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# An asymptotic theory for dynamic response of laminated piezoelectric shells

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**Summary.** A three-dimensional (3D) asymptotic theory for dynamic analysis of doubly curved laminated piezoelectric shells is formulated on the basis of 3D piezoelectricity. By using the direct elimination, we reduce the twenty-two basic equations of 3D piezoelectricity to eight differential equations in terms of eight primary variables of elastic and electric fields. In the formulation, multiple time scales are introduced to eliminate the secular terms so that the asymptotic expansion is uniform and feasible. By means of nondimensionalization, asymptotic expansion and successive integration, we finally can obtain recurrent sets of governing equations for various order problems. The classical laminated piezoelectric shell theory (CST) is derived as a first-order approximation to the 3D piezoelectricity. Higher-order corrections can be determined by considering the solvability and orthonormality conditions in a systematic and consistent way. Several benchmark solutions for various piezoelectric laminates are given to demonstrate the performance of the theory.

## 1 Introduction

In recent years, piezoelectric materials have been widely used in the engineering applications for sensing and actuation purposes. The static and dynamic analyses of piezoelectric laminates have therefore attracted considerable research interest. Since the number of basic equations within the framework of 3D piezoelectricity is twenty-two, which represents a complicated mathematic system to be solved, the 3D dynamic analysis of piezoelectric laminates is scarce in the literature. Determination of the exact solutions for the present problems becomes a challenging research subject.

Exact solutions for the free vibration behavior of laminated piezoelectric plates in cylindrical bending and of laminated piezoelectric cylinders were presented by Heyliger and Brooks [1] and Hussein and Heyliger [2], respectively. In their analyses, either the electrostatic potentials or the normal electric displacements at the upper and lower surfaces are specified to be zero. The primary field variables are expanded as Fourier series in the in-surface directions and are assumed as the harmonic functions in time variable. Since the discrete-layer model is used, the appropriate interface conditions are imposed to be satisfied. Natural frequencies and the distributions of modal variables in elastic and electric fields through the thickness direction for various piezoelectric laminates are presented. The previous discrete-layer model has also been used to study axisymmetric free vibration of homogeneous and laminated piezoelectric cylinders by Kharouf and Heyliger [3].

By using the general solution for coupled equations of piezoelectric media [4], Ding et al. [5] presented an exact 3D free vibration analysis of a transversely isotropic piezoelectric cylindrical panel. It was indicated that the natural frequencies of the piezoelectric panels are larger than those of non-piezoelectric ones. Sharma and Pathania [6] studied the previous problems by an alternative approach. Three displacement potential functions were introduced in their formulation to yield the simple form of the fundamental equations. The 3D free vibration solution was obtained by using modified Bessel functions with complex arguments. Tiersten [7], [8] presented the basic equations related to the static and dynamic analyses of piezoelectric plates and shells. Krommer [9] studied thermally induced vibration problems of laminated plates with attached or embedded piezoelectric layers, and proposed a novel approach to study the static and dynamic behaviors of piezoelectric plates with arbitrary polygonal edges [10]. Comprehensive reviews of theoretical analysis and numerical modeling for piezoelectric laminates have also been made in the literature [11]–[14].

Recently, several 3D solutions for the analyses of laminated doubly curved shells were presented on the basis of 3D elasticity [15]–[17]. After a close literature survey, however, we found that 3D solutions for the dynamic response of doubly curved laminated piezoelectric shells have not been studied.

In several papers [18]–[21], a 3D asymptotic theory was developed for the static, dynamic and buckling analyses of laminated composite shells by means of the method of perturbation. The purpose of this paper is to extend the asymptotic theory to the dynamic analysis of doubly curved laminated piezoelectric shells. Due to the coupling effect of elastic and electric fields, the present formulation is inherently more complicated than that in [19]. Nevertheless, we shall see that the derivation is consistent and the obtained asymptotic solution is feasible. A parametric study for determining the natural frequencies and the modal variables in elastic and electric fields has also been made to demonstrate the convergence and accuracy of the present asymptotic theory.

## 2 Basic three-dimensional equations

Consider a doubly curved laminated piezoelectric shell whose thickness is $2h$. For such a shell, the lines of principal curvature are denoted as $\alpha$ and $\beta$, as shown in Fig. 1. $R_\alpha$ and $R_\beta$ denote the curvature radii to the middle surface, $a_\alpha$ and $a_\beta$ are the curvilinear dimensions in $\alpha$- and $\beta$-directions, respectively.

The linear constitutive equations of the piezoelectric material are given by

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\zeta \\ \tau_{\beta\zeta} \\ \tau_{\alpha\zeta} \\ \tau_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\zeta \\ \gamma_{\beta\zeta} \\ \gamma_{\alpha\zeta} \\ \gamma_{\alpha\beta} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & e_{33} \\ e_{14} & e_{24} & 0 \\ e_{15} & e_{25} & 0 \\ 0 & 0 & e_{36} \end{bmatrix} \begin{Bmatrix} E_\alpha \\ E_\beta \\ E_\zeta \end{Bmatrix}, \tag{1}$$

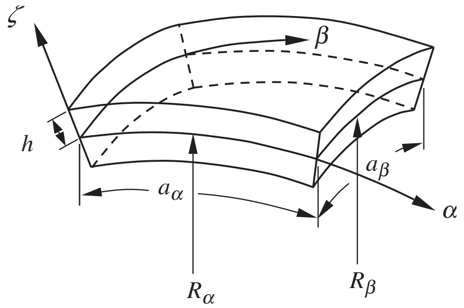


**Fig. 1.** The geometry and coordinates of a typical doubly curved shell

$$\begin{Bmatrix} D_\alpha \\ D_\beta \\ D_\zeta \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & e_{14} & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & e_{25} & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & e_{36} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\zeta \\ \gamma_{\beta\zeta} \\ \gamma_{\alpha\zeta} \\ \gamma_{\alpha\beta} \end{Bmatrix} + \begin{bmatrix} \eta_{11} & \eta_{12} & 0 \\ \eta_{12} & \eta_{22} & 0 \\ 0 & 0 & \eta_{33} \end{bmatrix} \begin{Bmatrix} E_\alpha \\ E_\beta \\ E_\zeta \end{Bmatrix}, \tag{2}$$

where $\sigma_\alpha$, $\sigma_\beta$, $\sigma_\zeta$, $\tau_{\alpha\zeta}$, $\tau_{\beta\zeta}$, $\tau_{\alpha\beta}$ and $\varepsilon_\alpha$, $\varepsilon_\beta$, $\varepsilon_\zeta$, $\gamma_{\alpha\zeta}$, $\gamma_{\beta\zeta}$, $\gamma_{\alpha\beta}$ denote the stress and strain components, respectively. $D_\alpha$, $D_\beta$, $D_\zeta$ and $E_\alpha$, $E_\beta$, $E_\zeta$ denote the components of electric displacement and electric field, respectively. $c_{ij}$, $e_{ij}$ and $\eta_{ij}$ are the elastic coefficients, piezoelectric coefficients and dielectric coefficients, respectively, relative to the geometrical axes of the shell. The material is regarded to be heterogeneous through the thickness (i.e., $c_{ij}(\zeta)$, $e_{ij}(\zeta)$ and $\eta_{ij}(\zeta)$). For a laminated piezoelectric shell, the material properties are the layerwise step functions through the thickness direction.

The kinematic equations in terms of the curvilinear coordinates $\alpha, \beta$ and $\zeta$ are

$$\begin{aligned} &\varepsilon_\alpha = \frac{1}{\gamma_\alpha}\left(\frac{\partial u_\alpha}{\partial \alpha} + \frac{u_\zeta}{R_\alpha}\right), \quad \varepsilon_\beta = \frac{1}{\gamma_\beta}\left(\frac{\partial u_\beta}{\partial \beta} + \frac{u_\zeta}{R_\beta}\right), \quad \varepsilon_\zeta = \frac{\partial u_\zeta}{\partial \zeta}, \\ &\gamma_{\beta\zeta} = \frac{1}{\gamma_\beta}\frac{\partial u_\zeta}{\partial \beta} + \frac{\partial u_\beta}{\partial \zeta} - \frac{u_\beta}{\gamma_\beta R_\beta}, \quad \gamma_{\alpha\zeta} = \frac{1}{\gamma_\alpha}\frac{\partial u_\zeta}{\partial \alpha} + \frac{\partial u_\alpha}{\partial \zeta} - \frac{u_\alpha}{\gamma_\alpha R_\alpha}, \\ &\gamma_{\alpha\beta} = \frac{1}{\gamma_\alpha}\frac{\partial u_\beta}{\partial \alpha} + \frac{1}{\gamma_\beta}\frac{\partial u_\alpha}{\partial \beta}, \end{aligned} \tag{3.1}$$

in which $\gamma_\alpha = 1 + \frac{\zeta}{R_\alpha}$, $\gamma_\beta = 1 + \frac{\zeta}{R_\beta}$, and $u_\alpha$, $u_\beta$ and $u_\zeta$ are the displacement components.

The equations of motion are given by

$$\gamma_\beta \frac{\partial \sigma_\alpha}{\partial \alpha} + \gamma_\alpha \frac{\partial \tau_{\alpha\beta}}{\partial \beta} + \gamma_\alpha\gamma_\beta \frac{\partial \tau_{\alpha\zeta}}{\partial \zeta} + \left(\frac{2}{R_\alpha} + \frac{1}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right)\tau_{\alpha\zeta} = \gamma_\alpha\gamma_\beta\rho\frac{\partial^2 u_\alpha}{\partial t^2}, \tag{4}$$

$$\gamma_\alpha \frac{\partial \sigma_\beta}{\partial \beta} + \gamma_\beta \frac{\partial \tau_{\alpha\beta}}{\partial \alpha} + \gamma_\alpha\gamma_\beta \frac{\partial \tau_{\beta\zeta}}{\partial \zeta} + \left(\frac{1}{R_\alpha} + \frac{2}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right)\tau_{\beta\zeta} = \gamma_\alpha\gamma_\beta\rho\frac{\partial^2 u_\beta}{\partial t^2}, \tag{5}$$

$$\gamma_\beta \frac{\partial \tau_{\alpha\zeta}}{\partial \alpha} + \gamma_\alpha \frac{\partial \tau_{\beta\zeta}}{\partial \beta} + \gamma_\alpha\gamma_\beta \frac{\partial \sigma_\zeta}{\partial \zeta} + \left(\frac{1}{R_\alpha} + \frac{1}{R_\beta} + \frac{2\zeta}{R_\alpha R_\beta}\right)\sigma_\zeta - \frac{\gamma_\beta}{R_\alpha}\sigma_\alpha - \frac{\gamma_\alpha}{R_\beta}\sigma_\beta = \gamma_\alpha\gamma_\beta\rho\frac{\partial^2 u_\zeta}{\partial t^2}. \tag{6}$$

The charge equation of the piezoelectric material is

$$\gamma_\beta \frac{\partial D_\alpha}{\partial \alpha} + \gamma_\alpha \frac{\partial D_\beta}{\partial \beta} + \gamma_\alpha\gamma_\beta \frac{\partial D_\zeta}{\partial \zeta} + \left(\frac{\gamma_\beta}{R_\alpha} + \frac{\gamma_\alpha}{R_\beta}\right)D_\zeta = 0. \tag{7}$$

The relations between the electric field and electric potential are

$$E_\alpha = -\frac{1}{\gamma_\alpha}\frac{\partial \Phi}{\partial \alpha}, \quad E_\beta = -\frac{1}{\gamma_\beta}\frac{\partial \Phi}{\partial \beta}, \quad E_\zeta = -\frac{\partial \Phi}{\partial \zeta}, \tag{8.1–3}$$

where $\Phi$ denotes the electric potential.

According to Eqs. (1)–(8), it is listed that there are twenty-two basic equations for the present dynamic analysis of doubly curved laminated piezoelectric shells. In order to make the previous complicated formulation suitable for mathematical treatment, we eliminate the in-surface stresses ($\sigma_\alpha$, $\sigma_\beta$, $\tau_{\alpha\beta}$) and electric displacements ($D_\alpha$ and $D_\beta$), the components of strain ($\varepsilon_\alpha$, $\varepsilon_\beta$, $\varepsilon_\zeta$, $\gamma_{\alpha\zeta}$, $\gamma_{\beta\zeta}$, $\gamma_{\alpha\beta}$) and the electric field ($E_\alpha$, $E_\beta$, $E_\zeta$) from Eqs. (1)–(8), and express the basic equations in terms of the transverse stresses ($\tau_{\alpha\zeta}$, $\tau_{\beta\zeta}$, $\sigma_\zeta$) and electric displacement ($D_\zeta$), the displacements ($u_\alpha$, $u_\beta$, $u_\zeta$) and the electric potential ($\Phi$) as follows:

$$\frac{\partial u_\zeta}{\partial \zeta} = -[l_{31} \quad l_{32}]\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} - l_{33}u_\zeta + l_{34}\sigma_\zeta + l_{35}D_\zeta, \tag{9}$$

$$\begin{Bmatrix} \dfrac{\partial u_\alpha}{\partial \zeta} \\ \dfrac{\partial u_\beta}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} l_{11} & 0 \\ 0 & l_{22} \end{bmatrix}\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} - \begin{bmatrix} l_{13} \\ l_{23} \end{bmatrix} u_\zeta + \begin{bmatrix} l_{14} & l_{15} \\ l_{15} & l_{25} \end{bmatrix}\begin{Bmatrix} \tau_{\alpha\zeta} \\ \tau_{\beta\zeta} \end{Bmatrix} + \begin{bmatrix} l_{16} \\ l_{26} \end{bmatrix}\Phi, \tag{10}$$

$$\frac{\partial D_\zeta}{\partial \zeta} = -[\, l_{13} \quad l_{23}\,]\begin{Bmatrix} D_\alpha \\ D_\beta \end{Bmatrix} - l_{71}D_\zeta, \tag{11}$$

$$\begin{Bmatrix} \dfrac{\partial \tau_{\alpha\zeta}}{\partial \zeta} \\ \dfrac{\partial \tau_{\beta\zeta}}{\partial \zeta} \end{Bmatrix} = -\begin{bmatrix} l_{41} & l_{42} \\ l_{51} & l_{52} \end{bmatrix}\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} - \begin{bmatrix} l_{43} \\ l_{53} \end{bmatrix} u_\zeta - \begin{bmatrix} l_{44} & 0 \\ 0 & l_{55} \end{bmatrix}\begin{Bmatrix} \tau_{\alpha\zeta} \\ \tau_{\beta\zeta} \end{Bmatrix} - \begin{bmatrix} l_{31} \\ l_{32} \end{bmatrix}\sigma_\zeta - \begin{bmatrix} l_{46} \\ l_{56} \end{bmatrix} D_\zeta + \begin{Bmatrix} \rho\dfrac{\partial^2 u_\alpha}{\partial t^2} \\ \rho\dfrac{\partial^2 u_\beta}{\partial t^2} \end{Bmatrix}, \tag{12}$$

$$\frac{\partial \sigma_\zeta}{\partial \zeta} = [l_{61} \quad l_{62}]\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} + l_{63}u_\zeta - [l_{13} \quad l_{23}]\begin{Bmatrix} \tau_{\alpha\zeta} \\ \tau_{\beta\zeta} \end{Bmatrix} - l_{64}\sigma_\zeta + l_{65}D_\zeta + \rho\frac{\partial^2 u_\zeta}{\partial t^2}, \tag{13}$$

$$\frac{\partial \Phi}{\partial \zeta} = -[\, l_{46} \quad l_{56}\,]\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} - l_{65}u_\zeta + l_{35}\sigma_\zeta + l_{81}D_\zeta, \tag{14}$$

where

$$l_{31} = \left(\frac{e_{31}e_{33} + c_{13}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{1}{\gamma_\alpha}\frac{\partial}{\partial \alpha} + \left(\frac{e_{33}e_{36} + c_{36}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{1}{\gamma_\beta}\frac{\partial}{\partial \beta},$$

$$l_{32} = \left(\frac{e_{36}e_{33} + c_{36}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{1}{\gamma_\alpha}\frac{\partial}{\partial \alpha} + \left(\frac{e_{33}e_{32} + c_{23}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{1}{\gamma_\beta}\frac{\partial}{\partial \beta},$$

$$l_{33} = \left(\frac{e_{31}e_{33} + c_{13}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{1}{\gamma_\alpha R_\alpha} + \left(\frac{e_{33}e_{32} + c_{23}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{1}{\gamma_\beta R_\beta},$$

$$l_{34} = \frac{\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}, \quad l_{35} = \frac{e_{33}}{e_{33}^2 + c_{33}\eta_{33}},$$

$$l_{11} = \frac{1}{\gamma_\alpha R_\alpha}, \quad l_{22} = \frac{1}{\gamma_\beta R_\beta}, \quad l_{13} = \frac{1}{\gamma_\alpha}\frac{\partial}{\partial\alpha}, \quad l_{23} = \frac{1}{\gamma_\beta}\frac{\partial}{\partial\beta},$$

$$l_{14} = \frac{c_{44}}{c_{44}c_{55} - c_{45}^2}, \quad l_{15} = -\frac{c_{45}}{c_{45}c_{55} - c_{45}^2}, \quad l_{25} = \frac{c_{55}}{c_{44}c_{55} - c_{45}^2},$$

$$l_{16} = \left(\frac{c_{45}e_{14} - c_{44}e_{15}}{c_{44}c_{55} - c_{45}^2}\right)\frac{1}{\gamma_\alpha}\frac{\partial}{\partial\alpha} + \left(\frac{c_{45}e_{24} - c_{44}e_{25}}{c_{44}c_{55} - c_{45}^2}\right)\frac{1}{\gamma_\beta}\frac{\partial}{\partial\beta},$$

$$l_{26} = \left(\frac{c_{45}e_{15} - c_{55}e_{14}}{c_{44}c_{55} - c_{45}^2}\right)\frac{1}{\gamma_\alpha}\frac{\partial}{\partial\alpha} + \left(\frac{c_{45}e_{25} - c_{55}e_{24}}{c_{44}c_{55} - c_{45}^2}\right)\frac{1}{\gamma_\beta}\frac{\partial}{\partial\beta},$$

$$l_{41} = \frac{Q_{11}}{\gamma_\alpha^2}\frac{\partial^2}{\partial\alpha^2} + \frac{(Q_{16} + Q_{61})}{\gamma_\alpha\gamma_\beta}\frac{\partial^2}{\partial\alpha\partial\beta} + \frac{Q_{66}}{\gamma_\beta^2}\frac{\partial^2}{\partial\beta^2},$$

$$l_{42} = \frac{Q_{16}}{\gamma_\alpha^2}\frac{\partial^2}{\partial\alpha^2} + \frac{(Q_{12} + Q_{66})}{\gamma_\alpha\gamma_\beta}\frac{\partial^2}{\partial\alpha\partial\beta} + \frac{Q_{62}}{\gamma_\beta^2}\frac{\partial^2}{\partial\beta^2},$$

