AN ASYMPTOTIC THEORY OF THIN PIEZOELECTRIC PLATES

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SUMMARY

Asymptotic integration theory in its variational form that exploits the 'zoom' technique, is used to establish the first two orders of an asymptotic theory of thin piezoelectric plane plates in the framework of electroelastostatics. The crystals considered belong to class 6 mm with sixth-order symmetry axis directed along the normal to the medium plane. At the first order of approximation a purely mechanical Love–Kirchhoff theory emerges while the electric potential satisfies a two-dimensional Poisson–Neumann problem with an effective dielectric constant accounting for electromechanical couplings. At the next order (which cannot be entirely solved because of a lack, at that order, of mechanical boundary conditions along the contour of the plate) electric and mechanical fields couple in a more symmetric manner with coupling coefficients characteristic of Bleustein–Gulayev surface waves. In particular, a Love–Kirchhoff scheme is also obtained at that order but with a deflection which includes an electric contribution. An appropriate functional framework is given for the validity of the asymptotic expansion.

1. Introduction

THE transition from three dimensions to two dimensions in the theory of *thin* elastic bodies such as shells and plates has been of serious concern in structural mechanics (for a review see Naghdi (**1**)). Depending on the way in which this is achieved more or less complicated two-dimensional models result from the analysis. A celebrated model is the Love–Kirchhoff model of plates in which the unit normal remains normal to the medium plane during deformation (Love (**2**), Timoshenko and Woinowsky-Krieger (**3**)). More sophisticated models include those developed by Mindlin, Reissner, Kármán and Ambartsumian; the *natural* theory of plates, etc. (see (**1**)) which all depend on the kinematic hypotheses made, the coupling or uncoupling between various stress effects envisaged, and the neglect or consideration of higher-order contributions in the three- to two-dimensional transition. The problem takes a special importance for finite elastic deformations. Some order was brought in this seemingly chaotic situation with the seminal papers of Gol'denveizer and coworkers (**4, 5**) in which a clear-cut mathematical method using asymptotics and known as the *asymptotic integration method* was introduced. A hierarchy of models can thus be derived by considering asymptotic expansions of all relevant fields throughout the thickness of this thin body. Later on, Ciarlet and Destuynder (**6** to **8**) were able to give a strict mathematical rigour to Gol'denveizer's approach by reformulating the problem in a variational form along with the specification of an adapted functional framework and the introduction of the *zoom technique* which allows one to work on a *fixed* domain while effecting the asymptotic expansion.

A new dimension is added to the above-stated problem when the geometrical object of interest (plate, shell, rod), not only deforms, but also is subjected to the action of electromagnetic fields because of its physical constitution. The problem becomes *electromagneto*-mechanical. While mechanics is still governed in three dimensions by the usual Cauchy equations, electromagnetism is governed by Maxwell's equations. Interaction terms appear both in the constitutive equations and in the form of forces and/or couples (**9, 10**), and the question naturally arises as to whether the above-mentioned methods generalize to this more complex, coupled framework. The reason for dealing with such a subject matter is manifold. It suffices to mention the salient role played by *ferromagnetic thin elastic plates* in high-field technology (see Moon and Pao (**11**), Moon (**12**), Wallerstein and Peach (**13**), Van de Ven (**14**), and Maugin and Goudjo (**15, 16**)), *normally conducting and superconducting elastic plates* in the technology of magnetically levitated vehicles (see the works edited by Moon (**17**), Maugin (**18**), and Yamamoto and Miya (**19**)) and *piezoelectric thin plates* used as dynamical capacitors and resonators (Mindlin (**20**)), Tiersten (**21**), Nelson (**22**), and Maugin (**23**)). The asymptotic integration method has been exploited in a rather intuitive way by Ambartsumian *et al.* (**24**, **25**) in the case of magnetoelastic, conducting or ferromagnetic, thin plates.

In the present work we aim at using the rigorous mathematical approach of Ciarlet and Destuynder in the simplest *electromechanical* case provided by linear *piezoelectric* (thus anisotropic) elastic plates in the *static* framework. A reminder on relevant three-dimensional equations is given in section 2. The three-dimensional plate problem is stated in section 3. Its functional framework and variational formulation are given in section 4. The method of asymptotic integration and the zoom technique are briefly recalled in section 5. The latter technique is implemented in the electroelastic case in section 6. The zeroth-order and first-order approximations are then given in sections 7 and 8, respectively; the former in its entirety, the latter only sketchily since the first-order mechanical approximation cannot be solved without accounting for boundary-layer effects along the contour of the plate (compare Gol'denveizer (**26**)). Section 9 concludes with a brief comparison with other approaches. The main results of this paper were previously presented in a short Note to the French Academy of Science (**27**).

2. Reminder: three-dimensional equations

The three-dimensional equations of the linear theory of piezoelectricity may be found in many books; for example, Auld (**28**), Dieulesaint and

Royer (29), Maugin (9, Chapter 4). They consist of the following two components.

(a) The *equilibrium equations* in any regular region Ω of Euclidean physical space $\mathbb{E}^3$:

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0} \qquad \text{(mechanical equilibrium)}, \tag{2.1}$$

$$\boldsymbol{\nabla} . \mathbf{D} = 0 \qquad \text{(electric equilibrium)}, \tag{2.2}$$

where $\boldsymbol{\sigma} = \{\sigma_{ij} = \sigma_{ji}\colon\ i, j = 1, 2, 3\}$ is the symmetric stress tensor and $\mathbf{D} = \{D_i\colon i = 1, 2, 3\}$ is the electric displacement vector. Equations (2.1) and (2.2) are written in the absence of body force (which is *not* linear in dielectrics) and inertia for the first one, and in the absence of electric-charge density in the second one (case of dielectrics).

(b) The *boundary conditions* at the regular boundary $\Gamma = \partial\Omega$ of Ω equipped with unit outward normal $\mathbf{n} = \{n_i\colon i = 1, 2, 3\}$:

$$\sigma_{ij} n_j = T_i, \tag{2.3}$$

$$D_i n_i = W, \tag{2.4}$$

where T_i is a prescribed surface traction and W is a surface-charge density. These conditions may hold only on parts of $\partial\Omega$, the dual (kinematic) conditions

$$u_i = u_i^d, \tag{2.5}$$

$$\phi = \phi^d \tag{2.6}$$

meaning imposed elastic displacement u_i and imposed electric potential ϕ, being valid on the complementary parts of $\partial\Omega$. This is mentioned as one possibility for simplicity. Many other combinations are possible.

