On the combination of asymptotic and direct approaches to the modeling of plates with piezoelectric actuators and sensors

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ABSTRACT

We suggest a method for the mathematical modeling of the response of a thin structure with integrated piezoelectric sensors and actuators under mechanical and electrical loads. The plate with attached piezoelectric patches is considered as a two-dimensional material surface with mechanical and electrical degrees of freedom of particles. The model is complemented with an asymptotic analysis of the three-dimensional problem. For the equations of a layered piezoelectric plate, we seek those terms in the solution, which dominate as the thickness tends to zero. The results serve as a basis for the analysis of geometrically nonlinear shell structures as well as for model-based monitoring and optimal sensor and actuator placement.

Keywords: Smart structures, thin plates and shells, piezoelectric patches, asymptotic analysis, direct approach

1. INTRODUCTION

Smart structure technology has become a key technology in the design of modern, so-called intelligent, civil, mechanical and aerospace systems. Similar to human beings, these intelligent or smart structures are capable to react to disturbances exerted upon them by the environment they are operating in. In the last few decades, rapid developments have been made in the modeling and control of smart structures. Reviews on the theory and application of smart structures have been presented by Crawley,[1] Tani et al[2] and Tzou.[3]

The control of smart structures requires two essential elements: sensors and actuators. While the actuators directly impose the action, which leads to a desired behavior of the structure, the sensors provide the controller with the required knowledge of the state the structure is in.[4–6] Both the problem of developing an optimal control law and of structural and health monitoring (when a qualitative change in the behavior of the structure needs to be detected) require the design of an optimal network of piezoelectric patch sensors and actuators for measuring particular mechanical entities and for their control.[7]

In the present paper we consider sensors and actuators, which are implemented with the help of piezoelectric materials. The application of idealized continuous strain-type sensors was recently considered by the authors for the geometrically nonlinear behavior of rod[8] and shell[9] structures. Concerning control of plate vibrations by piezoelectric sensors and actuators we refer to Krommer and Varadan[10,11] and Gattringer et al.[12] Putting these results into practice requires a reliable technique for the mathematical modeling of the coupled electromechanical behavior of the structure with the piezoelectric patches attached.

With respect to the modeling of plates composed of materials exhibiting the piezoelectric effect, Mindlin and Tiersten were the first to report on the dynamic behavior of single layer piezoelectric plates in their fundamental contributions,[13,14] in which higher order (wave propagation) theories have been utilized. For a vibration (low order) theory of composite structures with piezoelectric layers as structural members such a modeling is not suitable, as has been pointed out by Lee,[15] who introduced a classical lamination theory for piezoelectric plates, in which the lowest order approximation for the electric field is used. Higher order mechanical modeling up to the

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third order shear deformation theory of Reddy[16] can be found in Chandrashekhara and Agrawal.[17] Theories that incorporate both, higher order electric potential and mechanical displacement distributions were introduced by Tiersten[18] and Yang[19] in the restricted case of single layered piezoelectric plates. In case of composite structures, so-called hybrid or mixed formulations or discrete layerwise approximations for both, the mechanical and the electrical field are often utilized. For a review on modeling for laminated piezoelectric beams, plates, and shells see Saravanos and Heyliger.[20] Since the publication of the review paper by Saravanos and Heyliger[20] two main directions can be observed with respect to the ongoing research in the field of modeling of smart composite plates. One targets at developing more accurate plate models; e.g. Batra and Vidoli[21] and Xu and Wang.[22] The other direction is to incorporate coupling effects into equivalent single layer theories. The resulting theories are purely mechanical and characterized by effective stiffness and electric loadings. Examples of that kind can be found in Ling-Hui[23] and Krommer.[24]

In the present paper we discuss an alternative combined approach to the modeling of plates with layers or patches made of piezoelectric material. The direct approach to the plate as a material surface with additional field variables (voltage and total charge on the electrodes of the piezoelectric sensors and actuators) leads to a logically consistent system of equations. The asymptotic analysis of the three-dimensional formulation justifies the choice of degrees of freedom and the constraints of the direct approach. It also results in the expression of the enthalpy of the plate, which is necessary for practical solutions. These two logical steps provide a sound basis for the extension of the theory to the case of finite deformations of piezoelectric shell structures. A numerical example for the latter problem concludes the present paper.

2. THREE-DIMENSIONAL PROBLEM: VOIGT'S THEORY OF PIEZOELECTRICITY

In the present section we formulate the general equations and the variational principle, which govern the coupled electromechanical behavior of a plate.

2.1 Invariant form

The first group of the general system of equations of a piezoelectric continuum with the volume V is formed by the equation of balance for the stress tensor $\boldsymbol{\tau}_3$ (the index '3' will distinguish three-dimensional entities from their two-dimensional counterparts) and by the balance equation for the electric displacement vector $\boldsymbol{D}$:

$$\begin{aligned} &\nabla_3 \cdot \boldsymbol{\tau}_3 + \boldsymbol{f}_3 = 0, \\ &\nabla_3 \cdot \boldsymbol{D} = 0; \end{aligned} \tag{1}$$

∇_3 is Hamilton's operator and $\boldsymbol{f}_3$ is the vector of volumetric forces.

