}}_p)$ is the unique solution of $\bar{\mathsf{P}}(\Omega)_p$.*

Proof. Because of the proof in [4], it suffices to determine the asymptotic behavior of electrical fields like $\varepsilon^{-1}\hat{\nabla}\varphi_1(\varepsilon)$, $\partial_3\varphi_1(\varepsilon)$, $\hat{\nabla}\varphi_2(\varepsilon)$, $\varepsilon\partial_3\varphi_2(\varepsilon)$ by due account of the boundedness of $\nabla_p(\varepsilon,\varphi_p(\varepsilon))$ in $L^2(\Omega)^3$ and a classical characterization of fields of gradients. $\square$

4. Properties of the limit problems, reduced problems, models

Due to the V_p-ellipticity of the bilinear forms involved in $\bar{\mathsf{P}}(\Omega)_p$ the state variables $(\bar{v}_1, \bar{w}_1, \bar{\psi}_1)$ and $(\bar{\phi}_2, \bar{v}_2, \bar{w}_2)$ can be eliminated so that $\bar{s}_1$ and $\bar{u}_2$ solve *monodimensional* variational problems like

$$\bar{\mathcal{P}}(\Omega)_1\text{: Find } \bar{s}_1 \in (0,\varphi_0) + S_1;\, n_1(\bar{s}_1, s') = L(s'), \quad \forall s' \in S_1 := V_{\mathrm{BN}}(\Omega) \times H^1_{D_e}(0,L)$$

$$\bar{\mathcal{P}}(\Omega)_2\text{: Find } \bar{u}_2 \in V_{\mathrm{BN}}(\Omega);\, n_2(\bar{u}_2, u') = L_2(u'), \quad \forall u' \in V_{\mathrm{BN}}(\Omega)$$

Proceeding as in [3,4], tedious handlings of essentially algebraic nature which take advantage of the structures of Ω and V_p allow us to explicit n_p and L_2 as follows: we note $u^{\odot} := (\frac{\mathrm{d}^2 u_1^b}{\mathrm{d}x_3^2}, \frac{\mathrm{d}^2 u_2^b}{\mathrm{d}x_3^2}, \frac{\mathrm{d}u^s}{\mathrm{d}x_3})$ and we let $R(\Omega) \times RD_2^{\perp} \times \Psi_1 =: V_1^{\perp} \ni s_1^{\perp} := (v, w, \psi)$ and $\Phi_2 \times R(\Omega) \times RD_2^{\perp} =: V_2^{\perp} \ni s_2^{\perp} := (\phi, v, w)$, $k_1^{\perp}(s_1^{\perp})$, $k_2^{\perp}(s_2^{\perp}) \in \mathcal{H}$ being respectively represented by $(\hat{e}(w), e_{\alpha 3}(v), 0, \hat{\nabla}\psi, 0)$ and $(\hat{e}(w), e_{\alpha 3}(v), 0, \hat{\nabla}\phi, 0)$. We also define $V_{p3}^{\perp} := \{s_p^{\perp} \in V_p^{\perp};\ \hat{v} = 0\}$. Introducing $k^{\odot^{(q)}}, k^m, k^e \in \mathcal{H}$ respectively represented by $(0,0,x^{\odot^{(q)}},0,0)$, $(0,0,\frac{1}{2}x_\alpha^R,0,0)$, $(0,0,0,0,1)$, the solving of the problems

$$S_2^0\text{: Find } s_2^{0\perp} \in V_2^{\perp};\ \int_\Omega M k_2^{\perp}(s_2^{0\perp}) \cdot k_2^{\perp}(s^{\perp})\,\mathrm{d}x = L(0,\varphi), \quad \forall s^{\perp} \in V_2^{\perp}$$

$$S_p^{(q)}\text{: Find } s_p^{(q)\perp} \in V_{p3}^{\perp};\ \int_\Omega M k_p^{\perp}(s_p^{(q)\perp}) \cdot k_p^{\perp}(s^{\perp})\,\mathrm{d}x = -\int_\Omega M k^{\odot^{(q)}} \cdot k_p^{\perp}(s^{\perp})\,\mathrm{d}x, \quad \forall s^{\perp} \in V_{p3}^{\perp}$$

