An Asymptotic Theory for Vibrations of Inhomogeneous/Laminated Piezoelectric Plates

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***Abstract*—An asymptotic theory for the vibration analysis of inhomogeneous monoclinic piezoelectric plates is developed by using small parameter expansion. The theory includes the important special case of a laminated plate in which each layer is homogeneous and orthotropic, but distinct from the other layers by having a different material or a different orientation. A hierarchy of two-dimensional equations of the same homogeneous operator for each order is reduced from the three-dimensional framework of linear piezoelectricity. The elasticity version of the leading-order equation is the same as that of the classical Kirchhoff inhomogeneous plate theory and, therefore, is easily solvable. By contrast, it is not straightforward to find solutions of the higher-order equations. The solvability condition is thus established for this purpose, by which higher-order frequency parameters are derived. The present theoretical formulation is examined by comparing the present asymptotic results with an exact three-dimensional solution for a piezoelectric bimorph strip, and excellent agreement is reached. Some new results are presented.**

I. Introduction

Homogeneous or symmetric material properties in the thickness direction of piezoelectric plates simplify the analysis of piezoelectric ceramic actuators, sensors, and resonators. However, these functional devices can be excited only into extensional motions by an electric field parallel to the poling direction. In many applications, such as in the micro-electromechanical systems (MEMS), relatively large flexural motions are excited in the piezoelectric ceramic transducers with asymmetric material properties with respect to the midplane of a plate. The inhomogeneity of material properties across the thickness of piezoelectric plates introduces additional difficulties and thus draws additional attention. Recently, Lee and Yu [1] and Lee *et al.* [2], [3] derived a system of two-dimensional equations for vibrations of electroded piezoelectric crystal plates with thickness-graded material properties. Chang and Tung [4] presented a theoretical study on the vibrations of asymmetrically laminated piezoelectric plates (also see more references cited in [1]–[4]).

The series expansion method has been used in [1]–[3], in which physical quantities are expanded into power series of thickness coordinate, and a two-dimensional theory then is established. In addition, asymptotic expansion is an efficient alternative. This method has been used to develop new theories and to find higher-order asymptotic solutions of laminated piezoelectric plate problems [5]–[11].

The aforementioned asymptotic theories have been established only for the static analysis of electromechanical systems. The purpose of this work is to develop a new method for the vibration analysis of piezoelectric plates. The transfer matrix formulation is presented in combination with the asymptotic expansion to derive a hierarchy of two-dimensional equations from the three-dimensional framework of linear piezoelectricity. Instead of using multiple time scales expansion for an elastic plate [12], the asymptotic formulation is refined by expanding the frequency parameter in a rather simple way. The theory is examined by applying it to a simply supported rectangular piezoelectric bimorph to find any higher-order resonance frequencies with an established solvability condition. The present theory applies to any plates with inhomogeneous material properties in the thickness direction, though only a bimorph is illustrated in examples.

II. State-Space Equations

Consider an undeformed plate of uniform thickness h in a rectangular Cartesian coordinate system $\{x_i\}$ ($i = 1, 2, 3$), and the bottom plane of the plate coincides with $x_3 = 0$. The plate is composed of an inhomogeneous monoclinic piezoelectric material. Accordingly, the theory includes the important special case of a laminated cross-ply or angle-ply plate. Hereafter, a comma followed by a subscript i denotes the partial derivative with respect to x_i, and a repeated index implies summation over the range of the index with Latin indices ranging from 1 to 3 and Greek indices from 1 to 2.

In the absence of body force and electric charge density, the linear governing equations for steady-state deformations with time-harmonic dependence $\exp(i\omega t)$ and Gauss' law of electrostatics are [13]:

$$\tau_{ij,j} + \rho\omega^2 u_i = 0, \quad D_{i,i} = 0, \tag{1}$$

where ρ is the mass density, ω is an angular frequency, τ_{ij} is the stress tensor component, D_i is the electric displacement component, and u_k is the mechanical displacement. The time-harmonic factor $\exp(i\omega t)$ has been omitted, and

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each physical quantity refers to its spatial part. For a monoclinic piezoelectric material with reflectional symmetry in planes parallel to the surfaces of the plate, including the special classes of PZT and PVDF, the constitutive relations are:

$$\begin{aligned}
\tau_{\alpha\beta} &= c_{\alpha\beta\omega\rho}S_{\omega\rho} + c_{\alpha\beta 33}S_{33} - e_{3\alpha\beta}E_3,\\
\tau_{\alpha 3} &= 2c_{\alpha 3\omega 3}S_{\omega 3} - e_{\omega\alpha 3}E_\omega,\\
\tau_{33} &= c_{33\omega\rho}S_{\omega\rho} + c_{3333}S_{33} - e_{333}E_3,\\
D_\alpha &= 2e_{\alpha\omega 3}S_{\omega 3} + \varepsilon_{\alpha\omega}E_\omega,\\
D_3 &= e_{3\omega\rho}S_{\omega\rho} + e_{333}S_{33} + \varepsilon_{33}E_3.
\end{aligned} \tag{2}$$

