

Asymptotic analysis of piezoelectric energy harvester

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November 10, 2019

1 Summary of the interested equations

The dynamic equations for a typical piezoelectric composite cantilever beam is

$$B_p \frac{\partial^4 w(x, t)}{\partial x^4} + m_p \frac{\partial^2 w(x, t)}{\partial t^2} = 0, \quad (1)$$

where B_p is the equivalent bending stiffness and m_p is the line mass density of the piezoelectric cantilever beam. If the piezoelectric elements attached to the cantilever beam is connected to an external electrical load R_l , we have

$$\frac{dQ_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \quad (2)$$

For the underlying physics, we have the following constitutive equations

$$\begin{aligned} M_p(x, t) &= B_p \frac{\partial^2 w(x, t)}{\partial x^2} - e_p V_p(t), \\ q_p(x, t) &= e_p \frac{\partial^2 w(x, t)}{\partial x^2} + \varepsilon_p V_p(t), \end{aligned} \quad (3)$$

or equivalently,

$$\begin{cases} M_p(x, t) = B_p \frac{\partial^2 w(x, t)}{\partial x^2} - e_p V_p(t), \\ Q_p(x, t) = e_p \left[\frac{\partial w(x, t)}{\partial x} \right] \Big|_0^{l_p} + C_p V_p(t). \end{cases} \quad (4)$$

One end of the cantilever beam is fixed while the other end is free. So the boundary conditions are

$$\begin{cases} w(0, t) = w_b(t), \\ \frac{\partial w(0, t)}{\partial x} = 0, \end{cases} \quad (5)$$

and

$$\begin{cases} M_p(l_p, t) = B_p \frac{\partial^2 w(l_p, t)}{\partial x^2} - e_p V_p(t) = 0, \\ Q_p(l_p, t) = B_p \frac{\partial^3 w(l_p, t)}{\partial x^3} = 0. \end{cases} \quad (6)$$

In the classical energy harvesting applications, the cantilever beam is subject to a periodical base excitation $w_b(t)$. Thus the dynamic response of the cantilever beam is decomposed as

$$w(x, t) = w_b(t) + w_{rel}(x, t), \quad (7)$$

where $w_{rel}(x, t)$ is the relative displacement function of the cantilever beam. In this way, the system is converted into

$$B_p \frac{\partial^4 w_{rel}(x, t)}{\partial x^4} + m_p \frac{\partial^2 w_{rel}(x, t)}{\partial t^2} = -m_p \frac{\partial^2 w_b(x, t)}{\partial t^2}, \quad (8)$$

$$e_p \left[\frac{\partial^2 w_{rel}(x, t)}{\partial x \partial t} \right] \Big|_0^{l_p} + C_p \frac{dV_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \quad (9)$$

$$\begin{cases} w_{rel}(0, t) = 0, \\ \frac{\partial w_{rel}(0, t)}{\partial x} = 0, \end{cases} \quad (10)$$

and

$$\begin{cases} B_p \frac{\partial^2 w_{rel}(l_p, t)}{\partial x^2} - e_p V_p(t) = 0, \\ \frac{\partial^3 w_{rel}(l_p, t)}{\partial x^3} = 0. \end{cases} \quad (11)$$

Considering a sinusoidal base excitation

$$w_b(t) = \eta_b e^{j\sigma_b t} \quad (12)$$

where ξ_b is usually a real vibration amplitude, the steady state solution for the above system can be reasonably set as

$$w_{rel}(x, t) = \eta_{rel}(x) e^{j\sigma_b t}, \quad V_p(t) = \tilde{V}_p e^{j\sigma_b t}, \quad (13)$$

where $\eta_{rel}(x)$ and \tilde{V}_p are complex amplitudes. Then the above system is again simplified as

$$B_p \frac{\partial^4 \eta_{rel}(x)}{\partial x^4} - m_p \sigma_b^2 \eta_{rel}(x) = m_p \sigma_b^2 \eta_b, \quad (14)$$

$$\begin{cases} \eta_{rel}(0) = 0, \\ \frac{\partial \eta_{rel}(0)}{\partial x} = 0, \end{cases} \quad (15)$$

and

$$\begin{cases} B_p \frac{\partial^2 \eta_{rel}(l_p)}{\partial x^2} + \frac{j\sigma_b R_l}{1 + j\sigma_b C_p R_l} e_p^2 \frac{\partial \eta_{rel}(l_p)}{\partial x} = 0, \\ \frac{\partial^3 \eta_{rel}(l_p)}{\partial x^3} = 0. \end{cases} \quad (16)$$

