# Asymptotic analysis of the modes of wave propagation in a piezoelectric solid cylinder

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An asymptotic method due to "Achenbach" is used to analyze the longitudinal and circumferential modes of wave propagation in a piezoelectric solid circular cylinder of crystal class (6 mm) or ceramics ($\infty$m). Information obtained in this method is useful for the frequency spectrum at long wavelengths. In this method the displacement components, electric potential, and the frequency are expressed as power series of the dimensionless wavenumber, $\epsilon = 2\pi \times$ radius/wavelength. Substituting these expansions in the field equations and the boundary conditions, a system of coupled second-order inhomogeneous ordinary differential equations with radial coordinate as the independent variable is obtained by collecting the terms of same order $\epsilon^m$. Integration of such systems of differential equations yields the various terms in the series expansions for the above modes and for the whole range of frequencies, when the real valued dimensionless wavenumber is less than unity ($0<\epsilon<1$). To test the correctness of the present scheme, the roots of the exact frequency equation are computed and the results thus obtained are compared with the results obtained in the present analysis.

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## INTRODUCTION

Piezoelectrics, discovered a hundred years ago, are now used in many branches of science and engineering. The early developments of piezoelectricity and its applications up to about 1940 are authoritatively treated by Cady.[1] Detailed references to the current contributions on the vibration problems of piezoelectricity can be found in a recent survey article by Dokmeci.[2]

Piezoelectric crystals and ceramics are among the principal generators and detectors of acoustic power. Piezoceramic circular cylinders[3] are commonly used as hydrophones in undersea and geophysical investigations. Additionally they have been used as pressure transducers in biological experiments.

The exact frequency equation for piezoelectric circular cylindrical shell of hexagonal (6 mm) class was first obtained by Paul.[4] Ambardar and Ferris[5] treated harmonic wave propagation in a piezoelectric two-layered cylinder of dissimilar but transversely isotropic materials such as bone. The analysis of the frequency equation was not undertaken in the above studies, obviously, due to its complexity.

In this paper we present a study of dispersion relation in the neighborhood of cutoff frequencies using a method developed by Achenbach.[6-8] This method, which is based on an asymptotic integration of elastodynamic field equations is applicable to the whole range of frequencies, but for the range of the dimensionless wavenumber $0<\epsilon<1$, where $\epsilon=2\pi\times$radius/wavelength. The same method has been applied by the present authors to study the torsional modes of wave propagation in a piezoelectric solid circular cylinder of (622) class.[9]

In this method, the displacement components, the electric potential, and the frequency are first expressed as power series of the dimensionless wavenumber $\epsilon = 2\pi\times$ radius/wavelength. Substituting these expansions in the field equations and the boundary conditions, a system of coupled second-order, inhomogeneous, ordinary differential equations with radial coordinate as independent variable is obtained by collecting the terms of same order $\epsilon^m$. Integration of such systems of differential equations yields the various terms in the series expansions for the displacements, electric potential, and the frequency for the above modes and for the whole range of frequencies, but for $0<\epsilon<1$.

To test the correctness of the present scheme, the roots of the exact frequency equation are computed in double precision for $0<\epsilon\leqslant 0.1$ by using the bisection method and the results thus obtained are compared with the results obtained in the present analysis. Both the results fairly agree up to three decimal places.

Information obtained in this analysis is useful for the frequency spectrum at long wavelengths. The advantage of the present approach is that here all modes are included over the whole range of frequencies and the range of the wavenumber can be extended by including the terms of higher powers in $\epsilon$.

## I. FORMULATION OF THE PROBLEM

We use the cylindrical polar coordinate system $(r,\theta,z)$ with the axis of the cylinder as $z$ axis. The radius of the cylinder is taken to be $a$. The equations of motion and the Gaussian equation for the cylinder of hexagonal (6 mm) class or ceramics ($\infty$ m) are[4]

$$\begin{aligned}
&c_{11}(u_{,rr}+r^{-1}u_{,r}+r^{-2}u)+c_{66}r^{-2}u_{,\theta\theta}+c_{44}u_{,zz}\\
&\quad+(c_{66}+c_{12})r^{-1}v_{,r\theta}-(c_{11}+c_{66})r^{-2}v_{,\theta}\\
&\quad+(c_{44}+c_{13})w_{,rz}+(e_{31}+e_{15})\varphi_{,rz}=\rho u_{,tt},\\
&(c_{66}+c_{12})r^{-1}u_{,r\theta}+(c_{11}+c_{66})r^{-2}u_{,\theta}\\
&\quad+c_{66}(v_{,rr}+r^{-1}v_{,r}-r^{-2}v)+c_{44}v_{,zz}+c_{11}r^{-2}v_{,\theta\theta}\\
&\quad+(c_{44}+c_{13})r^{-1}w_{,\theta z}+(e_{31}+e_{15})r^{-1}\varphi_{,\theta z}=\rho v_{,tt}, \qquad (1)\\
&(c_{44}+c_{13})(u_{,rz}+r^{-1}u_{,z}+r^{-1}v_{,\theta z})\\
&\quad+c_{44}(w_{,rr}+r^{-1}w_{,r}+r^{-2}w_{,\theta\theta})+c_{33}w_{,zz}\\
&\quad+e_{33}\varphi_{,zz}+e_{15}(\varphi_{,rr}+r^{-1}\varphi_{,r}+r^{-2}\varphi_{,\theta\theta})=\rho w_{,tt},
\end{aligned}$$

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and

$$e_{15}(w_{,rr}+r^{-1}w_{,r}+r^{-2}w_{,\theta\theta})+(e_{31}+e_{15})$$
$$\times(u_{,rz}+r^{-1}u_{,z}+r^{-1}v_{,\theta z})+e_{33}w_{,zz}-\epsilon_{33}\varphi_{,zz}$$
$$-\epsilon_{11}(\varphi_{,rr}+r^{-1}\varphi_{,r}+r^{-2}\varphi_{,\theta\theta})=0\,,$$

where $u$, $v$, and $w$ are, respectively, the components of displacement in radial, circumferential, and the axial directions, $\varphi$ is the electric potential, $c_{ij}$, $e_{ij}$, and $\epsilon_{ij}$ are, respectively, the elastic, the piezoelectric, and the dielectric constants, and $\rho$ is the density of the material. The comma followed by a subscript denotes the partial derivative with respect to that subscript.

To study the modes of wave propagation of free harmonic waves we seek the mode shapes and the electric potential in the following form,

$$\begin{aligned}
u&=U(r)\cos n\theta\exp i(\alpha z-\omega t)\,,\\
v&=V(r)\sin n\theta\exp i(\alpha z-\omega t)\,,\\
w&=iW(r)\cos n\theta\exp i(\alpha z-\omega t)\,,\\
\varphi&=i(c_{44}/e_{33})\Phi(r)\cos n\theta\exp i(\alpha z-\omega t)\,,
\end{aligned}\tag{2}$$

where $\alpha$ is the wavenumber, $\omega$ is the circular frequency, and $n(\geqslant 0)$ is an integer. There are infinitely many modes of wave propagation for each value of $n$ and $\alpha$.

