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Asymptotic Representation for the Eigenvalues of a Non-selfadjoint Operator Governing the Dynamics of an Energy Harvesting Model

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Abstract We consider a well known model of a piezoelectric energy harvester. The harvester is designed as a beam with a piezoceramic layer attached to its top face (unimorph configuration). A pair of thin perfectly conductive electrodes is covering the top and the bottom faces of the piezoceramic layer. These electrodes are connected to a resistive load. The model is governed by a system consisting of two equations. The first of them is the equation of the Euler–Bernoulli model for the transverse vibrations of the beam and the second one represents the Kirchhoff's law for the electric circuit. Both equations are coupled due to the direct and converse piezoelectric effects. The boundary conditions for the beam equations are of clamped-free type. We represent the system as a single operator evolution equation in a Hilbert space. The dynamics generator of this system is a non-selfadjoint operator with compact resolvent. Our main result is an explicit asymptotic formula for the eigenvalues of this generator, i.e., we perform the modal analysis for electrically loaded (not short-circuit) system. We show that the spectrum splits into an infinite sequence of stable eigenvalues that approaches a vertical line in the left half plane and possibly of a finite number of unstable eigenvalues. This paper is the first in a series of three works. In the second one we will prove that the generalized eigenvectors of the dynamics generator form a Riesz basis (and, moreover, a Bari basis) in the energy space. In the third paper we will apply the results of the first two to control problems for this model.

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1 Statement of the Problem and Derivation of Model Equations

The present paper is a contribution to the volume dedicated to the memory of A.V. Balakrishnan, with whom the author has had a privilege to collaborate for more than a decade on mathematical problems arising in aeroelasticity. A significant part of that research is devoted to asymptotic and spectral analysis and control of an aircraft wing model in a subsonic inviscid airflow (see joint works [2–4], *author's work* [17] *and references therein.) One of A.V. Balakrishnan's recent projects was concerned with the energy harvesting from flutter and limit cycle oscillations induced by an ambient airfow* [1]. *The present paper is a natural continuation of the author's joint work with A.V. Balakrishnan. It is devoted to a rigorous spectral and asymptotic analysis of what can be called "ground vibrations" of an energy harvester.*

The goal of this paper is a rigorous mathematical analysis of a specific well known model of an energy harvester. The harvester considered in the paper is transforming the mechanical energy of ambient vibrations into electric energy by using the direct piezoelectric effect. The contemporary literature on energy harvesting involves numerous papers of engineers and numerical analysts across several engineering disciplines, such as mechanical and electrical engineers, and material science researchers (see [7] and references therein). For a detailed description of the model and derivation of the model equations we refer to [6,7].

The harvester model is governed by a system of two evolution partial differential equations with a set of boundary conditions. We reformulate the original initial-boundary problem as a single operator evolution equation in the state space of the model, which is a Hilbert space equipped with the energy metric (the energy space). This evolution equation generates a strongly continuous semigroup in the energy space (see Proposition 2.3 below). The infinitesimal generator of this semigroup is a non-selfadjoint operator, which we call the dynamics generator. The main result of the paper is an asymptotic representation for the eigenvalues of the dynamics generator. This paper is the first in a series of three works devoted to the harvester model. In the second paper we prove that the generalized eigenvectors of the dynamic generator form a Riesz basis (and, moreover, a Bari basis) in the energy space. In the third paper we use the results of first two papers to solve some control problem for the model via the spectral decomposition method.

Remark 1.1 We stress at this point that the equations and boundary conditions we use in this paper (see (1.1)–(1.4) below) are slightly different from the equations and boundary conditions presented in [6,7] and in engineering literature (see (1.11), (1.2), (1.24)–(1.25) below). Namely, Eq. (1.11) derived in [6,7] contains a singular distributional term while our Eq.(1.1) does not have this term. Instead, we introduce an extra term to the boundary conditions (see (1.4)). We prove in Proposition 1.5 below that both formulations of the model are equivalent. However, our formulation is more convenient for a rigorous Hilbert space reformulation of the problem.

Now we present a brief description of the model, the equations and boundary conditions that govern the model, and for the convenience of the reader we include a brief elementary derivation of the model equations.

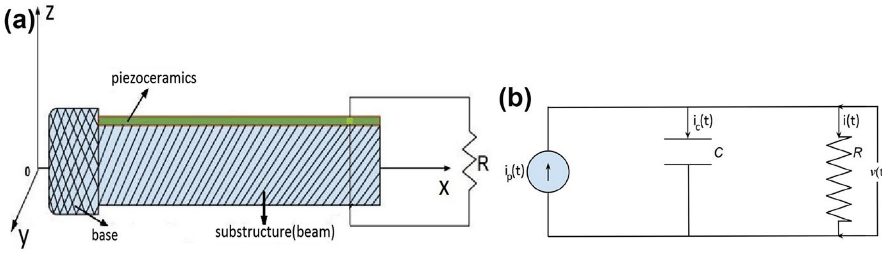

Fig. 1 Unimorph cantilever with a piezoceramic layer connected to a resistive load

Description of the Model The following sketch Fig. 1a shows the construction of the harvester. The electrical circuit of the harvester is presented on Fig. 1b.

We consider a piezoelectric energy harvester in the form of a thin cantilever (clamped-free) beam with a piezoceramic layer. We use the Euler–Bernoulli model to describe the transverse vibrations of the beam. The piezoceramic layer is attached to the top face of the beam. A pair of electrodes is covering the top and the bottom faces of the piezoceramic layer. The electrodes are thin enough for their thickness to be neglected; they are also perfectly conductive. The electrodes are connected to a resistive load. The load is considered in an electrical circuit along with the internal capacitance of the piezoceramic layer, which is assumed to be a perfect insulator. If the beam vibrates due to some external load then a dynamic strain appears in the piezoceramic layer. Due to the piezoelectric effect this strain results in an alternating voltage output across the electrodes. This output can be harvested for charging batteries or running low-powered electronic devices.

On the other hand, the electric potential difference between the electrodes generates an electric field in the piezoceramic layer. This electric field produces a stress on the piezoceramic due to the converse piezoelectric effect. So, the dynamics of the beam is affected by the electric circuit. As a result, the energy harvester is modeled by a coupled system of two differential equations. The first of them is a hyperbolic partial differential equation for the Euler–Bernoulli beam that contains an additional term depending on the voltage on the electrodes. This term represents the converse piezoelectric effect. In other words the backward piezoelectric coupling effect modifies the vibration response of the cantilever. The second equation is just the Kirchhoff's law for the electric circuit. This equation is a linear first order ordinary differential equation with respect to the voltage across the electrodes. The equation contains an additional integral term depending on the transverse displacement of the beam. This term represents the direct piezoelectric effect.

Remark 1.2 We consider a harvester with a single piezoceramic layer (a unimorph configuration). In practice most of piezoelectric energy harvesters have two piezoceramic layers (bimorph configuration). These layers can be connected to a resistive load either in a series or in a parallel manner. However, the equations of the model are the same for unimorph and for bimorph configurations, for series or parallel connection, except for the values of the parameters entering these equations. So, we consider the electromechanical model with the simplest electric circuit.

Statement of Initial Boundary-Value Problem We consider a system of two coupled partial differential equations for two unknown functions $w(x,t)$ and $v(t)$ with $0 \leq x \leq L < \infty$, $t \geq 0$

$$mw_{tt}(x,t) + c_a w_t(x,t) + YI\, w_{xxxx}(x,t) = 0, \tag{1.1}$$

$$Cv_t(t) + \frac{1}{R}v(t) + \kappa w_{tx}(L,t) = 0. \tag{1.2}$$

This system is equipped with the following set of boundary conditions:

$$w(0,t) = w_x(0,t) = 0, \tag{1.3}$$

$$YI\, w_{xx}(L,t) = \theta v(t), \tag{1.4}$$

$$w_{xxx}(L,t) = 0, \tag{1.5}$$

where $w(x,t)$—the transverse displacement of the beam; m— mass per unit length; c_a—viscous air damping coefficient; Y—the Young modulus; I—cross section moment of inertia with respect to the neutral axis; (YI—the bending stiffness); θ—converse piezoelectric effect backward coupling coefficient. $v(t)$—the output voltage across the electrodes of the piezoceramic layer; C—internal capacitance of the piezoceramic layer; R—resistance of the external load; κ—direct piezoelectric effect coupling coefficient.

Remark 1.3 a) The first and third terms in (1.1) represent an undamped beam. Eq. (1.1) contains in addition a viscous air damping term.

b) The left hand side of Eq. (1.1) may contain an additional Kelvin–Voigt (strain-rate) damping term $c_s I w_{txxx}$, where c_s is the strain—rate damping coefficient. In this paper we assume that this term is small enough to be neglected, i.e., we assume $c_s = 0$.

Brief derivation of the Model Equations (1) The equation of motion for the transverse vibrations of a thin beam has the form ([5]; [11], (4.4.1)):

$$m\, w_{tt}(x,t) = T\, w_{xx}(x,t) + N_x(x,t), \tag{1.6}$$

where m—mass per unit length, T—internal tension (we assume $T = 0$), $N(x,t)$—the shear force.

Remark 1.4 For brevity we did not include into (1.6) the viscous air damping term that is present in (1.1). This term can be easily added to the left hand side of (1.6).

Let $M(x,t)$ be the internal bending moment about y-axis. Conservation of the angular momentum implies ([5]; [11, (4.4.3)]):

$$N(x,t) = M_x(x,t). \tag{1.7}$$

The following constitutive relation holds ([5]; [11], (4.4.4))

$$M(x,t) = -YI\ w_{xx}(x,t). \tag{1.8}$$

Equation (1.8) says that M is proportional to the curvature of the beam, which for small displacements is approximately equal to w_{xx}. The proportionality coefficient is the bending stiffness. In (1.8) we use the standard sign agreement: positive bending moment creates negative curvature. Substituting (1.7), (1.8) into (1.6) and taking $T = 0$ we get the classical Euler–Bernoulli thin beam equation

$$m\ w_{tt}(x,t) + YI\, w_{xxxx}(x,t) = 0. \tag{1.9}$$

In the case of a beam with attached piezoceramic layer the constitutive relation between the displacement and the internal bending moment (1.8) contains an additional term:

$$M(x,t) = -YI\ w_{xx}(x,t) + \theta v(t)\,[H(x) - H(x-L)]\,. \tag{1.10}$$

here $v(t)$—voltage across the electrodes of the piezoceramic layer, θ—backward coupling coefficient, $H(x)$—Heaviside step-function. The second term in (1.10) is due to the converse piezoelectric effect. Namely, the voltage $v(t)$ produces a uniform electric field $E = -\frac{v(t)}{h}$ (h—the thickness of the layer) in the piezoceramic layer. Due to the converse piezoelectric effect, this field generates an additional bending moment that is constant along the beam ($0 \le x \le L$). The additional term is multiplied by the characteristic function of the interval $[0, L] : \mathscr{X}(x) = H(x) - H(x-L)$.