Obviously we can have the following dimensionless scheme:

$$\eta_{rel} \sim u \eta_b, \quad x \sim z l_p \quad (17)$$

and therefore the following dimensionless parameters

$$\sigma = \sigma_b \sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \delta = \frac{e_p^2 l_p}{C_p B_p}. \quad (18)$$

Now, we reach the following dimensionless system of boundary value problem

$$\begin{cases} u'''' - \sigma^2 u = \sigma^2, \\ u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \frac{j\sigma\beta}{1 + j\sigma\beta} \delta u'(1) = 0, \\ u'''(1) = 0, \end{cases} \quad (19)$$

where the prime denotes the derivative with respect to z .

Here we are interested in the classical model of a piezoelectric cantilever beam energy harvester, whose model is described using the following set of equations:

$$u'''' - \lambda^2 u = 0, \quad (20)$$

and the accompanying boundary conditions:

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) + \frac{j\lambda\beta\alpha^2}{j\lambda\beta + 1} u'(1) = 0 \\ u'''(1) = 0 \end{cases}, \quad (21)$$

where λ is the eigenvalues for the problem, u denotes the displace function of the cantilever beam, β is the dimensionless externally connected resistance, and α is the dimensionless piezoelectric coefficient. They can be expressed as follows

$$\lambda = \omega \sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \alpha = e_p \sqrt{\frac{l_p}{C_p B_p}}, \quad (22)$$

where ω is angular frequency, m_p is line mass density, l_p is the length of the cantilever beam, B_p is the bending stiffness, C_p is the inherent capacitance of the piezoelectric layer, e_p is the charge accumulation number, R_l is the externally connected resistance. In practical applications, dielectric property of piezoelectric materials indicate that the parameter β is changed from a very small value, which is close to a short-circuit condition to a very large value, which corresponds to an open-circuit condition. Thus we have that $0 \leq \beta \leq \infty$.

2 Asymptotic analysis when β is small

Here we seek to find the behavior of the above system at a small value of connected resistance, i.e., $\beta \rightarrow 0$. In this case, we set β to be the parameter for asymptotic expansion, and

$$\begin{aligned} \lambda^{(k)} &= \lambda_0^{(k)} + \beta \lambda_1^{(k)} + \beta^2 \lambda_2^{(k)} + \dots \\ u^{(k)} &= u_0^{(k)} + \beta u_1^{(k)} + \beta^2 u_2^{(k)} + \dots \end{aligned} \quad (23)$$

where $\lambda^{(k)}$ and $u^{(k)}$ are the k th eigenvalue and eigenfunction respectively of the above mentioned system under perturbation. $\lambda_0^{(k)}$ and $u_0^{(k)}$ are the corresponding eigenvalue and eigenfunction of the unperturbed system at $\beta = 0$:

$$u'''' - \lambda_0^2 u = 0, \quad (24)$$

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) = 0 \\ u'''(1) = 0 \end{cases}. \quad (25)$$

Obviously, the unperturbed system is a classical eigenvalue problem with the eigenvalues determined by

$$1 + \cosh(\sqrt{\lambda_0}) \cos(\sqrt{\lambda_0}) = 0 \quad (26)$$

whose first several values are

$$\frac{\sqrt{\lambda_0^{(1)}}}{\pi} = 0.59686, \quad \frac{\sqrt{\lambda_0^{(2)}}}{\pi} = 1.49418, \quad \frac{\sqrt{\lambda_0^{(3)}}}{\pi} = 2.50025, \quad \frac{\sqrt{\lambda_0^{(4)}}}{\pi} = 3.49999, \quad \dots \quad (27)$$