The field of displacements $\boldsymbol{u}_3$ and the field of strains $\boldsymbol{\varepsilon}_3$ are kinematically related, and the electric field vector $\boldsymbol{E}$ is related to the field of the electric potential φ_3:

$$\begin{aligned} \boldsymbol{\varepsilon}_3 &= \nabla_3 \boldsymbol{u}_3^S, \\ \boldsymbol{E} &= -\nabla_3 \varphi_3; \end{aligned} \tag{2}$$

$(\ldots)^S$ defines the symmetric part of a tensor.

In formulating the constitutive relations we choose the strain $\boldsymbol{\varepsilon}_3$ and the electric field $\boldsymbol{E}$ as independent variables; the stress $\boldsymbol{\tau}_3$ and the electric displacement $\boldsymbol{D}$ are dependent:

$$\begin{aligned} H_3 &= \frac{1}{2}\boldsymbol{\varepsilon}_3 \cdot\cdot {}^4\mathbf{C} \cdot\cdot \boldsymbol{\varepsilon}_3 - \boldsymbol{E} \cdot {}^3\mathbf{e} \cdot\cdot \boldsymbol{\varepsilon}_3 - \frac{1}{2}\boldsymbol{E} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{E}, \\ \boldsymbol{\tau} &= \frac{\partial H_3}{\partial \boldsymbol{\varepsilon}_3} = {}^4\mathbf{C} \cdot\cdot \boldsymbol{\varepsilon}_3 - \boldsymbol{E} \cdot {}^3\mathbf{e}, \\ \boldsymbol{D} &= -\frac{\partial H_3}{\partial \boldsymbol{E}} = {}^3\mathbf{e} \cdot\cdot \boldsymbol{\varepsilon}_3 + \boldsymbol{\epsilon} \cdot \boldsymbol{E}. \end{aligned} \tag{3}$$

Here H_3 is the enthalpy per unit volume, ${}^4\mathbf{C}$ is the fourth-rank tensor of material elastic stiffnesses at constant $\boldsymbol{E}$, ${}^3\mathbf{e}$ is the tensor of piezoelectric constants and the components of $\boldsymbol{\epsilon}$ are the dielectric constants at constant $\boldsymbol{\varepsilon}_3$.

The conditions at the boundary $\Omega = \partial V$ with the outer normal $\boldsymbol{n}$ read:

$$\begin{aligned} \boldsymbol{u}_3 = \boldsymbol{u}_0 \quad &\text{or} \quad \boldsymbol{n}\cdot\boldsymbol{\tau}_3 = \boldsymbol{t}, \\ \varphi_3 = \varphi_0 \quad &\text{or} \quad \boldsymbol{n}\cdot\boldsymbol{D} = -q. \end{aligned} \tag{4}$$

For the mechanical part, either displacements or tractions $\boldsymbol{t}$ should be specified on the boundary. For the electrical part, either the potential φ_0 or the charge density q must be given.

When a particular problem is being solved, one seeks for the field of displacements $\boldsymbol{u}_3$ and for the distribution of the electric potential φ_3. Sometimes it is more efficient not to consider displacements explicitly and to seek for the field of strains. The condition of compatibility means that for a given field of $\boldsymbol{\varepsilon}_3$ one can find such a field of displacements $\boldsymbol{u}_3$ that (2) holds; hence,

$$\nabla_3 \times (\nabla_3 \times \boldsymbol{\varepsilon}_3)^T = 0. \tag{5}$$

Another mathematically equivalent form of this condition can be advantageous in some cases:

$$\Delta_3\boldsymbol{\varepsilon}_3 + \nabla_3\nabla_3 \operatorname{tr}\boldsymbol{\varepsilon}_3 = 2\left(\nabla_3\nabla_3\cdot\boldsymbol{\varepsilon}_3\right)^S, \quad \Delta_3 \equiv \nabla_3\cdot\nabla_3. \tag{6}$$

2.2 Variational principles in three dimensions

In the book by Nowacki[25] one can find a formulation of Hamilton's principle for piezoelectric continua with the following counterpart to the potential energy function:

$$\Pi = \int\limits_V \left(H_3 - \boldsymbol{f}_3\cdot\boldsymbol{u}_3\right)\, dV + \int\limits_\Omega \left(q\varphi_3 - \boldsymbol{t}\cdot\boldsymbol{u}_3\right)\, d\Omega. \tag{7}$$

