

Revisit to the theoretical analysis of a classical piezoelectric cantilever energy harvester

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November 18, 2019

1 Summary of the interested equations

The dynamic equations for a typical piezoelectric composite cantilever beam is

$$B_p \frac{\partial^4 w(x, t)}{\partial x^4} + m_p \frac{\partial^2 w(x, t)}{\partial t^2} = 0, \quad (1)$$

where B_p is the equivalent bending stiffness and m_p is the line mass density of the piezoelectric cantilever beam. If the piezoelectric elements attached to the cantilever beam is connected to an external electrical load R_l , we have

$$\frac{dQ_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \quad (2)$$

For the underlying physics, we have the following constitutive equations

$$\begin{aligned} M_p(x, t) &= B_p \frac{\partial^2 w(x, t)}{\partial x^2} - e_p V_p(t), \\ q_p(x, t) &= e_p \frac{\partial^2 w(x, t)}{\partial x^2} + \varepsilon_p V_p(t), \end{aligned} \quad (3)$$

or equivalently,

$$\begin{cases} M_p(x, t) = B_p \frac{\partial^2 w(x, t)}{\partial x^2} - e_p V_p(t), \\ Q_p(x, t) = e_p \left[\frac{\partial w(x, t)}{\partial x} \right] \Big|_0^{l_p} + C_p V_p(t). \end{cases} \quad (4)$$

One end of the cantilever beam is fixed while the other end is free. So the boundary conditions are

$$\begin{cases} w(0, t) = w_b(t), \\ \frac{\partial w(0, t)}{\partial x} = 0, \end{cases} \quad (5)$$

and

$$\begin{cases} M_p(l_p, t) = B_p \frac{\partial^2 w(l_p, t)}{\partial x^2} - e_p V_p(t) = 0, \\ N_p(l_p, t) = \frac{\partial M_p(l_p, t)}{\partial x} = B_p \frac{\partial^3 w(l_p, t)}{\partial x^3} = 0. \end{cases} \quad (6)$$

In the classical energy harvesting applications, the cantilever beam is subject to a periodical base excitation $w_b(t)$. Thus the dynamic response of the cantilever beam is decomposed as

$$w(x, t) = w_b(t) + w_{rel}(x, t), \quad (7)$$

where $w_{rel}(x, t)$ is the relative displacement function of the cantilever beam. In this way, the system is converted into

$$B_p \frac{\partial^4 w_{rel}(x, t)}{\partial x^4} + m_p \frac{\partial^2 w_{rel}(x, t)}{\partial t^2} = -m_p \frac{\partial^2 w_b(t)}{\partial t^2}, \quad (8)$$

$$e_p \left[\frac{\partial^2 w_{rel}(x, t)}{\partial x \partial t} \right] \Big|_0^{l_p} + C_p \frac{dV_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \quad (9)$$

$$\begin{cases} w_{rel}(0, t) = 0, \\ \frac{\partial w_{rel}(0, t)}{\partial x} = 0, \end{cases} \quad (10)$$

and

$$\begin{cases} B_p \frac{\partial^2 w_{rel}(l_p, t)}{\partial x^2} - e_p V_p(t) = 0, \\ \frac{\partial^3 w_{rel}(l_p, t)}{\partial x^3} = 0. \end{cases} \quad (11)$$

Considering a sinusoidal base excitation

$$w_b(t) = \eta_b e^{j\sigma_b t} \quad (12)$$

where ξ_b is usually a real vibration amplitude, the steady state solution for the above system can be reasonably set as

$$w_{rel}(x, t) = \eta_{rel}(x) e^{j\sigma_b t}, \quad V_p(t) = \tilde{V}_p e^{j\sigma_b t}, \quad (13)$$

where $\eta_{rel}(x)$ and \tilde{V}_p are complex amplitudes. Then the above system is again simplified as

$$B_p \frac{\partial^4 \eta_{rel}(x)}{\partial x^4} - m_p \sigma_b^2 \eta_{rel}(x) = m_p \sigma_b^2 \eta_b, \quad (14)$$