Replacing constitutive relation (1.8) by (1.10), combining with (1.7), substituting into equation of motion (1.6) (taking into account that $H'(x) = \delta(x)$), and adding to the left hand side the above mentioned damping term we arrive at the following equation of motion (see [6,7]):

$$m\ w_{tt} + c_a\ w_t + YI\ w_{xxxx}(x,t) - \theta v(t)\left[\delta'(x) - \delta'(x-L)\right] = 0. \tag{1.11}$$

2) Explanation of equation (1.2). In Fig. 1b the piezoelectric element from Fig. 1a is represented as a current source in parallel with its internal capacitance connected to external load. Notations: $i_p(t)$—alternating current across the piezoceramic layer; $i_c(t)$—alternating current across the capacitor; $i(t)$—total current across the external resistive load R. Kirchhoff's law for the electrical circuit shown in Fig. 1b has the form:

$$i_c(t) + i(t) - i_p(t) = 0. \tag{1.12}$$

(Here we used the directions of the currents chosen on Fig. 1b). Now we derive Eq. (1.3) from the Kirchhoff's law (1.12). To do this we present formulas for all three terms in (1.12).

a) The charge of the capacitor C and the current across the capacitor are

$$q(t) = Cv(t), \qquad i_C(t) = \frac{dq(t)}{dt} = C\ \frac{dv(t)}{dt}. \tag{1.13}$$

(Sign "plus" in the second formula of (1.13) is due to the current direction shown in Fig. 1b).

b) According to Ohm's law for the resistive load, R, we have

$$i(t) = \frac{1}{R}\, v(t). \tag{1.14}$$

c) The third term requires more work. Additional notations: h—thickness of the piezoelectric layer (in z-direction); b—width of the piezoelectric layer (in y-direction); L—length of the piezoelectric layer (in x-direction); $A = bL$—area of the top face of the beam and of each of the electrodes; h_1—distance from the neutral axis of the beam (the x-axis) to the center of the piezoceramic layer; $D(x, z, t)$—electric displacement (directed along z-axis) inside the piezoelectric layer; $E(t)$—electric field inside the piezoceramic layer; $S(x, z, t)$—the strain at level z from the neutral axis produced by the displacement of the beam; d—the piezoelectric constant (of the direct piezoelectric effect); ε—the permittivity of the piezoceramic material. We have the following relations. According to a well known formula for the capacitance of a flat capacitor [15]

$$C = \frac{\varepsilon A}{h} = \frac{\varepsilon b L}{h}. \tag{1.15}$$

The electric field

$$E(t) = -\frac{v(t)}{h}. \tag{1.16}$$

Strain inside piezoelectric layer and the beam is proportional to the beam curvature:

$$S(x, z, t) = -z\, w_{xx}(x, t). \tag{1.17}$$

The constitutive relation between D, S, and E due to the direct piezoelectric effect is [12, Sect. 17]

$$D(x, z, t) = dS(x, z, t) + \varepsilon E(t). \tag{1.18}$$

Substituting (1.16), (1.17) into (1.18) we have

$$D(x, z, t) = -d\, z\, w_{xx}(x, t) - \frac{\varepsilon}{h} v(t). \tag{1.19}$$

According to Gauss's law [15], the total charge on the electrodes attached to the top and bottom faces of the piezoceramic layer is

$$Q(t) = \int_0^L \int_0^b D(x, h_1, t)dydx = b \int_0^L D(x, h_1, t)dx. \tag{1.20}$$

Substituting (1.19) with $z = h_1$ into (1.20) we get

$$Q(t) = -bdh_1 \int_0^L w_{xx}(x, t)\, dx - \frac{\varepsilon b L}{h} v(t). \tag{1.21}$$

Accounting for (1.15) we obtain an alternative formula for the total current across the load

$$i(t) = \frac{dQ(t)}{dt} = -\kappa \int_0^L w_{xxt}(x,t)dt - C\,\frac{dv(t)}{dt}\ , \tag{1.22}$$

where we denoted $\kappa = bdh_1$. Now we substitute (1.22), and (1.13) into (1.12) to get

$$i_p(t) = i(t) + i_c(t) = -\kappa \int_0^L w_{xxt}(x,t)dx = -\kappa[w_{tx}(L,t) - w_{tx}(0,t)]. \tag{1.23}$$

Substituting (1.13), (1.14), and (1.23) into (1.12) yields Eq.(1.2). Notice, the last term in (1.23): $w_{tx}(0,t) = 0$ due to the second boundary condition (1.3). Hence this term is omitted in (1.2).

3) Boundary conditions for system (1.11), (1.2). The left end $x = 0$ of the beam is clamped:

$$w(0,t) = 0, \quad w_x(0,t) = 0, \tag{1.24}$$

The right end of the beam is free:

$$w_{xx}(L,t) = 0, \qquad w_{xxx}(L,t) = 0. \tag{1.25}$$

Proposition 1.5 *The boundary-value problem (1.11), (1.2), (1.24)–(1.25) is equivalent to problem (1.1)–(1.5).*

Remark 1.6 a) The only difference between (1.1)–(1.5) and (1.11), (1.2), (1.24)–(1.25) is that the distributional term $\theta v(t)\left[\delta'(x) - \delta'(x-L)\right]$ in (1.11) is replaced by an additional term $\theta v(t)$ in the right hand side of boundary condition (1.4).

b) If we had the Kelvin–Voight damping term $c_s I w_{txxx}$ in both (1.1) and (1.11) then the boundary conditions (1.4), (1.5) and (1.25) would be different. They would contain additional left-hand side terms: $c_s I w_{txx}(L,t)$ for (1.4), $\frac{c_s I}{YI} w_{txx}(L,t)$ for (1.25), and $\frac{c_s I}{YI} w_{txxx}(L,t)$ for (1.5) and (1.25). However, in this case Proposition 1.5 is still valid.

Proof of Proposition. To show the equivalence of the problems we check that their weak (variational) formulations coincide. Let $\mathscr{P} = \left\{\varphi \in C^\infty[0,L] : \ \varphi(0) = \varphi'(0) = 0\right\}$ be the class of test functions. We multiply both (1.1) and (1.11) by $\varphi \in \mathscr{P}$ and integrate over $[0,L]$. Integrating by parts twice in the third terms of (1.1) and (1.11) and taking into account the distributional terms in (1.1) we arrive at the same weak formulations of both problems:

$$\int_0^L (m\ w_{tt} + c_a\ w_t)\,\varphi dx + \int_0^L w_{xx}\varphi'' dx - \theta v(t)\varphi'(L) = 0, \qquad \varphi \in \mathscr{P}. \tag{1.26}$$

In addition to (1.26) we assume that w satisfies boundary conditions (1.24). Since the weak formulation (1.26), (1.24) is the same for both problems, one gets the desired equivalence. □

2 Operator Reformulation of the Problem

In the analysis presented below it will be convenient to use scaled physical quantities. Let

$$G = \frac{c_a}{m}, \qquad E = \frac{YI}{m}, \qquad H = \frac{1}{CR}, \qquad h = \frac{\kappa}{C}, \qquad \Theta = \frac{\theta}{m}. \tag{2.1}$$

Thus, the equations (1.1), (1.2) in new notation have the form

$$\begin{aligned} &w_{tt}(x,t) + G\, w_t(x,t) + E\, w_{xxxx}(x,t) = 0, \\ &v_t(t) + H\, v(t) + h w_{tx}(L,t) = 0. \end{aligned} \tag{2.2}$$

The boundary conditions are

$$w(0,t) = w_x(0,t) = 0, \qquad E\, w_{xx}(L,t) = \Theta v(t), \qquad w_{xxx}(L,t) = 0. \tag{2.3}$$

We also introduce a set of initial conditions in a standard manner

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad v(0) = v_0. \tag{2.4}$$

Our goal is to rewrite the problem (2.2)–(2.4) as an operator evolution equation in the state space of the system, which is a Hilbert space of the Cauchy data equipped with the energy norm.

Definition 2.1 Let $\mathscr{A}_a$ be the differential operation defined on a smooth function $f(x)$ by

$$\mathscr{A}_a[f] = f'(a), \tag{2.5}$$

Now let $w(x,t)$ and $v(t)$ be a solution of the above system. Introduce a three-component vector

$$U(x,t) = (w(x,t), w_t(x,t), v(t))^T. \tag{2.6}$$

Direct verification shows that system (2.2)–(2.4) can be written as an operator equation

$$\frac{d}{dt} U(x,t) = i\,(\mathscr{L}U)(x,t). \qquad U(x,0) = U_0(x), \tag{2.7}$$

where $\mathscr{L}$ is a matrix differential operator given by the differential expression

$$\mathscr{L} = -i \begin{bmatrix} 0 & 1 & 0 \\ -E\frac{d^4}{dx^4} & -G & 0 \\ 0 & -h\ \mathscr{A}_L & -H \end{bmatrix}. \tag{2.8}$$

Now we define the state space of the system (the energy space) and the domain of the operator $\mathscr{L}$. Let us fix t in (2.6) and consider a three-component vector function of x. (Without misunderstanding we use the same notation U.)

$$U(x) = (u_0(x), u_1(x), u_2(x))^T. \tag{2.9}$$

Definition 2.2 The energy space $\mathscr{H} = \widetilde{H}_0^2(0, L) \times L^2(0, L) \times \mathbb{C}$ is the closure of smooth vector—valued functions (2.9) satisfying the conditions $u_0(0) = u_0'(0) = 0$ in the norm

$$\|U\|_{\mathscr{H}}^2 = \frac{1}{2}\left[\int_0^L \left(E\left|u_0''(x)\right|^2 + |u_1(x)|^2\right) dx + |u_2|^2\right]. \tag{2.10}$$

We use the notation $\widetilde{H}_0^2(0, L)$ for the subspace of the Sobolev space $H^2(0, L)$ consisting of functions u satisfying $u(0) = u'(0) = 0$. The tilde is used to distinguish this space from the subspace $H_0^2(0, L)$ consisting of functions satisfying similar conditions at both ends of $[0, L]$.