This is a functional over the fields of displacements and the electric potential: $\Pi = \Pi[\boldsymbol{u}_3, \varphi_3]$. Transforming the expression for the variation of the enthalpy,

$$\begin{aligned} \delta H_3 &= \boldsymbol{\tau}_3\cdot\cdot\,\delta\boldsymbol{\varepsilon}_3 - \boldsymbol{D}\cdot\delta\boldsymbol{E} = \boldsymbol{\tau}_3\cdot\cdot\,\nabla\delta\boldsymbol{u}_3 + \boldsymbol{D}\cdot\nabla\delta\varphi_3 = \\ &= \nabla\cdot(\boldsymbol{\tau}_3\cdot\delta\boldsymbol{u}_3) + \nabla\cdot(\boldsymbol{D}\,\delta\varphi_3) - \nabla\cdot\boldsymbol{\tau}_3\cdot\delta\boldsymbol{u}_3 - \nabla\cdot\boldsymbol{D}\,\delta\varphi_3, \end{aligned} \tag{8}$$

we write the condition of static equilibrium as a condition of stationarity of (7):

$$\begin{aligned} 0 = -\delta\Pi = &\int\limits_V \left((\nabla\cdot\boldsymbol{\tau}_3 + \boldsymbol{f}_3)\cdot\delta\boldsymbol{u}_3 + \nabla\cdot\boldsymbol{D}\,\delta\varphi_3\right)\, dV + \\ &+ \int\limits_\Omega \left((\boldsymbol{n}\cdot\boldsymbol{\tau}_3 - \boldsymbol{t})\cdot\delta\boldsymbol{u}_3 + (\boldsymbol{n}\cdot\boldsymbol{D} + q)\,\delta\varphi_3\right)\, d\Omega. \end{aligned} \tag{9}$$

As the variations $\delta\boldsymbol{u}_3$ and $\delta\varphi_3$ are independent, the balance equations (1) and the boundary conditions (4) follow from (9).

2.3 Constitutive relations in components

We consider a material of crystal class 2mm (see Eringen and Maugin[26]) with a plane of isotropy; assume the coordinate axes $1, 2$ are in the plane, and 3 is orthogonal to it. Then the relations (3) can be written in the following component form:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{13} \\ \tau_{23} \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{11}-C_{12} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{12} \end{pmatrix} - $$
$$ - \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \tag{10}$$
$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{12} \end{pmatrix} + $$
$$ + \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}.$$

For the case of an isotropic material with Young's modulus E and Poisson's ratio ν the elastic coefficients are

$$C_{11} = C_{33} = \frac{E(1-\nu)}{1-\nu-2\nu^2}, \quad C_{12} = C_{13} = \frac{E\nu}{1-\nu-2\nu^2}, \quad C_{44} = \frac{E}{2(1+\nu)}. \tag{11}$$

The corresponding quadratic form of the enthalpy becomes

$$\begin{aligned} 2H_3 = {} & C_{11}\left(\varepsilon_1^2 + 2\varepsilon_{12}^2 + \varepsilon_2^2\right) - 2C_{12}\left(\varepsilon_{12}^2 - \varepsilon_1\varepsilon_2\right) + 2C_{13}\varepsilon_3(\varepsilon_1+\varepsilon_2) + \\ & + C_{33}\varepsilon_3^2 + 4C_{44}\left(\varepsilon_{13}^2 + \varepsilon_{23}^2\right) - \epsilon_{11}\left(E_1^2 + E_2^2\right) - \\ & - 2e_{15}(E_1\varepsilon_{23} + E_2\varepsilon_{13}) - 2E_3 e_{31}(\varepsilon_1+\varepsilon_2) - E_3(E_3\epsilon_{33} + 2e_{33}\varepsilon_3). \end{aligned} \tag{12}$$

3. DIRECT APPROACH TO THE CLASSICAL THEORY OF PIEZOELECTRIC PLATES

The direct approach to a plate as a two-dimensional material surface is simple and logically consistent. One chooses a set of degrees of freedom of a continuum, writes the principle of virtual work in an appropriate form, and then lets the formal mathematical apparatus of analytical mechanics to "flow" into a system of equations.

We consider a plate as a two-dimensional material continuum with particles, which can undergo in-plane translations $\boldsymbol{u} = u_\alpha \boldsymbol{e}_\alpha$, out-of-plane deflections w, and rotations of the attached unit normal vector $\boldsymbol{\theta} = \theta_\alpha \boldsymbol{e}_\alpha$; $\boldsymbol{e}_\alpha$ is the Cartesian basis in the plane. Those rotations do not have a component along the out-of-plane direction $\boldsymbol{k} = \boldsymbol{e}_1 \times \boldsymbol{e}_2$, as we do not wish to account for the moment effects in the plane. These five mechanical degrees of freedom, which are typical for the classical theory of plates, are accomplished by a field variable φ, which should be interpreted as a potential difference in the electroded piezoelectric layer of the plate.

Due to the simplicity of plate structures in comparison to shells, we can directly apply the classical constraint of Kirchhoff: the normal remains orthogonal to the surface after deformation, and

$$\boldsymbol{\theta} = \nabla w \times \boldsymbol{k}, \tag{13}$$

here ∇ is the two-dimensional differential operator with respect to the in-plane position vector $\boldsymbol{x} = x_\alpha \boldsymbol{e}_\alpha$.