$$l_{51} = \frac{Q_{61}}{\gamma_\alpha^2}\frac{\partial^2}{\partial\alpha^2} + \frac{(Q_{21} + Q_{66})}{\gamma_\alpha\gamma_\beta}\frac{\partial^2}{\partial\alpha\partial\beta} + \frac{Q_{26}}{\gamma_\beta^2}\frac{\partial^2}{\partial\beta^2},$$

$$l_{52} = \frac{Q_{66}}{\gamma_\alpha^2}\frac{\partial^2}{\partial\alpha^2} + \frac{(Q_{26} + Q_{62})}{\gamma_\alpha\gamma_\beta}\frac{\partial^2}{\partial\alpha\partial\beta} + \frac{Q_{22}}{\gamma_\beta^2}\frac{\partial^2}{\partial\beta^2},$$

$$l_{43} = \left(\frac{Q_{11}}{\gamma_\alpha^2 R_\alpha} + \frac{Q_{12}}{\gamma_\alpha\gamma_\beta R_\beta}\right)\frac{\partial}{\partial\alpha} + \left(\frac{Q_{61}}{\gamma_\alpha\gamma_\beta R_\alpha} + \frac{Q_{62}}{\gamma_\beta^2 R_\beta}\right)\frac{\partial}{\partial\beta},$$

$$l_{53} = \left(\frac{Q_{61}}{\gamma_\alpha^2 R_\alpha} + \frac{Q_{62}}{\gamma_\alpha\gamma_\beta R_\beta}\right)\frac{\partial}{\partial\alpha} + \left(\frac{Q_{21}}{\gamma_\alpha\gamma_\beta R_\alpha} + \frac{Q_{22}}{\gamma_\beta^2 R_\beta}\right)\frac{\partial}{\partial\beta},$$

$$l_{44} = \frac{1}{\gamma_\alpha\gamma_\beta}\left(\frac{2}{R_\alpha} + \frac{1}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right), \quad l_{55} = \frac{1}{\gamma_\alpha\gamma_\beta}\left(\frac{1}{R_\alpha} + \frac{2}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right),$$

$$l_{46} = \left(\frac{c_{13}e_{33} - e_{31}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{1}{\gamma_\alpha}\frac{\partial}{\partial\alpha} + \left(\frac{c_{36}e_{33} - e_{36}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{1}{\gamma_\beta}\frac{\partial}{\partial\beta},$$

$$l_{56} = \left(\frac{c_{36}e_{33} - e_{36}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{1}{\gamma_\alpha}\frac{\partial}{\partial\alpha} + \left(\frac{c_{23}e_{33} - e_{32}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{1}{\gamma_\beta}\frac{\partial}{\partial\beta},$$

$$l_{61} = \left(\frac{Q_{11}}{\gamma_\alpha^2 R_\alpha} + \frac{Q_{21}}{\gamma_\alpha\gamma_\beta R_\beta}\right)\frac{\partial}{\partial\alpha} + \left(\frac{Q_{16}}{\gamma_\alpha\gamma_\beta R_\alpha} + \frac{Q_{26}}{\gamma_\beta^2 R_\beta}\right)\frac{\partial}{\partial\beta},$$

$$l_{62} = \left(\frac{Q_{16}}{\gamma_\alpha^2 R_\alpha} + \frac{Q_{26}}{\gamma_\alpha\gamma_\beta R_\beta}\right)\frac{\partial}{\partial\alpha} + \left(\frac{Q_{12}}{\gamma_\alpha\gamma_\beta R_\alpha} + \frac{Q_{22}}{\gamma_\beta^2 R_\beta}\right)\frac{\partial}{\partial\beta},$$

$$l_{63} = \frac{Q_{11}}{\gamma_\alpha^2 R_\alpha^2} + \frac{(Q_{12} + Q_{21})}{\gamma_\alpha\gamma_\beta R_\alpha R_\beta} + \frac{Q_{22}}{\gamma_\beta^2 R_\beta^2},$$

$$l_{64} = \frac{1}{\gamma_\alpha \gamma_\beta}\left(\frac{1}{R_\alpha} + \frac{1}{R_\beta} + \frac{2\zeta}{R_\alpha R_\beta}\right) - \frac{1}{\gamma_\alpha R_\alpha}\left(\frac{e_{31}e_{33} + c_{13}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right) - \frac{1}{\gamma_\beta R_\beta}\left(\frac{e_{32}e_{33} + c_{23}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right),$$

$$l_{65} = \frac{1}{\gamma_\alpha R_\alpha}\left(\frac{c_{13}e_{33} - e_{31}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right) + \frac{1}{\gamma_\beta R_\beta}\left(\frac{c_{23}e_{33} - e_{32}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right),$$

$$l_{71} = \frac{1}{\gamma_\alpha R_\alpha} + \frac{1}{\gamma_\beta R_\beta}, \quad l_{81} = -\frac{c_{33}}{e_{33}^2 + c_{33}\eta_{33}},$$

$$Q_{ij} = c_{ij} - \left(\frac{e_{33}e_{3j} + c_{j3}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)c_{i3} - \left(\frac{e_{33}c_{j3} - c_{33}e_{3j}}{e_{33}^2 + c_{33}\eta_{33}}\right)e_{3i} \quad (i,j = 1,2,6),$$

$$Q_{ij} \neq Q_{ji}.$$

After the elimination process, we obtain the resulting equations (9)–(14) where all the differential operators on the left-hand sides are with respect to $\zeta$ only, whereas those on the right-hand sides are with respect to $\alpha$ and $\beta$.

The in-surface stresses and electric displacements are dependent field variables that can be expressed in terms of the primary variables in the following form:

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \tau_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} + \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} u_\zeta + \begin{bmatrix} b_{14} \\ b_{24} \\ b_{34} \end{bmatrix}\sigma_\zeta + \begin{bmatrix} b_{15} \\ b_{25} \\ b_{35} \end{bmatrix} D_\zeta, \tag{15}$$

$$\begin{Bmatrix} D_\alpha \\ D_\beta \end{Bmatrix} = \begin{bmatrix} b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix}\begin{Bmatrix} \tau_{\alpha\zeta} \\ \tau_{\beta\zeta} \end{Bmatrix} + \begin{bmatrix} b_{43} \\ b_{53} \end{bmatrix}\Phi, \tag{16}$$

where

$$b_{11} = \frac{Q_{11}}{\gamma_\alpha}\frac{\partial}{\partial\alpha} + \frac{Q_{16}}{\gamma_\beta}\frac{\partial}{\partial\beta}, \quad b_{12} = \frac{Q_{16}}{\gamma_\alpha}\frac{\partial}{\partial\alpha} + \frac{Q_{12}}{\gamma_\beta}\frac{\partial}{\partial\beta}, \quad b_{21} = \frac{Q_{21}}{\gamma_\alpha}\frac{\partial}{\partial\alpha} + \frac{Q_{26}}{\gamma_\beta}\frac{\partial}{\partial\beta},$$

$$b_{22} = \frac{Q_{26}}{\gamma_\alpha}\frac{\partial}{\partial\alpha} + \frac{Q_{22}}{\gamma_\beta}\frac{\partial}{\partial\beta}, \quad b_{31} = \frac{Q_{61}}{\gamma_\alpha}\frac{\partial}{\partial\alpha} + \frac{Q_{66}}{\gamma_\beta}\frac{\partial}{\partial\beta}, \quad b_{32} = \frac{Q_{66}}{\gamma_\alpha}\frac{\partial}{\partial\alpha} + \frac{Q_{62}}{\gamma_\beta}\frac{\partial}{\partial\beta},$$

$$b_{13} = \frac{Q_{11}}{\gamma_\alpha R_\alpha} + \frac{Q_{12}}{\gamma_\beta R_\beta}, \quad b_{23} = \frac{Q_{21}}{\gamma_\alpha R_\alpha} + \frac{Q_{22}}{\gamma_\beta R_\beta}, \quad b_{33} = \frac{Q_{61}}{\gamma_\alpha R_\alpha} + \frac{Q_{62}}{\gamma_\beta R_\beta},$$

$$b_{14} = \frac{e_{31}e_{33} + c_{13}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}, \quad b_{24} = \frac{e_{32}e_{33} + c_{23}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}, \quad b_{34} = \frac{e_{36}e_{33} + c_{36}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}},$$

$$b_{15} = \frac{c_{13}e_{33} - e_{31}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}, \quad b_{25} = \frac{c_{23}e_{33} - e_{32}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}, \quad b_{35} = \frac{c_{36}e_{33} - e_{36}c_{33}}{e_{33}^2 + c_{33}\eta_{33}},$$

$$b_{41} = \frac{c_{44}e_{15} - c_{45}e_{14}}{c_{44}c_{55} - c_{45}^2}, \quad b_{42} = \frac{c_{55}e_{14} - c_{45}e_{15}}{c_{44}c_{55} - c_{45}^2},$$

$$b_{51} = \frac{c_{44}e_{25} - c_{45}e_{24}}{c_{44}c_{55} - c_{45}^2}, \quad b_{52} = \frac{c_{55}e_{24} - c_{45}e_{25}}{c_{44}c_{55} - c_{45}^2},$$

$$b_{43} = \left[\left(\frac{c_{45}e_{15} - c_{55}e_{14}}{c_{44}c_{55} - c_{45}^2}\right)e_{14} + \left(\frac{c_{45}e_{14} - c_{44}e_{15}}{c_{44}c_{55} - c_{45}^2}\right)e_{15} - \eta_{11}\right]\frac{1}{\gamma_\alpha}\frac{\partial}{\partial\alpha}$$
$$+ \left[\left(\frac{c_{45}e_{25} - c_{55}e_{24}}{c_{44}c_{55} - c_{45}^2}\right)e_{14} + \left(\frac{c_{45}e_{24} - c_{44}e_{25}}{c_{44}c_{55} - c_{45}^2}\right)e_{15} - \eta_{12}\right]\frac{1}{\gamma_\beta}\frac{\partial}{\partial\beta},$$

$$b_{53} = \left[\left(\frac{c_{45}e_{15} - c_{55}e_{14}}{c_{44}c_{55} - c_{45}^2}\right)e_{24} + \left(\frac{c_{45}e_{14} - c_{44}e_{15}}{c_{44}c_{55} - c_{45}^2}\right)e_{25} - \eta_{12}\right]\frac{1}{\gamma_\alpha}\frac{\partial}{\partial\alpha}$$
$$+ \left[\left(\frac{c_{45}e_{25} - c_{55}e_{24}}{c_{44}c_{55} - c_{45}^2}\right)e_{24} + \left(\frac{c_{45}e_{24} - c_{44}e_{25}}{c_{44}c_{55} - c_{45}^2}\right)e_{25} - \eta_{22}\right]\frac{1}{\gamma_\beta}\frac{\partial}{\partial\beta}.$$

The boundary conditions of the problem are specified as follows:

On the lateral surface the transverse load $\bar{q}_\zeta^\pm(\alpha, \beta, t)$ and electric potential $\bar{\Phi}_\zeta^\pm(\alpha, \beta, t)$ are prescribed,

$$[\tau_{\alpha\zeta} \quad \tau_{\beta\zeta}] = [0 \quad 0] \quad \text{on} \quad \zeta = \pm h, \tag{17.1}$$

$$\sigma_\zeta = \bar{q}_\zeta^\pm(\alpha, \beta, t) \qquad \text{on} \quad \zeta = \pm h, \tag{17.2}$$

$$\Phi = \bar{\Phi}_\zeta^\pm(\alpha, \beta, t) \qquad \text{on} \quad \zeta = \pm h. \tag{18}$$

The edge boundary conditions require one member of each pair of the following quantities to be satisfied:

$$n_1\sigma_\alpha + n_2\tau_{\alpha\beta} = p_1, \quad \text{or} \quad u_\alpha = \bar{u}_\alpha; \tag{19.1}$$

$$n_1\tau_{\alpha\beta} + n_2\sigma_\beta = p_2, \quad \text{or} \quad u_\beta = \bar{u}_\beta; \tag{19.2}$$

$$n_1\tau_{\alpha\zeta} + n_2\tau_{\beta\zeta} = p_3, \quad \text{or} \quad u_\zeta = \bar{u}_\zeta; \tag{19.3}$$

where $p_1$, $p_2$, $p_3$ are applied edge loads; $\bar{u}_\alpha$, $\bar{u}_\beta$ and $\bar{u}_\zeta$ are the prescribed edge displacements; $n_1$ and $n_2$ denote the outward unit normal at a point along the edge.

In addition, the edges are suitably grounded so that the electric potential $\Phi$ at the edges is zero and given by

$$\Phi = 0. \tag{20}$$

## 3 Nondimensionalization

A set of dimensionless coordinates and variables is defined as

$$\begin{aligned}
&x = \alpha/\sqrt{Rh}, \quad && y = \beta/\sqrt{Rh}, \quad && z = \zeta/h; \\
&u = u_\alpha/\sqrt{Rh}, && v = u_\beta/\sqrt{Rh}, && w = u_\zeta/R; \\
&R_x = R_\alpha/R, && R_y = R_\beta/R; && \\
&\sigma_x = \sigma_\alpha/Q, && \sigma_y = \sigma_\beta/Q, && \tau_{xy} = \tau_{\alpha\beta}/Q; \\
&\tau_{xz} = \tau_{\alpha\zeta}/Q\varepsilon, && \tau_{yz} = \tau_{\beta\zeta}/Q\varepsilon, && \sigma_z = \sigma_\zeta/Q\varepsilon^2; \\
&D_x = D_\alpha/e\varepsilon, && D_y = D_\beta/e\varepsilon, && D_z = D_\zeta/e; \\
&\phi = \Phi e/RQ\varepsilon^2; && &&
\end{aligned} \tag{21}$$

where $\varepsilon^2 = h/R$; $R$, $Q$ and $e$ denote a characteristic length of the shell, a reference elastic modulus and a reference piezoelectric modulus, respectively.

The method of multiple scales is used in the present formulation to eliminate the secular terms raised in a regular asymptotic approach. The dimensionless multiple time scales are defined by

$$\tau_k = \frac{\varepsilon^{2k}}{R}\sqrt{\frac{Q}{\rho_0}}t \quad (k = 0, 1, 2, \ldots), \tag{22}$$

where $\rho_0$ denotes a reference mass density.

After introducing the set of dimensionless coordinates and variables (21), (22) into (9)–(16), we obtain the following set of dimensionless equations:

$$w_{,z} = -\varepsilon^2 \mathbf{L}_1\ \mathbf{u} - \varepsilon^2\ \tilde{l}_{33} w + \varepsilon^4 \tilde{l}_{34} \sigma_z + \varepsilon^2 \tilde{l}_{35}\ D_z, \tag{23}$$

$$\mathbf{u}_{,z} = -\mathbf{D}\ w + \varepsilon^2 \mathbf{L}_2\ \mathbf{u} + \varepsilon^2 \mathbf{S}\ \boldsymbol{\sigma}_s + \varepsilon^4 \mathbf{L}_3\ \boldsymbol{\sigma}_s + \varepsilon^2 \mathbf{L}_4\ \phi, \tag{24}$$

$$D_{z,z} = -\varepsilon^2\ \mathbf{L}_{11}\ \mathbf{d} - \varepsilon^2\ \tilde{l}_{71} D_z, \tag{25}$$

$$\begin{aligned}\boldsymbol{\sigma}_{s,z} = {} & -\mathbf{L}_5\ \mathbf{u} - \mathbf{L}_6\ w - \varepsilon^2\ \mathbf{L}_7\ \boldsymbol{\sigma}_s - \varepsilon^2 (\gamma_\alpha \gamma_\beta) \mathbf{L}_1^T\ \sigma_z - \mathbf{L}_8\ D_z \\ & + \rho_1 \left[\frac{\partial^2}{\partial \tau_0^2} + 2\varepsilon^2 \frac{\partial^2}{\partial \tau_0 \partial \tau_1} + \varepsilon^4 \left(2 \frac{\partial^2}{\partial \tau_0 \partial \tau_2} + \frac{\partial^2}{\partial \tau_1^2}\right) + \ldots\right],\end{aligned} \tag{26}$$

$$\begin{aligned}\sigma_{z,z} = {} & \mathbf{L}_9\ \mathbf{u} + \tilde{l}_{63}\ w - \mathbf{D}^T\ \boldsymbol{\sigma}_s - \varepsilon^2\ \mathbf{L}_{10}\ \boldsymbol{\sigma}_s - \varepsilon^2\ \tilde{l}_{64}\ \sigma_z + \tilde{l}_{65}\ D_z \\ & + \rho_2 \left[\frac{\partial^2}{\partial \tau_0^2} + 2\varepsilon^2 \frac{\partial^2}{\partial \tau_0 \partial \tau_1} + \varepsilon^4 \left(2 \frac{\partial^2}{\partial \tau_0 \partial \tau_2} + \frac{\partial^2}{\partial \tau_1^2}\right) + \ldots\right],\end{aligned} \tag{27}$$

$$\phi_{,z} = -(\gamma_\alpha \gamma_\beta) \mathbf{L}_8^T\ \mathbf{u} - (1/\gamma_\alpha \gamma_\beta) \tilde{l}_{65}\ w + \varepsilon^2\ \tilde{l}_{35}\ \sigma_z + \tilde{l}_{81}\ D_z, \tag{28}$$

where

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \mathbf{D} = \begin{Bmatrix} \partial_x \\ \partial_y \end{Bmatrix}, \quad \mathbf{S} = Q \begin{bmatrix} l_{14} & l_{15} \\ l_{15} & l_{25} \end{bmatrix}, \quad \sigma_s = \begin{Bmatrix} \tau_{x\,z} \\ \tau_{y\,z} \end{Bmatrix}, \quad \mathbf{d} = \begin{Bmatrix} D_x \\ D_y \end{Bmatrix},$$

$$\mathbf{L}_1 = \begin{bmatrix} \tilde{l}_{31} & \tilde{l}_{32} \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} \tilde{l}_{11} & 0 \\ 0 & \tilde{l}_{22} \end{bmatrix}, \quad \mathbf{L}_3 = \begin{bmatrix} \tilde{l}_{16} & \tilde{l}_{17} \\ \tilde{l}_{26} & \tilde{l}_{27} \end{bmatrix}, \quad \mathbf{L}_4 = \begin{bmatrix} \tilde{l}_{18} \\ \tilde{l}_{28} \end{bmatrix},$$

$$\mathbf{L}_5 = \begin{bmatrix} \tilde{l}_{41} & \tilde{l}_{42} \\ \tilde{l}_{51} & \tilde{l}_{52} \end{bmatrix}, \quad \mathbf{L}_6 = \begin{bmatrix} \tilde{l}_{43} \\ \tilde{l}_{53} \end{bmatrix}, \quad \mathbf{L}_7 = \begin{bmatrix} \tilde{l}_{44} & 0 \\ 0 & \tilde{l}_{55} \end{bmatrix}, \quad \mathbf{L}_8 = \begin{bmatrix} \tilde{l}_{46} \\ \tilde{l}_{56} \end{bmatrix},$$