The domain of the operator $\mathscr{L}$ given in (2.7) is defined as follows

$$\begin{aligned}\mathscr{D}(\mathscr{L}) = \{U = (u_0, u_1, u_2)^T \in \mathscr{H} :\ & u_0 \in H^4(0, L) \cap \widetilde{H}_0^2(0, L), \\ u_1 \in \widetilde{H}_0^2(0, L),\ u_2 \in \mathbb{C};\ & u_0'''(L) = 0, \quad E u_0''(L) = \Theta u_2\}.\end{aligned} \tag{2.11}$$

Notice due to Definition 2.2 and formula (2.11) the domain $\mathscr{D}(\mathscr{L})$ is dense in $\mathscr{H}$.

Proposition 2.3 *The evolution problem (2.2)–(2.4) or, equivalently, (2.7) defines a C_0-semigroup in the energy space $\mathscr{H}$. The operator $\mathbb{L} = i\mathscr{L}$, with $\mathscr{L}$ defined in (2.8), (2.11), is the generator of this semigroup.*

This proposition is an immediate corollary of the result that will be proven in our next paper on the energy harvester model. Namely, it follows from the fact that $\mathscr{L}$ is a non-selfadjoint operator with purely discrete spectrum whose generalized eigenvectors form a Riesz basis in $\mathscr{H}$. Due to this fact the solution of (2.7) can be represented in the form of a generalized eigenvector expansion which is sufficient for a justification of the well-posedness and the C_0-semigroup property.

Recall, the main result of this paper is the asymptotics of the spectrum of $\mathscr{L}$. Proposition 2.3 has been presented only to point out that our evolution problem is well-posed and that $\mathbb{L} = i\mathscr{L}$ can be called its dynamics generator. Without misunderstanding, we will use the same term "dynamics generator" for the operator $\mathscr{L}$ as well. In the sequel, it will be more convenient to work with $\mathscr{L}$ rather than with $\mathbb{L}$, which is addressed in the remark below.

Remark 2.4 As was mentioned above, in our next paper we prove the Riesz basis property of the generalized eigenvectors of $\mathscr{L}$. Clearly $\mathbb{L} = i\mathscr{L}$ and $\mathscr{L}$ have the same generalized eigenvectors. However, as will be explained in that paper, it is $\mathscr{L}$ (not $\mathbb{L}$) that can be represented as a perturbation of a dissipative operator (see the definition in Sec. 3). There exists a significant number of fundamental results on the completeness and Riesz basis property of the generalized eigenvectors for dissipative operators. These results are collected in the monograph [10]. To make direct references to those results we have to deal with $\mathscr{L}$, not $\mathbb{L}$. Also, $\mathscr{L}$ can be represented in the form $\mathscr{L} = \mathscr{M} + ia\mathscr{I} + \mathscr{K}$, where $\mathscr{M}$ is selfadjoint, $a > 0$ is a constant, $\mathscr{I}$ is the identity operator and $\mathscr{K}$ is a small perturbation. This makes the analysis of Riesz basis property more convenient for $\mathscr{L}$ than for $\mathbb{L}$. Finally, it is convenient that the spectrum of $\mathscr{L}$ (except for may be a finite set of eigenvalues) is located in the upper half-plane. In the third paper on the harvester model we reduce one of the control problems to an interpolation problem in the Hardy space H_+^2. The interpolation points will be the eigenvalues of the dynamics generator. However, the space H_+^2 consists of the functions analytic in the upper half-plane. So, it is important that the interpolation points are also in the upper half-plane.

3 Properties of the Dynamics Generator

Compactness of the Operator $\mathscr{L}^{-1}$ In this section we prove that the operator $\mathscr{L}$ has a purely discrete spectrum consisting of normal eigenvalues (i.e., each eigenvalue is an isolated point of the spectrum whose multiplicity is finite). To this end, we start with the equation

$$\mathscr{L}F = Q, \qquad Q = (q_0, q_1, q_2)^T \in \mathscr{H} \tag{3.1}$$

and show that this equation has a unique solution $F = (f_0, f_1, f_2)^T \in \mathscr{D}(\mathscr{L})$ for each $Q \in \mathscr{H}$.

Theorem 3.1 *$\mathscr{L}^{-1}$ exists and is a compact operator in $\mathscr{H}$ given by the following formula:*

$$\mathscr{L}^{-1} = i \begin{bmatrix} G\mathbb{R} & \mathbb{R} + \frac{h\Theta x^2}{2EH}\mathscr{A}_L & \frac{\Theta x^2}{2EH} \\ 1 & 0 & 0 \\ -\frac{h}{H}\mathscr{A}_L & 0 & -\frac{1}{H} \end{bmatrix}, \tag{3.2}$$

where $\mathbb{R}$ is a Volterra integral operator given by

$$(\mathbb{R}\varphi)(x) = \frac{1}{E}\int_0^x d\xi \int_0^\xi d\eta \int_\eta^L d\tau \int_\tau^L \varphi(w)dw, \tag{3.3}$$

and $\frac{h\Theta x^2}{2EH}$ in (3.2) is understood as an operator of multiplication by this function.

Proof 1) Let us rewrite Eq. (3.1) component-wise and have

$$f_1(x) = iq_0(x), \tag{3.4}$$

$$Ef_0''''(x) + Gf_1(x) = -iq_1(x), \tag{3.5}$$

$$h\left[f_1'(L) - f_1'(0)\right] + Hf_2 = -iq_2. \tag{3.6}$$

Since $q_0 \in H_0^2(0, L)$, from (3.4) we get $f_1(0) = f_1'(0) = 0$. Counting (3.4) and keeping in mind that $f_0'''(L) = 0$, we integrate Eq. (3.5) and have

$$Ef_0'''(x) = i\int_x^L \left[Gq_0(\xi) + q_1(\xi)\right] d\xi. \tag{3.7}$$

Let $q_3(x) = Gq_0(x)+q_1(x)$, then $Ef_0'''(x) = i\int_x^L q_3(x)d\xi$. Integrate this equation once to have

$$Ef_0''(x) = -i\int_x^L d\xi \int_\xi^L q_3(\eta)d\eta + A, \tag{3.8}$$

with A being an arbitrary constant. We choose A in such a way that the condition $Ef_0''(L) = \Theta f_2$ is satisfied. Since by (3.6), $f_2 = -iH^{-1}q_2 - hH^{-1}f_1'(L) = -iH^{-1}\left(q_2 + hq_0'(L)\right)$, we get $\left(Ef_0''(L) = -i\Theta H^{-1}\left(q_2 + q_1'(L)\right)\right)$

$$Ef_0''(x) = -i\int_x^L d\xi \int_\xi^L q_3(\eta)d\eta - i\Theta H^{-1}\left(q_2 + hq_1'(L)\right). \tag{3.9}$$

Integrating twice Eq. (3.9) and keeping in mind that $f_0(0) = f_0'(0) = 0$, we finally obtain

$$f_0(x) = -iE^{-1}\int_0^x d\xi \int_0^\xi d\eta \int_\eta^L d\tau \int_\tau^L q_3(w)dw - \frac{i\Theta(EH)^{-1}}{2}\left(q_2 + hq_1'(L)\right)x^2. \tag{3.10}$$

Let $\mathbb{R}$ be a Volterra integral operator given by the formula (3.3). The inverse operator $\mathscr{L}^{-1}$ can be given as follows: $F(x) = \left(\mathscr{L}^{-1}Q\right)(x)$,

$$\begin{aligned} f_0(x) &= -\ i\ G\left(\mathbb{R}q_0\right)(x) -\ i\ \left(\mathbb{R}q_1\right)(x) - \frac{i\Theta}{2EH}\left(q_2 + hq_1'(L)\right)x^2, \\ f_1(x) &= iq_0(x), \qquad f_2 = -iH^{-1}\left(q_2 + hq_0'(L)\right). \end{aligned} \tag{3.11}$$

Obviously, by construction all boundary conditions are satisfied for the components of the vector F. In matrix form, the solution (3.11) can be given as $F = \mathscr{L}^{-1}Q$, where $\mathscr{L}^{-1}$ is given in (3.2). It is clear that $\mathscr{L}^{-1}$ is defined for each $Q \in \mathscr{H}$. To show that $\mathscr{L}^{-1}$ is compact, we note that the domain of $\mathscr{L}^{-1}$ is a closed subspace of the space $\mathscr{H}_1 = H^2(0, L) \times L^2(0, L) \times \mathbb{C}$ and the range is a closed subspace of the space $\mathscr{H}_2 = H^4(0, L) \times H^2(0, L) \times \mathbb{C}$. It follows from the above proof that $\mathscr{L}^{-1}$ is a bounded operator from $\mathscr{H}$ into $\mathscr{D}(\mathscr{L})$ if $\mathscr{D}(\mathscr{L})$ is equipped with the norm of $\mathscr{H}_2$. The fact that the embedding $\mathscr{H}_2 \hookrightarrow \mathscr{H}_1$ is compact yields the compactness of $\mathscr{L}^{-1}$. □

Non-dissipativity of the Operator $\mathscr{L}$. Recall (see, e.g., [10,13,14]), a linear operator T in a Hilbert space H is said to be *dissipative* if for any $F \in \mathscr{D}(T)$, one has $\Im(TF, F)_H \geq 0$.

Lemma 3.1 *The operator* $\mathscr{L}$ *is not dissipative unless* $\Theta = h$.

