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A high order model for piezoelectric rods: An asymptotic approach

J.M. Viaño[a], C. Ribeiro[b], J. Figueiredo[b], Á. Rodríguez-Arós[c,*]

[a] Departamento de Matemática Aplicada, Universidade de Santiago de Compostela, Spain
[b] Departamento de Matemática e Aplicações and Centro de Matemática, Universidade do Minho, Portugal
[c] Departamento de Métodos Matemáticos e Representación, Universidade da Coruña, Spain

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ABSTRACT

This paper is devoted to the study of an electromechanical model for a linear transversely inhomogeneous anisotropic rod made of piezoelectric symmetry class 2 materials. The materials in this class include, among many others, Barium Titanate (BT), Lead Zirconium Titanate (PZT) and Polyvinylidene Fluoride (PVDF).

For that purpose, we use asymptotic analysis in order to derive from the 3D piezoelectricity problem a reduced model, without making any *a priori* assumptions of geometrical or mechanical nature. This process involves considering the diameter of the cross section h as *small parameter*, the assumption of existence of asymptotic expansions of the unknowns (displacements field and electric potential) and the characterization of their respective leading terms as the unique solutions of limit reduced coupled equations. Finally, the validity of the limit reduced model is supported by providing rigorous convergence results.

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1. Introduction

The so-called "intelligent structures", i.e. those that are able to adapt in an automatic way when submitted to external actions in order to provide an optimal behavior, play an increasing role in the framework of nowadays security and wellbeing tools, and their importance will certainly increase in the near future. The use of piezoelectric materials is a strategy that has been used with considerable success in the development of new materials and technologies for the production of these structures.

The piezoelectric effect is an electromechanical property that is exhibited by many materials: when the material is deformed by the action of external forces or pressure, an electric field appears – direct piezoelectric effect – and conversely, when subject to an electric potential, the material suffers a deformation – inverse piezoelectric effect. This reciprocity provides both an efficient sensor capacity and the possibility of acting over large portions of structures without a considerable weight increase. For more complete information on the properties and use of piezoelectric materials see, for example, Banks et al. (1996); Ikeda (1990); Nye (1985); Royer and Dieulesaint (2000); Taylor et al. (1985). More specifically, information about traditional piezoelectric beam theories can be consulted in Bellis and Imperiale (2014); Bisegna (1998); Rovenski and Abramovich (2009).

From the above considerations one easily concludes on the relevance of designing new multifunctional materials (e.g. piezoelectric materials) and, consequently, on the importance of accurately knowing the electromechanical behavior of structures in which these materials are applied. Some important results in this field were successfully obtained by applying two different modeling techniques: homogenization and asymptotic analysis (see Cagnol et al., 2008; Miara et al., 2007 for a review of the state of the art). See in particular, Castillero et al. (1998); Ghergu et al. (2007); Mechkour (2004); Miara et al. (2005) for homogenization of piezoelectric plates and shells and Collard and Miara (2002); Figueiredo and Leal (2006); Rahmoune et al. (1998); Sabu (2002); Sène (2001); Weller and Licht (2002); Weller and Licht (2004) for formal asymptotic analysis to justify lower dimensional constitutive laws for piezoelectric plates, shells and rods. The above cited works were based on relevant and pioneer work in applying homogenization and asymptotic analysis techniques in elasticity problems (linear and nonlinear). See in particular Ciarlet (1980); Collard and Miara (1997); Le Dret and Raoult (1995); Lewiński and Telega (2000); Miara (1994); Sanchez-Hubert and Sanchez-Palencia (1997) for plates, Bermúdez and Viaño (1984a); Cimetière et al. (1988); Mascarenhas and Trabucho (1990); Trabucho and Viaño (1996); Tutek (1987); Tutek and Aganović (1986) for rods, and Caillerie and Sanchez-Palencia (1995); Ciarlet (2000); Ciarlet et al. (1996); Pitkaranta and Sanchez-Palencia (1997) for shells.

As it is well known, the classical theories of thin structures obtained from classical formulations are intuitive for understanding, but the *a priori* assumptions are often mechanically unjustified and can reveal inadequate for the analysis of real structures if those

* Corresponding author. Tel.: +34 981167000.
E-mail addresses: juan.viano@usc.es (J.M. Viaño), cribeiro@math.uminho.pt (C. Ribeiro), jmfiguei@math.uminho.pt (J. Figueiredo), angel.aros@udc.es (Á. Rodríguez-Arós).

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a priori assumptions are not met. Although the amount of research in the asymptotic analysis field is quite large, only a small number of works exist (as far as we know) devoted to the derivation of models for piezoelectric rods (see Figueiredo and Leal, 2006; Weller and Licht, 2004). Despite the progress made in those papers in order to derive reduced models for piezoelectric beams, the subject is still open to further research.

In this framework, our contribution consists in the obtention of a reduced model of a linear transversely inhomogeneous anisotropic rod made of piezoelectric symmetry class 2 materials, by using the asymptotic method introduced by Lions (see Lions, 1973), taking the diameter of the rod cross section h as small parameter, without any *a priori* hypothesis of geometrical or mechanical nature. The model obtained in this way allows to, simultaneously, justify and/or complement the traditional piezoelectric beam theories (such as those of Euler–Bernoulli, Timoshenko, and Vlasov). Additionally, in the reduced model, to our knowledge for the first time, the electric potential is coupled with first order terms of the asymptotic expansion of the displacements.

The present work is organized in the following manner. In Section 2 we show the 3D piezoelectric problem posed in its domain of reference Ω^h (the details on the notation used will be specified in the corresponding sections and can be found in tables in the appendix as well) and formulate it in the form of the virtual work principle (see (2.13)):

$$\text{Find } (\boldsymbol{u}^h,\varphi^h)\in\boldsymbol{V}^h(\Omega^h)\times H^1(\Omega^h),\quad \text{such that } \varphi^h=\hat{\varphi}^h \text{ on } \Gamma^h_{eD},$$
$$\int_{\Omega^h}\sigma^h_{ij}(\boldsymbol{u}^h,\varphi^h)e^h_{ij}(\boldsymbol{v}^h)d\boldsymbol{x}^h+\int_{\Omega^h}D^h_k(\boldsymbol{u}^h,\varphi^h)E^h_k(\psi^h)d\boldsymbol{x}^h$$
$$=\int_{\Omega^h}f^h_i v^h_i d\boldsymbol{x}^h+\int_{\Gamma^h_N}g^h_i v^h_i d\Gamma^h,$$
$$\text{for all } (\boldsymbol{v}^h,\psi^h)\in\boldsymbol{V}^h(\Omega^h)\times\Psi^h(\Omega^h),$$

where $\boldsymbol{u}^h=(u^h_i)$ and φ^h are the displacements field and the electric potential, respectively, to be found in the corresponding admissible spaces $\boldsymbol{V}^h(\Omega^h)$ and $H^1(\Omega^h)$. Also, $\boldsymbol{\sigma}^h=(\sigma^h_{ij})$ and $\boldsymbol{D}^h=(D^h_i)$ represent the stress tensor field and the electric displacements field, respectively, while $\boldsymbol{e}^h(\boldsymbol{v}^h)=(e^h_{ij}(\boldsymbol{v}^h))=(\frac{1}{2}(\frac{\partial v^h_i}{\partial x^h_j}+\frac{\partial v^h_j}{\partial x^h_i}))$ and $\boldsymbol{E}^h=(E^h_i(\psi^h))=(-\frac{\partial\psi^h}{\partial x^h_i})$ are the deformation operator and minus the gradient operator. The constitutive equations are (see (2.3)):

$$\begin{cases}\sigma^h_{ij}(\boldsymbol{u}^h,\varphi^h)=C^h_{ijkl}e^h_{kl}(\boldsymbol{u}^h)-P^h_{kij}E^h_k(\varphi^h), & \text{in } \Omega^h,\\ D^h_i(\boldsymbol{u}^h,\varphi^h)=P^h_{ikl}e^h_{kl}(\boldsymbol{u}^h)+\varepsilon^h_{ij}E^h_j(\varphi^h), & \text{in } \Omega^h,\end{cases}$$

where $\boldsymbol{C}^h=(C^h_{ijkl})$, $\boldsymbol{P}^h=(P^h_{kij})$ and $\boldsymbol{\varepsilon}^h=(\varepsilon^h_{ij})$ are the stiffness, piezoelectric and dielectric tensors, respectively.

In Section 3, we perform a change of variable and a scaling of the unknowns (to obtain the displacements field $\boldsymbol{u}(h)$ and the electric potential $\varphi(h)$) and data, so that the virtual work formulation is now posed in a domain independent on h, denoted by Ω.

In Section 4, we assume that the scaled unknowns can be written in the form of asymptotic developments, allowing to identify and characterize the corresponding leading terms; this procedure yields the new reduced model, which consists on a set of 1D variational problems for the displacements field and a variational problem for the electric potential and the first order term of the displacement, where some terms are not local.

The strong convergence of the unknowns as the small parameter tends to zero is proved in Section 5. This is a key step in order to give a rigorous mathematical justification of the reduced models of Section 4.

In Section 6 we undo the change of variable and de-scale the quantities appearing in the reduced model and constitutive equations derived in Section 4 in order to express both in the original reference domain Ω^h and thus have a true physical meaning. We find that the asymptotic expansion of the true displacements field $\boldsymbol{u}^h=(u^h_i)$ is of the form:

$$u^h_\alpha=\xi^h_\alpha+u^{1h}_\alpha+O(h),\quad u^h_3(\boldsymbol{x}^h)=\xi^h_3-x^h_\beta\partial^h_3\xi^h_\beta+u^{1h}_3+O(h^2),$$

where the leading term $\boldsymbol{u}^{0h}=(u^{0h}_i)$ is characterized by functions $\xi^h_\alpha\in H^2_0(0,L)$, $\xi^h_3\in H^1_0(0,L)$, satisfying the purely mechanical stretching and beam equations (see (6.5) and (6.6)):

$$\xi^h_\alpha\in H^2_0(0,L),\quad \int_0^L YI^h_\alpha\partial^h_{33}\xi^h_\alpha\partial^h_{33}\chi^h_\alpha dx^h_3$$
$$=\int_0^L F^h_\alpha\chi^h_\alpha dx^h_3-\int_0^L M^h_\alpha\partial^h_3\chi^h_\alpha dx^h_3,\quad (\text{no sum on } \alpha),$$
$$\text{for all}\quad \chi^h_\alpha\in H^2_0(0,L),$$
$$\xi^h_3\in H^1_0(0,L),\quad \int_0^L YA(\omega^h)\partial^h_3\xi^h_3\partial^h_3\chi^h_3 dx^h_3$$
$$=\int_0^L F^h_3\chi^h_3 dx^h_3,\ \text{for all } \chi^h_3\in H^1_0(0,L).$$

On the other hand, $\boldsymbol{u}^{1h}=(u^{1h}_i)$, the following order term of $\boldsymbol{u}^h$, is of the following form:

$$u^{1h}_\alpha=z^h_\alpha+\delta^h_\alpha z^h,\quad u^{1h}_3=-r^h-z^h_3-x^h_\alpha(z^h_\alpha)'-w^h(z^h)',$$

where $w^h\in R(\Omega)$ is the warping function, depending only on the geometry and stiffness tensor, while $z^h_\alpha, z^h\in H^1_0(0,L)$, $z^h_3\in L^2(0,L)$, and $r^h\in R(\Omega)$ is a function depending on the electric potential, thus giving the coupling between the mechanical and electrical quantities. More specifically, we find that the asymptotic expansion of the true electric potential φ^h is

$$\varphi^h=\hat{\varphi}^{0h}+\bar{\varphi}^{0h}+O(h^2),$$

where $\bar{\varphi}^{0h}\in\Psi_{l0}(\Omega)$ and the coupled equations for z^h, r^h and $\bar{\varphi}^{0h}$ are given by (see (6.8)–(6.10)):

$$\int_0^L J^h\partial^h_3 z^h\partial^h_3\zeta\, dx_3-\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\left(\partial^h_\alpha w^h-\delta^h_\alpha\right)\partial^h_\beta\bar{\varphi}^{0h}\,d\hat{\boldsymbol{x}}^h\right)\partial^h_3\zeta^h\,dx_3$$
$$=\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\partial^h_\beta\hat{\varphi}^h\left(\partial^h_\alpha w^h-\delta^h_\alpha\right)d\hat{\boldsymbol{x}}^h\right)\partial^h_3\zeta^h\,dx_3\quad\forall\zeta^h\in H^1_0(0,L),$$
$$\int_0^L\left(\int_{\omega^h}C^h_{3\alpha3\beta}\partial^h_\beta r^h\partial^h_\alpha\rho^h_\sharp\,d\hat{\boldsymbol{x}}^h\right)\rho^h_3\,dx_3$$
$$-\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\partial_\beta\bar{\varphi}^{0h}\partial_\alpha\rho^h_\sharp\,d\hat{\boldsymbol{x}}^h\right)\rho^h_3\,dx_3$$
$$=\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\partial_\beta\hat{\varphi}^h\partial_\alpha\rho^h_\sharp\,d\hat{\boldsymbol{x}}^h\right)\rho^h_3\,dx_3\ \forall\rho^h_\sharp\in Q^h(\omega^h),$$
$$\times\rho^h_3\in L^2(0,L),$$
$$\int_0^L\left(\int_{\omega^h}\varepsilon^h_{\alpha\beta}\partial^h_\beta\bar{\varphi}^{0h}\partial^h_\alpha\psi^h_\sharp\,d\hat{\boldsymbol{x}}^h\right)\psi^h_3\,dx_3$$
$$+\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\partial^h_\alpha r^h\partial^h_\beta\psi^h_\sharp\,d\hat{\boldsymbol{x}}^h\right)\psi^h_3\,dx_3$$
$$+\int_0^L\left(\int_{\omega^h}P^h_{\beta3\alpha}\left(\partial^h_\alpha w^h-\delta^h_\alpha\right)\partial^h_\beta\psi^h_\sharp\,d\hat{\boldsymbol{x}}^h\right)\partial^h_3 z^h\psi^h_3\,dx_3$$
$$=-\int_0^L\left(\int_{\omega^h}\varepsilon^h_{\alpha\beta}\partial^h_\beta\hat{\varphi}^h\partial^h_\alpha\psi^h_\sharp\,d\hat{\boldsymbol{x}}^h\right)\psi^h_3\,dx_3\ \forall\psi^h_\sharp\in S(\omega^h),$$
$$\times\psi^h_3\in L^2(0,L).$$

Notice that even though we know the existence of z^h_α and z^h_3, these functions are not characterized at this stage. Nevertheless, they are not needed in order to obtain φ^{0h}, which depends only on $\boldsymbol{u}^{1h}$ through r^h.

We then briefly comment on some particular cases, specifically when the cross-section is square, which leads to some practical simplifications, and we finish by giving some conclusions.

Table 1
List of relevant sets.

Notation	Description
ω	Domain in $\mathbb{R}^2$ with boundary γ,
Ω	Reference configuration of the scaled beam/rod defined as $\Omega = \omega \times (0, L)$,
$\boldsymbol{x} = (x_i)$	A point in $\bar{\Omega}$,
ω^h	Cross section of the beam/rod defined as $h\omega$, with boundary γ^h,
Ω^h	Reference configuration of the beam/rod defined as $\Omega^h = \omega^h \times (0, L)$,
Γ^h	Boundary of Ω^h, that is $\Gamma^h = \partial\Omega^h = [0, L] \times \gamma^h$,
$\boldsymbol{x}^h = (x_i^h)$	A point in $\bar{\Omega}^h$,
Γ_0^h	Left end of the rod, that is $\Gamma_0^h = \bar{\omega}^h \times \{0\}$
Γ_L^h	Right end of the rod, that is $\Gamma_L^h = \bar{\omega}^h \times \{L\}$
Γ_{eD}^h	Part of Γ^h where the electric potential is given,
Γ_{eN}^h	Part of Γ^h which is complementary of Γ_{eD}^h,
Γ_D^h	Part of Γ^h with a condition of weak clampling. $\Gamma_D^h = \Gamma_0^h \cup \Gamma_L^h$,
Γ_N^h	Part of Γ^h where tractions are given.

Table 2
List of functions.

Notation	Description
$\boldsymbol{n}^h = (n_i^h) : \Gamma^h \to \mathbb{R}^3$	Unit outer normal vector to Γ^h,
$\boldsymbol{f}^h = (f_i^h) : \bar{\Omega}^h \to \mathbb{R}^3$	Density of volume forces,
$\boldsymbol{g}^h = (g_i^h) : \bar{\Gamma}_N^h \to \mathbb{R}^3$	Density of tractions,
$\boldsymbol{u}^h = (u_i^h) : \bar{\Omega}^h \to \mathbb{R}^3$	Displacements field,
$\varphi^h : \bar{\Omega}^h \to \mathbb{R}$	Electric potential,
$\hat{\varphi}^h : \Gamma_{eD}^h \to \mathbb{R}$	Electric potential given in Γ_{eD}^h and its trace lifting to $H^1(\Omega^h)$,
$\bar{\varphi}^h : \bar{\Omega}^h \to \mathbb{R}$	Homogeneous part of φ^h, therefore given as $\varphi^h - \hat{\varphi}^h$,
$\boldsymbol{\sigma}^h = (\sigma_{ij}^h) : \Omega^h \to \mathbb{S}^3$	Stress tensor,
$\boldsymbol{D}^h = (D_i^h) : \Omega^h \to \mathbb{R}^3$	Electric displacement,
$\boldsymbol{e}^h(\boldsymbol{u}^h) = (e_{ij}^h) : \Omega^h \to \mathbb{S}^3$	Linearized strain tensor,
$\boldsymbol{E}(\varphi^h) = (E_i^h) : \Omega^h \to \mathbb{R}^3$	Electric field,
$\boldsymbol{C}^h = (C_{ijkl}^h)$	Stiffness tensor,
$\boldsymbol{P}^h = (P_{kij}^h)$	Piezoelectric tensor,
$\boldsymbol{\varepsilon}^h = (\varepsilon_{ij}^h)$	Dielectric tensor,
$\bar{\boldsymbol{C}}^h, \boldsymbol{M}^h, \boldsymbol{N}^h$	Auxiliary matrices defined from components of $\boldsymbol{C}^h$ in (2.7)–(2.8),
$\delta_\alpha : \omega \to \mathbb{R}$	Auxiliary functions $\delta_1(x_1, x_2) = x_2$, $\delta_2(x_1, x_2) = -x_1$.

For the sake of readability we have also included Tables 1–5. where we list the main functions, constants, sets and functional spaces involved in this paper.

2. The three-dimensional problem

Throughout this paper, Latin indices vary over the set {1, 2, 3} and Greek indices over the set {1, 2} for the components of both vectors and tensors. The summation convention on repeated indices will be also used. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, while " $\cdot$ " will represent the inner product on $\mathbb{S}^d$ and $\mathbb{R}^d$ $(d = 2, 3, \ldots)$.

Let ω be an open, bounded, connected subset of $\mathbb{R}^2$ with Lipschitz continuous boundary $\gamma = \partial\omega$, having area $A(\omega)$. We will study a prismatic rod that we identify with a solid occupying the reference configuration

$$\bar{\Omega}^h = \bar{\omega}^h \times [0, L],$$

where $h \in (0, 1]$ is very small with respect to the length of the rod L and $\omega^h = h\omega$ so that $A(\omega^h) = h^2 A(\omega)$ is the area of the rod cross section ω^h, with boundary $\gamma^h = \partial\omega^h$.

Denoting the boundary of Ω^h by $\Gamma^h = \partial\Omega^h$, we consider two partitions for $\partial\Omega^h$: firstly, let us define a partition given by the measurable parts Γ_{eN}^h and $\Gamma_{eD}^h = \gamma_{eD}^h \times (0, L)$, where $\gamma_{eD}^h \subset \gamma^h$ and $meas(\gamma_{eD}^h) > 0$; secondly, we consider a partition given by the measurable parts Γ_N^h and $\Gamma_D^h = \Gamma_0^h \cup \Gamma_L^h$ such that $meas(\Gamma_D^h) > 0$, where $\Gamma_0^h = \bar{\omega}^h \times \{0\}$ and $\Gamma_L^h = \bar{\omega}^h \times \{L\}$ are the ends of the rod. An arbitrary

Table 3
List of scaled functions.

Notation	Description
$\boldsymbol{u}(h) = (u_i(h)) : \bar{\Omega} \to \mathbb{R}^3$	Scaled displacements, with $u_\alpha(h)(\boldsymbol{x}) = h u_\alpha^h(\boldsymbol{x}^h)$ and $u_3(h)(\boldsymbol{x}) = u_3^h(\boldsymbol{x}^h)$; $\boldsymbol{u}(h) \in \boldsymbol{V}(\Omega)$,
$\bar{\varphi}(h) : \bar{\Omega} \to \mathbb{R}$	Scaled homogeneous electric potential, with $\bar{\varphi}(h)(\boldsymbol{x}) = h^{-1}\bar{\varphi}^h(\boldsymbol{x}^h)$; $\bar{\varphi}(h) \in \Psi(\Omega)$,
$\hat{\varphi} : \Gamma_{eD} \to \mathbb{R}$	Scaled electric potential in Γ_{eD} and its trace lifting to $H^1(\Omega)$; Also, $\hat{\varphi}^h(\boldsymbol{x}^h) = h\hat{\varphi}(\boldsymbol{x})$,
$\varphi(h) : \bar{\Omega} \to \mathbb{R}$	Scaled electric potential, $\varphi(h) = \bar{\varphi}(h) + \hat{\varphi}$,
$\boldsymbol{\sigma}(h) = (\sigma_{ij}(h)) : \Omega \to \mathbb{S}^3$	Scaled stress tensor, with $\sigma_{\alpha\beta}^h(\boldsymbol{x}^h) = h^2\sigma_{\alpha\beta}(h)(\boldsymbol{x})$, $\sigma_{3\alpha}^h(\boldsymbol{x}^h) = h\sigma_{3\alpha}(h)(\boldsymbol{x})$, and $\sigma_{33}^h(\boldsymbol{x}^h) = \sigma_{33}(h)(\boldsymbol{x})$,
$\boldsymbol{D}(h) = (D_i(h)) : \Omega \to \mathbb{R}^3$	Scaled electric displacement, with $D_\alpha^h(\boldsymbol{x}^h) = D_\alpha(h)(\boldsymbol{x})$, and $D_3^h(\boldsymbol{x}^h) = h^{-1}D_3(h)(\boldsymbol{x})$,
$\boldsymbol{e}(\boldsymbol{u}(h)) = (e_{ij}(h)) : \Omega \to \mathbb{S}^3$	Scaled linearized strain tensor, with $e_{\alpha\beta}^h(\boldsymbol{x}^h) = h^{-2}e_{\alpha\beta}(h)(\boldsymbol{x})$, $e_{3\beta}^h(\boldsymbol{x}^h) = h^{-1}e_{3\beta}(h)(\boldsymbol{x})$, and $e_{33}^h(\boldsymbol{x}^h) = e_{33}(h)(\boldsymbol{x})$,
$\boldsymbol{E}(\varphi(h)) = (E_i(h)) : \Omega \to \mathbb{R}^3$	Scaled electric field, with $E_\alpha^h(\varphi^h)(\boldsymbol{x}^h) = E_\alpha(\varphi(h))(\boldsymbol{x})$, and $E_3^h(\psi^h)(\boldsymbol{x}^h) = hE_3(\psi(h))(\boldsymbol{x})$.