Adapting (7), we write the corresponding variational principle for a two-dimensional continuum:

$$\delta\Pi_2 = 0, \quad \Pi_2 = \int\limits_\Omega (H_2 - \boldsymbol{f}\cdot\boldsymbol{u} - pw + q\varphi)\, d\Omega. \tag{14}$$

For simplicity we do not account for the mechanical forces on the boundary of the plate $\delta\Omega$ and for the distributed moment force factors, which produce virtual work on the rotations $\boldsymbol{\theta}$. The term $q\varphi$, which appeared in the integration over the boundary in (7), has now "migrated" into the integral over the domain Ω.

At the part of the plate, which is free from piezoelectric patches, H_2 is just the strain energy in the cross-section, and φ is not defined. In another part of the domain some voltage v is prescribed in the cross-section. This can be considered as a kinematic constraint: $\varphi = v$. There might exist other domains Ω_i in the structure, where piezoelectric layers are present, and the electric circuit is left open. Because of the conductivity of the electrodes, the value of the potential should be considered equal for the whole subdomain: $\varphi = \varphi_i$, $\delta\varphi = \delta\varphi_i$. The functional can be rewritten as

$$\Pi_2 = \int\limits_\Omega (H_2 - \boldsymbol{f}\cdot\boldsymbol{u} - pw)\, d\Omega + \sum_i Q_i\varphi_i, \quad Q_i = \int\limits_{\Omega_i} q\, d\Omega. \tag{15}$$

The first variation of the functional, which depends on the fields of $\boldsymbol{u}$, w and φ_i, is zero in the state of static equilibrium:

$$\delta\Pi_2[\boldsymbol{u}, w, \varphi_i] = \int\limits_\Omega (\delta H_2 - \boldsymbol{f}\cdot\delta\boldsymbol{u} - p\,\delta w)\, d\Omega + \sum_i Q_i\,\delta\varphi_i = 0. \tag{16}$$

The standard approach with Lagrange multipliers can be applied. We introduce two plate strain measures

$$\boldsymbol{\varepsilon} = \nabla\boldsymbol{u}^S, \quad \boldsymbol{\kappa} = \nabla\nabla w. \tag{17}$$

What is quite difficult mathematically for shells[27,28] appears to be much simpler for plates: it can easily be proven, that the constraints

$$\delta\boldsymbol{\varepsilon} = 0, \quad \delta\boldsymbol{\kappa} = 0 \tag{18}$$

together with (13) are equivalent to the variation of the actual state of the kind of a rigid body motion, i.e.

$$\delta(\boldsymbol{u} + w\boldsymbol{k}) = \delta\boldsymbol{\vartheta}\times\boldsymbol{x} + \text{const}, \quad \delta\boldsymbol{\theta} = \delta\boldsymbol{\vartheta}\cdot(\mathbf{I} - \boldsymbol{k}\boldsymbol{k}), \quad \delta\boldsymbol{\vartheta} = \text{const}; \tag{19}$$

the term $\delta\boldsymbol{\vartheta}\times\boldsymbol{x}$ answers to a virtual rotation of the plate as a rigid body; the identity tensor is denoted as $\mathbf{I}$. We assume H_2 to remain constant under rigid body motion, when the normals remain orthogonal and when the potential differences do not change:

$$\delta H_2 = 0 \quad \Leftrightarrow \quad (13), \quad (18), \quad \delta\varphi_i = 0. \tag{20}$$

We rewrite (16), setting $\delta H_2 = 0$ and introducing Lagrange multipliers $\mathbf{M}$, $\mathbf{T}$ and $\tilde{Q}_i$ according to the constraints (18) and $\delta\varphi_i = 0$:

$$\int\limits_\Omega (\mathbf{M}\cdot\cdot\,\delta\boldsymbol{\kappa} + \mathbf{T}\cdot\cdot\,\delta\boldsymbol{\varepsilon} - \boldsymbol{f}\cdot\delta\boldsymbol{u} - p\,\delta w)\, d\Omega + \sum_i (Q_i - \tilde{Q}_i)\,\delta\varphi_i = 0. \tag{21}$$

The tensors $\mathbf{M}$ and $\mathbf{T}$ are symmetric, as well as the left-hand sides of the corresponding constraints (18). With certain mathematical manipulations we "extract" variations $\delta\boldsymbol{u}$, δw and $\delta\varphi_i$, which should be now considered as independent because of the Lagrange multipliers. Equating the coefficients at the independent variations to zero, we arrive at the following equations:

$$\nabla\cdot\nabla\cdot\mathbf{M} - p = 0, \quad \nabla\cdot\mathbf{T} + \boldsymbol{f} = 0, \quad \tilde{Q}_i = Q_i. \tag{22}$$

The first two equalities here can easily be identified as the equations of balance of moments $\mathbf{M}$ and in-plane forces $\mathbf{T}$. The third equality in (22) means that the total free charge in the domains with the open electric circuit should remain equal to a given value. The natural boundary conditions at $\partial\Omega$, which also follow from (21), appear to be equivalent to the classical boundary conditions of Kirchhoff's theory of plates.