Here $\mathbf{c}$ is the fourth-order elasticity tensor, $\mathbf{e}$ is the third-order piezoelectric tensor, and $\boldsymbol{\varepsilon}$ is the second-order dielectric tensor. These material moduli exhibit the following symmetries:

$$c_{ijkl} = c_{jikl} = c_{klij}, \quad e_{kij} = e_{kji}, \quad \varepsilon_{ik} = \varepsilon_{ki}. \tag{3}$$

The inhomogeneity of the material is only in the plate thickness direction, i.e., the material properties are functions of x_3. For a laminated plate comprised of different homogeneous materials, the material moduli are piecewise constant functions of x_3. The components of infinitesimal strain tensor S_{kl} and electric field E_k are related to u_k and the electric potential φ through the relations:

$$S_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \quad E_k = -\varphi_{,k}. \tag{4}$$

Eq. (1), (2), and (4) are rewritten in the following state-space equation:

$$\partial_z \begin{bmatrix}\mathbf{F}\\ \mathbf{G}\end{bmatrix} = \chi \begin{bmatrix}\mathbf{0} & \mathbf{A}\\ \mathbf{B} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{F}\\ \mathbf{G}\end{bmatrix} - \chi\rho\Psi \begin{bmatrix}\mathbf{0} & \tilde{\mathbf{A}}\\ \tilde{\mathbf{B}} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{F}\\ \mathbf{G}\end{bmatrix}, \tag{5}$$

where $\Psi = \omega^2$, $z = x_3/\chi$, $\partial_z \equiv \partial/\partial z$, $\chi = h/a$, a is a typical in-plane dimension. The thickness coordinate x_3 has been scaled to z extending from 0 to a. The state-space functions are:

$$\mathbf{F} = \begin{bmatrix}u_1\\ u_2\\ \tau_{33}\\ D_3\end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix}\tau_{13}\\ \tau_{23}\\ u_3\\ \varphi\end{bmatrix}. \tag{6}$$

In the case of a laminated plate in perfect bonding and without internal electrodes, $\mathbf{F}$ and $\mathbf{G}$ are continuous across each interlaminar interface. The only nonzero elements of the 4×4 constant matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are $\tilde{A}_{33} = \tilde{B}_{11} = \tilde{B}_{22} = 1$. The 4×4 operator matrices $\mathbf{A}$ and $\mathbf{B}$ contain the in-plane differential operator $\partial_a \equiv \partial/\partial x_a$ and depend on z only through the material moduli:

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix}\mathbf{I} & -\mathbf{J}_\beta\partial_\beta\\ -\mathbf{J}_\beta^{\mathrm{T}}\partial_\beta & \mathbf{K}_{\beta\rho}\partial_\beta\partial_\rho\end{bmatrix},\\
\mathbf{B} &= \begin{bmatrix}-\mathbf{L}_{\beta\rho}\partial_\beta\partial_\rho & -\mathbf{M}_\beta\partial_\beta\\ -\mathbf{M}_\beta^{\mathrm{T}}\partial_\beta & \mathbf{N}\end{bmatrix}.
\end{aligned} \tag{7}$$

The elements of the above submatrices can be scalars, vectors, and tensors; and they are expressed in terms of the material moduli as:

$$\begin{aligned}
\mathbf{I} &= (I^{\omega\alpha}) = \begin{bmatrix}c_{1313} & c_{1323}\\ c_{1323} & c_{2323}\end{bmatrix}^{-1},\\
\mathbf{N} &= (N^{\alpha\omega}) = \begin{bmatrix}c_{3333} & e_{333}\\ e_{333} & -\varepsilon_{33}\end{bmatrix}^{-1},\\
\left[J_\beta^{\omega 1} \; J_\beta^{\omega 2}\right] &= \left[\delta_{\omega\beta} \; I^{\omega\alpha}e_{\beta\alpha 3}\right],\\
\left[M_\beta^{\alpha 1} \; M_\beta^{\alpha 2}\right] &= \left[c_{\alpha\beta 33} \; e_{3\alpha\beta}\right]\mathbf{N},\\
K_{\beta\rho}^{11} &= K_{\beta\rho}^{12} = K_{\beta\rho}^{21} = 0,\\
K_{\beta\rho}^{22} &= J_\beta^{\omega 2}e_{\rho\omega 3} + \varepsilon_{\beta\rho},\\
L_{\beta\rho}^{\alpha\omega} &= c_{\alpha\beta\omega\rho} - M_\beta^{\alpha 1}c_{33\omega\rho} - M_\beta^{\alpha 2}e_{3\omega\rho},
\end{aligned} \tag{8}$$

where $\delta_{\omega\beta}$ is the Kronecker delta symbol. Superscripts are used to denote the location (i.e., row and column) of matrix elements, and the Greek subscripts are for the usual tensor notation. The in-plane stresses and in-plane electric displacements, which may be discontinuous in x_3, are given by:

$$\begin{aligned}
\tau_{\alpha\beta} &= L_{\beta\rho}^{\alpha\omega}u_{\omega,\rho} + M_\beta^{\alpha 1}\tau_{33} + M_\beta^{\alpha 2}D_3,\\
D_\rho &= J_\rho^{\alpha 2}\tau_{\alpha 3} - K_{\beta\rho}^{22}\varphi_{,\beta}.
\end{aligned} \tag{9}$$

The static version of (5) has been given in [5]–[9]. The matrix on the right-hand side of (5) also is consistent with the 8×8 matrix [14] widely used in the theory of surface acoustic waves.