$$S_p^m\text{: Find } s_p^{m\perp} \in V_{p3}^{\perp};\ \int_\Omega M k_p^{\perp}(s_p^{m\perp}) \cdot k_p^{\perp}(s^{\perp})\,\mathrm{d}x = -\int_\Omega M k^m \cdot k_p^{\perp}(s^{\perp})\,\mathrm{d}x, \quad \forall s^{\perp} \in V_{p3}^{\perp}$$

$$S_1^e\text{: Find } s_1^{e\perp} \in V_{13}^{\perp};\ \int_\Omega M k_1^{\perp}(s_1^{e\perp}) \cdot k_1^{\perp}(s^{\perp})\,\mathrm{d}x = -\int_\Omega M k^e \cdot k_1^{\perp}(s^{\perp})\,\mathrm{d}x, \quad \forall s^{\perp} \in V_{13}^{\perp}$$

allows us to define

$$a_p := \int_\omega M\big(k_p^{\perp}(s_p^{m\perp}) + k^m\big) \cdot k^m\,\mathrm{d}\hat{x}, \quad b_p^{(q)} := \int_\omega M\big(k_p^{\perp}(s_p^{(q)\perp}) + k^{\odot^{(q)}}\big) \cdot k^m\,\mathrm{d}\hat{x}$$

$$b_1^e := \int_\omega M\big(k_1^{\perp}(s_1^{e\perp}) + k^e\big) \cdot k^m\,\mathrm{d}\hat{x}$$

$$\mathsf{a}_p := \left(a_p \int_0^L a_p^{-1}\,\mathrm{d}x_3\right)^{-1}, \quad \mathsf{b}_p^{(q)} := b_p^{(q)}/a_p \quad \text{and} \quad \mathsf{b}_1^e := b_1^e/a_1$$

Then, if

$$\kappa_1 = \kappa_1(u,\phi) := \mathsf{a}_1 \int_0^L \left(\mathsf{b}_1^q u^{\odot^{(q)}} + \mathsf{b}_1^e \frac{\mathrm{d}\phi}{\mathrm{d}x_3}\right)\mathrm{d}l, \quad \kappa_2 = \kappa_2(u) := \mathsf{a}_2 \int_0^L \mathsf{b}_2^{(q)} u^{\odot^{(q)}}\,\mathrm{d}l$$

and

$$T_p^m := M\big(k_p^\perp\big(s_p^{m\perp}\big) + k^m\big), \qquad T_p^{(q)} := M\big(k_p^\perp\big(s_p^{(q)\perp}\big) + k^{\odot(q)}\big) - \mathsf{b}_p^{(q)} T_p^m$$
$$T_1^e := M\big(k_1^\perp\big(s_1^{e\perp}\big) + k^e\big) - \mathsf{b}_1^e T_p^m, \qquad T_2^0 := M k_2^\perp\big(s_2^{0\perp}\big)$$

we have

$$n_1(s,s') = \int_\Omega \left[\left(u^{\odot(q)} T^{(q)}_{1_{(m,3)}} + \frac{\mathrm{d}\phi}{\mathrm{d}x_3} T^e_{1_{(m,3)}} + \kappa_1 T^m_{1_{(m,3)}}\right) e_{33}(u') + \left(u^{\odot(q)} T^{(q)}_{1_{(e,3)}} + \frac{\mathrm{d}\phi}{\mathrm{d}x_3} T^e_{1_{(e,3)}} + \kappa_1 T^m_{1_{(e,3)}}\right)\frac{\mathrm{d}\phi}{\mathrm{d}x_3}\right] \mathrm{d}x$$

$$n_2(u,u') = \int_\Omega \big(u^{\odot(q)} T^{(q)}_{2_{(m,3)}} + \kappa_2 T^m_{2_{(m,3)}}\big) e_{33}(u')\,\mathrm{d}x, \qquad L_2(u') = L(u',0) - \int_\Omega T^0_{2_{(m,3)}} e_{33}(u')\,\mathrm{d}x$$

