Revisit to the theoretical analysis of a classical piezoelectric cantilever energy harvester

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1 Introduction

The soaring depletion of fossil fuels and the growing awareness of environmental protection renders the challenge of renewable and sustainable energy sources for scientists and engineers. Apart from nuclear energy, wind energy, and solar energy, ambient energy harvesting has been proposed as an alternative to batteries for more than twenty years. Numerous principles, mechanisms, implementations, and applications have been put forward to deepen the understanding of energy harvesting, to find the routine for the development of new devices, to apply the devices to new situations. Among all these devices, piezoelectric energy harvesting devices have attracted much attention due to their simplicity of structure and superiority of performance.

Generally a piezoelectric energy harvesting device consists of some piezoelectric elements and some vibration transduction structure for the hosting of these elements. A key problem in the research of piezoelectric element based energy harvesting is to improve the output of those devices. A key point is to understand the mechanics and coupling behind the devices.

2 Summary of the interested equations

The dynamic equations for a typical piezoelectric composite cantilever beam is

$$B_p \frac{\partial^4 w(x,t)}{\partial x^4} + m_p \frac{\partial^2 w(x,t)}{\partial t^2} = 0, \tag{1}$$

where B_p is the equivalent bending stiffness and m_p is the line mass density of the piezoelectric cantilever beam. If the piezoelectric elements attached to the cantilever beam is connected to an external electrical load R_l, we have

$$\frac{dQ_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{2}$$

For the underlying physics, we have the following constitutive equations

$$\begin{aligned} M_p(x,t) &= B_p \frac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\ q_p(x,t) &= e_p \frac{\partial^2 w(x,t)}{\partial x^2} + \varepsilon_p V_p(t), \end{aligned} \tag{3}$$

or equivalently,

$$\begin{cases} M_p(x,t) = B_p \dfrac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\ Q_p(x,t) = e_p \left[\dfrac{\partial w(x,t)}{\partial x} \right] \Bigg|_0^{l_p} + C_p V_p(t). \end{cases} \tag{4}$$

One end of the cantilever beam is fixed while the other end is free. So the boundary conditions are

$$\begin{cases} w(0,t) = w_b(t), \\ \dfrac{\partial w(0,t)}{\partial x} = 0, \end{cases} \tag{5}$$

and

$$\begin{cases} M_p(l_p,t) = B_p\dfrac{\partial^2 w(l_p,t)}{\partial x^2} - e_pV_p(t) = 0, \\ N_p(l_p,t) = \dfrac{\partial M_p(l_p,t)}{\partial x} = B_p\dfrac{\partial^3 w(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{6}$$

3 Theoretical solution to the problem

In the classical energy harvesting applications, the cantilever beam is subject to a periodical base excitation $w_b(t)$. Thus the dynamic response of the cantilever beam is decomposed as

$$w(x,t) = w_b(t) + w_{rel}(x,t), \tag{7}$$

where $w_{rel}(x,t)$ is the relative displacement function of the cantilever beam. In this way, the system is converted into

$$B_p\frac{\partial^4 w_{rel}(x,t)}{\partial x^4} + m_p\frac{\partial^2 w_{rel}(x,t)}{\partial t^2} = -m_p\frac{\partial^2 w_b(t)}{\partial t^2}, \tag{8}$$

$$e_p\left[\frac{\partial^2 w_{rel}(x,t)}{\partial x\partial t}\right]\Bigg|_0^{l_p} + C_p\frac{dV_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{9}$$

$$\begin{cases} w_{rel}(0,t) = 0, \\ \dfrac{\partial w_{rel}(0,t)}{\partial x} = 0, \end{cases} \tag{10}$$

and

$$\begin{cases} B_p\dfrac{\partial^2 w_{rel}(l_p,t)}{\partial x^2} - e_pV_p(t) = 0, \\ \dfrac{\partial^3 w_{rel}(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{11}$$

