Revisit to the theoretical analysis of a classical piezoelectric cantilever energy harvester

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1 Summary of the interested equations

The dynamic equations for a typical piezoelectric composite cantilever beam is

$$B_p \frac{\partial^4 w(x,t)}{\partial x^4} + m_p \frac{\partial^2 w(x,t)}{\partial t^2} = 0, \tag{1}$$

where B_p is the equivalent bending stiffness and m_p is the line mass density of the piezoelectric cantilever beam. If the piezoelectric elements attached to the cantilever beam is connected to an external electrical load R_l, we have

$$\frac{dQ_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{2}$$

For the underlying physics, we have the following constitutive equations

$$\begin{aligned} M_p(x,t) &= B_p \frac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\ q_p(x,t) &= e_p \frac{\partial^2 w(x,t)}{\partial x^2} + \varepsilon_p V_p(t), \end{aligned} \tag{3}$$

or equivalently,

$$\begin{cases} M_p(x,t) = B_p \dfrac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\ Q_p(x,t) = e_p \left[\dfrac{\partial w(x,t)}{\partial x} \right] \Bigg|_0^{l_p} + C_p V_p(t). \end{cases} \tag{4}$$

One end of the cantilever beam is fixed while the other end is free. So the boundary conditions are

$$\begin{cases} w(0,t) = w_b(t), \\ \dfrac{\partial w(0,t)}{\partial x} = 0, \end{cases} \tag{5}$$

and

$$\begin{cases} M_p(l_p,t) = B_p \dfrac{\partial^2 w(l_p,t)}{\partial x^2} - e_p V_p(t) = 0, \\ N_p(l_p,t) = \dfrac{\partial M_p(l_p,t)}{\partial x} = B_p \dfrac{\partial^3 w(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{6}$$

In the classical energy harvesting applications, the cantilever beam is subject to a periodical base excitation $w_b(t)$. Thus the dynamic response of the cantilever beam is decomposed as

$$w(x,t) = w_b(t) + w_{rel}(x,t), \tag{7}$$

where $w_{rel}(x,t)$ is the relative displacement function of the cantilever beam. In this way, the system is converted into

$$B_p \frac{\partial^4 w_{rel}(x,t)}{\partial x^4} + m_p \frac{\partial^2 w_{rel}(x,t)}{\partial t^2} = -m_p \frac{\partial^2 w_b(t)}{\partial t^2}, \tag{8}$$

$$e_p \left[\frac{\partial^2 w_{rel}(x,t)}{\partial x \partial t} \right] \Bigg|_0^{l_p} + C_p \frac{dV_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{9}$$

$$
\begin{cases} w_{rel}(0,t) = 0, \\ \dfrac{\partial w_{rel}(0,t)}{\partial x} = 0, \end{cases} \tag{10}
$$

and

$$
\begin{cases} B_p \dfrac{\partial^2 w_{rel}(l_p,t)}{\partial x^2} - e_p V_p(t) = 0, \\ \dfrac{\partial^3 w_{rel}(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{11}
$$

Considering a sinusoidal base excitation

$$
w_b(t) = \eta_b e^{j\sigma_b t} \tag{12}
$$

where ξ_b is usually a real vibration amplitude, the steady state solution for the above system can be reasonably set as

$$
w_{rel}(x,t) = \eta_{rel}(x) e^{j\sigma_b t}, \quad V_p(t) = \tilde{V}_p e^{j\sigma_b t}, \tag{13}
$$

where $\eta_{rel}(x)$ and $\tilde{V}_p$ are complex amplitudes. Then the above system is again simplified as

$$
B_p \frac{\partial^4 \eta_{rel}(x)}{\partial x^4} - m_p \sigma_b^2 \eta_{rel}(x) = m_p \sigma_b^2 \eta_b, \tag{14}
$$

$$
\begin{cases} \eta_{rel}(0) = 0, \\ \dfrac{\partial \eta_{rel}(0)}{\partial x} = 0, \end{cases} \tag{15}
$$

and

$$
\begin{cases} B_p \dfrac{\partial^2 \eta_{rel}(l_p)}{\partial x^2} + \dfrac{j\sigma_b R_l}{1 + j\sigma_b C_p R_l} e_p^2 \dfrac{\partial \eta_{rel}(l_p)}{\partial x} = 0, \\ \dfrac{\partial^3 \eta_{rel}(l_p)}{\partial x^3} = 0. \end{cases} \tag{16}
$$

Note that here we assume a sinusoidal steady state response, which is not actually validated theoretically.

