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Asymptotic expressions for piezoelectric surface waves excited by the buried mechanical and electrical point sources

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The asymptotic expressions for the piezoelectric surface waves excited by the buried mechanical and electrical point sources are obtained by using the stationary-phase method developed by Lighthill. It is found that the amplitudes of the piezoelectric surface waves propagating in the directions which correspond to the points of zero curvature on the slowness surface are subject to a comparatively small decay at the rate $O(r^{-1/3})$ with distance, as opposed to a decay of $O(r^{-1/2})$ for the ordinary surface waves.

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I. INTRODUCTION

In general, the direction of the phase velocity vector of piezoelectric surface waves propagating on a semi-infinite free surface of an anisotropic crystal differs from that of the energy velocity vector, and it has been reported[1] that the wave surface of piezoelectric surface waves shows the cusps associated with the existence of points of inflection on the slowness surface. This phenomenon implies that a few separate surface waves with different phase velocities and displacement vectors propagate in the same direction. Therefore, it is important to investigate the propagation properties of the piezoelectric surface waves excited by the mechanical and electrical point sources buried in the semi-infinite piezoelectric crystal. In the case of isotropic medium, this problem has been analyzed in connection with the investigation of earth tremors.[2] In the case of the anisotropic crystal, the asymptotic expressions for the elastic surface waves excited by the mechanical point source placed on the surface of the nonpiezoelectric transversely isotropic crystal has been obtained by Buchwald.[3] However, according to our numerical analysis, it is found that the asymptotic expressions for the elastic waves obtained by him vanish identically. The reason is that he has used the incorrect mechanical boundary condition that the stress is free at each point of the crystal surface.

In this paper, we derive the dyadic Green's function[4] for the semi-infinite piezoelectric crystal which occupies the region $x_3 > 0$ shown in Fig. 1. An infinitely thin perfectly conducting plate lies at $x_3 = -h$, and the mechanical and electrical point sources are located at $\mathbf{r}' = (x_1', x_2', x_3')$. Then, making use of the stationary-phase method developed by Lighthill,[5] we derive the asymptotic expressions for the piezoelectric surface waves from the dyadic Green's function, and it is shown that the amplitudes of the piezoelectric surface waves propagating in the directions which correspond to the points of zero curvature on the slowness surface are subject to a comparatively small decay with distance, as opposed to a decay of the amplitudes of ordinary surface waves.

II. DYADIC GREEN'S FUNCTION IN THE INFINITELY EXTENDED PIEZOELECTRIC CRYSTAL

In this section, we derive the dyadic Green's function in the infinitely extended piezoelectric crystal. In the following, we use Einstein's summation convention, and our notation shall consist of using Latin letters for the range (1, 2, 3) and Greek letters for the range (1, 2, 3, 4). The linear piezoelectric equations consist of the following equations:

stress equation of motion,

$$\partial_j \tau_{ij} - \rho_m \frac{\partial^2 u_i}{\partial t^2} = -f_i; \tag{1}$$

charge equation of electrostatics,

$$\partial_j D_j = \rho_e; \tag{2}$$

electric field/electric potential relation,

$$E_j = -\partial_j \phi; \tag{3}$$

linear piezoelectric constitutive relations,

$$\tau_{ij} = c_{ijpq}\,\partial_q u_p - e_{pij} E_p, \tag{4}$$

$$D_i = e_{ipq}\,\partial_q u_p + \epsilon_{ip} E_p, \tag{5}$$

where ∂_j denotes differentiation with respect to a space coordinate x_j; u_i, τ_{ij}, D_i, and E_i are the components of mechanical displacement, stress, electric displacement, and electric field, respectively; ρ_m, ρ_e, f_i, and ϕ are the mass density, the charge density, the body force, and the electric potential, respectively; and c_{ijpq}, e_{ijq}, and ϵ_{iq} are the elastic, piezoelectric, and dielectric constants, respectively.

The crystal is assumed to be excited by the mechanical and electrical point sources,

$$f_i(\mathbf{r}) = a_i \delta(\mathbf{r}) \exp(-i\omega t) \quad (i = 1, 2, 3), \tag{6}$$

$$\rho_e(\mathbf{r}) = a_4 \delta(\mathbf{r}) \exp(-i\omega t), \tag{7}$$

respectively, where a_α is an arbitrary vector and $\delta(\mathbf{r})$ is the Dirac delta function. We postulate the mechanical displacement and the electric potential as follows:

$$u_i(\mathbf{r}) = G_{i\alpha}(\mathbf{r}) a_\alpha \quad (i = 1, 2, 3), \tag{8}$$

$$u_4(\mathbf{r}) = \phi(\mathbf{r}) = G_{4\alpha}(\mathbf{r}) a_\alpha, \tag{9}$$

where $G_{\alpha\beta}(\mathbf{r})$ is the dyadic Green's function in the infinitely extended piezoelectric crystal and is assumed to be

$$G_{\alpha\beta}(\mathbf{r}) = [1/(2\pi)^3] \int\int_{-\infty}^{+\infty}\int \tilde{G}_{\alpha\beta}(\mathbf{r}) \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]\, dk_1\, dk_2\, dk_3 \tag{10}$$

