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Asymptotic analysis of a multimaterial with a thin piezoelectric interphase

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Abstract We study the electromechanical behavior of a multimaterial constituted by a linear piezoelectric transversely isotropic plate-like body with high rigidity embedded between two generic three-dimensional piezoelectric bodies by means of the asymptotic expansion method. After defining a small real dimensionless parameter ε, which will tend to zero, we characterize the limit model and the associated limit problem. We give also a mathematical justification of the model by means of a functional convergence argument. Moreover, we identify the non classical electromechanical transmission conditions between the two three-dimensional bodies.

Keywords Asymptotic analysis · Piezoelectric interphase · Plate models

Mathematics Subject Classification 74K20 · 74K30 · 74K35 · 74F15

1 Introduction

The technology of smart structures provides a new degree of design flexibility for advanced composite structural members of flight vehicles, such as helicopter rotor blades or aircraft wings. The key to the technology is the ability to allow the structure to sense and react in a desired fashion, improving its performances for what concerns with structural vibrations, acoustic signature, and aerodynamic stability. The new concept of adaptive structure requires, for instance, the use of piezoelectric sensors and actuators for controlling the mechanical behavior of structural systems. Piezoelectric materials exhibit both direct and converse piezoelectric effects. The direct effect (electric field generation as a response to mechanical strains) is used in piezoelectric sensors; the converse effect (mechanical strain is produced as a result of an electric field) is used in piezoelectric actuators. Piezoelectric materials may be integrated into a host structure to change its shape and to enhance its mechanical properties with different configurations: for instance, a piezoelectric transducer can be embedded into the structure to be controlled or it can be glued on it, as in the case of piezo-patches. Moreover, the same piezoelectric actuators are often obtained by alternating different thin layers of material with highly contrasted electromechanical properties. This generates different types of complex multimaterial assemblies, in which each phase interacts with the others. Each of the different designs have their own application, showing advantages and disadvantages [21]. Various are the works on piezoelectric interface models obtained by classical variational tools, such as in [1, 8].

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The successful application of the asymptotic methods to obtain a mathematical justification of linear and non linear plate models in elasticity [7] has stimulated the research toward a rational simplification of the modeling of complex structures obtained joining elements of different dimensions and/or materials of highly contrasted properties. The asymptotic analysis has been also used to formally derive simplified models for piezoelectric plates and shells, taking into account both sensor and actuator functions, see, for instance, [11, 16–18], as well as magnetic effects, see [15, 19], or pyroelectric and pyroelastic effects, see [14]. The direct solution of a complex multimaterial problem by a standard finite element method is too expensive from a computational point of view and the presence of strong contrasts in the geometry and mechanical properties causes numerical instabilities. That is why specific asymptotic expansions are used and allow to replace the original problem by a set of problems in which the thin layer, for instance, is substituted by a two-dimensional surface. The thin inclusion of a third material between two other ones when the rigidity properties of the inclusion are highly contrasted with respect to those of the surrounding materials has been deeply investigated in different functional frameworks in the case of linear elasticity, see [3–6, 12, 13], and, also, in the case of thin conductor plates embedded into a piezoelectric matrix [9].

In this work we consider a particular piezoelectric multimaterial, constituted by two generic three-dimensional piezoelectric bodies separated by a thin piezoelectric transversely isotropic plate-like body with high rigidity. By defining a small real parameter ε, associated with the thickness and the electromechanical properties of the middle layer, we perform an asymptotic analysis by letting ε tend to zero, following the approach by Ciarlet [7]. Then we characterize the limit model and its associated limit problem. We give also a mathematical justification of the limit model by means of a functional convergence argument. In the simplified model the intermediate plate-like body "disappears" linearly piezoelectric materials, whose constitutive laws are and it is replaced by a specific electromechanical surface energy defined over the middle plane of the plate. This surface energy is then traduced in ad hoc transmission conditions at the interface between the two piezoelectric bodies in terms of the jump of stresses, electric displacements and electric potentials.

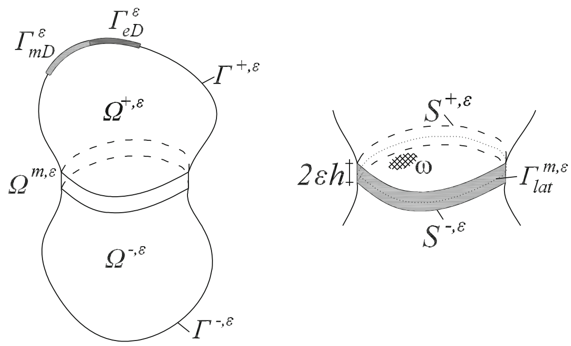

Fig. 1 The reference configuration of the multimaterial and the geometry of the interphase

The paper is organized as follows. In Sect. 2 we define the notation and the position of the problem. The limit model is then deduced through an asymptotic analysis, as shown in Sect. 3. In Sect. 4 we justify from a mathematical point of view the limit model by virtue of the functional convergence.

2 Statement of the problem

In the sequel, Greek indices range in the set $\{1,2\}$, Latin indices range in the set $\{1,2,3\}$, and the Einstein's summation convention with respect to the repeated indices is adopted.

Let us consider a three-dimensional Euclidian space identified by $\mathbb{R}^3$ and such that the three vectors $\mathbf{e}_i$ form an orthonormal basis. Let Ω^+ and Ω^- be two disjoint open domains with smooth boundaries $\partial\Omega^+$ and $\partial\Omega^-$. Let $\omega := \left\{\partial\Omega^+ \cap \partial\Omega^-\right\}^\circ$ be the interior of the common part of the boundaries which is assumed to be a non empty domain in $\mathbb{R}^2$ having a positive two-dimensional measure. We consider the assembly constituted by two solids bonded together by an intermediate thin plate-like body $\Omega^{m,\varepsilon}$ of thickness $2h^\varepsilon$, where $0<\varepsilon<1$ is a dimensionless small real parameter which will tend to zero. We suppose that the thickness h^ε of the middle layer depends linearly on ε, so that $h^\varepsilon = \varepsilon h$.

More precisely, we denote respectively with $\Omega^{\pm,\varepsilon} := \{x^\varepsilon := x \pm \varepsilon h \mathbf{e}_3; x \in \Omega^\pm\}$, the translation of Ω^+ (resp. Ω^-) along the direction $\mathbf{e}_3$ (resp. $-\mathbf{e}_3$) of the quantity εh, with $\Omega^{m,\varepsilon} := \omega \times (-\varepsilon h, \varepsilon h)$, the central plate-like domain, and with $\Omega^\varepsilon := \Omega^{+,\varepsilon} \cup \Omega^{m,\varepsilon} \cup \Omega^{-,\varepsilon}$, the reference configuration of the assembly, see Fig. 1.

Moreover, we define with $S^{\pm,\varepsilon} := \omega \times \{\pm\varepsilon h\} = \Omega^{\pm,\varepsilon} \cap \Omega^{m,\varepsilon}$, the upper and lower faces of the intermediate plate-like domain, $\Gamma^{\pm,\varepsilon} := \partial\Omega^{\pm,\varepsilon}/S^{\pm,\varepsilon}$, and $\Gamma^{m,\varepsilon}_{lat} := \partial\omega \times (-\varepsilon h, \varepsilon h)$, its lateral surface.

Let $(\Gamma^{\varepsilon}_{mD}, \Gamma^{\varepsilon}_{mN})$ and $(\Gamma^{\varepsilon}_{eD}, \Gamma^{\varepsilon}_{eN})$ be two suitable partitions of $\partial\Omega^{\varepsilon} := \Gamma^{\pm,\varepsilon} \cup \Gamma^{m,\varepsilon}_{lat}$, with both $\Gamma^{\varepsilon}_{mD}$ and $\Gamma^{\varepsilon}_{eD}$ of strictly positive Lebesgue measure. The multimaterial is, on one hand, clamped along $\Gamma^{\varepsilon}_{mD}$ and at an electrical potential $\varphi^{\varepsilon}_0 = 0$ on $\Gamma^{\varepsilon}_{eD}$ and, on the other hand, subjected to surface forces g^{ε}_i on $\Gamma^{\varepsilon}_{mN}$ and electrical displacement d^{ε} on $\Gamma^{\varepsilon}_{eN}$. The assembly is also subjected to body forces f^{ε}_i and electrical loadings F^{ε} acting in $\Omega^{\pm,\varepsilon}$. We suppose, without loss of generality, that $\Omega^{m,\varepsilon}$ and $\Gamma^{m,\varepsilon}_{lat}$ are both free of mechanical and electrical charges. The work of the external electromechanical loadings takes then the following form:

$$L^{\varepsilon}(r^{\varepsilon}) := \int_{\Omega^{\pm,\varepsilon}} (f^{\varepsilon}_i v^{\varepsilon}_i + F^{\varepsilon}\psi^{\varepsilon})dx^{\varepsilon} + \int_{\Gamma^{\varepsilon}_{mN}} g^{\varepsilon}_i v^{\varepsilon}_i d\Gamma^{\varepsilon} + \int_{\Gamma^{\varepsilon}_{eN}} d^{\varepsilon}\psi^{\varepsilon} d\Gamma^{\varepsilon}$$

We suppose that $f^{\varepsilon}_i \in L^2(\Omega^{\pm,\varepsilon})$, $F^{\varepsilon} \in L^2(\Omega^{\pm,\varepsilon})$, $g^{\varepsilon}_i \in L^2(\Gamma^{\varepsilon}_{mN})$ and $d^{\varepsilon} \in L^2(\Gamma^{\varepsilon}_{eN})$. We finally assume that $\Omega^{\pm,\varepsilon}$ are constituted by two homogeneous linearly piezoelectric materials, whose constitutive laws are defined as follows:

$$\begin{cases} \sigma^{\pm,\varepsilon}_{ij}(\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}) = A^{\pm,\varepsilon}_{ijk\ell} e^{\varepsilon}_{k\ell}(\mathbf{u}^{\varepsilon}) - P^{\pm,\varepsilon}_{ijk} E^{\varepsilon}_k(\varphi^{\varepsilon}), \\ D^{\pm,\varepsilon}_i(\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}) = P^{\pm,\varepsilon}_{ijk} e^{\varepsilon}_{jk}(\mathbf{u}^{\varepsilon}) + H^{\pm,\varepsilon}_{ij} E^{\varepsilon}_j(\varphi^{\varepsilon}), \end{cases}$$

where $(\sigma^{\varepsilon}_{ij})$ is the classical Cauchy stress tensor, $(e^{\varepsilon}_{ij}(\mathbf{u}^{\varepsilon})) := \left(\frac{1}{2}\left(\partial^{\varepsilon}_i u^{\varepsilon}_j + \partial^{\varepsilon}_j u^{\varepsilon}_i\right)\right)$ is the linearized strain tensor, (D^{ε}_i) is the electrical displacement field, φ^{ε} is the electrical potential and $E^{\varepsilon}_i(\varphi^{\varepsilon}) := -\partial^{\varepsilon}_i\varphi^{\varepsilon}$ its associated electrical field. $(A^{\pm,\varepsilon}_{ijk\ell})$, $(P^{\pm,\varepsilon}_{ijk})$ and $(H^{\pm,\varepsilon}_{ij})$ represent, respectively, the classical fourth order elasticity tensor, the third order piezoelectric coupling tensor and the second order dielectric tensor related to $\Omega^{\pm,\varepsilon}$.