First let us introduce the new dimensionless radial coordinate $x$ and the dimensionless wavenumber $\epsilon$ given by

$$x=r/a \text{ and } \epsilon=2\pi a/\text{wavelength}\,.\tag{3}$$

If we substitute Eqs. (2) into Eqs. (1) and use Eqs. (3), the following equations will result:

$$\begin{aligned}
&\bar{c}_{11}\left(\frac{d^2U}{dx^2}+x^{-1}\frac{dU}{dx}-x^{-2}U\right)-(\bar{c}_{66}x^{-2}n^2+\epsilon^2)U\\
&+(\bar{c}_{11}+\bar{c}_{66})x^{-2}nV+(\bar{c}_{66}+\bar{c}_{12})x^{-1}n\frac{dV}{dx}\\
&-(1+\bar{c}_{13})\epsilon\frac{dW}{dx}-(\bar{e}_{31}+\bar{e}_{15})\epsilon\frac{d\Phi}{dx}=-(ca)^2U\,,
\end{aligned}\tag{4}$$

$$\begin{aligned}
&-(\bar{c}_{11}+\bar{c}_{66})x^{-2}nU-(\bar{c}_{66}+\bar{c}_{12})x^{-1}n\frac{dU}{dx}\\
&+\bar{c}_{66}\left(\frac{d^2V}{dx^2}+x^{-1}\frac{dV}{dx}-x^{-2}V\right)-(\epsilon^2+\bar{c}_{11}x^{-2}n^2)V\\
&+(1+\bar{c}_{13})x^{-1}n\epsilon W+(\bar{e}_{31}+\bar{e}_{15})x^{-1}n\epsilon\Phi=-(ca)^2V\,,
\end{aligned}\tag{5}$$

$$\begin{aligned}
&(1+\bar{c}_{13})\epsilon\left(x^{-1}U+\frac{dU}{dx}\right)+x^{-1}n\epsilon(1+\bar{c}_{13})V+\frac{d^2W}{dx^2}+x^{-1}\frac{dW}{dx}\\
&-(\bar{c}_{33}\epsilon^2+x^{-2}n^2)W+\bar{e}_{15}\left(\frac{d^2\Phi}{dx^2}+x^{-1}\frac{d\Phi}{dx}\right)\\
&-(\epsilon^2+\bar{e}_{15}x^{-2}n^2)\Phi=-(ca)^2W\,,
\end{aligned}\tag{6}$$

$$\begin{aligned}
&(\bar{e}_{31}+\bar{e}_{15})\epsilon\left(\frac{dU}{dx}+x^{-1}U\right)+(\bar{e}_{31}+\bar{e}_{15})x^{-1}n\epsilon V\\
&+\bar{e}_{15}\left(\frac{d^2W}{dx^2}+x^{-1}\frac{dW}{dx}\right)-(\bar{e}_{15}x^{-2}n^2+\epsilon^2)W\\
&+k_1^{-2}\left(\frac{d^2\Phi}{dx^2}+x^{-1}\frac{d\Phi}{dx}\right)+(k_3^{-2}\epsilon^2+x^{-2}n^2k_1^{-2})\Phi=0\,,
\end{aligned}\tag{7}$$

where

$$\begin{aligned}
&c^2=\rho\omega^2/c_{44},\quad \bar{c}_{ij}=c_{ij}/c_{44},\\
&\bar{e}_{ij}=e_{ij}/e_{33}\,,\text{ and } k_i^2=e_{33}^2/(c_{44}\epsilon_{ii})\,.
\end{aligned}\tag{8}$$

We write

$$[U(x),V(x),W(x),\Phi(x)]=\sum_{m=0}^{\infty}[U^m(x),V^m(x),W^m(x),\Phi^m(x)]\epsilon^m\,,\tag{9}$$

where $\epsilon<1$. Computing frequencies as functions of wavenumbers being our aim, we also write

$$(ca)=\sum_{m=0}^{\infty}\omega_m\epsilon^m\,.\tag{10}$$

After substituting the Eqs. (9) and (10) into Eqs. (4)–(7) and using a theorem on product of two power series which is given as Eq. (18) in Ref. 6, the coefficients of $\epsilon^m$ are collected. By equating the coefficients of $\epsilon^m$ to zero, we get the following system of equations:

$$\begin{aligned}
L_1[U^{(m)},V^{(m)}]=U^{(m-2)}&+(1+\bar{c}_{13})\frac{dW^{(m-1)}}{dx}\\
&+(\bar{e}_{31}+\bar{e}_{15})\frac{d\Phi^{(m-1)}}{dx}-T_u^{(m-1)}\,,
\end{aligned}\tag{11}$$

$$\begin{aligned}
L_2[U^{(m)},V^{(m)}]=V^{(m-2)}&-(1+\bar{c}_{13})x^{-1}nW^{(m-1)}\\
&-(\bar{e}_{15}+\bar{e}_{31})nx^{-1}\Phi^{(m-1)}-T_v^{(m-1)}\,,
\end{aligned}\tag{12}$$

$$\begin{aligned}
M_1[W^{(m)},\Phi^{(m)}]=&-(1+\bar{c}_{13})\left(\frac{dU^{(m-1)}}{dx}+x^{-1}U^{(m-1)}\right)\\
&-(1+\bar{c}_{13})nx^{-1}V^{(m-1)}\\
&+\bar{c}_{33}W^{(m-2)}+\Phi^{(m-2)}-T_w^{(m-1)},
\end{aligned}\tag{13}$$

$$\begin{aligned}
M_2[W^{(m)},\Phi^{(m)}]=&-k_3^{-2}\Phi^{(m-2)}+W^{(m-2)}-(\bar{e}_{31}+\bar{e}_{15})\\
&\times\left(\frac{dU^{(m-1)}}{dx}+x^{-1}U^{(m-1)}\right)\\
&-(\bar{e}_{31}+\bar{e}_{15})nx^{-1}V^{(m-1)}\,,
\end{aligned}\tag{14}$$

where $L_1[\ ]$, $L_2[\ ]$, $M_1[\ ]$, and $M_2[\ ]$ in the Eqs. (11)–(14) are given by

$$L_1[U^{(m)}, V^{(m)}]$$

$$= \bar{c}_{11}\left(\frac{d^2U^{(m)}}{dx^2} + x^{-1}\frac{dU^{(m)}}{dx}\right) + [\omega_0^2 - x^{-2}(\bar{c}_{11} + n^2\bar{c}_{66})]U^{(m)}$$

$$- (\bar{c}_{11} + \bar{c}_{66})nx^{-2}V^{(m)} + (\bar{c}_{12} + \bar{c}_{66})nx^{-1}\frac{dV^{(m)}}{dx}, \qquad (15)$$

$$L_2[U^{(m)}, V^{(m)}]$$

$$= \bar{c}_{66}\left(\frac{d^2V^{(m)}}{dx^2} + x^{-1}\frac{dV^{(m)}}{dx}\right) + [\omega_0^2 - x^{-2}(\bar{c}_{66} + n^2\bar{c}_{11})]V^{(m)}$$

$$- (\bar{c}_{11} + \bar{c}_{66})nx^{-2}U^{(m)} - (\bar{c}_{12} + \bar{c}_{66})nx^{-1}\frac{dU^{(m)}}{dx}, \qquad (16)$$

$$M_1[W^{(m)}, \Phi^{(m)}] = \frac{d^2W^{(m)}}{dx^2} + x^{-1}\frac{dW^{(m)}}{dx} + (\omega_0^2 - x^{-2}n^2)W^{(m)}$$

$$+ \bar{e}_{15}\left(\frac{d^2\Phi^{(m)}}{dx^2} + x^{-1}\frac{d\Phi^{(m)}}{dx} - n^2x^{-2}\Phi^{(m)}\right), \qquad (17)$$

$$M_2[W^{(m)}, \Phi^{(m)}] = -k_1^{-2}\left(\frac{d^2\Phi^{(m)}}{dx^2} + x^{-1}\frac{d\Phi^{(m)}}{dx} - n^2x^{-2}\Phi^{(m)}\right)$$

$$+ \bar{e}_{15}\left(\frac{d^2W^{(m)}}{dx^2} + x^{-1}\frac{dW^{(m)}}{dx} - n^2x^{-2}W^{(m)}\right), \qquad (18)$$

and

$$T_u^{(m-1)} = \sum_{i=0}^{(m-1)} F_{m-i}U^{(i)}, \qquad (19)$$

with analogous definitions to $T_v^{(m-1)}$ and $T_w^{(m-1)}$ where

$$F_{m-i} = \sum_{j=0}^{m-i} \omega_{m-i-j}\omega_j. \qquad (20)$$

Equations (11)–(20) are valid for all $m \geq 0$ with an understanding that the quantities with negative superscripts vanish identically.