III. Asymptotic Approach

The two bounding surfaces of the plate are covered with very thin conducting electrodes. For simplicity, the thickness of each electrode is neglected, and it is modeled as a mathematical surface with a specified electric potential. The state-space functions $\mathbf{F}$ and $\mathbf{G}$ and the frequency parameter Ψ are expanded in terms of the small parameter χ as:

$$\begin{bmatrix}\mathbf{F}\\ \mathbf{G}\end{bmatrix} = \sum_{n=0}^{\infty}\chi^{2n}\begin{bmatrix}\chi\mathbf{f}^{(n)}\\ \mathbf{g}^{(n)}\end{bmatrix}, \quad \Psi = \sum_{n=0}^{\infty}\chi^{2n}\psi^{(n)}, \tag{10}$$

then substituted into (5) to give the following recursion relations:

$$\begin{aligned}
&\partial_z\mathbf{g}^{(0)} = \mathbf{0},\\
&\partial_z\mathbf{g}^{(n+1)} = \mathbf{B}\mathbf{f}^{(n)} - \sum_{k=0}^{n}\rho\psi^{(k)}\tilde{\mathbf{B}}\mathbf{f}^{(n-k)},\\
&\partial_z\mathbf{f}^{(0)} = \mathbf{A}\mathbf{g}^{(0)},\\
&\partial_z\mathbf{f}^{(n+1)} = \mathbf{A}\mathbf{g}^{(n+1)} - \sum_{k=0}^{n}\chi^{-2}\rho\psi^{(k)}\tilde{\mathbf{A}}\mathbf{g}^{(n-k)}, \quad (n \geq 0).
\end{aligned} \tag{11}$$

Note that the above expansion terms only contain odd powers of the small parameter χ for $\mathbf{F}$ and even powers for $\mathbf{G}$. This is because all of the complemented expansion terms, even powers for $\mathbf{F}$ and odd powers for $\mathbf{G}$, have a trivial contribution and thus are omitted.

As far as the free vibration problem is concerned in this paper, the condition of zero tractions and electric potentials on the two bounding surfaces may be expressed by:

$$g_\alpha^{(n)}(0) = \tau_{\alpha 3}^{(n)}(0) = 0, \quad f_3^{(n)}(0) = \tau_{33}^{(n)}(0) = 0, \quad g_4^{(n)}(0) = \varphi^{(n)}(0) = 0, \tag{12}$$

$$g_\alpha^{(n)}(a) = \tau_{\alpha 3}^{(n)}(a) = 0, \quad f_3^{(n)}(a) = \tau_{33}^{(n)}(a) = 0, \quad g_4^{(n)}(a) = \varphi^{(n)}(a) = 0. \tag{13}$$

Denoting:

$$Q\Pi \equiv \int_0^z \Pi \mathrm{d}z, \quad \overline{Q}\Pi \equiv \int_0^a \Pi \mathrm{d}z, \tag{14}$$

$$U_i^{(n)} \equiv u_i^{(n)}(x_\rho, 0), \quad D_0^{(n)} \equiv D_3^{(n)}(x_\rho, 0^+), \tag{15}$$

where Q and $\overline{Q}$ are the integral operators and Π symbolizes the associated integrand; a solution of different orders can be obtained by integrating the differential equations (11) with respect to z, using (12), as:

$$\begin{aligned}
\mathbf{g}^{(0)} &= \begin{bmatrix} 0 \\ 0 \\ U_3^{(0)} \\ 0 \end{bmatrix}, \\
\mathbf{g}^{(n+1)} &= \begin{bmatrix} 0 \\ 0 \\ U_3^{(n+1)} \\ 0 \end{bmatrix} + Q\mathbf{B}\mathbf{f}^{(n)} - \sum_{k=0}^{n} \psi^{(k)} \begin{bmatrix} Q\rho f_1^{(n-k)} \\ Q\rho f_2^{(n-k)} \\ 0 \\ 0 \end{bmatrix}, \\
\mathbf{f}^{(0)} &= \begin{bmatrix} U_1^{(0)} - zU_{3,1}^{(0)} \\ U_2^{(0)} - zU_{3,2}^{(0)} \\ 0 \\ D_0^{(0)} \end{bmatrix}, \\
\mathbf{f}^{(n+1)} &= \begin{bmatrix} U_1^{(n+1)} \\ U_2^{(n+1)} \\ 0 \\ D_0^{(n+1)} \end{bmatrix} + Q\mathbf{A}\mathbf{g}^{(n+1)} \\
&\quad - \sum_{k=0}^{n} \chi^{-2} \psi^{(k)} \begin{bmatrix} 0 \\ 0 \\ Q\rho g_3^{(n-k)} \\ 0 \end{bmatrix}, \quad (n \geq 0).
\end{aligned} \tag{16}$$