Obviously we can have the following dimensionless scheme:

$$
\eta_{rel} \sim u\eta_b, \quad x \sim z l_p \tag{17}
$$

and therefore the following dimensionless parameters

$$
\sigma = \sigma_b \sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \delta = \frac{e_p^2 l_p}{C_p B_p}. \tag{18}
$$

Now, we reach the following dimensionless system of boundary value problem

$$
\begin{cases} u'''' - \sigma^2 u = \sigma^2, \\ u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \dfrac{j\beta\sigma}{1 + j\beta\sigma} \delta u'(1) = 0, \\ u'''(1) = 0, \end{cases} \tag{19}
$$

where the prime denotes the derivative with respect to z. The analytical solution to this problem can be formulated as

$$
u(z;\delta) = A_\delta \cos\sqrt{\sigma}z + B_\delta \sin\sqrt{\sigma}z + C_\delta \cosh\sqrt{\sigma}z + D_\delta \sinh\sqrt{\sigma}z - 1 \tag{20}
$$

and hence

$$
\begin{aligned} u'(z;\delta) &= \sigma^{1/2} \left(-A_\delta \sin\sqrt{\sigma}z + B_\delta \cos\sqrt{\sigma}z + C_\delta \sinh\sqrt{\sigma}z + D_\delta \cosh\sqrt{\sigma}z\right), \\ u''(z;\delta) &= \sigma \left(-A_\delta \cos\sqrt{\sigma}z - B_\delta \sin\sqrt{\sigma}z + C_\delta \cosh\sqrt{\sigma}z + D_\delta \sinh\sqrt{\sigma}z\right), \\ u'''(z;\delta) &= \sigma^{3/2} \left(A_\delta \sin\sqrt{\sigma}z - B_\delta \cos\sqrt{\sigma}z + C_\delta \sinh\sqrt{\sigma}z + D_\delta \cosh\sqrt{\sigma}z\right). \end{aligned} \tag{21}
$$

The coefficients A_δ, B_δ, C_δ, and D_δ are then subject to the following linear system of equations:

$$\begin{cases} A_\delta + C_\delta = 1, \\ B_\delta + D_\delta = 0, \\ \left(-A_\delta \cos\sqrt{\sigma} - B_\delta \sin\sqrt{\sigma} + C_\delta \cosh\sqrt{\sigma} + D_\delta \sinh\sqrt{\sigma}\right) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\delta\left(-A_\delta \sin\sqrt{\sigma} + B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma}\right) = 0, \\ A_\delta \sin\sqrt{\sigma} - B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma} = 0. \end{cases} \tag{22}$$

Analytically, we can directly obtain the solution to this problem as

$$\begin{cases} A_\delta = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ B_\delta = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ C_\delta = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ D_\delta = \dfrac{-\cos\sqrt{\sigma}\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}. \end{cases} \tag{23}$$

According to equations (20) and (23), the dimensionless displacement amplitude function $u(z)$ is totally determined by the three dimensionless parameters σ, β, and δ introduced before. Among the dimensionless parameters, σ is the dimensionless base excitation frequency, β is the dimensionless electrical resonant frequency, and δ is the dimensionless electromechanical coupling strength for the structure. As σ and β is determined by the base excitation and externally connected circuit respectively, only the parameter δ is fully determined by the structure itself. Hence we would like to investigate the influence of parameter δ upon the solution displacement function $u(z)$. By taking different values of δ , we calculate the displacement amplitude function $u(z)$ and plot the results in Figure 1.