Returning to the original form of the variational equation and considering an arbitrary virtual deformation with no constraints, one can see that

$$\int\limits_{\Omega} \delta H_2 d\Omega = \int\limits_{\Omega} \left(\mathbf{M}\cdot\cdot\,\delta\boldsymbol{\kappa} + \mathbf{T}\cdot\cdot\,\delta\boldsymbol{\varepsilon}\right) d\Omega + \sum_i \tilde{Q}_i \delta\varphi_i. \tag{23}$$

Because the variations $\delta\boldsymbol{\varepsilon}$, $\delta\boldsymbol{\kappa}$ and $\delta\varphi_i$ are independent, and according to the principle of locality, the enthalpy is a function of these agruments:

$$H_2 = H_2(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}, \varphi). \tag{24}$$

The general form of the constitutive relations for a plate is determined:

$$\mathbf{T} = \frac{\partial H_2}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{M} = \frac{\partial H_2}{\partial \boldsymbol{\kappa}}, \quad \tilde{Q}_i = \int\limits_{\Omega_i} \tilde{q}\, d\Omega, \quad \tilde{q} \equiv \frac{\partial H_2}{\partial \varphi}. \tag{25}$$

Determination of the particular expression for H_2 requires a three-dimensional analysis. Numerical solutions of the problem can be based on the direct minimization of the functional (15) with the help of the Ritz method for the approximation of the fields w and $\boldsymbol{u}$; the variables φ_i are already discrete.

3.1 Solution of the coupled plate problem: workflow

In a general case, the equation for the electric circuit between the electrodes on the opposite sides of a plate connects the total charge Q and the voltage v, and complements the structural equations. The two typical cases are the following:

- Actuation. The voltage v is prescribed, which results in the generalized forces, and, consequently, in the deformation of the structure. A proper distribution of those generalized forces can be used to assign to the plate a desired deformation.
- Sensing. The measured voltage v is interpreted by an observer in terms of structural entities: displacements, amplitudes, etc. Although unknown, the voltage is the same for the whole pair of opposing electrodes. The system of equations is completed by the condition for the total charge: $\tilde{Q} = Q = 0$, as the external electric circuit is open. As in the case of actuation a proper distribution of the sensing can be used to measure arbitrary kinematic entities of interest (see[29] for the three-dimensional elastic case; see[8] and[9] for the application to the geometrically nonlinear behavior of rod and shell structures).

4. ASYMPTOTIC SPLITTING IN THE COUPLED THREE-DIMENSIONAL PROBLEM FOR PIEZOELECTRIC PLATES

There exists a vast literature on the three-dimensional analysis of the problem, which was already considered above with the help of the direct approach for a two-dimensional formulation. Recently a new asymptotic technique has been suggested,[30] which appears to be both simple and trustworthy. In the present section we briefly discuss some of the results of the latter work.

4.1 Three-dimensional problem for a piezoelectric plate

A plate is a three-dimensional body with the position vector of a point

$$\boldsymbol{r} = \boldsymbol{x} + z\boldsymbol{k}, \quad -\frac{h}{2} \le z \le \frac{h}{2}, \quad \boldsymbol{x} \in \Omega. \tag{26}$$

Here, the out-of-plane unit vector is denoted as $\boldsymbol{k}$, the corresponding Cartesian coordinate is z, and $\boldsymbol{x}$ is the plane part of the position vector; h is the thickness and Ω is the domain in the plane of the plate.

Without the loss of generality, we consider a plate with free upper and lower surfaces:

$$\boldsymbol{k}\cdot\boldsymbol{\tau}_3\big|_{z=\pm\frac{h}{2}} = 0. \tag{27}$$

The plate has two electrodes; the voltage between them is the potential difference:

$$v = \varphi\big|_{z=-\frac{h}{2}}^{\frac{h}{2}}. \tag{28}$$

The consideration of an electroded piezoelectric layer, bonded to a conducting substrate plate, requires a similar procedure.

Due to the structure of our problem, it appears to be convenient to separate the in-plane and out-of-plane parts of vectors and tensors. The in-plane part will be denoted with an index $_\perp$:

$$\mathbf{I}_\perp = \mathbf{I} - \boldsymbol{kk}, \quad \boldsymbol{f}_\perp = \mathbf{I}_\perp\cdot\boldsymbol{f}, \quad \boldsymbol{\tau}_{3\perp} \equiv \boldsymbol{\tau}_\perp = \mathbf{I}_\perp\cdot\boldsymbol{\tau}_3\cdot\mathbf{I}_\perp, \quad \boldsymbol{r}_\perp = \boldsymbol{x}. \tag{29}$$