$\mathbf{f}^{(n)}$ can be alternatively written as:

$$\mathbf{f}^{(n)} = \mathbf{X}^{(n)} + \mathbf{H}^{(n)}, \tag{17}$$

where

$$\begin{aligned}
\mathbf{X}^{(n)} &= \begin{bmatrix} U_1^{(n)} - zU_{3,1}^{(n)} \\ U_2^{(n)} - zU_{3,2}^{(n)} \\ 0 \\ D_0^{(n)} \end{bmatrix}, \\
\mathbf{H}^{(0)} &= \mathbf{0}, \\
\mathbf{H}^{(n+1)} &= Q\mathbf{A}Q\mathbf{B}\mathbf{f}^{(n)} \\
&\quad - \sum_{k=0}^{n} \psi^{(k)} \left\{ Q\mathbf{A} \begin{bmatrix} Q\rho f_1^{(n-k)} \\ Q\rho f_2^{(n-k)} \\ 0 \\ 0 \end{bmatrix} + \chi^{-2} \begin{bmatrix} 0 \\ 0 \\ Q\rho g_3^{(n-k)} \\ 0 \end{bmatrix} \right\}.
\end{aligned} \tag{18}$$

The conditions (13) for zero tractions and electric potential on the top surface of the plate may be expressed through (16) as:

$$\begin{aligned}
\overline{Q}B_{\alpha L} f_L^{(n)} &= \sum_{k=0}^{n} \psi^{(k)} \overline{Q}\rho f_\alpha^{(n-k)}, \\
\overline{Q}zB_{\alpha L} f_{L,\alpha}^{(n)} &= \sum_{k=0}^{n} \psi^{(k)} \overline{Q}\rho \left[z f_{\alpha,\alpha}^{(n-k)} + \chi^{-2} g_3^{(n-k)} \right], \\
\overline{Q}B_{4L} f_L^{(n)} &= 0,
\end{aligned} \tag{19}$$

where an upper case subscript L takes the values from 1 to 4, and the usual summation convention applies to L. Furthermore, it follows by using (17) that:

$$\left[\tilde{\mathbf{R}} - \psi^{(0)}\mathbf{m}\right] \tilde{\mathbf{X}}^{(n)} = \begin{cases} \mathbf{0}, & (n = 0) \\ \psi^{(n)}\mathbf{m}\tilde{\mathbf{X}}^{(0)} + \mathbf{Z}^{(n)}, & (n \geq 1) \end{cases}, \tag{20}$$

where

$$\tilde{\mathbf{X}}^{(n)} = \left[U_1^{(n)} \; U_2^{(n)} \; U_3^{(n)} \; D_0^{(n)} \right]^{\mathrm{T}}, \tag{21}$$

$$\begin{aligned}
\mathbf{Z}^{(n)} &= -\mathbf{R}\mathbf{H}^{(n)} + \sum_{k=1}^{n-1} \psi^{(k)} \mathbf{m} \tilde{\mathbf{X}}^{(n-k)} \\
&\quad + \sum_{k=0}^{n-1} \psi^{(k)} \overline{Q}\rho \begin{bmatrix} H_1^{(n-k)} \\ H_2^{(n-k)} \\ zH_{\alpha,\alpha}^{(n-k)} + \chi^{-2} QB_{3L} f_L^{(n-k-1)} \\ 0 \end{bmatrix},
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\tilde{R}_{\alpha\omega} = R_{\alpha\omega} = -\overline{Q}L_{\beta\rho}^{\alpha\omega}\partial_\beta\partial_\rho, \; \tilde{R}_{\alpha 3} = \overline{Q}zL_{\beta\rho}^{\alpha\omega}\partial_\beta\partial_\omega\partial_\rho, \\
&R_{\alpha 3} = -\overline{Q}M_\beta^{\alpha 1}\partial_\beta, \; \tilde{R}_{\alpha 4} = R_{\alpha 4} = -\overline{Q}M_\beta^{\alpha 2}\partial_\beta, \\
&\tilde{R}_{3\omega} = R_{3\omega} = -\overline{Q}zL_{\beta\rho}^{\alpha\omega}\partial_\alpha\partial_\beta\partial_\rho, \; \tilde{R}_{33} = \overline{Q}z^2L_{\beta\rho}^{\alpha\omega}\partial_\alpha\partial_\beta\partial_\omega\partial_\rho, \\
&R_{33} = -\overline{Q}zM_\beta^{\alpha 1}\partial_\alpha\partial_\beta, \; \tilde{R}_{34} = R_{34} = -\overline{Q}zM_\beta^{\alpha 2}\partial_\alpha\partial_\beta, \\
&\tilde{R}_{4\omega} = R_{4\omega} = \overline{Q}M_\beta^{\omega 2}\partial_\beta, \; \tilde{R}_{43} = -\overline{Q}zM_\beta^{\omega 2}\partial_\beta\partial_\omega, \\
&R_{43} = -\overline{Q}N^{21}, \; \tilde{R}_{44} = R_{44} = -\overline{Q}N^{22},
\end{aligned} \tag{23}$$