$$\mathbf{L}_9 = \begin{bmatrix} \tilde{l}_{61} & \tilde{l}_{62} \end{bmatrix}, \quad \mathbf{L}_{10} = \begin{bmatrix} \frac{z}{R_y} \partial_x & \frac{z}{R_x} \partial_y \end{bmatrix}, \quad \mathbf{L}_{11} = \begin{bmatrix} \frac{1}{\gamma_\alpha} \partial_x & \frac{1}{\gamma_\beta} \partial_y \end{bmatrix},$$

$$\tilde{l}_{31} = \left(\frac{e_{31} e_{33} + c_{13} \eta_{33}}{e_{33}^2 + c_{33} \eta_{33}}\right) \frac{1}{\gamma_\alpha} \frac{\partial}{\partial x} + \left(\frac{e_{33} e_{36} + c_{36} \eta_{33}}{e_{33}^2 + c_{33} \eta_{33}}\right) \frac{1}{\gamma_\beta} \frac{\partial}{\partial y},$$

$$\tilde{l}_{32} = \left(\frac{e_{36} e_{33} + c_{36} \eta_{33}}{e_{33}^2 + c_{33} \eta_{33}}\right) \frac{1}{\gamma_\alpha} \frac{\partial}{\partial x} + \left(\frac{e_{33} e_{32} + c_{23} \eta_{33}}{e_{33}^2 + c_{33} \eta_{33}}\right) \frac{1}{\gamma_\beta} \frac{\partial}{\partial y},$$

$$\tilde{l}_{33} = \left(\frac{e_{31} e_{33} + c_{13} \eta_{33}}{e_{33}^2 + c_{33} \eta_{33}}\right) \frac{1}{\gamma_\alpha R_x} + \left(\frac{e_{33} e_{32} + c_{23} \eta_{33}}{e_{33}^2 + c_{33} \eta_{33}}\right) \frac{1}{\gamma_\beta R_y},$$

$$\tilde{l}_{34} = l_{34}Q, \quad \tilde{l}_{35} = l_{35}e, \quad \tilde{l}_{11} = (1 - z\partial_z)/R_x, \quad \tilde{l}_{22} = (1 - z\partial_z)/R_y,$$

$$\tilde{l}_{16} = l_{14}Qz/R_x, \quad \tilde{l}_{17} = l_{15}Qz/R_x, \quad \tilde{l}_{26} = l_{15}Qz/R_y, \quad \tilde{l}_{27} = l_{25}Qz/R_y,$$

$$\tilde{l}_{18} = \left(\frac{c_{45}e_{14} - c_{44}e_{15}}{c_{44}c_{55} - c_{45}^2}\right)\frac{Q}{e}\frac{\partial}{\partial x} + \left(\frac{c_{45}e_{24} - c_{44}e_{25}}{c_{44}c_{55} - c_{45}^2}\right)\frac{Q\gamma_\alpha}{e\gamma_\beta}\frac{\partial}{\partial y},$$

$$\tilde{l}_{28} = \left(\frac{c_{45}e_{15} - c_{55}e_{14}}{c_{44}c_{55} - c_{45}^2}\right)\frac{Q\ \gamma_\beta}{e\gamma_\alpha}\frac{\partial}{\partial\ x} + \left(\frac{c_{45}e_{25} - c_{55}e_{24}}{c_{44}c_{55} - c_{45}^2}\right)\frac{Q}{e}\frac{\partial}{\partial\ y},$$

$$\tilde{l}_{41} = \frac{\tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha}\frac{\partial^2}{\partial x^2} + (\tilde{Q}_{16} + \tilde{Q}_{61})\frac{\partial^2}{\partial x\partial y} + \frac{\tilde{Q}_{66}\gamma_\alpha}{\gamma_\beta}\frac{\partial^2}{\partial\ y^2},$$

$$\tilde{l}_{42} = \frac{\tilde{Q}_{16}\gamma_\beta}{\gamma_\alpha}\frac{\partial^2}{\partial x^2} + (\tilde{Q}_{12} + \tilde{Q}_{66})\frac{\partial^2}{\partial x\partial y} + \frac{\tilde{Q}_{62}\ \gamma_\alpha}{\gamma_\beta}\frac{\partial^2}{\partial\ y^2},$$

$$\tilde{l}_{51} = \frac{\tilde{Q}_{61}\gamma_\beta}{\gamma_\alpha}\frac{\partial^2}{\partial x^2} + (\tilde{Q}_{21} + \tilde{Q}_{66})\frac{\partial^2}{\partial x\partial y} + \frac{\tilde{Q}_{26}\ \gamma_\alpha}{\gamma_\beta}\frac{\partial^2}{\partial\ y^2},$$

$$\tilde{l}_{52} = \frac{\tilde{Q}_{66}\gamma_\beta}{\gamma_\alpha}\frac{\partial^2}{\partial x^2} + (\tilde{Q}_{26} + \tilde{Q}_{62})\frac{\partial^2}{\partial x\partial y} + \frac{\tilde{Q}_{22}\ \gamma_\alpha}{\gamma_\beta}\frac{\partial^2}{\partial\ y^2},$$

$$\tilde{l}_{43} = \left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y}\right)\frac{\partial}{\partial x} + \left(\frac{\tilde{Q}_{61}}{R_x} + \frac{\tilde{Q}_{62}\gamma_\alpha}{R_y\gamma_\beta}\right)\frac{\partial}{\partial y},$$

$$\tilde{l}_{53} = \left(\frac{\tilde{Q}_{61}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{62}}{R_y}\right)\frac{\partial}{\partial\ x} + \left(\frac{\tilde{Q}_{21}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta}\right)\frac{\partial}{\partial y},$$

$$\tilde{l}_{44} = \frac{2}{R_x} + \frac{1}{R_y} + \frac{3hz}{R_xR_yR} + \left(\frac{z}{R_x} + \frac{z}{R_y}\right)\frac{\partial}{\partial z} + \left(\frac{hz^2}{R_xR_yR}\right)\frac{\partial}{\partial z},$$

$$\tilde{l}_{55} = \frac{1}{R_x} + \frac{2}{R_y} + \frac{3hz}{R_xR_yR} + \left(\frac{z}{R_x} + \frac{z}{R_y}\right)\frac{\partial}{\partial z} + \left(\frac{hz^2}{R_xR_yR}\right)\frac{\partial}{\partial z},$$

$$\tilde{l}_{46} = \left(\frac{c_{13}e_{33} - e_{31}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{\gamma_\beta e}{Q}\frac{\partial}{\partial x} + \left(\frac{c_{36}e_{33} - e_{36}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{\gamma_\alpha e}{Q}\frac{\partial}{\partial y},$$

$$\tilde{l}_{56} = \left(\frac{c_{36}e_{33} - e_{36}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{\gamma_\beta e}{Q}\frac{\partial}{\partial\ x} + \left(\frac{c_{23}e_{33} - e_{32}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{\gamma_\alpha e}{Q}\frac{\partial}{\partial y},$$

$$\tilde{l}_{61} = \left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{21}}{R_y}\right)\frac{\partial}{\partial\ x} + \left(\frac{\tilde{Q}_{16}}{R_x} + \frac{\tilde{Q}_{26}\gamma_\alpha}{R_y\gamma_\beta}\right)\frac{\partial}{\partial\ y},$$

$$\tilde{l}_{62} = \left(\frac{\tilde{Q}_{16}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{26}}{R_y}\right)\frac{\partial}{\partial\ x} + \left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta}\right)\frac{\partial}{\partial\ y},$$

$$\tilde{l}_{63} = \frac{\tilde{Q}_{11}\gamma_\beta}{R_x^2\gamma_\alpha} + \frac{(\tilde{Q}_{12} + \tilde{Q}_{21})}{R_xR_y} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y^2\gamma_\beta},$$

$$\tilde{l}_{64} = \left(\frac{1}{R_x} + \frac{1}{R_y} + \frac{2hz}{R_x R_y R}\right) - \frac{\gamma_\beta}{R_x}\left(\frac{e_{31}e_{33} + c_{13}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right) - \frac{\gamma_\alpha}{R_y}\left(\frac{e_{32}e_{33} + c_{23}\eta_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)$$

$$+ \left(\frac{z}{R_x} + \frac{z}{R_y}\right)\frac{\partial}{\partial z} + \frac{h}{R}\left(\frac{z^2}{R_x R_y}\right)\frac{\partial}{\partial z},$$

$$\tilde{l}_{65} = \frac{\gamma_\beta e}{R_x Q}\left(\frac{c_{13}e_{33} - e_{31}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right) + \frac{\gamma_\alpha e}{R_y Q}\left(\frac{c_{23}e_{33} - e_{32}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right),$$

$$\tilde{l}_{71} = \frac{1}{\gamma_\alpha R_x} + \frac{1}{\gamma_\beta R_y}, \quad \tilde{l}_{81} = -\left(\frac{c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\frac{e^2}{Q},$$

$$\tilde{Q}_{ij} = \frac{Q_{ij}}{Q}, \quad \rho_1 = \gamma_\alpha\gamma_\beta \frac{h}{R}\frac{\rho}{\rho_0}, \quad \rho_2 = \rho_1/\varepsilon^2.$$

Following the similar derivation process, we rewrite the in-surface stresses and electric displacements in the dimensionless form as

$$\boldsymbol{\sigma}_p = \mathbf{B}_1\mathbf{u} + \mathbf{B}_2 w + \varepsilon^2\mathbf{B}_3\boldsymbol{\sigma}_z + \mathbf{B}_4 D_z, \tag{29}$$

$$\mathbf{d} = \mathbf{B}_5\boldsymbol{\sigma}_s + \mathbf{B}_6\phi, \tag{30}$$

where

$$\boldsymbol{\sigma}_p = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ \tilde{b}_{21} & \tilde{b}_{22} \\ \tilde{b}_{31} & \tilde{b}_{32} \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} \tilde{b}_{13} \\ \tilde{b}_{23} \\ \tilde{b}_{33} \end{bmatrix}, \quad \mathbf{B}_3 = \begin{bmatrix} \tilde{b}_{14} \\ \tilde{b}_{24} \\ \tilde{b}_{34} \end{bmatrix},$$

$$\mathbf{B}_4 = \begin{bmatrix} \tilde{b}_{15} \\ \tilde{b}_{25} \\ \tilde{b}_{35} \end{bmatrix}, \quad \mathbf{B}_5 = \begin{bmatrix} \tilde{b}_{41} & \tilde{b}_{42} \\ \tilde{b}_{51} & \tilde{b}_{52} \end{bmatrix}, \quad \mathbf{B}_6 = \begin{bmatrix} \tilde{b}_{43} \\ \tilde{b}_{53} \end{bmatrix},$$

$$\tilde{b}_{11} = \frac{\tilde{Q}_{11}}{\gamma_\alpha}\frac{\partial}{\partial x} + \frac{\tilde{Q}_{16}}{\gamma_\beta}\frac{\partial}{\partial y}, \quad \tilde{b}_{12} = \frac{\tilde{Q}_{16}}{\gamma_\alpha}\frac{\partial}{\partial x} + \frac{\tilde{Q}_{12}}{\gamma_\beta}\frac{\partial}{\partial y},$$

$$\tilde{b}_{21} = \frac{\tilde{Q}_{21}}{\gamma_\alpha}\frac{\partial}{\partial x} + \frac{\tilde{Q}_{26}}{\gamma_\beta}\frac{\partial}{\partial y}, \quad \tilde{b}_{22} = \frac{\tilde{Q}_{26}}{\gamma_\alpha}\frac{\partial}{\partial x} + \frac{\tilde{Q}_{22}}{\gamma_\beta}\frac{\partial}{\partial y},$$

$$\tilde{b}_{31} = \frac{\tilde{Q}_{61}}{\gamma_\alpha}\frac{\partial}{\partial x} + \frac{\tilde{Q}_{66}}{\gamma_\beta}\frac{\partial}{\partial y}, \quad \tilde{b}_{32} = \frac{\tilde{Q}_{66}}{\gamma_\alpha}\frac{\partial}{\partial x} + \frac{\tilde{Q}_{62}}{\gamma_\beta}\frac{\partial}{\partial y},$$

$$\tilde{b}_{13} = \frac{\tilde{Q}_{11}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{12}}{\gamma_\beta R_y}, \quad \tilde{b}_{23} = \frac{\tilde{Q}_{21}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{22}}{\gamma_\beta R_y}, \quad \tilde{b}_{33} = \frac{\tilde{Q}_{61}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{62}}{\gamma_\beta R_y},$$

$$\tilde{b}_{14} = b_{14}, \quad \tilde{b}_{24} = b_{24}, \quad \tilde{b}_{34} = b_{34},$$

$$\tilde{b}_{15} = (e/Q)b_{15}, \qquad \tilde{b}_{25} = (e/Q)b_{25}, \qquad \tilde{b}_{35} = (e/Q)b_{35},$$

$$\tilde{b}_{41} = (e/Q)b_{41}, \qquad \tilde{b}_{42} = (e/Q)b_{42}, \qquad \tilde{b}_{51} = (e/Q)b_{51}, \qquad \tilde{b}_{52} = (e/Q)b_{52},$$

$$\tilde{b}_{43} = \left[\left(\frac{c_{45}e_{15} - c_{55}e_{14}}{c_{44}c_{55} - c_{45}^2}\right)e_{14} + \left(\frac{c_{45}e_{14} - c_{44}e_{15}}{c_{44}c_{55} - c_{45}^2}\right)e_{15} - \eta_{11}\right]\frac{Q}{e^2\gamma_\alpha}\frac{\partial}{\partial x}$$

$$+ \left[\left(\frac{c_{45}e_{25} - c_{55}e_{24}}{c_{44}c_{55} - c_{45}^2}\right)e_{14} + \left(\frac{c_{45}e_{24} - c_{44}e_{25}}{c_{44}c_{55} - c_{45}^2}\right)e_{15} - \eta_{12}\right]\frac{Q}{e^2\gamma_\beta}\frac{\partial}{\partial y},$$

$$\tilde{b}_{53} = \left[\left(\frac{c_{45}e_{15} - c_{55}e_{14}}{c_{44}c_{55} - c_{45}^2}\right)e_{24} + \left(\frac{c_{45}e_{14} - c_{44}e_{15}}{c_{44}c_{55} - c_{45}^2}\right)e_{25} - \eta_{12}\right]\frac{Q}{e^2\gamma_\alpha}\frac{\partial}{\partial x}$$

$$+ \left[\left(\frac{c_{45}e_{25} - c_{55}e_{24}}{c_{44}c_{55} - c_{45}^2}\right)e_{24} + \left(\frac{c_{45}e_{24} - c_{44}e_{25}}{c_{44}c_{55} - c_{45}^2}\right)e_{25} - \eta_{22}\right]\frac{Q}{e^2\gamma_\beta}\frac{\partial}{\partial y}.$$

The dimensionless forms of boundary conditions of the problem are specified as follows:

On the lateral surface the transverse load and electric potential are prescribed,

$$[\tau_{xz} \quad \tau_{yz}] = [0 \quad 0] \quad \text{on} \quad z = \pm 1, \tag{31.1}$$

$$\sigma_z = \bar{q}_z^{\pm}(x, y, t) \qquad \text{on} \quad z = \pm 1, \tag{31.2}$$

$$\phi = \bar{\phi}_z^{\pm}(x, y, t) \qquad \text{on} \quad z = \pm 1. \tag{32}$$

At the edges one member of each pair of the following quantities is satisfied:

$$n_1\sigma_x + n_2\tau_{xy} = p_{nx}, \quad \text{or} \quad u = \bar{u}; \tag{33.1}$$

$$n_1\tau_{xy} + n_2\sigma_y = p_{ny}, \quad \text{or} \quad v = \bar{v}; \tag{33.2}$$

$$n_1\tau_{xz} + n_2\tau_{yz} = p_{nz}, \quad \text{or} \quad w = \bar{w}; \tag{33.3}$$

In addition,

$$\phi = 0; \tag{34}$$

where

$$\bar{q}_z^{\pm} = \bar{q}_\zeta^{\pm}/Q\varepsilon^2, \quad \phi_z^{\pm} = \bar{\Phi}_\zeta^{\pm}e/RQ\varepsilon^2, \quad (p_{nx}, p_{ny}, p_{nz}) = (p_1/Q,\, p_2/Q,\, p_3/Q\varepsilon),$$

$$(\bar{u},\, \bar{v},\, \bar{w}) = \left(\bar{u}_\alpha/\sqrt{Rh},\, \bar{u}_\beta/\sqrt{Rh},\, \bar{u}_\zeta/R\right).$$

## 4 Asymptotic expansion

By observation of Eqs. (23)–(28), it is found that those equations contain terms involving only even powers of $\varepsilon$. Hence, we asymptotically expand the primary variables in the powers $\varepsilon^2$ as given by

$$f = f^{(0)}(x, y, z, \tau_0, \tau_1, \ldots) + \varepsilon^2 f^{(1)}(x, y, z, \tau_0, \tau_1, \ldots) + \varepsilon^4 f^{(2)}(x, y, z, \tau_0, \tau_1, \ldots) + \ldots . \tag{35}$$

Substituting Eq. (35) into Eqs. (23)–(28) and collecting coefficients of equal powers of $\varepsilon$, we obtain the following sets of recurrence equations:

Order $\varepsilon^0$:

$$w^{(0)}{}_{,z} = 0, \tag{36}$$

$$\mathbf{u}^{(0)}{}_{,z} = -\mathbf{D}w^{(0)}, \tag{37}$$

$$D_z^{(0)},_z = 0, \tag{38}$$

$$\sigma_s^{(0)},_z = -\mathbf{L}_5\mathbf{u}^{(0)} - \mathbf{L}_6 w^{(0)} - \mathbf{L}_8 D_z^{(0)} + \rho_1 \frac{\partial^2}{\partial \tau_0^2}\mathbf{u}^{(0)}, \tag{39}$$

$$\sigma_z^{(0)},_z = \mathbf{L}_9\mathbf{u}^{(0)} + \tilde{l}_{63} w^{(0)} - \mathbf{D}^T \sigma_s^{(0)} + \tilde{l}_{65} D_z^{(0)} + \rho_2 \frac{\partial^2}{\partial \tau_0^2} w^{(0)}, \tag{40}$$

$$\phi^{(0)},_z = -(\gamma_\alpha\gamma_\beta)\mathbf{L}_8^T\mathbf{u}^{(0)} - (1/\gamma_\alpha\gamma_\beta)\tilde{l}_{65} w^{(0)} + \tilde{l}_{81} D_z^{(0)}, \tag{41}$$

$$\sigma_p^{(0)} = \mathbf{B}_1\mathbf{u}^{(0)} + \mathbf{B}_2 w^{(0)} + \mathbf{B}_4 D_z^{(0)}, \tag{42}$$

$$\mathbf{d}^{(0)} = \mathbf{B}_5\sigma_s^{(0)} + \mathbf{B}_6\phi^{(0)}. \tag{43}$$