Table 4
List of functions and constants in the scaled domain Ω.

Notation	Description
$\boldsymbol{f} = (f_i) : \bar{\Omega} \to \mathbb{R}^3$	Density of scaled volume forces, with $f_\alpha^h(\boldsymbol{x}^h) = hf_\alpha(\boldsymbol{x})$, and $f_3^h(\boldsymbol{x}^h) = f_3(\boldsymbol{x})$,
$\boldsymbol{g} = (g_i) : \bar{\Gamma}_N \to \mathbb{R}^3$	Density of scaled tractions, with $g_\alpha^h(\boldsymbol{x}^h) = h^2 g_\alpha(\boldsymbol{x})$, and $g_3^h(\boldsymbol{x}^h) = hg_3(\boldsymbol{x})$,
$\boldsymbol{C} = (C_{ijkl})$	Scaled stiffness tensor zero-th order term,
$\boldsymbol{C}^\sharp = (C_{ijkl}^\sharp)$	Scaled stiffness tensor first order term,
$\boldsymbol{P} = (P_{kij})$	Scaled piezoelectric tensor,
$\boldsymbol{\varepsilon} = (\varepsilon_{ij})$	Scaled dielectric tensor,
$\bar{\boldsymbol{C}}, \boldsymbol{M}, \boldsymbol{N}, \boldsymbol{M}^\sharp, \boldsymbol{N}^\sharp$	Auxiliary matrices defined from components of $\boldsymbol{C}$ and $\boldsymbol{C}^\sharp$,
$\boldsymbol{u}^p : \Omega \to \mathbb{R}^3, p \geq 0$	Terms of the asymptotic expansion of $\boldsymbol{u}(h)$,
$\bar{\varphi}^p : \Omega \to \mathbb{R}, p \geq 0$	Terms of the asymptotic expansion of $\bar{\varphi}(h)$,
$\boldsymbol{\sigma}^p : \Omega \to \mathbb{S}^3, p \geq -4$	Terms of the asymptotic expansion of $\boldsymbol{\sigma}(h)$,
$\boldsymbol{D}^p : \Omega \to \mathbb{R}^3, p \geq -1$	Terms of the asymptotic expansion of $\boldsymbol{D}(h)$,
$\boldsymbol{\xi} = (\xi_i) : (0, L) \to \mathbb{R}^3$	Components in the characterization of $\boldsymbol{u}^0 \in V_{BN}(\Omega)$,
$\hat{\boldsymbol{u}}^2(x_3) = (\hat{u}_\alpha^2(x_3)) : \omega \to \mathbb{R}^2$	For a.e. $x_3 \in (0, L)$, first two components of $\boldsymbol{u}^2$,
$\boldsymbol{M}_{\alpha\beta}$	Auxiliary matrices defined from components of $\boldsymbol{C}$,
$\Lambda_\alpha : \Omega \to \mathbb{R}$	Functions defined as $\Lambda_\alpha(\boldsymbol{x}) = -\frac{\det \boldsymbol{M}_{\alpha\beta}}{\det \boldsymbol{M}} x_\beta$,
$\Phi_{\alpha\beta} : \Omega \to \mathbb{R}$	Functions defined using $\boldsymbol{M}_{\alpha\beta}$,
I_α	Inertia moment $I_\alpha = \int_\omega x_\alpha^2 d\omega$,
$X_{\alpha\beta}, Y_\alpha, Z : (0, L) \to \mathbb{R}$	Functions $X_{\alpha\beta} = \int_\omega \Phi_{\alpha\beta} d\omega$, $Y_\alpha = \int_\omega \Phi_{\alpha\beta}\delta_\beta d\omega$, $Z = \int_\omega \Lambda_\alpha\delta_\alpha d\omega$,
$s, s_\alpha : (0, L) \to \mathbb{R}$	Functions $s_\alpha = -\frac{1}{A(\omega)} X_{\alpha\beta}\xi_\beta''$, $s = -\frac{1}{I_1 + I_2}\left(Y_\beta \xi_\beta'' + Z\xi_3'\right)$,
$Y : (0, L) \to \mathbb{R}$	Function $Y = \frac{\det \bar{\boldsymbol{C}}}{\det \boldsymbol{M} \det \boldsymbol{N}}$,
$F_i, M_\alpha : (0, L) \to \mathbb{R}$	Resultants of forces and moments,
$\bar{P} : \Omega \to \mathbb{R}$	Function $\bar{P} = P_{333} - \frac{\det \boldsymbol{M}_{\alpha\beta}}{\det \boldsymbol{M}} P_{3\alpha\beta}$,
w: $(0, L) \to R(\Omega)$	Warping function,
$J : (0, L) \to \mathbb{R}$	Torsion function,
$z_\alpha, z : (0, L) \to \mathbb{R}$	Functions in $H_0^1(0, L)$ characterizing (u_α^1), see (4.49),
$r : (0, L) \to Q(\omega), z_3 : (0, L) \to \mathbb{R}$	Functions which characterize u_3^1, see (4.56),
$\tilde{\boldsymbol{C}} = (\tilde{C}_{ijkl}), \tilde{\boldsymbol{P}} = (\tilde{P}_{kij}), \tilde{\boldsymbol{\varepsilon}} = (\tilde{\varepsilon}_{ij})$	Inverse constitutive tensors of $\boldsymbol{C}$, $\boldsymbol{P}$ and $\boldsymbol{\varepsilon}$.

Table 5
List of relevant functional spaces.

Notation	Description
$\boldsymbol{V}^h(\Omega^h)=\{\boldsymbol{v}^h\in\left[H^1(\Omega^h)\right]^3 : \langle\boldsymbol{v}^h\rangle=\boldsymbol{0} \text{ on } \Gamma_D^h\}$	Space of admissible displacements,
$\Psi^h(\Omega^h)=\{\psi^h\in H^1(\Omega^h) : \psi^h=0 \text{ on } \Gamma_{eD}^h\}$	Space of admissible electric potentials,
$\boldsymbol{V}(\Omega)=\{\boldsymbol{v}\in[H^1(\Omega)]^3 : \langle\boldsymbol{v}\rangle=\boldsymbol{0} \text{ on } \Gamma_D\}$	Space of scaled admissible displacements,
$\Psi(\Omega)=\{\psi\in H^1(\Omega) : \psi=0 \text{ on } \Gamma_{eD}\}$	Space of scaled admissible electric potentials,
$\hat{\Psi}(\Omega)=\hat{\varphi}+\Psi(\Omega)$	Set consisting in $\{\psi\in\Psi(\Omega) : \psi=\hat{\varphi}$ on $\Gamma_{eD}\}$,
$\boldsymbol{V}_{BN}(\Omega)=\left\{\boldsymbol{v}\in\boldsymbol{V}(\Omega) : e_{\alpha\beta}(\boldsymbol{v})=e_{3\beta}(\boldsymbol{v})=0\right\}$	Space of Bernoulli-Navier displacements,
$\boldsymbol{V}_m^2(\omega)=\left\{\hat{\boldsymbol{v}}=(\hat{v}_\alpha)\in\left[H^1(\omega)\right]^2 : \int_\omega\hat{v}_\alpha\,d\omega=\int_\omega(x_1\hat{v}_2-x_2\hat{v}_1)\,d\omega=0\right\}$	Subspace where 2D Korn inequality holds,
$\boldsymbol{V}_1(\Omega)=\left\{\boldsymbol{v}\in\boldsymbol{V}(\Omega) : e_{\alpha\beta}(\boldsymbol{v})=0\right\}$	Subspace of transversal rigid displacements,
$Q=Q(\omega)=\left\{\rho\in H^1(\omega) : \int_\omega\rho\,d\omega=0\right\}$,	
$S=S(\omega)=\{\psi\in H^1(\omega) : \psi=0 \text{ on } \gamma_{eD}\}$,	
$\Psi_l(\Omega)=\{\psi\in L^2(\Omega) : \partial_\alpha\psi\in L^2(\omega)\}$	Space isomorph to $L^2(0,L;H^1(\omega))$,
$R(\Omega)=L^2(0,L;Q(\omega))$,	
$\Psi_{l0}(\Omega)=L^2(0,L;S(\omega))$,	
$\boldsymbol{T}(\Omega)=\Psi_{l0}(\Omega)\times R(\Omega)\times H_0^1(0,L)$	Product space,
$\boldsymbol{X}(\Omega)=\boldsymbol{V}(\Omega)\times\Psi(\Omega)$,	We also define the set $\hat{\boldsymbol{X}}(\Omega)=\boldsymbol{V}(\Omega)\times\hat{\Psi}(\Omega)$,
$\boldsymbol{M}_1(\Omega)=\left\{\boldsymbol{\tau}=(\tau_{ij})\in[L^2(\Omega)]^{3\times3} : \tau_{ij}=\tau_{ji}\right\}$,	
$\boldsymbol{M}_2(\Omega)=[L^2(\Omega)]^3$,	
$\boldsymbol{M}(\Omega)=\boldsymbol{M}_1(\Omega)\times\boldsymbol{M}_2(\Omega)$,	
$\boldsymbol{V}_{RD}(\Omega)=\{\boldsymbol{v}\in\boldsymbol{V}_1(\Omega) : v_3\in L^2(0,L;Q(\omega))\}$,	
$\boldsymbol{V}_R(\Omega)=\{\boldsymbol{v}\in\boldsymbol{V}_{RD}(\Omega) : v_\alpha=\delta_\alpha s,\ s\in H_0^1(0,L)\}$,	Subspace of admissible rotations,
$\boldsymbol{V}_T(\Omega)=\{\boldsymbol{v}\in\boldsymbol{V}_{RD}(\Omega) : v_\alpha=s_\alpha,\ v_3=-x_\alpha s_\alpha',\ s_\alpha\in H_0^1(0,L)\}$,	Subspace of admissible translations,
$\dot{\boldsymbol{V}}_{RD}(\Omega):=\boldsymbol{V}_{RD}(\Omega)/\boldsymbol{V}_T(\Omega)$,	Quotient space,
$\boldsymbol{V}_H(\Omega)=H^1(0,L;\boldsymbol{V}_m^2(\omega))$.	

point of $\bar{\Omega}^h$ will be denoted by $\boldsymbol{x}^h=(x_i^h)$ and the unit outer normal vector to the boundary $\partial\Omega^h$ by $\boldsymbol{n}^h=(n_i^h)$. The coordinate system $Ox_1^hx_2^hx_3^h$ will be assumed a principal system of inertia associated to ω^h, which means that

$$\int_{\omega^h}x_\alpha^h d\omega^h=0,\quad \int_{\omega^h}x_1^hx_2^h d\omega^h=0. \tag{2.1}$$

We define the following differential operators: $\partial_i^h=\partial^h/\partial x_i^h$ and $\partial_i=\partial/\partial x_i$. For a function $\chi:[0,L]\to\mathbb{R}$ we note $\chi',\chi'',\ldots$ its first, second, ... derivative with respect to the (only) variable x_3 (or x_3^h).

Now, assuming that $\bar{\Omega}^h$ is a perfect dielectric body, so that there is neither volume nor surface electric charge, and that volume forces of density $\boldsymbol{f}^h=(f_i^h)$ act in Ω^h, the resulting displacement field $\boldsymbol{u}^h=(u_i^h):\bar{\Omega}^h\to\mathbb{R}^3$ and electric potential $\varphi^h:\bar{\Omega}^h\to\mathbb{R}$ solve the equilibrium equations (see e.g. Ciarlet, 1988; Jackson, 1975):

$$\begin{cases}-div^h\boldsymbol{\sigma}^h(\boldsymbol{u}^h,\varphi^h)=\boldsymbol{f}^h, & \text{in } \Omega^h,\\ -div^h\boldsymbol{D}^h(\boldsymbol{u}^h,\varphi^h)=0, & \text{in } \Omega^h,\end{cases} \tag{2.2}$$

the stress tensor $\boldsymbol{\sigma}^h=(\sigma_{ij}^h)$ and the electric displacement vector $\boldsymbol{D}^h=(D_i^h)$ being related to the linear strain tensor $(e_{ij}^h(\boldsymbol{u}^h)=\frac{1}{2}(\partial_i^hu_j^h+\partial_j^hu_i^h))$ and the gradient of the electric potential $(E_k^h(\varphi^h)=-\partial_k^h\varphi^h)$ through the constitutive law (see e.g. Ikeda, 1990; Jackson, 1975):

$$\begin{cases}\sigma_{ij}^h(\boldsymbol{u}^h,\varphi^h)=C_{ijkl}^he_{kl}^h(\boldsymbol{u}^h)-P_{kij}^hE_k^h(\varphi^h), & \text{in } \Omega^h,\\ D_i^h(\boldsymbol{u}^h,\varphi^h)=P_{ikl}^he_{kl}^h(\boldsymbol{u}^h)+\varepsilon_{ij}^hE_j^h(\varphi^h), & \text{in } \Omega^h,\end{cases} \tag{2.3}$$

where $\boldsymbol{C}^h=(C_{ijkl}^h)$, $\boldsymbol{P}^h=(P_{kij}^h)$ and $\boldsymbol{\varepsilon}^h=(\varepsilon_{ij}^h)$ are the stiffness, piezoelectric and dielectric tensors, respectively. We assume that $C_{ijkl}^h, P_{kij}^h, \varepsilon_{ij}^h\in L^\infty(\Omega^h)$, implying the boundedness of these tensors, and also the following coercive relations: there exists a constant $c>0$ such that for all symmetric second order tensors $\boldsymbol{M}=(M_{ij})\in\mathbb{S}^3$ and every vector $\boldsymbol{\xi}=(\xi_i)\in\mathbb{R}^3$ one has

$$C_{ijkl}^h(\boldsymbol{x}^h)M_{ij}M_{kl}\geq c\sum_{i,j=1}^3(M_{ij})^2,\quad \varepsilon_{ij}^h(\boldsymbol{x}^h)\xi_i\xi_j\geq c\sum_{i=1}^3(\xi_i)^2, \tag{2.4}$$

for every $\boldsymbol{x}^h\in\bar{\Omega}^h$. Furthermore, the following symmetries hold:

$$C_{ijkl}^h=C_{klij}^h=C_{jikl}^h,\quad \varepsilon_{ij}^h=\varepsilon_{ji}^h,\quad P_{mij}^h=P_{mji}^h,\quad \text{in } \bar{\Omega}^h. \tag{2.5}$$

We will assume that the rod is made of piezoelectric materials of symmetry class 2, which comprises the following crystal systems (see e.g. Nye, 1985; Royer and Dieulesaint, 2000): monoclinic (except class m), orthorhombic, tetragonal, hexagonal (except classes $\bar{6}$ and $\bar{6}m2$), and cubic. For this class of materials, which includes among many other piezoelectric materials Barium Titanate (BT), Lead Zirconium Titanate (PZT) and Polyvinylidene Fluoride (PVDF), the following conditions hold:

$$C_{3\rho33}^h=C_{3\theta\alpha\beta}^h=0,\quad P_{\theta\rho\sigma}^h=P_{33\alpha}^h=P_{\beta33}^h=0,\quad \varepsilon_{3\theta}^h=0,\quad \text{in } \bar{\Omega}^h. \tag{2.6}$$

Accordingly, it is useful to define the following symmetric matrices:

$$\bar{\boldsymbol{C}}^h=\bar{\boldsymbol{C}}^h(\boldsymbol{x}^h):=\begin{pmatrix}C_{1111}^h & C_{1211}^h & C_{1122}^h & 0 & 0 & C_{1133}^h\\ C_{1211}^h & C_{1212}^h & C_{1222}^h & 0 & 0 & C_{1233}^h\\ C_{1122}^h & C_{1222}^h & C_{2222}^h & 0 & 0 & C_{2233}^h\\ 0 & 0 & 0 & C_{3131}^h & C_{3132}^h & 0\\ 0 & 0 & 0 & C_{3132}^h & C_{3232}^h & 0\\ C_{1133}^h & C_{1233}^h & C_{2233}^h & 0 & 0 & C_{3333}^h\end{pmatrix}, \tag{2.7}$$

$$\boldsymbol{M}^h=\boldsymbol{M}^h(\boldsymbol{x}^h):=\begin{pmatrix}C_{1111}^h & C_{1211}^h & C_{1122}^h\\ C_{1211}^h & C_{1212}^h & C_{1222}^h\\ C_{1122}^h & C_{1222}^h & C_{2222}^h\end{pmatrix},$$

$$\boldsymbol{N}^h=\boldsymbol{N}^h(\boldsymbol{x}^h):=\begin{pmatrix}C_{3131}^h & C_{3132}^h\\ C_{3132}^h & C_{3232}^h\end{pmatrix}. \tag{2.8}$$

From properties (2.4) it follows that these matrices are definite positive and therefore $\det\bar{\boldsymbol{C}}^h>0$, $\det\boldsymbol{M}^h>0$ and $\det\boldsymbol{N}^h>0$. For the moment, and until not explicited otherwise, we will not assume that the conditions traduced by (2.6) hold.

We suppose that the rod is weakly clamped at the extremities $\Gamma_0^h\cup\Gamma_L^h$, which means that the displacement field $\boldsymbol{u}^h$ is such that (see e.g. Trabucho and Viaño, 1996)

$$\int_{\Gamma_a^h}u_i^hd\omega^h=0,\quad \int_{\Gamma_a^h}\left(x_j^hu_i^h-x_i^hu_j^h\right)d\omega^h=0,\quad a=0,L. \tag{2.9}$$

In the following we will represent this boundary condition by $\langle\boldsymbol{u}^h\rangle=\boldsymbol{0}$ on Γ_D^h.

Remark 2.1. As we will see, the use of the " average clamping" condition (2.9) instead of a strong clamping condition such as $\boldsymbol{u}^h=\boldsymbol{0}$ on Γ_D^h allows to avoid the " boundary layer problem" (see Ref. Lions, 1973) that arises when a strong clamping condition is used - which turns out to be related to the fact that in general a strong clamping of the rod is physically impossible (see e.g. Trabucho and Viaño, 1996 and references therein).

We also assume that surface tractions of density $\boldsymbol{g}^h=(g_i^h)$ act on Γ_N^h and that an electric potential $\hat{\varphi}^h$ is applied on Γ_{eD}^h. In the following we suppose that $f_i^h = L^2(\Omega^h)$, $g_i^h \in L^2(\Gamma_N^h)$, and $\hat{\varphi}^h \in H^{1/2}(\Gamma_{eD}^h)$ are given quantities. Thus, the boundary conditions satisfied by $(\boldsymbol{u}^h, \varphi^h)$ are:

$$\begin{cases} \boldsymbol{\sigma}^h(\boldsymbol{u}^h, \varphi^h)\boldsymbol{n}^h = \boldsymbol{g}^h, & \text{on } \Gamma_N^h, \\ \langle \boldsymbol{u}^h \rangle = \boldsymbol{0}, & \text{on } \Gamma_D^h, \\ \boldsymbol{D}^h(\boldsymbol{u}^h, \varphi^h) \cdot \boldsymbol{n}^h = 0, & \text{on } \Gamma_{eN}^h, \\ \varphi^h = \hat{\varphi}^h, & \text{on } \Gamma_{eD}^h. \end{cases} \tag{2.10}$$

Let us now define the following closed subspaces of the Sobolev space of order one $[H^1(\Omega^h)]^m$, $m \geq 1$:

$$\begin{cases} \boldsymbol{V}^h(\Omega^h) := \left\{ \boldsymbol{v}^h \in [H^1(\Omega^h)]^3 : \langle \boldsymbol{v}^h \rangle = \boldsymbol{0} \text{ on } \Gamma_D^h \right\}, \\ \Psi^h(\Omega^h) := \left\{ \psi^h \in H^1(\Omega^h) : \psi^h = 0 \text{ on } \Gamma_{eD}^h \right\}. \end{cases} \tag{2.11}$$

Since $meas(\Gamma_D^h) > 0$ and $meas(\Gamma_{eD}^h) > 0$, it follows from Korn's and Poincaré–Friedrichs' inequalities that the following norms are equivalent to the standard Sobolev $\| \cdot \|_{1,\Omega^h}$ norms on $\boldsymbol{V}^h$ and Ψ^h, respectively,

$$\boldsymbol{v}^h \in \boldsymbol{V}^h(\Omega^h) \to |\boldsymbol{e}^h(\boldsymbol{v}^h)|_{0,\Omega^h}, \quad \psi^h \in \Psi^h(\Omega^h) \to |\boldsymbol{E}^h(\psi^h)|_{0,\Omega^h},$$

where $| \cdot |_{0,\Omega^h}$ stands for the usual norm in $[L^2(\Omega^h)]^m$, $m \geq 1$.

Defining $\bar{\varphi}^h = \varphi^h - \hat{\varphi}^h$, where now $\hat{\varphi}^h$ represents a trace lifting in $H^1(\Omega^h)$ of the boundary potential acting on Γ_{eD}^h, the variational problem associated to (2.2), (2.3) and (2.10) is

Find $(\boldsymbol{u}^h, \bar{\varphi}^h) \in \boldsymbol{V}^h(\Omega^h) \times \Psi^h(\Omega^h)$ such that

$$\begin{aligned} &\times \int_{\Omega^h} C_{ijkl}^h e_{kl}^h(\boldsymbol{u}^h) e_{ij}^h(\boldsymbol{v}^h) d\boldsymbol{x}^h + \int_{\Omega^h} \varepsilon_{ij}^h \partial_i^h \bar{\varphi}^h \partial_j^h \psi^h d\boldsymbol{x}^h \\ &+ \int_{\Omega^h} P_{mij}^h (\partial_m^h \bar{\varphi}^h e_{ij}^h(\boldsymbol{v}^h) - e_{ij}^h(\boldsymbol{u}^h) \partial_m^h \psi^h) d\boldsymbol{x}^h \\ &= \int_{\Omega^h} f_i^h v_i^h d\boldsymbol{x}^h + \int_{\Gamma_N^h} g_i^h v_i^h d\Gamma^h - \int_{\Omega^h} \varepsilon_{ij}^h \partial_i^h \hat{\varphi}^h \partial_j^h \psi^h d\boldsymbol{x}^h \\ &- \int_{\Omega^h} P_{mij}^h \partial_m^h \hat{\varphi}^h e_{ij}^h(\boldsymbol{v}^h) d\boldsymbol{x}^h, \quad \text{for all } (\boldsymbol{v}^h, \psi^h) \in \boldsymbol{V}^h(\Omega^h) \\ &\times \Psi^h(\Omega^h). \end{aligned} \tag{2.12}$$

It can be easily shown that the boundedness of $\boldsymbol{C}^h$, $\boldsymbol{P}^h$, $\boldsymbol{\varepsilon}^h$, $\boldsymbol{f}^h$ and $\boldsymbol{g}^h$ together with properties (2.4) ensure the existence and uniqueness of solution of problem (2.12) as a result of Lax–Milgram Lemma.

Remark 2.2. The derivation of (2.12) follows closely the steps presented in Bernadou and Haenel (1999) for the strong clamping case with minor changes to accommodate the weak clamping condition adopted here.

