# Revisit to the theoretical analysis of a classical piezoelectric cantilever energy harvester

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## 1 Summary of the interested equations

The dynamic equations for a typical piezoelectric composite cantilever beam is

$$B_p \frac{\partial^4 w(x,t)}{\partial x^4} + m_p \frac{\partial^2 w(x,t)}{\partial t^2} = 0, \tag{1}$$

where $B_p$ is the equivalent bending stiffness and $m_p$ is the line mass density of the piezoelectric cantilever beam. If the piezoelectric elements attached to the cantilever beam is connected to an external electrical load $R_l$, we have

$$\frac{dQ_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{2}$$

For the underlying physics, we have the following constitutive equations

$$\begin{aligned} M_p(x,t) &= B_p \frac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\ q_p(x,t) &= e_p \frac{\partial^2 w(x,t)}{\partial x^2} + \varepsilon_p V_p(t), \end{aligned} \tag{3}$$

or equivalently,

$$\begin{cases} M_p(x,t) = B_p \dfrac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\ Q_p(x,t) = e_p \left[ \dfrac{\partial w(x,t)}{\partial x} \right] \Bigg|_0^{l_p} + C_p V_p(t). \end{cases} \tag{4}$$

One end of the cantilever beam is fixed while the other end is free. So the boundary conditions are

$$\begin{cases} w(0,t) = w_b(t), \\ \dfrac{\partial w(0,t)}{\partial x} = 0, \end{cases} \tag{5}$$

and

$$\begin{cases} M_p(l_p,t) = B_p \dfrac{\partial^2 w(l_p,t)}{\partial x^2} - e_p V_p(t) = 0, \\ N_p(l_p,t) = \dfrac{\partial M_p(l_p,t)}{\partial x} = B_p \dfrac{\partial^3 w(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{6}$$

In the classical energy harvesting applications, the cantilever beam is subject to a periodical base excitation $w_b(t)$. Thus the dynamic response of the cantilever beam is decomposed as

$$w(x,t) = w_b(t) + w_{rel}(x,t), \tag{7}$$

where $w_{rel}(x,t)$ is the relative displacement function of the cantilever beam. In this way, the system is converted into

$$B_p \frac{\partial^4 w_{rel}(x,t)}{\partial x^4} + m_p \frac{\partial^2 w_{rel}(x,t)}{\partial t^2} = -m_p \frac{\partial^2 w_b(t)}{\partial t^2}, \tag{8}$$

$$e_p \left[ \frac{\partial^2 w_{rel}(x,t)}{\partial x \partial t} \right] \Bigg|_0^{l_p} + C_p \frac{dV_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{9}$$

$$
\begin{cases} w_{rel}(0,t) = 0, \\ \dfrac{\partial w_{rel}(0,t)}{\partial x} = 0, \end{cases} \tag{10}
$$

and

$$
\begin{cases} B_p \dfrac{\partial^2 w_{rel}(l_p,t)}{\partial x^2} - e_p V_p(t) = 0, \\ \dfrac{\partial^3 w_{rel}(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{11}
$$

Considering a sinusoidal base excitation

$$
w_b(t) = \eta_b e^{j\sigma_b t} \tag{12}
$$

where $\xi_b$ is usually a real vibration amplitude, the steady state solution for the above system can be reasonably set as

$$
w_{rel}(x,t) = \eta_{rel}(x) e^{j\sigma_b t}, \quad V_p(t) = \tilde{V}_p e^{j\sigma_b t}, \tag{13}
$$

where $\eta_{rel}(x)$ and $\tilde{V}_p$ are complex amplitudes. Then the above system is again simplified as

$$
B_p \frac{\partial^4 \eta_{rel}(x)}{\partial x^4} - m_p \sigma_b^2 \eta_{rel}(x) = m_p \sigma_b^2 \eta_b, \tag{14}
$$