Proof If $\mathbb{L} = i\mathscr{L}$, then let us evaluate $\Re\left(\mathbb{L}U, U\right)_{\mathscr{H}}$ for $U \in \mathscr{D}(\mathscr{L})$. We have

$$\begin{aligned} 2\left(\mathbb{L}U, U\right)_{\mathscr{H}} &= \left(\begin{pmatrix} u_1(x) \\ -Eu_0''''(x) - Gu_1(x) \\ -h\left(u_0'(L) - u_0'(0)\right) - Hu_2 \end{pmatrix}, \begin{pmatrix} u_0(x) \\ u_1(x) \\ u_2 \end{pmatrix}\right)_{\mathscr{H}} \\ &= \int_0^L Eu_1''(x)\overline{u_0''(x)}dx - \int_0^L \left[Eu_0''''(x)\overline{u_1(x)} + G\left|u_1(x)\right|^2\right]dx \\ &\quad - \left[hu_1'(L) + Hu_2\right]\overline{u_2}. \end{aligned} \tag{3.12}$$

Since $U \in \mathscr{D}(\mathscr{L})$, i.e., $u_1(0) = u_1'(0) = u_0'''(L) = 0$, we simplify the integral containing u_0'''' as

$$\int_0^L E u_0''''(x)\overline{u_1(x)}dx = -E u_0''(L)\overline{u_1'(L)} + \int_0^L E u_0''(x)\overline{u_1''(x)}dx. \tag{3.13}$$

Substituting (3.13) into (3.12), we get

$$\begin{aligned} 2\,(\mathbb{L}U, U)_{\mathscr{H}} = & \int_0^L E u_1''(x)\overline{u_0''(x)}dx - \int_0^L E u_0''(x)u_1''(x)dx \\ & -G\int_0^L |u_1(x)|^2\,dx - hu_1'(L)\overline{u_2} - H\,|u_2|^2 + E u_0''(L)\overline{u_1'(L)}. \end{aligned} \tag{3.14}$$

Using $E u_0''(L) = \Theta u_2$, we finally simplify (3.14) and have

$$2\Re\,(\mathbb{L}U, U)_{\mathscr{H}} = -G\,\|u_1\|^2 - H\,|u_2|^2 + \Re\left\{\Theta\overline{u_1'(L)}u_2 - hu_1'(L)\overline{u_2}\right\}. \tag{3.15}$$

Clearly, if $\Theta = h$ then $\Re\,(\mathbb{L}U, U)_{\mathscr{H}} \le 0$, and therefore $\Im\,(\mathscr{L}U, U)_{\mathscr{H}} \ge 0$. □

The Adjoint Operator Recall that the adjoint operator $\mathscr{L}^*$ satisfies

$$(\mathscr{L}U, V)_{\mathscr{H}} = (U, \mathscr{L}^*V)_{\mathscr{H}} \text{ for any } \quad U \in \mathscr{D}(\mathscr{L}), \quad V \in \mathscr{D}(\mathscr{L}^*). \tag{3.16}$$

Lemma 3.2 *The operator $\mathscr{L}^*$ adjoint to $\mathscr{L}$ is given by the following differential expression*

$$\mathscr{L}^* = -i\begin{bmatrix} 0 & 1 & 0 \\ -E\frac{d^4}{dx^4} & G & 0 \\ 0 & -\Theta A_L & H \end{bmatrix}, \tag{3.17}$$

on the domain

$$\begin{aligned} \mathscr{D}(\mathscr{L}^*) = \Big\{ & V \in \mathscr{H}: \;\; v_0 \in H^4(0,L) \cap \widetilde{H}_0^2(0,L),\; v_1 \in \widetilde{H}_0^2(0,L), \\ & v_1(0) = v_1'(0) = v_0'''(L) = 0; \;\; E v_0''(L) = h v_2. \Big\} \end{aligned} \tag{3.18}$$

Proof For $U \in \mathscr{D}(\mathscr{L})$ and $V \in \mathscr{D}(\mathscr{L}^*)$, we get

$$\begin{aligned} 2\,(\mathbb{L}U, V)_{\mathscr{H}} &= \left(\begin{pmatrix} u_1(x) \\ -E u_0''''(x) - G u_1(x) \\ -h\left(u_0'(L) - u_0'(0)\right) - H u_2 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}\right)_{\mathscr{H}} \\ &= \int_0^L \left[E u_1''(x)\overline{v_0''(x)} - E u_0''''(x)\overline{v_1(x)} - G u_1(x)\overline{v_1(x)}\right]dx - hu_1'(L)\overline{v_2} - H u_2\overline{v_2}. \end{aligned} \tag{3.19}$$

Let $\mathscr{I} = \int_0^L \left[Eu_1''(x)\overline{v_0''(x)} - Eu_0''''(x)\overline{v_1(x)} - Gu_1(x)\overline{v_1(x)}\right] dx$. Integrating by parts, using $v_1(0) = v_1'(0) = u_0'''(L) = 0$, and $Eu_0''(L) = \Theta u_2$, we obtain

$$\mathscr{I} = \int_0^L E\left(u_1''(x)\overline{v_0''(x)} - u_0''(x)\overline{v_1''(x)}\right) dx - G\int_0^L u_1(x)\overline{v_1(x)}dx + \Theta u_2\overline{v_1'(L)}.$$

Substituting this expression for $\mathscr{I}$ into (3.19), we obtain

$$\begin{aligned}2i\,(\mathscr{L}U, V)_{\mathscr{H}} = \int_0^L E\left(u_1''(x)\overline{v_0''(x)} - u_0''(x)\overline{v_1''(x)}\right) dx - G\int_0^L u_1(x)\overline{v_1(x)}dx \\ +\Theta u_2\overline{v_1'(L)} - hu_1'(L)\overline{v_2} - Hu_2\overline{v_2}.\end{aligned} \tag{3.20}$$

Now we consider the following expression in which $\mathscr{L}^*$ is given by (3.17):

$$2\,(U, \mathscr{L}^*V)_{\mathscr{H}} = 2i\left(\begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1(x) \\ -Ev_0''''(x) + Gv_1(x) \\ -\Theta v_1'(L) + Hv_2 \end{pmatrix}\right)_{\mathscr{H}},$$

which yields

$$\begin{aligned}-2i\left(U, \mathscr{L}^*V\right)_{\mathscr{H}} &= \int_0^L Eu_0''(x)\overline{v_1''(x)}dx - \int_0^L Eu_1(x)\overline{v_0''''(x)}dx \\ &\quad + G\int_0^L u_1(x)\overline{v_1(x)}dx - \Theta u_2\overline{v_1'(L)} + Hu_2\overline{v_2}.\end{aligned} \tag{3.21}$$

Integrating by parts the second integral in the right-hand side of (3.21), and taking into account

$$u_1(0) = u_1'(0) = 0, \qquad v_0'''(L) = 0, \qquad \text{and} \qquad Ev_0''(L) = hv_2,$$

we get

$$\begin{aligned}\int_0^L Eu_1(x)\overline{v_0''''(x)}dx = Eu_1(x)\,\overline{v_0'''(x)}\Big|_0^L - Eu_1'(x)\,\overline{v_0''(x)}\Big|_0^L + \int_0^L Eu_1''(x)\overline{v_0''(x)}dx \\ = hu_1'(L)\overline{v_2}.\end{aligned} \tag{3.22}$$

Substituting (3.22) into (3.21), we obtain

$$\begin{aligned}-2i\left(U, \mathscr{L}^*V\right)_{\mathscr{H}} &= \int_0^L E\left(u_0''(x)\overline{v_1''(x)} - u_1''(x)\overline{v_0''(x)}\right) dx \\ &\quad + G\int_0^L u_0(x)\overline{v_1(x)}dx + hu_1'(L)\overline{v_2} - \Theta u_2\overline{v_1'(L)} + Hu_2\overline{v_2}.\end{aligned} \tag{3.23}$$

Upon comparison of (3.20) and (3.23), we obtain that (3.16) holds. □

4 Spectral Equation

In this section we derive an equation whose solutions will be the eigenvalues of the operator $\mathscr{L}$. We start with reduction of the spectral problem for the operator $\mathscr{L}$ to the spectral problem for the corresponding polynomial pencil. We also prove several statements on the properties of the polynomial pencil eigenfunctions that may be of interest in themselves.

We consider the eigenvalue–eigenvector equation for $U = (u_0, u_1, u_2)^T$

$$\mathscr{L}U = \lambda U, \quad U \in \mathscr{D}(\mathscr{L}), \quad \lambda \in \mathbb{C}. \tag{4.1}$$

This equation generates the following system for the components of U:

$$\begin{aligned} &u_1(x) = i\lambda u_0(x), \qquad Eu_0''''(x) + Gu_1(x) = -i\lambda u_1(x), \\ &hu_1'(L) + Hu_2 = -i\lambda u_2. \end{aligned} \tag{4.2}$$

Eliminating u_1 from this system, we get

$$Eu_0''''(x) + i\lambda Gu_0(x) - \lambda^2 u_0(x) = 0, \qquad (H + i\lambda)u_2 = -i\lambda hu_0'(L). \tag{4.3}$$

Since $U \in \mathscr{D}(\mathscr{L})$, we have $u_2 = \Theta^{-1}Eu_0''(L)$, which yields the boundary-value problem for u_0:

$$Eu_0''''(\lambda, x) + i\lambda Gu_0(\lambda, x) - \lambda^2 u_0(\lambda, x) = 0, \tag{4.4}$$

$$u_0(\lambda, 0) = u_0'(\lambda, 0) = 0, \quad u_0'''(L) = 0, \tag{4.5}$$

$$\Theta^{-1}E(H + i\lambda)u_0''(\lambda, L) + i\lambda hu_0'(L) = 0. \tag{4.6}$$

Definition 4.1 [13] A polynomial operator pencil $A(\lambda)$ is an operator-valued function defined by the formula $A(\lambda) = \lambda^n + \lambda^{n-1}A_{n-1} + \ldots + A_0$, in which A_k are linear operators. These operators may be bounded or unbounded and selfadjoint or non-selfadjoint. The degree of the polynomial, n, is called the order of the pencil.

Let $\mathscr{P}(\cdot)$ be a quadratic operator pencil defined by the formula

$$\mathscr{P}(\lambda)\varphi = E\varphi'''' + i\lambda G\varphi - \lambda^2\varphi \tag{4.7}$$

on the domain

$$\begin{aligned} \mathscr{D}(\mathscr{P}) = \{\varphi \in H^4(0, L) : \varphi(0) = \varphi'(0) = \varphi''''(0) = 0, \\ \Theta^{-1}E(H + i\lambda)\varphi''(L) + i\lambda h\varphi'(L) = 0.\} \end{aligned} \tag{4.8}$$

Notice, $\mathscr{P}(\cdot)$ is a non-standard pencil since the spectral parameter λ enters the domain explicitly. This type of pencils have not been considered in monograph [13]. However, it is convenient to keep the terminology because there exists an extensive literature, in which the pencils with the spectral parameter-dependent boundary conditions appear naturally. A non-trivial solution $\varphi \in \mathscr{D}(\mathscr{P})$ of the pencil equation $\mathscr{P}(\lambda)\varphi = 0$ will be called an *eigenfunction* of the pencil and the corresponding value of λ will be called an *eigenvalue*. It is clear that having an eigenfunction of the pencil and using (4.2), we can find all components of the eigenvector of $\mathscr{L}$.