Here, classical piezoelectricity is considered within the *electrostatic* framework for which $\boldsymbol{\nabla} \wedge \mathbf{E} = \mathbf{0}$, and the electric-field vector $\mathbf{E}$ is derivable from the potential ϕ by

$$\mathbf{E} = -\boldsymbol{\nabla}\phi. \tag{2.7}$$

Finally, the *coupled linear constitutive equations* are given by

$$\sigma_{ij} = C^E_{ijkl} s_{kl} - e_{kij} E_k, \tag{2.8}$$

$$D_i = \varepsilon_{ij} E_j + e_{ijk} s_{jk}, \tag{2.9}$$

where

$$\mathbf{s} = \{s_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = s_{ji}\} \tag{2.10}$$

is the infinitesimally small strain. In (2.8), (2.9) $\mathbf{C}^E$ is the fourth-order tensor of elasticity coefficients or rigidities at zero electric field (it is a linear, real symmetric application of $\mathbb{R}^6$ onto itself), $\mathbf{e}$ is the third-order tensor of piezoelectricity coefficients (a linear real application of $\mathbb{R}^3$ onto $\mathbb{R}^6$ which is non-zero only for materials without centre of symmetry), and $\boldsymbol{\varepsilon}$ is the tensor of dielectric constants at vanishing strains (a linear, real symmetric application of $\mathbb{R}^3$ onto itself). Accounting for the symmetries just mentioned, we can use the *Voigt notation* which associates a Greek index with any symmetric couple of italic indices in agreement with the following rule.

(ij)	(11)	(22)	(33)	(23)	(13)	(12)
α	1	2	2	4	5	6

Then (2.8), (2.9) can also be written as

$$\left.\begin{aligned} \sigma_\alpha &= C^E_{\alpha\beta} s_\beta - e_{k\alpha} E_k, & C^E_{\alpha\beta} &= C^E_{\beta\alpha}, \\ D_i &= \varepsilon_{ij} E_j + e_{i\beta} s_\beta, & \varepsilon_{ij} &= \varepsilon_{ji} \end{aligned}\right\} \tag{2.11}$$

($\alpha, \beta = 1, 2, \ldots, 6$; $i, j, k = 1, 2, 3$).

We shall focus attention on piezoelectric materials which possess an axis of symmetry of order six (hexagonal symmetry). Let x_3 be this axis. Cadmium sulphide (CdS) and zinc oxide (ZnO) are examples of crystals that belong to this class denoted by 6 mm. Only the following material coefficients survive (cf. Maugin (9, p. 227)):

$$\begin{aligned}
&C^E_{\alpha\beta}\ (\text{5 coefficients}): && \begin{cases} c_{11} = c_{22}, \quad c_{13} = c_{23} = c_{31} = c_{32}, \\ c_{33}, \quad c_{44} = c_{55}, \quad c_{66} = \tfrac{1}{2}(c_{11} - c_{12}); \end{cases} \\
&e_{k\alpha}\ (\text{3 coefficients}): && e_{15} = e_{24}, \quad e_{31} = e_{32}, \quad e_{33}; \\
&\varepsilon_{ij}\ (\text{2 coefficients}): && \varepsilon_{11} = \varepsilon_{22}, \quad \varepsilon_{33}.
\end{aligned}$$

3. The problem of the plate

We consider a piezoelectric 3-dimensional object called a *plane plate* (Fig. 1) which occupies the open set Ω^ε of $\mathbb{R}^3$ in its reference configuration.

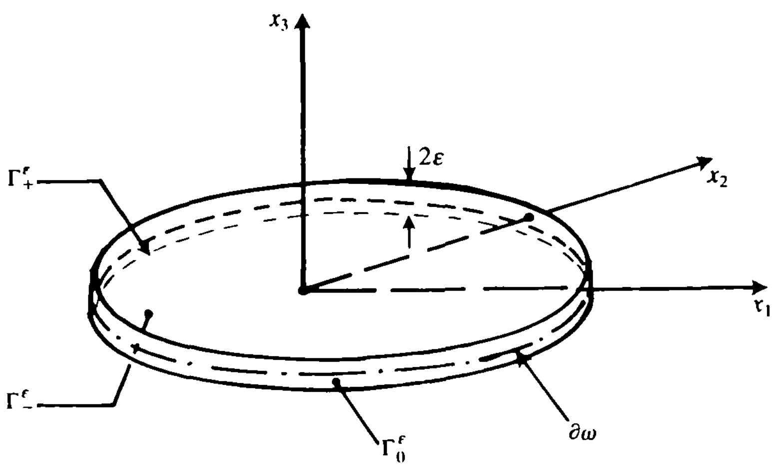

FIG. 1

Its boundary Γ^ε is assumed to be sufficiently smooth to allow for the forthcoming mathematical manipulations. We call $\Gamma^\varepsilon_\pm$ the upper and lower plane faces of the plate, Γ^ε_0 the lateral face, and ω the median plane chosen to be the $0x_1x_2$ coordinate plane (that is, $x_3=0$). We write γ for the boundary of ω in this plane. The thickness of the plate is 2ε and, in fact, we have set $\varepsilon = h/L$, where h is the actual, physical thickness of the plate and L is a typical in-plane dimension. The slenderness of the plate results in $\varepsilon \ll 1$, so that ε is the relevant smallness parameter in any eventual asymptotic expansion. More precisely, we note that

$$\left.\begin{aligned} \Omega^\varepsilon &= \omega \times]-\varepsilon, +\varepsilon[, \\ \Gamma^\varepsilon &= \Gamma^\varepsilon_0 \cup \Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-, \\ \Gamma^\varepsilon_0 &= \gamma \times]-\varepsilon, +\varepsilon[, \\ \Gamma^\varepsilon_\pm &= \omega \times \{\pm\varepsilon\}. \end{aligned}\right\} \tag{3.1}$$

In the plate problem we assume that the following boundary conditions hold:

$$u_i = 0 \quad \text{on } \Gamma^\varepsilon_0, \tag{3.2}$$

$$\sigma_{ij} n^\pm_j = T^\pm_i \quad \text{on } \Gamma^\varepsilon_\pm, \tag{3.3}$$

$$\mathbf{n} \,.\, [\![\mathbf{D}]\!]|_{\Gamma^\varepsilon_0} = q_0, \qquad \mathbf{n}^\pm \,.\, [\![\mathbf{D}]\!]|_{\Gamma^\varepsilon_\pm} = W_\pm, \tag{3.4}$$

$$[\![\phi]\!]|_{\Gamma^\varepsilon} = 0; \tag{3.5}$$

that is, the plate is clamped on its contour, there may be tractions acting on the upper and lower faces, electric charges may accumulate on both the upper and lower faces and the lateral face. The condition (3.5) usually replaces the continuity condition imposed on the tangential component of **E**, where $[\![\cdot]\!]$ denotes the jump of the enclosed quantity at the relevant surface. As a matter of fact, we shall assume that $q_0 = 0$, while $\Gamma^\varepsilon_\pm$ are covered with extremely thin electrodes of negligible thickness which play no role mechanically but allow one to impose a prescribed potential if necessary.