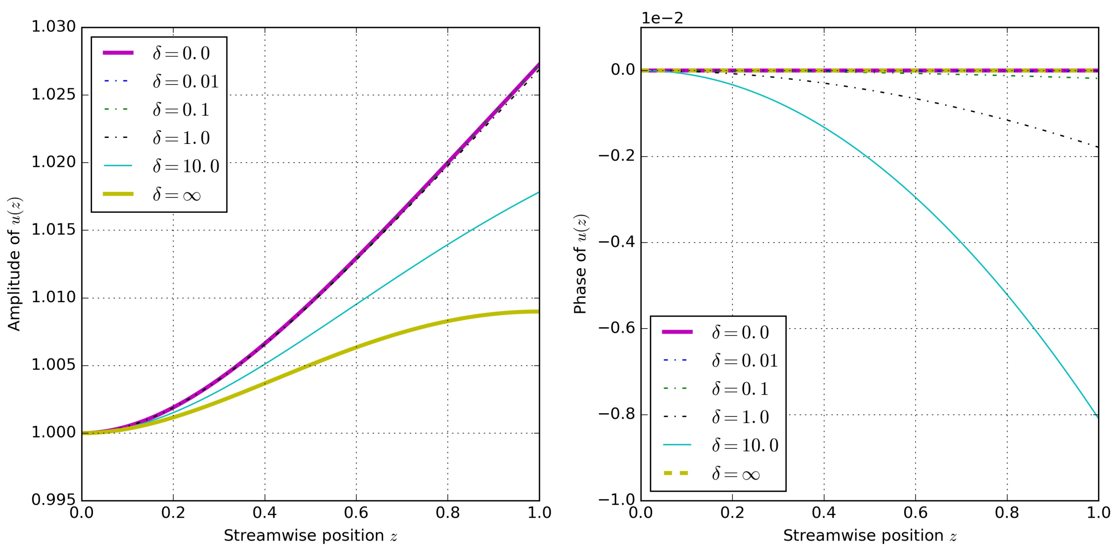

Figure 1: Amplitude and phase of the displacement function $u(z)$ for difference values of δ

It is shown in Figure 1, the parameter δ changes the function $u(z)$ through the change of the third boundary condition (to be inserted). When δ is zero, i.e., no electromechanical coupling is present, the system degenerates to the classical elastic cantilever beam problem, whose solution is a real function. That is to say, the phase of $u(z)$ is a constant across the whole beam (in the range of $0 \leq z \leq 1$). Analytical expressions for the coefficients are

$$\begin{cases} A_\varnothing = \dfrac{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}-\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\ B_\varnothing = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\ C_\varnothing = \dfrac{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\ D_\varnothing = \dfrac{-\cos\sqrt{\sigma}\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}. \end{cases} \tag{24}$$

and the resulting dimensionless displacement function $u_\varnothing(z)$ is represented as

$$u_\varnothing(z) = A_\varnothing \cos\sqrt{\sigma}z + B_\varnothing \sin\sqrt{\sigma}z + C_\varnothing \cosh\sqrt{\sigma}z + D_\varnothing \sinh\sqrt{\sigma}z - 1. \tag{25}$$

When the electromechanical coupling is extremely strong, and δ is extremely large and can be seen as ∞ in mathematical sense. In this situation, the solution $u_\infty(z)$ is again real without any phase difference in the z direction. The coefficients can be analytically expressed as

$$\begin{cases} A_\infty = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ B_\infty = \dfrac{\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ C_\infty = \dfrac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ D_\infty = \dfrac{-\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}. \end{cases} \tag{26}$$

and hence the dimensionless displacement function $u_\infty(z)$ is

$$u_\infty(z) = A_\infty \cos\sqrt{\sigma}z + B_\infty \sin\sqrt{\sigma}z + C_\infty \cosh\sqrt{\sigma}z + D_\infty \sinh\sqrt{\sigma}z - 1. \tag{27}$$

While a finite non-zero electromechanical coupling factor δ is present, which is expected in most applications, the resulting dimensionless displacement function $u(z)$ has varying magnitude and phase along the stream-wise direction or z direction. Nevertheless, it is seen from the right panel of Figure 1 that for different values of δ, the phase change of $u(z)$ is very small in the z direction, actually in the order 10^{-2}.