Proposition 2.3. *Problem (2.12) is formally equivalent to*

Find $(\boldsymbol{u}^h, \varphi^h) \in \boldsymbol{V}^h(\Omega^h) \times H^1(\Omega^h)$, such that $\varphi^h = \hat{\varphi}^h$ on Γ_{eD}^h,

$$\begin{aligned} &\times \int_{\Omega^h} \sigma_{ij}^h(\boldsymbol{u}^h, \varphi^h) e_{ij}^h(\boldsymbol{v}^h) d\boldsymbol{x}^h + \int_{\Omega^h} D_k^h(\boldsymbol{u}^h, \varphi^h) E_k^h(\psi^h) d\boldsymbol{x}^h \\ &= \int_{\Omega^h} f_i^h v_i^h d\boldsymbol{x}^h + \int_{\Gamma_N^h} g_i^h v_i^h d\Gamma^h, \\ &\times \text{for all } (\boldsymbol{v}^h, \psi^h) \in \boldsymbol{V}^h(\Omega^h) \times \Psi^h(\Omega^h), \end{aligned} \tag{2.13}$$

with the functions $\sigma_{ij}^h, D_k^h : \Omega \to \mathbb{R}$ *defined as in (2.3).*

The above formulation corresponds to the principle of virtual work.

3. Change of variable to the reference rod Ω

The major geometric feature of a three-dimensional rod is the fact that the largest cross sectional dimension is very small compared with its length ($h \ll L$), cause of ill-conditioning of the three-dimensional problem. We take advantage of this property to use an asymptotic expansion method (see Lions, 1973) with respect to the small parameter h as usually done in the elastic rod case (see e.g. Bermúdez and Viaño, 1984b; Trabucho and Viaño, 1996 and references therein). We will study the dependence of the solution $(\boldsymbol{u}^h, \bar{\varphi}^h)$ with respect to h. The technique of change of variable to a fixed domain and subsequent rescaling of the displacement and electric potential will allow us to derive variational problems equivalent to (2.12) and (2.13) where h shows up in an explicit way in the rescaled equations.

To start, we perform a change of variable to the reference domain Ω through the following transformation (see e.g. Trabucho and Viaño, 1996)

$$\begin{aligned} &\Pi^h : \boldsymbol{x} = (x_1, x_2, x_3) \in \bar{\Omega} \\ &\quad \to \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) = \left(x_1^h, x_2^h, x_3^h\right) = (hx_1, hx_2, x_3) \in \bar{\Omega}^h. \end{aligned} \tag{3.1}$$

Moreover,

$$\Gamma_N^h = \Pi^h(\Gamma_N), \quad \Gamma_D^h = \Pi^h(\Gamma_D), \quad \Gamma_0^h = \Pi^h(\Gamma_0), \quad \Gamma_L^h = \Pi^h(\Gamma_L),$$
$$\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{n}^h(\boldsymbol{x}^h), \quad \Gamma_{eN}^h = \Pi^h(\Gamma_{eN}), \quad \gamma_{eD}^h = \Pi^h(\gamma_{eD}), \quad \Gamma_{eD}^h = \Pi^h(\Gamma_{eD}),$$

where $\boldsymbol{n}$ is the unit outer normal to the set $\partial\Omega = \Gamma_D \cup \Gamma_N = \Gamma_{eD} \cup \Gamma_{eN}$ and

$$\Gamma_{eD} = \gamma_{eD} \times (0, L).$$

Furthermore, condition (2.1) becomes now

$$\int_\omega x_\alpha d\omega = 0, \quad \int_\omega x_1 x_2 d\omega = 0. \tag{3.2}$$

In view of (2.9) we will represent by $\langle \boldsymbol{v} \rangle = \boldsymbol{0}$ on Γ_D the boundary condition

$$\int_{\Gamma_a} v_i d\omega = 0, \quad \int_{\Gamma_a} (x_j v_i - x_i v_j) d\omega = 0, \quad a = 0, L,$$

that is, taking (3.2) into account,

$$\int_{\Gamma_a} v_i d\omega = 0, \quad \int_{\Gamma_a} \delta_\alpha v_\alpha d\omega = 0, \quad \int_{\Gamma_a} x_\alpha v_3 d\omega = 0, \quad a = 0, L, \tag{3.3}$$

where $\delta_1(x_1, x_2) = x_2$, $\delta_2(x_1, x_2) = -x_1$. We now define the following closed subspaces of $[H^1(\Omega)]^m$, $m \geq 1$ (cf. (2.11))

$$\boldsymbol{V}(\Omega) = \{\boldsymbol{v} \in [H^1(\Omega)]^3 : \langle \boldsymbol{v} \rangle = \boldsymbol{0} \text{ on } \Gamma_D\},$$
$$\Psi(\Omega) = \{\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_{eD}\},$$

endowed with the following norms equivalent to the usual Sobolev norms:

$$||\boldsymbol{v}||_{\boldsymbol{V}} = |\boldsymbol{e}(\boldsymbol{v})|_{0,\Omega}, \quad ||\psi||_\Psi = |\nabla\psi|_{0,\Omega},$$

where $| \cdot |_{0,\Omega}$ stands for the standard norm in $[L^2(\Omega)]^m$, $m \geq 1$, and

$$\boldsymbol{e}(\boldsymbol{v}) = (e_{ij}(\boldsymbol{v})), \quad e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i).$$

In order to obtain a problem in Ω equivalent to (2.12) we proceed as follows (see e.g. Viaño et al., 2005; Viaño et al., 2015). With the unknown and test displacement fields $\boldsymbol{u}^h$, $\boldsymbol{v}^h$ in $\boldsymbol{V}^h(\Omega^h)$, we associate the (unknown and test) scaled displacement fields $\boldsymbol{u}(h) = (u_i(h))$ and $\boldsymbol{v}(h) = (v_i(h))$ in $\boldsymbol{V}(\Omega)$ defined by the following scalings valid for all $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \bar{\Omega}$:

$$u_\alpha(h)(\boldsymbol{x}) = hu_\alpha^h(\boldsymbol{x}^h), \quad u_3(h)(\boldsymbol{x}) = u_3^h(\boldsymbol{x}^h), \tag{3.4}$$

$$v_\alpha(h)(\boldsymbol{x}) = hv_\alpha^h(\boldsymbol{x}^h), \quad v_3(h)(\boldsymbol{x}) = v_3^h(\boldsymbol{x}^h). \tag{3.5}$$

Similarly, the electric potential $\bar{\varphi}^h$ and the test function ψ^h in $\Psi^h(\Omega^h)$ are associated with the scaled potential $\bar{\varphi}(h)$ and the scaled (test) function $\psi(h)$ in $\Psi(\Omega)$ using the following scaling valid for all $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \bar{\Omega}$:

$$\bar{\varphi}(h)(\boldsymbol{x}) = h^{-1}\bar{\varphi}^h(\boldsymbol{x}^h), \tag{3.6}$$

$$\psi(h)(\boldsymbol{x}) = h^{-1}\psi^h(\boldsymbol{x}^h). \tag{3.7}$$

Furthermore, we consider the following hypothesis on the magnitude of the data with respect to the diameter of the rod cross-section h:

1. There exist functions $f_i \in L^2(\Omega)$ and $g_i \in L^2(\Gamma_N)$, independent of h, such that:

$$\begin{cases} f^h_\alpha(\boldsymbol{x}^h) = hf_\alpha(\boldsymbol{x}), \quad f^h_3(\boldsymbol{x}^h) = f_3(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Omega^h, \\ g^h_\alpha(\boldsymbol{x}^h) = h^2 g_\alpha(\boldsymbol{x}), \quad g^h_3(\boldsymbol{x}^h) = hg_3(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Gamma^h_N. \end{cases} \tag{3.8}$$

2. There exists a function $\hat{\varphi} \in H^{1/2}(\Gamma_{eD})$, independent of h, such that:

$$\hat{\varphi}^h(\boldsymbol{x}^h) = h\hat{\varphi}(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Gamma^h_{eD}. \tag{3.9}$$

We also denote by $\hat{\varphi} \in H^1(\Omega)$ a trace lifting of $\hat{\varphi}$ and define $\varphi(h) = \bar{\varphi}(h) + \hat{\varphi}$.

In addition to that, it is assumed that there exist functions $C_{ijkl} \in L^\infty(0, L)$, and $C^\sharp_{ijkl}, P_{kij}, \varepsilon_{ij} \in L^\infty(\Omega)$, independent of h, such that:

$$\begin{aligned} C^h_{ijkl}(x^h_1, x^h_2, x^h_3) &= C_{ijkl}(x_3) + hC^\sharp_{ijkl}(x_1, x_2, x_3), \\ P^h_{kij}(\boldsymbol{x}^h) &= P_{kij}(\boldsymbol{x}), \quad \varepsilon^h_{ij}(\boldsymbol{x}^h) = \varepsilon_{ij}(\boldsymbol{x}), \end{aligned} \tag{3.10}$$

for all $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, x_3) \in \bar{\Omega}$. We also assume that the properties (2.4)–(2.6) hold when we drop the superindex (take $h = 1$) and for $C^\sharp_{ijkl}$ as well. Consequently, there exist tensor fields $\mathbf{M}$, $\mathbf{N}$, $\mathbf{M}^\sharp$ and $\mathbf{N}^\sharp$ such that

$$\begin{aligned} \mathbf{M}^h(x^h_1, x^h_2, x^h_3) &= \mathbf{M}(x_3) + h\mathbf{M}^\sharp(x_1, x_2, x_3), \\ \mathbf{N}^h(\boldsymbol{x}^h) &= \mathbf{N}(x_3) + h\mathbf{N}^\sharp(x_1, x_2, x_3), \end{aligned} \tag{3.11}$$

where $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \bar{\Omega}^h$, $\boldsymbol{x} \in \bar{\Omega}$, with $\mathbf{M}$, $\mathbf{N}$, $\mathbf{M}^\sharp$ and $\mathbf{N}^\sharp$ independent of h (see (2.8)).

Remark 3.1. The Ansatz elastic tensor (3.10) includes the material's inhomogeneity effects in the piezoelectric response predominantly given by the leading term. However, the elastic tensor also accounts 3D structural parameters (such as grain size for PZT's, see e.g. Naceur et al., 2014; Su, 1997) effects on the piezoelectric response, through the perturbation introduced by the small parameter h. Note that other authors (e.g. Figueiredo and Leal, 2006) have assumed that there exist functions C_{ijkl} independent of h, such that:

$$C^h_{ijkl}(\boldsymbol{x}^h) = C_{ijkl}(\boldsymbol{x}) \; \forall \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}), \quad \boldsymbol{x} \in \bar{\Omega}. \tag{3.12}$$

We shall further comment on the implications of these two different assumptions (see Remark 4.8 and Ref. Viaño et al., 2015).

Combining transformation (3.1) with the notations (3.5) and (3.7) we have for all $\boldsymbol{v}^h \in \boldsymbol{V}^h(\Omega^h)$, $\psi^h \in \Psi^h(\Omega^h)$ and $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$:

$$\begin{cases} e^h_{\alpha\beta}(\boldsymbol{v}^h)(\boldsymbol{x}^h) = h^{-2} e_{\alpha\beta}(\boldsymbol{v}(h))(\boldsymbol{x}), \\ e^h_{3\beta}(\boldsymbol{v}^h)(\boldsymbol{x}^h) = h^{-1} e_{3\beta}(\boldsymbol{v}(h))(\boldsymbol{x}), \\ e^h_{33}(\boldsymbol{v}^h)(\boldsymbol{x}^h) = e_{33}(\boldsymbol{v}(h))(\boldsymbol{x}), \end{cases} \tag{3.13}$$

$$\begin{cases} E^h_\alpha(\psi^h)(\boldsymbol{x}^h) = -\partial^h_\alpha \psi^h(\boldsymbol{x}^h) = -\partial_\alpha(\psi(h))(\boldsymbol{x}) = E_\alpha(\psi(h))(\boldsymbol{x}), \\ E^h_3(\psi^h)(\boldsymbol{x}^h) = -\partial^h_3 \psi^h(\boldsymbol{x}^h) = -h\partial_3(\psi(h))(\boldsymbol{x}) = hE_3(\psi(h))(\boldsymbol{x}). \end{cases} \tag{3.14}$$

Using the scalings defined previously for the displacement vector and for the electric potential together with the above assumptions and results, we can reformulate the variational problem (2.12) into another variational problem posed in $\bar{\Omega}$ independent of h. Assuming from here on that we are dealing with piezoelectric materials of symmetry class 2 (i.e. properties (2.6) hold), we have the following result.

Proposition 3.2. *Let $(\boldsymbol{u}(h), \bar{\varphi}(h)) \in \boldsymbol{V}(\Omega) \times \Psi(\Omega)$ be the functions obtained from $(\boldsymbol{u}^h, \bar{\varphi}^h) \in \boldsymbol{V}^h(\Omega^h) \times \Psi^h(\Omega^h)$, the unique solution of problem (2.12), by transformation (3.1) and scalings (3.4) and (3.6). Bearing in mind (2.6) as well as scalings (3.5) and (3.7)–(3.13), then $(\boldsymbol{u}(h), \bar{\varphi}(h))$ is the unique solution of the following variational problem:*

$$\begin{aligned} &(\boldsymbol{u}(h), \bar{\varphi}(h)) \in \boldsymbol{V}(\Omega) \times \Psi(\Omega), \quad \hat{a}(h)((\boldsymbol{u}(h), \bar{\varphi}(h)), (\boldsymbol{v}, \psi)) \\ &\quad = \hat{l}(h)(\boldsymbol{v}, \psi) \; \forall (\boldsymbol{v}, \psi) \in \boldsymbol{V}(\Omega) \times \Psi(\Omega), \end{aligned} \tag{3.15}$$

where the bilinear form $\hat{a}(h)$ and the linear functional $\hat{l}(h)$ are defined, respectively, by

$$\begin{aligned} \hat{a}(h)((\boldsymbol{u}, \bar{\varphi}), (\boldsymbol{v}, \psi)) &= \int_\Omega \left(h^{-4} C_{\alpha\beta\theta\rho} + h^{-3} C^\sharp_{\alpha\beta\theta\rho}\right) e_{\theta\rho}(\boldsymbol{u}) e_{\alpha\beta}(\boldsymbol{v}) d\boldsymbol{x} \\ &+ \int_\Omega \left(4h^{-2} C_{3\alpha3\beta} + 4h^{-1} C^\sharp_{3\alpha3\beta}\right) e_{3\alpha}(\boldsymbol{u}) e_{3\beta}(\boldsymbol{v}) d\boldsymbol{x} \\ &+ \int_\Omega \left(h^{-2} C_{\alpha\beta33} + h^{-1} C^\sharp_{\alpha\beta33}\right) \left(e_{33}(\boldsymbol{u}) e_{\alpha\beta}(\boldsymbol{v}) + e_{\alpha\beta}(\boldsymbol{u}) e_{33}(\boldsymbol{v})\right) d\boldsymbol{x} \\ &+ h^{-1} \int_\Omega P_{3\alpha\beta} \left(\partial_3 \bar{\varphi} e_{\alpha\beta}(\boldsymbol{v}) - e_{\alpha\beta}(\boldsymbol{u}) \partial_3 \psi\right) d\boldsymbol{x} \\ &+ 2h^{-1} \int_\Omega P_{\beta3\alpha} \left(\partial_\beta \bar{\varphi} e_{3\alpha}(\boldsymbol{v}) - e_{3\alpha}(\boldsymbol{u}) \partial_\beta \psi\right) d\boldsymbol{x} \\ &+ \int_\Omega \left(C_{3333} + hC^\sharp_{3333}\right) e_{33}(\boldsymbol{u}) e_{33}(\boldsymbol{v}) d\boldsymbol{x} \\ &+ \int_\Omega \varepsilon_{\alpha\beta} \partial_\alpha \bar{\varphi} \partial_\beta \psi \, d\boldsymbol{x} + h \int_\Omega P_{333} (\partial_3 \bar{\varphi} e_{33}(\boldsymbol{v}) - e_{33}(\boldsymbol{u}) \partial_3 \psi) d\boldsymbol{x} \\ &+ h^2 \int_\Omega \varepsilon_{33} \partial_3 \bar{\varphi} \partial_3 \psi \, d\boldsymbol{x} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \hat{l}(h)((\boldsymbol{v}, \psi)) = &-2h^{-1} \int_\Omega P_{\beta3\alpha} \partial_\beta \hat{\varphi} e_{3\alpha}(\boldsymbol{v}) d\boldsymbol{x} \\ &+ \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i d\Gamma \\ &- \int_\Omega \varepsilon_{\alpha\beta} \partial_\alpha \hat{\varphi} \partial_\beta \psi \, d\boldsymbol{x} - h \int_\Omega P_{333} \partial_3 \hat{\varphi} e_{33}(\boldsymbol{v}) d\boldsymbol{x} \\ &- h^2 \int_\Omega \varepsilon_{33} \partial_3 \hat{\varphi} \partial_3 \psi \, d\boldsymbol{x}. \end{aligned} \tag{3.17}$$

The existence and uniqueness of solution of problem (3.15)–(3.17) is consequence of Lax–Milgram Lemma, since $\hat{a}(h)$ is $(\boldsymbol{V} \times \Psi)$-elliptic and both $\hat{a}(h)$ and $\hat{l}(h)$ are $(\boldsymbol{V} \times \Psi)$-continuous, as the reader can easily check having in mind properties (2.4), the boundness of $\boldsymbol{C}, \boldsymbol{C}^\sharp, \boldsymbol{P}, \boldsymbol{\varepsilon}, \boldsymbol{f}$ and $\boldsymbol{g}$, and that $0 < h \le 1$. We now consider the following scalings for the stress tensor $\boldsymbol{\sigma}(h) = (\sigma_{ij}(h))$ and the electric displacement vector $\boldsymbol{D}(h) = (D_i(h))$, valid for all $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \bar{\Omega}$,

$$\begin{cases} \sigma^h_{\alpha\beta}(\boldsymbol{x}^h) = h^2 \sigma_{\alpha\beta}(h)(\boldsymbol{x}), \quad \sigma^h_{3\alpha}(\boldsymbol{x}^h) = h\sigma_{3\alpha}(h)(\boldsymbol{x}), \\ \sigma^h_{33}(\boldsymbol{x}^h) = \sigma_{33}(h)(\boldsymbol{x}), \\ D^h_\alpha(\boldsymbol{x}^h) = D_\alpha(h)(\boldsymbol{x}), \quad D^h_3(\boldsymbol{x}^h) = h^{-1} D_3(h)(\boldsymbol{x}). \end{cases} \tag{3.18}$$

From transformation (3.1) and scalings (3.13)–(3.14) and (3.18), the variational problem (2.13) and the constitutive law (2.3) allow to deduce the following scaled principle of virtual work and scaled constitutive law posed in $\bar{\Omega}$.

Proposition 3.3. *Problem (3.15)–(3.17) is formally equivalent to*

$$\begin{aligned} &\text{Find } (\boldsymbol{u}(h), \varphi(h)) \in \boldsymbol{V}(\Omega) \times H^1(\Omega), \quad \text{such that } \varphi(h) = \hat{\varphi} \text{ on } \Gamma_{eD}, \\ &\int_\Omega \sigma_{ij}(h) e_{ij}(\boldsymbol{v}) d\boldsymbol{x} + \int_\Omega D_k(h) E_k(\psi) d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i d\Gamma, \\ &\text{for all } (\boldsymbol{v}, \psi) \in \boldsymbol{V}(\Omega) \times \Psi(\Omega), \end{aligned} \tag{3.19}$$

where $\boldsymbol{\sigma}(h) = \boldsymbol{\sigma}(h)(\boldsymbol{u}(h), \varphi(h))$ *and* $\boldsymbol{D}(h) = \boldsymbol{D}(h)(\boldsymbol{u}(h), \varphi(h))$ *are given by*

$$\begin{cases}\sigma_{\alpha\beta}(h) = \left(h^{-4}C_{\alpha\beta\theta\rho} + h^{-3}C^{\sharp}_{\alpha\beta\theta\rho}\right)e_{\theta\rho}(\boldsymbol{u}(h)) \\ \quad +\left(h^{-2}C_{\alpha\beta33} + h^{-1}C^{\sharp}_{\alpha\beta33}\right)e_{33}(\boldsymbol{u}(h)) - h^{-1}P_{3\alpha\beta}E_3(\varphi(h)), \\ \sigma_{3\alpha}(h) = 2\left(h^{-2}C_{3\alpha3\beta} + h^{-1}C^{\sharp}_{3\alpha3\beta}\right)e_{3\beta}(\boldsymbol{u}(h)) - h^{-1}P_{\beta3\alpha}E_\beta(\varphi(h)), \\ \sigma_{33}(h) = \left(h^{-2}C_{33\alpha\beta} + h^{-1}C^{\sharp}_{33\alpha\beta}\right)e_{\alpha\beta}(\boldsymbol{u}(h)) \\ \quad +\left(C_{3333} + hC^{\sharp}_{3333}\right)e_{33}(\boldsymbol{u}(h)) - hP_{333}E_3(\varphi(h)), \\ D_\alpha(h) = 2h^{-1}P_{\alpha3\beta}e_{3\beta}(\boldsymbol{u}(h)) + \varepsilon_{\alpha\beta}E_\beta(\varphi(h)), \\ D_3(h) = h^{-1}P_{3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}(h)) + hP_{333}e_{33}(\boldsymbol{u}(h)) + h^2\varepsilon_{33}E_3(\varphi(h)).\end{cases} \tag{3.20}$$