For materials in the 6 mm class and the sixth-order axis along the normal to our plate, the constitutive equations (2.11) take on the following explicit form (*caution*: here Greek indices no longer refer to the Voigt notation but to in-plane components, $\alpha, \beta = 1, 2$; underlined indices are *not* summed unless explicitly indicated by a summation sign):

$$\left.\begin{aligned} \sigma_{\underline{\beta}\underline{\beta}} &= \sum_{l=l}^{3} c_{\beta ll} u_{l,l} + e_{31}\phi_{,3}, \quad \beta = 1, 2, \ \beta \text{ fixed}, \\ \sigma_{33} &= \sum_{l=1}^{3} c_{3ll} u_{l,l} + e_{33}\phi_{,3}, \\ \sigma_{\alpha 3} &= \sigma_{3\alpha} = c_{44}(u_{\alpha,3} + u_{3,\alpha}), \quad \alpha = 1, 2, \\ \sigma_{12} &= \sigma_{21} = c_{66}(u_{1,2} + u_{2,1}), \end{aligned}\right\} \tag{3.6}$$

and

$$D_\alpha = \begin{cases} e_{15}(u_{\alpha,3} + u_{3,\alpha}) - \varepsilon_{11}\phi_{,\alpha}, & \alpha = 1, 2, \\ e_{31}(u_{1,1} + u_{2,2}) + e_{33}u_{3,3} - \varepsilon_{33}\phi_{,3}, & \alpha = 3. \end{cases} \tag{3.7}$$

These can be formally inverted to yield

$$\left.\begin{aligned} s_{ij} = u_{i,j} &= a_{ijkl}\sigma_{kl} + b_{jip}D_p, \\ \phi_{,k} &= \bar{a}_{kl_p}\sigma_{l_p} + \bar{b}_{k_j}D_j. \end{aligned}\right\} \tag{3.8}$$

We need not give precise expressions for these as only the symmetry on some of the indices is to be used shortly.

4. Functional and variational framework

Let $\mathbf{v}$, $\boldsymbol{\tau}$, $\boldsymbol{\delta}$ and φ denote test functions having the same tensorial and physical natures as $\mathbf{u}$, $\boldsymbol{\sigma}$, $\mathbf{D}$ and ϕ, respectively. On account of (3.2), for $\mathbf{v}$ the classical *Sobolev* space of mechanics can be introduced (cf. Raviart and Thomas (**30**), Kikuchi and Oden (**31**))

$$v^\varepsilon = \{\mathbf{v} \mid \mathbf{v} = \{v_i\},\ i = 1, 2, 3;\ v_i \in H^1(\Omega^\varepsilon),\ \mathbf{v} = \mathbf{0}|_{\Gamma^\varepsilon}\}. \tag{4.1}$$

This is a Hilbert space equipped with the norm induced by $H^1(\Omega^\varepsilon)$. For $\boldsymbol{\tau}$ we have

$$\Sigma^\varepsilon = \{\boldsymbol{\tau} \mid \boldsymbol{\tau} = \{\tau_{ij} = \tau_{ji}\},\ i, j = 1, 2, 3;\ \tau_{ij} \in [L^2(\Omega^\varepsilon)]^9\}. \tag{4.2}$$

This is a Hilbert space equipped with the norm product of the $L^2(\Omega^\varepsilon)$. Accounting for curl and divergence operators and following Duvaut and Lions (**32**), for $\boldsymbol{\delta}$ and φ we have the natural spaces

$$\Delta^\varepsilon = H(\mathrm{div}, \Omega^\varepsilon) = \{\boldsymbol{\delta} \mid \boldsymbol{\delta}[L^2(\Omega^\varepsilon)]^3,\ \boldsymbol{\nabla} \cdot \boldsymbol{\delta} \in L^2(\Omega^\varepsilon)\}, \tag{4.3}$$

$$\Phi^\varepsilon = \{\varphi \in H^1(\Omega^\varepsilon)\}. \tag{4.4}$$

The introduction of spaces (4.1) to (4.4) allows one to prove the *existence* and *uniqueness* of solutions of our 3-dimensional electroelastic problem by using a variant of Brezzi's theorem (**33**) for the correct *variational formulation*. The latter here is an *electromechanical* generalization of the celebrated two-field variational principle of Hellinger and Reissner used in mixed finite-element methods (see Fraeijs de Veubeke (**34**), Washizu (**35**)). Here the principle becomes a *four-field* variational principle that we can state as follows.

PROBLEM (P^ε). Find $(\boldsymbol{\sigma}, \mathbf{D}, \mathbf{u}, \phi)$ in $(\Sigma^\varepsilon \times \Delta^\varepsilon) \times (v^\varepsilon \times \Phi^\varepsilon)$ such that

$$\left.\begin{aligned} &\mathscr{A}^\varepsilon(\boldsymbol{\sigma}, \mathbf{D}; \boldsymbol{\tau}, \boldsymbol{\delta}) + \mathscr{B}^\varepsilon(\boldsymbol{\tau}, \boldsymbol{\delta}; \mathbf{u}, \phi) = 0 \quad \text{for all } (\boldsymbol{\tau}, \boldsymbol{\delta}) \in (\Sigma^\varepsilon \times \Delta^\varepsilon), \\ &\mathscr{B}^\varepsilon(\boldsymbol{\sigma}, \mathbf{D}; \mathbf{v}, \varphi) = \mathscr{F}^\varepsilon(\mathbf{v}, \varphi) \quad \text{for all } (\mathbf{v}, \varphi) \in (v^\varepsilon \times \Phi^\varepsilon), \end{aligned}\right\} \tag{4.5}$$