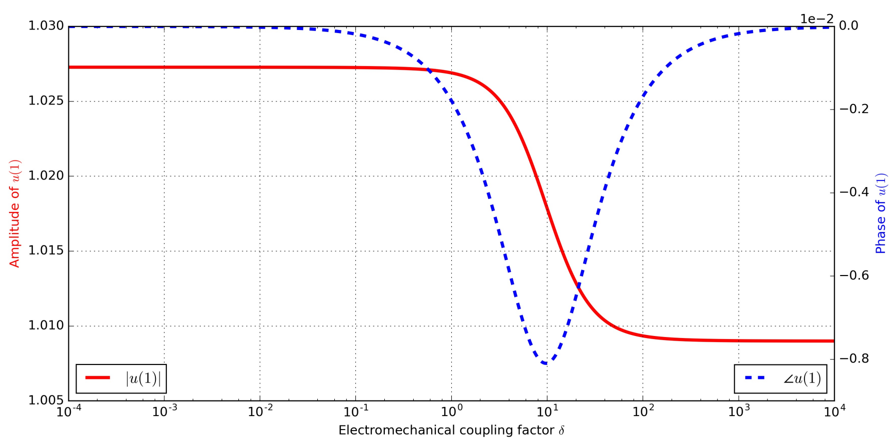

Figure 2: Amplitude and phase of the displacement function $u(z)$ at the position $z = 1$ versus electromechanical coupling factor δ.

To make it more clear, we plot the phase of $u(z)$ at $z = 1$ versus different values of δ in Figure 2. It is clear that with the increase of δ, amplitude of the end displamcent ($z = 1$) of the beam

$|u(z)|$ decreases, while its phase reaches a minimum at around $\delta = 10$. This also explains the fact expressed in Figure 1 that the amplitude of displacement function $u_\delta(z)$ with $0 < \delta < \infty$ is always between that of $u_\varnothing(z)$ and $u_\infty(z)$.

As for the output voltage $V_p(t)$, output current $I_p(t)$, and output power $P_p(t)$ for the classical piezoelectric cantilever energy harvester, their corresponding complex amplitudes $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ can be formulated as

$$\begin{cases} \tilde{V}_p = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}u'(1), \\ \quad = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}\sigma^{1/2}\left(-A_\delta \sin\sqrt{\sigma} + B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma}\right) \\ \quad = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}\dfrac{\sqrt{\sigma}\left(\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}\right)}{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)} \\ \quad = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\left(\dfrac{\eta_b}{l_p}\right)\left(\dfrac{e_p}{C_p}\right)\chi_p, \\ \tilde{I}_p = \tilde{V}_p/R_l = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\left(\dfrac{\eta_b}{l_p}\right)\left(\dfrac{e_p}{C_pR_l}\right)\chi_p, \\ \tilde{P}_p = \tilde{V}_p^2/R_l = \left(\dfrac{\eta_b}{l_p}\right)^2\left(\dfrac{e_p}{C_p}\right)\left(\dfrac{e_p}{C_pR_l}\right)\left(\dfrac{j\sigma\beta}{j\sigma\beta+1}\right)^2\chi_p^2, \end{cases} \tag{28}$$

in which we have used the notations that

$$\chi_p = u_1'(1) = \frac{\sqrt{\sigma}\left(\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}\right)}{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)}. \tag{29}$$

Clearly, The three output measures $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ are heavily dependent on another dimensionless parameter $r_d = \eta_b/l_p$. Formally, both $\tilde{V}_p$ and $\tilde{I}_p$ depend lineary upon r_d, while $\tilde{P}_p$ shows a quadratic dependence on r_d. However, it should be noted that parameter δ relies on e_p, l_p, C_p, and B_p, while the three measures $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ are dimensional values and depend on e_p, σ_b, and R_l. As a result, the change of parameter δ results in the change of reference voltage e_p/C_p, reference current $e_p/(C_pR_l)$, and reference power $(e_p/C_p)[e_p/(C_pR_l)]$. Hence, we can relate the change of value δ to that of e_p, which in turn determines all the three output measures. Then by taking a series of values of δ, we obtain the corresponding output measures and plot their amplitudes in Figure 3.