On the other hand, $\Omega^{m,\varepsilon}$ is made by a homogeneous linearly piezoelectric material, transversely isotropic with respect to $\mathbf{e}_3$, whose constitutive law is defined in the following expression:

$$\begin{cases} \sigma^{m,\varepsilon}_{\alpha\beta}(\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}) = \mathbf{2}\mu^{\varepsilon}\mathbf{e}^{\varepsilon}_{\alpha\beta}(\mathbf{u}^{\varepsilon}) + (\lambda^{\varepsilon}\mathbf{e}^{\varepsilon}_{\sigma\sigma}(\mathbf{u}^{\varepsilon}) \\ \quad + \tau^{\varepsilon}_2 e^{\varepsilon}_{33}(\mathbf{u}^{\varepsilon}))\delta_{\alpha\beta} - \delta^{\varepsilon}_{\mathbf{1}}\mathbf{E}^{\varepsilon}_{\mathbf{3}}(\varphi^{\varepsilon}), \\ \sigma^{m,\varepsilon}_{\alpha 3}(\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}) = \mathbf{2}\eta^{\varepsilon}\mathbf{e}^{\varepsilon}_{\alpha\mathbf{3}}(\mathbf{u}^{\varepsilon}) - \delta^{\varepsilon}_{\mathbf{2}}\mathbf{E}^{\varepsilon}_{\alpha}(\varphi^{\varepsilon}), \\ \sigma^{m,\varepsilon}_{33}(\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}) = \tau^{\varepsilon}_{\mathbf{1}}\mathbf{e}^{\varepsilon}_{\mathbf{33}}(\mathbf{u}^{\varepsilon}) + \tau^{\varepsilon}_{\mathbf{2}}\mathbf{e}^{\varepsilon}_{\sigma\sigma}(\mathbf{u}^{\varepsilon}) - \delta^{\varepsilon}_{\mathbf{3}}\mathbf{E}^{\varepsilon}_{\mathbf{3}}(\varphi^{\varepsilon}), \\ D^{m,\varepsilon}_{\alpha}(\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}) = \mathbf{2}\delta^{\varepsilon}_{\mathbf{2}}\mathbf{e}^{\varepsilon}_{\alpha\mathbf{3}}(\mathbf{u}^{\varepsilon}) + \gamma^{\varepsilon}_{\mathbf{1}}\mathbf{E}^{\varepsilon}_{\alpha}(\varphi^{\varepsilon}), \\ D^{m,\varepsilon}_3(\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}) = \delta^{\varepsilon}_{\mathbf{1}}\mathbf{e}^{\varepsilon}_{\sigma\sigma}(\mathbf{u}^{\varepsilon}) + \delta^{\varepsilon}_{\mathbf{3}}\mathbf{e}^{\varepsilon}_{\mathbf{33}}(\mathbf{u}^{\varepsilon}) + \gamma^{\varepsilon}_{\mathbf{2}}\mathbf{E}^{\varepsilon}_{\mathbf{3}}(\varphi^{\varepsilon}). \end{cases}$$

This material corresponds to the class of polarized ceramics, with crystal class C_{6v} or 6 mm, see [20]. Tensors $(A^{\varepsilon}_{ijk\ell})$, (H^{ε}_{ij}) and (P^{ε}_{ijk}) satisfy the following positive definiteness properties: for any symmetric matrix field (b_{ij}), there exists a constant $c > 0$ such that $A^{\varepsilon}_{ijk\ell} b_{k\ell} b_{ij} \geq c \sum_{i,j} |b_{ij}|^2$; for any vector field (a_i), there exists a constant $c > 0$ such that $H^{\varepsilon}_{ij} a_j a_i \geq c \sum_i |a_i|^2$. The above relations imply that

$$\mu^{\varepsilon} > 0,\ \tau^{\varepsilon}_1 > 0,\ \eta^{\varepsilon} > 0,\ \tau^{\varepsilon}_1(\lambda^{\varepsilon} + 2\mu^{\varepsilon}) - (\tau^{\varepsilon}_2)^2 > 0, \\ \gamma^{\varepsilon}_1 > 0,\ \gamma^{\varepsilon}_2 > 0.$$

Moreover, we have the symmetries $A^{\varepsilon}_{ijk\ell} = A^{\varepsilon}_{k\ell ij} = A^{\varepsilon}_{jik\ell}$, $H^{\varepsilon}_{ij} = H^{\varepsilon}_{ji}$ and $P^{\varepsilon}_{ijk} = P^{\varepsilon}_{kij}$.

The electromechanical state at the equilibrium is determined by the pair $s^{\varepsilon} := (\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon})$. We define the functional spaces

$$V^{\varepsilon} := \{\mathbf{v}^{\varepsilon} \in H^1(\Omega^{\varepsilon}; \mathbb{R}^3);\ \mathbf{v}^{\varepsilon} = \mathbf{0} \ \text{ on } \Gamma^{\varepsilon}_{\mathrm{mD}}\}, \\ \Psi^{\varepsilon} := \{\psi^{\varepsilon} \in H^1(\Omega^{\varepsilon});\ \psi^{\varepsilon} = 0 \ \text{ on } \Gamma^{\varepsilon}_{\mathrm{eD}}\}.$$

The physical variational problem defined over the variable domain Ω^{ε} reads as follows:

$$\begin{cases} \text{Find } s^{\varepsilon} \in \mathrm{V}^{\varepsilon} \times \Psi^{\varepsilon} \text{ such that} \\ A^{-,\varepsilon}(s^{\varepsilon}, r^{\varepsilon}) + A^{+,\varepsilon}(s^{\varepsilon}, r^{\varepsilon}) + A^{m,\varepsilon}(s^{\varepsilon}, r^{\varepsilon}) = L^{\varepsilon}(r^{\varepsilon}), \\ \text{for all } \ r^{\varepsilon} \in \mathrm{V}^{\varepsilon} \times \Psi^{\varepsilon}, \end{cases} \tag{1}$$

where the bilinear forms $A^{\pm,\varepsilon}$ and $A^{m,\varepsilon}$ are defined by

$$A^{\pm,\varepsilon}(s^{\varepsilon}, r^{\varepsilon}) := \int_{\Omega^{\pm,\varepsilon}} \Big\{A^{\pm,\varepsilon}_{ijk\ell} e^{\varepsilon}_{k\ell}(\mathbf{u}^{\varepsilon}) e^{\varepsilon}_{ij}(\mathbf{v}^{\varepsilon}) + H^{\pm,\varepsilon}_{ij} E^{\varepsilon}_j(\varphi^{\varepsilon}) E^{\varepsilon}_i(\psi^{\varepsilon}) \\ + P^{\pm,\varepsilon}_{ihk}(E^{\varepsilon}_i(\psi^{\varepsilon}) e^{\varepsilon}_{hk}(\mathbf{u}^{\varepsilon}) - E^{\varepsilon}_i(\varphi^{\varepsilon}) e^{\varepsilon}_{hk}(\mathbf{v}^{\varepsilon}))\Big\} dx^{\varepsilon},$$

$$A^{m,\varepsilon}(s^{\varepsilon}, r^{\varepsilon}) := \int_{\Omega^{m,\varepsilon}} \Big\{A^{m,\varepsilon}_{ijk\ell} e^{\varepsilon}_{k\ell}(\mathbf{u}^{\varepsilon}) e^{\varepsilon}_{ij}(\mathbf{v}^{\varepsilon}) + H^{m,\varepsilon}_{ij} E^{\varepsilon}_j(\varphi^{\varepsilon}) E^{\varepsilon}_i(\psi^{\varepsilon}) \\ + P^{m,\varepsilon}_{ihk}(E^{\varepsilon}_i(\psi^{\varepsilon}) e^{\varepsilon}_{hk}(\mathbf{u}^{\varepsilon}) - E^{\varepsilon}_i(\varphi^{\varepsilon}) e^{\varepsilon}_{hk}(\mathbf{v}^{\varepsilon}))\Big\} dx^{\varepsilon}.$$

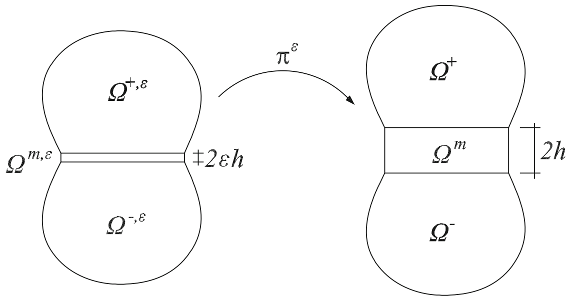

Fig. 2 Change of coordinates from the variable domain Ω^{ε} to Ω

By virtue of the $V^{\varepsilon} \times \Psi^{\varepsilon}$-coercivity of the bilinear forms and thanks to the Lax-Milgram lemma, problem (1) admits one and only one solution.