where we have defined the bilinear forms (using the notation of (3.8))

$$\mathscr{A}^{\varepsilon}(\boldsymbol{\sigma}, \mathbf{D}; \boldsymbol{\tau}, \boldsymbol{\delta}) = \int_{\Omega^{\varepsilon}} (a_{ijkl}\sigma_{ij}\tau_{kl} + b_{ijp}D_p\tau_{ij})\, d\Omega$$

$$+ \int_{\Omega^{\varepsilon}} (\bar{a}_{kl_p}\sigma_{l_p}\delta_k + \bar{b}_{k_j}D_j\delta_k)\, d\Omega, \tag{4.6}$$

$$\mathscr{B}^{\varepsilon}(\boldsymbol{\tau}, \boldsymbol{\delta}; \mathbf{u}, \phi) = -\int_{\Omega^{\varepsilon}} (\tau_{ji}u_{j,i} + \phi_{,k}\delta_k)\, d\Omega, \tag{4.7}$$

$$\mathscr{F}^{\varepsilon}(\mathbf{v}, \varphi) = -\int_{\Gamma^{\varepsilon}_{\pm}} (\mathbf{T}^{\pm} \,.\, \mathbf{v} + W_{\pm}\varphi)\, ds. \tag{4.8}$$

Solution. The expressions (4.5) to (4.8) are directly established from the field equations (3.8) and Green's formula, and noting the symmetry of $\boldsymbol{\tau}$ and the fact that $q_0 = 0$ on Γ^{ε}_0.

5. Asymptotic integration and zoom technique

It is salient to recall briefly Gol'denveizer's asymptotic method (**4, 5**) and the zoom technique of Ciarlet and Destuynder (**6** to **8**). The asymptotic-integration method is an asymptotic expansion method which, by taking account of the smallness of the parameter ε, allows one to establish the order of magnitude or singularity of all components of the mechanical fields in the process when the thickness of a slender body goes to zero. Without entering into details of the procedure, all fields Q are written in the form

$$Q = \varepsilon^{-q} \sum_{s=0}^{N} \varepsilon^{s} Q^{(s)}, \tag{5.1}$$

where q is an integer to be determined for each field. The method provides a hierarchy of problems which is solved step-by-step. The exponent q is found through the boundary conditions, which are not satisfied at the second order, and the method does not account for boundary-layer effects. Applied to linear elastic isotropic plates, Gol'denveizer's method provides the Love–Kirchhoff model of plates in the first approximation. To many the method appears intuitive and lacks a functional background.

The *zoom technique* of Ciarlet and Destuynder in fact constitutes a rigorous mathematical proof of the asymptotic-integration method by using a variational formulation in the adequate functional framework. The trick consists in applying Lions's formalism (**36**) of singular perturbations while bringing the problem back to a *fixed* reference open set of $\mathbb{E}^3$ by effecting an affinity along the normal coordinate. That is, for each ε we define

$$F^{\varepsilon}: \mathbf{x} = (x_1, x_2, x_3) \in \bar{\Omega} \rightarrow F^{\varepsilon}(\mathbf{x}) = \mathbf{x}^{\varepsilon} = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^{\varepsilon}. \tag{5.2}$$

Thus, with any function f defined on $\bar{\Omega}^\varepsilon$ there is associated a function f^ε on $\bar{\Omega}$ by

$$f^\varepsilon(\mathbf{x}) = f \circ F^\varepsilon(\mathbf{x}). \tag{5.3}$$

In our case the fixed reference domain is obtained by this zoom technique as

$$\left.\begin{aligned} &\Omega = \omega \times]-1, +1[, \\ &\Gamma_\pm = \omega \times \{\pm 1\}, \\ &\Gamma_0 = \gamma \times]-1, +1[, \end{aligned}\right\} \tag{5.4}$$

while we obviously have

$$\int_{\Omega^\varepsilon} f = \varepsilon \int_\Omega f^\varepsilon, \qquad \int_{\Gamma_0^\varepsilon} f = \varepsilon \int_{\Gamma_0} f, \qquad \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} f = \int_{\Gamma_+ \cup \Gamma_-} f^\varepsilon \tag{5.5}$$

and

$$\frac{\partial}{\partial x_\alpha} f(\mathbf{x}^\varepsilon) = \begin{cases} \dfrac{\partial}{\partial x_\alpha} f^\varepsilon(\mathbf{x}), & \alpha = 1, 2, \\ \varepsilon^{-1} \dfrac{\partial}{\partial x_3} f^\varepsilon(\mathbf{x}), & \alpha = 3. \end{cases} \tag{5.6}$$

Then the problem (P^ε) expressed by equations (4.5) can be rewritten as a problem (P) while $\mathscr{A}^\varepsilon$ is expanded in terms of powers of ε, and $\mathscr{B}^\varepsilon$ in fact reduces to $\mathscr{B}_0$.

6. Implementation in the electroelastic case

Change of unknowns (see (**6** to **8**))

For *mechanical quantities* we immediately have the following change of unknowns in agreement with (5.3) and Gol'denveizer's approach:

$$\left.\begin{aligned} &\sigma_{\alpha\beta}^\varepsilon = \sigma_{\alpha\beta} \circ F^\varepsilon, \quad u_\alpha^\varepsilon = u_\alpha \circ F^\varepsilon, \quad (\mathbf{T}_\pm^\varepsilon)_\alpha = \varepsilon^{-1} T_\alpha^\pm \circ F^\varepsilon, \quad \alpha, \beta = 1, 2, \\ &\sigma_{33}^\varepsilon = \varepsilon^{-2} \sigma_{33} \circ F^\varepsilon, \quad \sigma_{\alpha 3}^\varepsilon = \varepsilon^{-1} \sigma_{\alpha 3} \circ F^\varepsilon, \quad \alpha = 1, 2, \\ &u_3^\varepsilon = \varepsilon u_3 \circ F^\varepsilon, \quad T_{3\pm}^\varepsilon = \varepsilon^{-2} T_3 \circ F^\varepsilon. \end{aligned}\right\} \tag{6.1}$$

For *electrical entities*, a short reasoning on a plane capacitor (this is what the plate is!) shows that the E_3 or D_3 field varies like the inverse of the thickness for a prescribed potential difference between the electrodes sandwiching the plate. Thus ($q = 1$ in (5.1) for D_3)

$$D_3^\varepsilon = \varepsilon^{-1} D_3 \circ F^\varepsilon \tag{6.2}$$

and

$$\left.\begin{aligned} &D_\alpha^\varepsilon = D_\alpha \circ F^\varepsilon, \quad \alpha = 1, 2, \\ &\phi^\varepsilon = \phi \circ F^\varepsilon, \quad W_\pm^\varepsilon = \varepsilon^{-1} W_\pm \circ F^\varepsilon. \end{aligned}\right\} \tag{6.3}$$

One can now apply the asymptotic expansion to the variational principle (4.5) rewritten for a *fixed* domain Ω. We can state the following.