4. Asymptotic method

We now assume that the solution of problem (3.15)–(3.17) can be expressed in the form

$$\boldsymbol{u}(h) = \boldsymbol{u}^0 + h\boldsymbol{u}^1 + h^2\boldsymbol{u}^2 + \cdots, \quad \boldsymbol{u}^i \in \boldsymbol{V}(\Omega), \tag{4.1}$$

$$\bar{\varphi}(h) = \bar{\varphi}^0 + h\bar{\varphi}^1 + h^2\bar{\varphi}^2 + \cdots, \quad \bar{\varphi}^i \in \Psi(\Omega), \tag{4.2}$$

where the successive coefficients of the powers of h are independent of h . Since $\bar{\varphi}(h) = \varphi(h) - \hat{\varphi}$, then (4.2) leads to

$$\varphi(h) = \varphi^0 + h\bar{\varphi}^1 + h^2\bar{\varphi}^2 + \cdots, \tag{4.3}$$

with $\varphi^0 = \bar{\varphi}^0 + \hat{\varphi}$. In view of (3.20), the asymptotic developments (4.1) and (4.3) induce the following formal expansions for tensors $\boldsymbol{\sigma}(h)$ and $\boldsymbol{D}(h)$

$$\begin{aligned}\boldsymbol{\sigma}(h) &= h^{-4}\boldsymbol{\sigma}^{-4} + h^{-3}\boldsymbol{\sigma}^{-3} + h^{-2}\boldsymbol{\sigma}^{-2} + \cdots, \\ \boldsymbol{D}(h) &= h^{-1}\boldsymbol{D}^{-1} + \boldsymbol{D}^0 + h\boldsymbol{D}^1 + \cdots,\end{aligned} \tag{4.4}$$

with $\boldsymbol{\sigma}^q = (\sigma^q_{ij})$ and $\boldsymbol{D}^p = (D^p_j)$ independent of h. Inserting developments (4.1)–(4.4) into problem (3.19) results in a set of variational equations that must be satisfied for all $h > 0$ and consequently the terms at the successive powers of h must be zero. This procedure yields the following problems at the successive powers of h for all $(\boldsymbol{v}, \psi) \in \boldsymbol{V}(\Omega) \times \Psi(\Omega)$:

$$\int_\Omega \sigma^{-4}_{\alpha\beta}e_{\alpha\beta}(\boldsymbol{v})d\boldsymbol{x} = 0, \tag{4.5}$$

$$\int_\Omega \sigma^{-3}_{\alpha\beta}e_{\alpha\beta}(\boldsymbol{v})d\boldsymbol{x} = 0, \tag{4.6}$$

$$\int_\Omega \sigma^{-2}_{ij}e_{ij}(\boldsymbol{v})d\boldsymbol{x} = 0, \tag{4.7}$$

$$\int_\Omega \sigma^{-1}_{ij}e_{ij}(\boldsymbol{v})d\boldsymbol{x} + \int_\Omega D^{-1}_kE_k(\psi)d\boldsymbol{x} = 0, \tag{4.8}$$

$$\int_\Omega \sigma^0_{ij}e_{ij}(\boldsymbol{v})d\boldsymbol{x} + \int_\Omega D^0_kE_k(\psi)d\boldsymbol{x} = \int_\Omega f_iv_id\boldsymbol{x} + \int_{\Gamma_N} g_iv_id\Gamma, \tag{4.9}$$

where

$$\begin{aligned}\sigma^{-4}_{\alpha\beta} &= C_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^0), \\ \sigma^{-3}_{\alpha\beta} &= C_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^1) + C^{\sharp}_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^0),\end{aligned} \tag{4.10}$$

$$\sigma^{-2}_{\alpha\beta} = C_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^2) + C_{\alpha\beta33}e_{33}(\boldsymbol{u}^0) + C^{\sharp}_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^1), \tag{4.11}$$

$$\begin{aligned}\sigma^{-1}_{\alpha\beta} &= C_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^3) + C_{\alpha\beta33}e_{33}(\boldsymbol{u}^1) + C^{\sharp}_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^2) \\ &\quad + C^{\sharp}_{\alpha\beta33}e_{33}(\boldsymbol{u}^0) - P_{3\alpha\beta}E_3(\varphi^0), \\ \sigma^{-2}_{3\alpha} &= 2C_{3\alpha3\beta}e_{3\beta}(\boldsymbol{u}^0), \\ \sigma^{-1}_{3\alpha} &= 2C_{3\alpha3\beta}e_{3\beta}(\boldsymbol{u}^1) + 2C^{\sharp}_{3\alpha3\beta}e_{3\beta}(\boldsymbol{u}^0) - P_{\beta3\alpha}E_\beta(\varphi^0),\end{aligned} \tag{4.12}$$

$$\begin{aligned}\sigma^{-2}_{33} &= C_{33\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^0), \\ \sigma^{-1}_{33} &= C_{33\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^1) + C^{\sharp}_{33\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^0),\end{aligned} \tag{4.13}$$

$$\sigma^0_{33} = C_{33\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^2) + C^{\sharp}_{33\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^1) + C_{3333}e_{33}(\boldsymbol{u}^0), \tag{4.14}$$

$$\begin{aligned}D^{-1}_\alpha &= 2P_{\alpha3\beta}e_{3\beta}(\boldsymbol{u}^0), \\ D^0_\alpha &= 2P_{\alpha3\beta}e_{3\beta}(\boldsymbol{u}^1) + \varepsilon_{\alpha\beta}E_\beta(\varphi^0),\end{aligned} \tag{4.15}$$

$$\begin{aligned}D^{-1}_3 &= P_{3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^0), \\ D^0_3 &= P_{3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^1),\end{aligned} \tag{4.16}$$

$$\begin{aligned}D^1_3 &= P_{3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^2) + P_{333}e_{33}(\boldsymbol{u}^0), \\ D^2_3 &= P_{3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^4) + P_{333}e_{33}(\boldsymbol{u}^3) + \varepsilon_{33}(\varphi^1).\end{aligned} \tag{4.17}$$

4.1. Characterization of $\boldsymbol{u}^0$ and $\boldsymbol{u}^2$

Let $\boldsymbol{V}_{BN}(\Omega)$ be the space of Bernoulli–Navier displacements defined by

$$\boldsymbol{V}_{BN}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}(\Omega) : e_{\alpha\beta}(\boldsymbol{v}) = e_{3\beta}(\boldsymbol{v}) = 0\},$$

equipped with the norm $|\boldsymbol{e}(\boldsymbol{v})|_{0,\Omega}$. The space $\boldsymbol{V}_{BN}(\Omega)$ can be equivalently defined by (see e.g. Trabucho and Viaño, 1996)

$$\begin{aligned}\boldsymbol{V}_{BN}(\Omega) = \{&\boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega) : v_\alpha(x_1, x_2, x_3) = \chi_\alpha(x_3), \\ &\chi_\alpha \in H^2_0(0, L), \quad v_3(x_1, x_2, x_3) = \chi_3(x_3) - x_\beta\chi'_\beta(x_3), \\ &\chi_3 \in H^1_0(0, L)\}.\end{aligned} \tag{4.18}$$

Theorem 4.1. *Let* $\boldsymbol{u}(h)$ *be given by (4.1). The zeroth order term* $\boldsymbol{u}^0$ *is an element of the space of Bernoulli–Navier displacements:*

$$\begin{aligned}u^0_\alpha(x_1, x_2, x_3) &= \xi_\alpha(x_3), \quad u^0_3(x_1, x_2, x_3) = \xi_3(x_3) - x_\beta\xi'_\beta(x_3), \\ &\xi_\alpha \in H^2_0(0, L), \quad \xi_3 \in H^1_0(0, L).\end{aligned} \tag{4.19}$$

Proof. In problem (4.5), we take $\boldsymbol{v} = \boldsymbol{u}^0$, which yields:

$$e_{\alpha\beta}(\boldsymbol{u}^0) = 0, \quad \sigma^{-4}_{\alpha\beta} = \sigma^{-2}_{33} = D^{-1}_3 = 0. \tag{4.20}$$

Applying similar arguments to variational Eq. (4.7), we take $\boldsymbol{v} = \boldsymbol{u}^0$, yielding:

$$e_{3\alpha}(\boldsymbol{u}^0) = 0, \quad \sigma^{-2}_{3\alpha} = D^{-1}_\alpha = 0. \tag{4.21}$$

Taking into account that $e_{\alpha\beta}(\boldsymbol{u}^0) = e_{3\alpha}(\boldsymbol{u}^0) = 0$ will yield (4.19). □

Next, we go back to variational Eq. (4.6). Combining (4.6) with (4.10), using (4.20) and taking $\boldsymbol{v} = \boldsymbol{u}^1$ we get

$$e_{\alpha\beta}(\boldsymbol{u}^1) = 0. \tag{4.22}$$

Moreover, from (4.10), (4.13) and (4.16) we find that

$$\sigma^{-3}_{\alpha\beta} = \sigma^{-1}_{33} = 0, \quad D^0_3 = 0. \tag{4.23}$$

From (4.20) and (4.21), Eq. (4.7) can be written

$$\int_\Omega \sigma^{-2}_{\alpha\beta}e_{\alpha\beta}(\boldsymbol{v})d\boldsymbol{x} = 0, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega). \tag{4.24}$$

In view of (4.11), (4.20) and (4.22) the previous equality becomes

$$\int_\Omega (C_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^2) + C_{\alpha\beta33}e_{33}(\boldsymbol{u}^0))e_{\alpha\beta}(\boldsymbol{v})d\boldsymbol{x} = 0, \text{ for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega).$$

Using the same arguments as in Viaño et al. (2015), we define

$$\boldsymbol{V}^2_m(\omega) = \left\{\hat{\boldsymbol{v}} = (\hat{v}_\alpha) \in [H^1(\omega)]^2 : \int_\omega \hat{v}_\alpha\, d\omega = 0, \right. \\ \left. \int_\omega (x_1\hat{v}_2 - x_2\hat{v}_1)\, d\omega = 0\right\},$$

which is a subspace of $[H^1(\omega)]^2$ where the bi-dimensional Korn's inequality holds. Therefore $|(e_{\alpha\beta}(\hat{\boldsymbol{w}}))|_{0,\Omega}$ is a norm in the space $L^2(0,L;\boldsymbol{V}_m^2(\omega))$. As a consequence, the problem

$$\begin{cases} \text{Find } \hat{\boldsymbol{u}}^2(x_3) \in \boldsymbol{V}_m^2(\omega), \\ \int_\omega (C_{\alpha\beta\theta\rho} e_{\theta\rho}(\hat{\boldsymbol{u}}^2(x_3)) + C_{\alpha\beta33} e_{33}(\boldsymbol{u}^0)(x_3)) e_{\alpha\beta}(\hat{\boldsymbol{w}})\, d\omega = 0 \\ \forall \hat{\boldsymbol{w}} \in \boldsymbol{V}_m^2(\omega), \end{cases} \tag{4.25}$$

has a unique solution for a.e. $x_3 \in (0,L)$. We now define some functions that will be useful in the following results. We consider

$$\Lambda_\alpha(\boldsymbol{x}) = -\frac{\det \boldsymbol{M}_{\alpha\beta}}{\det \boldsymbol{M}} x_\beta,$$

and

$$\Phi_{11}(\boldsymbol{x}) = -\Phi_{22}(\boldsymbol{x}) = \frac{1}{2\det \boldsymbol{M}}\left(\det \boldsymbol{M}_{11} x_1^2 - \det \boldsymbol{M}_{22} x_2^2\right),$$

$$\Phi_{\alpha\beta}(\boldsymbol{x}) = \frac{\det \boldsymbol{M}_{\alpha\theta}}{\det \boldsymbol{M}} x_\theta x_\beta, \quad \alpha \neq \beta,$$

where $\boldsymbol{M}_{\alpha\beta}(x_3)$ are defined as follows:

$$\boldsymbol{M}_{11} = \begin{pmatrix} C_{1133} & C_{1112} & C_{1122} \\ C_{1233} & C_{1212} & C_{1222} \\ C_{2233} & C_{1222} & C_{2222} \end{pmatrix}, \quad \boldsymbol{M}_{22} = \begin{pmatrix} C_{1111} & C_{1112} & C_{1133} \\ C_{1112} & C_{1212} & C_{1233} \\ C_{1122} & C_{1222} & C_{2233} \end{pmatrix},$$

$$\boldsymbol{M}_{12} = \boldsymbol{M}_{21} = \frac{1}{2}\begin{pmatrix} C_{1111} & C_{1133} & C_{1122} \\ C_{1112} & C_{1233} & C_{1222} \\ C_{1122} & C_{2233} & C_{2222} \end{pmatrix}.$$

We also introduce the following geometric constant (see e.g. Trabucho and Viaño, 1996, p. 701)

$$I_\alpha = \int_\omega x_\alpha^2 d\omega$$

and we define, on each cross-section $\omega \times \{x_3\}$, the following functions

$$X_{\alpha\beta} = \int_\omega \Phi_{\alpha\beta} d\omega, \quad Y_\alpha = \int_\omega \Phi_{\alpha\beta}\delta_\beta d\omega, \quad Z = \int_\omega \Lambda_\alpha \delta_\alpha d\omega.$$

Theorem 4.2. *Let $\boldsymbol{u}^0 = (u_i^0) \in \boldsymbol{V}_{BN}(\Omega)$ be given by (4.19). Then, there exists $\tilde{\boldsymbol{u}} = (\tilde{u}_\alpha) \in L^2(0,L;\boldsymbol{V}_m^2(\omega))$ satisfying*

$$C_{\alpha\beta\theta\rho} e_{\theta\rho}(\tilde{\boldsymbol{u}}) + C_{\alpha\beta33} e_{33}(\boldsymbol{u}^0) = 0. \tag{4.26}$$

Moreover,

$$\tilde{u}_\alpha = s_\alpha + \delta_\alpha s + \Phi_{\alpha\beta}\xi_\beta'' + \Lambda_\alpha \xi_3', \tag{4.27}$$

where $s, s_\alpha \in L^2(0,L)$ verify

$$s_\alpha = -\frac{1}{A(\omega)} X_{\alpha\beta}\xi_\beta'', \quad s = -\frac{1}{I_1 + I_2}\left(Y_\beta \xi_\beta'' + Z\xi_3'\right) \quad \text{a.e. in } (0,L). \tag{4.28}$$

Proof. Let $\hat{\boldsymbol{u}}^2(x_3)$ be a particular solution of the equation

$$e_{\theta\rho}(\tilde{\boldsymbol{u}}(x_3)) = -\tilde{C}_{\theta\rho\alpha\beta}(x_3) C_{\alpha\beta33}(x_3) e_{33}(\boldsymbol{u}^0)(x_3), \quad \text{in } \omega \tag{4.29}$$

where $(\tilde{C}_{\theta\rho\alpha\beta})$ is defined as follows:

$$\begin{pmatrix} \tilde{C}_{1111} & 2\tilde{C}_{1112} & \tilde{C}_{1122} \\ \tilde{C}_{1211} & 2\tilde{C}_{1212} & \tilde{C}_{1222} \\ \tilde{C}_{2211} & 2\tilde{C}_{2212} & \tilde{C}_{2222} \end{pmatrix} = \begin{pmatrix} C_{1111} & 2C_{1112} & C_{1122} \\ C_{1211} & 2C_{1212} & C_{1222} \\ C_{2211} & 2C_{2212} & C_{2222} \end{pmatrix}^{-1}.$$

It is easy to show that (4.29) is equivalent to the Equation (4.26). Further, we verify that $\hat{\boldsymbol{u}}^2(x_3)$ also solves (4.25). On the other hand, as $\boldsymbol{u}^0 \in \boldsymbol{V}_{BN}(\Omega)$, Eq. (4.29) can be written in differential form

$$\begin{cases} \partial_1 \tilde{u}_1 = \dfrac{\det \boldsymbol{M}_{11}}{\det \boldsymbol{M}}\left(x_\alpha \xi_\alpha'' - \xi_3'\right), \\ \partial_2 \tilde{u}_2 = \dfrac{\det \boldsymbol{M}_{22}}{\det \boldsymbol{M}}\left(x_\alpha \xi_\alpha'' - \xi_3'\right), \\ \partial_1 \tilde{u}_2 + \partial_2 \tilde{u}_1 = \dfrac{\det \boldsymbol{M}_{12} + \det \boldsymbol{M}_{21}}{\det \boldsymbol{M}}\left(x_\alpha \xi_\alpha'' - \xi_3'\right), \end{cases} \tag{4.30}$$

and, taking into account that the matrix of coefficients $(C_{\alpha\beta\theta\rho})$ depends only on x_3 (see (3.10)), we have, by direct integration of the first two equations,

$$\begin{cases} \tilde{u}_1 = \dfrac{\det \boldsymbol{M}_{11}}{2\det \boldsymbol{M}}(x_1^2\xi_1'' + 2x_1x_2\xi_2'' - 2x_1\xi_3') + k_1(x_2,x_3), \\ \tilde{u}_2 = \dfrac{\det \boldsymbol{M}_{22}}{2\det \boldsymbol{M}}(x_2^2\xi_2'' + 2x_1x_2\xi_1'' - 2x_2\xi_3') + k_2(x_1,x_3). \end{cases} \tag{4.31}$$

Substituting these expressions in the third equation of (4.30), we obtain

$$\frac{\det \boldsymbol{M}_{22}}{\det \boldsymbol{M}} x_2\xi_1'' - \frac{\det \boldsymbol{M}_{12}}{\det \boldsymbol{M}}\left(2x_2\xi_2'' - \xi_3'\right) + \partial_2 k_1(x_2,x_3)$$
$$= -\frac{\det \boldsymbol{M}_{11}}{\det \boldsymbol{M}} x_1\xi_2'' + \frac{\det \boldsymbol{M}_{21}}{\det \boldsymbol{M}}\left(2x_1\xi_1'' - \xi_3'\right) - \partial_1 k_2(x_1,x_3).$$

Consequently, there exists a function s, depending only on x_3, such that

$$\partial_2 k_1(x_2,x_3) = s - \frac{\det \boldsymbol{M}_{22}}{\det \boldsymbol{M}} x_2\xi_1'' + \frac{\det \boldsymbol{M}_{12}}{\det \boldsymbol{M}}\left(2x_2\xi_2'' - \xi_3'\right),$$

$$\partial_1 k_2(x_1,x_3) = -s - \frac{\det \boldsymbol{M}_{11}}{\det \boldsymbol{M}} x_1\xi_2'' + \frac{\det \boldsymbol{M}_{21}}{\det \boldsymbol{M}}\left(2x_1\xi_1'' - \xi_3'\right).$$

Thus,

$$\begin{cases} k_1(x_2,x_3) = s_1 + x_2 s - \dfrac{\det \boldsymbol{M}_{22}}{2\det \boldsymbol{M}} x_2^2\xi_1'' + \dfrac{\det \boldsymbol{M}_{12}}{\det \boldsymbol{M}}\left(x_2^2\xi_2'' - x_2\xi_3'\right), \\ k_2(x_1,x_3) = s_2 - x_1 s - \dfrac{\det \boldsymbol{M}_{11}}{2\det \boldsymbol{M}} x_1^2\xi_2'' + \dfrac{\det \boldsymbol{M}_{21}}{\det \boldsymbol{M}}\left(x_1^2\xi_1'' - x_1\xi_3'\right), \end{cases} \tag{4.32}$$

where s_α are arbitrary functions depending only on x_3. Combining (4.31) with (4.32) leads to (4.27). Using (3.2) and the system (4.31) we have

$$\int_\omega \partial_\alpha \tilde{u}_\alpha d\omega = -\frac{\det \boldsymbol{M}_{\alpha\alpha}}{\det \boldsymbol{M}} A(\omega)\xi_3',$$
$$\int_\omega x_\alpha \partial_\alpha \tilde{u}_\alpha d\omega = \frac{\det \boldsymbol{M}_{\alpha\alpha}}{\det \boldsymbol{M}} I_\alpha \xi_\alpha'', \quad (\text{no sum on } \alpha). \tag{4.33}$$

Since $\tilde{u}_\alpha \in L^2(0,L;H^1(\omega))$, then $\int_\omega \partial_\alpha \tilde{u}_\alpha d\omega$, $\int_\omega x_\alpha \partial_\alpha \tilde{u}_\alpha d\omega$, ξ_α'', $\xi_3' \in L^2(0,L)$. On the other hand, identity (4.27) allows to write

$$\int_\omega \tilde{u}_\alpha d\omega = A(\omega)s_\alpha + X_{\alpha\beta}\xi_\beta'' \quad \text{a.e. in } (0,L), \tag{4.34}$$

$$\int_\omega \delta_\beta \tilde{u}_\beta d\omega = (I_1 + I_2)s + Y_\theta \xi_\theta'' + Z\xi_3' \quad \text{a.e. in } (0,L), \tag{4.35}$$

from which the average conditions required for $\tilde{u}_\alpha$ lead to (4.28). □

Corollary 4.3. *A necessary and sufficient condition for problem (4.26) to have solutions $\boldsymbol{u}^2 \in \boldsymbol{V}(\Omega)$ is that $\boldsymbol{u}^0 \in \boldsymbol{V}_{BN}(\Omega)$ be such that $\xi_\alpha \in H^3(0,L)$, $\xi_3 \in H^2(0,L)$ and that $s, s_\alpha \in H^1(0,L)$. In addition to that, it is verified that $\sigma_{\alpha\beta}^{-2} = 0$.*

Proof. By using (4.26) in (4.25) we find that

$$\int_\omega (C_{\alpha\beta\theta\rho} e_{\theta\rho}(\hat{\boldsymbol{u}}^2 - \tilde{\boldsymbol{u}})) e_{\alpha\beta}(\hat{\boldsymbol{w}})\, d\omega = 0 \quad \forall \hat{\boldsymbol{w}} \in \boldsymbol{V}_m^2(\omega),$$

allowing to conclude that $\hat{\boldsymbol{u}}^2 = \tilde{\boldsymbol{u}}$ and therefore (4.27) is in fact a characterization of u_α^2, the transverse components of the second order displacement $\boldsymbol{u}^2$. Now assume that $\boldsymbol{u}^2 \in \boldsymbol{V}(\Omega)$. Let $u_\alpha^2 \in H^1(\Omega)$. By relations (4.33)–(4.35) we deduce that $\int_\omega \partial_\alpha u_\alpha^2 d\omega$, $\int_\omega x_\alpha \partial_\alpha u_\alpha^2 d\omega$, s,

$s_\alpha \in H^1(0,L)$. Furthermore, the boundary condition for u^2_α (cf. (3.3)) leads to (4.28) for $x_3 = 0$ and $x_3 = L$. Conversely, taking $\boldsymbol{u}^0 \in \boldsymbol{V}_{BN}(\Omega)$ such that $\xi_\alpha \in H^3(0,L)$, $\xi_3 \in H^2(0,L)$, and s, $s_\alpha \in H^1(0,L)$ now of the form (4.28) for all $x_3 \in (0,L)$, then (4.27) implies that $u^2_\alpha \in H^1(\Omega)$ verifies the boundary condition (3.3). Finally, from (4.11), (4.22) and (4.26) we conclude that $\sigma^{-2}_{\alpha\beta} = 0$. □

Remark 4.4. From (4.27) it follows that in general u^2_α does not vanish at the rod ends, for that would require that both ξ''_β and ξ'_3 vanish there. Therefore, if we had chosen strong boundary conditions in the definition of $\boldsymbol{V}(\Omega)$, then we would not be able to ensure that $\boldsymbol{u}^2 \in \boldsymbol{V}(\Omega)$ (the well-known " boundary layer phenomenon") and consequently that $\sigma^{-2}_{\alpha\beta} = 0$.

As a result of the previous discussion, we will be able to obtain variational problems that have ξ_α and ξ_3 as unique solutions.

Theorem 4.5. *It is verified that $\boldsymbol{u}^0 \in \boldsymbol{V}_{BN}(\Omega)$ is the unique solution of the following variational problem:*

$$\int_\Omega \sigma^0_{33} e_{33}(\boldsymbol{v})d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i d\boldsymbol{\Gamma}, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}_{BN}(\Omega), \tag{4.36}$$

with σ^0_{33} given by

$$\sigma^0_{33} = Y e_{33}(\boldsymbol{u}^0) = Y(\xi'_3 - x_\alpha \xi''_\alpha), \tag{4.37}$$

where we can define the Young Modulus for the limit equations as

$$Y = \frac{\det \bar{\boldsymbol{C}}}{\det \boldsymbol{M} \det \boldsymbol{N}}. \tag{4.38}$$

Notice that it is positive and bounded (cf. (2.7) and (2.8)).