Now we return to Eq. (4.4) and construct the solution $\varphi(\lambda, x)$ of the equation

$$E\varphi''''(\lambda, x) + i\lambda G\varphi(\lambda, x) - \lambda^2\varphi(\lambda, x) = 0 \tag{4.9}$$

satisfying all boundary conditions. It is convenient to introduce scaled parameters

$$\widetilde{\lambda} = \frac{\lambda}{\sqrt{E}}, \qquad \widetilde{G} = \frac{G}{\sqrt{E}}, \qquad \widetilde{H} = \frac{H}{\sqrt{E}}, \qquad \widetilde{\Theta} = \frac{\Theta h}{\sqrt{E}}. \tag{4.10}$$

Problem (4.4)–(4.6) in the scaled parameters has the form

$$\begin{gathered}\varphi''''(\lambda, x) + i\widetilde{\lambda}\widetilde{G}\varphi(\lambda, x) - \left(\widetilde{\lambda}\right)^2\varphi(\lambda, x) = 0,\\ \varphi(\lambda, 0) = \varphi'(\lambda, 0) = \varphi'''(L) = 0, \quad \left(\widetilde{H} + i\widetilde{\lambda}\right)\varphi''(\lambda, L) + i\widetilde{\lambda}\widetilde{\Theta}\varphi'(\lambda, L) = 0.\end{gathered} \tag{4.11}$$

Omitting "tilde" from problem (4.11), we consider the following rescaled problem:

$$\varphi''''(\lambda, x) - \lambda^2\varphi(\lambda, x) + i\lambda G\varphi(\lambda, x) = 0, \tag{4.12}$$

$$\varphi(\lambda, 0) = \varphi'(\lambda, 0) = 0, \quad \varphi'''(\lambda, L) = 0, \tag{4.13}$$

$$(H + i\lambda)\,\varphi''(\lambda, L) + i\lambda\Theta\varphi'(\lambda, L) = 0. \tag{4.14}$$

The characteristic polynomial corresponding to Eq. (4.12), $z^4 = \lambda^2 - iG\lambda$, has four roots

$$z_{1,2} = \pm\mu(\lambda), \qquad z_{1,2} = \pm i\mu(\lambda) \quad \text{where} \quad \mu(\lambda) = \sqrt[4]{\lambda^2 - i\lambda G}. \tag{4.15}$$

The fourth order root is fixed by the condition that $\Im\sqrt[4]{\alpha} \geq 0$ for $\Im\alpha \geq 0$. The following asymptotic approximation is valid for $\mu(\lambda)$ as $|\lambda| \to \infty$:

$$\mu(\lambda) = \sqrt[4]{\lambda^2 - i\lambda G} = \sqrt{\lambda} - \frac{iG}{4\sqrt{\lambda}} + \frac{3G^2}{2^5\lambda^{3/2}} + O\left(\frac{1}{\lambda^{5/2}}\right). \tag{4.16}$$

The fundamental system for Eq. (4.12) is $\{\cos(\mu(\lambda)x), \sin(\mu(\lambda)x), \cosh(\mu(\lambda)x), \sinh(\mu(\lambda)x)\}$. The general solution of Eq. (4.12) is a linear combination of the above four functions. It can be readily checked that the following solution of (4.12) satisfies the left-end boundary conditions

$$\begin{aligned}\varphi(\lambda, x) = {} & \mathscr{A}(\lambda)\left[\cosh(\mu(\lambda)x) - \cos(\mu(\lambda)x)\right]\\ & + \mathscr{B}(\lambda)\left[\sinh(\mu(\lambda)x) - \sin(\mu(\lambda)x)\right]\end{aligned} \tag{4.17}$$

with $\mathscr{A}$ and $\mathscr{B}$ being arbitrary functions of λ. The first right-end boundary condition, $\varphi'''(\lambda, L) = 0$, can be written as follows:

$$\mathscr{A}(\lambda)\left[\sinh\left(\mu(\lambda)L\right) - \sin\left(\mu(\lambda)L\right)\right] + \mathscr{B}(\lambda)\left[\cosh\left(\mu(\lambda)L\right) + \cos\left(\mu(\lambda)L\right)\right] = 0.$$

Counting on this condition, we obtain that the solution satisfying all three conditions can be given in the form

$$\begin{aligned}\varphi(\lambda, x) = &\left[\cosh\left(\mu(\lambda)L\right) + \cos\left(\mu(\lambda)L\right)\right]\left[\cosh\left(\mu(\lambda)x\right) - \cos\left(\mu(\lambda)x\right)\right] \\ &- \left[\sinh\left(\mu(\lambda)L\right) - \sin\left(\mu(\lambda)L\right)\right]\left[\sinh\left(\mu(\lambda)x\right) - \sin\left(\mu(\lambda)x\right)\right].\end{aligned} \tag{4.18}$$

One can check that the following formulas are valid:

$$\begin{aligned}\varphi'(\lambda, L) &= 2\mu(\lambda)\left[\cosh\left(\mu(\lambda)L\right)\sin\left(\mu(\lambda)L\right) + \sinh\left(\mu(\lambda)L\right)\cos\left(\mu(\lambda)L\right)\right], \\ \varphi''(\lambda, L) &= 2\mu^2(\lambda)\left[1 + \cosh\left(\mu(\lambda)L\right)\cos\left(\mu(\lambda)L\right)\right].\end{aligned}$$

Substituting these formulas into the fourth boundary condition (4.14), we obtain the following *spectral equation*:

$$\begin{aligned}&\mu(\lambda)(H + i\lambda)\left[1 + \cosh\left(\mu(\lambda)L\right)\cos\left(\mu(\lambda)L\right)\right] \\ &+ i\lambda\Theta\left[\cosh\left(\mu(\lambda)L\right)\sin\left(\mu(\lambda)L\right) + \sinh\left(\mu(\lambda)L\right)\cos\left(\mu(\lambda)L\right)\right] = 0.\end{aligned} \tag{4.19}$$

We conclude this section with some general statements on the spectrum and the eigenfunctions.

Lemma 4.2 *The entire spectrum of the operator $\mathscr{L}$ is symmetric with respect to the imaginary axis on the complex λ—plane.*

Proof Let λ_0 and φ_0 be an eigenvalue and a corresponding eigenfunction of the problem (4.12)–(4.14). It can be readily checked that $\nu = -\overline{\lambda_0}$ and the function $\psi = \overline{\varphi_0}$ are also an eigenvalue and the corresponding eigenfunction of the same problem. □

Lemma 4.3 *For an eigenfunction φ_n of problem (4.12)—(4.14) corresponding to an eigenvalue λ_n the following result is valid:*

$$\left|\varphi_n'(L)\varphi_n''(L)\right| > 0. \tag{4.20}$$

Proof We have $\varphi_n(x) = \varphi(\lambda_n, x)$, where $\varphi(\lambda, x)$ is a solution of (4.12)–(4.14) with $\lambda = \lambda_n$. Using the contradiction argument, we assume that $\varphi_n'(L) = 0$. Then from the boundary condition (4.14), it follows that $\varphi_n''(L) = 0$ and from (4.13) we have $\varphi_n(0) = \varphi_n(0) = 0$, $\varphi'''(L) = 0$. Therefore, $\varphi_n(x)$ is a solution of the following spectral problem: the equation $\varphi''''(x) = \kappa\varphi(x)$, where $\kappa = \lambda^2 - i\lambda G$ (this is just Eq. (4.12)), and the boundary conditions $\varphi(0) = \varphi'(0) = 0$, $\varphi'(L) = \varphi''(L) = \varphi'''(L) = 0$. We notice that the above equation with the first, second, third, and fifth of the above conditions defines the eigenvalues $\{\sigma_i\}_{i=1}^{\infty}$ and the eigenfunctions of a clamped-sliding beam ([9], p.388). On the other hand, the above equation with the first four conditions defines the eigenvalues $\{\mu_i\}_{i=1}^{\infty}$ and the eigenfunctions of a clamped-free beam. It is known ([9], p. 380), that the following relations hold: $0 < \mu_1 < \sigma_1 < \mu_2 < \sigma_2 < \ldots$, which means that the two spectra do not have common points. According to our assumption, $\varphi_n(x)$ is an eigenfunction of both problems. This cannot happen since the two spectra have no intersection. The obtained contradiction proves the result for the case $\varphi_n'(L) = 0$. If we assume that $\varphi_n''(L) = 0$, a similar argument yields contradiction for that case. □

Lemma 4.4 *The geometric multiplicity of each eigenvalue is* 1*; the algebraic multiplicity is finite and can be greater than* 1.

Proof Using the contradiction argument, assume that there exist two linearly independent functions φ_1 and φ_2 and each one satisfies the boundary-value problem for some $\lambda = \lambda_0$. Since by Lemma 4.3, $\left|\varphi_1''(L)\varphi_2''(L)\right| > 0$ the following function is well-defined:

$$\psi(x) = \frac{\varphi_1(x)}{\varphi_1''(L)} - \frac{\varphi_2(x)}{\varphi_2''(L)}. \tag{4.21}$$

It can be easily seen that ψ satisfies the following problem:

$$\psi''''(x) = \kappa\psi(x), \quad \psi(0) = \psi'(0) = \psi''(L) = \psi'''(L) = 0, \quad \kappa = \lambda^2 - i\lambda G.$$

It means that ψ is an eigenfunction of a clamped-free beam and the corresponding value of κ is an eigenvalue. Now consider another function

$$f(x) = \frac{\varphi_1(x)}{\varphi_1'(x)} - \frac{\varphi_2(L)}{\varphi_2'(L)},$$

which is well-defined due to the same Lemma 4.3. This function, f, satisfies the problem

$$f''''(x) = \kappa f(x), \quad f(0) = f'(0) = f'(L) = f'''(L) = 0.$$

It means that the same value of κ is an eigenvalue and f is an eigenfunction of a clamped—sliding beam. Since the spectra of two problems (for a clamped-free and a clamped-sliding beam models) do not have common points, we get the contradiction. □

5 Spectral Asymptotics

In this section we derive an asymptotic approximation for the eigenvalues of the operator $\mathscr{L}$ as the number of an eigenvalue tends to infinity. Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}'}$ be the set of all eigenvalues of $\mathscr{L}$ (here and below $\mathbb{Z}' = \mathbb{Z} \setminus \{0\}$). Since the set Λ is symmetric with respect to the imaginary axis, the numeration can be made in such a way that $\lambda_{-n} = -\overline{\lambda_n}$ and it is convenient to enumerate the entire spectrum with the subindex $n \in \mathbb{Z}'$. Let $\left\{\lambda_n^+\right\}$ be the subset of all eigenvalues located in the closed upper half-plane: $\Im\lambda_n^+ \geq 0$; let $\left\{\lambda_n^-\right\}$, be the subset of all eigenvalues located in the open lower half-plane, i.e., $\Im\lambda_n^- < 0$.