PROBLEM (P). Find $(\boldsymbol{\sigma}^\varepsilon, \mathbf{D}^\varepsilon, \mathbf{u}, \phi)$ in $(\Sigma \times \Delta) \times (\upsilon \times \Phi)$ such that

$$\left.\begin{aligned} &[\mathscr{A}_0 + \varepsilon\mathscr{A}_1 + \varepsilon^2\mathscr{A}_2 + \ldots](\boldsymbol{\sigma}^\varepsilon, \mathbf{D}^\varepsilon; \boldsymbol{\tau}, \delta) + \mathscr{B}^\varepsilon(\boldsymbol{\tau}, \boldsymbol{\delta}; \mathbf{u}, \phi) = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } (\boldsymbol{\tau}, \boldsymbol{\delta}) \in (\Sigma \times \Delta), \\ &\mathscr{B}_0(\boldsymbol{\sigma}^\varepsilon, \mathbf{D}^\varepsilon; \mathbf{v}, \varphi) = \mathscr{F}_0(\mathbf{v}, \varphi) \quad \text{for all } (\mathbf{v}, \varphi) \in (\upsilon \times \Phi), \end{aligned}\right\} \tag{6.4}$$

where, after splitting normal, in-plane and mixed components, we have

$$\left.\begin{aligned} \mathscr{A}_0 &= \int_\Omega \Big(a_{\alpha\beta\gamma\delta}\sigma_{\alpha\beta}\tau_{\gamma\delta} + \sum_{\alpha=1}^{2} \tilde{b}_{\alpha\alpha}D_\alpha\delta_\alpha\Big)\, d\Omega, \\ \mathscr{A}_1 &= \int_\Omega \Big(\tilde{a}_{3\alpha\beta}\sigma_{\alpha\beta}\delta_3 + \sum_{\alpha=1}^{2} b_{3\alpha\alpha}D_\alpha\tau_{\alpha 3} \\ &\qquad + \sum_{\alpha=1}^{2} \tilde{a}_{\alpha\alpha 3}\sigma_{\alpha 3}\delta_\alpha + b_{\alpha\beta 3}D_3\tau_{\alpha\beta}\Big)\, d\Omega, \\ \mathscr{A}_2 &= \int_\Omega \{4a_{\alpha 3\beta 3}\sigma_{\alpha 3}\tau_{\beta 3} + a_{\alpha\beta 33}(\sigma_{33}\tau_{\alpha\beta} + \sigma_{\alpha\beta}\tau_{33}) + \tilde{b}_{33}D_3\delta_3\}\, d\Omega, \\ \mathscr{A}_3 &= \int_\Omega (\tilde{a}_{333}\sigma_{33}\delta_3 + b_{333}D_3\tau_{33})\, d\Omega, \\ \mathscr{A}_4 &= \int_\Omega a_{3333}\sigma_{33}\tau_{33}\, d\Omega \end{aligned}\right\} \tag{6.5}$$

and

$$\left.\begin{aligned} \mathscr{B}_0 &= -\int_\Omega (\tau_{ji}v_{i,j} + \phi_{,k}\delta_k)\, d\Omega, \\ \mathscr{F}_0 &= \int_{\Gamma_+\cup\Gamma_-} (\mathbf{T}_\pm \cdot \mathbf{v} + W_\pm\varphi)\, ds. \end{aligned}\right\} \tag{6.6}$$

In writing (6.5) and (6.6) and obtaining the statement (6.4) we have assumed that the coefficients a_{ijkl}, $\tilde{b}_{ij}$, $\tilde{a}_{ijk}$ and b_{ijk} do not depend on ε, that $\mathbf{T}^\varepsilon_\pm$ and $W^\varepsilon_\pm$ do not depend on ε, and that transverse shear effects uncouple from plane-stress effects (that is, $\sigma_{\alpha\beta}$, $\alpha, \beta = 1, 2$) and transverse normal effects (σ_{33}) so that

$$a_{33\alpha\beta} = a_{\alpha\beta\gamma 3} = 0, \qquad \alpha, \beta, \gamma = 1, 2, \ldots. \tag{6.7}$$

It remains to expand the fields $\boldsymbol{\sigma}^\varepsilon$, $\mathbf{D}^\varepsilon$, $\mathbf{u}^\varepsilon$ and ϕ^ε according to

$$\begin{aligned} (\boldsymbol{\sigma}^\varepsilon, \mathbf{D}^\varepsilon, \mathbf{u}^\varepsilon, \phi^\varepsilon) &= (\boldsymbol{\sigma}^{(0)}, \mathbf{D}^{(0)}, \mathbf{u}^{(0)}, \phi^{(0)}) + \varepsilon(\boldsymbol{\sigma}^{(1)}, \mathbf{D}^{(1)}, \mathbf{u}^{(1)}, \phi^{(1)}) \\ &\quad + \varepsilon^2(\boldsymbol{\sigma}^{(2)}, \mathbf{D}^{(2)}, \mathbf{u}^{(2)}, \phi^{(2)}) + \ldots \end{aligned} \tag{6.8}$$

and to deduce a hierarchy of problems from (6.4) to find the various orders of approximation, at least theoretically. With a rapidly increasing difficulty in this approach as the order increases, we give the first two orders in the next two sections.

7. Zeroth-order contributions

Substituting from (6.8) into (6.4), we obtain the following problem.