The boundary conditions are

(i) $T_{rr} = T_{r\theta} = T_{rz} = 0$, at $r = a$, (21)

i.e., the curved surface of the cylinder is kept traction free.

(ii) $\varphi = 0$, at $r = a$, (22)

i.e., the curved surface is coated with electrodes which are shorted.

In Eqs. (21) $T_{rr}$, $T_{r\theta}$, and $T_{rz}$ are the stresses given by

$$T_{rr} = c_{11}u_{,r} + c_{12}r^{-1}(u + v_{,\theta}) + c_{13}w_{,z} + e_{31}\varphi_{,z},$$
$$T_{r\theta} = c_{66}(r^{-1}u_{,\theta} + v_{,r} - r^{-1}v), \qquad (23)$$
$$T_{rz} = c_{44}(w_{,r} + u_{,z}) + e_{15}\varphi_{,r}.$$

Substituting Eqs. (2), (3), and (9) into the boundary condition Eqs. (21) and (22) and collecting the coefficients of $\epsilon^m$ and then equating them to zero, we get the relevant boundary conditions at $x = 1$ to Eqs. (11)–(14) as

$$\bar{c}_{11}\frac{dU^{(m)}}{dx} + \bar{c}_{12}(U^{(m)} + nV^{(m)}) = \bar{c}_{13}W^{(m-1)} + \bar{e}_{31}\Phi^{(m-1)}, \qquad (24)$$

$$nU^{(m)} - \frac{dV^{(m)}}{dx} + V^{(m)} = 0, \qquad (25)$$

$$\frac{dW^{(m)}}{dx} + \bar{e}_{15}\frac{d\Phi^{(m)}}{dx} = -U^{(m-1)}, \qquad (26)$$

$$\Phi^{(m)} = 0. \qquad (27)$$

For simplicity hereafter we shall drop the bars.

The Eqs. (24)–(27) are also valid for all integers $m \geq 0$ with the same understanding that the terms with negative superscripts vanish identically.

Since the quantities with negative superscripts vanish identically, for $m = 0$, the system of inhomogeneous coupled ordinary differential Eqs. (11)–(14) and the boundary conditions [Eqs. (24)–(27)] reduce to homogeneous ones. However, for $m \geq 1$, the Eqs. (11)–(14) and the boundary conditions will have lower order coefficients of expansions in their nonhomogeneous terms. For increasing $m$, the system can be solved by the method of integration. Thus, if we like to solve the system of equations for $m = m_1$, we have to solve first all systems of equations for $m < m_1$.

## II. SOLUTION OF THE PROBLEM

As a first step of solving Eqs. (11)–(14) subject to boundary conditions [Eqs. (24)–(27)] we write

$$L_1[U^{(m)}, V^{(m)}] = h_1^{(m)}(x)$$

and (28)

$$L_2[U^{(m)}, V^{(m)}] = h_2^{(m)}(x),$$

where $L_1[\ ]$ and $L_2[\ ]$ are given by Eqs. (11)–(12). We now decouple the Eqs. (11) and (12) by introducing potential functions $F^{(m)}(x)$ and $G^{(m)}(x)$ in terms of which the displacement components $U^{(m)}$ and $V^{(m)}$ are expressed as

$$U^{(m)}(x) = a_1^2\frac{dF^{(m)}}{dx} + a_2^2nx^{-1}G^{(m)},$$
$$V^{(m)}(x) = -a_1^2nx^{-1}F^{(m)} - a_2^2\frac{dG^{(m)}}{dx}, \qquad (29)$$

where

$$a_1^2 = 1/c_{11} \text{ and } a_2^2 = 1/c_{66}. \qquad (30)$$

Substitution of Eqs. (29) into Eqs. (28) results in

$$\frac{dP_1^{(m)}}{dx} + nx^{-1}P_2^{(m)} = h_1^{(m)}(x),$$
$$\frac{dP_2^{(m)}}{dx} + nx^{-1}P_1^{(m)} = -h_2^{(m)}(x), \qquad (31)$$

where

$$P_1^{(m)}(x) = \frac{d^2F^{(m)}}{dx^2} + x^{-1}\frac{dF^{(m)}}{dx} + [(a_1\omega_0)^2 - n^2x^{-2}]F^{(m)} \qquad (32)$$

and

$$P_2^{(m)}(x) = \frac{d^2G^{(m)}}{dx^2} + x^{-1}\frac{dG^{(m)}}{dx} + [(a_2\omega_0)^2 - n^2x^{-2}]G^{(m)}. \qquad (33)$$

Eliminating $P_2^{(m)}$ from Eqs. (31) we get

$$\frac{d^2P_1^{(m)}}{dx^2} + x^{-1}\frac{dP_1^{(m)}}{dx} - n^2x^{-2}P_1^{(m)} = x^{-2}H^{(m)}(x)\,, \tag{34}$$

where

$$H^{(m)}(x) = x^2\frac{dh_1^{(m)}}{dx} + xh_1^{(m)}(x) + nxh_2^{(m)}(x)\,. \tag{35}$$

Solution of Eq. (34) is given by

$$P_1^{(m)}(x) = B_1^{(m)}x^n + B_2^{(m)}x^{-n} + P_{1p}^{(m)}(x)\,, \tag{36}$$

where $B_1^{(m)}x^n + B_2^{(m)}x^{-n}$ is a complementary solution and $P_{1p}^{(m)}(x)$ is a particular solution. Substitution of Eq. (36) into the first of Eqs. (31) we obtain $P_2^{(m)}(x)$ as

$$P_2^{(m)}(x) = -\left(B_1^{(m)}x^n - B_2^{(m)}x^{-n} - n^{-1}xh_1^{(m)}(x) + n^{-1}x\frac{dP_{1p}^{(m)}}{dx}(x)\right). \tag{37}$$

We get $F^{(m)}$ and $G^{(m)}$ by solving Eqs. (32)–(33) in which $P_1^{(m)}$ and $P_2^{(m)}$ are given by Eqs. (36) and (37). The potential functions again contain a particular solution and a complementary solution. If we consider the particular solutions due to terms $x^{\pm m}$, for $\omega_0 \neq 0$, we find

$$\begin{aligned}(F_p^{(m)})_1 &= B_1^{(m)}x^n/(a_1\omega_0)^2\,,\\ (G_p^{(m)})_1 &= -B_1^{(m)}x^n/(a_2\omega_0)^2\,,\end{aligned} \tag{38}$$

where the terms with negative exponents are omitted because they give unbounded displacements for $x=0$.

If we substitute Eqs. (38) into Eqs. (29), we see that the displacements $U^{(m)}$ and $V^{(m)}$ vanish identically. Thus for $\omega_0 \neq 0$ we need not include the complementary solutions in computing $U^{(m)}$ and $V^{(m)}$. However, for $\omega_0 = 0$, we must include complementary solutions, at least those that give bounded displacements.