Proof. Problem (4.36) follows from variational Eq. (4.9) taking $\boldsymbol{v} \in \boldsymbol{V}_{BN}(\Omega)$ and $\psi = 0$. Furthermore, after some calculations (4.14) becomes (4.37). □

Corollary 4.6. *The transverse components ξ_α and the stretch component ξ_3 of the zeroth order displacement field solve respectively the following variational problems (no sum on α):*

Find $\xi_\alpha \in H^2_0(0,L)$ such that

$$\int_0^L Y I_\alpha \xi''_\alpha \chi''_\alpha dx_3 = \int_0^L F_\alpha \chi_\alpha dx_3 \tag{4.39}$$

$$- \int_0^L M_\alpha \chi'_\alpha dx_3, \quad \text{for all} \quad \chi_\alpha \in H^2_0(0,L), \tag{4.39}$$

and

Find $\xi_3 \in H^1_0(0,L)$ such that

$$\int_0^L YA(\omega)\xi'_3\chi'_3 dx_3 = \int_0^L F_3\chi_3 dx_3, \quad \text{for all} \quad \chi_3 \in H^1_0(0,L), \tag{4.40}$$

with the resultant forces and moments given by:

$$F_i(x_3) = \int_\omega f_i(x_1,x_2,x_3)d\omega + \int_{\partial\omega} g_i(x_1,x_2,x_3)d\gamma \in L^2(0,L), \tag{4.41}$$

$$M_\alpha(x_3) = \int_\omega x_\alpha f_3(x_1,x_2,x_3)d\omega + \int_{\partial\omega} x_\alpha g_3(x_1,x_2,x_3)d\gamma \in L^2(0,L). \tag{4.42}$$

Proof. We consider (4.36) with the test function $\boldsymbol{v} \in \boldsymbol{V}_{BN}(\Omega)$ of the form (4.18), implying $e_{33}(\boldsymbol{v}) = \chi'_3(x_3) - x_\theta \chi''_\theta(x_3)$. Then, taking successively $\chi_\theta = 0$ and $\chi_3 = 0$ in the resulting equation we obtain (4.39) and (4.40), respectively. □

Corollary 4.7. *The component D^1_3 of $\boldsymbol{D}^1$ is given by*

$$D^1_3 = \bar{P}e_{33}(\boldsymbol{u}^0) = \bar{P}\left(\xi'_3 - x_\alpha\xi''_\alpha\right) \in L^2(\Omega),$$

with

$$\bar{P} = P_{333} - \frac{\det \boldsymbol{M}_{\alpha\beta}}{\det \boldsymbol{M}} P_{3\alpha\beta}. \tag{4.43}$$

Proof. This result follows straightforwardly from (4.17) taking into account that $\boldsymbol{u}^2$ verifies (4.26). □

Remark 4.8. In the case of taking assumption (3.12) instead of (3.10), we can still formulate the problem (4.25) (the coefficients $C_{\alpha\beta\theta\rho}$ now depending also on x_1, x_2) which has an unique solution in $\boldsymbol{V}^2_m(\omega)$. The difference is that we cannot give explicitly $\boldsymbol{u}^2$ in terms of $\boldsymbol{u}^0$ as in (4.27). Following Theorem 3.1 in Girault and Raviart (1986), we have to introduce two auxiliary functions $\theta_\alpha \in H^1(\omega)$ a.e. in $(0,L)$ and

$$e_{\theta\rho}(\hat{\boldsymbol{u}}^2) = -\tilde{C}_{\theta\rho\alpha\beta}(C_{\alpha\beta 33}e_{33}(\boldsymbol{u}^0) + \Sigma_{\theta\rho}) \ \text{ in } \ \omega,$$

where

$$\Sigma_{\alpha\beta} = \begin{pmatrix} \partial_2\theta_1 & -\partial_1\theta_1 \\ \partial_2\theta_2 & -\partial_1\theta_2 \end{pmatrix}, \quad \partial_1\theta_1 = -\partial_2\theta_2.$$

In these conditions, the limit problems cannot be fully determined (see Figueiredo and Leal, 2006).

4.2. Characterization of $\boldsymbol{u}^1$ and φ^0

We start this section by introducing the space $\boldsymbol{V}_1(\Omega)$ formed by all the infinitesimal rigid displacements of the set Ω in the transversal directions:

$$\boldsymbol{V}_1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}(\Omega) : e_{\alpha\beta}(\boldsymbol{v}) = 0\}.$$

In Trabucho and Viaño (1996) it is proved that $\boldsymbol{V}_1(\Omega)$ coincides with the following space :

$$\boldsymbol{V}_1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}(\Omega) : v_\alpha(x_1,x_2,x_3) = \zeta_\alpha(x_3) + \delta_\alpha\zeta(x_3), \quad \zeta_\alpha, \zeta \in H^1_0(0,L)\}.$$

We also consider the spaces

$$Q = Q(\omega) = \left\{\rho \in H^1(\omega) : \int_\omega \rho \, d\omega = 0\right\},$$
$$S = S(\omega) = \{\psi \in H^1(\omega) : \psi = 0 \text{ on } \gamma_{eD}\}, \tag{4.44}$$

equipped with the (semi-)norms

$$\|\rho\|_Q = \left(\|\partial_1\rho\|^2_{0,\omega} + \|\partial_2\rho\|^2_{0,\omega}\right)^{1/2}, \quad \text{for all } \rho \in Q,$$
$$\|\psi\|_S = \left(\|\partial_1\psi\|^2_{0,\omega} + \|\partial_2\psi\|^2_{0,\omega}\right)^{1/2}, \quad \text{for all } \psi \in S.$$

With the aid of Poincaré–Friedrichs' inequality and Poincaré's inequality in ω, it can be shown that these norms are equivalent to the usual H^1-norm. Also, let $\Psi_l(\Omega)$ be the space

$$\Psi_l(\Omega) = \{\psi \in L^2(\Omega) : \partial_\alpha\psi \in L^2(\omega)\} \equiv L^2(0,L;H^1(\omega)), \tag{4.45}$$

and let us define its subspaces $R(\Omega) = L^2(0,L;Q(\omega))$ and $\Psi_{l0}(\Omega) = L^2(0,L;S(\omega))$. These spaces are equipped with the (semi-)norms

$$\|\psi\|_{\Psi_{l0}} = \left(\|\partial_1\psi\|^2_{0,\Omega} + \|\partial_2\psi\|^2_{0,\Omega}\right)^{1/2}, \quad \text{for all } \psi \in \Psi_{l0}(\Omega),$$
$$\|\rho\|_R = \left(\|\partial_1\rho\|^2_{0,\Omega} + |\partial_2\rho\|^2_{0,\Omega}\right)^{1/2}, \quad \text{for all } \rho \in R(\Omega).$$

Again, it can be shown that these norms are equivalent to the usual H^1-norm.

We now define the warping function $w \in R(\Omega)$ as the unique solution of the variational problem (see e.g. Trabucho and Viaño, 1996, p. 700)

$$w(x_3) \in Q(\omega) \text{ such that a.e. in} (0, L): \\ \int_\omega C_{3\alpha 3\beta} \partial_\beta w \partial_\alpha \upsilon d\omega = \int_\omega C_{3\alpha 3\beta} \delta_\beta \partial_\alpha \upsilon d\omega, \quad \text{for all } \upsilon \in H^1(\omega), \tag{4.46}$$

as well as the torsion function $J \in L^2(0, L)$, given by

$$J = \int_\omega C_{3\alpha 3\beta}(\delta_\beta - \partial_\beta w)(\delta_\alpha - \partial_\alpha w) d\omega \\ = \int_\omega C_{3\alpha 3\beta}(\delta_\beta - \partial_\beta w)\delta_\alpha d\omega > 0. \tag{4.47}$$

Finally, let $r \in R(\Omega)$ satisfy the following variational problem,

$$r(x_3) \in Q(\omega) \text{ such that for a.e.} x_3 \in (0, L): \\ \int_\omega C_{3\alpha 3\beta} \partial_\beta r(x_3) \partial_\alpha \rho d\omega = \int_\omega P_{\beta 3\alpha} \partial_\beta \varphi^0(x_3) \partial_\alpha \rho d\omega, \\ \text{for all } \rho \in Q(\omega). \tag{4.48}$$

Bearing in mind the previous definitions, we turn to the characterization of $\boldsymbol{u}^1$ and φ^0. In (4.22), we obtained that $e_{\alpha\beta}(\boldsymbol{u}^1) = 0$. Therefore, we can characterize the transverse components of $\boldsymbol{u}^1$ as follows (see e.g. Trabucho and Viaño, 1996)

$$u^1_\alpha = z_\alpha + \delta_\alpha z, \quad z_\alpha, z \in H^1_0(0, L). \tag{4.49}$$

Moreover, by taking $\boldsymbol{v} \in \boldsymbol{V}_1(\Omega)$ in (4.8) and $\boldsymbol{v} = \boldsymbol{0}$ in (4.9), and considering (4.12), (4.15) and (4.23) we get, respectively,

$$\int_\Omega \left(2C_{3\alpha 3\beta} e_{3\beta}(\boldsymbol{u}^1) + P_{\beta 3\alpha} \partial_\beta \varphi^0\right) e_{3\alpha}(\boldsymbol{v}) d\boldsymbol{x} = 0, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}_1(\Omega), \tag{4.50}$$

$$\int_\Omega \left(2P_{\beta 3\alpha} e_{3\alpha}(\boldsymbol{u}^1) - \varepsilon_{\alpha\beta} \partial_\alpha \varphi^0\right) \partial_\beta \psi d\boldsymbol{x} = 0, \quad \text{for all } \psi \in \Psi(\Omega), \tag{4.51}$$

which shows that $\boldsymbol{u}^1$ and φ^0 are coupled. In fact, we have the following result.

Theorem 4.9. *Let $\hat{\boldsymbol{u}} = (\hat{u}_i) \in [L^2(\Omega)]^3$ be of the form (4.49) and $\varphi^0 \in \Psi_l(\Omega)$. Then $\hat{\boldsymbol{u}}$ is a solution of (4.50) if and only if*

$$\hat{u}_3 = -r - z_3 - x_\alpha z'_\alpha - wz', \tag{4.52}$$

where $r \in R(\Omega)$ solves (4.48), w is the warping function and $z_3 \in L^2(0, L)$ is an arbitrary function of x_3.

Proof. Assuming that $\hat{\boldsymbol{u}} = (\hat{u}_i)$ is a solution of (4.50) and in view of (4.49) we conclude that $\hat{u}_3$ verifies

$$\int_\Omega (C_{3\alpha 3\beta}(\partial_\beta \hat{u}_3 + z'_\beta + \delta_\beta z') + P_{\beta 3\alpha} \partial_\beta \varphi^0) e_{3\alpha}(\boldsymbol{v}) d\boldsymbol{x} = 0, \\ \text{for all } \boldsymbol{v} \in \boldsymbol{V}_1(\Omega). \tag{4.53}$$

Choosing $\boldsymbol{v} = (0, 0, \chi(x_3)\upsilon(x_1, x_2))$, $\chi \in H^1_0(0, L)$, $\upsilon \in Q(\omega)$ in (4.53) and taking (4.46) into account we get *a.e.* $x_3 \in (0, L)$

$$\int_\omega C_{3\alpha 3\beta} \partial_\beta \hat{u}_3(x_3) \partial_\alpha \upsilon d\omega \\ = -\int_\omega C_{3\alpha 3\beta} \partial_\beta (x_\theta z'_\theta(x_3) + wz'(x_3)) \partial_\alpha \upsilon d\omega \\ - \int_\omega P_{\beta 3\alpha} \partial_\beta \varphi^0(x_3) \partial_\alpha \upsilon d\omega, \quad \text{for all } \upsilon \in Q(\omega).$$

Therefore,

$$U = \hat{u}_3 + x_\alpha z'_\alpha + wz' \tag{4.54}$$

is a solution a.e. $x_3 \in (0, L)$ of

$$\int_\omega C_{3\alpha 3\beta} \partial_\beta U(x_3) \partial_\alpha \upsilon d\omega \\ = -\int_\omega P_{\beta 3\alpha} \partial_\beta \varphi^0(x_3) \partial_\alpha \upsilon d\omega, \quad \text{for all } \upsilon \in Q(\omega). \tag{4.55}$$

Hence, if r is a solution of (4.48) then

$$U = -r - z_3, \quad \text{a.e. } x_3 \in (0, L),$$

where z_3 is an arbitrary function of x_3, leading to (4.52). From (4.52) we conclude that since $r \in L^2(0, L; H^1(\omega))$, $z_\alpha, z \in H^1_0(0, L)$ and $w \in H^1(\omega)$, then $z_3 \in L^2(0, L)$.

The converse result is immediate. □

Corollary 4.10. *Let u^1_α, the transverse component of $\boldsymbol{u}^1$, be of the form (4.49) and let $\boldsymbol{u}^1$ verify (4.50). Then the axial component u^1_3 is such that*

$$u^1_3 = -r - z_3 - x_\alpha z'_\alpha - wz', \tag{4.56}$$

where $z_3 \in L^2(0, L)$. Furthermore, (4.15) becomes

$$D^0_\alpha = -P_{\alpha 3\beta}(\partial_\beta r + z'(\partial_\beta w - \delta_\beta)) - \varepsilon_{\alpha\beta} \partial_\beta \varphi^0.$$

In view of the previous regularity result we conclude that although (4.49) implies that $u^1_\alpha \in H^1(\Omega)$, we can only ensure that $u^1_3 \in L^2(0, L; H^1(\omega))$. Furthermore, $\int_\omega u^1_3 d\omega$ vanishes at the rod ends as long as $z_3 \in H^1_0(0, L)$, but in general this is not the case for $\int_\omega x_\alpha u^1_3 d\omega$. Therefore, we cannot ensure that $\boldsymbol{u}^1 \in \boldsymbol{V}(\Omega)$, traducing a " boundary layer phenomenon" . On the other hand, the regularity result obtained for u^1_3 implies that $U \in L^2(0, L; H^1(\omega))$ - cf. (4.54) - and therefore from (4.55) we can only guarantee that φ^0 is in $\Psi_l(\Omega)$, not in $\Psi(\Omega)$. Hence, $\bar{\varphi}^0 = \varphi^0 - \hat{\varphi} \in \Psi_{l0}(\Omega)$.

Nevertheless, even when $\boldsymbol{u}^1 \notin \boldsymbol{V}(\Omega)$, the characterization of $\boldsymbol{u}^1$ done so far still holds, in particular as far as Eqs. (4.49) and (4.56) are concerned. This will allow to use Eqs. (4.50) and (4.51) to derive variational problems for $\bar{\varphi}^0$.

4.2.1. Variational problem for $\bar{\varphi}^0$, r and z

Theorem 4.11. *Element $z \in H^1_0(0, L)$ satisfies the variational problem*

$$z \in H^1_0(0, L), \int_0^L Jz'\zeta' dx_3 = \int_0^L \left(\int_\omega P_{\beta 3\alpha}(\partial_\alpha w - \delta_\alpha) \partial_\beta \varphi^0 d\omega\right) \zeta' dx_3, \\ \forall \zeta \in H^1_0(0, L). \tag{4.57}$$

Proof. We consider (4.49) and (4.56), yielding

$$2e_{3\alpha}(\boldsymbol{u}^1) = -\partial_\alpha r - z'(\partial_\alpha w - \delta_\alpha), \tag{4.58}$$

which plugged into (4.50) leads to

$$\int_\Omega (C_{3\alpha 3\beta}(z'(\delta_\beta - \partial_\beta w) - \partial_\beta r) + P_{\beta 3\alpha} \partial_\beta \varphi^0) e_{3\alpha}(\boldsymbol{v}) d\boldsymbol{x} = 0, \\ \text{for all } \boldsymbol{v} \in \boldsymbol{V}_1(\Omega).$$

Taking $\boldsymbol{v} = (\chi(x_3)x_2, -\chi(x_3)x_1, 0)$, $\chi \in H^1_0(0, L)$ in the previous equation we get

$$\int_\Omega (C_{3\alpha 3\beta}(z'(\delta_\beta - \partial_\beta w) - \partial_\beta r) + P_{\beta 3\alpha} \partial_\beta \varphi^0) \delta_\alpha \chi' d\boldsymbol{x} = 0, \tag{4.59}$$

that is (cf. (4.47)),

$$\frac{d}{dx_3}(Jz') = \frac{d}{dx_3} \int_\omega (C_{3\alpha 3\beta} \partial_\beta r - P_{\beta 3\alpha} \partial_\beta \varphi^0) \delta_\alpha d\omega, \quad \text{a.e. } x_3 \in (0, L),$$

or, in view of (4.46) and (4.48),

$$\frac{d}{dx_3}(Jz') = \frac{d}{dx_3} \int_\omega P_{\beta 3\alpha}(\partial_\alpha w - \delta_\alpha) \partial_\beta \varphi^0 d\omega, \quad \text{a.e. } x_3 \in (0, L). \tag{4.60}$$

Finally, from (4.60) we also obtain that z satisfies (4.57). We note that if φ^0 is given, then the solution of (4.57) is unique since J is bounded and positive. □

Remark 4.12. From the previous results we conclude that in general it is not possible to obtain the elasticity result $z = 0$ (see e.g. Álvarez-Dios and Viaño, 1993a; 1993b; Trabucho and Viaño, 1996). However, we can prove that this evidence happens when we take $\varphi^0 = \varphi^0(x_3)$.

Note that this implies $\varphi^0 = \hat{\varphi}$ and is only possible if $\hat{\varphi} = \hat{\varphi}(x_3)$. In this case, Eq. (4.59) takes the form

$$\int_0^L \Big(\int_\omega C_{3\alpha 3\beta}[z'(\delta_\beta - \partial_\beta w) - \partial_\beta r]\delta_\alpha d\omega \Big)\chi' dx_3 = 0,\ \forall \chi \in H_0^1(0, L),$$

and, by a density argument, we deduce

$$z' \int_\omega C_{3\alpha 3\beta}(\delta_\beta - \partial_\beta w)\delta_\alpha d\omega - \int_\omega C_{3\alpha 3\beta}\partial_\beta r \delta_\alpha d\omega = 0.$$

From (4.46)–(4.48), the previous relation becomes

$$Jz' = \int_\Omega C_{3\alpha 3\beta}\partial_\beta r \partial_\alpha w d\omega = \int_\Omega P_{\beta 3\alpha}\partial_\alpha w \partial_\beta \varphi^0 d\omega = 0,$$

and, therefore, we deduce that $Jz' = 0$ with $J > 0$. Thus, $z' = 0$ and since $z \in H_0^1(0, L)$ one gets $z = 0$ a.e in $(0, L)$.

Let us define the space

$$\boldsymbol{T}(\Omega) = \Psi_{l0}(\Omega) \times R(\Omega) \times H_0^1(0, L), \tag{4.61}$$

equipped with the norm

$$\|(\psi, \rho, \varsigma)\|^2_{\boldsymbol{T}(\Omega)} = |\partial_1\psi|^2_{0,\Omega} + |\partial_2\psi|^2_{0,\Omega} + |\partial_1\rho|^2_{0,\Omega} + |\partial_2\rho|^2_{0,\Omega} + |\varsigma'|^2_{0,(0,L)}. \tag{4.62}$$

Theorem 4.13. *The element $(\bar{\varphi}^0, r, z) \in \boldsymbol{T}(\Omega)$ is the unique solution of the following variational problem*

Find $\big(\bar{\varphi}^0, r, z\big) \in \boldsymbol{T}(\Omega)$ such that

$$\widetilde{a}((\bar{\varphi}^0, r, z), (\widetilde{\psi}, \widetilde{\rho}, \varsigma)) = \widetilde{l}(\widetilde{\psi}, \widetilde{\rho}, \varsigma),\ \text{for all } (\widetilde{\psi}, \widetilde{\rho}, \varsigma) \in \boldsymbol{T}(\Omega), \tag{4.63}$$

where the bilinear form $\widetilde{a}$ and the linear functional $\widetilde{l}$ are given by

$$\widetilde{a}((\bar{\varphi}^0, r, z), (\psi, \rho, \varsigma)) = \int_0^L Jz'\varsigma' dx_3 + \int_\Omega C_{3\alpha 3\beta}\partial_\beta r \partial_\alpha \rho d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta}\partial_\alpha\bar{\varphi}^0\partial_\beta\psi d\boldsymbol{x} + \int_\Omega P_{\beta 3\alpha}(\partial_\alpha r \partial_\beta\psi - \partial_\beta\bar{\varphi}^0\partial_\alpha\rho)d\boldsymbol{x} + \int_\Omega P_{\beta 3\alpha}(\partial_\alpha w - \delta_\alpha)(z'\partial_\beta\psi - \varsigma'\partial_\beta\bar{\varphi}^0)d\boldsymbol{x} \tag{4.64}$$

and

$$\widetilde{l}(\psi, \rho, \varsigma) = \int_\Omega P_{\beta 3\alpha}\partial_\beta\hat{\varphi}(\partial_\alpha\rho + \zeta'(\partial_\alpha w - \delta_\alpha))d\boldsymbol{x} - \int_\Omega \varepsilon_{\alpha\beta}\partial_\beta\hat{\varphi}\partial_\alpha\psi d\boldsymbol{x}, \tag{4.65}$$

respectively.

Proof. We consider the variational problem (4.51) and write it as (cf. (4.58))

$$\int_\Omega \big(P_{\beta 3\alpha}(\partial_\alpha r + z'(\partial_\alpha w - \delta_\alpha)) + \varepsilon_{\alpha\beta}\partial_\alpha\bar{\varphi}^0\big)\partial_\beta\psi d\boldsymbol{x} = -\int_\Omega \varepsilon_{\alpha\beta}\partial_\alpha\hat{\varphi}\partial_\beta\psi d\boldsymbol{x}\ \forall \psi \in \Psi(\Omega). \tag{4.66}$$

Combining (4.66) with (4.48) and (4.57), and taking into account that $\bar{\varphi}^0 = \varphi^0 - \hat{\varphi}$, we conclude that $\big(\bar{\varphi}^0, r, z\big) \in \boldsymbol{T}(\Omega)$ verifies:

$$\int_0^L Jz'\zeta' dx_3 + \int_\Omega C_{3\alpha 3\beta}\partial_\beta r \partial_\alpha\rho d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta}\partial_\beta\bar{\varphi}^0\partial_\alpha\psi d\boldsymbol{x} + \int_\Omega P_{\beta 3\alpha}(\partial_\alpha r\partial_\beta\psi - \partial_\beta\bar{\varphi}^0\partial_\alpha\rho)d\boldsymbol{x} + \int_\Omega P_{\beta 3\alpha}(\partial_\alpha w - \delta_\alpha)(z'\partial_\beta\psi - \partial_\beta\bar{\varphi}^0\zeta')d\boldsymbol{x} = \int_\Omega P_{\beta 3\alpha}\partial_\beta\hat{\varphi}(\partial_\alpha\rho + \zeta'(\partial_\alpha w - \delta_\alpha))d\boldsymbol{x} - \int_\Omega \varepsilon_{\alpha\beta}\partial_\beta\hat{\varphi}\partial_\alpha\psi d\boldsymbol{x},$$

for all $\psi \in \Psi(\Omega),\ \rho \in R(\Omega),\ \zeta \in H_0^1(0, L)$.