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using the Fourier integral. Substituting Eqs. (6)–(10) into (1)–(5), we obtain after some manipulations

$$\Gamma_{\alpha\beta}(\mathbf{k})\tilde{G}_{\beta\gamma}(\mathbf{k}) = \Delta_{\alpha\gamma}, \tag{11}$$

$$\begin{aligned}
&\Gamma_{ip}(\mathbf{k}) = c_{ijpq}k_j k_q - \rho_m \omega^2 \delta_{ip} \quad (i, p = 1, 2, 3),\\
&\Gamma_{i4}(\mathbf{k}) = \Gamma_{4i}(\mathbf{k}) = e_{ijp}k_j k_p \qquad (i = 1, 2, 3),\\
&\Gamma_{44}(\mathbf{k}) = -\epsilon_{ip}k_i k_p,
\end{aligned} \tag{12}$$

$$[\Delta_{1\gamma}, \Delta_{2\gamma}, \Delta_{3\gamma}, \Delta_{4\gamma}] = [\delta_{1\gamma}, \delta_{2\gamma}, \delta_{3\gamma}, -\delta_{4\gamma}], \tag{13}$$

where $\delta_{\alpha\beta}$ is the Kronecker's delta. Then, we obtain from Eq. (11)

$$\tilde{G}_{\alpha\beta}(\mathbf{k}) = D_{\alpha\beta}(\mathbf{k})/D(\mathbf{k}), \tag{14}$$

where

$$D(\mathbf{k}) = \det|\Gamma_{\alpha\beta}(\mathbf{k})|, \tag{15}$$

$D_{\alpha\beta}(\mathbf{k})$ is the determinant obtained by replacing the αth row of $D(\mathbf{k})$ by

$$[\delta_{1\beta}, \delta_{2\beta}, \delta_{3\beta}, -\delta_{4\beta}]^T, \tag{16}$$

and the symbol T denotes the transposed matrix. Therefore, the dyadic Green's function in the infinitely extended piezoelectric crystal is given by

$$G_{\alpha\beta}(\mathbf{r}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} \frac{D_{\alpha\beta}(\mathbf{k})}{D(\mathbf{k})} \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]\, dk_1\, dk_2\, dk_3. \tag{17}$$

III. DIMENSION REDUCTION OF THE DYADIC GREEN'S FUNCTION

When the point sources act on the point $\mathbf{r}' = (x_1', x_2', x_3')$ in the infinitely extended piezoelectric crystal, the dyadic Green's function is obtained from Eq. (17) by

$$G_{\alpha\beta}(\mathbf{r}|\mathbf{r}') = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} \frac{D_{\alpha\beta}(\mathbf{k})}{D(\mathbf{k})} \times \exp\{i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - \omega t]\}\, dk_1\, dk_2\, dk_3. \tag{18}$$

In this section, the three-dimensional integral of Eq. (18) is reduced to the two-dimensional one by a contour integral in the complex k_3 plane. The contour consists of the real axis and a large semicircle in the upper or the lower half-plane, and the contributions from this semicircle are vanishingly small (by Jordan's lemma). With this method, the integration with respect to k_3 can be replaced by appropriate contributions from the singularities of the integrand. The integration with respect to k_3 does not have any singularities except for the poles $k_3(k_1, k_2)$ which are the roots of the equation

$$D(k_1, k_2, k_3) = 0. \tag{19}$$

Since Eq. (19) is a polynomial of the eighth order [see Eqs. (12) and (15)], the complex roots, k_3 are given by complex conjugate pairs in the upper and the lower half-plane. Here we write the four poles in the upper half-plane as follows:

$$k_3 = iq_\gamma, \quad \mathrm{Re}[q_\gamma] > 0 \quad (\gamma = 1, 2, 3, 4). \tag{20}$$

Then, the poles in the lower half-plane are given by

$$k_3^* = -iq_\gamma^* \quad (\gamma = 1, 2, 3, 4), \tag{21}$$

where the asterisk denotes complex conjugate. We consider the semicircular contour consisting of the segment $-R < \mathrm{Re}[k_3] < R$, $R > 0$, and the semicircle C_R, $|k_3| = R$, in the upper half-plane for $x_3 > x_3'$, and the contour consisting of the segment $-R < \mathrm{Re}[k_3] < R$, $R > 0$, and the semicircle C_R, $|k_3| = R$, in the lower half-plane for $x_3 < x_3'$. Replacing the integration with respect to k_3 in Eq. (18) by the contributions from the appropriate poles included in the semicircular contours defined above in the limit of $R \to \infty$, the dyadic Green's function [Eq. (18)] becomes