$$\begin{cases} \eta_{rel}(0) = 0, \\ \frac{\partial \eta_{rel}(0)}{\partial x} = 0, \end{cases} \quad (15)$$

and

$$\begin{cases} B_p \frac{\partial^2 \eta_{rel}(l_p)}{\partial x^2} + \frac{j\sigma_b R_l}{1 + j\sigma_b C_p R_l} e_p^2 \frac{\partial \eta_{rel}(l_p)}{\partial x} = 0, \\ \frac{\partial^3 \eta_{rel}(l_p)}{\partial x^3} = 0. \end{cases} \quad (16)$$

Note that here we assume a sinusoidal steady state response, which is not actually validated theoretically.

Obviously we can have the following dimensionless scheme:

$$\eta_{rel} \sim u \eta_b, \quad x \sim z l_p \quad (17)$$

and therefore the following dimensionless parameters

$$\sigma = \sigma_b \sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \delta = \frac{e_p^2 l_p}{C_p B_p}. \quad (18)$$

Now, we reach the following dimensionless system of boundary value problem

$$\begin{cases} u'''' - \sigma^2 u = \sigma^2, \\ u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \frac{j\beta\sigma}{1 + j\beta\sigma} \delta u'(1) = 0, \\ u'''(1) = 0, \end{cases} \quad (19)$$

where the prime denotes the derivative with respect to z . The analytical solution to this problem can be formulated as

$$u(z; \delta) = A_\delta \cos \sqrt{\sigma} z + B_\delta \sin \sqrt{\sigma} z + C_\delta \cosh \sqrt{\sigma} z + D_\delta \sinh \sqrt{\sigma} z - 1 \quad (20)$$

and hence

$$\begin{aligned} u'(z; \delta) &= \sigma^{1/2} (-A_\delta \sin \sqrt{\sigma} z + B_\delta \cos \sqrt{\sigma} z + C_\delta \sinh \sqrt{\sigma} z + D_\delta \cosh \sqrt{\sigma} z), \\ u''(z; \delta) &= \sigma (-A_\delta \cos \sqrt{\sigma} z - B_\delta \sin \sqrt{\sigma} z + C_\delta \cosh \sqrt{\sigma} z + D_\delta \sinh \sqrt{\sigma} z), \\ u'''(z; \delta) &= \sigma^{3/2} (A_\delta \sin \sqrt{\sigma} z - B_\delta \cos \sqrt{\sigma} z + C_\delta \sinh \sqrt{\sigma} z + D_\delta \cosh \sqrt{\sigma} z). \end{aligned} \quad (21)$$

The coefficients A_δ , B_δ , C_δ , and D_δ are then subject to the following linear system of equations:

$$\begin{cases} A_\delta + C_\delta = 1, \\ B_\delta + D_\delta = 0, \\ (-A_\delta \cos \sqrt{\sigma} - B_\delta \sin \sqrt{\sigma} + C_\delta \cosh \sqrt{\sigma} + D_\delta \sinh \sqrt{\sigma}) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1} \delta (-A_\delta \sin \sqrt{\sigma} + B_\delta \cos \sqrt{\sigma} + C_\delta \sinh \sqrt{\sigma} + D_\delta \cosh \sqrt{\sigma}) = 0, \\ A_\delta \sin \sqrt{\sigma} - B_\delta \cos \sqrt{\sigma} + C_\delta \sinh \sqrt{\sigma} + D_\delta \cosh \sqrt{\sigma} = 0. \end{cases} \quad (22)$$