In order to study the asymptotic behavior of the solution of problem (1) when ε tends to zero, we rewrite the problem on a fixed domain Ω independent of ε. By using the approach of [7], we consider the change of coordinates $\pi^{\varepsilon} : x \in \overline{\Omega} \mapsto x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}$, see Fig. 2, given by

$$\begin{cases} \pi^{\varepsilon}(x_1, x_2, x_3) = (x_1, x_2, x_3 - (1 - \varepsilon h)), & \text{for all } \mathrm{x} \in \overline{\Omega}^{+}_{\mathrm{tr}}, \\ \pi^{\varepsilon}(x_1, x_2, x_3) = (x_1, x_2, \varepsilon x_3), & \text{for all } \mathrm{x} \in \overline{\Omega}^{\mathrm{m}}, \\ \pi^{\varepsilon}(x_1, x_2, x_3) = (x_1, x_2, x_3 + (1 - \varepsilon h)), & \text{for all } \mathrm{x} \in \overline{\Omega}^{-}_{\mathrm{tr}}, \end{cases}$$

where $\Omega^{\pm}_{tr} := \{x \pm \mathbf{e}_3,\ x \in \Omega^{\pm}\}$, $\Omega^m := \omega \times (-h, h)$ and $S^{\pm} := \omega \times \{\pm h\}$. In order to simplify the notation, we identify $\Omega^{\pm}_{tr}$ with $\Omega^{\pm}$, and $\overline{\Omega}$ with $\overline{\Omega}^{\pm} \cup \overline{\Omega}^{m}$. Likewise, we note $\Gamma_{\pm} := \partial\Omega^{\pm}/S^{\pm}$, $\Gamma^m_{lat} := \partial\omega \times (-h, h)$, $(\Gamma_{mD}, \Gamma_{mN})$ and $(\Gamma_{eD}, \Gamma_{eN})$, the partitions of $\partial\Omega := \Gamma^{\pm} \cup \Gamma^m_{lat}$.

Consequently, one has

$$\partial^{\varepsilon}_{\alpha} = \partial_{\alpha} \quad \text{and} \quad \partial^{\varepsilon}_3 = \frac{1}{\varepsilon}\partial_3 \quad \text{in } \Omega^{\mathrm{m}}.$$

With the unknown electromechanical state $s^{\varepsilon} = (\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon})$, we associate the scaled unknown electromechanical state $s(\varepsilon) := (\mathbf{u}(\varepsilon), \varphi(\varepsilon))$ defined by:

$$u^{\varepsilon}_{\alpha}(x^{\varepsilon}) = u_{\alpha}(\varepsilon)(x) \text{ and } \mathrm{u}^{\varepsilon}_3(\mathrm{x}^{\varepsilon}) = \varepsilon^{-1}\mathrm{u}_3(\varepsilon)(\mathrm{x}),$$
$$\varphi^{\varepsilon}(x^{\varepsilon}) = \varepsilon\varphi(\varepsilon)(x) \text{ for all } \mathrm{x}^{\varepsilon} = \pi^{\varepsilon}\mathrm{x} \in \overline{\Omega}^{\mathrm{m},\varepsilon}.$$

We likewise associate with any test functions $r^{\varepsilon} = (\mathbf{v}^{\varepsilon}, \psi^{\varepsilon})$, the scaled test functions $r = (\mathbf{v}, \psi)$, defined by the scalings:

$$v^{\varepsilon}_{\alpha}(x^{\varepsilon}) = v_{\alpha}(x) \text{ and } \mathrm{v}^{\varepsilon}_3(\mathrm{x}^{\varepsilon}) = \varepsilon^{-1}\mathrm{v}_3(\mathrm{x}),$$
$$\psi^{\varepsilon}(x^{\varepsilon}) = \varepsilon\psi(x) \text{ for all } \mathrm{x}^{\varepsilon} = \pi^{\varepsilon}\mathrm{x} \in \overline{\Omega}^{\mathrm{m},\varepsilon}.$$

For ε sufficiently small, we associate with the constant functions $A^{\pm,\varepsilon}_{ijk\ell}$, $H^{\pm,\varepsilon}_{ij}$, $P^{\pm,\varepsilon}_{ijk} : \overline{\Omega}^{\pm,\varepsilon} \to \mathbb{R}$ the constant functions $A^{\pm}_{ijk\ell}$, $H^{\pm}_{ij}$, $P^{\pm}_{ijk} : \overline{\Omega}^{\pm} \to \mathbb{R}$ defined by

$$A^{\pm,\varepsilon}_{ijk\ell} := A^{\pm}_{ijk\ell}, \quad H^{\pm,\varepsilon}_{ij} := H^{\pm}_{ij}, \quad P^{\pm,\varepsilon}_{ijk} := P^{\pm}_{ijk},$$
$$\text{for all } \mathrm{x}^{\varepsilon} = \pi^{\varepsilon}(\mathrm{x}) \in \overline{\Omega}^{\pm,\varepsilon},$$

and we associate with the constant functions $A^{m,\varepsilon}_{ijk\ell}$, $H^{m,\varepsilon}_{ij}$, $P^{m,\varepsilon}_{ijk} : \overline{\Omega}^{m,\varepsilon} \to \mathbb{R}$ the constant functions $A^{m}_{ijk\ell}$, H^{m}_{ij}, $P^{m}_{ijk} : \overline{\Omega}^{m} \to \mathbb{R}$ defined by

$$A^{m,\varepsilon}_{ijk\ell} := \varepsilon^{-1}A^{m}_{ijk\ell}, \quad H^{m,\varepsilon}_{ij} := \varepsilon^{-1}H^{m}_{ij}, \quad P^{m,\varepsilon}_{ijk} := \varepsilon^{-1}P^{m}_{ijk},$$
$$\text{for all } \mathrm{x}^{\varepsilon} = \pi^{\varepsilon}(\mathrm{x}) \in \overline{\Omega}^{\mathrm{m},\varepsilon}.$$

Hence, $\mu^{\varepsilon} = \varepsilon^{-1}\mu$, $\lambda^{\varepsilon} = \varepsilon^{-1}\lambda$, $\eta^{\varepsilon} = \varepsilon^{-1}\eta$, $\tau^{\varepsilon}_{\alpha} = \varepsilon^{-1}\tau_{\alpha}$, $\gamma^{\varepsilon}_{\alpha} = \varepsilon^{-1}\gamma_{\alpha}$ and $\delta^{\varepsilon}_i = \varepsilon^{-1}\delta_i$. We also make the following assumptions on the applied mechanical and electrical forces:

$$f^{\varepsilon}_i(x^{\varepsilon}) = f_i(x) \quad \text{and} \quad \mathrm{g}^{\varepsilon}_{\mathrm{i}}(\mathrm{x}^{\varepsilon}) = \mathrm{g}_{\mathrm{i}}(\mathrm{x}),$$
$$F^{\varepsilon}(x^{\varepsilon}) = F(x) \quad \text{and} \quad \mathrm{d}^{\varepsilon}(\mathrm{x}^{\varepsilon}) = \mathrm{d}(\mathrm{x}),$$
$$\text{for all } \mathrm{x}^{\varepsilon} = \pi^{\varepsilon}\mathrm{x} \in \overline{\Omega}^{\pm,\varepsilon},$$

where functions $f_i \in L^2(\Omega^{\pm})$, $F \in L^2(\Omega^{\pm})$, $g_i \in L^2(\Gamma_{mN})$ and $d \in L^2(\Gamma_{eN})$ are independent of ε. Thus $L^{\varepsilon}(r^{\varepsilon}) = L(r)$.

We define the spaces

$$V := \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^3);\ \mathbf{v} = \mathbf{0} \ \text{ on } \Gamma_{\mathrm{mD}}\},$$
$$\Psi := \{\psi \in H^1(\Omega);\ \psi = 0 \ \text{ on } \Gamma_{\mathrm{eD}}\}.$$

According to the previous assumptions, problem (1) can be reformulated on a fixed domain Ω independent of ε. Thus we obtain the following scaled problem:

$$\begin{cases} \text{Find} s(\varepsilon) \in \mathrm{V} \times \Psi \text{ such that} \\ A^{-}(s(\varepsilon), r) + A^{+}(s(\varepsilon), r) + A^{m}(s(\varepsilon), r) = L(r), \\ \text{for all } \ r \in \mathrm{V} \times \Psi, \end{cases} \tag{2}$$

where the bilinear forms $A^{\pm}$ and A^m are given by

$$A^{\pm}(s(\varepsilon),r) := \int_{\Omega^{\pm}} \Big\{ A^{\pm}_{ijk\ell} e_{k\ell}(\mathbf{u}(\varepsilon)) e_{ij}(\mathbf{v}) + H^{\pm}_{ij} \partial_j \varphi(\varepsilon) \partial_i \psi + P^{\pm}_{ihk}(E_i(\psi) e_{hk}(\mathbf{u}(\varepsilon)) - E_i(\varphi(\varepsilon)) e_{hk}(\mathbf{v})) \Big\} dx,$$
$$A^m(s(\varepsilon),r) := \frac{1}{\varepsilon^4} a^m_{-4}(s(\varepsilon),r) + \frac{1}{\varepsilon^2} a^m_{-2}(s(\varepsilon),r) + a^m_0(s(\varepsilon),r) + \varepsilon^2 a^m_2(s(\varepsilon),r),$$

with

$$a^m_{-4}(s(\varepsilon),r) := \int_{\Omega^m} \tau_1 e_{33}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v}) dx,$$
$$a^m_{-2}(s(\varepsilon),r) := \int_{\Omega^m} (\tau_2 e_{33}(\mathbf{u}(\varepsilon)) e_{\sigma\sigma}(\mathbf{v}) + (\tau_2 e_{\sigma\sigma}(\mathbf{u}(\varepsilon)) + \delta_3 \partial_3 \varphi(\varepsilon)) e_{33}(\mathbf{v}) + 2\eta e_{\alpha 3}(\mathbf{u}(\varepsilon)) e_{\alpha 3}(\mathbf{v}) - \delta_3 e_{33}(\mathbf{u}(\varepsilon)) \partial_3 \psi) dx,$$
$$a^m_0(s(\varepsilon),r) := \int_{\Omega^m} (2\mu e_{\alpha\beta}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + (\lambda e_{\sigma\sigma}(\mathbf{u}(\varepsilon)) + \delta_1 \partial_3 \varphi(\varepsilon)) e_{\tau\tau}(\mathbf{v}) + \delta_2 \partial_\alpha \varphi(\varepsilon) e_{\alpha 3}(\mathbf{v}) - 2\delta_2 e_{\alpha 3}(\mathbf{u}(\varepsilon)) \partial_\alpha \psi + (\gamma_2 \partial_3 \varphi(\varepsilon) - \delta_1 e_{\sigma\sigma}(\mathbf{u}(\varepsilon))) \partial_3 \psi) dx,$$
$$a^m_2(s(\varepsilon),r) := \int_{\Omega^m} \gamma_1 \partial_\alpha \varphi(\varepsilon) \partial_\alpha \psi dx.$$

The rescaled variational problem (2) has a unique solution in $V \times \Psi$ by virtue of the Lax-Milgram lemma. In the sequel, only if necessary, we will note, respectively, with $(\mathbf{v}^{\pm}, \psi^{\pm})$ and $(\mathbf{v}^m, \psi^m)$, the restrictions of functions $(\mathbf{v}, \psi)$ to $\Omega^{\pm}$ and Ω^m.