$$\begin{aligned}
G_{\alpha\beta}(\mathbf{r}|\mathbf{r}') &= \frac{-1}{(2\pi)^2} \sum_{\gamma=1}^{4} \iint_{-\infty}^{+\infty} \frac{D_{\alpha\beta}^{\gamma}(k_1, k_2)}{D_3^{\gamma}(k_1, k_2)} \exp[-q_\gamma(x_3 - x_3')]\\
&\quad \times \exp\{i[k_1(x_1 - x_1') + k_2(x_2 - x_2') - \omega t]\}\, dk_1\, dk_2 \quad (x_3 > x_3')\\
&= \frac{1}{(2\pi)^2} \sum_{\gamma=1}^{4} \iint_{-\infty}^{+\infty} \frac{D_{\alpha\beta}^{\gamma *}(k_1, k_2)}{D_3^{\gamma *}(k_1, k_2)} \exp[-q_\gamma^*(x_3' - x_3)]\\
&\quad \times \exp\{i[k_1(x_1 - x_1') + k_2(x_2 - x_2') - \omega t]\}\, dk_1\, dk_2 \quad (x_3 < x_3'),
\end{aligned} \tag{22}$$

where

$$D_3^{\gamma}(k_1, k_2) = \left(\frac{\partial D(k_1, k_2, iq)}{\partial q}\right)_{q=q_\gamma}, \tag{23}$$

$$D_{\alpha\beta}^{\gamma}(k_1, k_2) = D_{\alpha\beta}(k_1, k_2, iq_\gamma). \tag{24}$$

In the following, for convenience, we distinguish the variable related to the attenuation coefficient q_γ by attaching the superscript γ.

IV. DERIVATION OF THE FREE-VIBRATION TERM

In this section, assuming that the point sources do not exist, we obtain the free-vibration term in the form of surface waves. We write the mechanical displacement and the electric potential as follows:

$$u_i(\mathbf{r}) = A_{i\alpha}(\mathbf{r})a_\alpha \quad (i = 1, 2, 3), \tag{25}$$

$$u_4(\mathbf{r}) = \phi(\mathbf{r}) = A_{4\alpha}(\mathbf{r})a_\alpha, \tag{26}$$

where a_α is an arbitrary vector. We postulate $A_{\alpha\beta}(\mathbf{r})$ using the Fourier integral as follows:

$$A_{\alpha\beta}(\mathbf{r}) = [1/(2\pi)^2] \iint_{-\infty}^{+\infty} \tilde{A}_{\alpha\beta}(\mathbf{k}) \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]\, dk_1\, dk_2. \tag{27}$$

Substituting Eqs. (25)–(27) into Eqs. (1)–(5) in which the body force f_i and the charge density ρ_e are set to zero, we obtain

$$\Gamma_{\alpha\beta}(\mathbf{k})\tilde{A}_{\beta\gamma}(\mathbf{k}) = 0, \tag{28}$$

where $\Gamma_{\alpha\beta}(\mathbf{k})$ is given by Eq. (12). From the condition that $\tilde{A}_{\alpha\beta}(\mathbf{k})$ does not have the trivial solution, we obtain

$$D(\mathbf{k}) = \det|\Gamma_{\alpha\beta}(\mathbf{k})| = 0. \tag{29}$$

Since Eq. (29) is in the same form as Eq. (19), the complex roots k_3 are given by Eqs. (20) and (21). However, in order to obtain the free-vibration term in the form of surface waves which vanish exponentially at $x_3 = \infty$, we consider only the solutions given by Eq. (20).

Then, for each q_γ, we obtain from Eq. (28)

$$\frac{\tilde{A}_{1\alpha}}{\eta_1^\gamma}=\frac{\tilde{A}_{2\alpha}}{\eta_2^\gamma}=\frac{\tilde{A}_{3\alpha}}{\eta_3^\gamma}=\frac{\tilde{A}_{4\alpha}}{\eta_4^\gamma}=K_\alpha^\gamma, \tag{30}$$

where $K_\alpha^\gamma(k_1,k_2)$ is the weight function and $\eta_\alpha^\gamma(k_1,k_2)$ is given by

$$\eta_1^\gamma=-\begin{vmatrix}\Gamma_{14}^\gamma & \Gamma_{12}^\gamma & \Gamma_{13}^\gamma\\ \Gamma_{24}^\gamma & \Gamma_{22}^\gamma & \Gamma_{23}^\gamma\\ \Gamma_{34}^\gamma & \Gamma_{23}^\gamma & \Gamma_{33}^\gamma\end{vmatrix},\quad \eta_2^\gamma=-\begin{vmatrix}\Gamma_{11}^\gamma & \Gamma_{14}^\gamma & \Gamma_{13}^\gamma\\ \Gamma_{12}^\gamma & \Gamma_{24}^\gamma & \Gamma_{23}^\gamma\\ \Gamma_{13}^\gamma & \Gamma_{34}^\gamma & \Gamma_{33}^\gamma\end{vmatrix},$$
$$\eta_3^\gamma=-\begin{vmatrix}\Gamma_{11}^\gamma & \Gamma_{12}^\gamma & \Gamma_{14}^\gamma\\ \Gamma_{12}^\gamma & \Gamma_{22}^\gamma & \Gamma_{24}^\gamma\\ \Gamma_{13}^\gamma & \Gamma_{23}^\gamma & \Gamma_{34}^\gamma\end{vmatrix},\quad \eta_4^\gamma=\begin{vmatrix}\Gamma_{11}^\gamma & \Gamma_{12}^\gamma & \Gamma_{13}^\gamma\\ \Gamma_{12}^\gamma & \Gamma_{22}^\gamma & \Gamma_{23}^\gamma\\ \Gamma_{13}^\gamma & \Gamma_{23}^\gamma & \Gamma_{33}^\gamma\end{vmatrix}, \tag{31}$$