Analytically, we can directly obtain the solution to this problem as

$$\begin{cases} A_\delta = \frac{1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} - \sin \sqrt{\sigma} \sinh \sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma} \delta (\cos \sqrt{\sigma} \sinh \sqrt{\sigma})}{2 \left[1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \delta (\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}) \right]}, \\ B_\delta = \frac{\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma} \delta (\sin \sqrt{\sigma} \sinh \sqrt{\sigma})}{2 \left[1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \delta (\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}) \right]}, \\ C_\delta = \frac{1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \sin \sqrt{\sigma} \sinh \sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma} \delta (\sin \sqrt{\sigma} \cosh \sqrt{\sigma})}{2 \left[1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \delta (\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}) \right]}, \\ D_\delta = \frac{-\cos \sqrt{\sigma} \sinh \sqrt{\sigma} - \sin \sqrt{\sigma} \cosh \sqrt{\sigma} - \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma} \delta (\sin \sqrt{\sigma} \sinh \sqrt{\sigma})}{2 \left[1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \delta (\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}) \right]}. \end{cases} \quad (23)$$

The resulting complex amplitudes \tilde{V}_p , \tilde{I}_p , and \tilde{P}_p for output voltage $V_p(t)$, output current $I_p(t)$, and output power $P_p(t)$, respectively, can be formulated as follows

$$\begin{cases} \tilde{V}_p = -\frac{j\sigma\beta}{j\sigma\beta + 1} \frac{\eta_b}{l_p} \frac{e_p}{C_p} u'(1), \\ = -\frac{j\sigma\beta}{j\sigma\beta + 1} \frac{\eta_b}{l_p} \frac{e_p}{C_p} \sigma^{1/2} (-A_\delta \sin \sqrt{\sigma} + B_\delta \cos \sqrt{\sigma} + C_\delta \sinh \sqrt{\sigma} + D_\delta \cosh \sqrt{\sigma}) \\ = -\frac{j\sigma\beta}{j\sigma\beta + 1} \frac{\eta_b}{l_p} \frac{e_p}{C_p} \frac{\sqrt{\sigma} (\sinh \sqrt{\sigma} - \sin \sqrt{\sigma})}{1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \delta (\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma})} \\ = -\frac{j\sigma\beta}{j\sigma\beta + 1} \left(\frac{\eta_b}{l_p} \right) \left(\frac{e_p}{C_p} \right) \chi_p, \\ \tilde{I}_p = \tilde{V}_p / R_l = \frac{-j}{j\sigma\beta + 1} \left(\frac{\eta_b}{l_p} \right) (e_p \sigma_b) \chi_p, \\ \tilde{P}_p = \tilde{V}_p^2 / R_l = -\left(\frac{\eta_b}{l_p} \right)^2 \left(\frac{e_p}{C_p} \right) (e_p \sigma_b) \frac{\sigma\beta}{(j\sigma\beta + 1)^2} \chi_p^2, \end{cases} \quad (24)$$

in which we have used the notations that

$$\chi_p = u'_1(1) = \frac{\sqrt{\sigma} (\sinh \sqrt{\sigma} - \sin \sqrt{\sigma})}{1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \delta (\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma})}. \quad (25)$$

In view of the above process, it is shown that the problem is determined by the following four dimensionless groups:

$$\sigma = \sigma_b \sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \delta = \frac{e_p^2 l_p}{C_p B_p}, \quad r_d = \frac{\eta_b}{l_p}, \quad (26)$$

where σ is the dimensionless base excitation frequency, β dimensionless electrical resonant frequency, δ is the dimensionless electromechanical coupling strength for the structure, and r_d is the dimensionless base excitation amplitude. In fact, we have eight physical parameters in this problem: m_p , l_p , B_p , e_p , C_p , R_l , σ_b , and η_b , while four dimensions length [L], mass [M], time [T], and electrical current [I] are present. Thus according to the Buckingham's PI theorem, [1] we have four independent dimensionless groups, which is organized as above.

Nonetheless, the tuning effect is minuscule in the way that compared to the non-piezoelectric case, amplitude of the displacement $u(z)$ is disturbed to a small degree, while the phase is little offset from a value of 0.

In the following, we can obtain the corresponding displacement function $u(z)$ in terms of its amplitude and phase in Figure 1.