It should be noted that the *monodimensional* variational problems $\bar{\mathcal{P}}(\Omega)_p$ involve *non-local* terms κ_p. Similarly to the case of purely elastic rods [3], they appear only under the conjunction of the following three conditions: the symmetry class of the material is either 1 or (if $p=1$) m, its heterogeneity in the x_3 direction and the clamping condition on the two bases of the cylinder. Note that electrical boundary conditions do not affect non-local terms, nevertheless, for the class m, the non local terms are purely of electrical nature. When the symmetry class is neither 1 nor m and with a transversally homogeneous material if the cardinal of D_m, denoted by $\sharp D_m$, equals 1 or a homogeneous material if $\sharp D_m = 2$, the (necessarily local) bilinear forms n_p can be expressed just in terms of the entries of M. Let $\mathcal{H}_p^\clubsuit$ and $\mathcal{H}_p^\spadesuit$ be the subspaces of $\mathcal{H}$ and (if $p=2$) $\dot{\mathcal{H}} := \{(e,g) \in \mathcal{H};\ g_3 = 0\}$:

$$\mathcal{H}_1^\clubsuit := \big\{(e,g);\ e_{\alpha i} = g_\alpha = 0\big\}, \quad \mathcal{H}_1^\spadesuit = \mathcal{H}_2^\spadesuit := \big\{(e,g);\ e_{33} = g_3 = 0\big\}, \quad \mathcal{H}_2^\clubsuit := \big\{(e,g);\ e_{\alpha i} = g_i = 0\big\} \tag{6}$$

so that M can be decomposed in 4 elements $M_p^{\star\diamond} \in \mathcal{L}(\mathcal{H}_p^\diamond, \mathcal{H}_p^\star)$ with $\star, \diamond \in \{\clubsuit, \spadesuit\}$. Because $M_p^{\clubsuit\clubsuit}$ and $M_p^{\spadesuit\spadesuit}$ are positive operators on $\mathcal{H}_p^\clubsuit$ and $\mathcal{H}_p^\spadesuit$, the Schur complement $\tilde{M}_p := M_p^{\clubsuit\clubsuit} - M_p^{\clubsuit\spadesuit}(M_p^{\spadesuit\spadesuit})^{-1} M_p^{\spadesuit\clubsuit}$ is a positive element of $\mathcal{L}(\mathcal{H}_p^\clubsuit, \mathcal{H}_p^\clubsuit)$ which can actually be represented by a positively definite element of S^{3-p} still denoted $\tilde{M}_p$. Indeed, the conditions on the symmetry class imply $[(M_p^{\spadesuit\spadesuit})^{-1} M_p^{\spadesuit\clubsuit} k^{\odot(3)}]_{(e,3)} = 0$ and, because of the homogeneous properties of the rod, some systems involved by the definitions of n_p can be solved algebraically. Hence, the problems $\bar{\mathcal{P}}(\Omega)_p$ read as:

$$\bar{\mathcal{P}}(\Omega)_1 \quad \begin{cases} \big(\bar{u}_1^s, \bar{\varphi}_1\big) \in (0,\varphi_0) + S_1^s; \quad \tilde{n}_1^s\big(\bar{u}_1^s, \bar{\varphi}_1; u^s, \psi\big) = L_1^s\big(u^s, \psi\big) \\ \quad \forall \big(u^s, \psi\big) \in S_1^s := H^1_{D_m}(0,L) \times H^1_{D_e}(0,L) \\ \bar{u}^b_{1_\alpha} \in H^2_{D_m}(0,L); \quad \tilde{n}^b_{1_\alpha}\big(\bar{u}^b_{1_\alpha}, u^b_\alpha\big) = L^b_{1_\alpha}\big(u^b_\alpha\big), \quad \forall u^b_\alpha \in H^2_{D_m}(0,L) \end{cases}$$

$$\bar{\mathcal{P}}(\Omega)_2 \quad \begin{cases} \bar{u}_2^s \in H^1_{D_m}(0,L); \quad \tilde{n}_2^s\big(\bar{u}_2^s, u^s\big) = L_2^s\big(u^s\big), \quad \forall u^s \in H^1_{D_m}(0,L) \\ \bar{u}^b_{2_\alpha} \in H^2_{D_m}(0,L); \quad \tilde{n}_2^b\big(\bar{u}^b_{2_\alpha}, u^b\big) = L^b_{2_\alpha}\big(u^b\big), \quad \forall u' \in H^2_{D_m}(0,L) \end{cases}$$