and

$$\Gamma_{\alpha\beta}^\gamma(k_1,k_2)=\Gamma_{\alpha\beta}(k_1,k_2,iq_\gamma). \tag{32}$$

Therefore, the free-vibration term in the form of surface waves which vanish at $x_3=\infty$ is given by

$$A_{\alpha\beta}(\mathbf{r})=[1/(2\pi)^2]\sum_{\gamma=1}^{4}\int\!\!\int_{-\infty}^{+\infty}\eta_\alpha^\gamma(k_1,k_2)K_\beta^\gamma(k_1,k_2)\times\exp(-iq_\gamma x_3)\exp[i(k_1x_1+k_2x_2-\omega t)]\,dk_1\,dk_2. \tag{33}$$

V. DYADIC GREEN'S FUNCTION FOR THE SEMI-INFINITE PIEZOELECTRIC CRYSTAL

In this section, we obtain the dyadic Green's function for the semi-infinite piezoelectric crystal which occupies the region $x_3 \geq 0$. An infinitely thin perfectly conducting plate lies at $x_3=-h$, and the point sources are located at $\mathbf{r}'=(x_1',x_2',x_3')$. The geometry is depicted in Fig. 1. If we assume that the crystal surface is stress free, the mechanical boundary conditions at each point of the crystal are

$$\tau_{3j}\big|_{x_3=0}=c_{3jpq}\partial_q u_p+e_{p3j}\partial_p u_4\big|_{x_3=0}\quad (j=1,2,3). \tag{34}$$

From the conditions that the tangent components of the electric field and the normal component of the electric displacement are continuous across the surface $x_3=0$, the electrical boundary conditions at each point of the surface of the crystal are

$$\partial_1\phi^{\text{sub}}\big|_{x_3=0}=\partial_1\phi^{\text{air}}\big|_{x_3=0},\qquad \partial_2\phi^{\text{sub}}\big|_{x_3=0}=\partial_2\phi^{\text{air}}\big|_{x_3=0}, \tag{35}$$

$$e_{3pq}\partial_q u_p-\epsilon_{3q}\partial_q\phi^{\text{sub}}\big|_{x_3=0}=-\epsilon_0\partial_3\phi^{\text{air}}\big|_{x_3=0}, \tag{36}$$

where the superscript sub shows the field in the piezoelectric crystal $x_3>0$ and the superscript air shows the field in the air region $-h\leq x_3\leq 0$. Since the field should vanish at $x_3=-h$, the electrical boundary condition at $x_3=-h$ is given by

$$\phi^{\text{air}}\big|_{x_3=-h}=0. \tag{37}$$

Clearly, the dyadic Green's function for the infinitely extended piezoelectric crystal given by Eq. (22) does not satisfy the boundary conditions, and we must postulate the additional free-vibration term given by Eq. (33) as the perturbed term.[6] We write the mechanical displacement and the electric potential as follows:

$$u_i(\mathbf{r})=B_{i\alpha}(\mathbf{r})a_\alpha\quad (i=1,2,3), \tag{38}$$

$$u_4(\mathbf{r})=\phi(\mathbf{r})=B_{4\alpha}(\mathbf{r})a_\alpha, \tag{39}$$

where a_α is an arbitrary vector and $B_{\alpha\beta}(\mathbf{r})$ is the dyadic Green's function which satisfies the boundary conditions for the semi-infinite piezoelectric crystal. We postulate $B_{\alpha\beta}(\mathbf{r})$ using Eqs. (22) and (33) as follows:

$$\begin{aligned}B_{\alpha\beta}(\mathbf{r}\,|\,\mathbf{r}')&=\frac{1}{(2\pi)^2}\sum_{\gamma=1}^{4}\int\!\!\int_{-\infty}^{+\infty}\Big(\eta_\alpha^\gamma(k_1,k_2)K_\beta^\gamma(k_1,k_2)\exp(-q_\gamma x_3)\\&\quad+\frac{D_{\alpha\beta}^{\gamma *}(k_1,k_2)}{D_3^{\gamma *}(k_1,k_2)}\exp[-q_\gamma^*(x_3'-x_3)]\Big)\\&\quad\times\exp\{i[k_1(x_1-x_1')+k_2(x_2-x_2')-\omega t]\}\,dk_1\,dk_2\qquad (0\leq x_3<x_3'),\\&=\frac{1}{(2\pi)^2}\sum_{\gamma=1}^{4}\int\!\!\int_{-\infty}^{+\infty}\eta_\alpha^\gamma(k_1,k_2)K_\beta^\gamma(k_1,k_2)\exp(-q_\gamma x_3)\\&\quad-\frac{D_{\alpha\beta}^{\gamma}(k_1,k_2)}{D_3^{\gamma}(k_1,k_2)}\exp[-q_\gamma(x_3-x_3')]\\&\quad\times\exp\{i[k_1(x_1-x_1')+k_2(x_2-x_2')-\omega t]\}\,dk_1\,dk_2\qquad (x_3'<x_3).\end{aligned} \tag{40}$$