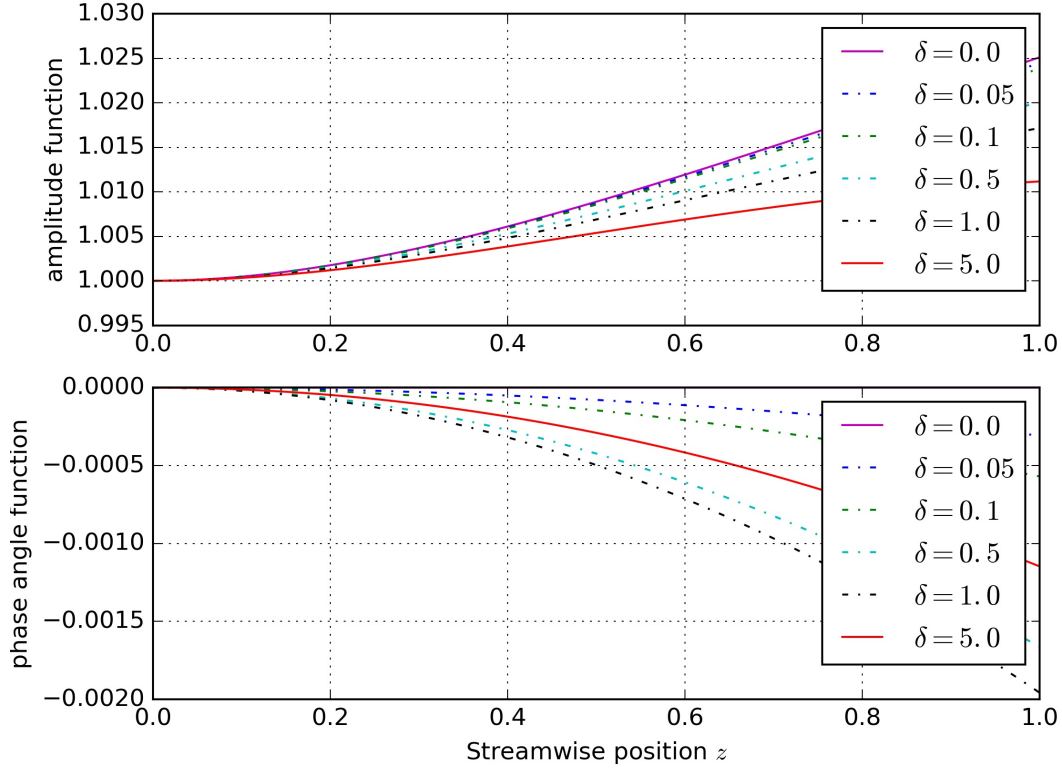


Figure 1: Amplitude and phase of the displacement function $u(z)$

Using the following regular expansion:

$$\begin{cases} A_\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \dots, \\ B_\epsilon = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \dots, \\ C_\epsilon = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \dots, \\ D_\epsilon = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots, \end{cases} \quad (27)$$

we obtain the successive expansion problem:

$O(\epsilon^0)$:

$$\begin{cases} A_0 + C_0 = 1, \\ B_0 + D_0 = 0, \\ -A_0 \cos \sqrt{\sigma} - B_0 \sin \sqrt{\sigma} + C_0 \cosh \sqrt{\sigma} + D_0 \sinh \sqrt{\sigma} = 0, \\ A_0 \sin \sqrt{\sigma} - B_0 \cos \sqrt{\sigma} + C_0 \sinh \sqrt{\sigma} + D_0 \cosh \sqrt{\sigma} = 0. \end{cases} \quad (28)$$

The solution is

$$\begin{cases} A_0 = \frac{1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} - \sin \sqrt{\sigma} \sinh \sqrt{\sigma}}{2 + 2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma}} \\ B_0 = \frac{\cosh \sqrt{\sigma} \sin \sqrt{\sigma} + \cos \sqrt{\sigma} \sinh \sqrt{\sigma}}{2 + 2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma}} \\ C_0 = \frac{1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \sin \sqrt{\sigma} \sinh \sqrt{\sigma}}{2 + 2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma}} \\ D_0 = -\frac{\cosh \sqrt{\sigma} \sin \sqrt{\sigma} + \cos \sqrt{\sigma} \sinh \sqrt{\sigma}}{2 + 2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma}} \end{cases} \quad (29)$$