Order $\varepsilon^2$:

$$w^{(1)},_z = -\mathbf{L}_1\mathbf{u}^{(0)} - \tilde{l}_{33} w^{(0)} + \tilde{l}_{35} D_z^{(0)}, \tag{44}$$

$$\mathbf{u}^{(1)},_z = -\mathbf{D} w^{(1)} + \mathbf{L}_2\mathbf{u}^{(0)} + \mathbf{S}\sigma_s^{(0)} + \mathbf{L}_4\phi^{(0)}, \tag{45}$$

$$\boldsymbol{\sigma}_s^{(1)},_z = -\mathbf{L}_5\mathbf{u}^{(1)} - \mathbf{L}_6 w^{(1)} - \mathbf{L}_7\sigma_s^{(0)} - (\gamma_\alpha\gamma_\beta)\mathbf{L}_1^T\sigma_z^{(0)} - \mathbf{L}_8 D_z^{(1)} + \rho_1\frac{\partial^2}{\partial\tau_0^2}\mathbf{u}^{(1)} + 2\rho_1\frac{\partial^2}{\partial\tau_0\partial\tau_1}\mathbf{u}^{(0)}, \tag{46}$$

$$\sigma_z^{(1)},_z = \mathbf{L}_9\mathbf{u}^{(1)} + \tilde{l}_{63} w^{(1)} - \mathbf{D}^T\boldsymbol{\sigma}_s^{(1)} - \mathbf{L}_{10}\boldsymbol{\sigma}_s^{(0)} - \tilde{l}_{64}\sigma_z^{(0)} + \tilde{l}_{65} D_z^{(1)} + \rho_2\frac{\partial^2}{\partial\tau_0^2} w^{(1)} + 2\rho_2\frac{\partial^2}{\partial\tau_0\partial\tau_1} w^{(0)}, \tag{47}$$

$$D_z^{(1)},_z = -\mathbf{L}_{11}\mathbf{d}^{(0)} - \tilde{l}_{71} D_z^{(0)}, \tag{48}$$

$$\phi^{(1)},_z = -(\gamma_\alpha\gamma_\beta)\mathbf{L}_8^T\mathbf{u}^{(1)} - (1/\gamma_\alpha\gamma_\beta)\tilde{l}_{65} w^{(1)} + \tilde{l}_{35}\sigma_z^{(0)} + \tilde{l}_{81} D_z^{(1)}, \tag{49}$$

$$\sigma_p^{(1)} = \mathbf{B}_1\mathbf{u}^{(1)} + \mathbf{B}_2 w^{(1)} + \mathbf{B}_3\sigma_z^{(0)} + \mathbf{B}_4 D_z^{(1)}, \tag{50}$$

$$\mathbf{d}^{(1)} = \mathbf{B}_5\sigma_s^{(1)} + \mathbf{B}_6\phi^{(1)}. \tag{51}$$

Order $\varepsilon^{2k}$ $(k = 2, 3, \ldots)$:

$$w^{(k)},_z = -\mathbf{L}_1\mathbf{u}^{(k-1)} - \tilde{l}_{33} w^{(k-1)} + \tilde{l}_{34}\sigma_z^{(k-2)} + \tilde{l}_{35} D_z^{(k-1)}, \tag{52}$$

$$\mathbf{u}^{(k)},_z = -\mathbf{D} w^{(k)} + \mathbf{L}_2\mathbf{u}^{(k-1)} + \mathbf{S}\boldsymbol{\sigma}_s^{(k-1)} + \mathbf{L}_3\boldsymbol{\sigma}_s^{(k-2)} + \mathbf{L}_4\phi^{(k-1)}, \tag{53}$$

$$\begin{aligned}\boldsymbol{\sigma}_s^{(k)},_z = {}& -\mathbf{L}_5\mathbf{u}^{(k)} - \mathbf{L}_6 w^{(k)} - \mathbf{L}_7\boldsymbol{\sigma}_s^{(k-1)} - (\gamma_\alpha\gamma_\beta)\mathbf{L}_1^T\sigma_z^{(k-1)} - \mathbf{L}_8 D_z^{(k)} \\ & + \left[\rho_1\frac{\partial^2}{\partial\tau_0^2}\mathbf{u}^{(k)} + 2\rho_1\frac{\partial^2}{\partial\tau_0\partial\tau_1}\mathbf{u}^{(k-1)} + \ldots + \rho_1\left(\frac{\partial^2}{\partial\tau_0\partial\tau_k} + \frac{\partial^2}{\partial\tau_1\partial\tau_{k-1}} + \ldots + \frac{\partial^2}{\partial\tau_k\partial\tau_0}\right)\mathbf{u}^{(0)}\right],\end{aligned} \tag{54}$$

$$\sigma_z^{(k)}{}_{,z} = \mathbf{L}_9\mathbf{u}^{(k)} + \tilde{l}_{63}\,w^{(k)} - \mathbf{D}^T\boldsymbol{\sigma}_s^{(k)} - \mathbf{L}_{10}\boldsymbol{\sigma}_s^{(k-1)} - \tilde{l}_{64}\sigma_z^{(k-1)} + \tilde{l}_{65}D_z^{(k)}$$
$$+ \left[\rho_2 \cdot \frac{\partial^2}{\partial\tau_0^2}w^{(k)} + 2\rho_2\frac{\partial^2}{\partial\tau_0\partial\tau_1}w^{(k-1)} + \ldots + \rho_2\left(\frac{\partial^2}{\partial\tau_0\partial\tau_k} + \frac{\partial^2}{\partial\tau_1\partial\tau_{k-1}} + \ldots + \frac{\partial^2}{\partial\tau_k\partial\tau_0}\right)w^{(0)}\right], \tag{55}$$

$$D_z^{(k)}{}_{,z} = -\mathbf{L}_{11}\mathbf{d}^{(k-1)} - \tilde{l}_{71}D_z^{(k-1)}, \tag{56}$$

$$\phi^{(k)}{}_{,z} = -(\gamma_\alpha\gamma_\beta)\mathbf{L}_8^T\mathbf{u}^{(k)} - (1/\gamma_\alpha\gamma_\beta)\tilde{l}_{65}w^{(k)} + \tilde{l}_{35}\sigma_z^{(k-1)} + \tilde{l}_{81}D_z^{(k)}, \tag{57}$$

$$\boldsymbol{\sigma}_p^{(k)} = \mathbf{B}_1\mathbf{u}^{(k)} + \mathbf{B}_2w^{(k)} + \mathbf{B}_3\sigma_z^{(k-1)} + \mathbf{B}_4D_z^{(k)}, \tag{58}$$

$$\mathbf{d}^{(k)} = \mathbf{B}_5\boldsymbol{\sigma}_s^{(k)} + \mathbf{B}_6\phi^{(k)}. \tag{59}$$

The transverse loads and electric potential at the lateral surfaces are given as:

Order $\varepsilon^0$:

$$\begin{bmatrix}\tau_{xz}^{(0)} & \tau_{yz}^{(0)}\end{bmatrix} = [0 \quad 0] \quad \text{on } z = \pm 1, \tag{60.1}$$

$$\sigma_z^{(0)} = \bar{q}_z^{\pm}(x, y, t) \qquad \text{on } z = \pm 1, \tag{60.2}$$

$$\phi^{(0)} = \bar{\phi}_z^{\pm}(x, y, t) \qquad \text{on } z = \pm 1. \tag{61}$$

Order $\varepsilon^{2k}$ $(k = 1, 2, 3, \ldots)$:

$$\begin{bmatrix}\tau_{xz}^{(k)} & \tau_{yz}^{(k)} & \sigma_z^{(k)}\end{bmatrix} = [0 \quad 0 \quad 0] \quad \text{on } z = \pm 1, \tag{62}$$

$$\phi^{(k)} = 0 \qquad \text{on } z = \pm 1. \tag{63}$$

Along the edges one member of each pair of the following quantities must be satisfied:

Order $\varepsilon^0$:

$$n_1\sigma_x^{(0)} + n_2\tau_{xy}^{(0)} = p_{nx} \quad \text{or } u^{(0)} = \bar{u}, \tag{64.1}$$

$$n_1\tau_{xy}^{(0)} + n_2\sigma_y^{(0)} = p_{ny} \quad \text{or } v^{(0)} = \bar{v}, \tag{64.2}$$

$$n_1\tau_{xz}^{(0)} + n_2\tau_{yz}^{(0)} = p_{nz} \quad \text{or } w^{(0)} = \bar{w}, \tag{64.3}$$

$$\phi^{(0)} = 0. \tag{65}$$

Order $\varepsilon^{2k}(k = 1, 2, 3, \ldots)$:

$$n_1\sigma_x^{(k)} + n_2\tau_{xy}^{(k)} = 0 \quad \text{or } u^{(k)} = 0, \tag{66.1}$$

$$n_1\tau_{xy}^{(k)} + n_2\sigma_y^{(k)} = 0 \quad \text{or } v^{(k)} = 0, \tag{66.2}$$

$$n_1\tau_{xz}^{(k)} + n_2\tau_{yz}^{(k)} = 0 \quad \text{or } w^{(k)} = 0, \tag{66.3}$$

$$\phi^{(k)} = 0. \tag{67}$$

## 5 Asymptotic integration

Examination of the sets of asymptotic equations reveals that the analysis can be carried on by integrating those equations through the thickness direction. We therefore integrate Eqs. (36)–(38) to obtain

$$w^{(0)} = w^0(x, y, \tau_0, \tau_1, \ldots), \tag{68}$$

$$\mathbf{u}^{(0)} = \mathbf{u}^0 - z\mathbf{D}w^0, \tag{69}$$

$$D_z^{(0)} = D_z^0(x, y, \tau_0, \tau_1, \ldots), \tag{70}$$

where $w^0$, $\mathbf{u}^0 = [u_0(x, y, \tau_0, \tau_1, \ldots) \quad v_0(x, y, \tau_0, \tau_1, \ldots)]^T$ and $D_z^0$ represent the displacements and piezoelectric displacement on the middle surface and are also of the Kirchhoff-Love type in CST.

With the lateral boundary conditions on $z = -1$ (Eqs. (60), (61)), we then proceed to successively integrate Eqs. (39)–(41), which yields

$$\boldsymbol{\sigma}_s^{(0)} = -\int_{-1}^{z} \left[\mathbf{L}_5\left(\mathbf{u}^0 - \eta\mathbf{D}w^0\right) + \mathbf{L}_6 w^0 + \mathbf{L}_8 D_z^0\right] d\eta + \frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z} \rho_1\left(\mathbf{u}^0 - \eta\mathbf{D}w^0\right) d\eta, \tag{71}$$

$$\begin{aligned}\sigma_z^{(0)} = \bar{q}_z^- &+ \int_{-1}^{z} \left[\mathbf{L}_9\left(\mathbf{u}^0 - \eta\mathbf{D}w^0\right) + \tilde{l}_{63} w^0\right] d\eta + \int_{-1}^{z} (z-\eta)\mathbf{D}^T\left[\mathbf{L}_5\left(\mathbf{u}^0 - \eta\mathbf{D}w^0\right) + \mathbf{L}_6 w^0 + \mathbf{L}_8 D_z^0\right] d\eta \\ &+ \left(\int_{-1}^{z} \rho_2 d\eta\right)\frac{\partial^2 w^0}{\partial\tau_0^2} - \frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z} \rho_1 (z-\eta)\mathbf{D}^T\left(\mathbf{u}^0 - \eta\mathbf{D}w^0\right) d\eta,\end{aligned} \tag{72}$$

$$\phi^{(0)} = \bar{\phi}_z^- - \int_{-1}^{z} \left[(\gamma_\alpha\gamma_\beta)\mathbf{L}_8^T\left(\mathbf{u}^0 - \eta\mathbf{D}w^0\right) + \left(1/\gamma_\alpha\gamma_\beta\right)\tilde{l}_{65} w^0 + \tilde{l}_{81} D_z^0\right] d\eta. \tag{73}$$

Imposing the remaining lateral boundary conditions on $z = 1$ (Eqs. (60), (61)) on Eqs. (71)–(73), we obtain

$$K_{11}u^0 + K_{12}v^0 + K_{13}w^0 + K_{14}D_z^0 = I_{10}\frac{\partial^2 u^0}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}\left(w^0{}_{,x}\right), \tag{74}$$

$$K_{21}u^0 + K_{22}v^0 + K_{23}w^0 + K_{24}D_z^0 = I_{10}\frac{\partial^2 v^0}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}\left(w^0{}_{,y}\right), \tag{75}$$

$$K_{31}u^0 + K_{32}v^0 + K_{33}w^0 + K_{34}D_z^0 = \bar{q}_z^+ - \bar{q}_z^- - I_{20}\frac{\partial^2 w^0}{\partial\tau_0^2} + I_{12}\frac{\partial^2}{\partial\tau_0^2}\left(w^0{}_{,xx} + w^0{}_{,yy}\right) - I_{11}\left(u^0{}_{,x} + v^0{}_{,y}\right), \tag{76}$$

$$K_{41}u^0 + K_{42}v^0 + K_{43}w^0 + K_{44}D_z^0 = \bar{\phi}_z^+ - \bar{\phi}_z^-, \tag{77}$$

in which

$$K_{11} = \hat{A}_{11}\partial_{xx} + \left(\tilde{A}_{16} + \tilde{A}_{61}\right)\partial_{xy} + \bar{A}_{66}\partial_{yy},$$

$$K_{12} = \hat{A}_{16}\partial_{xx} + (\tilde{A}_{12} + \tilde{A}_{66})\partial_{xy} + \bar{A}_{62}\partial_{yy},$$

$$K_{13} = -\hat{B}_{11}\partial_{xxx} - \left(\tilde{B}_{16} + \hat{B}_{16} + \tilde{B}_{61}\right)\partial_{xxy} - (\tilde{B}_{12} + \tilde{B}_{66} + \bar{B}_{66})\partial_{xyy}$$

$$- \bar{B}_{62}\partial_{yyy} + \left(\hat{A}_{11}/R_x + \tilde{A}_{12}/R_y\right)\partial_x + (\tilde{A}_{61}/R_x + \bar{A}_{62}/R_y)\partial_y,$$

$$K_{14} = \hat{E}_{31}\partial_x + \bar{E}_{36}\partial_y,$$

$$K_{21} = \hat{A}_{61}\partial_{xx} + (\tilde{A}_{21} + \tilde{A}_{66})\partial_{xy} + \bar{A}_{26}\partial_{yy},$$

$$K_{22} = \hat{A}_{66}\partial_{xx} + (\tilde{A}_{26} + \tilde{A}_{62})\partial_{xy} + \bar{A}_{22}\partial_{yy},$$

$$K_{23} = -\hat{B}_{16}\partial_{xxx} - \left(\tilde{B}_{21} + \tilde{B}_{66} + \hat{B}_{66}\right)\partial_{xxy} - (\bar{B}_{26} + \tilde{B}_{26} + \tilde{B}_{62})\partial_{xyy}$$

$$- \bar{B}_{22}\partial_{yyy} + \left(\hat{A}_{61}/R_x + \tilde{A}_{62}/R_y\right)\partial_x + (\tilde{A}_{21}/R_x + \bar{A}_{22}/R_y)\partial_y,$$

$$K_{24} = \hat{E}_{36}\partial_x + \bar{E}_{32}\partial_y,$$

$$K_{31} = -\hat{B}_{11}\partial_{xxx} - \left(\tilde{B}_{16} + \tilde{B}_{61} + \hat{B}_{61}\right)\partial_{xxy} - (\tilde{B}_{21} + \tilde{B}_{66} + \bar{B}_{66})\partial_{xyy}$$

$$- \bar{B}_{26}\partial_{yyy} + \left(\hat{A}_{11}/R_x + \tilde{A}_{21}/R_y\right)\partial_x + (\tilde{A}_{16}/R_x + \bar{A}_{26}/R_y)\partial_y,$$

$$K_{32} = -\hat{B}_{16}\partial_{xxx} - \left(\tilde{B}_{12} + \tilde{B}_{66} + \hat{B}_{66}\right)\partial_{xxy} - (\tilde{B}_{26} + \tilde{B}_{62} + \bar{B}_{62})\partial_{xyy}$$

$$- \bar{B}_{22}\partial_{yyy} + \left(\hat{A}_{16}/R_x + \tilde{A}_{26}/R_y\right)\partial_x + (\tilde{A}_{12}/R_x + \bar{A}_{22}/R_y)\partial_y,$$

$$K_{33} = \hat{D}_{11}\partial_{xxxx} + (\tilde{D}_{16} + \hat{D}_{16} + \tilde{D}_{61} + \hat{D}_{61})\partial_{xxxy} + \left(\tilde{D}_{12} + \tilde{D}_{21} + \bar{D}_{66} + 2\tilde{D}_{66} + \hat{D}_{66}\right)\partial_{xxyy}$$

$$+ (\bar{D}_{26} + \tilde{D}_{26} + \bar{D}_{62} + \tilde{D}_{62})\partial_{xyyy} + \bar{D}_{22}\partial_{yyyy} - \left[2\hat{B}_{11}/R_x + (\tilde{B}_{12} + \tilde{B}_{21})/R_y\right]\partial_{xx}$$

$$- \left[\left(\tilde{B}_{16} + \tilde{B}_{61} + \hat{B}_{16} + \hat{B}_{61}\right)/R_x + (\bar{B}_{26} + \bar{B}_{62} + \tilde{B}_{26} + \tilde{B}_{62})/R_y\right]\partial_{xy}$$

$$- \left[(\tilde{B}_{12} + \tilde{B}_{21})/R_x + 2\bar{B}_{22}/R_y\right]\partial_{yy} + \left[\hat{A}_{11}/R_x^2 + (\tilde{A}_{12} + \tilde{A}_{21})/R_xR_y + \bar{A}_{22}/R_y^2\right],$$

$$K_{34} = -\hat{F}_{31}\partial_{xx} - \left(\hat{F}_{36} + \bar{F}_{36}\right)\partial_{xy} - \bar{F}_{32}\partial_{yy} + \left(\hat{E}_{31}/R_x + \bar{E}_{32}/R_y\right),$$

$$K_{41} = -\tilde{E}_{31}\partial_x - \breve{E}_{36}\partial_y, \quad K_{42} = -\tilde{E}_{36}\partial_x - \breve{E}_{32}\partial_y,$$

$$K_{43} = \tilde{F}_{31}\partial_{xx} + \left(\breve{F}_{36} + \tilde{F}_{36}\right)\partial_{xy} + \breve{F}_{32}\partial_{yy} - \left(\tilde{E}_{31}/R_x + \breve{E}_{32}/R_y\right),$$

$$K_{44} = -E_{30},$$

$$\hat{A}_{ij} = \int_{-1}^{1} (\tilde{Q}_{ij}\gamma_\beta/\gamma_\alpha)\,dz, \quad \tilde{A}_{ij} = \int_{-1}^{1} \tilde{Q}_{ij}\,dz, \quad \bar{A}_{ij} = \int_{-1}^{1} (\tilde{Q}_{ij}\gamma_\alpha/\gamma_\beta)\,dz,$$

$$\hat{B}_{ij} = \int_{-1}^{1} z\big(\tilde{Q}_{ij}\gamma_\beta/\gamma_\alpha\big)\,dz, \quad \tilde{B}_{ij} = \int_{-1}^{1} z\tilde{Q}_{ij}\,dz, \quad \bar{B}_{ij} = \int_{-1}^{1} z\big(\tilde{Q}_{ij}\gamma_\alpha/\gamma_\beta\big)\,dz,$$

$$\hat{D}_{ij} = \int_{-1}^{1} z^2\big(\tilde{Q}_{ij}\gamma_\beta/\gamma_\alpha\big)\,dz, \quad \tilde{D}_{ij} = \int_{-1}^{1} z^2\tilde{Q}_{ij}\,dz, \quad \bar{D}_{ij} = \int_{-1}^{1} z^2\big(\tilde{Q}_{ij}\gamma_\alpha/\gamma_\beta\big)\,dz,$$

$$\big(\hat{E}_{3i} \quad \hat{F}_{3i}\big) = \int_{-1}^{1} (1 \quad z)\left(\frac{\gamma_\beta e}{Q}\right)\left(\frac{e_{33}c_{3i} - e_{3i}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)dz,$$

$$\big(\bar{E}_{3i} \quad \bar{F}_{3i}\big) = \int_{-1}^{1} (1 \quad z)\left(\frac{\gamma_\alpha e}{Q}\right)\left(\frac{e_{33}c_{3i} - e_{3i}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)dz,$$

$$\left(\breve{E}_{3i} \quad \breve{F}_{3i}\right) = \int_{-1}^{1} (1 \quad z)\left(\frac{e}{\gamma_\beta Q}\right)\left(\frac{e_{33}c_{3i} - e_{3i}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)dz,$$

$$\big(\tilde{E}_{3i} \quad \tilde{F}_{3i}\big) = \int_{-1}^{1} (1 \quad z)\left(\frac{e}{\gamma_\alpha Q}\right)\left(\frac{e_{33}c_{3i} - e_{3i}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)dz,$$

$$E_{30} = \int_{-1}^{1} \left(\frac{e^2}{Q}\right)\left(\frac{c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)dz, \quad I_{10} = \int_{-1}^{1} \rho_1 dz, \quad I_{11} = \int_{-1}^{1} \rho_1 z dz,$$

$$I_{12} = \int_{-1}^{1} \rho_1 z^2 dz, \quad I_{20} = \int_{-1}^{1} \rho_2 dz.$$