To obtain some terms in the expansions, we begin with homogeneous coupled differential equations for $U^{(0)}$, $V^{(0)}$, $W^{(0)}$, and $\Phi^{(0)}$ which are obtained by putting $m=0$ in Eqs. (11)–(19). We first consider the motion for $\omega_0 \neq 0$ as $\epsilon \to 0$. For $m=0$, the solutions of Eqs. (32) and (33) yielding bounded displacements at $x=0$ are given by

$$F^{(0)}(x) = C_1^{(n0)}J_n(a_1\omega_0x)$$

and (39)

$$G^{(0)}(x) = C_2^{(n0)}J_n(a_2\omega_0x)\,,$$

where $C_1^{(n0)}$ and $C_2^{(n0)}$ are constants. Substituting Eqs. (39) into Eqs. (29), we get the corresponding displacements as

$$U^{(0)}(x) = C_1^{(n0)}a_1^2(a_1\omega_0)J_n'(a_1\omega_0x) + C_2^{(n0)}a_2^2nx^{-1}J_n(a_2\omega_0x)\,, \tag{40}$$

$$V^{(0)}(x) = -C_1^{(n0)}a_1^2nx^{-1}J_n(a_2\omega_0x) - C_2^{(n0)}a_2^2(a_2\omega_0)J_n'(a_2\omega_0x)\,,$$

where a prime denotes a derivative with respect to the argument. Solving Eqs. (13) and (14) for $m=0$, we obtain

$$W^{(0)}(x) = C_3^{(n0)}(a_3\omega_0)^2J_n(a_3\omega_0x) \tag{41}$$

and

$$\Phi^{(0)}(x) = C_3^{(n0)}(a_3\omega_0)^2e_{15}k_1^2J_n(a_3\omega_0x) - C_4^{(n0)}k_1^2x^n\,, \tag{42}$$

where

$$a_3^2 = (1+e_{15}k_1^2)^{-1} \tag{43}$$

and $(a_3\omega_0)^2$ is introduced for convenience.

Substitution of Eqs. (40)–(42) in the corresponding boundary conditions, results in an equation of the form

$$A_{ij}C_j^{(n0)} = 0\,,\quad i,j = 1,2,3,4\,, \tag{44}$$

where the components of the matrix $A_{ij}$ are

$$\begin{aligned}
A_{11} &= a_1^2[2c_{66}(a_1\omega_0)J_{n+1}(a_1\omega_0) + 2c_{66}n(n-1)\\
&\quad\times J_n(a_1\omega_0) - c_{11}(a_1\omega_0)^2J_n(a_1\omega_0)]\,,\\
A_{12} &= 2n[(n-1)J_n(a_2\omega_0) - (a_2\omega_0)J_{n+1}(a_2\omega_0)]\,,\\
A_{21} &= 2na_1^2[(n-1)J_n(a_1\omega_0) - (a_1\omega_0)J_{n+1}(a_1\omega_0)]\,,\\
A_{22} &= a_2^2\{2(a_2\omega_0)J_{n+1}(a_2\omega_0) + [2n(n-1) - (a_2\omega_0)^2]J_n(a_2\omega_0)\}\,,\\
A_{23} &= A_{24} = 0\,,\\
A_{31} &= A_{32} = 0\,,\\
A_{33} &= -a_3\omega_0^3J_{n+1}(a_3\omega_0) + n\omega_0^2J_n(a_3\omega_0)\,,\\
A_{34} &= -nk_1^2e_{15}\,,\\
A_{41} &= A_{42} = 0\,,\\
A_{43} &= (a_3\omega_0)^2e_{15}k_1^2J_n(a_3\omega_0)\,,\\
A_{44} &= -k_1^2\,.
\end{aligned} \tag{45}$$

Equations (44) can be satisfied in two ways, either,

$$D_c = \begin{vmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{vmatrix} = 0 \text{ and } C_3^{(n0)} = C_4^{(n0)} = 0 \tag{46}$$

or

$$D_l = \begin{vmatrix} A_{33} & A_{34}\\ A_{43} & A_{44}\end{vmatrix} = 0 \text{ and } C_1^{(n0)} = C_2^{(n0)} = 0\,. \tag{47}$$

To investigate the circumferential modes we have to start with Eq. (46) whereas for longitudinal modes, the choice is given by Eq. (47).

For $\omega_0 \equiv 0$, since we have to include the complementary solutions of Eq. (36), we get the potential functions for $m=0$ as

$$F^{(0)} = C_1^{(n0)}x^n + [B_1^{(0)}/4(n+1)]x^{n+2}\,, \tag{48}$$

$$G^{(0)} = C_2^{(n0)}x^n - [B_1^{(0)}/4(n+1)]x^{n+2}\,, \tag{49}$$

substituting these equations in Eqs. (29), we get the displacement components $U^{(0)}$ and $V^{(0)}$ as

$$U^{(0)} = nD^{(n0)}x^{n-1} + \left(\frac{B_1^{(0)}}{4(n+1)}\right)[(n+2)a_1^2 - na_2^2]x^{n+1}\,, \tag{50}$$

$$V^{(0)} = -nD^{(n0)}x^{n-1} - \left(\frac{B_1^{(0)}}{4(n+1)}\right)[na_1^2 - (n+2)a_2^2]x^{n+1}\,, \tag{51}$$

where $D^{(n0)} = a_1^2C_1^{(n0)} + a_2^2C_2^{(n0)}$ is an arbitrary constant. From Eqs. (13) and (14) we obtain (for $\omega_0 \equiv 0$),

$$W^{(0)} = C_3^{(n0)}x^n\,, \tag{52}$$

$$\Phi^{(0)} = C_3^{(n0)} e_{15} k_1^2 x^n - C_4^{(n0)} k_1^2 x^n . \tag{53}$$

When Eqs. (50)–(53) are substituted in the relevant boundary conditions, we obtain

$$2a_2^{-2} n(n-1) D^{(n0)} + \{C_{12} + \tfrac{1}{2} a_2^{-2}[(n+2)a_1^2 - na_2^2]\} B_1^{(0)} = 0 , \tag{54}$$

$$2n(n-1)D^{(n0)} + \tfrac{1}{2} n(a_1^2 - a_2^2) B_1^{(0)} = 0 , \tag{55}$$

$$nC_3^{(n0)} + ne_{15}[e_{15}k_1^2 C_3^{(n0)} - C_4^{(n0)} k_1^2] = 0 , \tag{56}$$

$$C_3^{(n0)} k_1^2 e_{15} - C_4^{(n0)} k_1^2 = 0 . \tag{57}$$

Equations (54)–(57) are satisfied for

$$n = 0, \quad B_1^{(0)} = 0, \text{ and } C_3^{(n0)} e_{15} = C_4^{(n0)} \tag{58}$$

or

$$n = 1, \quad B_1^{(0)} = C_3^{(n0)} = C_4^{(n0)} = 0 . \tag{59}$$

For $n=0$, $C_3^{(n0)}$ and $C_4^{(n0)}$ remain arbitrary and the components of displacement and the electric potential are given by

$$U^{(0)} = 0, \quad V^{(0)} = 0, \quad W^{(0)} = C_3^{(0)}, \text{ and } \Phi^{(0)} = 0 . \tag{60}$$

For $n=1$, $B_1^{(0)} = C_3^{(n0)} = C_4^{(n0)} = 0$ and

$$U^{(0)} = D^{(10)}, \quad V^{(0)} = -D^{(10)}, \quad W^{(0)} = 0, \text{ and } \Phi^{(0)} = 0 . \tag{61}$$

Equations (60) and (61) show that the modes for which the frequency vanishes as the wavelength increases beyond bounds behave as if the medium is nonpiezoelectric.