Considering a sinusoidal base excitation

$$w_b(t) = \eta_b e^{j\sigma_b t} \tag{12}$$

where ξ_b is usually a real vibration amplitude, the steady state solution for the above system can be reasonably set as

$$w_{rel}(x,t) = \eta_{rel}(x)e^{j\sigma_b t}, \quad V_p(t) = \tilde{V}_p e^{j\sigma_b t}, \tag{13}$$

where $\eta_{rel}(x)$ and $\tilde{V}_p$ are complex amplitudes. Then the above system is again simplified as

$$B_p\frac{\partial^4 \eta_{rel}(x)}{\partial x^4} - m_p\sigma_b^2\eta_{rel}(x) = m_p\sigma_b^2\eta_b, \tag{14}$$

$$\begin{cases} \eta_{rel}(0) = 0, \\ \dfrac{\partial \eta_{rel}(0)}{\partial x} = 0, \end{cases} \tag{15}$$

and

$$\begin{cases} B_p\dfrac{\partial^2 \eta_{rel}(l_p)}{\partial x^2} + \dfrac{j\sigma_b R_l}{1 + j\sigma_b C_p R_l}e_p^2\dfrac{\partial \eta_{rel}(l_p)}{\partial x} = 0, \\ \dfrac{\partial^3 \eta_{rel}(l_p)}{\partial x^3} = 0. \end{cases} \tag{16}$$

Note that here we assume a sinusoidal steady state response, which is not actually validated theoretically.

Obviously we can have the following dimensionless scheme:

$$\eta_{rel} \sim u\eta_b, \quad x \sim zl_p \tag{17}$$

and therefore the following dimensionless parameters

$$\sigma = \sigma_b\sqrt{\frac{m_pl_p^4}{B_p}}, \quad \beta = R_lC_p\sqrt{\frac{B_p}{m_pl_p^4}}, \quad \delta = \frac{e_p^2l_p}{C_pB_p}. \tag{18}$$

Now, we reach the following dimensionless system of boundary value problem

$$\begin{cases} u'''' - \sigma^2 u = \sigma^2, \\ u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \dfrac{j\beta\sigma}{1+j\beta\sigma}\delta u'(1) = 0, \\ u'''(1) = 0, \end{cases} \tag{19}$$

where the prime denotes the derivative with respect to z. The analytical solution to this problem can be formulated as

$$u(z;\delta) = A_\delta \cos\sqrt{\sigma}z + B_\delta \sin\sqrt{\sigma}z + C_\delta \cosh\sqrt{\sigma}z + D_\delta \sinh\sqrt{\sigma}z - 1 \tag{20}$$

and hence

$$\begin{aligned} u'(z;\delta) &= \sigma^{1/2}\left(-A_\delta \sin\sqrt{\sigma}z + B_\delta \cos\sqrt{\sigma}z + C_\delta \sinh\sqrt{\sigma}z + D_\delta \cosh\sqrt{\sigma}z\right), \\ u''(z;\delta) &= \sigma\left(-A_\delta \cos\sqrt{\sigma}z - B_\delta \sin\sqrt{\sigma}z + C_\delta \cosh\sqrt{\sigma}z + D_\delta \sinh\sqrt{\sigma}z\right), \\ u'''(z;\delta) &= \sigma^{3/2}\left(A_\delta \sin\sqrt{\sigma}z - B_\delta \cos\sqrt{\sigma}z + C_\delta \sinh\sqrt{\sigma}z + D_\delta \cosh\sqrt{\sigma}z\right). \end{aligned} \tag{21}$$

The coefficients A_δ, B_δ, C_δ, and D_δ are then subject to the following linear system of equations:

$$\begin{cases} A_\delta + C_\delta = 1, \\ B_\delta + D_\delta = 0, \\ \left(-A_\delta \cos\sqrt{\sigma} - B_\delta \sin\sqrt{\sigma} + C_\delta \cosh\sqrt{\sigma} + D_\delta \sinh\sqrt{\sigma}\right) + \\ \dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\delta\left(-A_\delta \sin\sqrt{\sigma} + B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma}\right) = 0, \\ A_\delta \sin\sqrt{\sigma} - B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma} = 0. \end{cases} \tag{22}$$

Analytically, we can directly obtain the solution to this problem as

$$\begin{cases} A_\delta = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ B_\delta = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ C_\delta = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ D_\delta = \dfrac{-\cos\sqrt{\sigma}\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}. \end{cases} \tag{23}$$

According to equations (20) and (23), the dimensionless displacement amplitude function $u(z)$ is totally determined by the three dimensionless parameters σ, β, and δ introduced before. Among the dimensionless parameters, σ is the dimensionless base excitation frequency, β is the dimensionless electrical resonant frequency, and δ is the dimensionless electromechanical coupling strength for the structure. As σ and β is determined by the base excitation and externally connected circuit respectively, only the parameter δ is fully determined by the structure itself. Hence we would like to investigate the influence of parameter δ upon the solution displacement function $u(z)$. By taking different values of δ , we calculate the displacement amplitude function $u(z)$ and plot the results in Figure 1.