3 The limit model: asymptotic expansions

We can now perform an asymptotic analysis of the rescaled problem (2). Since the rescaled problem (2) has a polynomial structure with respect to the small parameter ε, we can look for the solution $s(\varepsilon) = (\mathbf{u}(\varepsilon), \varphi(\varepsilon))$ of the problem as a series of powers of ε:

$$s(\varepsilon) = s^0 + \varepsilon^2 s^2 + \varepsilon^4 s^4 + \ldots \Rightarrow \begin{cases} \mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon^2 \mathbf{u}^2 + \varepsilon^4 \mathbf{u}^4 + \ldots \\ \varphi(\varepsilon) = \varphi^0 + \varepsilon^2 \varphi^2 + \varepsilon^4 \varphi^4 + \ldots \end{cases}, \tag{3}$$

with $s^q = (\mathbf{u}^q, \varphi^q) \in V \times \Psi, q \geq 0$. By substituting (3) into the rescaled problem (2), and by identifying the terms with identical power of ε, we obtain, as customary, the following set of problems, defined for all $r \in V \times \Psi$:

$$\mathcal{P}_{-4} : a^m_{-4}(s^0, r) = 0,$$
$$\mathcal{P}_{-2} : a^m_{-4}(s^2, r) + a^m_{-2}(s^0, r) = 0,$$
$$\mathcal{P}_2 : a^m_{-4}(s^4, r) + a^m_{-2}(s^2, r) + a^m_0(s^0, r) + A^+(s^0, r) + A^-(s^0, r) = L(r),$$
$$\mathcal{P}_{2p} : a^m_{-4}(s^{2p+4}, r) + a^m_{-2}(s^{2p+2}, r) + a^m_0(s^{2p}, r) + a^m_2(s^{2p-2}, r) + A^+(s^{2p}, r) + A^-(s^{2p}, r) = 0,\ p \geq 2.$$

To proceed with the asymptotic analysis we need to solve each variational subproblem above and characterize the limit electromechanical state $s^0 = (\mathbf{u}^0, \varphi^0)$ and its associated limit problem. We state in Theorem 1 below the formulation of the limit problem.

Theorem 1 The leading term $s^0 = (\mathbf{u}^0, \varphi^0)$ of the asymptotic expansion (3) satisfies the following variational problem:

$$\begin{cases} \text{Find } \mathrm{s}^0 \in \widetilde{\mathrm{V}} \times \widetilde{\Psi} \text{ such that} \\ A^+(s^0, r) + A^-(s^0, r) + A^m_{KL}(s^0, r) = L(r), \\ \text{for all } \mathrm{r} \in \widetilde{\mathrm{V}} \times \widetilde{\Psi}, \end{cases}$$

where

$$\widetilde{V} := \{\mathbf{v}^{\pm} \in V,\ \mathbf{v}^m \in V_{KL};\ \mathbf{v}^{\pm}|_{S^{\pm}} = \mathbf{v}^m|_{S^{\pm}}\},$$
$$V_{KL} := \{\mathbf{v} \in H^1(\Omega^m; \mathbb{R}^3);\ e_{i3}(\mathbf{v}) = 0,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma^0_{\mathrm{mD}}\},$$
$$\widetilde{\Psi} := \{\psi^{\pm} \in \Psi,\ \psi^m \in \Psi^m;\ \psi^{\pm}|_{S^{\pm}} = \psi^m|_{S^{\pm}}\},$$
$$\Psi^m := \{\psi \in L^2(\Omega^m);\ \partial_3 \psi \in L^2(\Omega^m)\},$$

and

$$A^m_{KL}(s^0, r) := \int_{\Omega^m} \Big\{ 2\mu e_{\alpha\beta}(\mathbf{u}^0) e_{\alpha\beta}(\mathbf{v}) + (\mathcal{B} e_{\sigma\sigma}(\mathbf{u}^0) + \mathcal{C} \partial_3 \varphi^0) e_{\tau\tau}(\mathbf{v}) + (-\mathcal{C} e_{\sigma\sigma}(\mathbf{u}^0) + \mathcal{D} \partial_3 \varphi^0) \partial_3 \psi \Big\} dx.$$

Proof For the sake of clarity, the proof is divided into three steps numbered from (i) to (iii).

(i) We start by solving problem $\mathcal{P}_{-4}$. Let us choose test functions $r = s^0 \in V \times \Psi$:

$$\int_{\Omega^m} \tau_1 e_{33}(\mathbf{u}^0) e_{33}(\mathbf{u}^0) dx = 0.$$

Since $\tau_1 > 0$, we have $e_{33}(\mathbf{u}^0) = 0$ and, thus, $u_3^{m,0} = w^0(\tilde{x})$, with $\tilde{x} = (x_\alpha) \in \omega$.

(ii) Let us consider problem $\mathcal{P}_{-2}$. Since $e_{33}(\mathbf{u}^0) = 0$, we get

$$\int_{\Omega^m} \Big\{ (\tau_1 e_{33}(\mathbf{u}^2) + \tau_2 e_{\sigma\sigma}(\mathbf{u}^0) + \delta_3 \partial_3 \varphi^0) e_{33}(\mathbf{v}) + 2\eta e_{\alpha 3}(\mathbf{u}^0) e_{\alpha 3}(\mathbf{v}) \Big\} dx = 0.$$

By choosing test functions $r = (\mathbf{u}^0, 0) \in V \times \Psi$, we obtain that

$$\int_{\Omega^m} 2\eta e_{\alpha 3}(\mathbf{u}^0) e_{\alpha 3}(\mathbf{u}^0) dx = 0,$$

and, since $\eta > 0$, $e_{\alpha 3}(\mathbf{u}^0) = 0$, which implies that $u_\alpha^{m,0}(\tilde{x}, x_3) = \bar{u}_\alpha^0(\tilde{x}) - x_3 \partial_\alpha w^0(\tilde{x})$. Thus,

$$\int_{\Omega^m} (\tau_1 e_{33}(\mathbf{u}^2) + \tau_2 e_{\sigma\sigma}(\mathbf{u}^0) + \delta_3 \partial_3 \varphi^0) e_{33}(\mathbf{v}) dx = 0,$$

which is verified when $e_{33}(\mathbf{u}^2) = -\frac{1}{\tau_1}(\tau_2 e_{\sigma\sigma}(\mathbf{u}^0) + \delta_3 \partial_3 \varphi^0)$. From this relation we obtain a characterization of $u_3^{m,2}$. Indeed, one has

$$u_3^{m,2}(\tilde{x}, x_3) = \bar{u}_3^2(\tilde{x}) - \frac{\tau_2}{\tau_1}\left(x_3 \partial_\sigma \bar{u}_\sigma^0(\tilde{x}) - \frac{x_3^2}{2} \partial_{\sigma\sigma} w^0(\tilde{x}) \right) - \frac{\delta_3 \tau_2}{\tau_1} \varphi^0(\tilde{x}, x_3).$$

The displacement field $\mathbf{u}^{m,0}$ verifies the Kirchhoff–Love kinematical assumptions and it belongs to $V_{KL} := \{\mathbf{v} \in H^1(\Omega^m; \mathbb{R}^3);\ e_{i3}(\mathbf{v}) = 0,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma^0_{\mathrm{mD}}\} = \{\mathbf{v} \in H^1(\omega; \mathbb{R}^2) \times H^2(\omega);\ v_i = \partial_\nu v_3 = 0 \text{ on } \gamma_0\}$, where $\Gamma^0_{mD} := \gamma_0 \times (-h, h) \subset \Gamma^m_{lat}$ denotes the fixed part of the boundary Γ^m_{lat} of the plate-like body.