Since $\Psi(\Omega)$ is dense in $\Psi_{l0}(\Omega)$, we obtain (4.63)–(4.65). One can easily show that both $\widetilde{a}$ and $\widetilde{l}$ are $\boldsymbol{T}$-continuous. The coercivity of $\widetilde{a}$ is shown as follows. From (4.64) one has

$$\widetilde{a}((\psi, \rho, \varsigma), (\psi, \rho, \varsigma)) = \int_0^L J(\varsigma')^2 dx_3 + \int_\Omega C_{3\alpha 3\beta}\partial_\beta\rho\partial_\alpha\rho d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi d\boldsymbol{x},$$

for all $(\psi,\ \rho,\ \varsigma) \in \boldsymbol{T}(\Omega)$. On the other hand, since J is positive and bounded, we conclude that there exists a constant $c_1 > 0$ such that

$$\int_0^L J(\varsigma')^2 dx_3 > c_1|\varsigma'|^2_{0,(0,L)}, \quad \text{for all } \varsigma \in H_0^1(0, L).$$

Bearing in mind properties (2.4), there exists constants $c_2, c_3 > 0$ such that

$$\int_\Omega C_{3\alpha 3\beta}\partial_\beta\rho\partial_\alpha\rho d\boldsymbol{x} \geq c_2 \int_\Omega ((\partial_1\rho)^2 + (\partial_2\rho)^2)d\boldsymbol{x} = c_2(|\partial_1\rho|^2_{0,\Omega} + |\partial_2\rho|^2_{0,\Omega}), \quad \forall \rho \in R(\Omega),$$

$$\int_\Omega \varepsilon_{\alpha\beta}\partial_\beta\psi\partial_\alpha\psi d\boldsymbol{x} \geq c_3 \int_\Omega ((\partial_1\psi)^2 + (\partial_2\psi)^2)d\boldsymbol{x} = c_3(|\partial_1\psi|^2_{0,\Omega} + |\partial_2\psi|^2_{0,\Omega}), \quad \forall \psi \in \Psi_{l0}(\Omega).$$

Thus, there exists a constant $c_4 > 0$ such that (cf. (4.62))

$$\widetilde{a}((\psi, \rho, \varsigma), (\psi, \rho, \varsigma)) \geq c_4\|(\psi, \rho, \varsigma)\|^2_T,$$

that is, $\widetilde{a}$ is $\boldsymbol{T}$-elliptic. Hence, the uniqueness of solution of the variational problem (4.63)–(4.65) follows from Lax–Milgram Lemma. □

5. Convergence analysis

In this section we will provide a strong convergence result. Since we will need a mixed formulation to prove some results, in this section we have to invert the constitutive laws (2.3). Therefore, in order to simplify the exposition we assume throughout this section that in (3.10) we have $C^\sharp_{ijkl} = 0$.

We start with the scaled variational problem associated to the primal variational problem (3.15)–(3.17): Find $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h))) \in \boldsymbol{M}(\Omega) \times \hat{\boldsymbol{X}}(\Omega)$ such that

$$a_H((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) + b_H((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}(h), \varphi(h))) = 0, \quad \forall(\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{M}(\Omega), \tag{5.1}$$

$$b_H((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{v}, \psi)) = l_H(\boldsymbol{v}, \psi), \quad \forall(\boldsymbol{v}, \psi) \in \boldsymbol{X}(\Omega), \tag{5.2}$$

where

$$\boldsymbol{X}(\Omega) = \boldsymbol{V}(\Omega) \times \Psi(\Omega), \quad \hat{\boldsymbol{X}}(\Omega) = \boldsymbol{V}(\Omega) \times \hat{\Psi}(\Omega),$$
$$\boldsymbol{M}(\Omega) = \boldsymbol{M}_1(\Omega) \times \boldsymbol{M}_2(\Omega),$$
$$\boldsymbol{M}_1(\Omega) = \big\{\boldsymbol{\tau} = (\tau_{ij}) \in [L^2(\Omega)]^9 : \tau_{ij} = \tau_{ji}\big\}, \quad \boldsymbol{M}_2(\Omega) = [L^2(\Omega)]^3,$$
$$\hat{\Psi}(\Omega) = \hat{\varphi} + \Psi(\Omega) = \{\psi \in \Psi(\Omega) : \psi = \hat{\varphi} \text{ on } \Gamma_{eD}\},$$

and the forms a_H, l_H and b_H are defined as follows:

$$a_H((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) = h^{-2}a_{-2}((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) + h^{-1}a_{-1}((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) + a_0((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) + ha_1((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) + h^2a_2((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) + h^4a_4((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})),$$

$$l_H(\boldsymbol{v}, \psi) = -\int_\Omega f_i v_i d\boldsymbol{x} - \int_{\Gamma_{dN}} g_i v_i d\Gamma,$$

$$b_H((\boldsymbol{\sigma}, \boldsymbol{D}), (\boldsymbol{v}, \psi)) = -\int_\Omega \sigma_{ij}e_{ij}(\boldsymbol{v})d\boldsymbol{x} - \int_\Omega D_k E_k(\psi)d\boldsymbol{x},$$

with

$$a_{-2}((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})) = \int_\Omega \bar{\varepsilon}_{33}D_3 d_3 d\boldsymbol{x},$$
$$a_{-1}((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})) = \int_\Omega \bar{P}_{333}D_3\tau_{33}d\boldsymbol{x} - \int_\Omega \bar{P}_{333}\sigma_{33}d_3 d\boldsymbol{x},$$
$$a_0((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})) = \int_\Omega \bar{C}_{3333}\sigma_{33}\tau_{33}d\boldsymbol{x} + \int_\Omega \bar{\varepsilon}_{\alpha\beta}D_\beta d_\alpha d\boldsymbol{x},$$
$$a_1((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})) = \int_\Omega \bar{P}_{3\alpha\beta}D_3\tau_{\alpha\beta}d\boldsymbol{x} - \int_\Omega \bar{P}_{3\alpha\beta}\sigma_{\alpha\beta}d_3 d\boldsymbol{x}$$
$$- 2\int_\Omega \bar{P}_{\alpha 3\theta}\sigma_{3\theta}d_\alpha d\boldsymbol{x} + 2\int_\Omega \bar{P}_{\beta 3\alpha}D_\beta \tau_{3\alpha}d\boldsymbol{x},$$
$$a_2((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})) = \int_\Omega \bar{C}_{33\alpha\beta}\sigma_{\alpha\beta}\tau_{33}d\boldsymbol{x} + 4\int_\Omega \bar{C}_{3\alpha 3\beta}\sigma_{3\beta}\tau_{3\alpha}d\boldsymbol{x}$$
$$+ \int_\Omega \bar{C}_{\alpha\beta 33}\sigma_{33}\tau_{\alpha\beta}d\boldsymbol{x},$$
$$a_4((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})) = \int_\Omega \bar{C}_{\alpha\beta\theta\rho}\sigma_{\theta\rho}\tau_{\alpha\beta}d\boldsymbol{x}, \tag{5.3}$$

where $\bar{\boldsymbol{C}} = (\bar{C}_{ijkl})$, $\bar{\boldsymbol{P}} = (\bar{P}_{kij})$, $\bar{\boldsymbol{\varepsilon}} = (\bar{\varepsilon}_{ij})$ are inverse constitutive tensors of $\boldsymbol{C} = (C_{ijkl})$, $\boldsymbol{P} = (P_{kij})$ and $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ and inherit their properties of ellipticity and boundedness (see e.g. Viaño et al., 2015). We also define the following auxiliary tensor $(\tilde{e}_{ij}(h))$ and vector $(\tilde{E}_i(h))$:

$$\tilde{e}_{\alpha\beta}(h) := h^{-2}e_{\alpha\beta}(\boldsymbol{u}(h))(\boldsymbol{x}), \quad \tilde{e}_{3\beta}(h) := h^{-1}e_{3\beta}(\boldsymbol{u}(h))(\boldsymbol{x}),$$
$$\tilde{e}_{33}(h) := e_{33}(\boldsymbol{u}(h))(\boldsymbol{x}),$$
$$\tilde{E}_\alpha(h) := E_\alpha(\varphi(h))(\boldsymbol{x}), \quad \tilde{E}_3(h) := hE_3(\varphi(h))(\boldsymbol{x}),$$

verifying the constitutive equations (cf. (3.20))

$$S_{\alpha\beta}(h) := h^2\sigma_{\alpha\beta}(h) = C_{\alpha\beta\theta\rho}\tilde{e}_{\theta\rho}(h) + C_{\alpha\beta 33}\tilde{e}_{33}(h) - P_{3\alpha\beta}\tilde{E}_3(h), \tag{5.4}$$

$$S_{3\alpha}(h) := h\sigma_{3\alpha}(h) = 2C_{\alpha 33\theta}\tilde{e}_{3\theta}(h) - P_{\theta\alpha 3}\tilde{E}_\theta(h),$$
$$S_{33}(h) := \sigma_{33}(h) = C_{33\theta\rho}\tilde{e}_{\theta\rho}(h) + C_{3333}\tilde{e}_{33}(h) - P_{333}\tilde{E}_3(h),$$
$$T_\theta(h) := D_\theta(h) = 2P_{\theta 3\alpha}\tilde{e}_{3\alpha}(h) + \varepsilon_{\theta\alpha}\tilde{E}_\alpha(h),$$
$$T_3(h) := h^{-1}D_3(h) = P_{3\alpha\beta}\tilde{e}_{\alpha\beta}(h) + P_{333}\tilde{e}_{33}(h) + \varepsilon_{33}\tilde{E}_3(h). \tag{5.5}$$

Let us now define

$$\boldsymbol{u}^1(h) := h^{-1}(\boldsymbol{u}(h) - \boldsymbol{u}^0) = \boldsymbol{u}^1 + h.o.t.,$$
$$\hat{u}^2_\alpha(h) := h^{-2}\left(u_\alpha(h) - u^0_\alpha - hu^1_\alpha\right) = u^2_\alpha + h.o.t.$$

In order to have strong convergence we need the additional hypotheses:

$$\boldsymbol{u}^1(h) \in \boldsymbol{V}_{RD}(\Omega), \quad \hat{\boldsymbol{u}}^2(h) \in L^2\left(0,L;\boldsymbol{V}^2_m(\omega)\right), \tag{5.6}$$

where

$$\boldsymbol{V}_{RD}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}_1(\Omega) : v_3 \in L^2(0,L;Q(\omega))\},$$

is the space of admissible rigid displacements. We also define

$$\boldsymbol{V}_R(\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}_{RD}(\Omega) : v_\alpha = \delta_\alpha s, \ \ s \in H^1_0(0,L)\},$$
$$\boldsymbol{V}_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}_{RD}(\Omega) : v_\alpha = s_\alpha, \ v_3 = -x_\alpha s'_\alpha, \ s_\alpha \in H^1_0(0,L)\},$$

which are the the subspaces of admissible rotations and translations, respectively. We note that $|e_{3\beta}(\cdot)|_{0,\Omega}$ is a norm for $\boldsymbol{V}_R(\Omega)$ but not for $\boldsymbol{V}_{RD}(\Omega)$. In order to overcome this difficulty, we can think of using the quotient space $\dot{\boldsymbol{V}}_{RD}(\Omega) := \boldsymbol{V}_{RD}(\Omega)/\boldsymbol{V}_T(\Omega)$, where the induced application

$$|||\dot{\boldsymbol{v}}||| = |e_{3\beta}(\boldsymbol{v})|_{0,\Omega}, \quad \forall\, \dot{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{V}_T(\Omega) \in \dot{\boldsymbol{V}}_{RD}(\Omega),$$

is a norm.

Theorem 5.1. *Let $(\boldsymbol{u}(h),\bar{\varphi}(h)) \in \boldsymbol{V}(\Omega)\times\Psi(\Omega)$ be the solution of the scaled problem (3.15)–(3.17) and assume that (5.6) holds. Then, there exists a subsequence, still denoted by $(\boldsymbol{u}(h),\bar{\varphi}(h))_{h>0}$ which satisfies*

$$\boldsymbol{u}(h) \to \boldsymbol{u}^0 \quad \text{in } \boldsymbol{V}(\Omega), \quad \bar{\varphi}(h) \to \varphi^0 - \hat{\varphi} \quad \text{in } \hat{\Psi}(\Omega),$$
$$\tilde{\boldsymbol{e}}(h) \to \tilde{\boldsymbol{e}} \quad \text{in } \boldsymbol{M}_1(\Omega) \quad \tilde{\boldsymbol{E}}(h) \to \tilde{\boldsymbol{E}} \quad \text{in } \boldsymbol{M}_2(\Omega).$$

Furthermore

$$\boldsymbol{u}^0 \in \boldsymbol{V}_{BN}(\Omega) : \begin{cases} u^0_\alpha(x_1,x_2,x_3) = \xi_\alpha(x_3), & \xi_\alpha \in H^2_0(0,L), \\ u^0_3(x_1,x_2,x_3) = \xi_3(x_3) - x_\rho \xi'_\rho(x_3), & \xi_3 \in H^1_0(0,L), \end{cases}$$
$$\boldsymbol{u}^1 \in \boldsymbol{V}_1(\Omega) : \begin{cases} u^1_\alpha = z_\alpha + \delta_\alpha z, & z_\alpha, z \in H^1_0(0,L) \\ u^1_3 = -r - z_3 - x_\alpha z'_\alpha - wz', \end{cases}$$
$$\tilde{\boldsymbol{e}} = \begin{pmatrix} e_{\alpha\beta}(\boldsymbol{u}^2) & e_{3\alpha}(\boldsymbol{u}^1) \\ e_{3\alpha}(\boldsymbol{u}^1) & e_{33}(\boldsymbol{u}^0) \end{pmatrix}, \quad \tilde{\boldsymbol{E}} = -(\partial_1\varphi^0, \partial_2\varphi^0, 0),$$

where $z_3 \in L^2(0,L)$, functions ξ_α, ξ_3, z solve the variational problems (4.39), (4.40) and (4.57), respectively, and w, $r \in Q(\omega)$ solve the variational problems (4.46) and (4.48).

Proof. The proof is divided into 7 steps, numbered (i) to (vii).

Step (i): *We first show that there exists a constant $c(\hat{\varphi}) > 0$, depending only the applied potential $\hat{\varphi}$, such that, for all $h \in (0,1)$:*

$$\|\boldsymbol{u}(h)\|^2_{1,\Omega} + |\tilde{\boldsymbol{e}}(h)|^2_{0,\Omega} + |\bar{\boldsymbol{E}}(h)|^2_{0,\Omega} \le c(\hat{\varphi}), \tag{5.7}$$

$$\|\bar{\varphi}(h)\|_{0,\Omega} \le c(\hat{\varphi}), \tag{5.8}$$

where $\bar{\boldsymbol{E}}(h) = \tilde{\boldsymbol{E}}(\bar{\boldsymbol{\varphi}}(h)) = -(\partial_1\bar{\varphi}(h), \partial_2\bar{\varphi}(h), h\partial_3\bar{\varphi}(h))$.

The proof of this step is straightforward and can be found in Ref. Figueiredo and Leal (2006).

Step (ii): *There exists a subsequence of $(\boldsymbol{u}(h),\tilde{\boldsymbol{e}}(h),\varphi(h),\boldsymbol{E}(h))_{0<h<1}$, still parameterized by h, and there exist $\boldsymbol{u} \in \boldsymbol{V}(\Omega)$, $\tilde{\boldsymbol{e}} \in \boldsymbol{M}_1(\Omega)$, $\varphi \in L^2(\Omega)$ and $\tilde{\boldsymbol{E}} \in \boldsymbol{M}_2(\Omega)$, such that the following weak convergence results hold when h tends to zero:*

$$\boldsymbol{u}(h) \rightharpoonup \boldsymbol{u} \quad \text{in}[H^1(\Omega)]^3, \tag{5.9}$$

$$\tilde{\boldsymbol{e}}(h) \rightharpoonup \tilde{\boldsymbol{e}} \quad \text{in } \boldsymbol{M}_1(\Omega), \tag{5.10}$$

$$\varphi(h) \rightharpoonup \varphi \quad \text{in } L^2(\Omega), \tag{5.11}$$

$$\tilde{\boldsymbol{E}}(h) \rightharpoonup \tilde{\boldsymbol{E}} \quad \text{in } \boldsymbol{M}_2(\Omega). \tag{5.12}$$

From step (i) it is easy to show that there exists subsequences $(\boldsymbol{u}(h))_{h>0}$, $(\tilde{\boldsymbol{e}}(h))_{h>0}$, $(\bar{\varphi}(h))_{h>0}$, $(\bar{\boldsymbol{E}}(h))_{h>0}$ and functions $\boldsymbol{u}$, $\tilde{\boldsymbol{e}}$, $\bar{\varphi}$ and $\bar{\boldsymbol{E}}$ satisfying (5.9) and (5.10),

$$\bar{\varphi}(h) \rightharpoonup \bar{\varphi} \quad \text{in } L^2(\Omega), \ \text{ and } \ \bar{\boldsymbol{E}}(h) \rightharpoonup \bar{\boldsymbol{E}} \quad \text{in } [L^2(\Omega)]^3, \tag{5.13}$$

respectively. The weak convergence results (5.11) and (5.12) follow straightforwardly from (5.13) taking into account that $\varphi(h) = \bar{\varphi}(h) + \hat{\varphi}$, $\tilde{\boldsymbol{E}}(h) = \bar{\boldsymbol{E}}(h) + \hat{\boldsymbol{E}}(h)$ and $\hat{\boldsymbol{E}}(h) = (-\partial_1\hat{\varphi}, -\partial_2\hat{\varphi}, -h\partial_3\hat{\varphi}) \to (-\partial_1\hat{\varphi}, -\partial_2\hat{\varphi}, 0)$ as $h \to 0$.

Step (iii): *There exist subsequences of $(\boldsymbol{u}(h))_{h>0}$ and $(\varphi(h))_{h>0}$ still denoted the same way, and $\tilde{\boldsymbol{e}} \in \boldsymbol{M}_1(\Omega)$, $\tilde{\boldsymbol{E}} \in \boldsymbol{M}_2(\Omega)$ such that*

$$e_{33}(\boldsymbol{u}(h)) \rightharpoonup \tilde{e}_{33}, \quad h^{-1}e_{\alpha 3}(\boldsymbol{u}(h)) \rightharpoonup \tilde{e}_{\alpha 3}, \quad h^{-2}e_{\alpha\beta}(\boldsymbol{u}(h)) \rightharpoonup \tilde{e}_{\alpha\beta},$$
$$e_{\alpha 3}(\boldsymbol{u}(h)) \rightharpoonup 0, \quad e_{\alpha\beta}(\boldsymbol{u}(h)) \rightharpoonup 0,$$
$$\tilde{E}_\alpha(\varphi(h)) \rightharpoonup \tilde{E}_\alpha, \quad \tilde{E}_3(\varphi(h)) \rightharpoonup \tilde{E}_3, \quad hE_\alpha(\varphi(h)) \rightharpoonup 0, \tag{5.14}$$

in $L^2(\Omega)$. Moreover,

$$\tilde{e}_{33} = e_{33}(\boldsymbol{u}), \quad e_{3\alpha}(\boldsymbol{u}) = e_{\alpha\beta}(\boldsymbol{u}) = 0 \ (\text{i.e., } \boldsymbol{u} \in \boldsymbol{V}_{BN}(\Omega)), \tag{5.15}$$

$$\tilde{E}_\alpha = -\partial_\alpha\varphi, \quad \tilde{E}_3 = 0. \tag{5.16}$$

The obtention of (5.14) is straightforward. Indeed, from (5.10) and (5.12) we deduce that there exist $(\tilde{e}_{\alpha\beta}, \tilde{e}_{3\alpha}, \tilde{e}_{33}, \tilde{E}_\alpha, \tilde{E}_3) \in \boldsymbol{M}(\Omega)$, which is the weak limit of a subsequence of

$$(h^{-2}e_{\alpha\beta}(\boldsymbol{u}(h)), h^{-1}e_{3\alpha}(\boldsymbol{u}(h)), e_{33}(\boldsymbol{u}(h)), E_\alpha(\varphi(h)), hE_3(\varphi(h))),$$

still denoted by the same way. Also, from (5.9) and the uniqueness of the limit as $h \to 0$, we deduce that (5.15) holds. Similarly, from (5.11) we obtain (5.16).

Step (iv): *Let $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h)))$ be the solution of the mixed variational problem (5.1)–(5.3). Then, there exists a constant $C = C(\hat{\varphi}) > 0$, independent of h, such that, for all $0 < h \leq 1$:*

$$\|S_{33}(h)\|_{0,\Omega} \leq C, \quad \|S_{\alpha 3}(h)\|_{0,\Omega} \leq C, \quad \|S_{\alpha\beta}(h)\|_{0,\Omega} \leq C, \tag{5.17}$$

$$\|T_\alpha(h)\|_{0,\Omega} \leq C, \quad \|T_3(h)\|_{0,\Omega} \leq C. \tag{5.18}$$

Moreover, there exists a subsequence, still parameterized by h, and there exist $\boldsymbol{\Sigma} \in \boldsymbol{M}_1(\Omega)$ and $\boldsymbol{T} \in \boldsymbol{M}_2(\Omega)$, such that:

$$S_{33}(h) \rightharpoonup \Sigma_{33}, \quad S_{\alpha 3}(h) \rightharpoonup \Sigma_{\alpha 3}, \quad S_{\alpha\beta}(h) \rightharpoonup \Sigma_{\alpha\beta}, \quad \text{in } L^2(\Omega) \tag{5.19}$$

$$T_\alpha(h) \rightharpoonup T_\alpha, \quad T_3(h) \rightharpoonup T_3, \quad \text{in } L^2(\Omega) \tag{5.20}$$

when h goes to zero. Furthermore,

$$\tilde{e}_{\alpha\beta} = \bar{C}_{\theta\alpha\beta\rho}\Sigma_{\theta\rho} + \bar{C}_{33\alpha\beta}\Sigma_{33} + \bar{P}_{3\alpha\beta}T_3, \tag{5.21}$$

$$\tilde{e}_{3\alpha} = 2\bar{C}_{3\alpha 3\theta}\Sigma_{3\theta} + \bar{P}_{\theta 3\alpha}T_\theta, \tag{5.22}$$

$$\tilde{e}_{33} = \bar{C}_{3333}\Sigma_{33} + \bar{P}_{333}T_3 + \bar{C}_{33\theta\rho}\Sigma_{\theta\rho}, \tag{5.23}$$

$$\tilde{E}_\alpha = -2\bar{P}_{\theta 3\alpha}\Sigma_{3\theta} - \bar{\varepsilon}_{\theta\alpha}T_\theta, \tag{5.24}$$

$$\tilde{E}_3 = -\bar{P}_{3\theta\rho}\Sigma_{\theta\rho} - \bar{P}_{333}\Sigma_{33} + \bar{\varepsilon}_{33}T_3. \tag{5.25}$$

Indeed, (5.17) and (5.18) follow from (5.7) and (5.8) and (5.4) and (5.5). Moreover, (5.19) and (5.20) is an immediate consequence of (5.17) and (5.18). Furthermore, by choosing $\tau_{33} = \tau_{3\alpha} = d_3 = d_\alpha = 0$ as test functions in (5.1), multiplying by h^{-2} and passing to the limit, we deduce (5.21) in $L^2(\Omega)$. Proceeding similarly, we obtain (5.22)–(5.25).