$$
\begin{cases} \eta_{rel}(0) = 0, \\ \dfrac{\partial \eta_{rel}(0)}{\partial x} = 0, \end{cases} \tag{15}
$$

and

$$
\begin{cases} B_p \dfrac{\partial^2 \eta_{rel}(l_p)}{\partial x^2} + \dfrac{j\sigma_b R_l}{1 + j\sigma_b C_p R_l} e_p^2 \dfrac{\partial \eta_{rel}(l_p)}{\partial x} = 0, \\ \dfrac{\partial^3 \eta_{rel}(l_p)}{\partial x^3} = 0. \end{cases} \tag{16}
$$

Note that here we assume a sinusoidal steady state response, which is not actually validated theoretically.

Obviously we can have the following dimensionless scheme:

$$
\eta_{rel} \sim u\eta_b, \quad x \sim z l_p \tag{17}
$$

and therefore the following dimensionless parameters

$$
\sigma = \sigma_b \sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \delta = \frac{e_p^2 l_p}{C_p B_p}. \tag{18}
$$

Now, we reach the following dimensionless system of boundary value problem

$$
\begin{cases} u'''' - \sigma^2 u = \sigma^2, \\ u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \dfrac{j\beta\sigma}{1 + j\beta\sigma} \delta u'(1) = 0, \\ u'''(1) = 0, \end{cases} \tag{19}
$$

where the prime denotes the derivative with respect to $z$. The analytical solution to this problem can be formulated as

$$
u(z;\delta) = A_\delta \cos\sqrt{\sigma} z + B_\delta \sin\sqrt{\sigma} z + C_\delta \cosh\sqrt{\sigma} z + D_\delta \sinh\sqrt{\sigma} z - 1 \tag{20}
$$

and hence

$$
\begin{aligned}
u'(z;\delta) &= \sigma^{1/2} \left( -A_\delta \sin\sqrt{\sigma} z + B_\delta \cos\sqrt{\sigma} z + C_\delta \sinh\sqrt{\sigma} z + D_\delta \cosh\sqrt{\sigma} z \right), \\
u''(z;\delta) &= \sigma \left( -A_\delta \cos\sqrt{\sigma} z - B_\delta \sin\sqrt{\sigma} z + C_\delta \cosh\sqrt{\sigma} z + D_\delta \sinh\sqrt{\sigma} z \right), \\
u'''(z;\delta) &= \sigma^{3/2} \left( A_\delta \sin\sqrt{\sigma} z - B_\delta \cos\sqrt{\sigma} z + C_\delta \sinh\sqrt{\sigma} z + D_\delta \cosh\sqrt{\sigma} z \right).
\end{aligned} \tag{21}
$$

The coefficients $A_\delta$, $B_\delta$, $C_\delta$, and $D_\delta$ are then subject to the following linear system of equations:

$$\begin{cases} A_\delta + C_\delta = 1, \\ B_\delta + D_\delta = 0, \\ \left(-A_\delta \cos\sqrt{\sigma} - B_\delta \sin\sqrt{\sigma} + C_\delta \cosh\sqrt{\sigma} + D_\delta \sinh\sqrt{\sigma}\right) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\delta\left(-A_\delta \sin\sqrt{\sigma} + B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma}\right) = 0, \\ A_\delta \sin\sqrt{\sigma} - B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma} = 0. \end{cases} \tag{22}$$

Analytically, we can directly obtain the solution to this problem as

$$\begin{cases} A_\delta = \dfrac{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ B_\delta = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ C_\delta = \dfrac{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ D_\delta = \dfrac{-\cos\sqrt{\sigma}\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}. \end{cases} \tag{23}$$

According to equations (20) and (23), the dimensionless displacement amplitude function $u(z)$ is totally determined by the three dimensionless parameters $\sigma$, $\beta$, and $\delta$ introduced before. Among the dimensionless parameters, $\sigma$ is the dimensionless base excitation frequency, $\beta$ is the dimensionless electrical resonant frequency, and $\delta$ is the dimensionless electromechanical coupling strength for the structure. As $\sigma$ and $\beta$ is determined by the base excitation and externally connected circuit respectively, only the parameter $\delta$ is fully determined by the structure itself. Hence we would like to investigate the influence of parameter $\delta$ upon the solution displacement function $u(z)$. By taking different values of $\delta$ , we calculate the displacement amplitude function $u(z)$ and plot the results in Figure 1.