$$\begin{aligned}
&m_{11} = m_{22} = \overline{Q}\rho, \; m_{\alpha 3} = -m_{3\alpha} = -\overline{Q}z\rho\partial_\alpha, \\
&m_{33} = \chi^{-2}\overline{Q}\rho - \overline{Q}z^2\rho\partial_\beta\partial_\beta, \\
&m_{12} = m_{21} = m_{i4} = m_{4i} = m_{44} = 0.
\end{aligned} \tag{24}$$

The key equation in the asymptotic theory is (20), from which the frequency parameter and the modal vector (21) of each order are to be solved with specified edge conditions.

IV. Solvability Condition

It can be recognized that the operator matrix $[\tilde{\mathbf{R}} - \psi^{(0)}\mathbf{m}]$ on the left-hand side of (20), when degenerated from piezoelectricity to elasticity, is identical with that of the classical Kirchhoff theory for the vibrations of a thin plate made of a monoclinic inhomogeneous/laminated elastic material [15]. The leading-order frequency parameter $\psi^{(0)}$ and its corresponding modal vector $\tilde{\mathbf{X}}^{(0)}$ can be determined by standard methods. However, a method for solving the higher-order equation (20) $(n \geq 1)$ is not straightforward. The operator matrix $[\tilde{\mathbf{R}} - \psi^{(0)}\mathbf{m}]$ for the higher-order equation $(n \geq 1)$ is same as that of the leading-order equation $(n = 0)$, whereas terms $(n \geq 1)$ on the right-hand side of (20) do not vanish. The nonhomogeneous terms $(n \geq 1)$ in (20) involve lower-order solutions that may generate secular terms in the higher-order equations. To ensure a uniformly valid expansion of the asymptotic approach, it is necessary to study the solvability condition under which the higher-order equations possess solutions that are bounded and free from secular terms.

Premultiplying $[\tilde{\mathbf{R}} - \psi^{(0)}\mathbf{m}]\tilde{\mathbf{X}}^{(n)}$ by $\tilde{\mathbf{X}}^{(0)^{\mathrm{T}}}$ and integrating it over the domain of the plate surface Ω leads to, after lengthy but straightforward manipulation:

$$
\begin{aligned}
&\int_\Omega \tilde{\mathbf{X}}^{(0)^{\mathrm{T}}}[\tilde{\mathbf{R}} - \psi^{(0)}\mathbf{m}]\tilde{\mathbf{X}}^{(n)}\mathrm{d}\Omega = \int_\Omega \tilde{\mathbf{X}}^{(n)^{\mathrm{T}}}[\tilde{\mathbf{R}} - \psi^{(0)}\mathbf{m}]\tilde{\mathbf{X}}^{(0)}\mathrm{d}\Omega \\
&- \int_\Gamma U_\alpha^{(0)}\mathcal{N}_{\alpha\beta}^{(n)}n_\beta \mathrm{d}\Gamma + \int_\Gamma U_\alpha^{(n)}\mathcal{N}_{\alpha\beta}^{(0)}n_\beta \mathrm{d}\Gamma \\
&+ \int_\Gamma U_{3,\alpha}^{(0)}\mathcal{M}_{\alpha\beta}^{(n)}n_\beta \mathrm{d}\Gamma - \int_\Gamma U_{3,\alpha}^{(n)}\mathcal{M}_{\alpha\beta}^{(0)}n_\beta \mathrm{d}\Gamma \\
&- \int_\Gamma U_3^{(0)}\left[\mathcal{M}_{\alpha\beta,\alpha}^{(n)} + \psi^{(0)}\overline{Q}z\rho U_\beta^{(n)} - \psi^{(0)}\overline{Q}z^2\rho U_{3,\beta}^{(n)}\right]n_\beta \mathrm{d}\Gamma \\
&+ \int_\Gamma U_3^{(n)}\left[\mathcal{M}_{\alpha\beta,\alpha}^{(0)} + \psi^{(0)}\overline{Q}z\rho U_\beta^{(0)} - \psi^{(0)}\overline{Q}z^2\rho U_{3,\beta}^{(0)}\right]n_\beta \mathrm{d}\Gamma,
\end{aligned} \tag{25}
$$

where Γ denotes the boundary of the domain Ω, n_β is an outward normal vector to the boundary, and:

$$
\begin{aligned}
&\left[\mathcal{N}_{\alpha\beta}^{(n)},\ \mathcal{M}_{\alpha\beta}^{(n)}\right] = \\
&\qquad \overline{Q}[1,\ z]\left[L_{\beta\rho}^{\alpha\omega}U_{\omega,\rho}^{(n)} - zL_{\beta\rho}^{\alpha\omega}U_{3,\omega\rho}^{(n)} + M_\beta^{\alpha 2}D_0^{(n)}\right].
\end{aligned} \tag{26}
$$