PROBLEM ($P^{(0)}$). Find ($\boldsymbol{\sigma}^{(0)}, \mathbf{D}^{(0)}, \mathbf{u}^{(0)}, \phi^{(0)}$) such that

$$\left.\begin{aligned}&\mathscr{A}_0(\boldsymbol{\sigma}^{(0)}, \mathbf{D}^{(0)}; \boldsymbol{\tau}, \boldsymbol{\delta}) + \mathscr{B}_0(\boldsymbol{\tau}, \boldsymbol{\delta}; \mathbf{u}^{(0)}, \phi^{(0)}) = 0 \quad \text{for all } (\boldsymbol{\tau}, \boldsymbol{\delta}) \in (\Sigma \times \Delta),\\ &\mathscr{B}_0(\boldsymbol{\sigma}^{(0)}, \mathbf{D}^{(0)}; \mathbf{v}, \varphi) = \mathscr{F}_0(\mathbf{v}, \varphi) \quad \text{for all } (\mathbf{v}, \varphi) \in (V \times \Phi).\end{aligned}\right\} \tag{7.1}$$

In *local* form the first of these reads

$$\left.\begin{aligned}&\sigma^{(0)}_{\gamma\delta} = A_{\delta\gamma\beta\alpha} u^{(0)}_{\beta,\alpha},\\ &u^{(0)}_{\alpha,3} + u^{(0)}_{3,\alpha} = 0, \qquad u^{(0)}_{3,3} = 0,\\ &\phi^{(0)}_{,\alpha} - \bar{b}_{\alpha\alpha} D^{(0)}_{\alpha} = 0, \qquad \phi^{(0)}_{,3} = 0.\end{aligned}\right\} \tag{7.2}$$

Integrating the second and third of these in the sense of distribution theory we obtain the classical displacement field of the *Love–Kirchhoff theory of plates* (see (**3**), for example)

$$\left.\begin{aligned}&u^{(0)}_3 = w(x_1, x_2),\\ &u^{(0)}_{\alpha} = \bar{u}^{(0)}_{\alpha} - x_3 u^{(0)}_{3,\alpha}.\end{aligned}\right\} \tag{7.3}$$

Ciarlet and Destuynder (**6**) have shown that

$$u^{(0)}_3 \in H^2_0(\omega), \qquad \bar{u}^{(0)}_{\alpha} \in H^1_0(\omega). \tag{7.4}$$

Equation $(7.2)_5$ shows that $\phi^{(0)}$ does *not* depend on x_3 and we can, therefore, identify it with an element of $H^1(\omega)$:

$$\phi^{(0)} = \varphi(x_1, x_2). \tag{7.5}$$

From $(7.2)_1$ and $(7.2)_4$ we can express the *plane* stresses $\sigma^{(0)}_{\alpha\beta}$ and the in-plane components of $\mathbf{D}^{(0)}$ as

$$\left.\begin{aligned}&\sigma^{(0)}_{\alpha\beta} = a^{-1}_{\alpha\beta\gamma\delta} u^{(0)}_{\gamma,\delta},\\ &D^{(0)}_{\alpha} = -\bar{\varepsilon}_{11}\varphi_{,\alpha},\end{aligned}\right\} \tag{7.6}$$

where φ is defined in (7.5) and we have set

$$\bar{\varepsilon}_{11} = \varepsilon_{11}(1 + K^2), \qquad K^2 = e^2_{15}/\varepsilon_{11} c_{44}. \tag{7.7}$$

Here K is none other than the coefficient of *electromechanical coupling* (cf. (**29**, p. 225; **9**, p. 252)). The equations governing $u^{(0)}_3$, $\bar{u}^{(0)}_{\alpha}$ and $\phi^{(0)}$ can be established. Of special interest are the *bending* equations governing the deflection w, and the equation governing φ.

Bending problem

There are two ways to obtain the bending equations. One consists, in the spirit of the variational formulation, in exploiting $(7.1)_2$ and choosing a smaller functional space than before for the displacement $\mathbf{u}$. For instance, v

can be replaced by V_{LK} (LK for Love–Kirchhoff) such that

$$V_{LK} = \{\mathbf{v} = \{v_i\} \mid v_3 \in H_0^2(\omega),\ v_\alpha = \bar{v}_\alpha - x_3 v_{3,\alpha},\ \bar{v}_\alpha \in H_0^1(\omega)\} \quad (7.8)$$

in such a way that $V_{LK} \subset v$ and it is a Hilbert space when equipped with the norm induced by v. For the electric potential we already have

$$\phi \in H^1(\omega). \quad (7.9)$$

The second way consists in exploiting directly the local equations in the median plane of the plate ω. In either case we find a bending equation for $u_3^{(0)} = w(x_1, x_2)$ in the form of the bending equation for a weakly anisotropic plate clamped on its contour (compare Lekhnitskii (**37**) for anisotropic plates). More precisely, we obtain

$$\left.\begin{aligned} &\tfrac{2}{3}[c_{11}(w_{,1111} + w_{,2222}) + (c_{11} + 2c_{12})(w_{,1112} + w_{,1222} + 2w_{,1122})] = F_3, \\ &F_3 = 2\langle T_3 + T_{\alpha,\alpha} \rangle, \end{aligned}\right\} \quad (7.10)$$

where the symbolism $\langle \cdot \rangle$ indicates the *mean* value on the upper and lower faces of the plate, together with the clamping conditions

$$w = 0 \quad \text{on } \gamma = \partial\omega, \qquad \partial w/\partial n = 0 \quad \text{on } \gamma. \quad (7.11)$$

As to the electric problem, one obtains the following *Poisson–Neumann* problem for the potential $\phi^{(0)} = \varphi$:

$$\bar{\varepsilon}_{11}\Delta_2\varphi = \langle W \rangle \quad \text{in } \omega, \qquad \bar{\varepsilon}_{11}\, \partial\varphi/\partial n = 0 \quad \text{on } \gamma, \quad (7.12)$$

where Δ_2 is the in-plane Laplacian operator. For the Neumann problem we must have that meas $\gamma > 0$, where meas is Lebesgue's measure, and the condition

$$\int_\omega \langle W \rangle = 0 \quad (7.13)$$

must be satisfied.