As obtained in earlier papers [18], [19], the CST governing equations are recovered from Eqs. (74)–(77) by introducing a geometry assumption of the thin shell: $z/R_\alpha \ll 1$ and $z/R_\beta \ll 1$. Thus, the CST has been derived as the first-order approximation to the 3D theory. Solution of Eqs. (74)–(77) must be supplemented with the edge boundary conditions Eqs. (64), (65) to constitute a well-posed boundary value problem. Once the modal variables of $u^0$, $v^0$, $w^0$ and $D_z^0$ are determined, the leading-order solutions of displacements are given by Eqs. (68)–(70), the transverse shear and normal stresses by Eqs. (71), (72), the in-surface stresses by Eq. (42), the in-surface electric displacements by Eq. (43) and the electric potential by Eq. (73).

Proceeding to order $\varepsilon^2$ following the same line as was done before, we readily obtain

$$w^{(1)} = w^1(x, y,\ \tau_0,\ \tau_1, \ldots) + \varphi_{31}(x, y, z,\ \tau_0,\ \tau_1, \ldots), \tag{78}$$

$$\mathbf{u}^{(1)} = \mathbf{u}^1(x,\ y,\ \tau_0,\ \tau_1, \ldots) - z\mathbf{D}w^1 + \varphi^1(x,\ y,\ z,\ \tau_0,\ \tau_1, \ldots), \tag{79}$$

$$D_z^{(1)} = D_z^1(x,\ y,\ \tau_0,\ \tau_1, \ldots) + \varphi_{41}(x, y, z,\ \tau_0,\ \tau_1, \ldots), \tag{80}$$

$$\boldsymbol{\sigma}_s^{(1)} = -\int_{-1}^{z} \big[\mathbf{L}_5\big(\mathbf{u}^1 - \eta\mathbf{D}w^1\big) + \mathbf{L}_6 w^1 + \mathbf{L}_8 D_z^1\big]d\eta + \mathbf{f}^1(x, y, z, \tau_0, \tau_1, \ldots) + \frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z} \rho_1\big(\mathbf{u}^1 - \eta\mathbf{D}w^1\big)d\eta, \tag{81}$$

$$\sigma_z^{(1)} = \int_{-1}^{z} \left[\mathbf{L}_9\left(\mathbf{u}^1 - \eta\mathbf{D}w^1\right) + \tilde{l}_{63}w^1\right]d\eta$$

$$+ \int_{-1}^{z} (z-\eta)\mathbf{D}^T\left[\mathbf{L}_5\left(\mathbf{u}^1 - \eta\mathbf{D}w^1\right) + \mathbf{L}_6 w^1 + \mathbf{L}_8 D_z^1\right]d\eta - f_{31}(x,y,z,\tau_0,\tau_1,\ldots) \tag{82}$$

$$-\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z} \rho_1(z-\eta)\mathbf{D}^T\left(\mathbf{u}^1 - \eta\mathbf{D}w^1\right)d\eta + \left(\int_{-1}^{z} \rho_2 d\eta\right)\frac{\partial^2 w^1}{\partial\tau_0^2},$$

$$\phi^{(1)} = -\int_{-1}^{z} \left[(\gamma_\alpha\gamma_\beta)\mathbf{L}_8^T\left(\mathbf{u}^1 - \eta\mathbf{D}w^1\right) + \left(1/\gamma_\alpha\gamma_\beta\right)\tilde{l}_{65}w^1 + \tilde{l}_{81}D_z^1\right]d\eta - f_{41}(x,y,z,\tau_0,\tau_1,\ldots), \tag{83}$$

where

$$\varphi_{31}(x,y,z,\tau_0,\tau_1,\ldots) = -\int_0^z \left[\mathbf{L}_1\mathbf{u}^{(0)} + \tilde{l}_{33}w^{(0)} - \tilde{l}_{35}D_z^{(0)}\right]d\eta,$$

$$\mathbf{u}^1 = \left[u^1(x,y,\tau_0,\tau_1,\ldots) \quad v^1(x,y,\tau_0,\tau_1,\ldots)\right]^T,$$

$$\boldsymbol{\varphi}^1 = \begin{Bmatrix} \varphi_{11}(x,y,z,\tau_0,\tau_1,\ldots) \\ \varphi_{21}(x,y,z,\tau_0,\tau_1,\ldots) \end{Bmatrix} = \int_0^z \left(\mathbf{L}_2\mathbf{u}^{(0)} + \mathbf{S}\boldsymbol{\sigma}_s^{(0)} + \mathbf{L}_4\phi^{(0)} - \mathbf{D}\varphi_{31}\right)d\eta,$$

$$\varphi_{41}(x,y,z,\tau_0,\tau_1,\ldots) = -\int_0^z \left(\mathbf{L}_{11}\mathbf{d}^{(0)} + \tilde{l}_{71}D_z^{(0)}\right)d\eta,$$

$$\mathbf{f}^1 = \begin{Bmatrix} f_{11}(x,y,z,\tau_0,\tau_1,\ldots) \\ f_{21}(x,y,z,\tau_0,\tau_1,\ldots) \end{Bmatrix} = -\int_{-1}^{z} \left[\mathbf{L}_5\varphi^1 + \mathbf{L}_6\varphi_{31} + \mathbf{L}_8\varphi_{41} + \mathbf{L}_7\sigma_s^{(0)} + (\gamma_\alpha\gamma_\beta)\mathbf{L}_1^T\sigma_z^{(0)}\right]d\eta$$

$$+\frac{\partial^2}{\partial\tau_0^2}\left(\int_{-1}^{z} \rho_1\boldsymbol{\varphi}^1 d\eta\right) + \frac{\partial^2}{\partial\tau_0\partial\tau_1}\left(2\int_{-1}^{z} \rho_1\mathbf{u}^{(0)}d\eta\right),$$

$$f_{31}(x,y,z,\tau_0,\tau_1,\ldots) = -\int_{-1}^{z} \left[\mathbf{L}_9\boldsymbol{\varphi}^1 + \tilde{l}_{63}\varphi_{31} - \mathbf{L}_{10}\boldsymbol{\sigma}_s^{(0)} - \tilde{l}_{64}\sigma_z^{(0)} + \tilde{l}_{65}\varphi_{41} - \mathbf{D}^T\mathbf{f}^1\right]d\eta$$

$$-\frac{\partial^2}{\partial\tau_0^2}\left(\int_{-1}^{z} \rho_2\varphi_{31}d\eta\right) - \left(2\int_{-1}^{z} \rho_2 d\eta\right)\frac{\partial^2 w^0}{\partial\tau_0\partial\tau_1},$$

$$f_{41}(x,y,z,\tau_0,\tau_1,\ldots) = \int_{-1}^{z} \left[(\gamma_\alpha\gamma_\beta)\mathbf{L}_8^T\boldsymbol{\varphi}^1 + (1/\gamma_\alpha\gamma_\beta)\tilde{l}_{65}\varphi_{31} - \tilde{l}_{35}\sigma_z^{(0)} - \tilde{l}_{81}\varphi_{41}\right]d\eta.$$

$w^1$, $\mathbf{u}^1$and $D_z^1$ represent the modifications to the elastic and electric displacements on the middle surface. By imposition of the associated lateral boundary conditions (62), (63) on Eqs. (81)–(83), we obtain again the CST governing equations, and the nonhomogeneous terms can be calculated using the leading-order solution. The resulting equations are

$$K_{11}u^1 + K_{12}v^1 + K_{13}w^1 + K_{14}D_z^1 = I_{10}\frac{\partial^2 u^1}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}\left(w^1{}_{,x}\right) + f_{11}(z=1), \tag{84}$$

$$K_{21}u^1 + K_{22}v^1 + K_{23}w^1 + K_{24}D_z^1 = I_{10}\frac{\partial^2 v^1}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}\left(w^1{}_{,y}\right) + f_{21}(z=1), \tag{85}$$

$$\begin{aligned}K_{31}u^1 + K_{32}v^1 + K_{33}w^1 + K_{34}D_z^1 = {} & -I_{20}\frac{\partial^2 \omega^1}{\partial\tau_0^2} + I_{12}\frac{\partial^2}{\partial\tau_0^2}\left(w^1{}_{,xx} + w^1{}_{,yy}\right) - I_{11}\frac{\partial^2}{\partial\tau_0^2}\left(u^1{}_{,x} + v^1{}_{,y}\right) \\ & + f_{31}(z=1) - \frac{\partial f_{11}(z=1)}{\partial x} - \frac{\partial f_{21}(z=1)}{\partial y},\end{aligned} \tag{86}$$

$$K_{41}u^1 + K_{42}v^1 + K_{43}w^1 + K_{44}D_z^1 = f_{41}(z=1). \tag{87}$$

The governing equation for the $\varepsilon^{2k}$-order modifications to $u^k(x,y,\tau_0,\tau_1,\ldots)$, $v^k(x,y,\tau_0,\tau_1,\ldots)$ and $w^k(x,y,\tau_0,\tau_1,\ldots)$, of order $\varepsilon^{2k}$ $(k=2,3,\ldots)$ are obtained in a similar way by integrating the higher-order equations (52)–(56) in succession. The equations are given by

$$K_{11}u^k + K_{12}v^k + K_{13}w^k + K_{14}D_z^k = I_{10}\frac{\partial^2 u^k}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}\left(w^k{}_{,x}\right) + f_{1k}(z=1), \tag{88}$$

$$K_{21}u^k + K_{22}v^k + K_{23}w^k + K_{24}D_z^k = I_{10}\frac{\partial^2 v^k}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}\left(w^k{}_{,y}\right) + f_{2k}(z=1), \tag{89}$$

$$\begin{aligned}K_{31}u^k + K_{32}v^k + K_{33}w^k + K_{34}D_z^k = {} & -I_{20}\frac{\partial^2 w^k}{\partial\tau_0^2} + I_{12}\frac{\partial^2}{\partial\tau_0^2}\left(w^k{}_{,xx} + w^k{}_{,yy}\right) - I_{11}\frac{\partial^2}{\partial\tau_0^2}\left(u^k{}_{,x} + v^k{}_{,y}\right) \\ & + f_{3k}(z=1) - \frac{\partial f_{1k}(z=1)}{\partial x} - \frac{\partial f_{2k}(z=1)}{\partial y},\end{aligned} \tag{90}$$

$$K_{41}u^k + K_{42}v^k + K_{43}w^k + K_{44}D_z^k = f_{4k}(z=1), \tag{91}$$

in which

$$\varphi_{3k}(x,y,z,\,\tau_0,\,\tau_1,\ldots) = -\int_0^z \left[\mathbf{L}_1\mathbf{u}^{(k-1)} + \tilde{l}_{33}w^{(k-1)} - \tilde{l}_{35}D_z^{(k-1)} - \tilde{l}_{34}\sigma_z^{(k-2)}\right]d\eta,$$

$$\mathbf{u}^k = \left[u^k(x,y,\,\tau_0,\,\tau_1,\ldots) \quad v^k(x,y,\,\tau_0,\,\tau_1,\ldots)\right]^T,$$

$$\boldsymbol{\varphi}^k = \begin{Bmatrix}\varphi_{1k}(x,y,z,\tau_0,\,\tau_1,\ldots)\\ \varphi_{2k}(x,y,z,\tau_0,\,\tau_1,\ldots)\end{Bmatrix} = -\int_0^z \left(\mathbf{D}\varphi_{3k} - \mathbf{L}_2\mathbf{u}^{(k)} - \mathbf{S}\boldsymbol{\sigma}_s^{(k-1)} - \mathbf{L}_3\boldsymbol{\sigma}_s^{(k-2)} - \mathbf{L}_4\phi^{(k-1)}\right)d\eta,$$

$$\varphi_{4k}(x,\,y,\,z,\,\tau_0,\,\tau_1,\ldots) = -\int_0^z \left(\mathbf{L}_{11}\mathbf{d}^{(k-1)} + \tilde{l}_{71}D_z^{(k-1)}\right)d\eta,$$

$$\begin{aligned}\mathbf{f}^k = \begin{Bmatrix}f_{1k}(x,\,y,\,z,\,\tau_0,\,\tau_1,\ldots)\\ f_{2k}(x,\,y,\,z,\,\tau_0,\,\tau_1,\ldots)\end{Bmatrix} = {} & -\int_{-1}^z \left[\mathbf{L}_5\boldsymbol{\varphi}^k + \mathbf{L}_6\varphi_{3k} + \mathbf{L}_8\varphi_{4k} + \mathbf{L}_7\boldsymbol{\sigma}_s^{(k-1)} + \left(\gamma_\alpha\gamma_\beta\right)\mathbf{L}_1^T\sigma_z^{(k-1)}\right]d\eta \\ & + \left[\frac{\partial^2}{\partial\tau_0^2}\left(\int_{-1}^z \rho_1\boldsymbol{\varphi}^k d\eta\right) + \frac{\partial^2}{\partial\tau_0\partial\tau_1}\left(2\int_{-1}^z \rho_1\mathbf{u}^{(k-1)}d\eta\right)\right. \\ & \left. + \ldots + \left(\frac{\partial^2}{\partial\tau_0\partial\tau_1} + \frac{\partial^2}{\partial\tau_1\partial\tau_{k-1}} + \ldots + \frac{\partial^2}{\partial\tau_k\partial\tau_0}\right)\left(\int_{-1}^z \rho_1\mathbf{u}^{(0)}d\eta\right)\right]\end{aligned}$$

$$f_{3k}(x,\,y,\,z,\,\tau_0,\,\tau_1,\ldots)=-\int_{-1}^{z}\Big[\mathbf{L}_9\boldsymbol{\varphi}^k+\tilde{l}_{63}\varphi_{3k}-\mathbf{L}_{10}\boldsymbol{\sigma}_s^{(k-1)}-\tilde{l}_{64}\sigma_z^{(k-1)}+\tilde{l}_{65}\varphi_{4k}-\mathbf{D}^T\mathbf{f}^k\Big]d\eta$$

$$-\left[\frac{\partial^2}{\partial\tau_0^2}\left(\int_{-1}^{z}\rho_2\varphi_{3k}d\eta\right)+\frac{\partial^2}{\partial\tau_0\partial\tau_k}\left(2\int_{-1}^{z}\rho_2 w^{(k-1)}d\eta\right)\right.$$

$$\left.+\ldots+\left(\frac{\partial^2}{\partial\tau_0\partial\tau_k}+\frac{\partial^2}{\partial\tau_1\partial\tau_{k-1}}+\ldots+\frac{\partial^2}{\partial\tau_k\partial\tau_0}\right)\left(\int_{-1}^{z}\rho_2 w^0 d\eta\right)\right],$$

$$f_{4k}(x,\,y,\,z,\,\tau_0,\,\tau_1,\ldots)=\int_{-1}^{z}\Big[(\gamma_\alpha\gamma_\beta)\mathbf{L}_8^T\boldsymbol{\phi}^k+(1/\gamma_\alpha\gamma_\beta)\tilde{l}_{65}\varphi_{3k}-\tilde{l}_{35}\sigma_z^{(k-1)}-\tilde{l}_{81}\varphi_{4k}\Big]d\eta.$$

## 6 Applications to benchmark problems

The benchmark free vibration problems of simply supported, doubly curved laminated piezoelectric shells are studied using the present asymptotic theory where $\bar{q}_z^{\pm}=\bar{\phi}_z^{\pm}=0$. The material of cross-ply laminated piezoelectric shells is considered so that the elastic moduli for orthotropic materials are

$$(Q_{16})_i=(Q_{26})_i=(Q_{36})_i=(Q_{45})_i=(e_{36})_i=0. \tag{92}$$

The boundary conditions on the four edges are of a shear diaphragm type specified by

$$\sigma_\alpha=u_\beta=u_\zeta=\phi=0\quad\text{on}\quad\alpha=0\quad\text{and}\quad\alpha=a_\alpha, \tag{93}$$

$$\sigma_\beta=u_\alpha=u_\zeta=\phi=0\quad\text{on}\quad\beta=0\quad\text{and}\quad\beta=a_\beta. \tag{94}$$

For this problem the governing equations of the leading-order problem (74)–(77) can be easily solved by letting

$$u^0=U_0\cos\tilde{m}x\,\sin\tilde{n}y\,\cos(\omega\tau_0-\psi), \tag{95}$$

$$v^0=V_0\sin\tilde{m}x\,\cos\tilde{n}y\,\cos(\omega\tau_0-\psi), \tag{96}$$

$$w^0=W_0\sin\tilde{m}x\,\sin\tilde{n}y\,\cos(\omega\tau_0-\psi), \tag{97}$$

$$D_z^0=D_0\sin\tilde{m}x\,\sin\tilde{n}y\,\cos(\omega\tau_0-\psi). \tag{98}$$

where $\tilde{m}=m\pi\sqrt{Rh}/a_\alpha$, $\tilde{n}=n\pi\sqrt{Rh}/a_\beta$ $(m,n=1,2,3,\ldots)$ and $\omega$ denotes the circular frequency of the motion. The phase angle $\psi$ is a function of the time scales $\tau_1,\tau_2,\ldots$ but not $\tau_0$.