Eq. (25) is identically satisfied for $n = 0$. The first integral on the right-hand side of (25) vanishes due to (20) $(n = 0)$. If the remaining boundary integrals on the right-hand side of (25) vanish (i.e., specifying on the plate edge Γ):

$$
\begin{aligned}
&U_\alpha^{(n)} = 0,\ \text{or}\ \mathcal{N}_{\alpha\beta}^{(n)}n_\beta = 0, \\
&U_{3,\alpha}^{(n)} = 0,\ \text{or}\ \mathcal{M}_{\alpha\beta}^{(n)}n_\beta = 0, \\
&U_3^{(n)} = 0,\ \text{or} \\
&\quad \left[\mathcal{M}_{\alpha\beta,\alpha}^{(n)} + \psi^{(0)}\overline{Q}z\rho U_\beta^{(n)} - \psi^{(0)}\overline{Q}z^2\rho U_{3,\beta}^{(n)}\right]n_\beta = 0, \\
&\quad (n \geq 0),
\end{aligned} \tag{27}
$$

then by substituting (20) $(n \geq 1)$ into (25), it follows that the higher-order frequency parameter is determined by:

$$
\psi^{(n)} = -\frac{\int_\Omega \tilde{\mathbf{X}}^{(0)^{\mathrm{T}}}\mathbf{Z}^{(n)}\mathrm{d}\Omega}{\int_\Omega \tilde{\mathbf{X}}^{(0)\mathrm{T}}\mathbf{m}\tilde{\mathbf{X}}^{(0)}\mathrm{d}\Omega}, \quad (n \geq 1). \tag{28}
$$

With this, (20) is solvable for higher-order modal vectors.

In addition that the elasticity counterpart of the operator matrix $[\tilde{\mathbf{R}} - \psi^{(0)}\mathbf{m}]$ in (20) is the same as for the free vibration problem using the two-dimensional classical plate theory, interestingly the edge conditions (27) for $n = 0$ are also the same as in the classical plate theory. $\mathcal{N}_{\alpha\beta}^{(0)}$ and $\mathcal{M}_{\alpha\beta}^{(0)}$ defined by (26) reduce to the conventional stress resultant and moment in that theory in the case of pure elasticity.

The conditions (27) for $n = 0$ are satisfied by the conventional edges such as simply supported edges and clamped edges. The edge conditions for higher-order equations $(n \geq 1)$ also can be satisfied provided that the edge conditions for higher orders are approximately given in the same way as given in the classical Kirchhoff plate theory. It should be noted that, in general, specifying the edge boundary conditions in the sense of the Kirchhoff plate theory only yields the accurate leading-order interior solution. The leading-order solution does not account for the through-thickness distribution of the edge boundary conditions and cannot be valid near to the edges. As far as the interior solution is concerned, it is generally expected by virtue of Saint-Venant's principle that the boundary layer effect will not be felt away from the local disturbance.

Wu *et al.* [16] proposed the solvability condition in order to eliminate secular terms in their multiple time scales expansion for elastic laminate problems. Note that minor formal errors, which do not affect their example results, appear in their work, in which the dynamic terms involving the frequency parameter should be incorporated as they are into (25) and (27) in this paper, and the solvability condition should be given in the integral form.

V. Normalization Condition

It is often convenient to render a modal vector unique by assigning a given value either to one of the components of the modal vector or to the magnitude of the modal vector. This process is known as normalization. Clearly, the normalization process is devoid of physical significance and should be regarded as a mere convenience. In this paper, the normalization condition is defined by:

$$
\int_\Omega \tilde{\mathbf{X}}^{\mathrm{T}}\tilde{\mathbf{X}}\mathrm{d}\Omega = 1, \quad \text{where } \tilde{\mathbf{X}} = \sum_{n=0}^{\infty}\chi^{2n}\tilde{\mathbf{X}}^{(n)}. \tag{29}
$$

More detailed expressions can be found in [16].