Once $\sigma_{\alpha\beta}^{(0)}$ and $D_\alpha^{(0)}$ are determined, there remains to find $\sigma_{\alpha 3}^{(0)}$, $\sigma_{33}^{(0)}$ and $D_3^{(0)}$. From (7.2) we have

$$\left.\begin{aligned} &\sigma_{\gamma\delta}^{(0)} = A_{\gamma\delta\beta\alpha} u_{\beta,\alpha}^{(0)} = A_{\gamma\delta\beta\alpha}(\bar{u}_\beta^{(0)} - x_3 w_{,\beta})_{,\alpha}, \\ &D_\alpha^{(0)} = -\bar{\varepsilon}_{11}\varphi_{,\alpha}. \end{aligned}\right\} \quad (7.14)$$

If membrane effects are discarded, the first of these yields

$$\sigma_{\gamma\delta}^{(0)} = -x_3 A_{\gamma\delta\beta\alpha} w_{,\beta\alpha}. \quad (7.15)$$

But when these effects are *not* neglected, then one must have recourse to the equation satisfied by $\bar{u}_\alpha^{(0)}$, $\alpha = 1, 2$. In variational form we have

$$\left.\begin{aligned} &\int_\omega a_{\alpha\beta\gamma\delta}\gamma_{\alpha\beta}(\bar{\mathbf{u}}^{(0)})\gamma_{\gamma\delta}(\bar{\mathbf{v}})\, ds = \int_\omega F_\alpha \bar{v}_\alpha\, ds, \\ &F_\alpha = \langle T_\alpha \rangle, \qquad \gamma_{\alpha\beta}(\bar{\mathbf{v}}) = \tfrac{1}{2}(\bar{v}_{\alpha,\beta} + \bar{v}_{\beta,\alpha}), \\ &\forall \bar{\mathbf{v}} = \{\bar{v}_\alpha\} \in [H_0^1(\omega)]^2. \end{aligned}\right\} \quad (7.16)$$

This determines the in-plane displacements $\bar{\mathbf{u}}^{(0)}$ and hence the plane stresses are entirely determined. The *shear* stresses $\sigma_{\alpha 3}^{(0)}$ and the transverse normal stress $\sigma_{33}^{(0)}$ are obtained through the equilibrium equations by integration as

$$\left.\begin{aligned}\sigma_{\alpha 3}^{(0)} &= -T_\alpha^- - \int_{-l}^{x_3} \sigma_{\alpha\beta,\beta}^{(0)}(x_1, x_2, \zeta)\, d\zeta,\\ \sigma_{33}^{(0)} &= -T_3^- - \int_{-1}^{x_3} \sigma_{\alpha 3,\alpha}^{(0)}(x_1, x_2, \zeta)\, d\zeta.\end{aligned}\right\} \tag{7.17}$$

Finally, the normal component of $\mathbf{D}^{(0)}$ is obtained by integration as

$$\begin{aligned}D_3^{(0)} &= -W^- - \int_{-1}^{x_3} D_{\alpha,\alpha}^{(0)}(x_1, x_2, \zeta)\, d\zeta\\ &= -W^- + (1 + x_3)\,\Delta_2\varphi.\end{aligned} \tag{7.18}$$

This completes the zeroth-order solution. Remarkably enough, if it were not for the introduction of the *effective* dielectric constant $\bar{\varepsilon}_{11}$ and the electromechanical coupling through K, the mechanical and electrical parts of this solution would be entirely independent. In effect, the practical problems are purely mechanical and electrical, abstraction being made in the definition of $\bar{\varepsilon}_{11}$. The reason for this is to be sought in the symmetry of the piezoelectric crystal (class 6 mm) and the fact that its preferred axis here coincides with the direction along which the zoom is applied, while we know that in 6 mm crystals the electromechanical coupling is essentially between the elastic displacement in the preferred direction and the spatial variation of the electric potential in the plane orthogonal to that direction (see Maugin (**9**, Chapter 4)). This is what is exploited in the existence of Bleustein–Gulyaev surface waves (see below). We obtain thus a mechanical influence in the equation governing $\varphi(x_1, x_2)$, through $\bar{\varepsilon}_{11}$ in agreement with this general remark, but *no* converse effect because D_3 and $\sigma_{\alpha\beta}$ do not have the same singular behaviour (compare (6.1) and (6.2)). The situation is different at the next order of approximation, where the electromechanical coupling becomes more symmetric.

8. First-order contributions

At order one in ε, the problem (6.4) yields the following problem.

PROBLEM ($P^{(1)}$). Find ($\boldsymbol{\sigma}^{(1)}$, $\mathbf{D}^{(1)}$, $\mathbf{u}^{(1)}$, $\phi^{(1)}$) such that

$$\left.\begin{aligned}&\mathcal{A}_0(\boldsymbol{\sigma}^{(1)}, \mathbf{D}^{(1)}; \boldsymbol{\tau}, \boldsymbol{\delta}) + \mathcal{B}_0(\boldsymbol{\tau}, \boldsymbol{\delta}; \mathbf{u}^{(1)}, \phi^{(1)})\\ &\quad = -\mathcal{A}_1(\boldsymbol{\sigma}^{(0)}, \mathbf{D}^{(0)}; \boldsymbol{\tau}, \boldsymbol{\delta}) \quad \text{for all } (\boldsymbol{\tau}, \boldsymbol{\delta}) \in (\Sigma \times \Delta),\\ &\mathcal{B}_0(\boldsymbol{\sigma}^{(1)}, \mathbf{D}^{(1)}; \mathbf{v}, \varphi) = 0 \quad \text{for all } (\mathbf{v}, \varphi) \in (\upsilon \times \Phi).\end{aligned}\right\} \tag{8.1}$$

In local form (8.1) yields

$$\left.\begin{aligned} &a_{\alpha\beta\gamma\delta}\sigma^{(1)}_{\alpha\beta} - u^{(1)}_{\delta,\gamma} + b_{\gamma\delta3}D^{(0)}_3 = 0,\\ &u^{(1)}_{\alpha,3} + u^{(1)}_{3,\alpha} = b_{3\alpha\alpha}D^{(0)}_{\alpha}, \qquad u^{(1)}_{3,3} = 0,\\ &\phi^{(1)}_{,\alpha} - \bar{b}_{\alpha\alpha}D^{(1)}_{\alpha} - \bar{a}_{\alpha\alpha3}\sigma^{(0)}_{\alpha3} = 0,\\ &\phi^{(1)}_{,3} - \bar{a}_{3\alpha\beta}\sigma^{(0)}_{\alpha\beta} = 0. \end{aligned}\right\} \tag{8.2}$$