Step (v): *The limit model is given by:*

$$\begin{aligned}&\int_\Omega \Sigma_{\alpha\beta}e_{\alpha\beta}(\hat{\boldsymbol{v}})d\boldsymbol{x} + \int_\Omega \Sigma_{3\alpha}e_{3\alpha}(\bar{\boldsymbol{v}})d\boldsymbol{x}\\ &\quad + \int_\Omega \Sigma_{33}e_{33}(\boldsymbol{v})d\boldsymbol{x} + \int_\Omega T_\alpha E_\alpha(\bar{\psi})d\boldsymbol{x}\\ &= \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i d\Gamma, \quad \forall(\hat{\boldsymbol{v}}, \bar{\boldsymbol{v}}, \boldsymbol{v}, \bar{\psi}) \in \boldsymbol{V}_H(\Omega)\\ &\quad \times \boldsymbol{V}_1(\Omega) \times \boldsymbol{V}_{BN}(\Omega) \times \Psi_{l0}(\Omega),\end{aligned} \tag{5.26}$$

where

$$\begin{cases}\Sigma_{\alpha\beta} = C_{\alpha\beta\theta\rho}\tilde{e}_{\theta\rho} + C_{\alpha\beta 33}\tilde{e}_{33}, \ \Sigma_{3\alpha} = 2C_{3\alpha 3\theta}\tilde{e}_{3\theta} - P_{\theta 3\alpha}\tilde{E}_\theta,\\ \Sigma_{33} = C_{33\theta\rho}\tilde{e}_{\theta\rho} + C_{3333}\tilde{e}_{33},\\ T_\alpha = 2P_{\alpha 3\beta}\tilde{e}_{3\beta} + \varepsilon_{\alpha\beta}\tilde{E}_\beta, \ T_3 = P_{3\alpha\beta}\tilde{e}_{\alpha\beta} + P_{333}\tilde{e}_{33}.\end{cases} \tag{5.27}$$

Taking $\boldsymbol{v} \in \boldsymbol{V}_{BN}(\Omega)$ in (5.2) and passing to the limit when $h \to 0$, we obtain

$$\begin{aligned}&\int_\Omega \Sigma_{33}e_{33}(\boldsymbol{v})d\boldsymbol{x} + \int_\Omega T_\alpha E_\alpha(\bar{\psi})d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i d\Gamma,\\ &\forall(\boldsymbol{v}, \bar{\psi}) \in \boldsymbol{V}_{BN}(\Omega) \times \Psi(\Omega).\end{aligned} \tag{5.28}$$

Since $\Psi(\Omega)$ is dense in $\Psi_{l0}(\Omega)$, the previous equation is also valid for all $(\boldsymbol{v}, \bar{\psi}) \in \boldsymbol{V}_{BN}(\Omega) \times \Psi_{l0}(\Omega)$. Now, we multiply Eq. (5.2) by h^2. Let us now define the space of horizontal displacements $\boldsymbol{V}_H(\Omega) = H^1(0, L; \boldsymbol{V}_m^2(\omega))$. By choosing $v_3 = 0$ and $(v_\alpha) \in \boldsymbol{V}_H(\Omega)$, and passing to the limit as $h \to 0$, we obtain

$$\int_\Omega \Sigma_{\alpha\beta}e_{\alpha\beta}(\hat{\boldsymbol{v}})d\boldsymbol{x} = 0, \quad \forall \hat{\boldsymbol{v}} \in \boldsymbol{V}_H(\Omega). \tag{5.29}$$

Multiplying Eq. (5.2) by h, taking the test function $\boldsymbol{v} \in \boldsymbol{V}_{RD}(\Omega)$ and passing to the limit as $h \to 0$, we find

$$\int_\Omega \Sigma_{3\alpha}e_{3\alpha}(\bar{\boldsymbol{v}})d\boldsymbol{x} = 0 \quad \forall \bar{\boldsymbol{v}} \in \boldsymbol{V}_{RD}(\Omega). \tag{5.30}$$

Therefore, from (5.28)–(5.30) we get (5.26). The relations (5.27) follow from the inversion of (5.21)–(5.25).

Step (vi): *The following characterizations hold:*

$$\tilde{e}_{\theta\rho} = e_{\theta\rho}(\boldsymbol{u}^2), \ \ \tilde{e}_{33} = e_{33}(\boldsymbol{u}^0), \ \ \tilde{e}_{3\beta} = e_{3\beta}(\boldsymbol{u}^1), \ \ \tilde{E}_\beta = E_\beta(\varphi^0). \tag{5.31}$$

Moreover,

$$\Sigma_{\alpha\beta} = \sigma^{-2}_{\alpha\beta}, \quad \Sigma_{33} = \sigma^0_{33}, \quad \Sigma_{3\alpha} = \sigma^{-1}_{3\alpha}, \quad \text{in } L^2(\Omega), \tag{5.32}$$

$$T_3 = D^1_3, \quad T_\alpha = D^0_\alpha, \quad \text{in } L^2(\Omega). \tag{5.33}$$

From (5.7) and (5.6) we find that there exist $\hat{\boldsymbol{u}}^2 \in L^2(0, L; \boldsymbol{V}_m^2(\omega))$ and $\dot{\boldsymbol{u}}^1 \in \dot{\boldsymbol{V}}_{RD}(\Omega)$ such that the following weak convergences hold:

$$\hat{\boldsymbol{u}}^2(h) \rightharpoonup \hat{\boldsymbol{u}}^2, \quad \boldsymbol{u}^1(h) \rightharpoonup \dot{\boldsymbol{u}}^1.$$

Moreover, since $e_{\alpha\beta} : \boldsymbol{V}_m^2(\omega) \to [L^2(\omega)]^{2\times 2}_s$ and $e_{3\beta} : \dot{\boldsymbol{V}}_{RD}(\Omega) \to [L^2(\Omega)]^2$ are weakly continuous applications, by using the closed graph theorem (see, for example Ref. Brézis, 1983) we find that

$$\tilde{e}_{\alpha\beta} = e_{\alpha\beta}(\hat{\boldsymbol{u}}^2), \quad \tilde{e}_{3\beta} = e_{3\beta}(\bar{\boldsymbol{u}}^1). \tag{5.34}$$

Now, taking into account Eq. (4.8) for test functions $\boldsymbol{v} \in \boldsymbol{V}_{RD}(\Omega)$ and $\psi = 0$, Eq. (4.9) with $\boldsymbol{v} = \boldsymbol{0}$ and $\psi \in \Psi(\Omega)$, dense in $\Psi_{l0}(\Omega)$, and relations (4.24), (4.36) and (5.26), we obtain

$$\begin{aligned}&\int_\Omega \left(\Sigma_{\alpha\beta} - \sigma^{-2}_{\alpha\beta}\right)e_{\alpha\beta}(\hat{\boldsymbol{v}})d\boldsymbol{x} + \int_\Omega \left(\Sigma_{3\alpha} - \sigma^{-1}_{3\alpha}\right)e_{3\alpha}(\bar{\boldsymbol{v}})d\boldsymbol{x}\\ &\quad + \int_\Omega \left(\Sigma_{33} - \sigma^0_{33}\right)e_{33}(\boldsymbol{v})d\boldsymbol{x} + \int_\Omega \left(T_\alpha - D^0_\alpha\right)E_\alpha(\bar{\psi})d\boldsymbol{x} = 0,\\ &\forall(\hat{\boldsymbol{v}}, \bar{\boldsymbol{v}}, \boldsymbol{v}, \bar{\psi}) \in \boldsymbol{V}_H(\Omega) \times \boldsymbol{V}_{RD}(\Omega) \times \boldsymbol{V}_{BN}(\Omega) \times \Psi_{l0}(\Omega),\end{aligned}$$

which can be equivalently written as follows, taking into account the constitutive Eqs. (4.11), (4.12), (4.14), (4.15) and (5.27),

$$\int_\Omega \begin{pmatrix} e_{\alpha\beta}(\hat{\boldsymbol{v}}) \\ e_{33}(\boldsymbol{v}) \end{pmatrix}^T \begin{pmatrix} C_{\alpha\beta\theta\rho} & C_{\alpha\beta 33} \\ C_{33\theta\rho} & C_{3333} \end{pmatrix} \begin{pmatrix} \tilde{e}_{\theta\rho} - e_{\theta\rho}(\boldsymbol{u}^2) \\ \tilde{e}_{33} - e_{33}(\boldsymbol{u}^0) \end{pmatrix} d\boldsymbol{x} = 0, \tag{5.35}$$

$$\int_\Omega \begin{pmatrix} e_{3\alpha}(\bar{\boldsymbol{v}}) \\ E_\alpha(\bar{\psi}) \end{pmatrix}^T \begin{pmatrix} C_{3\alpha 3\beta} & -P_{\beta 3\alpha} \\ P_{\alpha 3\beta} & \varepsilon_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \tilde{e}_{3\beta} - e_{3\beta}(\boldsymbol{u}^1) \\ \tilde{E}_\beta - E_\beta(\bar{\varphi}^0 + \hat{\varphi}) \end{pmatrix} d\boldsymbol{x} = 0, \tag{5.36}$$

for all $(\hat{\boldsymbol{v}}, \bar{\boldsymbol{v}}, \boldsymbol{v}, \bar{\psi}) \in \boldsymbol{V}_H(\Omega) \times \boldsymbol{V}_{RD}(\Omega) \times \boldsymbol{V}_{BN}(\Omega) \times \Psi_{l0}(\Omega)$. By using density arguments, the integral Eq. (5.35) is also valid when we replace $\boldsymbol{V}_H(\Omega)$ by $L^2(0, L; \boldsymbol{V}_m^2(\omega))$. Therefore, we can take as test functions

$$\hat{\boldsymbol{v}} = \hat{\boldsymbol{u}}^2 - \boldsymbol{u}^2, \quad \bar{\boldsymbol{v}} = \bar{\boldsymbol{u}}^1 - \boldsymbol{u}^1, \quad \boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}^0, \quad \bar{\psi} = \varphi - (\bar{\varphi}^0 + \hat{\varphi}).$$

Having in mind (5.15), (5.16) and (5.34), the ellipticity in (5.35)–(5.36) gives (5.31). Consequently, from (4.11), (4.12), (4.14), (4.15), (4.17) and (5.27), we can deduce (5.32)–(5.33).

Step (vii): *The following strong convergences hold:*

$$\tilde{\boldsymbol{e}}(h) \to \tilde{\boldsymbol{e}} \quad \text{in } \boldsymbol{M}_1(\Omega), \quad \tilde{\boldsymbol{E}}(h) \to \tilde{\boldsymbol{E}} \quad \text{in} \quad [L^2(\Omega)]^3,$$

$$\boldsymbol{u}(h) \to \boldsymbol{u}^0 \quad \text{in } \boldsymbol{V}(\Omega), \quad \varphi(h) \to \varphi^0 \quad \text{in } \hat{\Psi}(\Omega).$$

Let $X(h)$ be the norm of $(\tilde{\boldsymbol{e}}(h), -\tilde{\boldsymbol{E}}(h))$ in $\boldsymbol{M}(\Omega)$:

$$\begin{aligned}X(h) = &\left\{\int_\Omega C_{ijkl}\tilde{e}_{kl}(h)\tilde{e}_{ij}(h)d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta}\tilde{E}_\alpha(h)\tilde{E}_\beta(h)d\boldsymbol{x}\right.\\ &\left.\quad + \int_\Omega \varepsilon_{33}\tilde{E}_3(h)\tilde{E}_3(h)d\boldsymbol{x}\right\}^{1/2}.\end{aligned} \tag{5.37}$$

Taking $(\boldsymbol{v}, \psi) = (\boldsymbol{u}(h), \varphi(h))$ in (3.15)–(3.17) we find that $(\boldsymbol{u}(h), \varphi(h))_{h>0}$ verifies

$$\int_\Omega C_{ijkl}\tilde{e}_{kl}(h)\tilde{e}_{ij}(h)d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta}\tilde{E}_\alpha(h)\tilde{E}_\beta(h)d\boldsymbol{x} + \int_\Omega \varepsilon_{33}\tilde{E}_3(h)\tilde{E}_3(h)d\boldsymbol{x} = \int_\Omega f_i u_i(h)d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i(h)d\Gamma. \tag{5.38}$$

Combining (5.37) and (5.38) we have

$$(X(h))^2 = \int_\Omega f_i u_i(h)d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i(h)d\Gamma.$$

From the weak convergences (5.9) we get

$$\lim_{h\to 0}(X(h))^2 = \int_\Omega f_i u_i d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i d\Gamma.$$

Since we have already proved that $(\tilde{\boldsymbol{e}}(h), \partial_1\varphi(h), \partial_2\varphi(h), h\partial_3\varphi(h))$ converges weakly, it suffices to show that the limit of its norm tends to the norm of its weak limit to obtain strong convergence. We denote by

$$X = \left(\int_\Omega C_{ijkl}\tilde{e}_{kl}\tilde{e}_{ij}d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta}\tilde{E}_\alpha\tilde{E}_\beta d\boldsymbol{x}\right)^{1/2}$$

the L^2 norm of $(\tilde{\boldsymbol{e}}, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3)$. So, we want to show that

$$X^2 = \lim_{h\to 0}(X(h))^2,$$

or, equivalently,

$$\int_\Omega C_{ijkl}\tilde{e}_{kl}\tilde{e}_{ij}d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta}\tilde{E}_\alpha\tilde{E}_\beta d\boldsymbol{x} = \int_\Omega f_i u_i d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i d\Gamma.$$

It easy to check the following relation

$$\begin{aligned}\int_\Omega C_{ijkl}\tilde{e}_{kl}\tilde{e}_{ij}d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta}\tilde{E}_\alpha\tilde{E}_\beta d\boldsymbol{x} &= \int_\Omega (C_{\alpha\beta\theta\rho}\tilde{e}_{\theta\rho} + C_{\alpha\beta33}\tilde{e}_{33})\tilde{e}_{\alpha\beta}d\boldsymbol{x} \\ &+ 4\int_\Omega C_{3\alpha3\theta}\tilde{e}_{3\theta}\tilde{e}_{3\alpha}d\boldsymbol{x} + \int_\Omega (C_{33\theta\rho}\tilde{e}_{\theta\rho} + C_{3333}\tilde{e}_{33})\tilde{e}_{33}d\boldsymbol{x} \\ &+ \int_\Omega \varepsilon_{\alpha\beta}\tilde{E}_\alpha\tilde{E}_\beta d\boldsymbol{x}.\end{aligned}$$

and, having in mind the relations (5.27), one deduce

$$\begin{aligned}&\int_\Omega C_{ijkl}\tilde{e}_{kl}\tilde{e}_{ij}d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta}\tilde{E}_\alpha\tilde{E}_\beta d\boldsymbol{x} \\ &= \int_\Omega \underbrace{(C_{\alpha\beta\theta\rho}\tilde{e}_{\theta\rho} + C_{\alpha\beta33}\tilde{e}_{33})}_{\Sigma_{\alpha\beta}}\tilde{e}_{\alpha\beta}dx \\ &\quad + \int_\Omega \underbrace{(C_{33\theta\rho}\tilde{e}_{\theta\rho} + C_{3333}\tilde{e}_{33})}_{\Sigma_{33}}\tilde{e}_{33}d\boldsymbol{x} \\ &\quad + 2\int_\Omega \underbrace{2C_{3\alpha3\theta}\tilde{e}_{3\theta}}_{\Sigma_{3\alpha}+P_{\beta3\alpha}\tilde{E}_\beta}\tilde{e}_{3\alpha}d\boldsymbol{x} + \int_\Omega \underbrace{\varepsilon_{\alpha\beta}\tilde{E}_\alpha}_{T_\beta - 2P_{\alpha3\beta}\tilde{e}_{3\beta}}\tilde{E}_\beta\, d\boldsymbol{x}.\end{aligned}$$

By using (5.26) with test functions $\boldsymbol{v} = \boldsymbol{u}^0$, $\bar{\boldsymbol{v}} = \boldsymbol{u}^1$, $\hat{\boldsymbol{v}} = \hat{\boldsymbol{u}}^2$ and $\bar{\psi} = \varphi^0$ and the equality relations (5.31) and (5.32), we find

$$\begin{aligned}\int_\Omega C_{ijkl}\tilde{e}_{kl}\tilde{e}_{ij}d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta}\tilde{E}_\alpha\tilde{E}_\beta d\boldsymbol{x} &= \int_\Omega \Sigma_{\alpha\beta}\tilde{e}_{\alpha\beta}d\boldsymbol{x} + 2\int_\Omega \Sigma_{3\alpha}\tilde{e}_{3\alpha}d\boldsymbol{x} \\ &+ \int_\Omega \Sigma_{33}\tilde{e}_{33}d\boldsymbol{x} + \int_\Omega T_\beta\tilde{E}_\beta d\boldsymbol{x} = \int_\Omega f_i u_i^0 d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i^0 d\Gamma,\end{aligned}$$

as required. □

6. Leading terms and limits models for the reference rod Ω

The results obtained in the previous sections concerning the leading terms of the developments for the scaled displacement vector $(u_i(h))$, electric potential $\bar{\varphi}(h)$, stress tensor $(\sigma_{ij}(h))$ and electric displacement vector $(D_i(h))$, for the reference rod Ω, can be summarized as follows:

$$\begin{cases} u_\alpha(h) = \xi_\alpha + hu_\alpha^1 + O(h^2), \quad u_3(h) = \xi_3 - x_\beta\xi_\beta' + hu_3^1 + O(h^2), \\ u_\alpha^1 = z_\alpha + \delta_\alpha z, \quad u_3^1 = -r - z_3 - x_\alpha z_\alpha' - wz', \\ \bar{\varphi}(h) = \bar{\varphi}^0 + O(h), \end{cases} \tag{6.1}$$

and

$$\begin{cases} \sigma_{\alpha\beta}(h) = h^{-1}\sigma_{\alpha\beta}^{-1} + O(h^0), \\ \sigma_{\alpha\beta}(h) = h^{-1}\sigma_{\alpha\beta}^{-1} + O(h^0), \\ \sigma_{3\alpha}(h) = h^{-1}[P_{\beta3\alpha}\partial_\beta(\bar{\varphi}^0 + \hat{\varphi}) \\ \quad -C_{3\alpha3\beta}(\partial_\beta r + z'(\partial_\beta w - \delta_\beta))] + O(h^0), \\ \sigma_{33}(h) = Y(\xi_3' - x_\alpha\xi_\alpha'') + O(h), \\ D_\alpha(h) = -P_{\alpha3\beta}(\partial_\beta r + z'\left(\partial_\beta w - \delta_\beta\right)) - \varepsilon_{\alpha\beta}\partial_\beta(\bar{\varphi}^0 + \hat{\varphi}) + O(h), \\ D_3(h) = h\bar{P}(\xi_3' - x_\alpha\xi_\alpha'') + O(h^2), \end{cases} \tag{6.2}$$

where Y and $\bar{P}$ are given by (4.38) and (4.43), respectively, and the (Bernoulli–Navier) displacement components $\xi_\alpha \in H_0^2(0, L)$ and $\xi_3 \in H_0^1(0, L)$ are, respectively, the unique solutions of the 1D variational (limit) problems (4.39) and (4.40), just as in the elastic case. This result holds under the assumption (3.10).

Regarding the electric potential, we have showed that even if no conditions are imposed on the dependence of $\boldsymbol{C}, \boldsymbol{P}$ and $\boldsymbol{\varepsilon}$ with respect to $\boldsymbol{x}$, the element $\left(\bar{\varphi}^0, r, z\right) \in \boldsymbol{T}(\Omega)$ is the unique solution of the 3D variational (limit) problem (4.63)–(4.65).

6.1. The limit model on the original rod Ω^h

We now return to the original rod Ω^h and define the following spaces (cf. (4.44), (4.45) and (4.61)):

$$\begin{aligned}\Psi_l^h(\Omega^h) &= L^2(0, L; H^1(\omega^h)), \\ \Psi_{l0}^h(\Omega^h) &= \{\psi^h \in L^2(0, L; H^1(\omega^h)) : \psi^h = 0 \text{ on } \Gamma_{eD}^h\} \\ &\equiv L^2(0, L; S^h(\omega^h)), \\ R^h(\Omega^h) &= L^2(0, L; Q^h(\omega^h)), \quad Q^h(\omega^h) \\ &= \{q^h \in H^1(\omega^h) : \int_{\omega^h} q^h d\omega^h = 0\}, \\ \boldsymbol{T}^h(\Omega^h) &= \Psi_{l0}^h(\Omega^h) \times R^h(\Omega^h) \times H_0^1(0, L).\end{aligned}$$

Given the scalings (3.4), (3.6) and (3.18), the developments (4.1), (4.2) and (4.4) induce formal developments on $\boldsymbol{u}^h$, $\bar{\varphi}^h$, $\boldsymbol{\sigma}^h$ and $\boldsymbol{D}^h$, respectively, whose leading terms we will identify and characterize in the following. For that we will undo the change of variable $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$ and accordingly define the de-scaled quantities:

$$\begin{aligned}&\xi_\alpha^h := \xi_\alpha^h\left(x_3^h\right) = h^{-1}\xi_\alpha(x_3), \quad \xi_3^h := \xi_3^h\left(x_3^h\right) = \xi_3(x_3), \\ &\bar{\varphi}^{0h} := \bar{\varphi}^{0h}(\boldsymbol{x}^h) = h\bar{\varphi}^0(\boldsymbol{x}), \quad r^h := r^h(\boldsymbol{x}^h) = hr(\boldsymbol{x}), \\ &z_\alpha^h := z_\alpha, \quad z^h := h^{-1}z,\end{aligned}$$

as well as the warping function

$$w^h = w^h\left(x_1^h, x_2^h\right) = h^2 w(x_1, x_2),$$

which is the unique solution of (cf. (4.46))

$$\begin{aligned}&w^h \in Q^h(\omega^h) \text{ such that} \\ &\int_{\omega^h} C_{3\alpha3\beta}\partial_\beta^h w^h\partial_\alpha^h v^h d\omega^h = \int_{\omega^h} C_{3\alpha3\beta}\delta_\beta^h\partial_\alpha^h v^h d\omega^h, \\ &\text{for all } v^h \in H^1(\omega^h),\end{aligned} \tag{6.3}$$

the torsion constant (cf. (4.47))

$$\begin{aligned}J^h = h^4 J &= \int_{\omega^h} C_{3\alpha3\beta}(\delta_\beta^h - \partial_\beta^h w^h)(\delta_\alpha^h - \partial_\alpha^h w^h)d\omega^h \\ &= \int_{\omega^h} C_{3\alpha3\beta}(\delta_\beta^h - \partial_\beta^h w^h)\delta_\alpha^h d\omega^h,\end{aligned}$$

and the second moment of area of the cross section with respect to axis Ox^h_β ($\alpha \neq \beta$)

$$I^h_\alpha = \int_{\omega^h} (x^h_\alpha)^2 d\omega^h,$$

where $\delta^h_1(x^h_1, x^h_2) = x^h_2$, $\delta^h_2(x^h_1, x^h_2) = -x^h_1$. We also define the resultant of the applied loads and moments in each cross section (cf. (4.41) and (4.42)):

$$\begin{aligned}
F^h_\alpha(x^h_3) &= h^3 F_\alpha(x_3) = \int_{\omega^h} f^h_\alpha(x^h_1, x^h_2, x^h_3) d\omega^h \\
&\quad + \int_{\gamma^h_N} g^h_\alpha(x^h_1, x^h_2, x^h_3) d\gamma^h \in L^2(0,L), \\
F^h_3(x^h_3) &= h^2 F_3(x_3) = \int_{\omega^h} f^h_3(x^h_1, x^h_2, x^h_3) d\omega^h \\
&\quad + \int_{\gamma^h_N} g^h_3(x^h_1, x^h_2, x^h_3) d\gamma^h \in L^2(0,L), \\
M^h_\alpha(x^h_3) &= h^3 M_\alpha(x_3) = \int_{\omega^h} x^h_\alpha f^h_3(x^h_1, x^h_2, x^h_3) d\omega^h \\
&\quad + \int_{\gamma^h_N} x^h_\alpha g^h_3(x^h_1, x^h_2, x^h_3) d\gamma^h \in L^2(0,L).
\end{aligned}$$

Bearing in mind these definitions and in view of (6.1) and (6.2), one has the following result for the leading terms of the induced formal developments for $\boldsymbol{u}^h$, $\bar{\varphi}^h$, $\boldsymbol{\sigma}^h$ and $\boldsymbol{D}^h$.