## III. LONGITUDINAL MODES

Since, in the case of longitudinal modes $C_1^{(n0)} = C_2^{(n0)} = 0$ [Eq. (47)], the zeroth coefficients for $\omega_0 \neq 0$ are $U^{(0)} = V^{(0)} = 0$, and $W^{(0)}$ and $\Phi^{(0)}$ are given by Eqs. (41) and (42). $\omega_0$ is the solution of the equation $D_l = 0$, i.e.,

$$(a_3\omega_0) J_{n+1}(a_3\omega_0) - n(1 - a_3^2 e_{15}^2 k_1^2) J_n(a_3\omega_0) = 0 . \tag{62}$$

By using Eqs. (28)–(35), the system of Eqs. (11)–(14) can be solved. For $n=1$ and $\omega_0 \neq 0$ the solutions are

$$\begin{aligned} U^{(1)} = {} & C_1^{(n1)} a_1^2 (a_1\omega_0) J_n'(a_1\omega_0 x) + C_2^{(n1)} a_2^2 n x^{-1} \\ & \times J_n(a_2\omega_0 x) + C_3^{(n0)} [R_0 R_1 a_1^2 (a_3\omega_0) J_n'(a_3\omega_0 x) \\ & - (e_{31} + e_{15}) k_1^2 e_{15} a_3^2 J_n(a_3\omega_0) n x^{n-1}] , \end{aligned} \tag{63}$$

$$\begin{aligned} V^{(1)} = {} & -C_1^{(n1)} n x^{-1} a_1^2 J_n(a_1\omega_0 x) - C_2^{(n1)} a_2^2 (a_2\omega_0) \\ & \times J_n'(a_2\omega_0 x) - C_3^{(n0)} [R_0 R_1 n x^{-1} a_1^2 J_n(a_3\omega_0 x) \\ & - (e_{31} + e_{15}) k_1^2 e_{15} a_3^2 J_n(a_3\omega_0) n x^{n-1}] , \end{aligned} \tag{64}$$

$$\begin{aligned} W^{(1)} = {} & C_3^{(n1)} (a_3\omega_0)^2 J_n(a_3\omega_0 x) + C_3^{(n0)} \\ & \times a_3^2 \omega_0^2 \omega_1 x J_{n-1}(a_3\omega_0 x) , \end{aligned} \tag{65}$$

$$\begin{aligned} \Phi^{(1)} = {} & C_3^{(n1)} e_{15} k_1^2 (a_3\omega_0)^2 [J_n(a_3\omega_0 x) - J_n(a_3\omega_0) x^n] \\ & + C_3^{(n0)} e_{15} k_1^2 \omega_0^2 \omega_1 a_1^3 [x J_{n-1}(a_3\omega_0 x) - J_{n-1}(a_3\omega_0) x^n] , \end{aligned} \tag{66}$$

where

$$R_0 = (a_3^2 - a_1^2)^{-1} \text{ and } R_1 = [(1 + c_{13}) + (e_{31} + e_{15}) e_{15} k_1^2] . \tag{67}$$

We determine the constants $C_1^{(n1)}$ and $C_2^{(n1)}$ and the frequency coefficient $\omega_1$ by using Eqs. (63)–(66) into the boundary conditions [Eqs. (24)–(27)]. Substitution of Eqs. (63) and (64) into the corresponding boundary conditions [Eqs. (24) and (25)] results in

$$A_{ij}^{(1)} C_j^{(n1)} = D_i^{(1)} C_3^{(n0)}, \quad i, j = 1, 2 , \tag{68}$$

where $A_{ij}^{(1)}$ and $D_i^{(1)}$ are given by

$$\begin{aligned} A_{11}^{(1)} = {} & (1 - c_{12} a_1^2)(a_1\omega_0) J_{n+1}(a_1\omega_0) \\ & + [(1 - c_{12} a_1^2) n(n-1) - (a_1\omega_0)^2] J_n(a_1\omega_0) , \end{aligned} \tag{69}$$

$$\begin{aligned} A_{12}^{(1)} = {} & -2n[(a_2\omega_0) J_{n+1}(a_2\omega_0) - (n-1) J_n(a_2\omega_0)] , \\ A_{21}^{(1)} = {} & 2na_1^2 [(n-1) J_n(a_1\omega_0) - (a_1\omega_0) J_{n+1}(a_1\omega_0)] , \\ A_{22}^{(1)} = {} & a_2^2 \{2(a_2\omega_0) J_{n+1}(a_2\omega_0) \\ & + [2n(n-1) - (a_2\omega_0)^2] J_n(a_2\omega_0)\} , \end{aligned} \tag{70}$$

$$\begin{aligned} D_1^{(1)} = {} & [c_{13}(a_3\omega_0)^2 + (1 - c_{12} a_1^2) n(n-1)(e_{15} + e_{31}) \\ & \times e_{15} k_1^2 a_3^2 a_1^{-2}] J_n(a_3\omega_0) - R_0 R_1 \{(1 - c_{12} a_1^2)(a_3\omega_0) \\ & \times J_{n+1}(a_3\omega_0) + [(1 - c_{12} a_1^2) n(n-1) - (a_3\omega_0)^2] J_n(a_3\omega_0)\} , \end{aligned} \tag{71}$$

$$\begin{aligned} D_2^{(1)} = {} & 2n\{R_0 R_1 a_1^2 [(n-1) J_n(a_3\omega_0) - (a_3\omega_0) \\ & \times J_{n+1}(a_3\omega_0)] - (e_{31} + e_{15}) e_{15} k_1^2 (n-1) J_n(a_3\omega_0)\} . \end{aligned} \tag{72}$$

Now $C_1^{(n1)}$ and $C_2^{(n1)}$ can be obtained in terms of $C_3^{(n0)}$ by solving Eqs. (68).

If we substitute $W^{(1)}$ and $\Phi^{(1)}$ given by Eqs. (65) and (66) into boundary conditions [Eq. (26) at $m=1$], we get

$$\begin{aligned} & C_3^{(n1)} \omega_0^2 [(a_3\omega_0) J_{n+1}(a_3\omega_0) - n(1 - a_3^2 e_{15}^2 k_1^2) \\ & \times J_n(a_3\omega_0)] + C_3^{(n0)} [J_{n-1}(a_3\omega_0) + (a_3\omega_0) \\ & \times J_{n-1}'(a_3\omega_0) - a_3^2 e_{15}^2 k_1^2 n J_n(a_3\omega_0)] a_3 \omega_1 \omega_0^2 = 0 . \end{aligned} \tag{73}$$

Since, the coefficient of $C_3^{(n1)}$ is zero [from Eq. (62)] and $\omega_0 \neq 0$ in the above equation, we conclude that

$$\omega_1 \equiv 0 . \tag{74}$$

To compute the frequency coefficients $\omega_2$, only $W^{(2)}$ and $\Phi^{(2)}$ are needed. Solving Eqs. (13) and (14) for $n=2$, we obtain $W^{(2)}$ and $\Phi^{(2)}$ as

$$\begin{aligned} W^{(2)} = {} & C_3^{(n2)} (a_3\omega_0)^2 J_n(a_3\omega_0 x) + C_1^{(n1)} R_0 R_1 a_1^4 a_3^2 J_n(a_1\omega_0 x) + C_3^{(n0)} [(k_1^2 k_3^{-2} - 1) e_{15} a_3^2 k_1^2 J_n(a_3\omega_0) x^n \\ & - S_1 (a_3\omega_0) a_3^2 x J_{n-1}(a_3\omega_0 x) + \omega_0 \omega_2 a_3^2 (a_3\omega_0 x) J_{n-1}(a_3\omega_0 x)] , \end{aligned} \tag{75}$$

$$\begin{aligned} \Phi^{(2)} = {} & C_3^{(n2)} e_{15} k_1^2 (a_3\omega_0)^2 [J_n(a_3\omega_0 x) - J_n(a_3\omega_0) x^n] + C_1^{(n1)} a_1^2 k_1^2 [R_0 R_1 a_1^2 a_3^2 + (e_{31} + e_{15})] [J_n(a_1\omega_0 x) \\ & - J_n(a_1\omega_0) x^n] + C_3^{(n0)} \{R_2 [J_n(a_3\omega_0 x) - J_n(a_3\omega_0) x^n] - S_1 e_{15} k_1^2 (a_3\omega_0) a_3^2 [x J_{n-1}(a_3\omega_0 x) - J_{n-1}(a_3\omega_0) x^n] \\ & - [4(n+1)]^{-1} k_3^{-2} k_1^4 (a_3\omega_0)^2 J_n(a_3\omega_0)(x^{n+2} - x^n)\} + C_3^{(n0)} e_{15} k_1^2 \omega_0 \omega_2 a_3^2 (a_3\omega_0) [x J_{n-1}(a_3\omega_0 x) - J_{n-1}(a_3\omega_0) x^n] , \end{aligned} \tag{76}$$

where

$$R_2=\{(e_{31}+e_{15})[(1+c_{13})+(e_{31}+e_{15})]R_0+1-k_3^{-2}e_{15}k_1^2\}k_1^2\,, \tag{77}$$

$$2S_1=R_2(a_3\omega_0)^{-3}-R_1(1+c_{13})(R_0a_3/\omega_0)-c_{33}(a_3\omega_0)-e_{15}(a_3\omega_0)k_1^2\,. \tag{78}$$