VI. Application

To examine the present asymptotic theory, a rectangular piezoelectric plate is considered. The following mixed boundary conditions are used to model simply supported edges:

$$\begin{aligned} &u_2 = u_3 = \tau_{11} = \varphi = 0, \text{ at } x_1 = 0, a, \\ &u_1 = u_3 = \tau_{22} = \varphi = 0, \text{ at } x_2 = 0, b. \end{aligned} \tag{30}$$

The edges conditions (30) can be satisfied on a pointwise basis by assuming

$$\tilde{\mathbf{X}}^{(n)} = \begin{bmatrix} U_1^{(n)} \\ U_2^{(n)} \\ U_3^{(n)} \\ D_0^{(n)} \end{bmatrix} = \begin{bmatrix} \hat{U}_1^{(n)} \cos l_1 x_1 \sin l_2 x_2 \\ \hat{U}_2^{(n)} \sin l_1 x_1 \cos l_2 x_2 \\ \hat{U}_3^{(n)} \sin l_1 x_1 \sin l_2 x_2 \\ \hat{D}_0^{(n)} \sin l_1 x_1 \sin l_2 x_2 \end{bmatrix}. \tag{31}$$

where

$$l_1 = \frac{m_1 \pi}{a}, \quad l_2 = \frac{m_2 \pi}{b}, \tag{32}$$

and a quantity with a superimposed hat denotes the amplitude of the corresponding physical quantity. A solution of each order may be obtained in the way described earlier, and numerical results can be computed to any desired degree of accuracy for the specific problem.

In the example problem, a rectangular piezoelectric bimorph is considered. The thickness of the upper and lower layers are h_2 and h_1. The upper and lower layers of the bimorph plate are made of the same material but are polarized in opposite x_3 directions. Hence, the piezoelectric coefficients are of opposite signs for the upper and lower layers, whereas all the other material properties are the same across the thickness. The material of the bimorph is taken as barium titanate ceramic ($BaTiO_3$) with the following material properties [17]:

$$\begin{aligned} &c_{1111} = 150 \times 10^9 \,\mathrm{N/m^2}, \\ &c_{1122} = 66 \times 10^9 \,\mathrm{N/m^2}, \\ &c_{1133} = 66 \times 10^9 \,\mathrm{N/m^2}, \\ &c_{3333} = 146 \times 10^9 \,\mathrm{N/m^2}, \\ &c_{2323} = 44 \times 10^9 \,\mathrm{N/m^2}, \\ &c_{1212} = 43 \times 10^9 \,\mathrm{N/m^2}, \\ &e_{311} = -4.35 \mathrm{C/m^2}, \\ &e_{333} = 17.5 \mathrm{C/m^2}, \\ &e_{113} = 11.4 \mathrm{C/m^2}, \\ &\varepsilon_{11}/\varepsilon_0 = 1115, \\ &\varepsilon_{33}/\varepsilon_0 = 1260, \\ &\varepsilon_0 = 8.854185 \times 10^{-12} \,\mathrm{F/m}, \\ &\rho = 5700 \,\mathrm{kg/m^3}. \end{aligned} \tag{33}$$

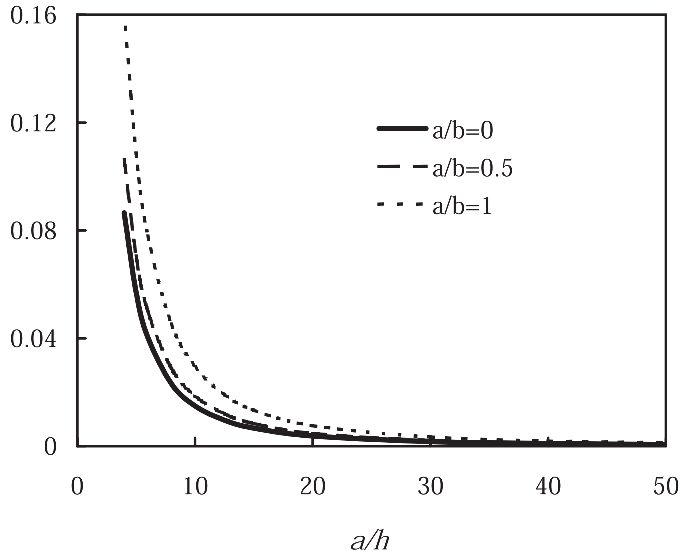

Fig. 1. Frequency versus span-to-thickness ratio for different aspect ratios ($h_2/h_1 = 1$).

The dimensionless frequency is defined by $\overline{\omega} = \sqrt{\rho h^2/\pi^2 c_{2323}}\omega$. The frequency solution of various orders is calculated from:

$$\Psi = \omega^2 = \left(\sum_{n=0}^{\infty} \chi^{2n} \omega^{(n)} \right)^2. \tag{34}$$

Table I presents a convergence study on the asymptotic approach. The span-to-thickness ratio a/h of the plate is taken as 10 (moderately thick) through 100 (thin), the aspect ratio is $a/b = 0$, i.e., an infinite bimorph strip. The three-dimensional exact solution for the flexural-mode-predominant fundamental frequency, obtained by Lee and Yu [1], is given for comparison. It is seen that the leading-order solution ($n = 0$) gives good results. Virtually the same results as the exact solution can be generated by the first-order asymptotic solution for $a/h = 50$, 800, 100, by the second-order asymptotic solution for $a/h = 20$, 30, 40, and by the third-order asymptotic solution for $a/h = 10$.