The main result of the paper is the following statement.

Theorem 5.1 (1) *There are infinitely many eigenvalues λ_n^+ located in the upper half-plane and, therefore, they can be renumbered by the index $n \in \mathbb{Z}'$: $\{\lambda_n^+\}_{n\in\mathbb{Z}'}$. The following asymptotic approximation holds as $|n| \to \infty$, $\quad n \in \mathbb{Z}'$:*

$$\lambda_n^+ = \sqrt{E}\frac{\pi^2}{L^2}\left(n^2+n\right)+\left(\frac{\pi^2}{4L^2}\sqrt{E}+\frac{2\Theta h}{L}\right)+i\ \frac{G}{2}+O\left(\frac{1}{n}\right), \qquad \lambda_{-n}^+ = -\overline{\lambda_n^+}. \tag{5.1}$$

(2) *There can be only a finite number of eigenvalues λ_n^- located in the lower half-plane.*

Remark 5.1 Recall that by Proposition 2.3 the operator $\mathbb{L} = i\mathscr{L}$ is the dynamics generator of our system. The eigenvalues of $\mathbb{L}$ are $\{i\ \lambda_n\}_{n\in\mathbb{Z}'}$. It follows from (5.1) that $\{i\ \lambda_n^+\}_{n\in\mathbb{Z}'}$ are asymptotically close to the vertical line $\Re\lambda = -G/2$ in the left half-plane. Only a finite number of unstable eigenvalues $\{i\ \lambda_n^-\}$ can be located in the open right half-plane. There may be a finite number of eigenvalues $i\ \lambda_n^+$ on the imaginary axis.

The rest of this section is devoted to the proof of Theorem 5.1. This proof will be obtained as a corollary of Theorems 5.2 and 5.3 below. We begin with some notations. In establishing these notations we assume that the set $\{\lambda_n^+\}_{n\in\mathbb{Z}'}$ is infinite. This fact will be justified in the course of the proof of Theorem 5.3.

First of all, from this point until the end of the proof we will work with the parameters defined in (4.10). However, the "tilde" over these parameters will be omitted. Only at the very end of the derivation of our main asymptotic formula (5.1), right after (5.20), we use (4.10) to return to our original parameters. The main formula (5.1) is written in terms of the original parameters introduced in (2.1). So, again, from this point $\lambda_n^\pm$ actually means $\lambda_n^\pm/\sqrt{E}$ as defined in (4.10).

Let $\nu_n^\pm = \sqrt{\lambda_n^\pm}$. It is clear that the set of points $\left\{\nu_n^+\right\}_{n\in\mathbb{Z}'}$ is located in the closed first coordinate angle, i.e., $\Re\nu_n^+ \geq 0$ and $\Im\nu_n^+ \geq 0$. Respectively, the set of points $\left\{\nu_n^-\right\}$ is located in the open second coordinate angle, i.e., $\Re\nu_n^- < 0$ and $\Im\nu_n^- > 0$. It can be easily seen that due to the above numeration, $\nu_{-n}^+ = i\ \overline{\nu_n^+}$, which means that the set $\left\{\nu_n^+\right\}_{n\in\mathbb{Z}'}$ is symmetric with respect to the bisector of the first coordinate angle. Similarly, the set $\left\{\nu_n^-\right\}$ is symmetric with respect to the bisector of the second coordinate angle.

Theorem 5.2 *The set $\left\{\nu_n^+\right\}_{n=1}^\infty$ is located in a strip S^+ of the first quadrant such that if $\nu \in S^+$, then $0 \leq \Im\nu \leq M < \infty$ and $0 \leq \Re\nu < \infty$. The set $\left\{\nu_n^+\right\}_{n=-\infty}^{-1}$ is located in a strip of the first quadrant that is symmetric to the strip S^+ with respect to the bisector of the first coordinate angle.*

Proof It suffices to show that the set $\left\{\nu_n^+\right\}_{n=1}^\infty$ is located in a strip parallel to the positive real semi-axis. Arguing by contradiction, we assume that there exists a subsequence $\{\widetilde{\nu_n}\}_{n=1}^\infty$ of the sequence $\left\{\nu_n^+\right\}_{n=1}^\infty$ such that $\widetilde{\nu_n} = x_n + iy_n$ and $x_n \geq y_n \to \infty$ as

$n \to \infty$. For such subsequence the following estimates are valid:

$$\begin{aligned} &(i) \;\; \cosh(\widetilde{\nu_n} L) = \tfrac{1}{2} e^{(x_n + i y_n)L} \left(1 + O\left(e^{-2x_n L}\right)\right), \\ &(ii) \;\; \cos(\widetilde{\nu_n} L) = \tfrac{1}{2} e^{(y_n - i x_n)L} \left(1 + O\left(e^{-2y_n L}\right)\right), \\ &(iii) \;\; \sinh(\widetilde{\nu_n} L) = \tfrac{1}{2} e^{(x_n + i y_n)L} \left(1 + O\left(e^{-2x_n L}\right)\right), \\ &(iv) \;\; \sin(\widetilde{\nu_n} L) = \tfrac{i}{2} e^{(y_n - i x_n)L} \left(1 + O\left(e^{-2y_n L}\right)\right). \end{aligned} \tag{5.2}$$

Let $\widetilde{\lambda_n} = \widetilde{\nu_n}^2$ and $\widetilde{\mu_n} = \mu(\widetilde{\lambda_n})$, where the function $\mu(\lambda)$ is defined in (4.15). Since $\widetilde{\lambda_n}$ are the eigenvalues of $\mathscr{L}$ they are solutions of the spectral equation (4.19). Asymptotic formula (4.16) for $\mu(\lambda)$ yields the following approximation for the sequence $\{\widetilde{\mu_n}\}_{n=1}^{\infty}$:

$$\begin{aligned} \widetilde{\mu_n} &= (x_n + i y_n) - \frac{iG(x_n - i y_n)}{4\left(x_n^2 + y_n^2\right)} + O\left(\frac{1}{\left(x_n^2 + y_n^2\right)^{3/2}}\right) \\ &= (x_n + i y_n) + O\left(\frac{1}{x_n + y_n}\right), \quad n \to \infty. \end{aligned} \tag{5.3}$$

At the last step, we took into account that $x_n \geq y_n > 0$. Now we use (5.3) to modify formulas $(i) - (iv)$ from (5.2) and have

$$\begin{aligned} &(\alpha) \;\; \cosh(\widetilde{\mu_n} L) = \cosh\left(\widetilde{\nu_n} L + O\left((x_n + y_n)^{-1}\right)\right) = \cosh(\widetilde{\nu_n} L)\left(1 + O\left((x_n + y_n)^{-2}\right)\right) \\ &\quad + \sinh(\widetilde{\nu_n} L) \;\; O\left((x_n + y_n)^{-1}\right) = \tfrac{1}{2} e^{(x_n + i y_n)L}\left(1 + O\left((x_n + y_n)^{-1}\right)\right), \\ &(\beta) \;\; \sinh(\widetilde{\mu_n} L) = \sinh\left(\widetilde{\nu_n} L + O\left((x_n + y_n)^{-1}\right)\right) = \sinh(\widetilde{\nu_n} L)\left(1 + O\left((x_n + y_n)^{-2}\right)\right) \\ &\quad + \cosh(\widetilde{\nu_n} L) \;\; O\left((x_n + y_n)^{-1}\right) = \tfrac{1}{2} e^{(x_n + i y_n)L}\left(1 + O\left((x_n + y_n)^{-1}\right)\right), \\ &(\gamma) \;\; \cos(\widetilde{\mu_n} L) = \cos\left(\widetilde{\nu_n} L + O\left((x_n + y_n)^{-1}\right)\right) = \cos(\widetilde{\nu_n} L)\cos\left(O\left((x_n + y_n)^{-1}\right)\right) \\ &\quad - \sin(\widetilde{\nu_n} L)\sin\left(O\left((x_n + y_n)^{-1}\right)\right) = \tfrac{1}{2} e^{(y_n - i x_n)L}\left(1 + O\left((x_n + y_n)^{-1}\right)\right), \\ &(\delta) \;\; \sin(\widetilde{\mu_n} L) = \tfrac{i}{2} e^{(y_n - i x_n)L}\left(1 + O\left((x_n + y_n)^{-1}\right)\right). \end{aligned} \tag{5.4}$$

Combining the above formulas, we get

$$\begin{aligned} &(a) \;\; \cosh(\widetilde{\mu_n} L)\cos(\widetilde{\mu_n} L) = \tfrac{1}{4} e^{(x_n + y_n)L + i(-x_n + y_n)L}\left(1 + O\left((x_n + y_n)^{-1}\right)\right), \\ &(b) \;\; \cosh(\widetilde{\mu_n} L)\sin(\widetilde{\mu_n} L) = \tfrac{i}{4} e^{(x_n + y_n)L + i(-x_n + y_n)L}\left(1 + O\left((x_n + y_n)^{-1}\right)\right), \\ &(c) \;\; \sinh(\widetilde{\mu_n} L)\cos(\widetilde{\mu_n} L) = \tfrac{1}{4} e^{(x_n + y_n)L + i(-x_n + y_n)L}\left(1 + O\left((x_n + y_n)^{-1}\right)\right). \end{aligned} \tag{5.5}$$