Substitution of Eqs. (75) and (76) in the corresponding boundary condition [Eq. (26) at $m=2$] yields

$$\begin{aligned}\omega_2=&(\beta_1^{(n1)}/Q)\{a_1^2[nJ_n(a_1\omega_0)-(a_1\omega_0)J_{n+1}(a_1\omega_0)]+[n(1-e_{15}^2k_1^2a_3^2)J_n(a_1\omega_0)-(a_1\omega_0)J_{n+1}(a_1\omega_0)]R_0R_1a_1^4\\&-a_1^2e_{15}k_1^2(e_{31}+e_{15})(a_3\omega_0)J_{n+1}(a_3\omega_0)\}+(\beta_2^{(n1)}/Q)a_2^2nJ_n(a_2\omega_0)-(1/Q)((e_{31}+e_{15})e_{15}k_1^2a_3^2nJ_n(a_3\omega_0)\\&-R_0R_1a_1^2[nJ_n(a_3\omega_0)-(a_3\omega_0)J_{n+1}(a_3\omega_0)]-(k_1^2k_3^{-2}-1)e_{15}a_3^2k_1^2nJ_n(a_3\omega_0)-S_1e_{15}^2k_1^2(a_3\omega_0)^2\\&\times a_3^2n[2nJ_n(a_3\omega_0)-(a_3\omega_0)J_{n+1}(a_3\omega_0)]+S_1\{[2n^2-(a_3\omega_0)^2]J_n(a_3\omega_0)-n(a_3\omega_0)J_{n+1}(a_3\omega_0)\}\\&+[2(n+1)]^{-1}k_3^{-2}k_1^4e_{15}(a_3\omega_0)^2J_n(a_3\omega_0)+R_2e_{15}(a_3\omega_0)J_{n+1}(a_3\omega_0))\,,\end{aligned} \tag{79}$$

where

$$Q=\omega_0\{(a_3\omega_0)^2J_n(a_3\omega_0)-n(1-e_{15}^2k_1^2a_3^2)\times[2nJ_n(a_3\omega_0)-(a_3\omega_0)J_{n+1}(a_3\omega_0)]\}\,, \tag{80}$$

$$\beta_1^{(n1)}=\frac{(D_1^{(1)}A_{22}^{(1)}-D_2^{(1)}A_{12}^{(1)})}{(A_{11}^{(1)}A_{22}^{(1)}-A_{21}^{(1)}A_{12}^{(1)})}\,, \tag{81}$$

and

$$\beta_2^{(n1)}=\frac{(A_{11}^{(1)}D_2^{(1)}-A_{21}^{(1)}D_1^{(1)})}{(A_{11}^{(1)}A_{22}^{(1)}-A_{21}^{(1)}A_{12}^{(1)})}\,. \tag{82}$$

In the case of $\omega_0\equiv 0$, we see from Eqs. (60) that the lowest longitudinal mode is defined by $n=0$, $U^{(0)}=0$, $V^{(0)}=0$, and $\Phi^{(0)}=0$. The easier way of obtaining the higher order coefficients for $n=0$ is to proceed directly with the following equations:

$$\begin{aligned}&a_1^{-2}\left(\frac{d^2U^{(m)}}{dx^2}+x^{-1}\frac{dU^{(m)}}{dx}-x^{-2}U^{(m)}\right)\\&=U^{(m-2)}+(1+c_{13})\frac{dW^{(m-1)}}{dx}\\&+(e_{31}+e_{15})\frac{d\Phi^{(m-1)}}{dx}-T_u^{(m-1)}\,,\end{aligned} \tag{83}$$

$$\begin{aligned}&\frac{d^2W^{(m)}}{dx^2}+x^{-1}\frac{dW^{(m)}}{dx}+e_{15}\left(\frac{d^2\Phi^{(m)}}{dx^2}+x^{-1}\frac{d\Phi^{(m)}}{dx}\right)\\&=-(1+c_{13})\left(\frac{dU^{(m-1)}}{dx}+x^{-1}U^{(m-1)}\right)\\&+c_{33}W^{(m-2)}+\Phi^{(m-2)}-T_w^{(m-1)}\,,\end{aligned} \tag{84}$$

$$\begin{aligned}&-k_1^{-2}\left(\frac{d^2\Phi^{(m)}}{dx^2}+x^{-1}\frac{d\Phi^{(m)}}{dx}\right)+e_{15}\left(\frac{d^2W^{(m)}}{dx^2}+x^{-1}\frac{dW^{(m)}}{dx}\right)\\&=-k_3^{-2}\Phi^{(m-2)}+W^{(m-2)}-(e_{31}+e_{15})\\&\times\left(\frac{dU^{(m-1)}}{dx}+x^{-1}U^{(m-1)}\right),\end{aligned} \tag{85}$$

with the boundary conditions at $x=1$:

$$c_{11}\frac{dU^{(m)}}{dx}+c_{12}U^{(m)}=c_{13}W^{(m-1)}+e_{31}\Phi^{(m-1)}\,, \tag{86}$$

$$\frac{dW^{(m)}}{dx}+e_{15}\frac{d\Phi^{(m)}}{dx}=-U^{(m-1)}\,, \tag{87}$$

$$\Phi^{(m)}=0\,. \tag{88}$$

Solving the Eqs. (82)–(88) for $m=1$, we get

$$U^{(1)}=[c_{13}/(c_{11}+c_{12})]C_3^{(0)}x \text{ and } W^{(1)}=C_3^{(1)}\,. \tag{89}$$

If we solve the Eqs. (84) and (85) for $m=2$ subject to boundary condition [Eq. (87)] we obtain $\omega_1$ as

$$\omega_1^2=[c_{33}(c_{11}+c_{12})-2(c_{13})^2]/(c_{11}+c_{12})\,. \tag{90}$$

We note that this equation agrees with Eq. (44) of Ref. 10. The next step yields

$$\omega_2\equiv 0\,, \tag{91}$$

which shows that for small $\epsilon$, the lowest longitudinal mode behaves as if the medium is nonpiezoelectric.