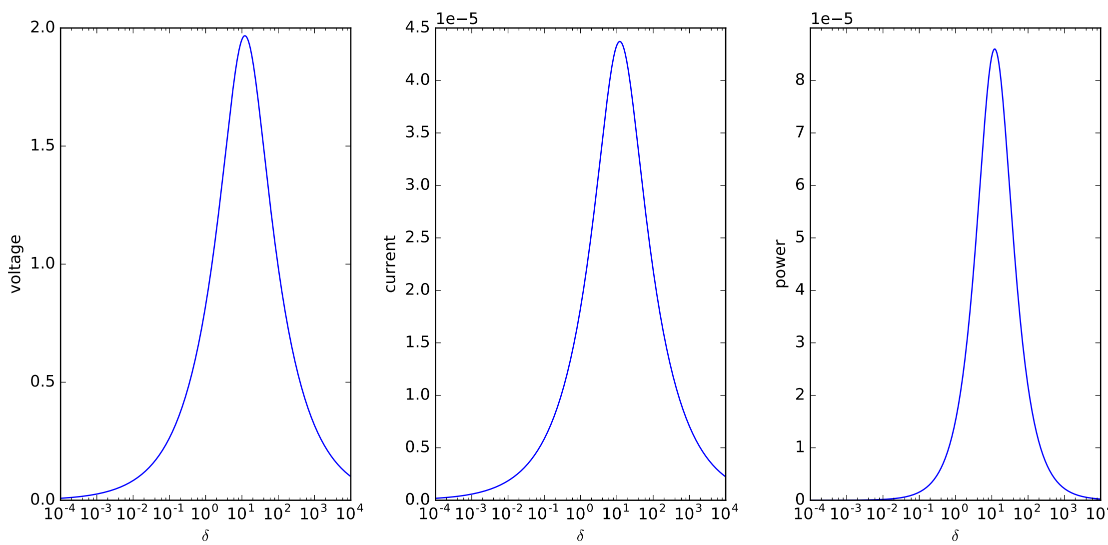

Figure 3: Voltage, current and power output for the piezoelectric cantilever energy harvester

It is seen from Figure 3 that all the three measures show a maximum peak with the increase of δ at the approximate value of $\delta = 10$.

Using the following regular expansion:

$$
\begin{cases}
A_\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots, \\
B_\epsilon = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \cdots, \\
C_\epsilon = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \cdots, \\
D_\epsilon = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots,
\end{cases}
\tag{30}
$$

we obtain the successive expansion problem:
$O(\epsilon^0)$:

$$
\begin{cases}
A_0 + C_0 = 1, \\
B_0 + D_0 = 0, \\
-A_0 \cos\sqrt{\sigma} - B_0 \sin\sqrt{\sigma} + C_0 \cosh\sqrt{\sigma} + D_0 \sinh\sqrt{\sigma} = 0, \\
A_0 \sin\sqrt{\sigma} - B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma} = 0.
\end{cases}
\tag{31}
$$

The solution is

$$
\begin{cases}
A_0 = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\
B_0 = \dfrac{\cosh\sqrt{\sigma}\sin\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\
C_0 = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\
D_0 = -\dfrac{\cosh\sqrt{\sigma}\sin\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}}
\end{cases}
\tag{32}
$$

Hence we have

$$
-A_0 \sin\sqrt{\sigma} + B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma} = \frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}
\tag{33}
$$

$O(\epsilon^1)$:

$$
\begin{cases}
A_1 + C_1 = 0, \\
B_1 + D_1 = 0, \\
\left(-A_1 \cos\sqrt{\sigma} - B_1 \sin\sqrt{\sigma} + C_1 \cosh\sqrt{\sigma} + D_1 \sinh\sqrt{\sigma}\right) + \\
\dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1}\left(-A_0 \sin\sqrt{\sigma} + B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma}\right) = 0, \\
A_1 \sin\sqrt{\sigma} - B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma} = 0.
\end{cases}
\tag{34}
$$

The solution is

$$
\begin{cases}
A_1 = \dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\
B_1 = \dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\
C_1 = \dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(-\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\
D_1 = \dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right)
\end{cases}
\tag{35}
$$

Then we have

$$
\begin{aligned}
&-A_1 \sin\sqrt{\sigma} + B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma} \\
&= \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\frac{\sin\sqrt{\sigma} - \sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)
\end{aligned}
\tag{36}
$$