It is shown in Figure 1, the parameter δ changes the function $u(z)$ through the change of the third boundary condition (to be inserted). When δ is zero, i.e., no electromechanical coupling is present, the system degenerates to the classical elastic cantilever beam problem, whose solution is

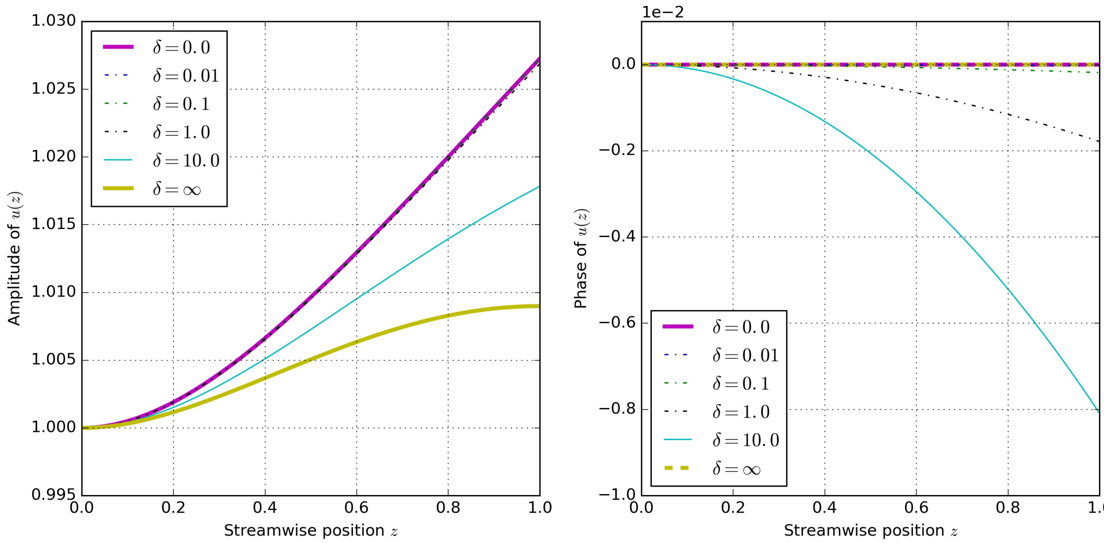

Figure 1: Amplitude and phase of the displacement function $u(z)$ for difference values of δ

a real function. That is to say, the phase of $u(z)$ is a constant across the whole beam (in the range of $0 \le z \le 1$). Analytical expressions for the coefficients are

$$\begin{cases} A_\varnothing = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\ B_\varnothing = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\ C_\varnothing = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\ D_\varnothing = \dfrac{-\cos\sqrt{\sigma}\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}. \end{cases} \tag{24}$$

and the resulting dimensionless displacement function $u_\varnothing(z)$ is represented as

$$u_\varnothing(z) = A_\varnothing \cos\sqrt{\sigma}z + B_\varnothing \sin\sqrt{\sigma}z + C_\varnothing \cosh\sqrt{\sigma}z + D_\varnothing \sinh\sqrt{\sigma}z - 1. \tag{25}$$

When the electromechanical coupling is extremely strong, and δ is extremely large and can be seen as ∞ in mathematical sense. In this situation, the solution $u_\infty(z)$ is again real without any phase difference in the z direction. The coefficients can be analytically expressed as

$$\begin{cases} A_\infty = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ B_\infty = \dfrac{\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ C_\infty = \dfrac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ D_\infty = \dfrac{-\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}. \end{cases} \tag{26}$$

and hence the dimensionless displacement function $u_\infty(z)$ is

$$u_\infty(z) = A_\infty \cos\sqrt{\sigma}z + B_\infty \sin\sqrt{\sigma}z + C_\infty \cosh\sqrt{\sigma}z + D_\infty \sinh\sqrt{\sigma}z - 1. \tag{27}$$

While a finite non-zero electromechanical coupling factor δ is present, which is expected in most applications, the resulting dimensionless displacement function $u(z)$ has varying magnitude and phase along the stream-wise direction or z direction. Nevertheless, it is seen from the right panel of Figure 1 that for different values of δ, the phase change of $u(z)$ is very small in the z direction, actually in the order 10^{-2}.