Let $\varepsilon_n^{(a)}, \varepsilon_n^{(b)}, \varepsilon_n^{(c)}$ be the notations for the factors $\left(1 + O\left((x_n + y_n)^{-1}\right)\right)$ from approximations (a), (b), and (c) respectively. Substituting approximations (a) – (c) into Eq.(4.19), we obtain the following asymptotic approximation for the spectral equation:

$$\begin{aligned} &\widetilde{\mu_n}\left(H + i(\widetilde{\nu_n})^2\right)\left[1 + \frac{1}{4} e^{(x_n + y_n)L + i(-x_n + y_n)L} \varepsilon_n^{(a)}\right] \\ &\quad = -\frac{i}{4}(\widetilde{\nu_n})^2 \,\Theta e^{(x_n + y_n)L + i(-x_n + y_n)L}\left[i\varepsilon_n^{(b)} + \varepsilon_n^{(c)}\right]. \end{aligned} \tag{5.6}$$

Dividing both sides of Eq.(5.6) by $(\widetilde{\nu_n})^2\,\frac{1}{4}e^{(x_n+y_n)L+i(-x_n+y_n)L}$, we get

$$\widetilde{\mu_n}\left(i+\frac{H}{(\widetilde{\nu_n})^2}\right)\left[4e^{-(x_n+y_n)L+i(x_n-y_n)L}+\varepsilon_n^{(a)}\right]=-i\Theta\left[i\varepsilon_n^{(b)}+\varepsilon_n^{(c)}\right]. \tag{5.7}$$

One can see that when $n\to\infty$, the left hand-side tends to infinity since $\widetilde{\mu}_n\to\infty$ while the right hand-side of (5.7) remains bounded. The obtained contradiction proves the result. □

Our main result Theorem 5.1 is an immediate corollary of the following statement.

Theorem 5.3 *1) The following asymptotic approximation holds for* $\{\nu_n^+\}_{n=1}^{\infty}$ *as* $n\to\infty$.

$$\nu_n=\frac{\pi\, n}{L}+\frac{\pi}{2L}+\frac{iGL+4\Theta}{4\,\pi n}+O\left(\frac{1}{n^2}\right). \tag{5.8}$$

2) There could be only a finite number of the roots $\lambda_n^-=(\nu_n^-)^2$ *of the spectral equation (4.19) such that* ν_n^- *are located in the second quadrant of the* ν*-complex plane.*

Proof Let us consider the strip S^+ of a width d parallel to the positive real semi-axis in the first coordinate angle of the ν-plane. In this strip, for $\nu=x+iy$, we have $0<x<\infty$ and $0\le y<d<\infty$. The following approximations are valid for $\nu=x+iy\in S^+$ as $x\to\infty$ (here $\lambda=\nu^2$):

$$\begin{aligned}
&(a)\quad \cosh(\mu(\lambda)L)=\cosh\left(\nu L-\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)=\cosh(\nu L)\cosh\left(\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)\\
&-\sinh(\nu L)\sinh\left(\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)=\cosh(\nu L)\left(1+O\left(\tfrac{1}{\nu^2}\right)\right)\\
&-\left[\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right]\sinh(\nu L)=\cosh(\nu L)-\tfrac{iGL}{4\nu}\sinh(\nu L)+O\left(e^{\nu L}\nu^{-2}\right),\\
&(b)\quad \sinh(\mu(\lambda)L)=\sinh\left(\nu L-\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)=\sinh(\nu L)\cosh\left(\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)\\
&-\cosh(\nu L)\sinh\left(\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)=\sinh(\nu L)-\tfrac{iGL}{4\nu}\cosh(\nu L)+O\left(e^{\nu L}\nu^{-2}\right),\\
&\\
&(c)\quad \cos(\mu(\lambda)L)=\cos\left(\nu L-\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)=\cos(\nu L)\cos\left(\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)\\
&+\sin(\nu L)\sin\left(\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)=\cos(\nu L)+\tfrac{iGL}{4\nu}\sin(\nu L)+O\left(\tfrac{1}{\nu^2}\right),\\
&(d)\quad \sin(\mu(\lambda)L)=\sin\left(\nu L-\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)=\sin(\nu L)\cos\left(\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)\\
&-\cos(\nu L)\sin\left(\tfrac{iGL}{4\nu}+O\left(\tfrac{1}{\nu^3}\right)\right)=\sin(\nu L)-\tfrac{iGL}{4\nu}\cos(\nu L)+O\left(\tfrac{1}{\nu^2}\right).
\end{aligned} \tag{5.9}$$

Recalling that for $\nu=x+iy\in S^+$, $x\to\infty$ and $0\le y<d$, we modify the approximations (a), (b) as follows:

$$\begin{aligned}
&(a')\quad \cosh(\mu(\lambda)L)=\tfrac{1}{2}e^{(x+iy)L}\left[1-\tfrac{iGL}{4\nu}\right]+O\left(e^{\nu L}\nu^{-2}\right).\\
&(b')\quad \sinh(\mu(\lambda)L)=\tfrac{1}{2}e^{(x+iy)L}\left[1-\tfrac{iGL}{4\nu}\right]+O\left(e^{\nu L}\nu^{-2}\right),
\end{aligned} \tag{5.10}$$

The estimates for $\cos(\mu(\lambda)L)$ and $\sin(\mu(\lambda)L)$ remain the same as (c) and (d) in (5.9). The following approximations are valid for the products:

$$\begin{aligned}
(i)\quad \cosh(\mu(\lambda)L)\cos(\mu(\lambda)L) &= \tfrac{1}{2}e^{(x+iy)L}\left[1-\tfrac{iGL}{4\nu}\right]\left(1+O\left(\tfrac{1}{\nu^2}\right)\right)\\
&\quad\times\left\{\left[\cos(\nu L)+\tfrac{iGL}{4\nu}\sin(\nu L)\right]+O\left(\tfrac{1}{\nu^2}\right)\right\},\\
(ii)\quad \cosh(\mu(\lambda)L)\sin(\mu(\lambda)L) &= \tfrac{1}{2}e^{(x+iy)L}\left[1-\tfrac{iGL}{4\nu}\right]\left(1+O\left(\tfrac{1}{\nu^2}\right)\right)\\
&\quad\times\left\{\left[\sin(\nu L)-\tfrac{iGL}{4\nu}\cos(\nu L)\right]+O\left(\tfrac{1}{\nu^2}\right)\right\},
\end{aligned}\tag{5.11}$$

$$\begin{aligned}
(iii)\quad \sinh(\mu(\lambda)L)\cos(\mu(\lambda)L) &= \tfrac{1}{2}e^{(x+iy)L}\left[1-\tfrac{iGL}{4\nu}\right]\left(1+O\left(\tfrac{1}{\nu^2}\right)\right)\\
&\quad\times\left\{\left[\cos(\nu L)+\tfrac{iGL}{4\nu}\sin(\nu L)\right]+O\left(\tfrac{1}{\nu^2}\right)\right\}.
\end{aligned}$$

Substituting (5.11) into Eq.(4.19), we obtain the following approximation to the spectral equation:

$$\begin{aligned}
&\mu(\lambda)\left[H+i\nu^2(\lambda)\right]\left\{1+\frac{1}{2}e^{(x+iy)L}\left[1-\frac{iGL}{4\nu}\right]\left[\cos(\nu L)+\frac{iGL}{4\nu}\sin(\nu L)\right]\left(1+\text{Ø}\left(\frac{1}{\nu^2}\right)\right)\right\}\\
&+i\nu^2(\lambda)\Theta\left\{\frac{1}{2}e^{(x+iy)L}\left[1-\frac{iGL}{4\nu}\right]\left[\sin(\nu L)-\frac{iGL}{4\nu}\cos(\nu L)\right]\left(1+O\left(\frac{1}{\nu^2}\right)\right)\right.\\
&\left.+\frac{1}{2}e^{(x+iy)L}\left[1-\frac{iGL}{4\nu}\right]\left[\cos(\nu L)+\frac{iGL}{4\nu}\sin(\nu L)\right]\left(1+O\left(\frac{1}{\nu^2}\right)\right)\right\}=0.
\end{aligned}\tag{5.12}$$

Dividing both sides of (5.12) by $\frac{i}{2}\ \nu^2\, e^{(x+iy)L}\left[1-\frac{iGL}{4\nu}\right]$ we obtain:

$$\begin{aligned}
&\mu(\lambda)\left[1-\frac{iH}{\nu^2}\right]\left[\cos(\nu L)+\frac{iGL}{4\nu}\sin(\nu L)+O\left(\frac{1}{\nu^2}\right)\right]\\
&\quad+\Theta\left[(\sin(\nu L)+\cos(\nu L))+\frac{iGL}{4\nu}(\sin(\nu L)-\cos(\nu L))+O\left(\frac{1}{\nu^2}\right)\right]=0.
\end{aligned}\tag{5.13}$$

Moving all terms of the order ν^{-2} to the right hand-side and using (4.16) with $\lambda=\nu^2$ we obtain the following asymptotic form of the spectral equation:

$$\cos(\nu L)+\tfrac{iGL}{4\nu}\sin(\nu L)+\tfrac{\Theta}{\nu}\left[\sin(\nu L)+\cos(\nu L)\right]=O\left(\tfrac{1}{\nu^2}\right).\tag{5.14}$$

From now on our main object of interest is Eq.(5.14). By replacing the right-hand side of Eq.(5.14) with zero, we obtain a new equation which we call the model equation for (5.14):

$$\cos\,\widetilde{\nu}+\tfrac{iGL^2}{4\widetilde{\nu}}\sin\,\widetilde{\nu}+\tfrac{\Theta L}{\widetilde{\nu}}\left[\sin\,\widetilde{\nu}+\cos\,\widetilde{\nu}\right]=0,\quad \widetilde{\nu}=\nu\,L.\tag{5.15}$$

Using Rouche's Theorem one can readily show that the roots of Eqs.(5.14) and (5.15) are asymptotically close, i.e., the distance between corresponding roots tends to zero at the rate $O(n^{-2})$ as the number of a root pair goes to infinity. (For a detailed exposition of the estimates needed for application of Rouche's Theorem, see [16].) As $\widetilde{\nu} \to \infty$, Eq.(5.15) has an asymptotic form $\cos \widetilde{\nu} = O(\widetilde{\nu}^{-1})$, which means that the set of solutions of the model equation in S^+ can be given by the formula:

$$\widetilde{\nu}_n = \frac{\pi(2n+1)}{2} + O\left(\frac{1}{n}\right). \tag{5.16}$$