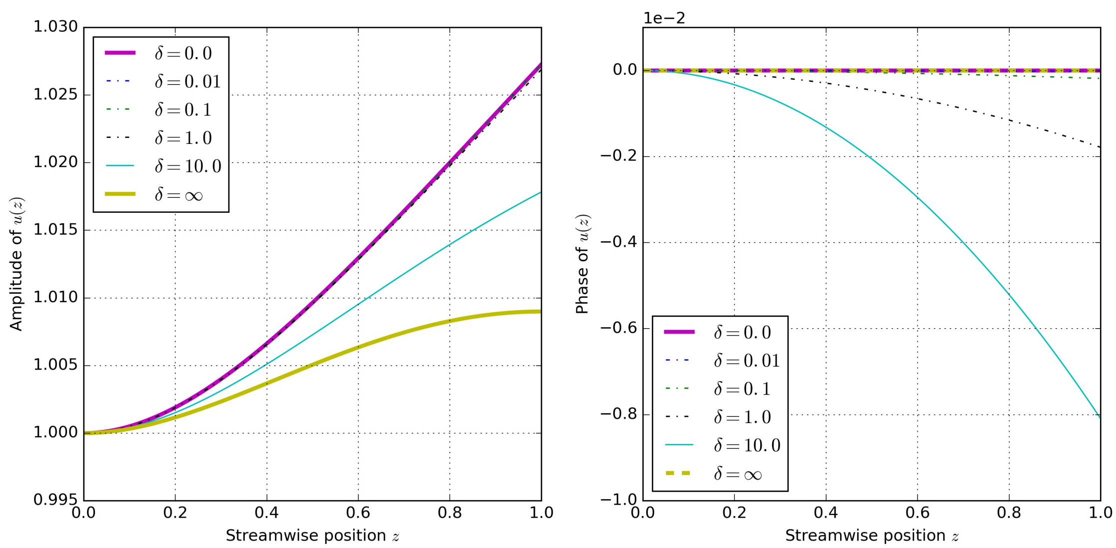


Figure 1: Amplitude and phase of the displacement function $u(z)$

It is shown in Figure 1, the parameter $\delta$ changes the function $u(z)$ through the change of the third boundary condition (to be inserted). When $\delta$ is zero, i.e., no electromechanical coupling is present, the system degenerates to the classical elastic cantilever beam problem, whose solution is a real function. That is to say, the phase of $u(z)$ is a constant across the whole beam (in the range of $0 \le z \le 1$). Analytical expressions for the coefficients are

$$\begin{cases} A_{\varnothing} = \dfrac{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}-\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\ B_{\varnothing} = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\ C_{\varnothing} = \dfrac{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\ D_{\varnothing} = \dfrac{-\cos\sqrt{\sigma}\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{2\left[1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}. \end{cases} \tag{24}$$

and the resulting dimensionless displacement function $u_{\varnothing}(z)$ is represented as

$$u_{\varnothing}(z) = A_{\varnothing}\cos\sqrt{\sigma}z + B_{\varnothing}\sin\sqrt{\sigma}z + C_{\varnothing}\cosh\sqrt{\sigma}z + D_{\varnothing}\sinh\sqrt{\sigma}z - 1. \tag{25}$$