Hence we have

$$-A_0 \sin \sqrt{\sigma} + B_0 \cos \sqrt{\sigma} + C_0 \sinh \sqrt{\sigma} + D_0 \cosh \sqrt{\sigma} = \frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \quad (30)$$

$O(\epsilon^1)$:

$$\left\{ \begin{array}{l} A_1 + C_1 = 0, \\ B_1 + D_1 = 0, \\ (-A_1 \cos \sqrt{\sigma} - B_1 \sin \sqrt{\sigma} + C_1 \cosh \sqrt{\sigma} + D_1 \sinh \sqrt{\sigma}) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1} (-A_0 \sin \sqrt{\sigma} + B_0 \cos \sqrt{\sigma} + C_0 \sinh \sqrt{\sigma} + D_0 \cosh \sqrt{\sigma}) = 0, \\ A_1 \sin \sqrt{\sigma} - B_1 \cos \sqrt{\sigma} + C_1 \sinh \sqrt{\sigma} + D_1 \cosh \sqrt{\sigma} = 0. \end{array} \right. \quad (31)$$

The solution is

$$\left\{ \begin{array}{l} A_1 = \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{\cos \sqrt{\sigma} + \cosh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \\ B_1 = \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{-\sinh \sqrt{\sigma} + \sin \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \\ C_1 = \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(-\frac{\cos \sqrt{\sigma} + \cosh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \\ D_1 = \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{-\sin \sqrt{\sigma} + \sinh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \end{array} \right. \quad (32)$$

Then we have

$$\begin{aligned} & -A_1 \sin \sqrt{\sigma} + B_1 \cos \sqrt{\sigma} + C_1 \sinh \sqrt{\sigma} + D_1 \cosh \sqrt{\sigma} \\ &= \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \left(\frac{\sin \sqrt{\sigma} - \sinh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \end{aligned} \quad (33)$$

$O(\epsilon^2)$:

$$\left\{ \begin{array}{l} A_2 + C_2 = 0, \\ B_2 + D_2 = 0, \\ (-A_2 \cos \sqrt{\sigma} - B_2 \sin \sqrt{\sigma} + C_2 \cosh \sqrt{\sigma} + D_2 \sinh \sqrt{\sigma}) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1} (-A_1 \sin \sqrt{\sigma} + B_1 \cos \sqrt{\sigma} + C_1 \sinh \sqrt{\sigma} + D_1 \cosh \sqrt{\sigma}) = 0, \\ A_2 \sin \sqrt{\sigma} - B_2 \cos \sqrt{\sigma} + C_2 \sinh \sqrt{\sigma} + D_2 \cosh \sqrt{\sigma} = 0. \end{array} \right. \quad (34)$$

The solution is

$$\left\{ \begin{array}{l} A_2 = \left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right)^2 \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{\cos \sqrt{\sigma} + \cosh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \\ B_2 = \left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right)^2 \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{-\sinh \sqrt{\sigma} + \sin \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \\ C_2 = \left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right)^2 \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(-\frac{\cos \sqrt{\sigma} + \cosh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \\ D_2 = \left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right)^2 \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{-\sin \sqrt{\sigma} + \sinh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \end{array} \right. \quad (35)$$

To get higher order expansions, we can use the following iteration method:

$O(\epsilon^{k+1})$ ($k \geq 1$):

$$\left\{ \begin{array}{l} A_{k+1} + C_{k+1} = 0, \\ B_{k+1} + D_{k+1} = 0, \\ (-A_{k+1} \cos \sqrt{\sigma} - B_{k+1} \sin \sqrt{\sigma} + C_{k+1} \cosh \sqrt{\sigma} + D_{k+1} \sinh \sqrt{\sigma}) + \\ \frac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1} (-A_k \sin \sqrt{\sigma} + B_k \cos \sqrt{\sigma} + C_k \sinh \sqrt{\sigma} + D_k \cosh \sqrt{\sigma}) = 0, \\ A_{k+1} \sin \sqrt{\sigma} - B_{k+1} \cos \sqrt{\sigma} + C_{k+1} \sinh \sqrt{\sigma} + D_{k+1} \cosh \sqrt{\sigma} = 0. \end{array} \right. \quad (36)$$