Proposition 6.1. *The leading terms of the formal developments induced by (4.1), (4.2) and (4.4) on the displacement vector* (u^h_i), *electric potential* $\bar{\varphi}^h$, *stress tensor* (σ^h_{ij}) *and electric displacement vector* (D^h_i) *are such that:*

$$\begin{cases}
u^h_\alpha = \xi^h_\alpha + u^{1h}_\alpha + O(h), \quad u^h_3(\boldsymbol{x}^h) = \xi^h_3 - x^h_\beta \partial^h_3 \xi^h_\beta + u^{1h}_3 + O(h^2), \\
u^{1h}_\alpha = z^h_\alpha + \delta^h_\alpha z^h, \quad u^{1h}_3 = -r^h - z^h_3 - x^h_\alpha (z^h_\alpha)' - w^h (z^h)', \\
\bar{\varphi}(h) = \bar{\varphi}^0 + O(h), \\
\bar{\varphi}^h = \bar{\varphi}^{0h} + O(h^2), \\
\sigma^h_{\alpha\beta} = O(h), \\
\sigma^h_{3\alpha} = P_{\beta 3\alpha} \partial^h_\beta (\bar{\varphi}^{0h} + \hat{\varphi}) - C_{3\alpha 3\beta}(\partial^h_\beta r^h + \partial^h_3 z^h (\partial^h_\beta w^h - \delta^h_\beta)) + O(h), \\
\sigma^h_{33} = Y(\partial^h_3 \xi^h_3 - x^h_\alpha \partial^h_{33} \xi^h_\alpha) + O(h), \\
D^h_\alpha = -P_{\alpha 3\beta}(\partial^h_\beta r^h + \partial^h_3 z^h (\partial^h_\beta w^h - \delta^h_\beta)) - \varepsilon_{\alpha\beta} \partial^h_\beta (\bar{\varphi}^{0h} + \hat{\varphi}) + O(h), \\
D^h_3 = \bar{P}(\partial^h_3 \xi^h_3 - x^h_\alpha \partial^h_{33} \xi^h_\alpha) + O(h),
\end{cases} \tag{6.4}$$

(we note that $\xi^h_\alpha \sim O(h^{-1})$, $\bar{\varphi}^{0h} \sim O(h)$, $z^h_\alpha \sim O(h^0)$, $z^h \sim O(h^{-1})$ *and* $z^h_3 \sim O(h)$*) where* $\xi^h_\alpha \in H^2_0(0,L)$, $\xi^h_3 \in H^1_0(0,L)$ *and* $(\bar{\varphi}^{0h}, r^h, z^h) \in \boldsymbol{T}^h$ *are the unique solutions of the following variational problems,*

$$\begin{aligned}
&\xi^h_\alpha \in H^2_0(0,L), \\
&\int_0^L Y^h I^h_\alpha \partial^h_{33} \xi^h_\alpha \partial^h_{33} \chi^h_\alpha dx^h_3 = \int_0^L F^h_\alpha \chi^h_\alpha dx^h_3 - \int_0^L M^h_\alpha \partial^h_3 \chi^h_\alpha dx^h_3, \\
&\quad (\text{no sum on } \alpha),
\end{aligned} \tag{6.5}$$

for all $\chi^h_\alpha \in H^2_0(0,L)$,

$$\begin{aligned}
&\xi^h_3 \in H^1_0(0,L), \\
&\int_0^L Y^h A(\omega^h) \partial^h_3 \xi^h_3 \partial^h_3 \chi^h_3 dx^h_3 = \int_0^L F^h_3 \chi^h_3 dx^h_3,
\end{aligned} \tag{6.6}$$

for all $\chi^h_3 \in H^1_0(0,L)$,

$$(\bar{\varphi}^{0h}, r^h, z^h) \in \boldsymbol{T}^h(\Omega^h),$$

$$\begin{aligned}
&\int_0^L J^h \partial^h_3 z^h \partial^h_3 \varsigma^h dx^h_3 + \int_{\Omega^h} C^h_{3\alpha 3\beta} \partial^h_\beta r^h \partial^h_\alpha \rho^h d\boldsymbol{x}^h \\
&\quad + \int_{\Omega^h} \varepsilon^h_{\alpha\beta} \partial^h_\alpha \bar{\varphi}^{0h} \partial^h_\beta \psi^h d\boldsymbol{x}^h + \int_{\Omega^h} P^h_{\beta 3\alpha}(\partial_\alpha r^h \partial_\beta \psi^h - \partial^h_\beta \bar{\varphi}^{0h} \partial^h_\alpha \rho^h) d\boldsymbol{x}^h \\
&\quad + \int_{\Omega^h} P^h_{\beta 3\alpha}(\partial^h_\alpha w^h - \delta^h_\alpha)(\partial^h_3 z^h \partial^h_\beta \psi^h - \partial^h_3 \varsigma^h \partial^h_\beta \bar{\varphi}^{0h}) d\boldsymbol{x}^h \\
&= \int_{\Omega^h} P^h_{\beta 3\alpha} \partial^h_\beta \hat{\varphi}^h (\partial^h_\alpha \rho^h + \partial^h_3 \varsigma^h (\partial^h_\alpha w^h - \delta^h_\alpha)) d\boldsymbol{x}^h \\
&\quad - \int_{\Omega^h} \varepsilon^h_{\alpha\beta} \partial^h_\beta \hat{\varphi}^h \partial^h_\alpha \psi^h d\boldsymbol{x}^h,
\end{aligned} \tag{6.7}$$

for all $(\psi^h, \rho^h, \varsigma^h) \in \boldsymbol{T}^h(\Omega^h)$,

whereas $w^h \in Q^h(\omega^h)$ *is the unique solution of (6.3).*

The variational problems obtained for the transverse displacement ξ^h_α, the axial displacement ξ^h_3 and the electric potential $\bar{\varphi}^{0h}$ constitute the limit model for the original rod. On the other hand, the limit constitutive equations are (cf. (6.4))

$$\begin{aligned}
\sigma^h_{\alpha\beta} &= 0, \\
\sigma^h_{3\alpha} &= P^h_{\beta 3\alpha} \partial^h_\beta (\bar{\varphi}^{0h} + \hat{\varphi}) - C^h_{3\alpha 3\beta}(\partial^h_\beta r^h + \partial^h_3 z^h (\partial^h_\beta w^h - \delta^h_\beta)), \\
\sigma^h_{33} &= Y^h(\partial^h_3 \xi^h_3 - x^h_\alpha \partial^h_{33} \xi^h_\alpha), \\
D^h_\alpha &= -P^h_{\alpha 3\beta}(\partial^h_\beta r^h + \partial^h_3 z^h (\partial^h_\beta w^h - \delta^h_\beta)) - \varepsilon^h_{\alpha\beta} \partial^h_\beta (\bar{\varphi}^{0h} + \hat{\varphi}), \\
D^h_3 &= \bar{P}^h(\partial^h_3 \xi^h_3 - x^h_\alpha \partial^h_{33} \xi^h_\alpha).
\end{aligned}$$

From (6.7), by taking appropriate choices of test functions we obtain the following set of independent equations (note that $\hat{\boldsymbol{x}} = (x^h_1, x^h_2)$):

$$\begin{aligned}
&\int_0^L J^h \partial^h_3 z^h \partial^h_3 \zeta \, dx_3 - \int_0^L \left(\int_{\omega^h} P^h_{\beta 3\alpha} (\partial^h_\alpha w^h - \delta^h_\alpha) \partial^h_\beta \bar{\varphi}^{0h} \, d\hat{\boldsymbol{x}}^h \right) \partial^h_3 \zeta^h \, dx_3 \\
&= \int_0^L \left(\int_{\omega^h} P^h_{\beta 3\alpha} \partial^h_\beta \hat{\varphi}^h (\partial^h_\alpha w^h - \delta^h_\alpha) \, d\hat{\boldsymbol{x}}^h \right) \partial^h_3 \zeta^h \, dx_3 \quad \forall \zeta^h \in H^1_0(0,L),
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
&\int_0^L \left(\int_{\omega^h} C^h_{3\alpha 3\beta} \partial^h_\beta r^h \partial^h_\alpha \rho^h_\sharp \, d\hat{\boldsymbol{x}}^h \right) \rho^h_3 \, dx_3 \\
&\quad - \int_0^L \left(\int_{\omega^h} P^h_{\beta 3\alpha} \partial_\beta \bar{\varphi}^{0h} \partial_\alpha \rho^h_\sharp \, d\hat{\boldsymbol{x}}^h \right) \rho^h_3 \, dx_3 \\
&= \int_0^L \left(\int_{\omega^h} P^h_{\beta 3\alpha} \partial_\beta \hat{\varphi}^h \partial_\alpha \rho^h_\sharp \, d\hat{\boldsymbol{x}}^h \right) \rho^h_3 \, dx_3 \\
&\quad \forall \rho^h_\sharp \in Q^h(\omega^h), \ \rho^h_3 \in L^2(0,L),
\end{aligned} \tag{6.9}$$

$$\begin{aligned}
&\int_0^L \left(\int_{\omega^h} \varepsilon^h_{\alpha\beta} \partial^h_\beta \bar{\varphi}^{0h} \partial^h_\alpha \psi^h_\sharp \, d\hat{\boldsymbol{x}}^h \right) \psi^h_3 \, dx_3 \\
&\quad + \int_0^L \left(\int_{\omega^h} P^h_{\beta 3\alpha} \partial^h_\alpha r^h \partial^h_\beta \psi^h_\sharp \, d\hat{\boldsymbol{x}}^h \right) \psi^h_3 \, dx_3 \\
&\quad + \int_0^L \left(\int_{\omega^h} P^h_{\beta 3\alpha} (\partial^h_\alpha w^h - \delta^h_\alpha) \partial^h_\beta \psi^h_\sharp \, d\hat{\boldsymbol{x}}^h \right) \partial^h_3 z^h \psi^h_3 \, dx_3 \\
&= -\int_0^L \left(\int_{\omega^h} \varepsilon^h_{\alpha\beta} \partial^h_\beta \hat{\varphi}^h \partial^h_\alpha \psi^h_\sharp \, d\hat{\boldsymbol{x}}^h \right) \psi^h_3 \, dx_3 \\
&\quad \forall \psi^h_\sharp \in S(\omega^h), \ \psi^h_3 \in L^2(0,L).
\end{aligned} \tag{6.10}$$

Now, let us assume the particular case where $\omega^h = [-a^h, a^h] \times [-b^h, b^h]$ and $\gamma^h_{eD} = (-a^h, a^h) \times \{-b^h\}$. In this case we can take as transversal test functions $\rho^h_\sharp(x^h_1, x^h_2) = x^h_1 x^h_2$ and $\psi^h_\sharp(x^h_1, x^h_2) = x^h_1(x^h_2 + b^h)$. In addition to that, we define the following quantities:

$$\begin{aligned}
&\theta^h_1 = x^h_2, \theta^h_2 = x^h_1, \hat{\theta}^h_1 = x^h_2 + b^h, \hat{\theta}^h_2 = x^h_1, \ C^{\sharp h}_\beta = C^h_{3\alpha 3\beta} \theta^h_\alpha, \\
&P^{\sharp h}_\beta = P^h_{\beta 3\alpha} \theta^h_\alpha, \ P^{\dagger h}_\alpha = P^h_{\beta 3\alpha} \theta^h_\beta, \ \varepsilon^{\sharp h}_\beta = \varepsilon^h_{\alpha\beta} \hat{\theta}^h_\alpha.
\end{aligned}$$

Therefore, the latter two variational equations can be reformulated as follows:

$$\int_0^L \left(\int_{\omega^h} C_\beta^{\sharp h}\partial_\beta^h r^h\, d\hat{\boldsymbol{x}}^h\right)\rho_3^h\, dx_3 - \int_0^L \left(\int_{\omega^h} P_\beta^{\sharp h}\partial_\beta^h\bar{\varphi}^{0h}\, d\hat{\boldsymbol{x}}^h\right)\rho_3^h\, dx_3$$
$$= \int_0^L \left(\int_{\omega^h} P_\beta^{\sharp h}\partial_\beta^h\hat{\varphi}^h\, d\hat{\boldsymbol{x}}^h\right)\rho_3^h\, dx_3\ \forall\rho_3^h \in L^2(0,L),$$
$$\int_0^L \left(\int_{\omega^h} \varepsilon_\beta^{\sharp h}\partial_\beta^h\bar{\varphi}^{0h}\, d\hat{\boldsymbol{x}}^h\right)\psi_3^h\, dx_3 + \int_0^L \left(\int_{\omega^h} P_\alpha^{\dagger h}\partial_\alpha^h r^h\, d\hat{\boldsymbol{x}}^h\right)\psi_3^h\, dx_3$$
$$+\int_0^L \left(\int_{\omega^h} P_\alpha^{\dagger h}(\partial_\alpha^h w^h - \delta_\alpha^h)\, d\hat{\boldsymbol{x}}^h\right)\partial_3^h z^h\psi_3^h\, dx_3$$
$$= -\int_0^L \left(\int_{\omega^h} \varepsilon_\beta^{\sharp h}\partial_\beta^h\hat{\varphi}^h\, d\hat{\boldsymbol{x}}^h\right)\psi_3^h\, dx_3\ \forall\psi_3^h \in L^2(0,L).$$

Moreover, the strong formulation of our problem can be formulated as follows:

$$\left\{\begin{aligned}
&\partial_3^h\big(J^h\partial_3^h z^h\big) - \partial_3^h\Big(\int_{\omega^h} P_{\beta 3\alpha}(\partial_\alpha^h w^h - \delta_\alpha^h)\partial_\beta^h\bar{\varphi}^{0h}\, d\hat{\boldsymbol{x}}^h\Big)\\
&\quad = \partial_3^h\Big(\textstyle\int_{\omega^h} P_{\beta 3\alpha}\partial_\beta^h\hat{\varphi}^h(\partial_\alpha^h w^h - \delta_\alpha^h)\, d\hat{\boldsymbol{x}}^h\Big),\\
&z^h(a) = 0,\ (J^h\partial_3^h z^h)(a) = 0,\ \int_{\omega^h\times\{a\}} P_{\beta 3\alpha}(\partial_\alpha^h w^h\\
&\quad -\delta_\alpha^h)\partial_\beta^h(\bar{\varphi}^{0h} + \hat{\varphi}^h)\, d\hat{\boldsymbol{x}}^h = 0,\ \text{for } a\in\{0,L\}\\
&\int_{\omega^h} C_\beta^{\sharp h}\partial_\beta^h r^h\, d\hat{\boldsymbol{x}}^h - \int_{\omega^h} P_\beta^{\sharp h}\partial_\beta^h\bar{\varphi}^{0h}\, d\hat{\boldsymbol{x}}^h = \int_{\omega^h} P_\beta^{\sharp h}\partial_\beta^h\hat{\varphi}^h\, d\hat{\boldsymbol{x}}^h,\\
&\int_{\omega^h} \varepsilon_\beta^{\sharp h}\partial_\beta^h\bar{\varphi}^{0h}\, d\hat{\boldsymbol{x}}^h + \int_{\omega^h} P_\alpha^{\dagger h}\partial_\alpha^h r^h\, d\hat{\boldsymbol{x}}^h + \int_{\omega^h} P_\alpha^{\dagger h}(\partial_\alpha^h w^h\\
&\quad -\delta_\alpha^h)\, d\hat{\boldsymbol{x}}^h\partial_3^h z^h = -\textstyle\int_{\omega^h} \varepsilon_\beta^{\sharp h}\partial_\beta^h\hat{\varphi}^h\, d\hat{\boldsymbol{x}}^h,\\
&\text{a.e. in } (0,L)
\end{aligned}\right. \tag{6.11}$$

Remark 6.2. Assume that $\partial_\beta^h\hat{\varphi}^h = \partial_\beta^h\bar{\varphi}^{0h} = 0$, i.e., both the applied potential and the first order component of the electric potential φ^h distributed on the beam depend only on x_3. In this case, the previous equations give

$$\left\{\begin{aligned}
&\partial_3^h\big(J^h\partial_3^h z^h\big) = 0 \text{ a.e. in } (0,L),\ z^h(a) = \partial_3^h(J^h z^h)(a) = 0,\\
&\int_{\omega^h} C_\beta^{\sharp h}\partial_\beta^h r^h\, d\hat{\boldsymbol{x}}^h = 0 \text{ a.e. in } (0,L),\\
&\int_{\omega^h} P_\alpha^{\dagger h}\partial_\alpha^h r^h\, d\hat{\boldsymbol{x}}^h + \int_{\omega^h} P_\alpha^{\dagger h}(\partial_\alpha^h w^h - \delta_\alpha^h)\, d\hat{\boldsymbol{x}}^h\partial_3^h z^h = 0 \text{ a.e. in } (0,L),
\end{aligned}\right.$$

from which we deduce that $z^h = r^h = 0$ and $\bar{\varphi}^{0h} = \hat{\varphi}^h$, i.e., the first order component of the displacements $\boldsymbol{u}^{1h}$ is of the Bernoulli–Navier kind and the zeroth order component of the electric potential φ^{0h} coincides with the applied potential $\hat{\varphi}^h$.

Remark 6.3. For materials having an isotropic mechanical behavior, such as PVDF, the limit model and constitutive equations are somewhat simplified since in this case the stiffness tensor $\boldsymbol{C}$ has the form

$$C_{ijkl} = \frac{\nu Y}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{Y}{2(1+\nu)}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big),$$

where Y is the Young's modulus, ν the Poisson's ratio and δ the Kronecker's delta. As a consequence, Y defined by (4.38) is indeed the Young's modulus and

$$C_{3\alpha 3\beta} = \frac{Y}{2(1+\nu)}\delta_{\alpha\beta}.$$

Therefore, the variational problems (6.5) and (6.6) remain unchanged, while in the variational problem for $\bar{\varphi}^{0h}$ the bilinear form becomes

$$\tilde{a}_{iso}^h((z^h, r^h, \bar{\varphi}^{0h}), (\varsigma^h, \rho^h, \psi^h)) = \int_0^L J^h\partial_3^h z^h\partial_3^h\varsigma^h dx_3^h$$
$$+\frac{Y}{2(1+\nu)}\int_{\Omega^h}\partial_\alpha^h r^h\partial_\alpha^h\rho^h d\boldsymbol{x}^h$$
$$+\int_{\Omega^h}\varepsilon_{\alpha\beta}\partial_\alpha^h\bar{\varphi}^{0h}\partial_\beta^h\psi^h d\boldsymbol{x}^h + \int_{\Omega^h}P_{\beta 3\alpha}(\partial_\alpha r^h\partial_\beta\psi^h - \partial_\beta^h\bar{\varphi}^{0h}\partial_\alpha^h\rho^h)d\boldsymbol{x}^h$$
$$+\int_{\Omega^h}P_{\beta 3\alpha}(\partial_\alpha^h w^h - \delta_\alpha^h)(\partial_3^h z^h\partial_\beta^h\psi^h - \partial_3^h\varsigma^h\partial_\beta^h\bar{\varphi}^{0h})d\boldsymbol{x}^h,$$

with, from (4.46) and (4.47)

$$J^h = \frac{Y}{2(1+\nu)}(I_1^h + I_2^h - \int_{\omega^h}(\partial_1^h w^h)^2 - \int_{\omega^h}(\partial_2^h w^h)^2)$$

and

$$w^h \in Q^h(\omega^h),$$
$$\int_{\omega^h}\partial_\alpha^h w^h\partial_\alpha^h\psi^h d\omega^h = \int_{\omega^h}(x_2^h\partial_1^h\psi^h - x_1^h\partial_2^h\psi^h)d\omega^h,$$
$$\text{for all } \psi^h \in H^1(\omega^h).$$

7. Conclusions

The main result in this paper is the model shown in Eqs. (6.5)–(6.7). From (6.5)–(6.6) we observe that the electric potential has no effect on the zeroth order components of the bendings ξ_α^h and the stretching ξ_3^h. This effect only appears on higher order terms as we can see from (6.7). Unfortunately, unless a specific dependence of φ^h on (x_1^h, x_2^h) is established, we cannot reduce (6.7) to a set of one-dimensional expressions as we did for the zeroth order displacements. Still, by taking particular cases of transversal section ω^h we can give a strong formulation of (6.7) as a set of two-dimensional expressions, as shown in (6.11). There we see that some of the terms are still global since they involve an integration over the whole transversal section. This is a remarkable difference with our previous work Viaño et al. (2015) (the electric potential was applied at the beam ends), where in the limit model the stretching was coupled with the electric potential, even for the zeroth order terms, and that the set of equations was reduced to one-dimensional expressions (despite having some global terms acting as coefficients).

We ended this work by showing two interesting particular cases. In Remark 6.2 we show that a trivial case is obtained when the potential depends only on the axial coordinate, and in Remark 6.3 we show the equations for the homogeneous isotropic case.

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For the sake of readability we give a list of the functional spaces and more important functions and notations involved in this paper.

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Juan Viaño, born in 1955, got a Ph.D. in Mathematics at the University of Santiago de Compostela and a Ph.D. in Numerical Analysis at the University Pierre et Marie Curie (Paris 6). He is Professor of Applied Mathematics at the University of Santiago de Compostela since 1988, member of the Promoter Group of the Node in Galicia of National Spanish Centre of Mathematics (IEMath-Galicia) since 2008 and member of the Promoter Group of Consortium of Technological Institute of Industrial Mathematics since 2012. He has been Head of the Department of Applied Mathematics ath the University of Santiago de Compostela for 6 years, Dean of the Faculty of Mathematics at the University of Santiago de Compostela for 8 years and Vice-rector of Teaching and Academic Organisation of the University of Santiago de Compostela for 2 years. He is and has been assesor and member of a number of committes of national and international level. He has published more than 70 papers in international journals, several books and directed more than 10 doctoral theses. He has also directed more than 10 national and international research projects.

Jorge Figueiredo, born in 1966, has a Bachelor degree in Physics/Applied Mathematics (Porto University, 1988) and a M.Sc. in Astrophysics and Spatial Techniques (Paris University, 1989). He got his Ph.D. in Applied Mathematical Sciences form Porto University (1996), under the theme A new numerical code to model pre-main-sequence low-mass stars. He joined the Dep. of Mathematics and Applications of Minho University as a Junior Teaching Assistant in 1993 and became Assistant Professor in 1996. The main research interests concern PDEs with particular emphasis on deriving reduced beam models from 3D piezoelectricity and developing high-order finite volume schemes for the shallow-water equations.

Carolina Ribeiro received her Bachelor degree in Mathematics from University of Porto and her Ph.D. degree in Mathematical Science, under the theme Asymptotic derivation of models for anisotropic piezoelectric beams and shallow arches, from University of Minho in 2009. She joined the Department of Mathematics and Applications of the University of Minho, as a Junior Teaching Assistant in 2000 and was promoted to Assistant Professor in 2009. Carolina's research interests focus in the area of partial differential equations with a particular emphasis on deriving beam models from 3D piezoelectricity and developing high-order finite volume schemes for the shallow water equations.

Ángel Rodríguez-Arós, born in 1976, has a Ph.D. in Applied Mathematics from the University of Santiago de Compostela, Spain, with European special mention (2005) and spent a post-doctoral year at the ONERA-site Toulouse, France (2008). Currently, he is Lecturer at the University of A Coruña. Over the years he has published over 20 papers in international journals, given around 20 talks in national and international meetings and has been member of nearly 10 national and international research projects. He has also published a text book in spherical trigonometry with applications to navigation (2012).