(iii) Let us consider problem $\mathcal{P}_0$. By choosing test functions $r \in \widetilde{V} \times \widetilde{\Psi} \subset V \times \Psi$, and by means of the relations among $\mathbf{u}^2$, $\mathbf{u}^0$ and φ^0, obtained above, one gets the limit problem, announced in Theorem 1:

$$\begin{cases} \text{Find } \mathrm{s}^0 \in \widetilde{\mathrm{V}} \times \widetilde{\Psi} \text{ such that} \\ A^+(s^0, r) + A^-(s^0, r) + A^m_{KL}(s^0, r) = L(r), \\ \text{for all } \mathrm{r} \in \widetilde{\mathrm{V}} \times \widetilde{\Psi}, \end{cases}$$

where

$$A^m_{KL}(s^0, r) := \int_{\Omega^m} \Big\{ 2\mu e_{\alpha\beta}(\mathbf{u}^0) e_{\alpha\beta}(\mathbf{v}) + (\mathcal{B} e_{\sigma\sigma}(\mathbf{u}^0) + \mathcal{C} \partial_3 \varphi^0) e_{\tau\tau}(\mathbf{v}) + (-\mathcal{C} e_{\sigma\sigma}(\mathbf{u}^0) + \mathcal{D} \partial_3 \varphi^0) \partial_3 \psi \Big\} dx, \tag{4}$$

and,

$$\mathcal{B} := \frac{\lambda \tau_1 - \tau_2^2}{\tau_1}, \quad \mathcal{C} := \frac{\delta_1 \tau_1 - \delta_3 \tau_2}{\tau_1}, \quad \mathcal{D} := \frac{\delta_3^2 + \tau_1 \gamma_2}{\tau_1}.$$

This completes the proof. □

The limit electric potential $\varphi^{m,0}$ can be explicitly characterized as a second order polynomial function of x_3. Indeed let us consider the limit problem (4) and choose test functions $r = (\mathbf{0}, \psi) \in \widetilde{V} \times \widetilde{\Psi}$. By integrating by parts, we obtain the following equation with its associated continuity conditions of the electric potential at the interfaces $S^\pm$:

$$\begin{cases} \partial_{33} \varphi^{m,0}(\tilde{x}, x_3) = -\dfrac{\mathcal{C}}{\mathcal{D}} \Delta w^0(\tilde{x}) & \text{in } \Omega^{\mathrm{m}}, \\ \varphi^{m,0}(\tilde{x}, h) = \varphi^{+,0}, \quad \varphi^{m,0}(\tilde{x}, -h) = \varphi^{-,0} & \text{on } \mathrm{S}^\pm, \end{cases}$$

where $\varphi^{\pm,0} := \varphi^{\pm,0}(\tilde{x}, \pm h)$ and $\Delta = \partial_{\sigma\sigma}$ is the two-dimensional Laplacian operator. The electric potential can be written as follows

$$\varphi^{m,0}(\tilde{x}, x_3) = \sum_{k=0}^{2} \phi^k(\tilde{x}) x_3^k \tag{5}$$

with

$$\phi^0 = \frac{\varphi^{+,0} + \varphi^{-,0}}{2} + \frac{\mathcal{C}h^2}{2\mathcal{D}} \Delta w^0, \quad \phi^1 = \frac{\varphi^{+,0} - \varphi^{-,0}}{2h},$$
$$\phi^2 = -\frac{\mathcal{C}}{2\mathcal{D}} \Delta w^0.$$

Remark 1 The previous characterization of the electric potential $\varphi^{m,0}$ is a rigorous justification of the a priori assumptions conjectured in [2]. We can also notice that the regularity of the electric potential only depends on the regularities of $\varphi^{\pm,0}$ and of w^0. Hence, $\varphi^{\pm,0} \in L^2(\omega)$ and $w^0 \in H^2(\omega)$ imply that $\varphi^{m,0} \in \Psi^m$. The space Ψ^m with the norm $\|\psi\|^2_{\Psi^m} := |\psi|^2_{0,\Omega^m} + |\partial_3\psi|^2_{0,\Omega^m}$ can be identified with the space $H^1(-h,h;L^2(\omega))$ endowed with the usual norm. Therefore, the trace of the elements of Ψ^m on $S^\pm$ makes sense in $L^2(S^\pm)$.

Theorem 2 The limit problem (4) admits one and only one solution in $\widetilde{V} \times \widetilde{\Psi}$.

Proof In order to prove the uniqueness of the solution of the limit problem (4), we assume that there exist two solution $s^1 = (\mathbf{u}^1, \varphi^1)$ and $s^2 = (\mathbf{u}^2, \varphi^2)$. By letting $\mathbf{w} := \mathbf{u}^1 - \mathbf{u}^2$ and $\vartheta := \varphi^1 - \varphi^2$, we define the electromechanical state $z = (\mathbf{w}, \vartheta) \in \widetilde{V} \times \widetilde{\Psi}$. Then

$$A^+(z,z) + A^-(z,z) + A^m_{KL}(z,z) = 0, \quad \text{for all } z \in \widetilde{V} \times \widetilde{\Psi},$$

where

$$A^m_{KL}(z,z) := \int_{\Omega^m} \{ (2\mu\delta_{\alpha\sigma}\delta_{\beta\tau} + \mathcal{B}\delta_{\alpha\beta}\delta_{\sigma\tau}) e_{\alpha\beta}(\mathbf{w}) e_{\sigma\tau}(\mathbf{w}) + \mathcal{D}\partial_3\vartheta\partial_3\vartheta \} dx.$$

Thanks to the positivity of the elastic coefficients, we obtain that $\mathbf{w}^\pm = \mathbf{0} \in V(\Omega^\pm)$, $\vartheta^\pm = 0 \in \Psi$, $\mathbf{w}^m = \mathbf{0} \in V_{KL}$, $\partial_3\vartheta^m = 0 \in L^2(\Omega^m)$. The continuity conditions at the interfaces imply also that $\vartheta^m = 0 \in \Psi^m$. □

We justify mathematically the simplified limit model in Appendix by means of a strong convergence argument. The strong convergence represents an important formal result according to which the energy of the initial physical problem converges towards the energy of the simplified limit model.

3.1 A different form of the limit problem

By taking advantage of the explicit form (5) of the electric potential $\varphi^{m,0}$, we can rewrite the limit problem in a different way, as shown in Theorem 3.

Theorem 3 The limit electromechanical state $s^0 = (\mathbf{u}^0, \varphi^0)$ satisfies the following limit problem

$$\begin{cases} \text{Find } s^0 \in \widetilde{V} \times \widetilde{\Psi} \text{ such that} \\ A^+(s^0, r) + A^-(s^0, r) + \widetilde{A}^m_{KL}(s^0, r) = L(r), \\ \text{for all } r \in \widetilde{V} \times \widetilde{\Psi}, \end{cases} \tag{6}$$

with

$$\begin{aligned} \widetilde{A}^m_{KL}(s^0, r) := 2h \int_\omega & \Big\{ 2\mu e_{\alpha\beta}(\bar{\mathbf{u}}^0) e_{\alpha\beta}(\bar{\mathbf{v}}) \\ & + \left(\mathcal{B} e_{\sigma\sigma}(\bar{\mathbf{u}}^0) + \mathcal{C}\frac{[\![\varphi^0]\!]}{2h} \right) e_{\tau\tau}(\bar{\mathbf{v}}). \\ & + \left(-\mathcal{C} e_{\sigma\sigma}(\bar{\mathbf{u}}^0) \mathcal{D}\frac{[\![\varphi^0]\!]}{2h} \right) \frac{[\![\psi]\!]}{2h} \Big\} d\tilde{x} \\ & + \frac{2h^3}{3} \int_\omega \Big\{ 2\mu \partial_{\alpha\beta} w^0 \partial_{\alpha\beta} v_3 \\ & + \left(\frac{\mathcal{B}\mathcal{D} + \mathcal{C}^\in}{\mathcal{D}} \right) \Delta w^0 \Delta v_3 \Big\} d\tilde{x}. \end{aligned}$$

where $\bar{\mathbf{u}}^0 = (\bar{u}^0_\alpha)$ represents the in-plane displacement field and $[\![f]\!] := f^+ - f^-$ denotes the jump function at the interface ω between Ω^+ and Ω^-.

Proof In order to prove the result of Theorem 3, we just insert expression (5) for the limit electric potential $\varphi^{m,0} \in \Psi^m$ in (4) and we choose some specific test functions belonging to $\widetilde{V} \times \widetilde{\Psi}$, having the same form of the limit electromechanical state, i.e., $v^m_\alpha(\tilde{x}, x_3) = \bar{v}_\alpha(\tilde{x}) - x_3\partial_\alpha v_3(\tilde{x})$, $v^m_3(\tilde{x}, x_3) = v_3(\tilde{x}) \in V_{KL}$, $\psi^m(\tilde{x}, x_3) = \psi^0(\tilde{x}) + x_3\psi^1(\tilde{x}) + x_3^2\psi^2(\tilde{x}) \in \Psi^m$, with ψ^0, ψ^2, $\psi^1 := \frac{[\![\psi]\!]}{2h} \in L^2(\omega)$ and $[\![\psi]\!] = \psi^+ - \psi^-$. By integrating through the thickness along x_3, we obtain that the bilinear form $A^m_{KL}(s^0, r)$, defined over Ω^m, can be identified with an equivalent two-dimensional bilinear form $\widetilde{A}^m_{KL}(s^0, r)$, defined over the middle plane ω. □

3.2 The electromechanical interface problems

The aim of this section is to derive a coupled electromechanical interface problem between the two piezoelectric bodies Ω^+ and Ω^- with some ad hoc transmission conditions at the interface ω. By virtue of the asymptotic methods we replace the three-dimensional electromechanical energy of the intermediate piezoelectric layer with a specific two-dimensional surface energy defined over the middle plane of the plate. This surface energy generates non classical transmission conditions between the two three-dimensional bodies. We distinguish, respectively, between the electric and the mechanical interface problems, each one with their associated appropriate transmission conditions. By rewriting problem (6) in its differential form after an integration by parts, we obtain:

Electrostatic problems in $\Omega^{\pm}$

$$\begin{cases} \partial_i D_i^{\pm}(\mathbf{u}^0, \varphi^0) = F & \text{in } \Omega^{\pm}, \\ D_i^{\pm}(\mathbf{u}^0, \varphi^0) n_i = d & \text{on } \Gamma_{\mathrm{eN}}, \\ \varphi^0 = 0 & \text{on } \Gamma_{\mathrm{eD}}, \end{cases}$$

Elasticity problems in $\Omega^{\pm}$

$$\begin{cases} -\partial_j \sigma_{ij}^{\pm}(\mathbf{u}^0, \varphi^0) = f_i & \text{in } \Omega^{\pm}, \\ \sigma_{ij}^{\pm}(\mathbf{u}^0, \varphi^0) n_j = g_i & \text{on } \Gamma_{\mathrm{mN}}, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \Gamma_{\mathrm{mD}}, \end{cases}$$

Transmission conditions on ω

$$\begin{cases} [\![D_3(\mathbf{u}^0, \varphi^0)]\!] = 0 & \text{on } \omega, \\ [\![\sigma_{\alpha 3}(\mathbf{u}^0, \varphi^0)]\!] - \mathcal{C}[\![\partial_\alpha \varphi^0]\!] = \partial_\beta n_{\alpha\beta}(\bar{\mathbf{u}}^0) & \text{on } \omega, \\ [\![\sigma_{33}(\mathbf{u}^0, \varphi^0)]\!] = \partial_{\alpha\beta} m_{\alpha\beta}(w^0) & \text{on } \omega, \\ [\![\mathbf{u}^0]\!] = \mathbf{0} & \text{on } \omega, \end{cases}$$

where $D_i^{\pm}(\mathbf{u}^0, \varphi^0)$ and $\sigma_{ij}^{\pm}(\mathbf{u}^0, \varphi^0)$ represent, respectively, the electric displacement components and the Cauchy stress tensor components on $\Omega^{\pm}$, $n_{\alpha\beta}(\bar{\mathbf{u}}^0) := 2h(2\mu e_{\alpha\beta}(\bar{\mathbf{u}}^0) + \mathcal{B} e_{\sigma\sigma}(\bar{\mathbf{u}}^0)\delta_{\alpha\beta})$ denotes the membrane stress tensor components, while $m_{\alpha\beta}(w^0) := -\frac{2h^3}{3}(2\mu \partial_{\alpha\beta} w^0 + (\mathcal{C} - \mathcal{B})\Delta w^0 \delta_{\alpha\beta})$ denote the moment tensor components.