Take the asymptotic expansions and substitute them into the previously derived system of equations, we have the following asymptotic expansions to different orders of β :

$O(\beta^0)$:

$$\begin{cases} u_0'''' - \lambda_0^2 u_0 = 0 \\ u_0(0) = 0 \\ u_0'(0) = 0 \\ u_0''(1) = 0 \\ u_0'''(1) = 0 \end{cases} \quad (28)$$

$O(\beta^1)$:

$$\begin{cases} u_1'''' - (\lambda_0^2 u_1 + 2\lambda_0 u_0 \lambda_1) = 0 \\ u_1(0) = 0 \\ u_1'(0) = 0 \\ u_1''(1) + j\alpha^2 \lambda_0 u_0'(1) = 0 \\ u_1'''(1) = 0 \end{cases} \quad (29)$$

$O(\beta^2)$:

$$\begin{cases} u_2'''' - (\lambda_0^2 u_2 + 2\lambda_0 u_1 \lambda_1 + \lambda_1^2 u_0 + 2\lambda_0 u_0 \lambda_2) = 0 \\ u_2(0) = 0 \\ u_2'(0) = 0 \\ u_2''(1) + \alpha^2 \lambda_0 u_0'(1) + j\alpha^2 [\lambda_0 u_1'(1) + \lambda_1 u_0'(1)] = 0 \\ u_2'''(1) = 0 \end{cases} \quad (30)$$

3 Asymptotic analysis when β is large

Here we seek to find the behavior of the above system at a large value of connected resistance, i.e., $\beta \rightarrow \infty$. In this case, we set $\frac{1}{\beta}$ to be the parameter for asymptotic expansion and

$$\begin{aligned} \lambda^{(k)} &= \tilde{\lambda}_0^{(k)} + \left(\frac{1}{\beta}\right) \tilde{\lambda}_1^{(k)} + \left(\frac{1}{\beta}\right)^2 \tilde{\lambda}_2^{(k)} + \dots \\ u^{(k)} &= \tilde{u}_0^{(k)} + \left(\frac{1}{\beta}\right) \tilde{u}_1^{(k)} + \left(\frac{1}{\beta}\right)^2 \tilde{u}_2^{(k)} + \dots \end{aligned} \quad (31)$$

where $\tilde{\lambda}^{(k)}$ and $\tilde{u}^{(k)}$ are the k th eigenvalue and eigenfunction respectively of the above mentioned system under perturbation. $\tilde{\lambda}_0^{(k)}$ and $\tilde{u}_0^{(k)}$ are the corresponding eigenvalue and eigenfunction of the unperturbed system at $\beta = \infty$:

$O(\frac{1}{\beta^0})$:

$$\begin{cases} \tilde{u}_0'''' - \tilde{\lambda}_0^2 \tilde{u}_0 = 0 \\ \tilde{u}_0(0) = 0 \\ \tilde{u}_0'(0) = 0 \\ \tilde{u}_0''(1) + \alpha^2 \tilde{u}_0'(1) = 0 \\ \tilde{u}_0'''(1) = 0 \end{cases} \quad (32)$$

$O(\frac{1}{\beta^1})$:

$$\begin{cases} \tilde{u}_1'''' - (\tilde{\lambda}_0^2 u_1 + 2\tilde{\lambda}_0 \tilde{u}_0 \tilde{\lambda}_1) = 0 \\ \tilde{u}_1(0) = 0 \\ \tilde{u}_1'(0) = 0 \\ \tilde{u}_1''(1) + \alpha^2 \tilde{u}_1'(1) + \frac{j\alpha^2}{\tilde{\lambda}_0} \tilde{u}_0'(1) = 0 \\ \tilde{u}_1'''(1) = 0 \end{cases} \quad (33)$$