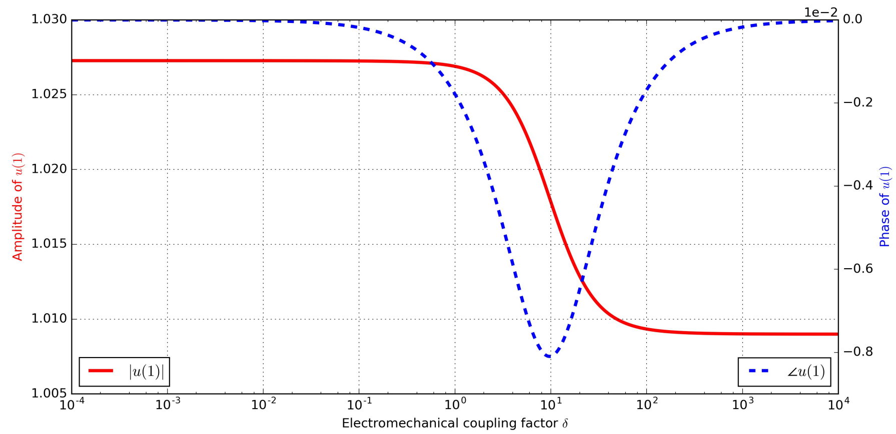

Figure 2: Amplitude and phase of the displacement function $u(z)$ at the position $z = 1$ versus electromechanical coupling factor δ.

To make it more clear, we plot the phase of $u(z)$ at $z = 1$ versus different values of δ in Figure 2. It is clear that with the increase of δ, amplitude of the end displamcent ($z = 1$) of the beam $|u(z)|$ decreases, while its phase reaches a minimum at around $\delta = 10$. This also explains the fact expressed in Figure 1 that the amplitude of displacement function $u_\delta(z)$ with $0 < \delta < \infty$ is always between that of $u_\varnothing(z)$ and $u_\infty(z)$.

As for the output voltage $V_p(t)$, output current $I_p(t)$, and output power $P_p(t)$ for the classical piezoelectric cantilever energy harvester, their corresponding complex amplitudes $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ can be formulated as

$$\begin{cases} \tilde{V}_p = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}u'(1), \\ \quad = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}\sigma^{1/2}\left(-A_\delta\sin\sqrt{\sigma}+B_\delta\cos\sqrt{\sigma}+C_\delta\sinh\sqrt{\sigma}+D_\delta\cosh\sqrt{\sigma}\right) \\ \quad = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}\dfrac{\sqrt{\sigma}\,(\sinh\sqrt{\sigma}-\sin\sqrt{\sigma})}{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\,(\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma})} \\ \quad = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\left(\dfrac{\eta_b}{l_p}\right)\left(\dfrac{e_p}{C_p}\right)\chi_p, \\ \tilde{I}_p = \tilde{V}_p/R_l = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\left(\dfrac{\eta_b}{l_p}\right)\left(\dfrac{e_p}{C_pR_l}\right)\chi_p, \\ \tilde{P}_p = \tilde{V}_p^2/R_l = \left(\dfrac{\eta_b}{l_p}\right)^2\left(\dfrac{e_p}{C_p}\right)\left(\dfrac{e_p}{C_pR_l}\right)\left(\dfrac{j\sigma\beta}{j\sigma\beta+1}\right)^2\chi_p^2, \end{cases} \tag{28}$$

in which we have used the notations that

$$\chi_p = u_1'(1) = \frac{\sqrt{\sigma}\,(\sinh\sqrt{\sigma}-\sin\sqrt{\sigma})}{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\,(\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma})}. \tag{29}$$

Clearly, The three output measures $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ are heavily dependent on another dimensionless parameter $r_d = \eta_b/l_p$. Formally, both $\tilde{V}_p$ and $\tilde{I}_p$ depend lineary upon r_d, while $\tilde{P}_p$ shows a quadratic dependence on r_d. The only dependence upon δ is introduced in χ_p. However, it should be noted that the parameter δ relies on e_p, l_p, C_p, and B_p, while the three measures $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ are dimensional values and depend on e_p, σ_b, and R_l. As a result, the change of parameter δ results in the change of reference voltage e_p/C_p, reference current $e_p/(C_pR_l)$, and reference power $(e_p/C_p)[e_p/(C_pR_l)]$, and therefore the corresponding values of $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$. Hence, we may establish a bijective relation between δ and e_p, and relate the change of δ to that of e_p. In this way, we calculate the output measures at different values of δ and plot their amplitudes in Figure 3.