Remark 2 The previous electromechanical interface problem can be considered as a generalization in the case of piezoelectric assemblies of the transmission problem obtained in [3–5] for thin elastic inclusions with high rigidity. The particular jump conditions at the interface yield to a non standard transmission problem which can be solved by an adapted Neumann-Neumann domain decomposition algorithm [10].

4 Concluding remarks

In the present work we derive an interface model corresponding to a piezoelectric multimaterial with a piezoelectric interphase by means of an asymptotic analysis. The intermediate thin piezoelectric plate-like body is replaced by a particular surface energy associated with ad hoc transmission conditions at the interface depending on the jump of stresses and normal electric displacement. Besides, we give a formal justification of the limit model by virtue of a strong convergence argument.

The limit model is extremely versatile for the applications: indeed, by changing the nature of the bodies constituting the multimaterial, we can obtain different configurations, modeling various assemblies. The more general situation is the one described in this work with two piezoelectric bodies divided by a thin piezoelectric layer. We can adapt the limit model to other situations: for instance, we can choose the case of two elastic conductor bodies with a piezoelectric intermediate layer, or the case of two elastic conductor bodies with an elastic conductor intermediate layer, or all other different combinations playing with the electromechanical characteristics of the involved materials.

As future developments, it would be interesting to implement numerically the limit model by using a domain decomposition algorithm as in [10].

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Appendix

In the section we justify from mathematical point of view the obtained limit model through a variational convergence result. We prove that the solution of the physical problem strongly converges towards the solution of the limit model. The strong convergence corresponds to the convergence of the energies.

In the sequel we denote by $\|\cdot\|_{s,\Omega}$ the norm of the Sobolev space $H^s(\Omega;\mathbb{R}^d)$ for all $d\geq 1$ and $|\cdot|_{0,\Omega}$ stands for the norm in $L^2(\Omega;\mathbb{R}^d)$. Obviously, the same holds in $\Omega^{\pm}$, Ω^m and ω.

To begin with, we introduce the following notation. We let

$$\mathbf{a}\cdot\mathbf{b} := a_i b_i, \qquad \mathbf{A}:\mathbf{B} := a_{ij}b_{ij},$$

for, respectively, all vectors $\mathbf{a}=(a_i)$ and $\mathbf{b}=(b_i)$, and for all symmetric second order matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$.

With the scaled electromechanical state $s(\varepsilon)=(\mathbf{u}(\varepsilon),\varphi(\varepsilon))\in V\times\Psi$, we associate, respectively, the following scaled strain tensor field $\boldsymbol{\kappa}(\varepsilon)=(\kappa_{ij}(\varepsilon))$, with $\kappa_{ij}(\varepsilon)\in L^2(\Omega)$, and the scaled electric potential vector field $\boldsymbol{\theta}(\varepsilon)=(\theta_i(\varepsilon))$, with $\theta_i(\varepsilon)\in L^2(\Omega)$, defined by

$$\kappa^{\pm}_{ij}(\varepsilon) := e_{ij}(\mathbf{u}^{\pm}(\varepsilon)), \quad \theta^{\pm}_i(\varepsilon) := \partial_i\varphi^{\pm}(\varepsilon)$$

$$\kappa^{m}_{\alpha\beta}(\varepsilon) := e_{\alpha\beta}(\mathbf{u}^{m}(\varepsilon)), \quad \kappa^{m}_{\alpha 3}(\varepsilon) := \frac{1}{\varepsilon}e_{\alpha 3}(\mathbf{u}^{m}(\varepsilon)),$$

$$\kappa^{m}_{33}(\varepsilon) := \frac{1}{\varepsilon^2}e_{33}(\mathbf{u}^{m}(\varepsilon)),$$

$$\theta^{m}_{\alpha}(\varepsilon) := \varepsilon\partial_\alpha\varphi^{m}(\varepsilon), \quad \theta^{m}_{3}(\varepsilon) := \partial_3\varphi^{m}(\varepsilon).$$

With an arbitrary electromechanical state $r=(\mathbf{v},\psi)\in V\times\Psi$, we associate, respectively, the following tensor field $\boldsymbol{\kappa}(\varepsilon;\mathbf{v})=(\kappa_{ij}(\varepsilon;\mathbf{v}))$ and vector field $\boldsymbol{\theta}(\varepsilon;\psi)=(\theta_i(\varepsilon;\psi))$. In particular, one has $\boldsymbol{\kappa}(\varepsilon)=\boldsymbol{\kappa}(\varepsilon;\mathbf{u}(\varepsilon))$ and $\boldsymbol{\theta}(\varepsilon)=\boldsymbol{\theta}(\varepsilon;\varphi(\varepsilon))$. Then the rescaled problem (2) can be rewritten in the following condensed form:

$$\begin{cases} \text{Find} s(\varepsilon)\in \mathrm{V}\times\Psi \text{ such that} \\ A^{-}(s(\varepsilon),r)+A^{+}(s(\varepsilon),r)+A^{m}(s(\varepsilon),r)=L(r), \\ \text{for all } \mathrm{r}\in\mathrm{V}\times\Psi, \end{cases} \tag{7}$$

where the bilinear forms $A^{\pm}$ and A^m are given by

$$A^{\pm}(s(\varepsilon),r) := \int_{\Omega^{\pm}}\{\mathsf{A}^{\pm}\boldsymbol{\kappa}^{\pm}(\varepsilon):\boldsymbol{\kappa}^{\pm}(\varepsilon;\mathbf{v})+\mathsf{H}^{\pm}\boldsymbol{\theta}^{\pm}(\varepsilon)\cdot\boldsymbol{\theta}^{\pm}(\varepsilon;\psi)$$
$$+\mathsf{P}^{\pm}\boldsymbol{\kappa}^{\pm}(\varepsilon;\mathbf{v})\cdot\boldsymbol{\theta}^{\pm}(\varepsilon)-\mathsf{P}^{\pm}\boldsymbol{\kappa}^{\pm}(\varepsilon)\cdot\boldsymbol{\theta}^{\pm}(\varepsilon;\psi)\}dx,$$
$$A^{m}(s(\varepsilon),r) := \int_{\Omega^{m}}\{\mathsf{A}^{m}\boldsymbol{\kappa}^{m}(\varepsilon):\boldsymbol{\kappa}^{m}(\varepsilon;\mathbf{v})+\mathsf{H}^{m}\boldsymbol{\theta}^{m}(\varepsilon)\cdot\boldsymbol{\theta}^{m}(\varepsilon;\psi)$$
$$+\mathsf{P}^{m}\boldsymbol{\kappa}^{m}(\varepsilon;\mathbf{v})\cdot\boldsymbol{\theta}^{m}(\varepsilon)-\mathsf{P}^{m}\boldsymbol{\kappa}^{m}(\varepsilon)\cdot\boldsymbol{\theta}^{m}(\varepsilon;\psi)\}dx.$$

$\mathsf{A}=(A_{ijk\ell})$, $\mathsf{P}=(P_{ijk})$ and $\mathsf{H}=(H_{ij})$ represent, respectively, the elasticity tensor, the coupling tensor and the dielectric tensor.

The strong convergence result is stated in the following theorem.

Theorem 4 The sequence $(s(\varepsilon))_{\varepsilon>0}=(\mathbf{u}(\varepsilon),\varphi(\varepsilon))_{\varepsilon>0}$ strongly converges to $s^0=(\mathbf{u}^0,\varphi^0)$ in $H^1(\Omega,\mathbb{R}^3)\times L^2(\Omega)$, the unique solution of (4).

Proof For convenience, the proof is divided into three steps, numbered from (i) to (iii).