$O(\frac{1}{\beta^2})$:

$$\begin{cases} \tilde{u}_2'''' - (\tilde{\lambda}_0^2 \tilde{u}_2 + 2\tilde{\lambda}_0 \tilde{u}_1 \tilde{\lambda}_1 + \tilde{\lambda}_1^2 \tilde{u}_0 + 2\tilde{\lambda}_0 \tilde{u}_0 \tilde{\lambda}_2) = 0 \\ \tilde{u}_2(0) = 0 \\ \tilde{u}_2'(0) = 0 \\ \tilde{u}_2''(1) + \left[\alpha^2 \tilde{u}_2'(1) - \frac{\alpha^2}{\tilde{\lambda}_0^2} \tilde{u}_0'(1) \right] + j \left[\frac{\alpha^2}{\tilde{\lambda}_0} \tilde{u}_1'(1) - \frac{\alpha^2 \tilde{\lambda}_1}{\tilde{\lambda}_0^2} \tilde{u}_0'(1) \right] = 0 \\ \tilde{u}_2'''(1) = 0 \end{cases} \quad (34)$$

4 Asymptotic analysis in terms of small α^2

Directly using the eigenvalue analysis method for linear boundary value problem, we arrive at the equation for the eigenvalue λ :

$$\sqrt{\lambda} \left[1 + \left(\frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{2} \right) \cos \sqrt{\lambda} \right] + \frac{j\beta\lambda\alpha^2}{1+j\beta\lambda} \left[\left(\frac{e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}}{2} \right) \cos \sqrt{\lambda} + \left(\frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{2} \right) \sin \sqrt{\lambda} \right] = 0 \quad (35)$$

or

$$\sqrt{\lambda} \left[1 + \cosh \sqrt{\lambda} \cos \sqrt{\lambda} \right] + \frac{j\beta\lambda\alpha^2}{1+j\beta\lambda} \left[\sinh \sqrt{\lambda} \cos \sqrt{\lambda} + \cosh \sqrt{\lambda} \sin \sqrt{\lambda} \right] = 0 \quad (36)$$

Taking the parameter α^2 as the small parameter ϵ and expanding the eigenvalue λ in terms of this ϵ , we have

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \dots \quad (37)$$

and therefore:

$O(\epsilon^0)$:

$$1 + \cosh \sqrt{\lambda_0} \cos \sqrt{\lambda_0} = 0 \quad (38)$$

$O(\epsilon^1)$:

$$2j\beta\lambda_0 \left(\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} + \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0} \right) + (1+j\beta\lambda_0)\lambda_1 \left(-\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} + \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0} \right) = 0 \quad (39)$$

or equivalently

$$\lambda_1 = \frac{2j\beta\lambda_0}{(1+j\beta\lambda_0)} \frac{(\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} + \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0})}{(\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} - \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0})} \quad (40)$$

5 Asymptotic analysis in terms of small α^2

The forced vibration problem of a piezoelectric cantilever bimorph is described by

$$u'''' - \lambda^2 u = \lambda^2, \quad (41)$$

and the accompanying boundary conditions:

$$\begin{cases} u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \frac{j\lambda\beta}{j\lambda\beta + 1} \epsilon u'(1) = 0, \\ u'''(1) = 0. \end{cases} \quad (42)$$

This problem can readily be solved using a conventional boundary value problem solver. However, here we would like to develop an asymptotic expansion of the solution for the system. Using ϵ as a parameter, we have

$$u(x; \epsilon) = A_\epsilon \cos \sqrt{\lambda} x + B_\epsilon \sin \sqrt{\lambda} x + C_\epsilon \cosh \sqrt{\lambda} x + D_\epsilon \sinh \sqrt{\lambda} x - 1 \quad (43)$$

As a result, we have

$$\begin{aligned} u'(x; \epsilon) &= \sqrt{\lambda} \left(-A_\epsilon \sin \sqrt{\lambda} x + B_\epsilon \cos \sqrt{\lambda} x + C_\epsilon \sinh \sqrt{\lambda} x + D_\epsilon \cosh \sqrt{\lambda} x \right) \\ u''(x; \epsilon) &= \lambda \left(-A_\epsilon \cos \sqrt{\lambda} x - B_\epsilon \sin \sqrt{\lambda} x + C_\epsilon \cosh \sqrt{\lambda} x + D_\epsilon \sinh \sqrt{\lambda} x \right) \\ u'''(x; \epsilon) &= \lambda \sqrt{\lambda} \left(A_\epsilon \sin \sqrt{\lambda} x - B_\epsilon \cos \sqrt{\lambda} x + C_\epsilon \sinh \sqrt{\lambda} x + D_\epsilon \cosh \sqrt{\lambda} x \right) \end{aligned} \quad (44)$$