Substituting Eqs. (95)–(98) into Eqs. (74)–(77) gives

$$\begin{bmatrix}k_{11}&k_{12}&k_{13}&k_{14}\\k_{21}&k_{22}&k_{23}&k_{24}\\k_{31}&k_{32}&k_{33}&k_{34}\\k_{41}&k_{42}&k_{43}&k_{44}\end{bmatrix}\begin{Bmatrix}U_0\\V_0\\W_0\\D_0\end{Bmatrix}=\omega^2\begin{bmatrix}-I_{10}&0&\tilde{m}I_{11}&0\\0&-I_{10}&\tilde{n}I_{11}&0\\\tilde{m}I_{11}&\tilde{n}I_{11}&-\big[I_{20}+(\tilde{m}^2+\tilde{n}^2)I_{12}\big]&0\\0&0&0&0\end{bmatrix}\begin{Bmatrix}U_0\\V_0\\W_0\\D_0\end{Bmatrix}, \tag{99}$$

where

$$k_{11} = -\tilde{m}^2 \hat{A}_{11} - \tilde{n}^2 \bar{A}_{66},$$

$$k_{12} = -\tilde{m}\,\tilde{n}\left(\tilde{A}_{12} + \tilde{A}_{66}\right),$$

$$k_{13} = \tilde{m}^3 \hat{B}_{11} + \tilde{m}\tilde{n}^2\left(\tilde{B}_{12} + \bar{B}_{66} + \tilde{B}_{66}\right) + \tilde{m}\Big(\hat{A}_{11}/R_x + \tilde{A}_{12}/R_y\Big),$$

$$k_{14} = \tilde{m}\hat{E}_{31},$$

$$k_{21} = -\tilde{m}\tilde{n}\left(\tilde{A}_{21} + \tilde{A}_{66}\right),$$

$$k_{22} = -\tilde{m}^2 \hat{A}_{66} - \tilde{n}^2 \bar{A}_{22},$$

$$k_{23} = \tilde{m}^2\tilde{n}\left(\tilde{B}_{21} + \tilde{B}_{66} + \hat{B}_{66}\right) + \tilde{n}^3\bar{B}_{22} + \tilde{n}\left(\tilde{A}_{21}/R_x + \bar{A}_{22}/R_y\right),$$

$$k_{24} = \tilde{n}\bar{E}_{32},$$

$$k_{31} = \tilde{m}^3 \hat{B}_{11} + \tilde{m}\tilde{n}^2\left(\tilde{B}_{21} + \bar{B}_{66} + \tilde{B}_{66}\right) + \tilde{m}\Big(\hat{A}_{11}/R_x + \tilde{A}_{21}/R_y\Big),$$

$$k_{32} = \tilde{m}^2\tilde{n}\left(\tilde{B}_{12} + \tilde{B}_{66} + \hat{B}_{66}\right) + \tilde{n}^3\bar{B}_{22} + \tilde{n}\left(\tilde{A}_{12}/R_x + \bar{A}_{22}/R_y\right),$$

$$\begin{aligned} k_{33} = &- \tilde{m}^4\hat{D}_{11} - \tilde{m}^2\,\tilde{n}^2\Big(\tilde{D}_{12} + \tilde{D}_{21} + \bar{D}_{66} + 2\tilde{D}_{66} + \hat{D}_{66}\Big) - \tilde{n}^4\bar{D}_{22} \\ &- \tilde{m}^2\Big[2\hat{B}_{11}/R_x + \left(\tilde{B}_{12} + \tilde{B}_{21}\right)/R_y\Big] - \tilde{n}^2\left[\left(\tilde{B}_{12} + \tilde{B}_{21}\right)/R_x + 2\bar{B}_{22}/R_y\right] \\ &- \Big[\hat{A}_{11}/R_x^2 + \left(\tilde{A}_{12} + \tilde{A}_{21}\right)/R_xR_y + \bar{A}_{22}/R_y^2\Big], \end{aligned}$$

$$k_{34} = -\tilde{m}^2\hat{F}_{31} - \tilde{n}^2\bar{F}_{32} - \Big(\hat{E}_{31}/R_x + \bar{E}_{32}/R_y\Big),$$

$$k_{41} = \tilde{m}\tilde{E}_{31},$$

$$k_{42} = \tilde{n}\breve{E}_{32},$$

$$k_{43} = -\tilde{m}^2\tilde{F}_{31} - \tilde{n}^2\breve{F}_{32} - \Big(\tilde{E}_{31}/R_x + \breve{E}_{32}/R_y\Big),$$

$$k_{44} = -E_{30}.$$

From the last equation of Eq. (99), we can express $D_0$ in term of $U_0, V_0$ and $W_0$ as

$$D_0 = -\frac{k_{41}}{k_{44}}U_0 - \frac{k_{42}}{k_{44}}V_0 - \frac{k_{43}}{k_{44}}W_0. \tag{100}$$

Substituting Eq. (100) into Eq. (99), we can rewrite Eq. (99) in the form

$$\begin{bmatrix} k_{11}-\dfrac{k_{14}k_{41}}{k_{44}} & k_{12}-\dfrac{k_{14}k_{42}}{k_{44}} & k_{13}-\dfrac{k_{14}k_{43}}{k_{44}} \\ k_{21}-\dfrac{k_{24}k_{41}}{k_{44}} & k_{22}-\dfrac{k_{24}k_{42}}{k_{44}} & k_{23}-\dfrac{k_{24}k_{43}}{k_{44}} \\ k_{31}-\dfrac{k_{34}k_{41}}{k_{44}} & k_{32}-\dfrac{k_{34}k_{42}}{k_{44}} & k_{33}-\dfrac{k_{34}k_{43}}{k_{44}} \end{bmatrix} \begin{Bmatrix} U_0 \\ V_0 \\ W_0 \end{Bmatrix}$$

$$= \omega^2 \begin{bmatrix} -I_{10} & 0 & \tilde{m}I_{11} \\ 0 & -I_{10} & \tilde{n}I_{11} \\ \tilde{m}I_{11} & \tilde{n}I_{11} & -\left[I_{20}+\left(\tilde{m}^2+\tilde{n}^2\right)I_{12}\right] \end{bmatrix} \begin{Bmatrix} U_0 \\ V_0 \\ W_0 \end{Bmatrix}. \tag{101}$$

Equation (101) is an eigenvalue problem. A nontrivial solution of Eq. (101) exists if the determinant of the coefficient matrix vanishes:

$$\begin{vmatrix} k_{11}-\dfrac{k_{14}k_{41}}{k_{44}}+I_{10}\omega^2 & k_{12}-\dfrac{k_{14}k_{42}}{k_{44}} & k_{13}-\dfrac{k_{14}k_{43}}{k_{44}}-\tilde{m}I_{11}\omega^2 \\ k_{21}-\dfrac{k_{24}k_{41}}{k_{44}} & k_{22}-\dfrac{k_{24}k_{42}}{k_{44}}+I_{10}\omega^2 & k_{23}-\dfrac{k_{24}k_{43}}{k_{44}}-\tilde{n}I_{11}\omega^2 \\ k_{31}-\dfrac{k_{34}k_{41}}{k_{44}}-\tilde{m}I_{11}\omega^2 & k_{32}-\dfrac{k_{34}k_{42}}{k_{44}}-\tilde{n}I_{11}\omega^2 & k_{33}-\dfrac{k_{34}k_{43}}{k_{44}}-\left[I_{20}+\left(\tilde{m}^2+\tilde{n}^2\right)I_{12}\right]\omega^2 \end{vmatrix} = 0. \tag{102}$$

From this, three eigenvalues $\omega_i$ ($i = 1, 2, 3$) that represent the leading-order natural frequencies for a specific set of values of $m$ and $n$ are obtained.

To make the solution unique, the model displacements are normalized by imposing the orthonormality condition:

$$\begin{aligned} &\left[\left(U_0+\varepsilon^2U_1+\varepsilon^4U_2+\ldots\right) \quad \left(V_0+\varepsilon^2V_1+\varepsilon^4V_2+\ldots\right) \quad \left(W_0+\varepsilon^2W_1+\varepsilon^4W_2+\ldots\right)\right] \\ &\quad \left[\left(U_0+\varepsilon^2U_1+\varepsilon^4U_2+\ldots\right) \quad \left(V_0+\varepsilon^2V_1+\varepsilon^4V_2+\ldots\right) \quad \left(W_0+\varepsilon^2W_1+\varepsilon^4W_2+\ldots\right)\right]^T = 1. \end{aligned} \tag{103}$$

The normalization conditions at each levels are

$$\varepsilon^0 - \text{ order}: \quad U_0^2+V_0^2+W_0^2 = 1; \tag{104}$$

$$\begin{aligned} \varepsilon^2 - \text{ order}: \quad & U_0^2+V_0^2+W_0^2 = 1, \\ & U_0U_1+V_0V_1+W_0W_1 = 0, \end{aligned} \tag{105}$$

$$\begin{aligned} \varepsilon^4 - \text{ order}: \quad & U_0^2+V_0^2+W_0^2 = 1, \\ & U_0U_1+V_0V_1+W_0W_1 = 0, \\ & U_1^2+2U_0U_2+V_1^2+2\,V_0\,V_2+W_1^2+2\,W_1\,W_2 = 0; \\ & \ldots \\ & \text{etc.} \end{aligned} \tag{106}$$

At the $\varepsilon^0$-order level, the normalized eigenvectors corresponding to $\omega_i$ $(i = 1,\ 2,\ 3)$ are written as $\left\{ U_0^{(i)} \quad V_0^{(i)} \quad W_0^{(i)} \right\}^T$. Once the normalized modal displacements on the mid-surface at $\varepsilon^0$-order level (i.e., $\left\{ U_0^{(i)} \quad V_0^{(i)} \quad W_0^{(i)} \right\}^T$) have been determined, the corresponding modal displacements and piezoelectric displacements can be obtained using Eqs. (68)–(70), and the corresponding stresses and electric potential using Eqs. (71)–(73).

Carrying on the solution to order $\varepsilon^2$, we find that the nonhomogeneous terms for fixed values of $m$ and $n$ in the $\varepsilon^2$-order equations are

$$f_{11}(x,y,1) = \left(\hat{f}_{11}(1)\frac{\partial\psi_i}{\partial\tau_1} + \tilde{f}_{11}(1)\right)\cos\tilde{m}x\,\sin\tilde{n}y\,\cos(\omega_i\,\tau_0 - \psi_i), \tag{107}$$

$$f_{21}(x,y,1) = \left(\hat{f}_{21}(1)\frac{\partial\psi_i}{\partial\tau_1} + \tilde{f}_{21}(1)\right)\sin\tilde{m}x\,\cos\tilde{n}y\,\cos(\omega_i\,\tau_0 - \psi_i), \tag{108}$$

$$f_{31}(x,y,1) = \left(\hat{f}_{31}(1)\frac{\partial\psi_i}{\partial\tau_1} + \tilde{f}_{31}(1)\right)\sin\tilde{m}x\,\sin\tilde{n}y\,\cos(\omega_i\,\tau_0 - \psi_i), \tag{109}$$

$$f_{41}(x,y,1) = \left(\hat{f}_{41}(1)\frac{\partial\psi_i}{\partial\tau_1} + \tilde{f}_{41}(1)\right)\sin\tilde{m}x\,\sin\tilde{n}y\,\cos(\omega_i\,\tau_0 - \psi_i), \tag{110}$$

where $\hat{f}_{11}, \hat{f}_{21}, \hat{f}_{31}, \hat{f}_{41}, \tilde{f}_{11}, \tilde{f}_{21}, \tilde{f}_{31}, \tilde{f}_{41}$ are given in Appendix A.

In view of the recurrence of the equations, the $\varepsilon^2$-order solution can be obtained by letting

$$u^1 = U_1\cos\tilde{m}x\sin\tilde{n}y\,\cos(\omega_i\,\tau_0 - \psi_i), \tag{111}$$

$$v^1 = V_1\sin\tilde{m}x\cos\tilde{n}y\,\cos(\omega_i\,\tau_0 - \psi_i), \tag{112}$$

$$w^1 = W_1\sin\tilde{m}x\sin\tilde{n}y\,\cos(\omega_i\,\tau_0 - \psi_i), \tag{113}$$

$$D_z^1 = D_1\sin\tilde{m}x\sin\tilde{n}y\,\cos(\omega_i\,\tau_0 - \psi_i). \tag{114}$$

Substituting Eqs. (111)–(114) into Eqs. (84)–(87) gives

$$\begin{bmatrix} k_{11} + \omega^2 I_{10} & k_{12} & k_{13} - \omega^2\tilde{m}I_{11} & k_{14} \\ k_{21} & k_{22} + \omega^2 I_{10} & k_{23} - \omega^2\tilde{n}I_{11} & k_{24} \\ k_{31} - \omega^2\tilde{m}I_{11} & k_{32} - \omega^2\tilde{n}I_{11} & k_{33} + \omega^2\left[I_{20} + \left(\tilde{m}^2 + \tilde{n}^2\right)I_{12}\right] & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ W_1 \\ D_1 \end{Bmatrix}$$
$$= \begin{Bmatrix} \hat{f}_{11}(1)\dfrac{\partial\psi_i}{\partial\tau_1} + \tilde{f}_{11}(1) \\ \hat{f}_{21}(1)\dfrac{\partial\psi_i}{\partial\tau_1} + \tilde{f}_{21}(1) \\ \left[\hat{f}_{31}(1) + \tilde{m}\hat{f}_{11}(1) + \tilde{n}\hat{f}_{21}(1)\right]\dfrac{\partial\psi_i}{\partial\tau_1} + \left[\tilde{f}_{31}(1) + \tilde{m}\tilde{f}_{11}(1) + \tilde{n}\tilde{f}_{21}(1)\right] \\ \tilde{f}_{41}(1) \end{Bmatrix}. \tag{115}$$

From the last equation in Eq. (115), we have

$$D_1 = -\frac{k_{41}}{k_{44}}U_1 - \frac{k_{42}}{k_{44}}V_1 - \frac{k_{43}}{k_{44}}W_1 + \frac{\tilde{f}_{41}(1)}{k_{44}}. \tag{116}$$

Substituting Eq. (116) into Eq. (115), we may rewrite Eq. (115) in the form

$$\begin{bmatrix} k_{11}-\frac{k_{14}k_{41}}{k_{44}}+\omega^2 I_{10} & k_{12}-\frac{k_{14}k_{42}}{k_{44}} & k_{13}-\frac{k_{14}k_{43}}{k_{44}}-\omega^2\tilde{m}I_{11} \\ k_{21}-\frac{k_{24}k_{41}}{k_{44}} & k_{22}-\frac{k_{24}k_{42}}{k_{44}}+\omega^2 I_{10} & k_{23}-\frac{k_{24}k_{43}}{k_{44}}-\omega^2\tilde{n}I_{11} \\ k_{31}-\frac{k_{34}k_{41}}{k_{44}}-\omega^2\tilde{m}I_{11} & k_{32}-\frac{k_{34}k_{42}}{k_{44}}-\omega^2\tilde{n}I_{11} & k_{33}-\frac{k_{34}k_{43}}{k_{44}}+\omega^2\left[I_{20}+\left(\tilde{m}^2+\tilde{n}^2\right)I_{12}\right] \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ W_1 \end{Bmatrix}$$

$$= \begin{Bmatrix} \hat{f}_{11}(1)\frac{\partial\psi_i}{\partial\tau_1}+\left[\tilde{f}_{11}(1)-\frac{k_{14}}{k_{44}}\tilde{f}_{41}(1)\right] \\ \hat{f}_{21}(1)\frac{\partial\psi_i}{\partial\tau_1}+\left[\tilde{f}_{21}(1)-\frac{k_{24}}{k_{44}}\tilde{f}_{41}(1)\right] \\ \left[\hat{f}_{31}(1)+\tilde{m}\hat{f}_{11}(1)+\tilde{n}\hat{f}_{21}(1)\right]\frac{\partial\psi_i}{\partial\tau_1}+\left[\tilde{f}_{31}(1)+\tilde{m}\tilde{f}_{11}(1)+\tilde{n}\tilde{f}_{21}(1)-\frac{k_{34}}{k_{44}}\tilde{f}_{41}(1)\right] \end{Bmatrix}. \tag{117}$$

The solvability condition for Eq. (117) is given by [22]

$$U_0^{(i)}\left\{\hat{f}_{11}(1)\frac{\partial\psi_i}{\partial\tau_1}+\left[\tilde{f}_{11}(1)-\frac{k_{14}}{k_{44}}\tilde{f}_{41}(1)\right]\right\}+V_0^{(i)}\left\{\hat{f}_{21}(1)\frac{\partial\psi_i}{\partial\tau_1}+\left[\tilde{f}_{21}(1)-\frac{k_{24}}{k_{44}}\tilde{f}_{41}(1)\right]\right\}$$
$$+W_0^{(i)}\left\{\left[\hat{f}_{31}(1)+\tilde{m}\hat{f}_{11}(1)+\tilde{n}\hat{f}_{21}(1)\right]\frac{\partial\psi_i}{\partial\tau_1}+\left[\tilde{f}_{31}(1)+\tilde{m}\tilde{f}_{11}(1)+\tilde{n}\tilde{f}_{21}(1)-\frac{k_{34}}{k_{44}}\tilde{f}_{41}(1)\right]\right\}=0. \tag{118}$$

Equation (117) is solvable if and only if the solvability condition (118) is satisfied. The dependence of $\psi_i$ upon $\tau_1$ can then be determined as

$$\psi_i = -\lambda_i\,\tau_1 + \bar{\psi}_i(\tau_2,\,\tau_3,\ldots), \tag{119}$$

where

$$\lambda_i=\frac{U_0^{(i)}\left[\tilde{f}_{11}(1)-\frac{k_{14}}{k_{44}}\tilde{f}_{41}(1)\right]+V_0^{(i)}\left[\tilde{f}_{21}(1)-\frac{k_{24}}{k_{44}}\tilde{f}_{41}(1)\right]+W_0^{(i)}\left[\tilde{f}_{31}(1)+\tilde{m}\tilde{f}_{11}(1)+\tilde{n}\tilde{f}_{21}(1)-\frac{k_{34}}{k_{44}}\tilde{f}_{41}(1)\right]}{U_0^{(i)}\hat{f}_{11}(1)+V_0^{(i)}\hat{f}_{21}(1)+W_0^{(i)}\left[\hat{f}_{31}(1)+\tilde{m}\hat{f}_{11}(1)+\tilde{n}\hat{f}_{21}(1)\right]},$$

and $\bar{\psi}_i$ represents integration functions of the scales $\tau_2,\tau_3,\ldots$, and so on.