Fig. 1 shows the flexural-mode-predominant fundamental frequency of the bimorph plate for various span-to-thickness ratios and aspect ratios. Comparison of overtones with the fundamental frequency is given in Fig. 2 for various aspect ratios of the plate ($a/h = 10$). The tenth-order solution has been given to ensure numerical convergence.

The exact solution [1] is obtained only for an infinite bimorph strip ($a/b = 0$). Except the case of $a/b = 0$, the results shown in Figs. 1 and 2 are not available in [1]. We also present new results in Table II for the flexural-mode-predominant fundamental frequency of an asymmetric bimorph with $h_2/h_1 = 2$.

VII. Conclusions

This work has presented an asymptotic theory for vibrations of inhomogeneous/laminated piezoelectric plates

TABLE I
CONVERGENCE OF VARIOUS ORDERS OF THE FUNDAMENTAL FREQUENCY 1000ϖ CORRESPONDING TO THE PREDOMINANTLY FLEXURAL MODE OF BIMORPH STRIPS ($a/b = 0$, $h_2/h_1 = 1$, $m_1 = 1$, $m_2 = 1$).

a/h	Exact [1]	Leading-order	1st-order	2nd-order	3rd-order
10	14.91876	15.10104	14.91539	14.91883	14.91876
20	3.77513	3.78686	3.77507	3.77513	
30	1.68168	1.68401	1.68167	1.68168	
40	0.94671	0.94744	0.94670	0.94671	
50	0.60612	0.60642	0.60612		
80	0.23686	0.23691	0.23686		
100	0.15161	0.15162	0.15161		

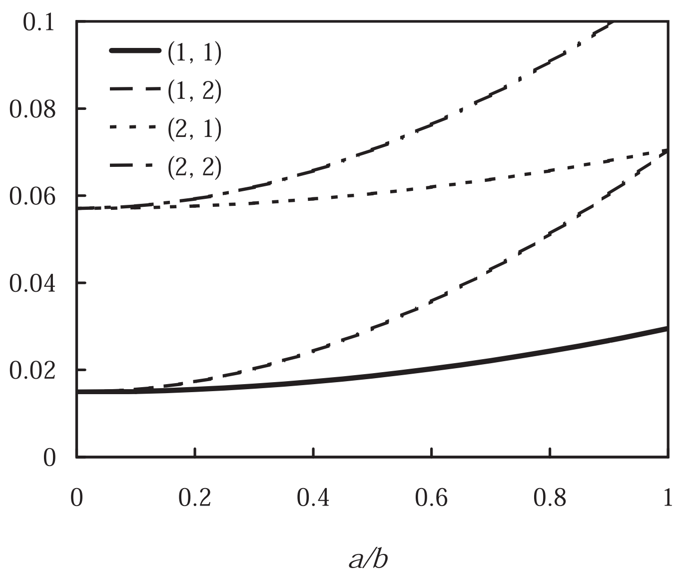

Fig. 2. Fundamental frequency and overtones (m_1, m_2) versus aspect ratio ($a/h = 10$, $h_2/h_1 = 1$).

TABLE II
FUNDAMENTAL FREQUENCY 1000ϖ CORRESPONDING TO THE PREDOMINANTLY FLEXURAL MODE OF AN ASYMMETRIC BIMORPH ($h_2/h_1 = 2$, $m_1 = 1$, $m_2 = 1$).

a/h	$a/b = 0$	$a/b = 0.5$	$a/b = 1$
10	15.01866	18.74456	29.67963
20	3.80117	4.75857	7.60073
50	0.61033	0.76474	1.22472
100	0.15266	0.19131	0.30649

by means of small parameter expansion. The asymptotic theory reduces a three-dimensional problem to a hierarchy of two-dimensional plate equations. The elasticity counterpart of the two-dimensional operator matrix is precisely that for the classical Kirchhoff elastic plate theory. Solvability condition has been developed for higher-order equations. Numerical results show excellent agreement with the available three-dimensional exact solution, and new results have been presented.

The theoretical approach shows that the completely arbitrarily inhomogeneous material properties included in this theory do not present any difficulty in the study of piezoelectric plate problems. Only a hierarchy of two-dimensional equations of the same homogeneous operator for each order is to be solved, yet a three-dimensional solution is achieved.

A finite element analysis could be developed on the basis of the proposed asymptotic methodology. Because the two-dimensional differential equation (20) has the same homogeneous operator, the stiffness matrix in the asymptotic finite element scheme would be the same. Thus, the finite element model would require only two-dimensional discretization on the reference plane of the plate. Three-dimensional results could be obtained by solving the two-dimensional equations hierarchically, with the same stiffness matrix and different effective load vectors.

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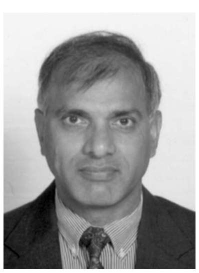

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