When the electromechanical coupling is extremely strong, and $\delta$ is extremely large and can be seen as $\infty$ in mathematical sense. In this situation, the solution $u_{\infty}(z)$ is again real without any phase difference in the $z$ direction. The coefficients can be analytically expressed as

$$\begin{cases} A_{\infty} = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ B_{\infty} = \dfrac{\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ C_{\infty} = \dfrac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\ D_{\infty} = \dfrac{-\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}. \end{cases} \tag{26}$$

and hence the dimensionless displacement function $u_{\infty}(z)$ is

$$u_{\infty}(z) = A_{\infty}\cos\sqrt{\sigma}z + B_{\infty}\sin\sqrt{\sigma}z + C_{\infty}\cosh\sqrt{\sigma}z + D_{\infty}\sinh\sqrt{\sigma}z - 1. \tag{27}$$

While a finite non-zero electromechanical coupling factor $\delta$ is present, which is expected in most applications, the resulting dimensionless displacement function $u(z)$ has varying magnitude and phase along the stream-wise direction or $z$ direction. However, it is seen from the right panel of Figure 1 that for different values of $\delta$, the phase change of $u(z)$ is very small in the $z$ direction, actually in the order $10^{-2}$.

The resulting complex amplitudes $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ for output voltage $V_p(t)$, output current $I_p(t)$, and output power $P_p(t)$, respectively, can be formulated as follows

$$\begin{cases} \tilde{V}_p = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}u'(1), \\ \quad = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}\sigma^{1/2}\left(-A_\delta\sin\sqrt{\sigma}+B_\delta\cos\sqrt{\sigma}+C_\delta\sinh\sqrt{\sigma}+D_\delta\cosh\sqrt{\sigma}\right) \\ \quad = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}\dfrac{\sqrt{\sigma}\left(\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}\right)}{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)} \\ \quad = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\left(\dfrac{\eta_b}{l_p}\right)\left(\dfrac{e_p}{C_p}\right)\chi_p, \\ \tilde{I}_p = \tilde{V}_p/R_l = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\left(\dfrac{\eta_b}{l_p}\right)\left(\dfrac{e_p}{C_pR_l}\right)\chi_p, \\ \tilde{P}_p = \tilde{V}_p^2/R_l = \left(\dfrac{\eta_b}{l_p}\right)^2\left(\dfrac{e_p}{C_p}\right)\left(\dfrac{e_p}{C_pR_l}\right)\left(\dfrac{j\sigma\beta}{j\sigma\beta+1}\right)^2\chi_p^2, \end{cases} \tag{28}$$

in which we have used the notations that

$$\chi_p = u_1'(1) = \frac{\sqrt{\sigma}\left(\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}\right)}{1+\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)}. \tag{29}$$

Clearly, output performance $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ of a classical piezoelectric cantilever energy harvester is heavily dependent on another dimensionless parameter $r_d = \eta_b/l_p$. To be more explicit, both $\tilde{V}_p$ and $\tilde{I}_p$ are linearly dependent on $r_d$ and as a result, $\tilde{P}_p$ shows a quadratic dependence on $r_d$.

In the following, we can obtain the corresponding displacement function $u(z)$ in terms of its amplitude and phase in Figure 1.

For a typical piezoelectric cantilever energy harvester in the literature [1, 2], this parameter $\delta$ is rather small