Then, the electric potential ϕ^{air} which satisfies the boundary conditions (35) and (37) is given by

$$\phi^{\text{air}}(\mathbf{r})=C_{4\alpha}(\mathbf{r}\,|\,\mathbf{r}')a_\alpha, \tag{41}$$

where

$$\begin{aligned}C_{4\alpha}(\mathbf{r}\,|\,\mathbf{r}')&=\frac{1}{(2\pi)^2}\sum_{\gamma=1}^{4}\int\!\!\int_{-\infty}^{+\infty}\frac{\sinh[kh(x_3+h)]}{\sinh(kh)}\\&\quad\times\Big(\eta_4^\gamma(k_1,k_2)K_\alpha^\gamma(k_1,k_2)+\frac{D_{4\alpha}^{\gamma *}(k_1,k_2)}{D_3^{\gamma *}(k_1,k_2)}\exp(-q_\gamma^* x_3)\Big)\\&\quad\times\exp\{i[k_1(x_1-x_1')+k_2(x_2-x_2')-\omega t]\}\,dk_1\,dk_2.\end{aligned} \tag{42}$$

Substituting Eqs. (38)–(42) into Eqs. (34) and (36), we obtain

$$f_\alpha^\gamma(k_1,k_2)K_\beta^\gamma(k_1,k_2)=h_{\alpha\beta}(k_1,k_2), \tag{43}$$

where

$$\begin{aligned}f_i^\gamma&=(c_{i3p1}k_1+c_{i3p2}k_2+ic_{i3p3}q_\gamma)\eta_p^\gamma\\&\quad+(e_{13i}k_1+e_{23i}k_2+ie_{33i}q_\gamma)\eta_4^\gamma\quad (i=1,2,3),\\f_4^\gamma&=(e_{3p1}k_1+e_{3p2}k_2+ie_{3p3}q_\gamma)\eta_p^\gamma\\&\quad-[\epsilon_{31}k_1+\epsilon_{32}k_2+i\epsilon_{33}q_\gamma+i\epsilon_0k\coth(kh)]\eta_4^\gamma\quad (kh\neq 0),\\f_4^\gamma&=\eta_4^\gamma\quad (kh=0),\end{aligned} \tag{44}$$

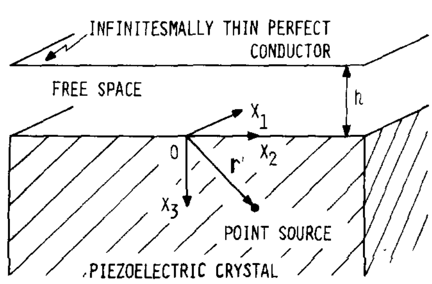

FIG. 1. Geometry of the problem.

$$h_{i\alpha} = -\sum_{\gamma=1}^{4} [(c_{i3p1}k_1 + c_{i3p2}k_2 - ic_{i3p3}q_\gamma^*)(D_{p\alpha}^{\gamma*}/D_3^{\gamma*})$$
$$\times \exp(-q_\gamma^* x_3') + (e_{13i}k_1 + e_{23i}k_2 - ie_{33i}q_\gamma^*)$$
$$\times (D_{4\alpha}^{\gamma*}/D_3^{\gamma*}) \exp(-q_\gamma^* x_3')] \quad (i=1,2,3), \tag{45}$$

$$h_{4\alpha} = -\sum_{\gamma=1}^{4} \{(e_{3p1}k_1 + e_{3p2}k_2 - ie_{3p3}q_\gamma^*)(D_{p\alpha}^{\gamma*}/D_3^{\gamma*})$$
$$\times \exp(-q_\gamma^* x_3') - [\epsilon_{31}k_1 + \epsilon_{32}k_2 + i\epsilon_{33}q_\gamma^* + i\epsilon_0 k \coth(kh)]$$
$$\times (D_{4\alpha}^{\gamma*}/D_3^{\gamma*}) \exp(-q_\gamma^* x_3')\} \quad (kh \neq 0),$$
$$h_{4\alpha} = -\sum_{\gamma=1}^{4} (D_{4\alpha}^{\gamma*}/D_3^{\gamma*}) \exp(-q_\gamma^* x_3') \quad (kh = 0).$$