where

$$\tilde{n}_1^s\big(\bar{u}_1^s, \bar{\varphi}_1; u^s, \psi\big) := |\omega| \int_0^L \tilde{M}_1\left(\frac{\mathrm{d}\bar{u}_1^s}{\mathrm{d}x_3}, \frac{\mathrm{d}\bar{\varphi}_1}{\mathrm{d}x_3}\right)\cdot\left(\frac{\mathrm{d}u^s}{\mathrm{d}x_3}, \frac{\mathrm{d}\psi}{\mathrm{d}x_3}\right)\mathrm{d}x_3, \quad \tilde{n}^b_{1_\alpha}\big(\bar{u}^b_{1_\alpha}, \zeta\big) := I^2_\alpha(\omega)\int_0^L \tilde{M}_{1_{11}} \frac{\mathrm{d}^2\bar{u}^b_{1_\alpha}}{\mathrm{d}x_3^2}\frac{\mathrm{d}^2\zeta}{\mathrm{d}x_3^2}\,\mathrm{d}x_3$$

$$\tilde{n}_2^s\big(\bar{u}_2^s, u^s\big) := |\omega| \int_0^L \tilde{M}_2 \frac{\mathrm{d}\bar{u}_2^s}{\mathrm{d}x_3}\cdot\frac{\mathrm{d}u^s}{\mathrm{d}x_3}\,\mathrm{d}x_3, \quad \tilde{n}^b_{2_\alpha}(\xi,\zeta) := I^2_\alpha(\omega)\int_0^L \tilde{M}_2 \frac{\mathrm{d}^2\xi}{\mathrm{d}x_3^2}\frac{\mathrm{d}^2\zeta}{\mathrm{d}x_3^2}\,\mathrm{d}x_3, \quad I^2_\alpha(\omega) = \int_\omega x_\alpha^2\,\mathrm{d}\hat{x}$$

$$L_1^s(u^s,\psi) := \int_0^L \left[\int_\omega f_3\,\mathrm{d}\hat{x} + \int_{\gamma_{mN}} g_3\,\mathrm{d}\hat{s}\right] u^s\,\mathrm{d}x_3 + \int_0^L \left[\int_{\gamma_{eN}} w\,\mathrm{d}\hat{s}\right]\psi\,\mathrm{d}x_3 + (2 - \sharp D_m)\int_\omega g_3(\hat{x}, l_m) u^s(l_m)\,\mathrm{d}\hat{s}$$
$$+ (2 - \sharp D_e)\int_\omega w(\hat{x}, l_e)\,\mathrm{d}\hat{x}, \quad l_m \in \{0,L\}\setminus D_m,\ l_e \in \{0,L\}\setminus D_e$$

$$L^b_{1_\alpha}(\zeta) := \int_0^L \left[\int_\omega f_\alpha \,\mathrm{d}\hat{x} + \int_{\gamma_{mN}} g_\alpha \,\mathrm{d}\hat{s} \right] \zeta \,\mathrm{d}x_3 - \left[\int_\omega x_\alpha f_3 \,\mathrm{d}\hat{x} + \int_{\gamma_{mN}} x_\alpha g_3 \,\mathrm{d}\hat{s} \right] \frac{\mathrm{d}\zeta}{\mathrm{d}x_3} \,\mathrm{d}x_3$$
$$+ (2 - \sharp D_m) \int_\omega \left[g_\alpha(\hat{x}, l_m)\zeta(l_m) - x_\alpha g_3(\hat{x}, l_m) \frac{\mathrm{d}\zeta}{\mathrm{d}x_3}(l_m) \right] \mathrm{d}\hat{s}$$

$$L^s_2(u^s) := \int_0^L \left[\int_\omega f_3 \,\mathrm{d}\hat{x} + \int_{\gamma_{mN}} g_3 \,\mathrm{d}\hat{s} \right] u^s \,\mathrm{d}x_3 + (2 - \sharp D_m) \int_\omega g_3(\hat{x}, l_m) u^s(l_m) \,\mathrm{d}\hat{s} - \int_\Omega T^0_{2(m,3)} \frac{\mathrm{d}u^s}{\mathrm{d}x_3} \,\mathrm{d}x$$

$$L^b_{2_\alpha}(\zeta) := \int_0^L \left[\int_\omega f_\alpha \,\mathrm{d}\hat{x} + \int_{\gamma_{mN}} g_\alpha \,\mathrm{d}\hat{s} \right] \zeta \,\mathrm{d}x_3 - \left[\int_\omega x_\alpha f_3 \,\mathrm{d}\hat{x} + \int_{\gamma_{mN}} x_\alpha g_3 \,\mathrm{d}\hat{s} \right] \frac{\mathrm{d}\zeta}{\mathrm{d}x_3} \,\mathrm{d}x_3$$
$$+ (2 - \sharp D_m) \int_\omega \left[g_\alpha(\hat{x}, l_m)\zeta(l_m) - x_\alpha g_3(\hat{x}, l_m) \frac{\mathrm{d}\zeta}{\mathrm{d}x_3}(l_m) \right] \mathrm{d}\hat{s} + \int_\Omega x_\alpha T^0_{2(m,3)} \frac{\mathrm{d}^2\zeta}{\mathrm{d}x_3^2} \,\mathrm{d}x$$