Then, we introduce

$$\begin{aligned}
\boldsymbol{\varepsilon}_3 &= \varepsilon_z\boldsymbol{kk} + \boldsymbol{\gamma k} + \boldsymbol{k\gamma} + \boldsymbol{\varepsilon}_\perp,\\
\boldsymbol{\tau}_3 &= \sigma_z\boldsymbol{kk} + \boldsymbol{sk} + \boldsymbol{ks} + \boldsymbol{\tau}_\perp,\\
\boldsymbol{u}_3 &= u_z\boldsymbol{k} + \boldsymbol{u}_\perp,
\end{aligned} \tag{30}$$

in which $\boldsymbol{\gamma}$ and $\boldsymbol{s}$ are out-of-plane shear strain and stress vectors. The constitutive relations (3) can then be written in an invariant form; thus, in the absense of piezoelectric effects we would have

$$\begin{aligned}
&\boldsymbol{\tau}_3 = {}^4\mathbf{C}\cdot\cdot\boldsymbol{\varepsilon}_3, \quad \boldsymbol{\tau}_\perp = C_1\,\mathrm{tr}\,\boldsymbol{\varepsilon}_\perp\mathbf{I}_\perp + C_2\boldsymbol{\varepsilon}_\perp + C_3\varepsilon_z\mathbf{I}_\perp,\\
&\boldsymbol{s} = C_4\boldsymbol{\gamma}, \quad \sigma_z = C_5\varepsilon_z + C_3\,\mathrm{tr}\,\boldsymbol{\varepsilon}_\perp;
\end{aligned} \tag{31}$$

the elastic moduli C_i are functions of z.

4.2 Asymptotic splitting for stresses

In order to indicate the thinness of the plate, we introduce a formal small parameter λ in the expression of the position vector of a point of the plate: instead of (26) we write

$$\boldsymbol{r} = \lambda^{-1}\boldsymbol{x} + z\boldsymbol{k}. \tag{32}$$

The corresponding form of Hamilton's operator will be

$$\nabla_3 = \lambda\nabla + \boldsymbol{k}\partial_z. \tag{33}$$

Here ∇ is the differential operator with respect to the in-plane position vector $\boldsymbol{x}$.

Another possible argumentation to (33) is that we seek for solutions, which vary in the plane much slower than over the thickness (i.e. z is a "fast" variable), and therefore the derivatives with respect to the in-plane coordinates $\boldsymbol{x}$ acquire a corresponding order of smallness.

The procedure of asymptotic splitting has successfully been applied to the derivation of the theory of spatial rods with a structure,[31] as well as to the theory of thin-walled rods of open profile.[32] According to this approach, we seek for the unknown field of stresses in the form of a power series in the small parameter:

$$\boldsymbol{\tau}_3 = \lambda^{-2}\overset{0}{\boldsymbol{\tau}} + \lambda^{-1}\overset{1}{\boldsymbol{\tau}} + \ldots \tag{34}$$

Our goal is to find those terms in the solution, which dominate as the plate is getting thinner and $\lambda \to 0$. Only the term $\overset{0}{\boldsymbol{\tau}}$ is of real interest to us, and the role of the rest of the series is to let the principal term to be determined from the conditions of solvability for the minor terms. The leading power λ^{-2} in (34) should provide successful application of the procedure.

Considering the equation of balance of stresses (1) with the boundary conditions (27), after two steps of the procedure we arrive at the following results for the principal terms:[30]

$$\begin{aligned}
&\overset{0}{\sigma}_z = 0, \quad \overset{0}{\boldsymbol{s}} = 0;\\
&\nabla\cdot\mathbf{T} = 0, \quad \mathbf{T} = h\left\langle \overset{0}{\boldsymbol{\tau}}_\perp \right\rangle;\\
&\nabla\cdot\nabla\cdot\mathbf{M} - p = 0, \quad \mathbf{M} = -h\left\langle z\overset{0}{\boldsymbol{\tau}}_\perp \right\rangle,
\end{aligned} \tag{35}$$

here the standard notation for the mean value over the thickness $\langle \ldots \rangle \equiv \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \ldots\, dz$ is introduced. The force factors in terms of the plate theory are now related to the in-plane part of the principal term in the series expansion of the stress tensor $\overset{0}{\boldsymbol{\tau}}_\perp$, and the out-of-plane part of this principal term is zero. Derivation of the static conditions at the boundary $\delta\Omega$ in the framework of the asymptotic approach requires additional analysis of an edge layer near the boundary.[30]

4.3 Constitutive relations

The field of strains must have the same asymptotic behavior with respect to the formal small parameter:

$$\boldsymbol{\varepsilon}_3 = \lambda^{-2}\overset{0}{\boldsymbol{\varepsilon}} + \lambda^{-1}\overset{1}{\boldsymbol{\varepsilon}} + \ldots; \tag{36}$$

from the condition of compatibility (6) it then follows that the in-plane part of the principal term of the strain tensor is linearly distributed over the thickness:

$$\overset{0}{\boldsymbol{\varepsilon}}_\perp = -\boldsymbol{\kappa} z + \boldsymbol{\varepsilon}. \tag{37}$$

The coefficients $\boldsymbol{\kappa}(\boldsymbol{x})$ and $\boldsymbol{\varepsilon}(\boldsymbol{x})$ are functions of the in-plane coordinates.