(i) By letting $r=s(\varepsilon)$, i.e., $\mathbf{v}=\mathbf{u}(\varepsilon)$ and $\psi=\varphi(\varepsilon)$, the rescaled problem (7) becomes, as customary,

$$\int_{\Omega^{\pm}}\{\mathsf{A}^{\pm}\boldsymbol{\kappa}^{\pm}(\varepsilon):\boldsymbol{\kappa}^{\pm}(\varepsilon)+\mathsf{H}^{\pm}\boldsymbol{\theta}^{\pm}(\varepsilon)\cdot\boldsymbol{\theta}^{\pm}(\varepsilon)\}dx$$
$$+\int_{\Omega^{m}}\{\mathsf{A}^{m}\boldsymbol{\kappa}^{m}(\varepsilon):\boldsymbol{\kappa}^{m}(\varepsilon)+\mathsf{H}^{m}\boldsymbol{\theta}^{m}(\varepsilon)\cdot\boldsymbol{\theta}^{m}(\varepsilon)\}$$
$$dx=L(s(\varepsilon)).$$

By virtue of the positive definiteness of A and H and by means of the Korn's inequality and the Poincaré's inequality, we derive that

$$\int_{\Omega^{\pm}}\{\mathsf{A}^{\pm}\boldsymbol{\kappa}^{\pm}(\varepsilon):\boldsymbol{\kappa}^{\pm}(\varepsilon)+\mathsf{H}^{\pm}\boldsymbol{\theta}^{\pm}(\varepsilon)\cdot\boldsymbol{\theta}^{\pm}(\varepsilon)\}dx$$
$$+\int_{\Omega^{m}}\{\mathsf{A}^{m}\boldsymbol{\kappa}^{m}(\varepsilon):\boldsymbol{\kappa}^{m}(\varepsilon)+\mathsf{H}^{m}\boldsymbol{\theta}^{m}(\varepsilon)\cdot\boldsymbol{\theta}^{m}(\varepsilon)\}dx$$
$$\geq C\left\{|\boldsymbol{\kappa}^{\pm}(\varepsilon)|^2_{0,\Omega^{\pm}}+|\boldsymbol{\theta}^{\pm}(\varepsilon)|^2_{0,\Omega^{\pm}}+|\boldsymbol{\kappa}^{m}(\varepsilon)|^2_{0,\Omega^{m}}+|\boldsymbol{\theta}^{m}(\varepsilon)|^2_{0,\Omega^{m}}\right\}$$
$$\geq C\left\{|\boldsymbol{\kappa}^{\pm}(\varepsilon)|^2_{0,\Omega^{\pm}}+|\boldsymbol{\theta}^{\pm}(\varepsilon)|^2_{0,\Omega^{\pm}}+\sum_{\alpha,\beta}|e_{\alpha\beta}(\mathbf{u}(\varepsilon))|^2_{0,\Omega^{m}}\right.$$
$$+\frac{1}{\varepsilon^2}\sum_{\alpha}|e_{\alpha 3}(\mathbf{u}(\varepsilon))|^2_{0,\Omega^{m}}+\frac{1}{\varepsilon^4}|e_{33}(\mathbf{u}(\varepsilon))|^2_{0,\Omega^{m}}$$
$$\left.+\varepsilon^2\sum_{\alpha}|\partial_\alpha\varphi(\varepsilon)|^2_{0,\Omega^{m}}+|\partial_3\varphi(\varepsilon)|^2_{0,\Omega^{m}}\right\}$$
$$\geq\left\{\|\mathbf{u}(\varepsilon)\|^2_{1,\Omega^{\pm}}+|\varphi(\varepsilon)|^2_{0,\Omega^{\pm}}+\|\mathbf{u}(\varepsilon)\|^2_{1,\Omega^{m}}\right.$$
$$\left.+\varepsilon^2\sum_{\alpha}|\partial_\alpha\varphi(\varepsilon)|^2_{0,\Omega^{m}}+|\partial_3\varphi(\varepsilon)|^2_{0,\Omega^{m}}\right\}.$$

On the other side, by virtue of the continuity of the linear form, one has:

$$L(s(\varepsilon)) \le C\Big\{\|\mathbf{u}(\varepsilon)\|_{1,\Omega^\pm} + |\varphi(\varepsilon)|_{0,\Omega^\pm}\Big\}$$
$$\le \Big\{\|\mathbf{u}(\varepsilon)\|^2_{1,\Omega^\pm} + |\varphi(\varepsilon)|^2_{0,\Omega^\pm} + \|\mathbf{u}(\varepsilon)\|^2_{1,\Omega^m}$$
$$+\varepsilon^2\sum_\alpha |\partial_\alpha\varphi(\varepsilon)|^2_{0,\Omega^m} + |\partial_3\varphi(\varepsilon)|^2_{0,\Omega^m}\Big\}^{1/2}.$$

The inequalities above imply that the norms $|\boldsymbol{\kappa}^\pm(\varepsilon)|_{0,\Omega^\pm}$, $|\boldsymbol{\theta}^\pm(\varepsilon)|_{0,\Omega^\pm}$, $|\boldsymbol{\kappa}^m(\varepsilon)|_{0,\Omega^m}$ and $|\boldsymbol{\theta}^m(\varepsilon)|_{0,\Omega^m}$ are bounded independently of ε in $L^2(\Omega)$. This means that $\kappa^\pm_{ij}(\varepsilon) \rightharpoonup \kappa^\pm_{ij}$ in $L^2(\Omega^\pm)$, $\theta^\pm_i(\varepsilon) \rightharpoonup \theta^\pm_i$ in $L^2(\Omega^\pm)$, $\kappa^m_{ij}(\varepsilon) \rightharpoonup \kappa^m_{ij}$ in $L^2(\Omega^m)$ and $\theta^m_i(\varepsilon) \rightharpoonup \theta^m_i$ in $L^2(\Omega^m)$. Moreover, the norms $\|\mathbf{u}(\varepsilon)\|_{1,\Omega^\pm}$, $|\varphi(\varepsilon)|_{0,\Omega^\pm}$ and $\|\mathbf{u}(\varepsilon)\|_{1,\Omega^m}$ are also bounded. Then, from definition of $\kappa_{ij}(\varepsilon)$ and $\theta_i(\varepsilon)$, there exists a constant c such that

$$\begin{aligned}&|e_{ij}(\mathbf{u}(\varepsilon))|_{0,\Omega^\pm} \le c, \quad \|\mathbf{u}^\pm(\varepsilon)\|_{1,\Omega^\pm} \le c,\\ &|\partial_i\varphi^\pm(\varepsilon)|_{0,\Omega^\pm} \le c, \quad |\varphi^\pm(\varepsilon)|_{0,\Omega^\pm} \le c,\end{aligned} \tag{8}$$

and, also,

$$\begin{aligned}&|e_{\alpha\beta}(\mathbf{u}(\varepsilon))|_{0,\Omega^m} \le c, \qquad |e_{\alpha3}(\mathbf{u}(\varepsilon))|_{0,\Omega^m} \le c\varepsilon,\\ &|e_{33}(\mathbf{u}(\varepsilon))|_{0,\Omega^m} \le c\varepsilon^2, \quad \|\mathbf{u}(\varepsilon)\|_{1,\Omega^m} \le c,\\ &|\partial_\alpha\varphi^m(\varepsilon)|_{0,\Omega^m} \le \tfrac{c}{\varepsilon}, \qquad |\partial_3\varphi^m(\varepsilon)|_{0,\Omega^m} \le c.\end{aligned} \tag{9}$$

From the first set of inequalities (8), we obtain that $\mathbf{u}^\pm(\varepsilon) \rightharpoonup \mathbf{u}^\pm$ in $H^1(\Omega^\pm;\mathbb{R}^3)$ and, consequently, $e_{ij}(\mathbf{u}(\varepsilon)) \rightharpoonup e_{ij}(\mathbf{u}^\pm)$ in $L^2(\Omega^\pm)$, i.e., $\kappa^\pm = e_{ij}(\mathbf{u}^\pm)$; besides, since both $|\partial_i\varphi^\pm(\varepsilon)|_{0,\Omega^\pm}$ and $|\varphi^\pm(\varepsilon)|_{0,\Omega^\pm}$ are bounded in $L^2(\Omega^\pm)$, we have that $\varphi^\pm(\varepsilon) \rightharpoonup \varphi^\pm$ in $H^1(\Omega^\pm)$.
From the second set of inequalities (9), we get that $\mathbf{u}^m(\varepsilon) \rightharpoonup \mathbf{u}^m$ in $H^1(\Omega^m;\mathbb{R}^3)$ and $e_{i3}(\mathbf{u}(\varepsilon)) \to 0$ in $L^2(\Omega^m)$. This implies that $\partial_3 u^m_3(\varepsilon) \to 0$ and, thus, by continuity of the derivative operator, we obtain that $\partial_3 u^m_3 = 0$, i.e., $u^m_3(\tilde{x}, x_3) = u^m_3(\tilde{x})$ is independent of x_3. We also have that $\partial_3 u^m_\alpha(\varepsilon) + \partial_\alpha u^m_3(\varepsilon) \to 0$, meaning that $\partial_3 u^m_\alpha = -\partial_\alpha u^m_3$, i.e., $u^m_\alpha(\tilde{x}, x_3) = \bar{u}^m_\alpha(\tilde{x}) - x_3\partial_\alpha u^m_3(\tilde{x})$. Consequently, $\mathbf{u}^m \in V_{KL}$. Moreover, we obtain that $e_{\alpha\beta}(\mathbf{u}(\varepsilon)) \rightharpoonup e_{\alpha\beta}(\mathbf{u}^m)$ in $L^2(\Omega^m)$, $e_{\alpha3}(\frac{\mathbf{u}(\varepsilon)}{\varepsilon}) \rightharpoonup \kappa^m_{\alpha3}$ in $L^2(\Omega^m)$, $e_{33}(\frac{\mathbf{u}(\varepsilon)}{\varepsilon}) \to 0$ in $L^2(\Omega^m)$ and also $e_{33}(\frac{\mathbf{u}(\varepsilon)}{\varepsilon^2}) \rightharpoonup \kappa^m_{33}$ in $L^2(\Omega^m)$.
As already shown, the vector $\boldsymbol{\theta}^m(\varepsilon) := (\varepsilon\partial_1\varphi^m(\varepsilon), \varepsilon\partial_2\varphi^m(\varepsilon), \partial_3\varphi^m(\varepsilon)) \rightharpoonup (\theta^m_1, \theta^m_2, \theta^m_3)$ in $L^2(\Omega^m;\mathbb{R}^3)$. Thanks to the H^1-boundedness of the sequences $(\varphi^\pm(\varepsilon))_{\varepsilon>0}$ and by virtue of the continuity of the trace operator, from expression

$$\varphi^m(\varepsilon)(\tilde{x}, x_3) = \varphi^m(\varepsilon)|_{S^-} + \int_{-h}^{x_3} \partial_3\varphi^m(\varepsilon)(\tilde{x}, \xi)d\xi,$$

it follows that there exist two constants, c_1 and c_2, such that $|\varphi^m(\varepsilon)|_{0,\Omega^m} \le c_1\{|\varphi^-(\varepsilon)|_{0,S^-} + |\partial_3\varphi^m(\varepsilon)|_{0,\Omega^m}\} \le c_2$. This implies that $\varphi^m(\varepsilon) \rightharpoonup \varphi^m$ in $L^2(\Omega^m)$ and, thus, $(\varepsilon\partial_1\varphi^m(\varepsilon), \varepsilon\partial_2\varphi^m(\varepsilon), \partial_3\varphi^m(\varepsilon)) \rightharpoonup (0, 0, \partial_3\varphi^m)$.
Hence, the weak limit $(\mathbf{u}, \varphi)$ belongs to $\widetilde{V} \times \widetilde{\Psi}$.