## IV. CIRCUMFERENTIAL MODES

In the case of circumferential modes $C_3^{(n0)}=C_4^{(n0)}=0$ [from Eq. (46)]. The zeroth coefficients for $\omega_0\neq 0$ are

$$U^{(0)}=C_1^{(n0)}[a_1^2(a_1\omega_0)J_n'(a_1\omega_0x)+\beta^{(n0)}a_2^2nx^{-1}J_n(a_2\omega_0x)]\,, \tag{92}$$

$$V^{(0)}=-C_1^{(n0)}[nx^{-1}a_1^2J_n(a_1\omega_0x)+\beta^{(n0)}a_2^2(a_2\omega_0)J_n'(a_2\omega_0x)]\,, \tag{93}$$

$$W^{(0)}=0\,, \tag{94}$$

and

$$\Phi^{(0)}=0\,, \tag{95}$$

where $\omega_0$ is the solution of the equation $D_c=0$ [Eq. (46)] and

$$\beta^{(n0)}=C_2^{(n0)}/C_1^{(n0)}=-A_{21}/A_{22}. \tag{96}$$

Now for $m=1$, Eqs. (11)–(14) can be solved using Eqs. (28)–(35). Substitution of solutions in the boundary conditions [Eqs. (24)–(27)] result in as the main result,

$$\omega_1\equiv 0\,. \tag{97}$$

Next, we solve Eqs. (11)–(14) for $m=2$. If we substitute the solutions in the corresponding boundary conditions, we get

$$\omega_2=\Omega_1/\Omega_2\,, \tag{98}$$

where

$$\Omega_1 = 2n[(n-1)J_n(a_2\omega_0) - (a_2\omega_0)J_{n+1}(a_2\omega_0)](R_5(n/\omega_0^2)\{[2n(n-1) - (a_1\omega_0)^2]J_{n+1}(a_3\omega_0) - (n-1)J_{n+1}(a_1\omega_0)\}$$
$$-2R_6 n[(n-1)J_n(a_3\omega_0) - (a_3\omega_0)J_{n+1}(a_3\omega_0)] - \beta^{(n0)}(a_2/2\omega_0)[(a_2\omega_0)^2 - 2n(n-1)][2nJ_n(a_2\omega_0)$$
$$-(a_2\omega_0)J_{n+1}(a_2\omega_0)]) + \{[2n - (a_2\omega_0)^2]J_n(a_2\omega_0) + 2(a_2\omega_0)J_{n+1}(a_2\omega_0)\}[(R_5/2)\ \{J_{n+1}(a_1\omega_0) - 2nJ_n(a_1\omega_0)$$
$$+[2/(a_1a_2)^2(a_1\omega_0)][(a_2^2 - a_1^2)(a_1\omega_0)^2 - 2n^2a_1^2(n-1)]J_n(a_1\omega_0) - [2n(n-1)/a_2^2]J_{n+1}(a_1\omega_0)\}$$
$$+R_6(\{[(a_3\omega_0)^2/a_1^2] - [2n(n-1)/a_2^2]\}J_n(a_3\omega_0) - [2(a_3\omega_0)/a_2^2]J_{n+1}(a_3\omega_0)) + \beta^{(n0)}(n/\omega_0^2)\{[(n-1)$$
$$-(a_2\omega_0)][2nJ_n(a_2\omega_0) - (a_2\omega_0)J_{n+1}(a_2\omega_0)]\} + [(R_3 + R_4)(c_{13} + e_{31}e_{15}k_1^2) + (e_{31} + e_{15})e_{31}k_1^2a_1^2]J_n(a_1\omega_0)], \tag{99}$$

$$\Omega_2 = 2n[(n-1)J_n(a_2\omega_0) - (a_2\omega_0)J_{n+1}(a_2\omega_0)]((na_1^2/\omega_0)\{[2n(n-1) - (a_1\omega_0)^2]J_n(a_1\omega_0) - (a_1\omega_0)J_{n+1}(a_1\omega_0)\}$$
$$+\beta^{(n0)}(a_2^2/\omega_0)[(a_2\omega_0)^2 - 2n(n-1)][2nJ_n(a_2\omega_0) - (a_2\omega_0)J_{n+1}(a_2\omega_0)])$$
$$+\{[2(a_2\omega_0)J_{n+1}(a_2\omega_0)] - [2n(n-1) + (a_2\omega_0)^2]J_n(a_2\omega_0)\}(a_3^2\{(\omega_0/a_1)[(a_1\omega_0)J_{n+1}(a_1\omega_0) - 2nJ_n(a_1\omega_0)] + [2(a_2^2 - a_1^2)/a_1^2a_2^2]$$
$$\times(a_1\omega_0)J_n(a_1\omega_0) + [2n(n-1)/a_2^2(a_1\omega_0)][2nJ_n(a_1\omega_0) - (a_1\omega_0)J_{n+1}(a_1\omega_0)]\} + 2\beta^{(n0)}(n/\omega_0)\{[2n(n-1)$$
$$-(a_2\omega_0)^2]J_n(a_2\omega_0) - (n-1)(a_2\omega_0)J_{n+1}(a_2\omega_0)\}), \tag{100}$$

in which

$$R_3 = \{a_3/[(a_3\omega_0)J_{n+1}(a_3\omega_0) - nJ_n(a_3\omega_0)]\}\{R_0R_1a_1^5[nJ_n(a_3\omega_0) - J_{n+1}(a_3\omega_0)] - a_1^2a_3[(e_{31} + e_{15})e_{15}k_1^2 + 1]J_{n+1}(a_1\omega_0)$$
$$+na_1^2a_3[(e_{31} + e_{15})e_{15}k_1^2a_1^2a_3^2 + 1 + \beta^{(n0)}]J_n(a_2\omega_0)\}, \tag{101}$$

$$R_4 = R_0[(1 + c_{13}) + (e_{31} + e_{15})e_{15}]a_1^4a_3^2, \tag{102}$$

$$R_5 = [a_1^2 + R_1R_4 + (e_{31} + e_{15})k_1^2a_1^2], \tag{103}$$

and

$$R_6 = R_3R_0R_1a_1^2/\omega_0^2. \tag{104}$$

Equation (98) is valid for all values of $m$ including the case $m = 0$, which corresponds to the radial breathing motion of the cylinder.

For $\omega_0 \equiv 0$, the zeroth coefficients of lowest circumferential mode defined by $n = 1$ are given in Eq. (61). We can obtain the higher order coefficients by solving Eqs. (11)–(14) subject to corresponding boundary conditions. For $m = 1$, the solutions are

$$U^{(1)} = -V^{(1)} = D^{(11)}, \quad W^{(1)} = D^{(10)}x, \quad \Phi^{(1)} = 0. \tag{105}$$

From the next step we get

$$U^{(2)} = D^{(12)} + \tfrac{1}{4}D^{(10)}a_1^2a_2^2x^2, \tag{106}$$

$$V^{(2)} = -D^{(12)} + \tfrac{1}{4}D^{(10)}a_1^2a_2^2x^2, \tag{107}$$

$$W^{(2)} = -D^{(11)}x, \quad \Phi^{(2)} = 0, \text{ and } \omega_1 \equiv 0. \tag{108}$$

To obtain the frequency coefficient $\omega_2$ we have to solve Eqs. (11)–(14) up to $m = 4$. For $m = 4$, $U^{(4)}$ and $V^{(4)}$ are given by

$$U^{(4)} = D^{(14)} + (B_1^{(4)}/8)(3a_1^2 - a_2^2) + (5/192)[a_1^4a_2^2 - \tfrac{1}{5}a_1^2a_2^4 - (1 + c_{13})R_7 + (e_{31} + e_{15})k_1^2R_8]D^{(10)}x^4 - (x^2/64)$$
$$\times\{8D^{(12)}c_{13}a_2^2 - 2[(1 + c_{13})a_1^2a_2^4 + 2(1 + c_{13})a_2^2c_{13} + (1 + c_{13})R_9 - (e_{31} + e_{15})k_1^2R_{10} - 8\omega_2^2a_2^2]D^{(10)}\} \tag{109}$$

and

$$V^{(4)} = D^{(14)} + (B_1^{(4)}/8)(3a_2^2 - a_1^2) + (1/192)[-a_1^4a_2^2 + 5a_1^2a_2^4 + (1 + c_{13})R_7 - (e_{31} + e_{15})k_1^2R_8]D^{(10)}x^4 + (3x^2/64)$$
$$\times\{8c_{13}D^{(12)}a_2^2 - [2(1 + c_{13})c_{13}a_1^2a_2^4 + 2c_{33}a_2^2(1 + c_{13}) + (1 + c_{13})R_9 - (e_{31} + e_{15})k_1^2R_{10} - 8\omega_2^2a_2^2]D^{(10)}\}, \tag{110}$$

where

$$R_7 = a_1^2e_{15}k_1^2a_2^2 + a_1^4a_2^2a_3^2(1 + c_{13}) + a_1^2a_3^2c_{33} + (e_{31} + e_{15})e_{15}^2k_1a_1^4a_2^2a_3^2, \tag{111}$$

$$R_8 = a_1^2a_3^2 - (1 + c_{13})a_1^4a_2^2a_3^2e_{15} - c_{33}a_1^2a_3^2e_{15} + (e_{31} + e_{15})a_1^4a_2^2a_3^2, \tag{112}$$

$$R_9 = e_{15}k_1^2a_2^2a_3^2 + a_1^2a_2^4a_3^2(1 + c_{13}) + a_2^2a_3^2c_{33} + (e_{31} + e_{15})e_{15}k_1^2a_1^2a_2^4a_3^2, \tag{113}$$

$$R_{10} = a_2^2a_3^2 - (1 + c_{13})a_1^2a_2^4a_3^2e_{15} - c_{33}a_2^2a_3^2e_{15} + (e_{31} + e_{15})a_1^2a_2^4a_3^2. \tag{114}$$