The electric variables $\boldsymbol{D}$, φ_3 also begin their asymptotic expansions with λ^{-2}. From (1) and (33) it follows that

$$\partial_z \overset{0}{D}_z = 0 \quad \Rightarrow \quad \overset{0}{D}_z = \text{const} = D_{z0} = -q. \tag{38}$$

Analyzing further the consequences of the constitutive relations (3), the results of the asymptotic procedure for the stresses (35) and the law (37), we arrive at the relations between $\mathbf{T}$, $\mathbf{M}$, q and $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$, φ in the form (25) with the function of enthalpy

$$H_2 = \frac{1}{2}\boldsymbol{\varepsilon}\cdot\cdot{}^4\mathbf{A}\cdot\cdot\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\cdot\cdot{}^4\mathbf{B}\cdot\cdot\boldsymbol{\kappa} + \frac{1}{2}\boldsymbol{\kappa}\cdot\cdot{}^4\mathbf{D}\cdot\cdot\boldsymbol{\kappa} + v\mathbf{p}\cdot\cdot\boldsymbol{\varepsilon} + v\mathbf{m}\cdot\cdot\boldsymbol{\kappa} + \frac{1}{2}cv^2. \tag{39}$$

In the case of isotropy in the plane of the plate with the material relations (10) the function of enthalpy is

$$H_2 = \frac{1}{2}\left(A_1\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)^2 + A_2\boldsymbol{\varepsilon}\cdot\cdot\boldsymbol{\varepsilon}\right) + B_1\operatorname{tr}\boldsymbol{\varepsilon}\operatorname{tr}\boldsymbol{\kappa} + B_2\boldsymbol{\varepsilon}\cdot\cdot\boldsymbol{\kappa} + \frac{1}{2}\left(D_1\left(\operatorname{tr}\boldsymbol{\kappa}\right)^2 + D_2\boldsymbol{\kappa}\cdot\cdot\boldsymbol{\kappa}\right) + vp\operatorname{tr}\boldsymbol{\varepsilon} + vm\operatorname{tr}\boldsymbol{\kappa} + \frac{1}{2}cv^2. \tag{40}$$

4.4 Displacements

The asymptotic analysis is concluded by the study of the field of displacements, which also provides a straight connection to the degrees of freedom of the direct approach. Indeed, the asymptotic procedure for the field of displacements with the kinematic relation (2) results in the relations (18) for the principal terms $\overset{0}{u}_z = w$, $\overset{0}{\boldsymbol{u}}_\perp = \boldsymbol{u}$.

5. NUMERICAL MODELING OF GEOMETRICALLY NONLINEAR PIEZOELECTRIC SHELLS

For the case of purely elastic behavior the direct approach, presented in section 3, can successfully be applied to finite deformations of curved shells.[27,28,33] Resulting in the field equations of balance, boundary conditions, general form of elastic relations and kinematic relations, this is probably the only logically consistent way to

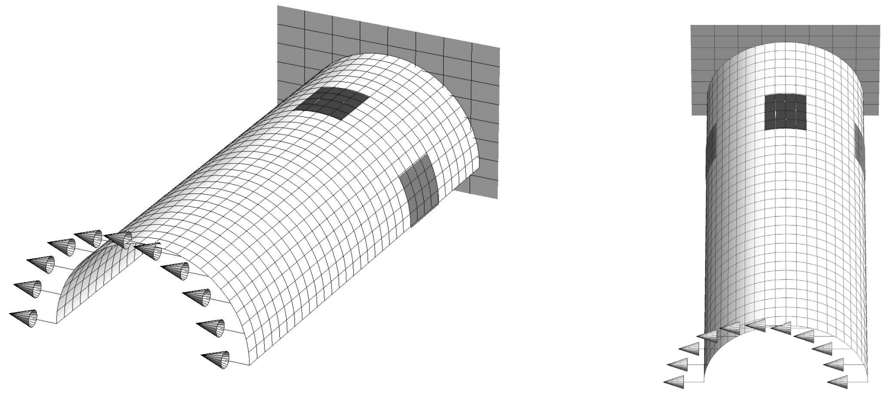

Figure 1. Clamped and transversely loaded cylindrical panel, equipped with three piezoelectric patches

the equations of geometrically nonlinear theory of shells. The inclusion of the additional electric field variable φ is simple in the framework of the direct approach, and the finite element scheme, which was initially presented by Eliseev and Vetyukov,[28] has been extended to account for the electromechanical coupling for the case of prescribed voltage on the piezoelectric patch.