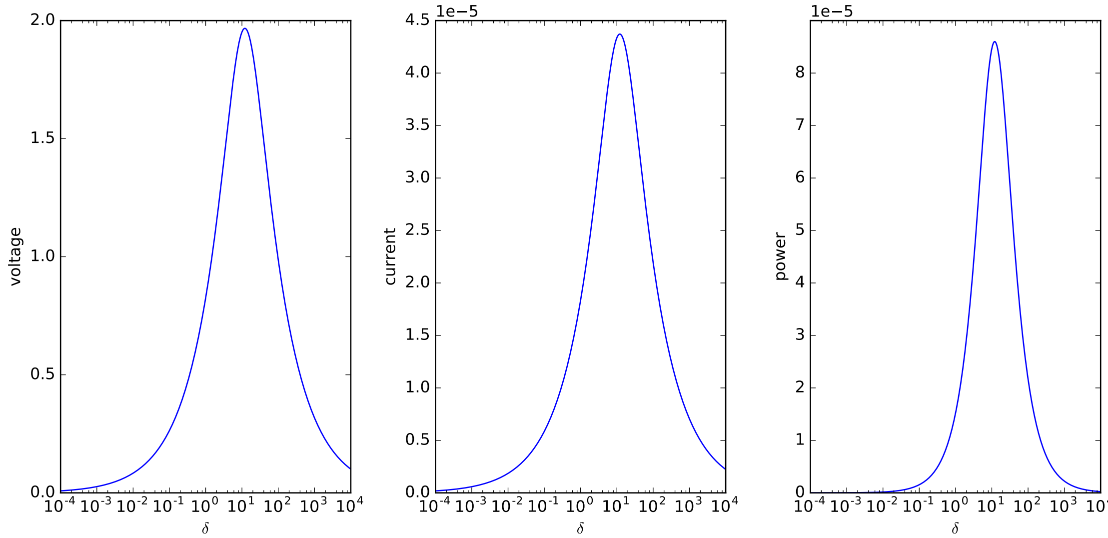


Figure 2: Voltage, current and power output for the piezoelectric cantilever energy harvester

Using the following regular expansion:

$$\begin{cases} A_\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots, \\ B_\epsilon = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \cdots, \\ C_\epsilon = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \cdots, \\ D_\epsilon = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \end{cases} \tag{30}$$

we obtain the successive expansion problem:
$O(\epsilon^0)$:

$$\begin{cases} A_0 + C_0 = 1, \\ B_0 + D_0 = 0, \\ -A_0 \cos\sqrt{\sigma} - B_0 \sin\sqrt{\sigma} + C_0 \cosh\sqrt{\sigma} + D_0 \sinh\sqrt{\sigma} = 0, \\ A_0 \sin\sqrt{\sigma} - B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma} = 0. \end{cases} \tag{31}$$

The solution is

$$\begin{cases} A_0 = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\ B_0 = \dfrac{\cosh\sqrt{\sigma}\sin\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\ C_0 = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\ D_0 = -\dfrac{\cosh\sqrt{\sigma}\sin\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \end{cases} \tag{32}$$

Hence we have

$$-A_0 \sin\sqrt{\sigma} + B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma} = \frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \tag{33}$$

$O(\epsilon^1)$:

$$\begin{cases} A_1 + C_1 = 0, \\ B_1 + D_1 = 0, \\ \left(-A_1 \cos\sqrt{\sigma} - B_1 \sin\sqrt{\sigma} + C_1 \cosh\sqrt{\sigma} + D_1 \sinh\sqrt{\sigma}\right) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\left(-A_0 \sin\sqrt{\sigma} + B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma}\right) = 0, \\ A_1 \sin\sqrt{\sigma} - B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma} = 0. \end{cases} \tag{34}$$

The solution is

$$\begin{cases} A_1 = \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ B_1 = \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{-\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ C_1 = \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(-\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ D_1 = \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{-\sin\sqrt{\sigma}+\sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \end{cases} \tag{35}$$

Then we have

$$\begin{aligned} & -A_1 \sin\sqrt{\sigma} + B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma} \\ = & \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\frac{\sin\sqrt{\sigma}-\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right) \end{aligned} \tag{36}$$

$O(\epsilon^2)$:

$$\begin{cases} A_2 + C_2 = 0, \\ B_2 + D_2 = 0, \\ \left(-A_2 \cos\sqrt{\sigma} - B_2 \sin\sqrt{\sigma} + C_2 \cosh\sqrt{\sigma} + D_2 \sinh\sqrt{\sigma}\right) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\left(-A_1 \sin\sqrt{\sigma} + B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma}\right) = 0, \\ A_2 \sin\sqrt{\sigma} - B_2 \cos\sqrt{\sigma} + C_2 \sinh\sqrt{\sigma} + D_2 \cosh\sqrt{\sigma} = 0. \end{cases} \tag{37}$$

