Asymptotic analysis of piezoelectric energy harvester

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1 Summary of the interested equations

The dynamic equations for a typical piezoelectric composite cantilever beam is

$$B_p \frac{\partial^4 w(x,t)}{\partial x^4} + m_p \frac{\partial^2 w(x,t)}{\partial t^2} = 0, \tag{1}$$

where B_p is the equivalent bending stiffness and m_p is the line mass density of the piezoelectric cantilever beam. If the piezoelectric elements attached to the cantilever beam is connected to an external electrical load R_l, we have

$$\frac{dQ_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{2}$$

For the underlying physics, we have the following constitutive equations

$$\begin{aligned} M_p(x,t) &= B_p \frac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\ q_p(x,t) &= e_p \frac{\partial^2 w(x,t)}{\partial x^2} + \varepsilon_p V_p(t), \end{aligned} \tag{3}$$

or equivalently,

$$\begin{cases} M_p(x,t) = B_p \dfrac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\ Q_p(x,t) = e_p \left[\dfrac{\partial w(x,t)}{\partial x} \right] \Bigg|_0^{l_p} + C_p V_p(t). \end{cases} \tag{4}$$

One end of the cantilever beam is fixed while the other end is free. So the boundary conditions are

$$\begin{cases} w(0,t) = w_b(t), \\ \dfrac{\partial w(0,t)}{\partial x} = 0, \end{cases} \tag{5}$$

and

$$\begin{cases} M_p(l_p,t) = B_p \dfrac{\partial^2 w(l_p,t)}{\partial x^2} - e_p V_p(t) = 0, \\ Q_p(l_p,t) = B_p \dfrac{\partial^3 w(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{6}$$

In the classical energy harvesting applications, the cantilever beam is subject to a periodical base excitation $w_b(t)$. Thus the dynamic response of the cantilever beam is decomposed as

$$w(x,t) = w_b(t) + w_{rel}(x,t), \tag{7}$$

where $w_{rel}(x,t)$ is the relative displacement function of the cantilever beam. In this way, the system is converted into

$$B_p \frac{\partial^4 w_{rel}(x,t)}{\partial x^4} + m_p \frac{\partial^2 w_{rel}(x,t)}{\partial t^2} = -m_p \frac{\partial^2 w_b(x,t)}{\partial t^2}, \tag{8}$$

$$e_p \left[\frac{\partial^2 w(x,t)}{\partial x \partial t} \right] \Bigg|_0^{l_p} + C_p \frac{dV_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{9}$$

$$\begin{cases} w_{rel}(0,t) = 0, \\ \dfrac{\partial w_{rel}(0,t)}{\partial x} = 0, \end{cases} \tag{10}$$

and

$$\begin{cases} B_p \dfrac{\partial^2 w_{rel}(l_p,t)}{\partial x^2} - e_p V_p(t) = 0, \\ \dfrac{\partial^3 w_{rel}(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{11}$$

$$w_b(t) = \xi_b e^{j\lambda t} \tag{12}$$

where ξ_b is usually a real vibration amplitude.

Here we are interested in the classical model of a piezoelectric cantilever beam energy harvester, whose model is described using the following set of equations:

$$u'''' - \lambda^2 u = 0, \tag{13}$$

and the accompanying boundary conditions:

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) + \dfrac{j\lambda\beta\alpha^2}{j\lambda\beta + 1} u'(1) = 0 \\ u'''(1) = 0 \end{cases}, \tag{14}$$

where λ is the eigenvalues for the problem, u denotes the displace function of the cantilever beam, β is the dimensionless externally connected resistance, and α is the dimensionless piezoelectric coefficient. They can be expressed as follows

$$\lambda = \omega\sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \alpha = e_p \sqrt{\frac{l_p}{C_p B_p}}, \tag{15}$$

where ω is angular frequency, m_p is line mass density, l_p is the length of the cantilever beam, B_p is the bending stiffness, C_p is the inherent capacitance of the piezoelectric layer, e_p is the charge accumulation number, R_l is the externally connected resistance. In practical applications, dielectric property of piezoelectric materials indicate that the parameter β is changed from a very small value, which is close to a short-circuit condition to a very large value, which corresponds to an open-circuit condition. Thus we have that $0 \leq \beta \leq \infty$.