If we substitute the Eqs. (109) and (110) into the boundary conditions [Eqs. (24) and (25)] we get $\omega_2$ as

$$\omega_2^2 = (1/24)(3a_1^{-2} - 2a_2^{-2})[2(1 + c_{13})c_{13}a_1^2a_2^4 + 2(1 + c_{13})a_2^2c_{33} + (1 + c_{13})R_9 - (e_{31} + e_{15})k_1^2R_{10}]$$
$$-(1/8)[2(1 + c_{13})a_1^2c_{13} + 2(1 + c_{13})c_{33} + (1 + c_{13})R_9a_2^{-2} - (e_{31} + e_{15})k_1^2R_{10}a_2^{-2} + a_1^2a_2^2 + 2c_{13}a_1^2a_2^2 + 2c_{13}]. \tag{115}$$

TABLE I. Longitudinal modes.

| $(ca) \longrightarrow$ | $n=0$ | | $n=1$ | | $n=2$ | |
|---|---|---|---|---|---|---|
| $\epsilon$ | Present analysis | Exact frequency equation | Present analysis | Exact frequency equation | Present analysis | Exact frequency equation |
| 0.01 | 4.6659 | 4.6655 | 1.8910 | 1.8911 | 3.1819 | 3.1816 |
| | 8.5430 | 8.5424 | 6.4147 | 6.4143 | 8.0356 | 8.0351 |
| | 12.3883 | 12.3875 | 10.3475 | 10.3469 | 12.0568 | 12.0561 |
| | 16.2243 | 16.2234 | 14.2202 | 14.2195 | 15.9759 | 15.9750 |
| 0.04 | 4.6667 | 4.6641 | 1.8912 | 1.8923 | 3.1854 | 3.1825 |
| | 8.5441 | 8.5425 | 6.4161 | 6.4142 | 8.0375 | 8.0351 |
| | 12.3892 | 12.3873 | 10.3487 | 10.3470 | 12.0577 | 12.0563 |
| | 16.2249 | 16.2234 | 14.2214 | 14.2195 | 15.9768 | 15.9751 |
| 0.07 | 4.6684 | 4.6611 | 1.8916 | 1.8950 | 3.1932 | 3.1842 |
| | 8.5466 | 8.5426 | 6.4193 | 6.4140 | 8.0417 | 8.0351 |
| | 12.3914 | 12.3867 | 10.3514 | 10.3472 | 12.0597 | 12.0567 |
| | 16.2262 | 16.2235 | 14.2240 | 14.2195 | 15.9788 | 15.9751 |
| 0.1 | 4.6712 | 4.6566 | 1.8923 | 1.8992 | 3.2052 | 3.1870 |
| | 8.5505 | 8.5427 | 6.4242 | 6.4137 | 8.0483 | 8.0351 |
| | 12.3947 | 12.3858 | 10.3555 | 10.3475 | 12.0628 | 12.0573 |
| | 16.2283 | 16.2235 | 14.2279 | 14.2195 | 15.9818 | 15.9756 |

## V. NUMERICAL RESULTS

For comparison purposes the roots of the exact frequency equation [Eq. (29) of Ref. 4 in the case of solid cylinder] are computed in double precision in the case of ceramic PZT−4 by using the bisection method for $0 < \epsilon \leq 0.1$ and then compared with the values of $(ca)$ obtained in the present analysis. The results are given in Tables I and II. It is observed that the exact frequency and the frequency obtained by the present analysis agree up to three decimal places for $0 < \epsilon < 0.1$. The elastic, piezoelectric, and dielectric constants of ceramic PZT−4 are taken from Ref. 11. For each value of $\epsilon$, first four roots in the case of both longitudinal and circumferential modes are listed in the tables. First three terms in the series expansions for $(ca)$ in the present analysis are considered. From the numerical results we can observe that the present analysis clearly displays the structure of the frequency spectrum near the cutoff frequencies. The numerical computations are carried out on IBM 370/155.

## VI. CONCLUSIONS

In this paper, longitudinal and circumferential modes of wave propagation in a piezoelectric solid circular cylinder of crystal class (6 mm) or ceramics ($\infty$m) are investigated. The present analysis yields information

TABLE II. Circumferential modes.

| $(ca) \longrightarrow$ | $n=0$ | | $n=1$ | | $n=2$ | |
|---|---|---|---|---|---|---|
| $\epsilon$ | Present analysis | Exact frequency equation | Present analysis | Exact frequency equation | Present analysis | Exact frequency equation |
| 0.01 | 5.1220 | 5.1221 | 3.1572 | 3.1573 | 2.5690 | 2.5690 |
| | 5.6147 | 5.6148 | 7.2247 | 7.2246 | 4.9306 | 4.9305 |
| | 9.2027 | 9.2026 | 8.6783 | 8.6783 | 8.7665 | 8.7665 |
| | 12.6731 | 12.6731 | 10.9111 | 10.9112 | 11.5781 | 11.5781 |
| 0.04 | 5.1222 | 5.1241 | 3.1562 | 3.1577 | 2.5686 | 2.5691 |
| | 5.6149 | 5.6149 | 7.2263 | 7.2247 | 4.9316 | 4.9308 |
| | 9.2036 | 9.2027 | 8.6784 | 8.6788 | 8.7668 | 8.7666 |
| | 12.6731 | 12.6737 | 10.9096 | 10.9113 | 11.5781 | 11.5782 |
| 0.07 | 5.1227 | 5.1284 | 3.1541 | 3.1584 | 2.5676 | 2.5679 |
| | 5.6149 | 5.6152 | 7.2299 | 7.2251 | 4.9339 | 4.9315 |
| | 9.2057 | 9.2029 | 8.6785 | 8.6797 | 8.7676 | 8.7668 |
| | 12.6731 | 12.6750 | 10.9063 | 10.9115 | 11.5781 | 11.5785 |
| 0.1 | 5.1235 | 5.1351 | 3.1507 | 3.1595 | 2.5661 | 2.5675 |
| | 5.6151 | 5.6157 | 7.2356 | 7.2255 | 4.9374 | 4.9324 |
| | 9.2090 | 9.2031 | 8.6788 | 8.6812 | 8.7688 | 8.7672 |
| | 12.6733 | 12.6771 | 10.9013 | 10.9117 | 11.5782 | 11.5790 |

on the frequency spectrum and the mode shapes of all modes for the whole range of frequencies, but for the wavelengths that are large compared to the circumference of the cylinder. The merit of this asymptotic method lies in its less computational work and also the fact that the frequency coefficients and the corresponding coefficients of the mode shapes are obtained simultaneously. Torsional modes are not studied because in the case of hexagonal (6 mm) crystal these modes behave as if the medium is nonpiezoelectric.


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