With Eq. (119) and the relation $\tau_1 = \varepsilon^2\,\tau_0 = (h/R)\tau_0$, the time functions of all field variables are now expressed in terms of $\cos\left[(\omega+\lambda h/R)\,\tau_0-\bar{\psi}\right]$. Therefore the natural frequencies at the $\varepsilon^2$-order level have been modified to

$$\omega_i+\lambda_i\,\frac{h}{R}\quad(i=1,2,3). \tag{120}$$

Substituting Eq. (120) into Eq. (117) yields

$$\begin{bmatrix} k_{11}-\dfrac{k_{14}k_{41}}{k_{44}}+\omega_i^2 I_{10} & k_{12}-\dfrac{k_{14}k_{42}}{k_{44}} & k_{13}-\dfrac{k_{14}k_{43}}{k_{44}}-\omega_i^2\tilde{m}I_{11} \\ k_{21}-\dfrac{k_{24}k_{41}}{k_{44}} & k_{22}-\dfrac{k_{24}k_{42}}{k_{44}}+\omega_i^2 I_{10} & k_{23}-\dfrac{k_{24}k_{43}}{k_{44}}-\omega_i^2\tilde{n}I_{11} \\ k_{31}-\dfrac{k_{34}k_{41}}{k_{44}}-\omega_i^2\tilde{m}I_{11} & k_{32}-\dfrac{k_{34}k_{42}}{k_{44}}-\omega_i^2\tilde{n}I_{11} & k_{33}-\dfrac{k_{34}k_{43}}{k_{44}}+\omega_i^2\left[I_{20}+\left(\tilde{m}^2+\tilde{n}^2\right)I_{12}\right] \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ W_1 \end{Bmatrix}$$

$$= \begin{Bmatrix} -\lambda_i\hat{f}_{11}(1)+\left[\tilde{f}_{11}(1)-\dfrac{k_{14}}{k_{44}}\tilde{f}_{41}(1)\right] \\ -\lambda_i\hat{f}_{21}(1)+\left[\tilde{f}_{21}(1)-\dfrac{k_{24}}{k_{44}}\tilde{f}_{41}(1)\right] \\ -\lambda_i\left[\hat{f}_{31}(1)+\tilde{m}\hat{f}_{11}(1)+\tilde{n}\hat{f}_{21}(1)\right]+\left[\tilde{f}_{31}(1)+\tilde{m}\tilde{f}_{11}(1)+\tilde{n}\tilde{f}_{21}(1)-\dfrac{k_{34}}{k_{44}}\tilde{f}_{41}(1)\right] \end{Bmatrix}, \tag{121}$$

from which $U_1$, $V_1$, $W_1$ can be uniquely determined by a simple solution of the algebraic equations along with the normalization conditions (105).

Once $U_1$, $V_1$ and $W_1$ are obtained, determination of the $\varepsilon^2$-order corrections of the modal stresses and displacements is straightforward. The solution procedure can be continued to higher levels in a similar way.

## 7 Illustrative examples

### *7.1 Single-layer piezoelectric plates*

The benchmark solutions of single-layer piezoelectric plates were presented by Heyliger [23]. The piezoelectric plates are regarded as a special case of doubly curved shells in the present formulation by letting $R_\alpha = R_\beta = \infty$ and $1/R_\alpha = 1/R_\beta = 0$. To facilitate numerical comparisons, we first compute the results for a single-layer plate composed of a piezoceramic material. The edge boundaries of the plate are considered as fully simple supports.

The elastic, piezoelectric and dielectric properties of piezoelectric materials are given in Table 1 [23]. For comparison purposes, the geometry parameters are taken as $a_\alpha/a_\beta = 1$, $a_\alpha/2h = 4,\ 10,\ 50$ and $2h = 0.01\,\mathrm{m}$. Table 2 shows the asymptotic solutions of natural frequencies for various orders when $m = n = 1$. The asymptotic solution is computed up to the $\varepsilon^6$-order level in order to closely examine the convergence of the present asymptotic theory. The results of non-piezoelectric plates are also presented in Table 2 by letting $e_{ij} = 0$ in the present formulation. It is shown that the convergent solutions of fundamental frequencies are obtained at the $\varepsilon^6$-order level in the case of thick plates ($a_\alpha/2h = 4$), $\varepsilon^4$-order level in the case of moderately thick plates ($a_\alpha/2h = 10$), and $\varepsilon^2$-order level in the case of thin plates ($a_\alpha/2h = 50$). The frequencies corresponding to the flexural modes are lower than those corresponding to extensional modes for the fixed values of $m$ and $n$. The deviation of the frequencies between

**Table 1.** Elastic, piezoelectric and dielectric properties of piezoelectric materials

| Moduli (PZT-4) (Heyliger et al. [23]) | | Moduli (PVDF) (Cheng et al. [24]) | |
|---|---|---|---|
| $E_1$ (Gpa) | 81.3 | $c_{11}$ (GPa) | 238.00 |
| $E_2$ (Gpa) | 81.3 | $c_{22}$ (GPa) | 23.60 |
| $E_3$ (Gpa) | 64.5 | $c_{33}$ (GPa) | 10.60 |
| $v_{12}$ | 0.329 | $c_{12}$ (GPa) | 3.98 |
| $v_{13}$ | 0.432 | $c_{13}$ (GPa) | 2.19 |
| $v_{23}$ | 0.432 | $c_{23}$ (GPa) | 1.92 |
| $G_{23}$ (Gpa) | 25.6 | $c_{44}$ (GPa) | 2.15 |
| $G_{13}$ (Gpa) | 25.6 | $c_{55}$ (GPa) | 4.40 |
| $G_{12}$ (Gpa) | 30.6 | $c_{66}$ (GPa) | 6.43 |
| $e_{24}$ (C/m$^2$) | 12.72 | $e_{24}$ (C/m$^2$) | −0.01 |
| $e_{15}$ (C/m$^2$) | 12.72 | $e_{15}$ (C/m$^2$) | −0.01 |
| $e_{31}$ (C/m$^2$) | −5.20 | $e_{31}$ (C/m$^2$) | −0.13 |
| $e_{32}$ (C/m$^2$) | −5.20 | $e_{32}$ (C/m$^2$) | −0.14 |
| $e_{33}$ (C/m$^2$) | 15.08 | $e_{33}$ (C/m$^2$) | −0.28 |
| $\eta_{11}/\varepsilon_0$ | 1475.0 | $\eta_{11}/\varepsilon_0$ | 12.50 |
| $\eta_{22}/\varepsilon_0$ | 1475.0 | $\eta_{22}/\varepsilon_0$ | 11.98 |
| $\eta_{33}/\varepsilon_0$ | 1300.0 | $\eta_{33}/\varepsilon_0$ | 11.98 |

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

flexural and extensional modes increases as the plate becomes thin. The natural frequencies of the piezoelectric plates are higher than those of the non-piezoelectric ones. The piezoelectric effect on the natural frequencies corresponding to flexural modes is much more significant than on the ones corresponding to extensional modes. The convergent solution of the present asymptotic theory is also compared with the 3D piezoelectricity solution [23] available in the

**Table 2.** Frequency parameters $\Omega$ for single-ply piezoelectric plates ($\Omega = \omega \times 100$)

| $a_\alpha/2h$ | Modes ($m = n = 1$) | Materials | Present asymptotic solutions | | | | Heyliger et al. [23] |
|---|---|---|---|---|---|---|---|
| | | | $\varepsilon^0$ | $\varepsilon^2$ | $\varepsilon^4$ | $\varepsilon^6$ | |
| 4 | Flexural mode | Piezoelectric | 111280 | 93967.7 | 97538.6 | 96922.4 | 96929.9 |
| | | Non-piezoelectric | 102405 | 85134.6 | 88821.4 | 88153.7 | N/A |
| | Extensional mode | Piezoelectric | 194255 | 194255 | 194255 | 194255 | 194255 |
| | | Non-piezoelectric | 194255 | 194255 | 194255 | 194255 | N/A |
| | Extensional mode | Piezoelectric | 335396 | 331568 | 331635 | 331568 | 327663 |
| | | Non-piezoelectric | 335396 | 328250 | 328060 | 327897 | N/A |
| 10 | Flexural mode | Piezoelectric | 18545.7 | 17989.8 | 18015.6 | 18014.3 | 18013.4 |
| | | Non-piezoelectric | 17066.7 | 16521.6 | 16546.9 | 16545.6 | N/A |
| | Extensional mode | Piezoelectric | 77702.1 | 77702.1 | 77702.1 | 77702.1 | 77702.1 |
| | | Non-piezoelectric | 77702.1 | 77702.1 | 77702.1 | 77702.1 | N/A |
| | Extensional mode | Piezoelectric | 134158 | 133907 | 133908 | 133907 | 133695 |
| | | Non-piezoelectric | 134158 | 133701 | 133699 | 133697 | N/A |
| 50 | Flexural mode | Piezoelectric | 747.660 | 746.731 | 746.733 | 746.733 | 746.752 |
| | | Non-piezoelectric | 688.033 | 687.125 | 687.126 | 687.126 | N/A |
| | Extensional mode | Piezoelectric | 15540.4 | 15540.4 | 15540.4 | 15540.4 | 15540.4 |
| | | Non-piezoelectric | 15540.4 | 15540.4 | 15540.4 | 15540.4 | N/A |
| | Extensional mode | Piezoelectric | 26831.7 | 26829.6 | 26829.6 | 26829.6 | 26828.0 |
| | | Non-piezoelectric | 26831.7 | 26828.0 | 26828.0 | 26828.0 | N/A |

literature. It is shown that the convergent solution of the present asymptotic theory is in excellent agreement with the 3D piezoelectricity solution.

### 7.2 Laminated piezoelectric plates

The free vibration problems of laminated PVDF piezoelectric plates are studied in Table 3. The elastic, piezoelectric and dielectric properties of piezoelectric materials used in [24] are adopted in the numerical applications and are listed in Table 1. The geometry parameters are taken as $a_\alpha/2h = 10$, 50; $a_\alpha/a_\beta = 1$, 5, 1000. The dimensionless frequency $\Omega$ is defined as $\Omega = 2\omega h\sqrt{\rho/c_{11}}$. The fundamental frequencies of various laminated piezoelectric plates are presented in Table 3. It is shown that the fundamental frequencies decrease as the aspect ratio $(a_\alpha/a_\beta)$ is getting large. The fundamental frequencies of thick plates are larger than those of thin plates. For the $[0/90]_n$ antisymmetric laminates, the fundamental frequencies are getting higher and then converge to the certain values, as the number of layers $(2n)$ increases.

**Table 3.** Fundamental frequencies $\Omega$ of laminated piezoelectric plates ($\Omega = 2\omega h\sqrt{\rho/c_{11}}$)

| | $a_\alpha/2h$ | $a_\alpha/a_\beta$ | Present asymptotic solutions | | | |
|---|---|---|---|---|---|---|
| | | | $\varepsilon^0$ | $\varepsilon^2$ | $\varepsilon^4$ | $\varepsilon^6$ |
| [0/90] | 10 | 1 | 2.29323 | 2.04275 | 2.08281 | 2.07597 |
| | | 5 | 1.48689 | 1.35107 | 1.36962 | 1.36685 |
| | | 1000 | 1.47308 | 1.33842 | 1.35675 | 1.35402 |
| | 50 | 1 | 9.28374e − 2 | 9.24114e − 2 | 9.24144e − 2 | 9.24144e − 2 |
| | | 5 | 5.98471e − 2 | 5.96231e − 2 | 5.96244e − 2 | 5.96244e − 2 |
| | | 1000 | 5.92730e − 2 | 5.90512e − 2 | 5.90524e − 2 | 5.90524e − 2 |
| $[0/90]_2$ | 10 | 1 | 2.95009 | 2.21172 | 2.49804 | 2.37748 |
| | | 5 | 1.97440 | 1.51934 | 1.68483 | 1.61904 |
| | | 1000 | 1.96228 | 1.50918 | 1.67341 | 1.60812 |
| | 50 | 1 | 1.19057e − 1 | 1.17828e − 1 | 1.17848e − 1 | 1.17848e − 1 |
| | | 5 | 7.93424e − 1 | 7.86003e − 1 | 7.86113e − 1 | 7.86111e − 1 |
| | | 1000 | 7.88396e − 1 | 7.81006e − 1 | 7.81115e − 1 | 7.81113e − 1 |
| $[0/90]_3$ | 10 | 1 | 3.05711 | 2.29891 | 2.58043 | 2.46703 |
| | | 5 | 2.05232 | 1.58860 | 1.74778 | 1.68784 |
| | | 1000 | 2.04032 | 1.57860 | 1.73671 | 1.67729 |
| | 50 | 1 | 1.23303e − 1 | 1.22042e − 1 | 1.22061e − 1 | 1.22061e − 1 |
| | | 5 | 8.24488e − 2 | 8.16920e − 2 | 8.17026e − 2 | 8.17024e − 2 |
| | | 1000 | 8.19524e − 2 | 8.11992e − 2 | 8.12098e − 2 | 8.12096e − 2 |
| $[0/90]_4$ | 10 | 1 | 3.09375 | 2.33248 | 2.61160 | 2.50054 |
| | | 5 | 2.07892 | 1.61474 | 1.77140 | 1.71336 |
| | | 1000 | 2.06696 | 1.60490 | 1.76045 | 1.70295 |
| | 50 | 1 | 1.24756e − 1 | 1.23489e − 1 | 1.23508e − 1 | 1.23508e − 1 |
| | | 5 | 8.35087e − 2 | 8.27511e − 2 | 8.27616e − 2 | 8.27615e − 2 |
| | | 1000 | 8.30144e − 2 | 8.22605e − 2 | 8.22709e − 2 | 8.22708e − 2 |
| $[0/90]_5$ | 10 | 1 | 3.11057 | 2.34852 | 2.62648 | 2.51647 |
| | | 5 | 2.09112 | 1.62717 | 1.78263 | 1.72544 |
| | | 1000 | 2.07918 | 1.61741 | 1.77174 | 1.71510 |
| | 50 | 1 | 1.25422e − 1 | 1.24154e − 1 | 1.24173e − 1 | 1.24173e − 1 |
| | | 5 | 8.39948e − 2 | 8.32375e − 2 | 8.32480e − 2 | 8.32478e − 2 |
| | | 1000 | 8.35013e − 2 | 8.27479e − 2 | 8.27582e − 2 | 8.27581e − 2 |

**Table 4.** Fundamental frequencies $\Omega$ of [0/90] laminated piezoelectric doubly curved shells ($a_\alpha/a_\beta = 1$, $\Omega = 2\omega h\sqrt{\rho/c_{11}}$)

| $2h/a_\alpha$ | $R_\alpha/a_\alpha$ | $R_\beta/a_\beta$ | Present asymptotic solutions | | | |
|---|---|---|---|---|---|---|
| | | | $\varepsilon^0$ | $\varepsilon^2$ | $\varepsilon^4$ | $\varepsilon^6$ |
| 0.05 | 1 | 1 | 1.54267 | 1.51690 | 1.51886 | 1.51881 |
| | | 5 | 1.06275 | 1.04380 | 1.04508 | 1.04505 |
| | | 10 | 1.00062 | 9.81068e − 1 | 9.82379e − 1 | 9.82356e − 1 |
| | | 20 | 9.69790e − 1 | 9.49757e − 1 | 9.51090e − 1 | 9.51067e − 1 |
| | −1 | 1 | 5.10404e − 1 | 4.38925e − 1 | 4.41560e − 1 | 4.41450e − 1 |
| | | 5 | 8.01111e − 1 | 7.64092e − 1 | 7.66391e − 1 | 7.66339e − 1 |
| | | 10 | 8.59691e − 1 | 8.24824e − 1 | 8.27040e − 1 | 8.26988e − 1 |
| | | 20 | 8.89804e − 1 | 8.55856e − 1 | 8.58032e − 1 | 8.57979e − 1 |
| 0.1 | 1 | 1 | 3.57510 | 3.30712 | 3.35404 | 3.34787 |
| | | 5 | 2.87050 | 2.60783 | 2.65033 | 2.64431 |
| | | 10 | 2.78372 | 2.51569 | 2.55858 | 2.55255 |
| | | 20 | 2.74116 | 2.46981 | 2.51295 | 2.50691 |
| | −1 | 1 | 1.99345 | 1.54690 | 1.60852 | 1.59862 |
| | | 5 | 2.42121 | 2.09410 | 2.14743 | 2.13947 |
| | | 10 | 2.50470 | 2.18802 | 2.24045 | 2.23267 |
| | | 20 | 2.54820 | 2.23627 | 2.28824 | 2.28056 |
| 0.15 | 1 | 1 | 6.36820 | 5.22758 | 5.60150 | 5.49769 |
| | | 5 | 5.67078 | 4.45440 | 4.84570 | 4.72909 |
| | | 10 | 5.58373 | 4.34671 | 4.74341 | 4.62504 |
| | | 20 | 5.54085 | 4.29249 | 4.69202 | 4.57278 |
| | −1 | 1 | 4.36765 | 2.87007 | 3.30270 | 3.16680 |
| | | 5 | 4.97195 | 3.69893 | 4.09933 | 3.97587 |
| | | 10 | 5.07490 | 3.82350 | 4.22049 | 4.09862 |
| | | 20 | 5.12811 | 3.88681 | 4.28209 | 4.16099 |

### *7.3 Laminated piezoelectric doubly curved shells*

We consider the free vibration problems of laminated PVDF piezoelectric doubly curved shells in Table 4. The elastic, piezoelectric and dielectric properties of piezoelectric materials used are identical to those used in Example 7.2 [24]. The geometry parameters are taken as $2h/a_\alpha = 0.05,\ 0.1,\ 0.15$; $a_\alpha/a_\beta = 1$; $R_\alpha/a_\alpha = 1,\ -1$; $R_\beta/a_\beta = 1,\ 5,\ 10,\ 20$. The dimensionless frequency $\Omega$ is defined as $\Omega = 2\omega h\sqrt{\rho/c_{11}}$. It is shown that the natural frequencies decrease as $R_\beta/a_\beta$ is getting larger and the principle curvature radii ($R_\alpha$ and $R_\beta$) are of the same sign. On the contrary, the natural frequenies increase as $R_\beta/a_\beta$ is getting larger and the principle curvature radii are of different sign. Table 5 shows the natural frequencies of $[0/90]_n$ laminated piezoelectric shells for various values of $(m, n)$. To the best of the authors' knowledge, the 3D piezoelectricity solution for the dynamic response of doubly curved laminated piezoelectric shells is lacking in the literature. The present solution can be provided as the benchmark solution for assessing various approximate 2D shell theories. To have a more clear picture in the laminates, we present the through-thickness distributions of the modal transverse shear and normal stresses and the modal electric potential in Figs. 2–5. The geometry parameters are taken as $a_\alpha/a_\beta = 1$, $a_\alpha/2h = 10$, $R_\alpha/R_\beta = 1$, $R_\alpha/a_\alpha = 10$. It is shown that the asymptotic solution yields continuous interlaminar mechanical and electric field variables. The lateral boundary conditions of tractions and electric potential on the outer surfaces are satisfied exactly.