From Eq. (43), the weight function is given by

$$K_\alpha^\gamma(k_1,k_2) = L_\alpha^\gamma(k_1,k_2)/L(k_1,k_2), \tag{46}$$

where

$$L = \begin{vmatrix} f_1^1 & f_1^2 & f_1^3 & f_1^4 \\ f_2^1 & f_2^2 & f_2^3 & f_2^4 \\ f_3^1 & f_3^2 & f_3^3 & f_3^4 \\ f_4^1 & f_4^2 & f_4^3 & f_4^4 \end{vmatrix}. \tag{47}$$

L_α^γ is the determinant obtained by replacing the γth row of L by

$$[h_{1\alpha}, h_{2\alpha}, h_{3\alpha}, h_{4\alpha}]^T. \tag{48}$$

Therefore, substituting Eq. (46) into Eq. (40), the dyadic Green's function for the semi-infinite piezoelectric crystal is given by

$$B_{\alpha\beta}(\mathbf{r}|\mathbf{r}')$$
$$= \frac{1}{(2\pi)^2} \sum_{\gamma=1}^{4} \iint_{-\infty}^{+\infty} \left(\frac{\eta_\alpha^\gamma(k_1,k_2) L_\beta^\gamma(k_1,k_2)}{L(k_1,k_2)} \exp(-q_\gamma x_3) + \frac{D_{\alpha\beta}^{\gamma*}(k_1,k_2)}{D_3^{\gamma*}(k_1,k_2)} \exp[-q_\gamma^*(x_3'-x_3)] \right)$$
$$\times \exp\{i[k_1(x_1-x_1') + k_2(x_2-x_2') - \omega t]\} dk_1\, dk_2 \quad (0 \leq x_3 \leq x_3'),$$
$$= \frac{1}{(2\pi)^2} \sum_{\gamma=1}^{4} \iint_{-\infty}^{+\infty} \left(\frac{\eta_\alpha^\gamma(k_1,k_2) L_\beta^\gamma(k_1,k_2)}{L(k_1,k_2)} \exp(-q_\gamma x_3) - \frac{D_{\alpha\beta}^{\gamma}(k_1,k_2)}{D_3^{\gamma}(k_1,k_2)} \exp[-q_\gamma(x_3-x_3')] \right)$$
$$\times \exp\{i[k_1(x_1-x_1') + k_2(x_2-x_2') - \omega t]\} dk_1\, dk_2 \quad (x_3' < x_3). \tag{49}$$

In general, when the body forces $f_j(\mathbf{r}')$ $(j=1,2,3)$ and the electric charge $\rho_e(\mathbf{r})$ are located in the region V_j' $(j=1,2,3)$ and V_4', respectively, the mechanical displacement and the electric potential are given by

$$u_i(\mathbf{r}) = \iiint_{V_j'} B_{ij}(\mathbf{r}|\mathbf{r}') f_j(\mathbf{r}')\, d\mathbf{r}' + \iiint_{V_4'} B_{i4}(\mathbf{r}|\mathbf{r}') \rho_e(\mathbf{r}')\, d\mathbf{r}' \quad (i=1,2,3), \tag{50}$$

$$\phi(\mathbf{r}) = \iiint_{V_j'} B_{4j}(\mathbf{r}|\mathbf{r}') f_j(\mathbf{r}')\, d\mathbf{r}' + \iiint_{V_4'} B_{44}(\mathbf{r}|\mathbf{r}') \rho_e(\mathbf{r}')\, d\mathbf{r}'. \tag{51}$$

VI. ASYMPTOTIC EXPRESSIONS FOR THE PIEZOELECTRIC SURFACE WAVES

In this section, we derive the asymptotic expressions for the piezoelectric surface waves from the dyadic Green's function for the semi-infinite piezoelectric crystal given by Eq. (49). In Eq. (49), for convenience, we put the point sources at $\mathbf{r}' = (0,0,x_3')$ in the crystal and introduce a new coordinate system x_1'', x_2'' which is defined by the rotation of the system x_1, x_2 around the x_3 axis, where the x_2'' direction is parallel to the vector $\mathbf{R}$ which is the projection of $\mathbf{r}$ on the x_1-x_2 plane as shown in Fig. 2. As a result of this transformation to the new coordinate axes, the dyadic Green's function [Eq. (49)] is rewritten as

