

# Asymptotic analysis of piezoelectric energy harvester

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## 1 Summary of the interested equations

Here we are interested in the classical model of a piezoelectric cantilever beam energy harvester, whose model is described using the following set of equations:

$$u'''' - \lambda^2 u = 0, \quad (1)$$

and the accompanying boundary conditions:

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) + \frac{j\lambda\beta\alpha^2}{j\lambda\beta + 1} u'(1) = 0 \\ u'''(1) = 0 \end{cases}, \quad (2)$$

where  $\lambda$  is the eigenvalues for the problem,  $u$  denotes the displace function of the cantilever beam,  $\beta$  is the dimensionless externally connected resistance, and  $\alpha$  is the dimensionless piezoelectric coefficient. They can be expressed as follows

$$\lambda = \omega \sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \alpha = e_p \sqrt{\frac{l_p}{C_p B_p}}, \quad (3)$$

where  $\omega$  is angular frequency,  $m_p$  is line mass density,  $l_p$  is the length of the cantilever beam,  $B_p$  is the bending stiffness,  $C_p$  is the inherent capacitance of the piezoelectric layer,  $e_p$  is the charge accumulation number,  $R_l$  is the externally connected resistance. In practical applications, dielectric property of piezoelectric materials indicate that the parameter  $\beta$  is changed from a very small value, which is close to a short-circuit condition to a very large value, which corresponds to an open-circuit condition. Thus we have that  $0 \leq \beta \leq \infty$ .

## 2 Asymptotic analysis when $\beta$ is small

Here we seek to find the behavior of the above system at a small value of connected resistance, i.e.,  $\beta \rightarrow 0$ . In this case, we set  $\beta$  to be the parameter for asymptotic expansion, and

$$\begin{aligned} \lambda^{(k)} &= \lambda_0^{(k)} + \beta \lambda_1^{(k)} + \beta^2 \lambda_2^{(k)} + \dots \\ u^{(k)} &= u_0^{(k)} + \beta u_1^{(k)} + \beta^2 u_2^{(k)} + \dots \end{aligned} \quad (4)$$

where  $\lambda^{(k)}$  and  $u^{(k)}$  are the  $k$ th eigenvalue and eigenfunction respectively of the above mentioned system under perturbation.  $\lambda_0^{(k)}$  and  $u_0^{(k)}$  are the corresponding eigenvalue and eigenfunction of the unperturbed system at  $\beta = 0$ :

$$u'''' - \lambda_0^2 u = 0, \quad (5)$$

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) = 0 \\ u'''(1) = 0 \end{cases}. \quad (6)$$

Obviously, the unperturbed system is a classical eigenvalue problem with the eigenvalues determined by

$$1 + \cosh(\sqrt{\lambda_0}) \cos(\sqrt{\lambda_0}) = 0 \quad (7)$$

whose first several values are

$$\frac{\sqrt{\lambda_0^{(1)}}}{\pi} = 0.59686, \quad \frac{\sqrt{\lambda_0^{(2)}}}{\pi} = 1.49418, \quad \frac{\sqrt{\lambda_0^{(3)}}}{\pi} = 2.50025, \quad \frac{\sqrt{\lambda_0^{(4)}}}{\pi} = 3.49999, \quad \dots \quad (8)$$

Take the asymptotic expansions and substitute them into the previously derived system of equations, we have the following asymptotic expansions to different orders of  $\beta$ :

$O(\beta^0)$ :

$$\begin{cases} u_0'''' - \lambda_0^2 u_0 = 0 \\ u_0(0) = 0 \\ u_0'(0) = 0 \\ u_0''(1) = 0 \\ u_0'''(1) = 0 \end{cases} \quad (9)$$

$O(\beta^1)$ :

$$\begin{cases} u_1'''' - (\lambda_0^2 u_1 + 2\lambda_0 u_0 \lambda_1) = 0 \\ u_1(0) = 0 \\ u_1'(0) = 0 \\ u_1''(1) + j\alpha^2 \lambda_0 u_0'(1) = 0 \\ u_1'''(1) = 0 \end{cases} \quad (10)$$

$O(\beta^2)$ :

$$\begin{cases} u_2'''' - (\lambda_0^2 u_2 + 2\lambda_0 u_1 \lambda_1 + \lambda_1^2 u_0 + 2\lambda_0 u_0 \lambda_2) = 0 \\ u_2(0) = 0 \\ u_2'(0) = 0 \\ u_2''(1) + \alpha^2 \lambda_0 u_0'(1) + j\alpha^2 [\lambda_0 u_1'(1) + \lambda_1 u_0'(1)] = 0 \\ u_2'''(1) = 0 \end{cases} \quad (11)$$