**Table 5.** Frequency parameters $\Omega$ of laminated piezoelectric doubly curved shells ($a_\alpha/a_\beta = 1$, $R_\alpha/R_\beta = 1$, $R_\alpha/a_\alpha = 10$, $2h/a_\alpha = 0.05$, $\Omega = 2\omega h\sqrt{\rho/c_{11}}$)

| $2h/a_\alpha$ | $(m,n)$ | Present asymptotic solutions | | | |
|---|---|---|---|---|---|
| | | $\varepsilon^0$ | $\varepsilon^2$ | $\varepsilon^4$ | $\varepsilon^6$ |
| [0/90] | (1, 1) | 5.98974e − 1 | 5.82478e − 1 | 5.83183e − 1 | 5.83150e − 1 |
| | (1, 2) | 1.60281 | 1.45613 | 1.47676 | 1.47359 |
| | (2, 1) | 1.60660 | 1.46517 | 1.48463 | 1.48167 |
| | (2, 2) | 2.29800 | 2.04752 | 2.08759 | 2.08075 |
| $[0/90]_2$ | (1, 1) | 7.58343e − 1 | 7.11113e − 1 | 7.15881e − 1 | 7.15354e − 1 |
| | (1, 2) | 2.09262 | 1.62735 | 1.79740 | 1.72880 |
| | (2, 1) | 2.09412 | 1.63297 | 1.80084 | 1.73330 |
| | (2, 2) | 2.95359 | 2.21572 | 2.50192 | 2.38142 |
| $[0/90]_3$ | (1, 1) | 7.84360e − 1 | 7.35808e − 1 | 7.40512e − 1 | 7.40013e − 1 |
| | (1, 2) | 2.17172 | 1.69653 | 1.86090 | 1.79822 |
| | (2, 1) | 2.17269 | 1.70036 | 1.86337 | 1.80128 |
| | (2, 2) | 3.06044 | 2.30269 | 2.58411 | 2.47079 |
| $[0/90]_4$ | (1, 1) | 7.93268e − 1 | 7.44497e − 1 | 7.49165e − 1 | 7.48675e − 1 |
| | (1, 2) | 2.19880 | 1.72288 | 1.88477 | 1.82404 |
| | (2, 1) | 2.19952 | 1.72578 | 1.88668 | 1.82636 |
| | (2, 2) | 3.09703 | 2.33618 | 2.61520 | 2.50422 |
| $[0/90]_5$ | (1, 1) | 7.97359e − 1 | 7.48527e − 1 | 7.53177e − 1 | 7.52691e − 1 |
| | (1, 2) | 2.21125 | 1.73548 | 1.89616 | 1.83632 |
| | (2, 1) | 2.21182 | 1.73781 | 1.89771 | 1.83818 |
| | (2, 2) | 3.11383 | 2.35219 | 2.63005 | 2.52012 |

## 8 Conclusions

The 3D asymptotic solution for the dynamic response of doubly curved laminated piezoelectric shells has been presented. The derivation is based on 3D piezoelectricity with neither kinematic nor static assumptions in advance. The method of multiple scales is used to eliminate the secular terms and make the regular asymptotic expansion feasible. The present asymptotic formulation can be reduced to that of laminated piezoelectric plates by letting $1/R_\alpha = 1/R_\beta = 0$. Applications to the benchmark problems of piezoelectric plates show that the present asymptotic solution converges rapidly and is in excellent agreement with the accurate solution available in the literature. It is noted that the frequencies of piezoelectric plates are higher than those of the non-piezoelectric ones. The piezoelectric effect on the natural frequencies corresponding to flexural modes is much more significant than on the ones corresponding to extensional modes. A parametric study for determining the frequencies and through-thickness distributions of modal variables of laminated piezoelectric plates and shells has also been presented.

In the present analysis, the electric potential is considered to be prescribed at the outer surfaces of the shell. This is due to the fact that several accuracy solutions for the present problem are available in the literature and can be used for comparisons. The case of electric displacement boundary conditions is an alternative boundary condition and is more feasible in practical applications. The problems of piezoelectric shells with electric displacement boundary conditions will be considered in forthcoming studies.

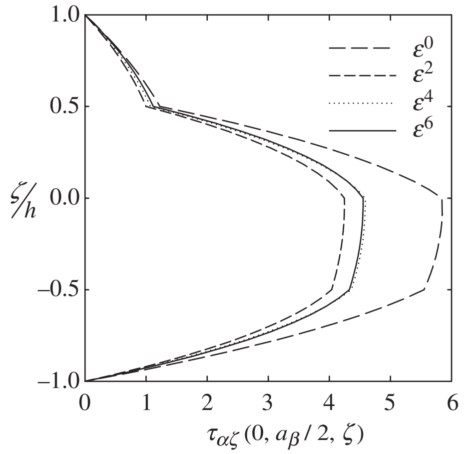


**Fig. 2.** The distributions of the transverse shear stress through the thickness of a [0/90/0/90] laminated piezoelectric shell

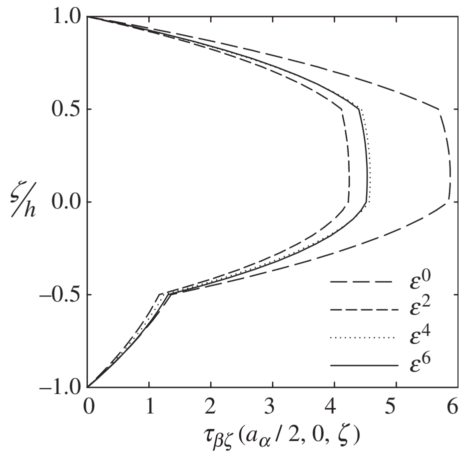


**Fig. 3.** The distributions of the transverse shear stress through the thickness of a [0/90/0/90] laminated piezoelectric shell

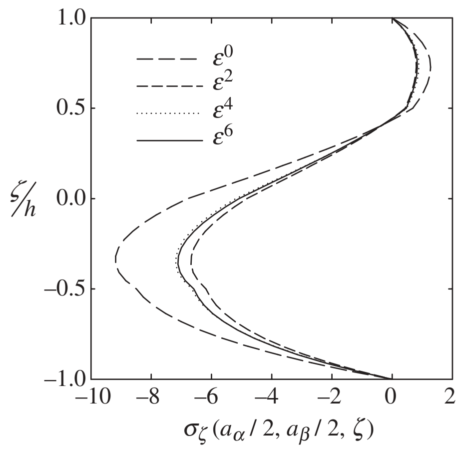


**Fig. 4.** The distributions of the transverse normal stress through the thickness of a [0/90/0/90] laminated piezoelectric shell

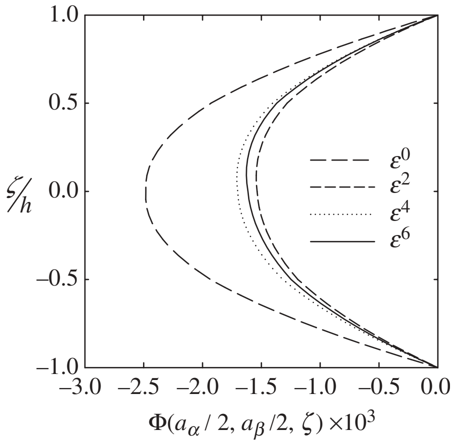


**Fig. 5.** The distributions of the electric potential through the thickness of a [0/90/0/90] laminated piezoelectric shell

## Appendix A

The relevant functions of $\hat{f}_{11}$, $\hat{f}_{21}$, $\hat{f}_{31}$, $\tilde{f}_{11}$, $\tilde{f}_{21}$, $\tilde{f}_{31}$ and $\tilde{f}_{41}$ are given by

$$\hat{f}_{11} = \int_{-1}^{z} 2\rho_1\omega\left(U_0 - z\tilde{m}\tilde{W}_0\right) dz, \tag{A.1}$$

$$\hat{f}_{21} = \int_{-1}^{z} 2\rho_1\omega\left(V_0 - z\tilde{n}W_0\right) dz, \tag{A.2}$$

$$\hat{f}_{31} = -\int_{-1}^{z} \left[\tilde{m}\hat{f}_{11} + \tilde{n}\hat{f}_{21} + 2\rho_2\omega W_0\right] dz, \tag{A.3}$$

$$\begin{aligned}\tilde{f}_{11} = &-\left(\frac{z}{R_x} + \frac{z}{R_y} + \frac{h}{R}\frac{z^2}{R_xR_y}\right)\tau_{xz}^{(0)} + \int_{-1}^{z}\left[\left(\tilde{m}^2\frac{\gamma_\beta\tilde{c}_{11}}{\gamma_\alpha Q} + \tilde{n}^2\frac{\gamma_\alpha\tilde{c}_{66}}{\gamma_\beta Q}\right)\varphi_{11}\right.\\ &+\tilde{m}\tilde{n}\left(\frac{\tilde{c}_{12}+\tilde{c}_{66}}{Q}\right)\varphi_{21} - \tilde{m}\left(\frac{\gamma_\beta\tilde{c}_{11}}{\gamma_\alpha QR_x} + \frac{\tilde{c}_{12}}{QR_y}\right)\varphi_{31} - \tilde{m}\gamma_\beta\frac{e}{Q}\left(\frac{e_{33}c_{13} - e_{31}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)\varphi_{41}\\ &\left.-\tilde{m}\gamma_\beta\left(\frac{e_{33}e_{31} + \eta_{33}c_{13}}{e_{33}^2 + \eta_{33}c_{33}}\right)\sigma_z^{(0)} - \left(\frac{1}{R_x} + \frac{hz}{RR_xR_y}\right)\tau_{xz}^{(0)} - \omega^2\rho_1\varphi_{11}\right] dz,\end{aligned} \tag{A.4}$$

$$\begin{aligned}\tilde{f}_{21} = &-\left(\frac{z}{R_x} + \frac{z}{R_y} + \frac{h}{R}\frac{z^2}{R_xR_y}\right)\tau_{yz}^{(0)} + \int_{-1}^{z}\left[\tilde{m}\tilde{n}\left(\frac{\tilde{c}_{66}+\tilde{c}_{21}}{Q}\right)\varphi_{11} + \left(\tilde{m}^2\frac{\gamma_\beta\tilde{c}_{66}}{\gamma_\alpha Q} + \tilde{n}^2\frac{\gamma_\alpha\tilde{c}_{22}}{\gamma_\beta Q}\right)\varphi_{21}\right.\\ &-\tilde{n}\left(\frac{\tilde{c}_{21}}{QR_x} + \frac{\gamma_\alpha\tilde{c}_{22}}{\gamma_\beta QR_y}\right)\varphi_{31} - \tilde{n}\gamma_\alpha\frac{e}{Q}\left(\frac{e_{33}c_{23} - e_{32}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)\varphi_{41}\\ &\left.-\tilde{n}\gamma_\alpha\left(\frac{e_{33}e_{32} + \eta_{33}c_{23}}{e_{33}^2 + \eta_{33}c_{33}}\right)\sigma_z^{(0)} - \left(\frac{1}{R_y} + \frac{hz}{RR_xR_y}\right)\tau_{yz}^{(0)} - \omega^2\rho_1\varphi_{21}\right]dz,\end{aligned} \tag{A.5}$$

$$\tilde{f}_{31} = \left(\frac{z}{R_x} + \frac{z}{R_y} + \frac{h}{R}\frac{z^2}{R_x R_y}\right)\sigma_z^{(0)} + \int_{-1}^{z} \left\{ \left(\tilde{m}\frac{\gamma_\beta \tilde{c}_{11}}{\gamma_\alpha Q R_x} + \frac{\tilde{c}_{21}}{Q R_y}\right)\varphi_{11} + \tilde{n}\left(\frac{\tilde{c}_{12}}{Q R_x} + \frac{\gamma_\alpha \tilde{c}_{22}}{\gamma_\beta Q R_y}\right)\varphi_{21} \right.$$

$$- \left[\frac{\gamma_\beta \tilde{c}_{11}}{\gamma_\alpha Q R_x^2} + \frac{(\tilde{c}_{12} + \tilde{c}_{21})}{Q R_x R_y} + \frac{\gamma_\alpha \tilde{c}_{22}}{\gamma_\beta Q R_y^2}\right]\varphi_{31}$$

$$- \left[\frac{e\gamma_\beta}{Q R_x}\left(\frac{e_{33}c_{13} - e_{31}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right) + \frac{e\gamma_\alpha}{Q R_y}\left(\frac{e_{33}c_{23} - e_{32}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)\right]\varphi_{41}$$

$$- \left(\frac{\tilde{m}z}{R_y}\right)\tau_{xz}^{(0)} - \left(\frac{\tilde{n}z}{R_x}\right)\tau_{yz}^{(0)} - \left[\frac{\gamma_\beta}{R_x}\left(\frac{e_{33}e_{31} + \eta_{33}c_{13}}{e_{33}^2 + \eta_{33}c_{33}}\right) + \frac{\gamma_\alpha}{R_y}\left(\frac{e_{33}e_{32} + \eta_{33}c_{23}}{e_{33}^2 + \eta_{33}c_{33}}\right)\right]\sigma_z^{(0)}$$

$$\left. - \tilde{m}\hat{f}_{11} - \tilde{n}\tilde{f}_{21} + \omega^2 \rho_2 \varphi_{31} \right\} dz, \tag{A.6}$$

$$\tilde{f}_{41} = \int_{-1}^{z} \left\{ -\frac{\tilde{m}e}{\gamma_\alpha Q}\left(\frac{e_{33}c_{13} - e_{31}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)\varphi_{11} - \frac{\tilde{n}e}{\gamma_\beta Q}\left(\frac{e_{33}c_{23} - e_{32}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)\varphi_{21} \right.$$

$$+ \left[\frac{e}{\gamma_\alpha R_x Q}\left(\frac{e_{33}c_{13} - e_{31}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right) + \frac{e}{\gamma_\beta R_y Q}\left(\frac{e_{33}c_{23} - e_{32}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)\right]\varphi_{31}$$

$$\left. + \left[\frac{c_{33}e^2}{(e_{33}^2 + \eta_{33}c_{33})Q}\right]\varphi_{41} - \left(\frac{e_{33}e}{e_{33}^2 + \eta_{33}c_{33}}\right)\sigma_z^{(0)} \right\} dz. \tag{A.7}$$

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## References


[1] Heyliger, P., Brooks, S.: Free vibration of piezoelectric laminates in cylindrical bending. Int. J. Solids Struct. **32**, 2945–2960 (1995).

[2] Hussein, M., Heyliger, P.: Three-dimensional vibrations of layered piezoelectric cylinders. J. Engng. Mech. ASCE **124**, 1294–1298 (1998).

[3] Kharouf, N., Heyliger, P. R.: Axisymmetric free vibrations of homogeneous and laminated piezoelectric cylinders. J. Sound Vibr. **174**, 539–561 (1994).

[4] Ding, H. J., Chen, B., Liang, J.: General solutions for coupled equations for piezoelectric media. Int. J. Solids Struct. **33**, 2283–2298 (1996).

[5] Ding, H. J., Xu, R. Q., Chen, W. Q.: Free vibration of transversely isotropic piezoelectric circular cylindrical panels. Int. J. Mech. Sci. **44**, 191–206 (2002).

[6] Sharma, J. N., Pathania, V.: Three-dimensional vibration analysis of a transversely isotropic piezoelectric cylindrical panel. Acta Mech. **166**, 119–129 (2003).

[7] Tiersten, H. F.: Linear piezoelectric plate vibrations. New York: Plenum Press 1969.

[8] Tiersten, H. F.: Electroelastic interactions and the piezoelectric equations. J. Acoust. Soc. Am. **70**, 1567–1576 (1981).

[9] Krommer, M.: On the influence of pyroelectricity upon thermally induced vibrations of piezothermoelastic plates. Acta Mech. **171**, 59–73 (2004).
[10] Krommer, M.: The significance of non-local constitutive relations for constitutive relations for composite thin plates including piezoelastic layers with prescribed electric charge. Smart Mater. Struct. **12**, 318–330 (2003).
[11] Rao, S. S., Sunar, M.: Piezoelectricity and its use in disturbance sensing and control of flexible structures: A survey. Appl. Mech. Rev. **47**, 113–123 (1994).
[12] Chee, C. Y. K., Tong, L., Steven G. P.: A review on the modeling of piezoelectric sensors and actuators incorporated in intelligent structures. J. Intell. Mater. Sys. Struct. **9**, 3–19 (1998).
[13] Saravanos, D. A., Heyliger, P. R.: Mechanics and computational models for laminated piezoelectric beams, plates and shells. Appl. Mech. Rev. **52**, 305–319 (1999).
[14] Gopinathan, S. V., Varadan, V. V., Varadan, V. K.: A review and critique of theories for piezoelectric laminates. Smart Mater. Struct. **9**, 24–48 (2000).
[15] Huang, N. N., Tauchert, T. R.: Thermal stresses in doubly-curved cross-ply laminates. Int. J. Solids Struct. **29**, 991–1000 (1992).
[16] Fan, J., Zhang, J.: Analytical solutions for thick doubly curved laminated shells. J. Engng. Mech., ASCE **118**, 1338–1356 (1992).
[17] Bhimaraddi, A.: Three-dimensional elasticity solution for static response of orthotropic doubly curved shallow shells on rectangular platform. Compos. Struct. **24**, 67–77 (1993).
[18] Wu, C. P., Tarn, J. Q., Chi, S. M.: Three-dimensional analysis of doubly curved laminated shells. J. Engng. Mech., ASCE **122**, 391–401 (1996).
[19] Wu, C. P., Tarn, J. Q., Chi, S. M.: An asymptotic theory for dynamic response of doubly curved laminated shells. Int. J. Solids Struct. **33**, 3813–3841 (1996).
[20] Wu, C. P., Chiu, S. J.: Thermoelastic buckling of laminated composite conical shells. J. Therm. Stresses **24**, 881–901 (2001).
[21] Wu, C. P., Chiu, S. J.: Thermally induced dynamic instability of laminated composite conical shells. Int. J. Solids Struct. **39**, 3001–3021 (2002).
[22] Nayfeh, A. H.: Introduction to perturbation techniques. New York: Wiley 1993.
[23] Heyliger, P.: Exact free-vibration analysis of laminated plates with embedded piezoelectric layers. J. Acoust. Soc. Am. **98**, 1547–1557 (1995).
[24] Cheng, Z. Q., Lim, C. W., Kitipornchai, S.: Three-dimensional asymptotic approach to inhomogeneous and laminated piezoelectric plates. Int. J. Solids Struct. **37**, 3153–3175 (2000).

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