As in [9] and [10], we can show that $\tilde{M}_1$ keeps the same structure than M^ε in spite of the dimension reduction process, i.e. $\tilde{M}_{1_{12}} = -\tilde{M}_{1_{21}}$. When $p = 1$, there is a decoupling between mechanical and electrical equations for the classes 222, 32, $\bar{4}$, 422, $\bar{4}2m$, $\bar{6}$, 622, $\bar{6}m2$ and 23, so that $\tilde{M}_1$ is symmetric. Moreover, $\tilde{M}_{1_{11}}$ is a purely mechanical entry, whereas $\tilde{M}_{1_{22}}$ is purely dielectric for classes 32, 422, $\bar{6}$, 622 and $\bar{6}m2$. For all symmetry classes, $\tilde{M}_2 = \tilde{M}_{1_{11}}$ and consequently is a purely mechanical coefficient.

As usual (see [9] and [10]), models of piezoelectric slender rods with cross section $\varepsilon\omega$ are obtained by a descaling of $\bar{\mathsf{P}}(\Omega)_p$ and $\bar{\mathcal{P}}(\Omega)_p$; obviously, these models have the same properties concerning the reduction of state variables and the decouplings. In [12], the case $p = 2$ (which corresponds to the situation when the rod is used as a sensor) has been treated in a particular class of anisotropy and with $w = 0$. Here, we prove the uniqueness of the solution of the limit problems $\bar{\mathsf{P}}(\Omega)_p$ and $\bar{\mathcal{P}}(\Omega)_p$, moreover we reduce the state variables to the sole displacement and exhibit the cases when the problem is local. We also propose an additional and more complex model which corresponds to actuators.

References

[1] P.G. Ciarlet, Mathematical Elasticity, vol. II, North-Holland, 1997.
[2] F. Murat, A. Sili, Comportement asymptotique des solutions du système de l'élasticité linéarisée anisotrope hétérogène dans des cylindres minces, C. R. Acad. Sci. Paris, Sér. I 328 (1999) 179–184.
[3] F. Murat, A. Sili, Effets non locaux dans le passage 3d-1d en élasticité linéarisée anisotrope hétérogène, C. R. Acad. Sci. Paris, Sér. I 330 (2000) 745–750.
[4] A. Sili, Analyse asymptotique de quelques problèmes de conduction et d'élasticité dans des cylindres minces, Mémoire d'habilitation à diriger des recherches, Université de Toulon et du Var, 2000.
[5] M. Lenczner, Modèle d'assemblage de plaques piézoélectriques, Rapport d'activité du Laboratoire de Calcul Scientifique, Equipe de Mathématiques, Université de Franche-Comté, Besançon, 1993.
[6] G.A. Maugin, D. Attou, An asymptotic theory of thin piezoelectric plates, Quart. J. Mech. Appl. Math. 43 (1993) 519–524.
[7] M. Rahmoune, A. Benjeddou, R. Ohayon, New thin piezoelectric plates models, J. Intelligent Material Systems and Structures 9 (1998) 1017–1029.
[8] A. Sène, Modélisation asymptotique de plaques: contrôlabilité exacte frontière, piezoélectricité, Thèse de l'Université Joseph Fourier – Grenoble I, 2000.
[9] T. Weller, C. Licht, Analyse asymptotique de plaques minces linéairement piézoélectriques, C. R. Acad. Sci. Paris, Sér. I 335 (2002) 309–314.
[10] T. Weller, C. Licht, Modeling of linearly electromagneto-elastic thin plates, C. R. Mecanique 335 (2007) 201–206.
[11] C. Maurini, Piezoelectric composites for distributed passive electric control: beam modelling, modal analysis, and experimental implementation, Thèse de l'Université Paris VI, 2005.
[12] I.M.N. Figueiredo, C.M.F. Franco Leal, A generalized piezoelectric Bernoulli–Navier anisotropic rod model, J. Elasticity 85 (2006) 85–106.