$$B_{\alpha\beta}(\mathbf{r}''|\mathbf{r}')$$
$$= \frac{1}{(2\pi)^2} \sum_{\gamma=1}^{4} \iint_{-\infty}^{+\infty} \left(\frac{\eta_\alpha^\gamma(k_1'',k_2'') L_\beta^\gamma(k_1'',k_2'')}{L(k_1'',k_2'')} \exp(-q_\gamma x_3) + \frac{D_{\alpha\beta}^{\gamma*}(k_1'',k_2'')}{D_3^{\gamma*}(k_1'',k_2'')} \exp[-q_\gamma^*(x_3'-x_3)] \right)$$
$$\times \exp[i(k_2''x_2'' - \omega t)]\, d_1''\, dk_2'' \quad (0 \leq x_3 < x_3'),$$
$$= \frac{1}{(2\pi)^2} \sum_{\gamma=1}^{4} \iint_{-\infty}^{+\infty} \left(\frac{\eta_\alpha^\gamma(k_1'',k_2'') L_\beta^\gamma(k_1'',k_2'')}{L(k_1'',k_2'')} \exp(-q_\gamma x_3) - \frac{D_{\alpha\beta}^{\gamma}(k_1'',k_2'')}{D_3^{\gamma}(k_1'',k_2'')} \exp[-q_\gamma(x_3-x_3')] \right)$$
$$\times \exp[i(k_2''x_2'' - \omega t)]\, dk_1''\, dk_2'' \quad (x_3' < x_3), \tag{52}$$

where k_1'' and k_2'' are the wave number in the new coordinates. In Eq. (52), the poles which contribute to the piezoelectric surface waves are given by the roots of the equation

$$L(k_1'', k_2'') = 0. \tag{53}$$

Assuming that x_2'' is very large, we neglect the contributions from the poles given by the complex roots k_2'' of Eq. (53) which correspond to the leaky surface waves. Therefore, we consider only the contributions from the poles on the real axis of the complex k_2'' plane. Then, in the k_1''-k_2'' plane, the real curve given by Eq. (53) represents the slowness surface.

The following process is necessary for obtaining the solution which satisfies the radiation condition. We replace ω by $\omega + i\epsilon$, where $\epsilon > 0$, and afterwards we let $\epsilon \to 0$. Then, Eq. (53) is approximated by

$$L + i\epsilon \frac{\partial L}{\partial \omega} = 0, \tag{54}$$

where L stands for the original L (with $\epsilon = 0$). Near a

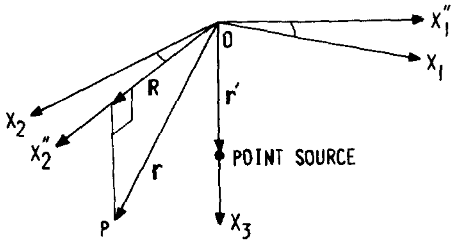

FIG. 2. Definition of new coordinate axes.

real simple zero $k_2''=k_{20}''$ of L, Eq. (54) is approximated by

$$(k_2''-k_{20}'')\frac{\partial L}{\partial k_2''}+i\epsilon\frac{\partial L}{\partial\omega}=0. \tag{55}$$

Therefore, the zero of Eq. (55) exists in the upper half-plane of the complex k_2'' plane if the group velocity satisfies

$$v_g=-\frac{\partial L}{\partial k_2''}\left(\frac{\partial L}{\partial\omega}\right)^{-1}>0, \tag{56}$$

and in the lower half-plane if the opposite inequality holds. As we postulate $x_2>0$, we consider the semicircular contour in the upper half-plane. When we integrate Eq. (52) along this contour, we can neglect the contribution from the semicircle C_R at $R=\infty$ by Jordan's lemma, and the contribution from the pole corresponding to the piezoelectric surface waves is given by

$$B_{\alpha\beta}(\mathbf{r}''|\mathbf{r})\sim\frac{i}{2\pi}\sum_{\gamma=1}^{4}\int_{-\infty}^{+\infty}\frac{\eta_\alpha^\gamma(k_1'')L_\beta^\gamma(k_1'')}{L_2(k_1'')}\times\exp(-q_\gamma x_3)\exp[i(k_2''x_2''-\omega t)]\,dk_1'', \tag{57}$$

in the limit of $\epsilon\to 0$, where

$$L_2(k_1'')=\left(\frac{\partial L(k_1'',k_2'')}{\partial k_2''}\right)_{k_2''=k_2''(k_1'')}. \tag{58}$$

Making use of the stationary-phase method developed by Lighthill,[5] the main contributions of the integral in Eq. (57) are from the neighborhood of those points on the slowness surface given by Eq. (53), where $k_2''>0$ and where the normal is parallel to the k_2'' direction. Let there be N such points, with coordinates (k_{1s}'',k_{2s}'') $(s=1,2,\ldots,N)$, at which k_2'' can be expanded in the Taylor series

$$k_2''=k_{2s}''+\tfrac{1}{2}\lambda_s(k_1''-k_{1s}'')^2+\cdots, \tag{59}$$

where λ_s is the Gaussian curvature given by

$$\lambda_s=\frac{L_1^2L_{22}-2L_1L_2L_{12}+L_2^2L_{11}}{(L_1^2+L_2^2)^{3/2}}, \tag{60}$$

$$L_j=\left(\frac{\partial L}{\partial k_j''}\right)_{k_1''=k_{1s}'',\,k_2''=k_{2s}''}. \tag{61}$$

$$L_{ij}=\left(\frac{\partial^2 L}{\partial k_i''\,\partial k_j''}\right)_{k_1''=k_{1s}'',\,k_2''=k_{2s}''}. \tag{62}$$