Thus the above boundary value problem is converted into the following linear equation systems:

$$\begin{cases} A_\epsilon + C_\epsilon = 1, \\ B_\epsilon + D_\epsilon = 0, \\ \left(-A_\epsilon \cos \sqrt{\lambda} - B_\epsilon \sin \sqrt{\lambda} + C_\epsilon \cosh \sqrt{\lambda} + D_\epsilon \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \epsilon \left(-A_\epsilon \sin \sqrt{\lambda} + B_\epsilon \cos \sqrt{\lambda} + C_\epsilon \sinh \sqrt{\lambda} + D_\epsilon \cosh \sqrt{\lambda} \right) = 0, \\ A_\epsilon \sin \sqrt{\lambda} - B_\epsilon \cos \sqrt{\lambda} + C_\epsilon \sinh \sqrt{\lambda} + D_\epsilon \cosh \sqrt{\lambda} = 0. \end{cases} \quad (45)$$

Analytically, we can directly obtain the solution to this problem as

$$\begin{cases} A_\epsilon = \frac{1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} - \sin \sqrt{\lambda} \sinh \sqrt{\lambda} + \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]}, \\ B_\epsilon = \frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\sin \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]}, \\ C_\epsilon = \frac{-1 - \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \sin \sqrt{\lambda} \sinh \sqrt{\lambda} - \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]}, \\ D_\epsilon = \frac{-\cos \sqrt{\lambda} \sinh \sqrt{\lambda} - \sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\sin \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]} \end{cases} \quad (46)$$

The resulting output voltage V_p , current I_p , and power P_p can be formulated as follows

$$\begin{cases} \tilde{V}_p = \frac{j\lambda\beta}{j\lambda\beta + 1} \frac{\xi_b}{l_p} \frac{e_p}{C_p} u'(1), \\ \tilde{I}_p = \tilde{V}_p / R_l, \\ \tilde{P}_p = \tilde{V}_p^2 / R_l. \end{cases} \quad (47)$$

Using the following regular expansion:

$$\begin{cases} A_\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \dots, \\ B_\epsilon = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \dots, \\ C_\epsilon = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \dots, \\ D_\epsilon = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots, \end{cases} \quad (48)$$

we obtain the successive expansion problem:

$O(\epsilon^0)$:

$$\begin{cases} A_0 + C_0 = 1, \\ B_0 + D_0 = 0, \\ -A_0 \cos \sqrt{\lambda} - B_0 \sin \sqrt{\lambda} + C_0 \cosh \sqrt{\lambda} + D_0 \sinh \sqrt{\lambda} = 0, \\ A_0 \sin \sqrt{\lambda} - B_0 \cos \sqrt{\lambda} + C_0 \sinh \sqrt{\lambda} + D_0 \cosh \sqrt{\lambda} = 0. \end{cases} \quad (49)$$

The solution is

$$\begin{cases} A_0 = \frac{1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} - \sin \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \\ B_0 = \frac{\cosh \sqrt{\lambda} \sin \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \\ C_0 = \frac{1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \sin \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \\ D_0 = -\frac{\cosh \sqrt{\lambda} \sin \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \end{cases} \quad (50)$$

Hence we have

$$-A_0 \sin \sqrt{\lambda} + B_0 \cos \sqrt{\lambda} + C_0 \sinh \sqrt{\lambda} + D_0 \cosh \sqrt{\lambda} = \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \quad (51)$$