$O(\epsilon^2)$:

$$\left\{\begin{aligned} A_2+C_2&=0,\\ B_2+D_2&=0,\\ \left(-A_2\cos\sqrt{\sigma}-B_2\sin\sqrt{\sigma}+C_2\cosh\sqrt{\sigma}+D_2\sinh\sqrt{\sigma}\right)&+\\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\left(-A_1\sin\sqrt{\sigma}+B_1\cos\sqrt{\sigma}+C_1\sinh\sqrt{\sigma}+D_1\cosh\sqrt{\sigma}\right)&=0,\\ A_2\sin\sqrt{\sigma}-B_2\cos\sqrt{\sigma}+C_2\sinh\sqrt{\sigma}+D_2\cosh\sqrt{\sigma}&=0.\end{aligned}\right. \tag{37}$$

The solution is

$$\left\{\begin{aligned} A_2&=\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)\\ B_2&=\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{-\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)\\ C_2&=\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(-\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)\\ D_2&=\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{-\sin\sqrt{\sigma}+\sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)\end{aligned}\right. \tag{38}$$

To get higher order expansions, we can use the following iteration method:
$O(\epsilon^{k+1})$ $(k\geq 1)$:

$$\left\{\begin{aligned} A_{k+1}+C_{k+1}&=0,\\ B_{k+1}+D_{k+1}&=0,\\ \left(-A_{k+1}\cos\sqrt{\sigma}-B_{k+1}\sin\sqrt{\sigma}+C_{k+1}\cosh\sqrt{\sigma}+D_{k+1}\sinh\sqrt{\sigma}\right)&+\\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\left(-A_k\sin\sqrt{\sigma}+B_k\cos\sqrt{\sigma}+C_k\sinh\sqrt{\sigma}+D_k\cosh\sqrt{\sigma}\right)&=0,\\ A_{k+1}\sin\sqrt{\sigma}-B_{k+1}\cos\sqrt{\sigma}+C_{k+1}\sinh\sqrt{\sigma}+D_{k+1}\cosh\sqrt{\sigma}&=0.\end{aligned}\right. \tag{39}$$

The solution is

$$\left\{\begin{aligned} A_{k+1}&=\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k)\\ B_{k+1}&=\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\frac{-\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k)\\ C_{k+1}&=\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(-\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k)\\ D_{k+1}&=\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\frac{-\sin\sqrt{\sigma}+\sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k)\end{aligned}\right. \tag{40}$$

where for $k\geq 2$

$$Q_k=-A_k\sin\sqrt{\sigma}+B_k\cos\sqrt{\sigma}+C_k\sinh\sqrt{\sigma}+D_k\cosh\sqrt{\sigma}, \tag{41}$$

and for $k\geq 0$

$$\begin{aligned} Q_{k+1}&=-A_{k+1}\sin\sqrt{\sigma}+B_{k+1}\cos\sqrt{\sigma}+C_{k+1}\sinh\sqrt{\sigma}+D_{k+1}\cosh\sqrt{\sigma}\\ &=-\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}+\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)Q_k,\end{aligned} \tag{42}$$

and

$$\begin{aligned} Q_1&=-A_1\sin\sqrt{\sigma}+B_1\cos\sqrt{\sigma}+C_1\sinh\sqrt{\sigma}+D_1\cosh\sqrt{\sigma}\\ &=\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\frac{\sin\sqrt{\sigma}-\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\end{aligned} \tag{43}$$

$$Q_0=\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1} \tag{44}$$

Hence it is shown that for $k\geq 0$

$$\begin{aligned} Q_k&=-\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}+\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)Q_k\\ &=\left[-\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}+\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\right]^k\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\end{aligned} \tag{45}$$

As a result, we obtain that for $k \geq 0$

$$\begin{cases} A_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1} \left(\frac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^{k} \left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right) \left(\frac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1} \left(\frac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^{k} \left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right) \left(\frac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1} \left(\frac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^{k} \left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right) \left(\frac{-\cos\sqrt{\sigma} - \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1} \left(\frac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^{k} \left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right) \left(\frac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \end{cases} \tag{46}$$

Reference

References