### 3 Asymptotic analysis when $\beta$ is large

Here we seek to find the behavior of the above system at a large value of connected resistance, i.e.,  $\beta \rightarrow \infty$ . In this case, we set  $\frac{1}{\beta}$  to be the parameter for asymptotic expansion and

$$\begin{aligned} \lambda^{(k)} &= \tilde{\lambda}_0^{(k)} + \left(\frac{1}{\beta}\right) \tilde{\lambda}_1^{(k)} + \left(\frac{1}{\beta}\right)^2 \tilde{\lambda}_2^{(k)} + \dots \\ u^{(k)} &= \tilde{u}_0^{(k)} + \left(\frac{1}{\beta}\right) \tilde{u}_1^{(k)} + \left(\frac{1}{\beta}\right)^2 \tilde{u}_2^{(k)} + \dots \end{aligned} \quad (12)$$

where  $\tilde{\lambda}^{(k)}$  and  $\tilde{u}^{(k)}$  are the  $k$ th eigenvalue and eigenfunction respectively of the above mentioned system under perturbation.  $\tilde{\lambda}_0^{(k)}$  and  $\tilde{u}_0^{(k)}$  are the corresponding eigenvalue and eigenfunction of the unperturbed system at  $\beta = \infty$ :

$O(\frac{1}{\beta^0})$ :

$$\begin{cases} \tilde{u}_0'''' - \tilde{\lambda}_0^2 \tilde{u}_0 = 0 \\ \tilde{u}_0(0) = 0 \\ \tilde{u}_0'(0) = 0 \\ \tilde{u}_0''(1) + \alpha^2 \tilde{u}_0'(1) = 0 \\ \tilde{u}_0'''(1) = 0 \end{cases} \quad (13)$$

$O(\frac{1}{\beta^1})$ :

$$\left\{ \begin{array}{l} \tilde{u}_1'''' - \left( \tilde{\lambda}_0^2 u_1 + 2\tilde{\lambda}_0 \tilde{u}_0 \tilde{\lambda}_1 \right) = 0 \\ \tilde{u}_1(0) = 0 \\ \tilde{u}_1'(0) = 0 \\ \tilde{u}_1''(1) + \alpha^2 \tilde{u}_1'(1) + \frac{j\alpha^2}{\tilde{\lambda}_0} \tilde{u}_0'(1) = 0 \\ \tilde{u}_1'''(1) = 0 \end{array} \right. \quad (14)$$

$O(\frac{1}{\beta^2})$ :

$$\left\{ \begin{array}{l} \tilde{u}_2'''' - \left( \tilde{\lambda}_0^2 \tilde{u}_2 + 2\tilde{\lambda}_0 \tilde{u}_1 \tilde{\lambda}_1 + \tilde{\lambda}_1^2 \tilde{u}_0 + 2\tilde{\lambda}_0 \tilde{u}_0 \tilde{\lambda}_2 \right) = 0 \\ \tilde{u}_2(0) = 0 \\ \tilde{u}_2'(0) = 0 \\ \tilde{u}_2''(1) + \left[ \alpha^2 \tilde{u}_2'(1) - \frac{\alpha^2}{\tilde{\lambda}_0^2} \tilde{u}_0'(1) \right] + j \left[ \frac{\alpha^2}{\tilde{\lambda}_0} \tilde{u}_1'(1) - \frac{\alpha^2 \tilde{\lambda}_1}{\tilde{\lambda}_0^2} \tilde{u}_0'(1) \right] = 0 \\ \tilde{u}_2'''(1) = 0 \end{array} \right. \quad (15)$$

## 4 Asymptotic analysis in terms of small $\alpha^2$

Directly using the eigenvalue analysis method for linear boundary value problem, we arrive at the equation for the eigenvalue  $\lambda$ :

$$\sqrt{\lambda} \left[ 1 + \left( \frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{2} \right) \cos \sqrt{\lambda} \right] + \frac{j\beta\lambda\alpha^2}{1 + j\beta\lambda} \left[ \left( \frac{e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}}{2} \right) \cos \sqrt{\lambda} + \left( \frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{2} \right) \sin \sqrt{\lambda} \right] = 0 \quad (16)$$