It follows from this that

$$\left.\begin{aligned} &u^{(1)}_3 = \eta(x_1, x_2),\\ &u^{(1)}_{\alpha} = \bar{u}^{(1)}_{\alpha} - x_3\left[u^{(1)}_{3,\alpha} + \frac{e_{15}}{c_{44}}\phi^{(0)}_{,\alpha}\right] \end{aligned}\right\} \tag{8.3}$$

in such a way that

$$u^{(1)}_{\alpha} = \bar{u}^{(1)}_{\alpha} - x_3\bar{\eta}_{,\alpha}, \qquad \bar{\eta} = \eta + \frac{e_{15}}{c_{44}}\phi^{(0)}. \tag{8.4}$$

That is, the first-order in-plane displacement again takes the Love–Kirchhoff form, but with an *effective deflection* $\bar{\eta}$ which accounts for the electromechanical coupling with the zeroth-order electrical solution. The same combination as that present in $(8.4)_2$ appears in the treatment of Bleustein–Gulyaev surface waves in crystals of the 6 mm class (cf. (**9**, equation $(4.11.14)_1$)). If we impose the condition that

$$\phi^{(1)}|_{\gamma} = 0, \tag{8.5}$$

then we can express $\phi^{(1)}$ and $D^{(1)}_{\alpha}$ by

$$\phi^{(1)} = -\tfrac{1}{2}x_3^2\bar{a}_{3\alpha\beta}a^{-1}_{\gamma\delta\beta\alpha}w_{,\delta\gamma} \tag{8.6}$$

and

$$D^{(1)}_{\alpha} = \bar{\varepsilon}_{11}\left[-\phi^{(1)}_{,\alpha} + \bar{a}_{\alpha\alpha3}\left\{-T^{-}_{\alpha} - \int_{-1}^{x_3}\sigma^{(0)}_{\alpha\beta,\beta}(x_1, x_2, \zeta)\, d\zeta\right\}\right], \tag{8.7}$$

which are obtained through (8.2), partly from the zeroth-order solution. We have set

$$\bar{a}_{\alpha\alpha3} = \frac{e_{15}}{\bar{\varepsilon}_1c_{44}} = \bar{K}^2 = \frac{K^2}{1+K^2}, \tag{8.8}$$

where $\bar{K}$ is none other than the electromechanical coupling coefficient that intervenes in the study of Bleustein–Gulyaev surface waves propagating along the electrically grounded surface of a piezoelectric half-space of crystal class 6 mm (cf. (**9**, p. 253)). The fact that surface-wave characteristics show up in the description of properties of *plates*, be they static, should not come as a surprise since this often occurs in plate theory (for example, limit speeds provided by the Rayleigh-wave speed in elastic plates and rods; cf. Graff (**38**)).

The equation governing η is a fourth-order partial differential equation with a source term due to the zeroth-order solution:

$$a^{-1}_{\alpha\beta\gamma\delta}\eta_{,\alpha\beta\gamma\delta} = \varepsilon_{11}K^2[a^{-1}_{\delta\alpha\beta\alpha}(\Delta_2\varphi)_{,\delta\beta} - a^{-1}_{\alpha\beta\gamma\delta}\varphi_{,\gamma\delta\alpha\beta}]. \tag{8.9}$$

Equations (8.6) and (8.9) show how the zeroth-order mechanical (respectively electrical) solution determines the first-order electrical (respectively mechanical) solution, thus re-establishing a kind of symmetry between electrical and mechanical properties. Unfortunately, we *cannot* reach the first-order *mechanical* solution for we have *no* boundary conditions imposed on $\mathbf{u}^{(1)}$ along the lateral face Γ_0.

Furthermore, the solution found for the problem $P^{(0)}$ does not account for, and cannot be valid near to, the edge $\partial\omega$. There must exist boundary-layer effects in the vicinity of the contour and these can be solved only by using techniques such as that of *correctors* of Lions (**35**) or *matched asymptotic expansions* as in all boundary-layer problems. Coutris and Monavon (**39, 40**) have done this for purely mechanical, linear isotropic plates. The algebra is somewhat formidable. Clearly, an electroelastic generalization of these in the anisotropic case is outside the scope of the present paper and we shall content ourselves with the above-stated results.

9. Concluding remarks

Previous works in the field along similar lines belong to the Soviet school of applied mechanics (**41** to **47**). In particular, Vekovishcheva (**41** to **43**) has effected averages throughout the thickness of the plate, both of field and constitutive equations, along the lines of the engineering approach to plate theory (see (**3**)). Then *plane* problems are solved by introducing a stress function and an electrical-displacement function (*not* a potential for the electric field!). Kosmodamianskii and Lozhkin (**44, 45**) have used the asymptotic-integration method in its ‘intuitive’ form (Gol’denveizer’s, as opposed to the zoom technique and variational formulation). Their result depends drastically on the loads and their symmetry with respect to the median plane. For symmetric loads, they agree with Vekovishcheva’s generalized electroelastic plane stress state. For antisymmetric loads corresponding to bending, the flexure equation reduces to one for anisotropic elastic plates in pure elasticity (compare with our equation (7.10) at the zeroth-order approximation) but, unlike the present paper, the authors have not continued to higher orders. Generalizations to piezoceramic shells (**46**) and elements of vibration theory (**47**) have been given on this basis. However, none of these papers gives the rationale of the procedure and exhibits the typical electromechanical couplings as does the present approach. The latter should also yield interesting results in the dynamical framework where the piezoelectric plate can be used as a resonator. In addition, we may also consider the *nonlinear* theory of piezoelectricity (equations in (**23**) or (**9**, Chapter 4)) and study parametric excitations. The boundary-layer effects in the *linear* electroelastic plate should also be a concern.

Finally, we must emphasize that the present approach differs totally from the one used previously for thin magnetoelastic plates by, among others, Maugin and Goudjo (**15, 16**), where an averaging procedure is applied to mechanical equations while the magnetostatic problem in the bent plate remains fully three-dimensional (but with boundary effects discarded) and provides source terms (themselves containing the unknown deflection) to the bending problem. In this paper, the electrostatic problem at zeroth-order is purely planar (cf. (7.12) and $(7.14)_2$), the third component being obtained by integration in the thickness direction (equation (7.18)). The next-order approximation is also deduced from the mechanical bending problem (cf. (8.6) and $(8.2)_5$). It is the purpose of future work to re-examine the magnetoelastic case (for ferromagnets or conductors) in the light of the present approach. However, an essential difference between electric and magnetic cases is that a *bias* field is necessarily present in the second case and this, a priori, may destroy the symmetry and coercivity of the operators needed in the variational and functional formulation.

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