(ii) Now we characterize the expression of the weak limit. Let us multiply the rescaled problem (7) by ε^2 and let ε tends to zero. If we choose $\psi = 0$, we find that

$$\int_{\Omega^m} \big(\tau_1\kappa^m_{33} + \tau_2 e_{\sigma\sigma}(\mathbf{u}^m) + \delta_3\partial_3\varphi^m\big)e_{33}(\mathbf{v})dx = 0,$$

for all $\mathbf{v} \in V$, which is satisfied when $\kappa^m_{33} = -\frac{1}{\tau_1}(\tau_2 e_{\sigma\sigma}(\mathbf{u}^m) + \delta_3\partial_3\varphi^m)$ in $L^2(\Omega^m)$. By multiplying by ε and by choosing test functions such that $v_3 = \psi = 0$, we obtain

$$\int_{\Omega^m} 2\eta\kappa^m_{\alpha3}\partial_3 v_\alpha dx = 0,$$

for all $v_\alpha \in V$, which implies that $\kappa^m_{\alpha3} = 0$ and, thus, $e_{\alpha3}(\frac{\mathbf{u}(\varepsilon)}{\varepsilon}) \to 0$ in $L^2(\Omega^m)$. Let us choose test functions $r \in \widetilde{V} \times \widetilde{\Psi}$ in the rescaled problem (7) and let ε tends to zero. We get

$$\begin{aligned}A^\pm(s, r) + \int_{\Omega^m} \big\{&2\mu e_{\alpha\beta}(\mathbf{u}^m)e_{\alpha\beta}(\mathbf{v}) + (\lambda e_{\sigma\sigma}(\mathbf{u}^m)\\ &+ \delta_1\partial_3\varphi^m + \tau_2\kappa^m_{33})e_{\tau\tau}(\mathbf{v}) + (-\delta_1 e_{\sigma\sigma}(\mathbf{u}^m)\\ &+ \gamma_2\partial_3\varphi^m - \delta_3\kappa^m_{33})\partial_3\psi\big\} = L(r).\end{aligned}$$

By means of the expression of κ^m_{33}, we obtain, as customary, the limit problem (4) and, hence, we can identify the weak limit with the limit electromechanical state $s^0 = (\mathbf{u}^0, \varphi^0)$. Besides, by integrating by parts we can obtain the same characterization (5) for φ^m, already shown in Sect. 3, which means that $\varphi^m \in H^1(-h, h; L^2(\omega))$. Since the variational equation (4) has a unique solution, not only a subsequence but the whole family $(s(\varepsilon))_{\varepsilon>0} = (\mathbf{u}(\varepsilon), \varphi(\varepsilon))_{\varepsilon>0}$ weakly converges to $s^0 = (\mathbf{u}^0, \varphi^0)$ in $H^1(\Omega, \mathbb{R}^3)\times L^2(\Omega)$.

(iii) To show that $(s(\varepsilon))_{\varepsilon>0} = (\mathbf{u}(\varepsilon), \varphi(\varepsilon))_{\varepsilon>0}$ strongly converges to $s^0 = (\mathbf{u}^0, \varphi^0)$ in $H^1(\Omega, \mathbb{R}^3) \times L^2(\Omega)$, it suffices to show that both $(\kappa_{ij}(\varepsilon))_{\varepsilon>0} = (e_{ij}(\mathbf{u}(\varepsilon)))_{\varepsilon>0}$ and $(\theta_i(\varepsilon))_{\varepsilon>0} = \partial_i\varphi(\varepsilon))_{\varepsilon>0}$, respectively, strongly converges to $\kappa_{ij} = e_{ij}(\mathbf{u})$ and $\theta_i = \partial_i\varphi$ in $L^2(\Omega)$, as a consequence of Korn's and Poincaré's inequalities with boundary conditions. Using the variational problem (7), we infer that

$$
\begin{aligned}
c\Big\{&|\boldsymbol{\kappa}^\pm(\varepsilon) - \boldsymbol{\kappa}^\pm|^2_{0,\Omega^\pm} + |\boldsymbol{\theta}^\pm(\varepsilon) - \boldsymbol{\theta}^\pm|^2_{0,\Omega^\pm}\\
&+ |\boldsymbol{\kappa}^m(\varepsilon) - \boldsymbol{\kappa}^m|^2_{0,\Omega^m} + |\boldsymbol{\theta}^m(\varepsilon) - \boldsymbol{\theta}^m|^2_{0,\Omega^m}\Big\}\\
&\le \int_{\Omega^\pm} \Big\{\mathsf{A}^\pm(\boldsymbol{\kappa}^\pm(\varepsilon) - \boldsymbol{\kappa}^\pm) : (\boldsymbol{\kappa}^\pm(\varepsilon) - \boldsymbol{\kappa}^\pm)\\
&\qquad + \mathsf{H}^\pm(\boldsymbol{\theta}^\pm(\varepsilon) - \boldsymbol{\theta}^\pm)\cdot(\boldsymbol{\theta}^\pm(\varepsilon) - \boldsymbol{\theta}^\pm)\Big\}dx\\
&+ \int_{\Omega^m} \Big\{\mathsf{A}^m(\boldsymbol{\kappa}^m(\varepsilon) - \boldsymbol{\kappa}^m) : (\boldsymbol{\kappa}^m(\varepsilon) - \boldsymbol{\kappa}^m)\\
&\qquad + \mathsf{H}^m(\boldsymbol{\theta}^m(\varepsilon) - \boldsymbol{\theta}^m)\cdot(\boldsymbol{\theta}^m(\varepsilon) - \boldsymbol{\theta}^m)\Big\}dx\\
&= \int_{\Omega^\pm} \Big\{\mathsf{A}^\pm\boldsymbol{\kappa}^\pm : (\boldsymbol{\kappa}^\pm - 2\boldsymbol{\kappa}^\pm(\varepsilon))\\
&\qquad + \mathsf{H}^\pm\boldsymbol{\theta}^\pm\cdot(\boldsymbol{\theta}^\pm - 2\boldsymbol{\theta}^\pm(\varepsilon))\Big\}dx + L(s(\varepsilon))\\
&+ \int_{\Omega^m} \Big\{\mathsf{A}^m\boldsymbol{\kappa}^m : (\boldsymbol{\kappa}^m - 2\boldsymbol{\kappa}^m(\varepsilon))\\
&\qquad + \mathsf{H}^m\boldsymbol{\theta}^m\cdot(\boldsymbol{\theta}^m - 2\boldsymbol{\theta}^m(\varepsilon))\Big\}dx.
\end{aligned}
$$

Since we already established in part (i) and (ii) the weak convergences

$$
\begin{aligned}
&\kappa^\pm_{ij}(\varepsilon) \rightharpoonup \kappa^\pm_{ij}, \quad \theta^\pm_i(\varepsilon) \rightharpoonup \theta^\pm_i \text{ in } \mathrm{L}^2(\Omega^\pm),\\
&\kappa^m_{ij}(\varepsilon) \rightharpoonup \kappa^m_{ij}, \quad \theta^m_i(\varepsilon) \rightharpoonup \theta^m_i \text{ in } \mathrm{L}^2(\Omega^\mathrm{m}),
\end{aligned}
$$

the right-hand side of the last inequality converges to

$$
\begin{aligned}
\Lambda := &-\int_{\Omega^\pm} \{\mathsf{A}^\pm\boldsymbol{\kappa}^\pm : \boldsymbol{\kappa}^\pm + \mathsf{H}^\pm\boldsymbol{\theta}^\pm\cdot\boldsymbol{\theta}^\pm\}dx\\
&- \int_{\Omega^m} \{\mathsf{A}^m\boldsymbol{\kappa}^m : \boldsymbol{\kappa}^m + \mathsf{H}^m\boldsymbol{\theta}^m\cdot\boldsymbol{\theta}^m\}dx + L(s)
\end{aligned}
$$

as ε tends to zero. From relations $\kappa^\pm_{ij} = e_{ij}(\mathbf{u}^\pm)$, $\theta^\pm_i = \partial_i\varphi^\pm$, $\kappa^m_{\alpha\beta} = e_{\alpha\beta}(\mathbf{u}^m)$, $\kappa^m_{\alpha3} = 0$, $\kappa^m_{33} = -\frac{1}{\tau_1}(\tau_2 e_{\sigma\sigma}(\mathbf{u}^m) + \delta_3\partial_3\varphi^m)$, $\theta^m_\alpha = 0$ and $\theta^m_3 = \partial_3\varphi^m$, we obtain that

$$
\begin{aligned}
&\int_{\Omega^\pm} \{\mathsf{A}^\pm\boldsymbol{\kappa}^\pm : \boldsymbol{\kappa}^\pm + \mathsf{H}^\pm\boldsymbol{\theta}^\pm\cdot\boldsymbol{\theta}^\pm\}dx\\
&\quad + \int_{\Omega^m} \{\mathsf{A}^m\boldsymbol{\kappa}^m : \boldsymbol{\kappa}^m + \mathsf{H}^m\boldsymbol{\theta}^m\cdot\boldsymbol{\theta}^m\}\\
&= \int_{\Omega^\pm} \Big\{A^\pm_{ijk\ell}e_{k\ell}(\mathbf{u}^\pm)e_{ij}(\mathbf{u}^\pm) + H^\pm_{ij}\partial_j\varphi^\pm\partial_i\varphi^\pm\Big\}dx\\
&\quad + \int_{\Omega^m} \Big\{2\mu e_{\alpha\beta}(\mathbf{u}^m)e_{\alpha\beta}(\mathbf{u}^m) + \mathcal{B}e_{\sigma\sigma}(\mathbf{u}^m)e_{\tau\tau}(\mathbf{u}^m)\\
&\quad + \mathcal{D}\partial_3\varphi^m\partial_3\varphi^m\Big\}dx\\
&= L(s)
\end{aligned}
$$

by step (ii). Hence $\Lambda = 0$, as was to be proved. $\square$

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