It is seen from Figure 3 that all the three measures show a maximum peak with the increase of δ at the approximate value of $\delta = 10$. When δ is small, or equivalently, e_p is small, amplitude of the

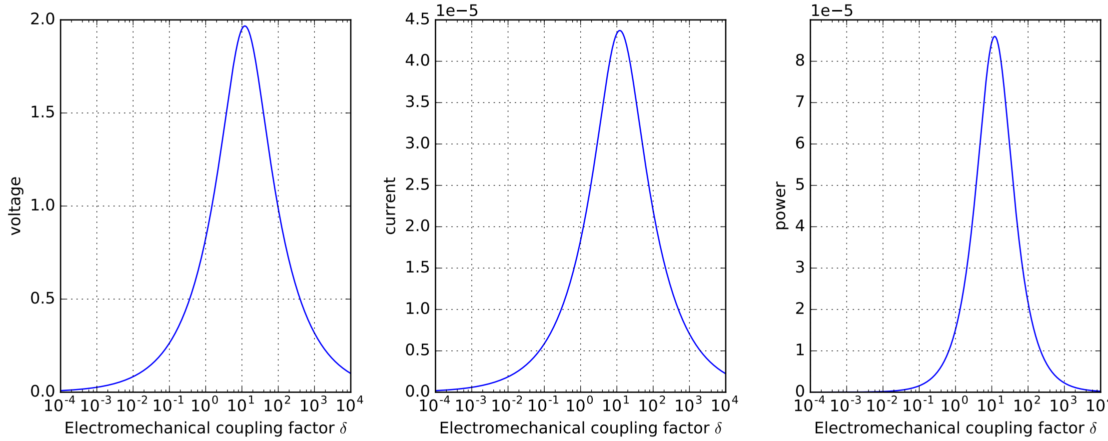

Figure 3: Voltage, current and power output for the piezoelectric cantilever energy harvester

three output measures $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ increase with the increase of δ. Then after the critical value of δ, a further increase of δ causes the decrease of output measures. Thus we come to a small conclusion that to obtain an optimal output performance, the electromechanical coupling factor δ should be set to an appropriate value. However, a direct calculation using the parameters introduced in the literature [1, 2] shows that the parameter δ is rather small for a typical piezoelectric cantilever energy harvester. For example, for a piezoelectric voltage constant $e_{31} = -5.35\ C/m^2$, the value of e_p is $-5.35 \times 10^{-5}\ C$, and the final value of δ is 0.028. According to the properties of commonly used piezoelectric materials, the parameter e_{31} is always in the range of several or several tens C/m^2 reference to be inserted. That is to say, the final value of δ can be seen always in the order of 10^{-2}, which is a rather small value according to the diagram. Hence we could present an asymptotic analysis of the performance of the classical piezoelectric energy harvester. This is the subject of the following section.

4 Asymptotic analysis of the problem

Considering that the parameter δ is small, we expand the theoretical solution to the problem in terms of the small parameter δ using the following regular expansion:

$$\begin{cases} A_\delta = A_0 + \delta A_1 + \delta^2 A_2 + \cdots, \\ B_\delta = B_0 + \delta B_1 + \delta^2 B_2 + \cdots, \\ C_\delta = C_0 + \delta C_1 + \delta^2 C_2 + \cdots, \\ D_\delta = D_0 + \delta D_1 + \delta^2 D_2 + \cdots. \end{cases} \tag{30}$$

As a result, we obtain the following successive expansion problem:
$O(\delta^0)$:

$$\begin{cases} A_0 + C_0 = 1, \\ B_0 + D_0 = 0, \\ -A_0 \cos\sqrt{\sigma} - B_0 \sin\sqrt{\sigma} + C_0 \cosh\sqrt{\sigma} + D_0 \sinh\sqrt{\sigma} = 0, \\ A_0 \sin\sqrt{\sigma} - B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma} = 0. \end{cases} \tag{31}$$