Substituting Eq. (59) into Eq. (57), and using the stationary-phase method, we obtain

$$B_{\alpha\beta}(\mathbf{r}''|\mathbf{r}')\sim\sum_{s=1}^{N}\sum_{\gamma=1}^{4}\left(\frac{i\eta_\alpha^\gamma(k_{1s}'')L_\beta^\gamma(k_{1s}'')}{2\pi L_2(k_{1s}'')}\exp(-q_\gamma x_3)\times\exp[i(k_{2s}''x_2''-\omega t)]\times\int_{-\infty}^{+\infty}\exp[i\tfrac{1}{2}\lambda_s(k_1''-k_{1s}'')^2x_2'']\,dk_1''\right). \tag{63}$$

With the formula

$$\int_{-\infty}^{+\infty}\exp(\tfrac{1}{2}iku^2)\,du=(2\pi/|k|)^{1/2}\exp(\tfrac{1}{4}\pi i\,\mathrm{sgn}k), \tag{64}$$

Eq. (63) can be expressed by

$$B_{\alpha\beta}(\mathbf{r}''|\mathbf{r}')\sim\sum_{s=1}^{N}\sum_{\gamma=1}^{4}\left(\frac{A_s\eta_\alpha^\gamma(k_{1s}'')L_\beta^\gamma(k_{1s}'')}{(2\pi x_2''|\lambda_s|)^{1/2}L_2(k_{1s}'')}\times\exp(-q_{\gamma s}x_3)\exp[i(k_{2s}''x_2''-\omega t)]\right), \tag{65}$$

where A_s is given by

$$A_s=\exp[\tfrac{1}{4}\pi i(2+\mathrm{sgn}\lambda_s)]. \tag{66}$$

If we express Eq. (65) in terms of the original coordinate system, the asymptotic form for the piezoelectric surface waves is given by

$$B_{\alpha\beta}(\mathbf{r}|\mathbf{r}')\sim\sum_{s=1}^{N}\sum_{\gamma=1}^{4}\left(\pm\frac{A_s\eta_\alpha^\gamma(k_{1s})L_\beta^\gamma(k_{1s})}{(2\pi|\mathbf{r}||\lambda_s|)^{1/2}|\mathrm{grad}L|_s}\times\exp(-q_{\gamma s}x_3)\exp[i(k_{1s}x_1+k_{2s}x_2-\omega t)]\right), \tag{67}$$

where the sign is positive if the function L is increasing, and negative if L is decreasing, with distance from the origin at the point (k_{1s},k_{2s}).

In the case of zero curvature of the slowness surface, the Gaussian curvature λ_s vanishes and k_2'' can be expanded in the Taylor series

$$k_2''=k_{2s}''+\tfrac{1}{6}\mu_s(k_1''-k_{1s}'')^3+\cdots \tag{68}$$

With the use of the formula

$$\int_{-\infty}^{+\infty}\exp(\tfrac{1}{6}iku^3)\,du=\sqrt{3}(6/|k|)^{1/3}(\tfrac{1}{3})!, \tag{69}$$

and in terms of the original coordinate system, the asymptotic form for the piezoelectric surface waves is given by

$$B_{\alpha\beta}(\mathbf{r}|\mathbf{r}')\sim\sum_{s=1}^{N}\sum_{\gamma=1}^{4}\left(\pm\frac{i(3)^{1/2}(6)^{1/3}(\frac{1}{3})!\,\eta_\alpha^\gamma(k_{1s})L_\beta^\gamma(k_{1s})}{2\pi(|\mathbf{r}||\mu_s|)^{1/3}|\mathrm{grad}L|_s}\times\exp(-q_{\gamma s}x_3)\exp[i(k_{1s}x_1+k_{2s}x_2-\omega t)]\right). \tag{70}$$

For the piezoelectric surface waves excited by the mechanical and electrical point sources placed on the piezoelectric crystal surface, we can obtain the asymptotic expressions in the limit of $x_3'\to 0$ in Eqs. (67) and (70). These expressions do not vanish, as opposed to the results obtained by Buchwald.[3] If we compare Eq. (67) with Eq. (70), it is found that the amplitudes of the piezoelectric surface waves propagating in the directions which correspond to the points of zero curvature are subject to a comparatively small decay at the rate $O(r^{-1/3})$ with distance, as opposed to a decay of $O(r^{-1/2})$ for the ordinary surface waves.

VII. CONCLUSION

We have obtained the asymptotic expressions for the piezoelectric surface waves excited by the buried mechanical and electrical point sources, using the stationary-phase method developed by Lighthill, and it is found that the amplitudes of the piezoelectric surface waves propagating in the directions which correspond to the points of zero curvature on the slowness surface are subject to a comparatively small decay at the rate $O(r^{-1/3})$ with distance, as opposed to a decay of

$O(r^{-1/2})$ for the ordinary surface waves. The detailed calculations of the radiation patterns for several crystals whose wave surfaces show the cusps are presently under way and will be reported later.

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