As an example, we consider bending of a cylindrical panel, which is clamped at one edge and loaded with a transversal distributed force at another edge. The plate is equipped with three piezoelectric patches, as it is shown in Fig. 1 from two different viewpoints. The radius of curvature of the shell is $80\,mm$, the length is $400\,mm$, the thickness is $2\,mm$ and the material properties answer to aluminium. The patches have the thickness $0.5\,mm$ and the dimensions $50\,mm \times 41.9\,mm$; we used the parameters of the piezoelectric material PZT-5A.

Subsequently increasing the applied force, we sought for static equilibrium configurations, preserving the zero voltage on the electrodes (all the electrodes are short-circuited). Initially, the panel behaves as a thin-walled rod of open profile, in which bending is strongly coupled with torsion. In the course of deformation, local buckling happens at one of the edges near the patch, and the overall stiffness of the structure is significantly reduced. The corresponding deformed configurations, which answer to the loading value $f = 1.5 \cdot 10^4\,N/m$, are presented from two viewpoints in Fig. 2. The simulation featured 767 finite elements; the 4-node finite elements with 9 degrees of freedom per node provide smooth approximations of the deformed surface, which is necessary for modeling classical shells.

Figure 2. Finite deformation of a cylindrical panel with piezoelectric sensors

In the absence of the patches the local buckling happens directly at the point on the edge, where one of the patches is placed; their presence shifts the point of the buckling, as it can be clearly seen from Fig. 2. To study the general effect of the patches on the structural behavior, in Fig. 3 we plotted the horizontal displacement u

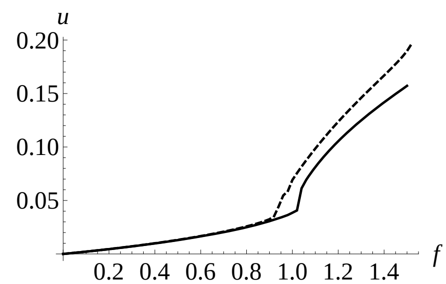

Figure 3. Horizontal displacement as a function of the force in the model with the patches (solid line) and without the patches (dashed line)

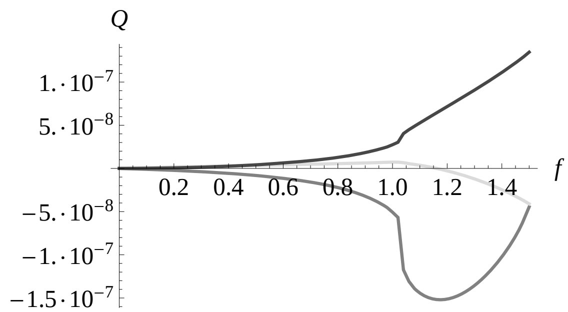

Figure 4. Variation of the signals of the three piezoelectric patches in the course of deformation

of the lower left corner point of the shell as a function of the distributed force f; both for the purely elastic structure (dashed line) and for the actual structure with the patches (solid line). The influence of the patches is nearly negligible in the geometrically linear regime, but their effect on possible buckling can become crucial.

Finally, we studied the total charge Q, accumulated on the electrodes of all the three patches. In practice, this quantity can be measured by integration of the electric current. Our future purpose is to interpret this charge (or voltage in case of an open circuit) as a signal, which is relevant to some structural entities. The dependencies $Q(f)$ for all three patches is presented in Fig. 4. One can conclude, that the evolution of the signal of a sensor strongly depends on its placing. Having these data, one should be able to solve the problem of detection of a qualitative change in the structural behavior e.g. as a result of local buckling, i.e. to design a system for structural health monitoring.

6. CONCLUSIONS

Motivated by our recent work on optimal strain-type sensing of kinematical entities in slender and thin structures in a geometrically nonlinear regime, we have started the present work on the mathematical modeling of thin structures with attached piezoelectric sensors and actuators. This is important to us, because optimal strain-type sensing needs to put into practice by means of, e.g. piezoelectric sensors, which constitute themselves as attached patches. For the translation of results of continuous strain-type sensing to practical problems accurate mathematical models must be available. Our approach is characterized by a combination of a direct approach to thin plates as material surfaces with an asymptotic analysis. The following aspects are of particular importance.

- The direct approach is especially efficient, if being complimented by the asymptotic analysis.
- The additional stiffness, brought into the structure by sensors and actuators, can significantly influence the behavior of the structure.

For the future we plan on using our mathematical model for thin plates and shells with piezoelectric patches for a kinematical interpretation of the signals of the members of a network of sensors to compute optimal weights;

the latter optimization should be based on the already available solutions for continuous strain-type sensors. To this end we will have to extend the simulation tool for the case, when voltage is not prescribed on the patches, but rather acts as an additional unknown in the course of computation. Moreover, we intend to extend our model to dynamic problems and to account for passve electric networks connected to the piezoelectric patches as a first step towards active vibration control in the geometrically nonlinear regime.

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Support of Yu. Vetyukov and M. Krommer from the Austrian Science Fund (FWF Translational Project L441-N41 “Sensor Systems for Structural and Health Monitoring”) and of M. Krommer from the K2 Austrian Center of Competence in Mechatronics (ACCM) is gratefully acknowledged.

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