The solution is

$$\begin{cases} A_0 = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ B_0 = \dfrac{\cosh\sqrt{\sigma}\sin\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ C_0 = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ D_0 = -\dfrac{\cosh\sqrt{\sigma}\sin\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}}. \end{cases} \tag{32}$$

Hence we have

$$-A_0 \sin\sqrt{\sigma} + B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma} = \frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \tag{33}$$

$O(\delta^1)$:

$$\begin{cases} A_1 + C_1 = 0, \\ B_1 + D_1 = 0, \\ \left(-A_1 \cos\sqrt{\sigma} - B_1 \sin\sqrt{\sigma} + C_1 \cosh\sqrt{\sigma} + D_1 \sinh\sqrt{\sigma}\right) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1}\left(-A_0 \sin\sqrt{\sigma} + B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma}\right) = 0, \\ A_1 \sin\sqrt{\sigma} - B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma} = 0. \end{cases} \tag{34}$$

The solution is

$$\begin{cases} A_1 = \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\ B_1 = \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\ C_1 = \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(-\frac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\ D_1 = \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \end{cases} \tag{35}$$

Then we have

$$\begin{aligned} & -A_1 \sin\sqrt{\sigma} + B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma} \\ = & \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\frac{\sin\sqrt{\sigma} - \sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right) \end{aligned} \tag{36}$$

$O(\delta^2)$:

$$\begin{cases} A_2 + C_2 = 0, \\ B_2 + D_2 = 0, \\ \left(-A_2 \cos\sqrt{\sigma} - B_2 \sin\sqrt{\sigma} + C_2 \cosh\sqrt{\sigma} + D_2 \sinh\sqrt{\sigma}\right) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1}\left(-A_1 \sin\sqrt{\sigma} + B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma}\right) = 0, \\ A_2 \sin\sqrt{\sigma} - B_2 \cos\sqrt{\sigma} + C_2 \sinh\sqrt{\sigma} + D_2 \cosh\sqrt{\sigma} = 0. \end{cases} \tag{37}$$

The solution is

$$\begin{cases} A_2 = \left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\ B_2 = \left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\ C_2 = \left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(-\frac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\ D_2 = \left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\frac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \end{cases} \tag{38}$$

Indeed, we can continue to obtain the coefficients for higher order (≥ 2) expansions, as shown in the appendices (To add some comments) using successive iteration method. Nevertheless, it suffices here to consider up to the second order expansion $u^{(0)}(z)$, $u^{(1)}(z)$, and $u^{(2)}(z)$, respectively:

$$\begin{cases} u^{(0)}(z) = u_0(z), \\ u^{(1)}(z) = u_0(z) + \delta u_1(z), \\ u^{(2)}(z) = u_0(z) + \delta u_1(z) + \delta^2 u_2(z), \end{cases} \tag{39}$$

where the terms $u_0(z)$, $u_1(z)$, and $u_2(z)$ are defined as

$$\begin{cases} u_0(z) = A_0 \cos\sqrt{\sigma}z + B_0 \sin\sqrt{\sigma}z + C_0 \cosh\sqrt{\sigma}z + D_0 \sinh\sqrt{\sigma}z - 1, \\ u_1(z) = A_1 \cos\sqrt{\sigma}z + B_1 \sin\sqrt{\sigma}z + C_1 \cosh\sqrt{\sigma}z + D_1 \sinh\sqrt{\sigma}z, \\ u_2(z) = A_2 \cos\sqrt{\sigma}z + B_2 \sin\sqrt{\sigma}z + C_2 \cosh\sqrt{\sigma}z + D_2 \sinh\sqrt{\sigma}z. \end{cases} \tag{40}$$

For different values of δ and σ, we calculate the three approximations $u^{(0)}(z)$, $u^{(1)}(z)$, and $u^{(2)}(z)$, and the closed form solution $u_\delta(z)$ In Figure 4.