The solution is

$$\begin{cases} A_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right) \left(\frac{\cos \sqrt{\sigma} + \cosh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) (Q_k) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right) \left(\frac{-\sinh \sqrt{\sigma} + \sin \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) (Q_k) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right) \left(-\frac{\cos \sqrt{\sigma} + \cosh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) (Q_k) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right) \left(\frac{-\sin \sqrt{\sigma} + \sinh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) (Q_k) \end{cases} \quad (37)$$

where for $k \geq 2$

$$Q_k = -A_k \sin \sqrt{\sigma} + B_k \cos \sqrt{\sigma} + C_k \sinh \sqrt{\sigma} + D_k \cosh \sqrt{\sigma}, \quad (38)$$

and for $k \geq 0$

$$\begin{aligned} Q_{k+1} &= -A_{k+1} \sin \sqrt{\sigma} + B_{k+1} \cos \sqrt{\sigma} + C_{k+1} \sinh \sqrt{\sigma} + D_{k+1} \cosh \sqrt{\sigma} \\ &= - \left(\frac{\sin \sqrt{\sigma} \cosh \sqrt{\sigma} + \cos \sqrt{\sigma} \sinh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right) Q_k, \end{aligned} \quad (39)$$

and

$$\begin{aligned} Q_1 &= -A_1 \sin \sqrt{\sigma} + B_1 \cos \sqrt{\sigma} + C_1 \sinh \sqrt{\sigma} + D_1 \cosh \sqrt{\sigma} \\ &= \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \left(\frac{\sin \sqrt{\sigma} - \sinh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \end{aligned} \quad (40)$$

$$Q_0 = \frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \quad (41)$$

Hence it is shown that for $k \geq 0$

$$\begin{aligned} Q_k &= - \left(\frac{\sin \sqrt{\sigma} \cosh \sqrt{\sigma} + \cos \sqrt{\sigma} \sinh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right) Q_k \\ &= \left[- \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right) \left(\frac{\sin \sqrt{\sigma} \cosh \sqrt{\sigma} + \cos \sqrt{\sigma} \sinh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \right]^k \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \end{aligned} \quad (42)$$

As a result, we obtain that for $k \geq 0$

$$\begin{cases} A_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right)^{k+1} \left(\frac{-\sin \sqrt{\sigma} \cosh \sqrt{\sigma} - \cos \sqrt{\sigma} \sinh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right)^k \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{\cos \sqrt{\sigma} + \cosh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right)^{k+1} \left(\frac{-\sin \sqrt{\sigma} \cosh \sqrt{\sigma} - \cos \sqrt{\sigma} \sinh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right)^k \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{-\sinh \sqrt{\sigma} + \sin \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right)^{k+1} \left(\frac{-\sin \sqrt{\sigma} \cosh \sqrt{\sigma} - \cos \sqrt{\sigma} \sinh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right)^k \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{-\cos \sqrt{\sigma} - \cosh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma} \right)^{k+1} \left(\frac{-\sin \sqrt{\sigma} \cosh \sqrt{\sigma} - \cos \sqrt{\sigma} \sinh \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right)^k \left(\frac{\sinh \sqrt{\sigma} - \sin \sqrt{\sigma}}{\cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 1} \right) \left(\frac{-\sin \sqrt{\sigma} + \sinh \sqrt{\sigma}}{2 \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + 2} \right) \end{cases} \quad (43)$$

Reference

References

- [1] Buckingham E. On physically similar systems; illustrations of the use of dimensional equations. Physical review. 1914;4(4):345.