The solution is

$$\begin{cases} A_2 = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ B_2 = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{-\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ C_2 = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(-\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ D_2 = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{-\sin\sqrt{\sigma}+\sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \end{cases} \tag{38}$$

To get higher order expansions, we can use the following iteration method:
$O(\epsilon^{k+1})$ $(k \geq 1)$:

$$\begin{cases} A_{k+1} + C_{k+1} = 0, \\ B_{k+1} + D_{k+1} = 0, \\ \left(-A_{k+1} \cos\sqrt{\sigma} - B_{k+1} \sin\sqrt{\sigma} + C_{k+1} \cosh\sqrt{\sigma} + D_{k+1} \sinh\sqrt{\sigma}\right) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\left(-A_k \sin\sqrt{\sigma} + B_k \cos\sqrt{\sigma} + C_k \sinh\sqrt{\sigma} + D_k \cosh\sqrt{\sigma}\right) = 0, \\ A_{k+1} \sin\sqrt{\sigma} - B_{k+1} \cos\sqrt{\sigma} + C_{k+1} \sinh\sqrt{\sigma} + D_{k+1} \cosh\sqrt{\sigma} = 0. \end{cases} \tag{39}$$

The solution is

$$
\begin{cases}
A_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\dfrac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k) \\
B_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\dfrac{-\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k) \\
C_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(-\dfrac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k) \\
D_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\dfrac{-\sin\sqrt{\sigma}+\sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k)
\end{cases}
\tag{40}
$$

where for $k \geq 2$

$$
Q_k = -A_k \sin\sqrt{\sigma} + B_k \cos\sqrt{\sigma} + C_k \sinh\sqrt{\sigma} + D_k \cosh\sqrt{\sigma}, \tag{41}
$$

and for $k \geq 0$

$$
\begin{aligned}
Q_{k+1} &= -A_{k+1}\sin\sqrt{\sigma} + B_{k+1}\cos\sqrt{\sigma} + C_{k+1}\sinh\sqrt{\sigma} + D_{k+1}\cosh\sqrt{\sigma} \\
&= -\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}+\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)Q_k,
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
Q_1 &= -A_1\sin\sqrt{\sigma} + B_1\cos\sqrt{\sigma} + C_1\sinh\sqrt{\sigma} + D_1\cosh\sqrt{\sigma} \\
&= \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\frac{\sin\sqrt{\sigma}-\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)
\end{aligned}
\tag{43}
$$

$$
Q_0 = \frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1} \tag{44}
$$

Hence it is shown that for $k \geq 0$

$$
\begin{aligned}
Q_k &= -\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}+\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)Q_k \\
&= \left[-\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}+\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\right]^k\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)
\end{aligned}
\tag{45}
$$

As a result, we obtain that for $k \geq 0$

$$
\begin{cases}
A_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1}\left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma}-\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^k\left(\dfrac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\
B_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1}\left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma}-\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^k\left(\dfrac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{-\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\
C_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1}\left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma}-\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^k\left(\dfrac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{-\cos\sqrt{\sigma}-\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\
D_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1}\left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma}-\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^k\left(\dfrac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{-\sin\sqrt{\sigma}+\sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)
\end{cases}
\tag{46}
$$

# Reference

# References


[1] Erturk A, Inman DJ. A distributed parameter electromechanical model for cantilevered piezoelectric energy harvesters. Journal of vibration and acoustics. 2008;130(4):041002.

[2] Erturk A, Inman DJ. An experimentally validated bimorph cantilever model for piezoelectric energy harvesting from base excitations. Smart materials and structures. 2009;18(2):025009.