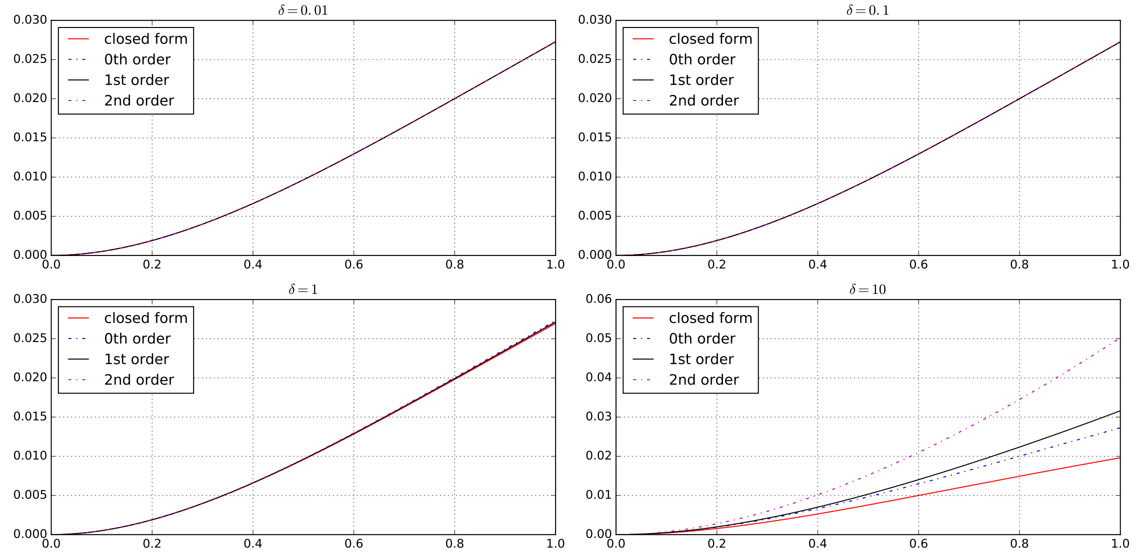

Figure 4: Comparison for the different orders of asymptotic expansion for the dimensionless relative displacement function $u_\delta(z)$.

Appendices

The asymptotic expansion of equation (23) can be found using an iterative method. In fact, for higher order expansions ($k \geq 1$), we have the following iterative relation:

$$\begin{cases} A_{k+1} + C_{k+1} = 0, \\ B_{k+1} + D_{k+1} = 0, \\ \left(-A_{k+1}\cos\sqrt{\sigma} - B_{k+1}\sin\sqrt{\sigma} + C_{k+1}\cosh\sqrt{\sigma} + D_{k+1}\sinh\sqrt{\sigma}\right) + \\ \dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\left(-A_k\sin\sqrt{\sigma} + B_k\cos\sqrt{\sigma} + C_k\sinh\sqrt{\sigma} + D_k\cosh\sqrt{\sigma}\right) = 0, \\ A_{k+1}\sin\sqrt{\sigma} - B_{k+1}\cos\sqrt{\sigma} + C_{k+1}\sinh\sqrt{\sigma} + D_{k+1}\cosh\sqrt{\sigma} = 0, \end{cases} \tag{41}$$

whose solution is expressed by

$$\begin{cases} A_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\dfrac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k), \\ B_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\dfrac{-\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k), \\ C_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(-\dfrac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k), \\ D_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\dfrac{-\sin\sqrt{\sigma}+\sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k), \end{cases} \tag{42}$$

in which

$$Q_k = -A_k\sin\sqrt{\sigma} + B_k\cos\sqrt{\sigma} + C_k\sinh\sqrt{\sigma} + D_k\cosh\sqrt{\sigma}. \tag{43}$$

In terms of Q_k ($k \geq 0$), we have the following iterative relation

$$Q_{k+1} = -\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}+\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)Q_k, \tag{44}$$

and the initial two values Q_0 and Q_1:

$$\begin{cases} Q_0 = \dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}, \\ Q_1 = \dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\dfrac{\sin\sqrt{\sigma} - \sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right). \end{cases} \tag{45}$$

Hence it is shown that for $k \geq 0$,

$$Q_k = \left[-\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\right]^k \left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right). \tag{46}$$

As a result, we obtain that for $k \geq 1$,

$$\begin{cases} A_k = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^k \left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^{k-1} \left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right), \\ B_k = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^k \left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^{k-1} \left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right), \\ C_k = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^k \left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^{k-1} \left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{-\cos\sqrt{\sigma} - \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right), \\ D_k = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^k \left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^{k-1} \left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right). \end{cases}$$

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