$O(\epsilon^1)$:

$$\begin{cases} A_1 + C_1 = 0, \\ B_1 + D_1 = 0, \\ \left(-A_1 \cos \sqrt{\lambda} - B_1 \sin \sqrt{\lambda} + C_1 \cosh \sqrt{\lambda} + D_1 \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \left(-A_0 \sin \sqrt{\lambda} + B_0 \cos \sqrt{\lambda} + C_0 \sinh \sqrt{\lambda} + D_0 \cosh \sqrt{\lambda} \right) = 0, \\ A_1 \sin \sqrt{\lambda} - B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} = 0. \end{cases} \quad (52)$$

The solution is

$$\begin{cases} A_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ B_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ C_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(-\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ D_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \end{cases} \quad (53)$$

Then we have

$$\begin{aligned} & -A_1 \sin \sqrt{\lambda} + B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} \\ &= \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sin \sqrt{\lambda} - \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \end{aligned} \quad (54)$$

$$O(\epsilon^2): \quad \left\{ \begin{array}{l} A_2 + C_2 = 0, \\ B_2 + D_2 = 0, \\ \left(-A_2 \cos \sqrt{\lambda} - B_2 \sin \sqrt{\lambda} + C_2 \cosh \sqrt{\lambda} + D_2 \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \left(-A_1 \sin \sqrt{\lambda} + B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} \right) = 0, \\ A_2 \sin \sqrt{\lambda} - B_2 \cos \sqrt{\lambda} + C_2 \sinh \sqrt{\lambda} + D_2 \cosh \sqrt{\lambda} = 0. \end{array} \right. \quad (55)$$

The solution is

$$\left\{ \begin{array}{l} A_2 = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right)^2 \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ B_2 = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right)^2 \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ C_2 = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right)^2 \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(-\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ D_2 = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right)^2 \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \end{array} \right. \quad (56)$$

To get higher order expansions, we can use the following iteration method:
 $O(\epsilon^{k+1})$ ($k \geq 1$):

$$\left\{ \begin{array}{l} A_{k+1} + C_{k+1} = 0, \\ B_{k+1} + D_{k+1} = 0, \\ \left(-A_{k+1} \cos \sqrt{\lambda} - B_{k+1} \sin \sqrt{\lambda} + C_{k+1} \cosh \sqrt{\lambda} + D_{k+1} \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \left(-A_k \sin \sqrt{\lambda} + B_k \cos \sqrt{\lambda} + C_k \sinh \sqrt{\lambda} + D_k \cosh \sqrt{\lambda} \right) = 0, \\ A_{k+1} \sin \sqrt{\lambda} - B_{k+1} \cos \sqrt{\lambda} + C_{k+1} \sinh \sqrt{\lambda} + D_{k+1} \cosh \sqrt{\lambda} = 0. \end{array} \right. \quad (57)$$

The solution is

$$\left\{ \begin{array}{l} A_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right) \left(\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right) \left(\frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right) \left(-\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right) \left(\frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \end{array} \right. \quad (58)$$

where for $k \geq 2$

$$Q_k = -A_k \sin \sqrt{\lambda} + B_k \cos \sqrt{\lambda} + C_k \sinh \sqrt{\lambda} + D_k \cosh \sqrt{\lambda}, \quad (59)$$

and for $k \geq 0$

$$\begin{aligned} Q_{k+1} &= -A_{k+1} \sin \sqrt{\lambda} + B_{k+1} \cos \sqrt{\lambda} + C_{k+1} \sinh \sqrt{\lambda} + D_{k+1} \cosh \sqrt{\lambda} \\ &= -\left(\frac{\sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right) Q_k, \end{aligned} \quad (60)$$

and

$$\begin{aligned} Q_1 &= -A_1 \sin \sqrt{\lambda} + B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} \\ &= \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\frac{\sin \sqrt{\lambda} - \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \end{aligned} \quad (61)$$

$$Q_0 = \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \quad (62)$$

Hence it is shown that for $k \geq 0$

$$\begin{aligned} Q_k &= - \left(\frac{\sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) Q_k \\ &= \left[- \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) \left(\frac{\sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \right]^k \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \end{aligned} \quad (63)$$

As a result, we obtain that for $k \geq 0$

$$\begin{cases} A_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left(\frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left(\frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left(\frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\cos \sqrt{\lambda} - \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left(\frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \end{cases} \quad (64)$$