Now we look for solutions of Eq.(5.15) in the form (see, e.g., [8])

$$\widetilde{\nu}_n = \frac{\pi(2n+1)}{2} + \frac{a}{n} + O\left(\frac{1}{n^2}\right), \qquad n \to \infty. \tag{5.17}$$

Taking into account that

$$\cos \widetilde{\nu}_n = (-1)^{n+1} \sin\left(\frac{a}{n} + O\left(\frac{1}{n^2}\right)\right) = (-1)^{n+1}\left(\frac{a}{n} + O\left(\frac{1}{n^2}\right)\right),$$
$$\sin \widetilde{\nu}_n = (-1)^{n} \cos\left(\frac{a}{n} + O\left(\frac{1}{n^2}\right)\right) = (-1)^{n}\left(1 + O\left(\frac{1}{n^2}\right)\right),$$

we substitute (5.17) into (5.15) and obtain

$$-\tfrac{a}{n} + O\left(\tfrac{1}{n^2}\right) + \tfrac{iGL^2}{4\pi n}\left(1 + O\left(\tfrac{1}{n^2}\right)\right) + \tfrac{\Theta L}{\pi n}\left[1 - \tfrac{a}{n} + O\left(\tfrac{1}{n^2}\right)\right] = 0, \tag{5.18}$$

which yields

$$a = \frac{iGL^2 + 4\Theta L}{4\pi} \quad . \tag{5.19}$$

So we have obtained the following asymptotic approximation for the root chain of (5.15) from S^+:

$$\widetilde{\nu}_n = \frac{\pi(2n+1)}{2} + \frac{iGL^2 + 4\Theta L}{4\pi n} + O\left(\frac{1}{n^2}\right), \qquad n \to \infty. \tag{5.20}$$

From $\widetilde{\nu} = \nu\, L$ we obtain (5.8), which yields the desired result for λ_n^+. Now we recall that, as was stated at the beginning of our proof right before Theorem 5.2, we actually worked with the parameters introduced in (4.10). Returning to the original parameters we arrive at (5.1).

Thus, statement 1) of Theorem 5.3 is shown.

Now we prove statement 2) of Theorem 5.3. Let S^- be the strip in the upper half-plane such that $\nu = -x + iy$, $0 < x < \infty$, $0 \le y < d < \infty$. We will show that the assumption that there exists an infinite sequence of roots ν_n^- of (5.14) such that $\nu_n^- \in S^-$ leads to a contradiction. For $\nu \in S^-$, the following asymptotic representations can be readily derived as $|\nu| \to \infty$ ($x \to \infty$):

$$\begin{aligned}
&(a)\ \cosh(\mu(\lambda)L) = \tfrac{1}{2}e^{(x-iy)L}\left[1+\tfrac{iGL}{4\nu}\right] + O\left(\tfrac{e^{xL}}{\nu^2}\right),\\
&(b)\ \sinh(\mu(\lambda)L) = -\tfrac{1}{2}e^{(x-iy)L}\left[1+\tfrac{iGL}{4\nu}\right] + O\left(\tfrac{e^{xL}}{\nu^2}\right),\\
&(c)\ \cos(\mu(\lambda)L) = \left[\cos(\nu L) + \tfrac{iGL}{4\nu}\sin(\nu L)\right] + O\left(\tfrac{1}{\nu^2}\right),\\
&(d)\ \sin(\mu(\lambda)L) = \left[\sin(\nu L) - \tfrac{iGL}{4\nu}\cos(\nu L)\right] + O\left(\tfrac{1}{\nu^2}\right).
\end{aligned} \tag{5.21}$$

Combining appropriate formulas of (5.21), we obtain approximations for the products

$$\begin{aligned}
(i)\quad & \cosh(\mu(\lambda)L)\cos(\mu(\lambda)L) = \tfrac{1}{2}e^{(x-iy)L}\left\{\left[1+\tfrac{iGL}{4\nu}\right] + O\left(\tfrac{1}{\nu^2}\right)\right\}\\
& \times\left\{\left[\cos(\nu L) + \tfrac{iGL}{4\nu}\sin(\nu L)\right] + O\left(\tfrac{1}{\nu^2}\right)\right\}\\
& = \tfrac{1}{2}e^{(x-iy)L}\left\{\left[1+\tfrac{iGL}{4\nu}\right]\cos(\nu L) + \tfrac{iGL}{4\nu}\sin(\nu L) + O\left(\tfrac{1}{\nu^2}\right)\right\},\\
(ii)\quad & \cosh(\mu(\lambda)L)\sin(\mu(\lambda)L) = \tfrac{1}{2}e^{(x-iy)L}\left\{\left[1+\tfrac{iGL}{4\nu}\right] + O\left(\tfrac{1}{\nu^2}\right)\right\}\\
& \times\left\{\left[\sin(\nu L) - \tfrac{iGL}{4\nu}\cos(\nu L)\right] + O\left(\tfrac{1}{\nu^2}\right)\right\}\\
& = \tfrac{1}{2}e^{(x-iy)L}\left\{\left[1+\tfrac{iGL}{4\nu}\right]\sin(\nu L) - \tfrac{iGL}{4\nu}\cos(\nu L) + O\left(\tfrac{1}{\nu^2}\right)\right\},\\
(iii)\quad & \sinh(\mu(\lambda)L)\cos(\mu(\lambda)L) = -\tfrac{1}{2}e^{(x-iy)L}\left\{\left[1+\tfrac{iGL}{4\nu}\right] + O\left(\tfrac{1}{\nu^2}\right)\right\}\\
& \times\left\{\left[\cos(\nu L) + \tfrac{iGL}{4\nu}\sin(\nu L)\right] + O\left(\tfrac{1}{\nu^2}\right)\right\}\\
& = -\tfrac{1}{2}e^{(x-iy)L}\left\{\left[1+\tfrac{iGL}{4\nu}\right]\cos(\nu L) + \tfrac{iGL}{4\nu}\sin(\nu L) + O\left(\tfrac{1}{\nu^2}\right)\right\}.
\end{aligned} \tag{5.22}$$

Substituting formulas (5.22) into Eq.(4.19), we obtain

$$\begin{aligned}
&\mu(\lambda)(H+i\lambda)\left[1+\tfrac{1}{2}e^{(x-iy)L}\left\{\left[1+\tfrac{iGL}{4\nu}\right]\cos(\nu L) + \tfrac{iGL}{4\nu}\sin(\nu L) + O\left(\tfrac{1}{\nu^2}\right)\right\}\right]\\
&+ i\lambda\Theta\left[\tfrac{1}{2}e^{(x-iy)L}\left\{\left[1+\tfrac{iGL}{4\nu}\right]\sin(\nu L) - \tfrac{iGL}{4\nu}\cos(\nu L) + O\left(\tfrac{1}{\nu^2}\right)\right\}\right.\\
&\left. - \tfrac{1}{2}e^{(x-iy)L}\left\{\left[1+\tfrac{iGL}{4\nu}\right]\cos(\nu L) + \tfrac{iGL}{4\nu}\sin(\nu L) + O\left(\tfrac{1}{\nu^2}\right)\right\}\right] = 0.
\end{aligned} \tag{5.23}$$

After dividing both sides by $\frac{i}{2}\ \lambda\ e^{(x-iy)L}$ and moving all terms of order ν^{-2} to the right-hand side, we obtain the following equation:

$$\begin{aligned}
&\mu(\lambda)\left(1-\tfrac{iH}{\lambda}\right)\left\{\cos(\nu L) + \tfrac{iGL}{4\nu}\left(\cos(\nu L)+\sin(\nu L)\right)\right\} +\\
&\Theta\left\{\sin(\nu L) - \left[1-\tfrac{iGL}{2\nu}\right]\cos(\nu L)\right\} = O\left(\tfrac{1}{\nu^2}\right),
\end{aligned}$$

which can be simplified to

$$\cos(\nu L) + \left(\tfrac{iGL}{4\nu} + \tfrac{\Theta}{\nu}\right)\sin(\nu L) = O\left(\tfrac{1}{\nu^2}\right). \tag{5.24}$$

As $|\nu| \to \infty$ Eq.(5.24) obtains an asymptotic form $\cos(\nu L) = O\left(\widetilde{\nu}^{-1}\right)$, which means that the set of roots of this equation can be given by the formula

$$\nu_m = \frac{\pi(1-2m)}{2L} + O\left(\frac{1}{m}\right), \qquad m \to \infty. \tag{5.25}$$

Now we look for solutions of (5.24) in the form [8]

$$\nu_m = \frac{\pi(1-2m)}{2L} + \frac{b}{mL} + O\left(\frac{1}{m^2}\right), \qquad m \to \infty. \tag{5.26}$$

Taking into account that

$$\cos(\nu_m L) = \cos\left(\frac{\pi(1-2m)}{2}\right)\cos\left(\frac{b}{m} + O\left(\frac{1}{m^2}\right)\right) - \sin\left(\frac{\pi(1-2m)}{2}\right)\sin\left(\frac{b}{m} + O\left(\frac{1}{m^2}\right)\right)$$
$$= (-1)^{m+1}\sin\left(\frac{b}{m} + O\left(\frac{1}{m^2}\right)\right) = (-1)^{m+1}\left(\frac{b}{m} + O\left(\frac{1}{m^2}\right)\right),$$

$$\sin(\nu_m L) = \sin\left(\frac{b}{m} + O\left(\frac{1}{m^2}\right)\right) = (-1)^m \cos\left(\frac{b}{m} + O\left(\frac{1}{m^2}\right)\right)$$
$$= (-1)^m \left(1 + O\left(\frac{1}{m^2}\right)\right),$$

we substitute the above approximations into Eq.(5.24) to obtain

$$(-1)^{m+1}\frac{b}{m} + (-1)^{m+1}\left(\frac{iGL^2}{4\pi m} + \frac{\Theta L}{\pi m}\right) + O\left(\frac{1}{m^2}\right) = O\left(\frac{1}{m^2}\right),$$

which can happen only for $b = -\left(\frac{iGL^2}{4\pi} + \frac{\Theta L}{\pi}\right)$. However, with such b, the set ν_n^- does not belong to the upper half-plane, which is a contradiction. □

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