2 Asymptotic analysis when β is small

Here we seek to find the behavior of the above system at a small value of connected resistance, i.e., $\beta \to 0$. In this case, we set β to be the parameter for asymptotic expansion, and

$$\begin{aligned} \lambda^{(k)} &= \lambda_0^{(k)} + \beta\lambda_1^{(k)} + \beta^2\lambda_2^{(k)} + \cdots \\ u^{(k)} &= u_0^{(k)} + \beta u_1^{(k)} + \beta^2 u_2^{(k)} + \cdots \end{aligned} \tag{16}$$

where $\lambda^{(k)}$ and $u^{(k)}$ are the kth eigenvalue and eigenfunction respectively of the above mentioned system under perturbation. $\lambda_0^{(k)}$ and $u_0^{(k)}$ are the corresponding eigenvalue and eigenfunction of the unperturbed system at $\beta = 0$:

$$u'''' - \lambda_0^2 u = 0, \tag{17}$$

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) = 0 \\ u'''(1) = 0 \end{cases}. \tag{18}$$

Obviously, the unperturbed system is a classical eigenvalue problem with the eigenvalues determined by

$$1 + \cosh(\sqrt{\lambda_0})\cos(\sqrt{\lambda_0}) = 0 \tag{19}$$

whose first several values are

$$\frac{\sqrt{\lambda_0^{(1)}}}{\pi} = 0.59686, \quad \frac{\sqrt{\lambda_0^{(2)}}}{\pi} = 1.49418, \quad \frac{\sqrt{\lambda_0^{(3)}}}{\pi} = 2.50025, \quad \frac{\sqrt{\lambda_0^{(4)}}}{\pi} = 3.49999, \quad \cdots \tag{20}$$

Take the asymptotic expansions and substitute them into the previously derived system of equations, we have the following asymptotic expansions to different orders of β:
$O(\beta^0)$:

$$\begin{cases} u_0'''' - \lambda_0^2 u_0 = 0 \\ u_0(0) = 0 \\ u_0'(0) = 0 \\ u_0''(1) = 0 \\ u_0'''(1) = 0 \end{cases} \tag{21}$$

$O(\beta^1)$:

$$\begin{cases} u_1'''' - \left(\lambda_0^2 u_1 + 2\lambda_0 u_0 \lambda_1\right) = 0 \\ u_1(0) = 0 \\ u_1'(0) = 0 \\ u_1''(1) + j\alpha^2 \lambda_0 u_0'(1) = 0 \\ u_1'''(1) = 0 \end{cases} \tag{22}$$

$O(\beta^2)$:

$$\begin{cases} u_2'''' - \left(\lambda_0^2 u_2 + 2\lambda_0 u_1 \lambda_1 + \lambda_1^2 u_0 + 2\lambda_0 u_0 \lambda_2\right) = 0 \\ u_2(0) = 0 \\ u_2'(0) = 0 \\ u_2''(1) + \alpha^2 \lambda_0 u_0'(1) + j\alpha^2 \left[\lambda_0 u_1'(1) + \lambda_1 u_0'(1)\right] = 0 \\ u_2'''(1) = 0 \end{cases} \tag{23}$$

3 Asymptotic analysis when β is large

Here we seek to find the behavior of the above system at a large value of connected resistance, i.e., $\beta \to \infty$. In this case, we set $\frac{1}{\beta}$ to be the parameter for asymptotic expansion and

$$\begin{aligned} \lambda^{(k)} &= \tilde{\lambda}_0^{(k)} + \left(\frac{1}{\beta}\right)\tilde{\lambda}_1^{(k)} + \left(\frac{1}{\beta}\right)^2 \tilde{\lambda}_2^{(k)} + \cdots \\ u^{(k)} &= \tilde{u}_0^{(k)} + \left(\frac{1}{\beta}\right)\tilde{u}_1^{(k)} + \left(\frac{1}{\beta}\right)^2 \tilde{u}_2^{(k)} + \cdots \end{aligned} \tag{24}$$

where $\tilde{\lambda}^{(k)}$ and $\tilde{u}^{(k)}$ are the kth eigenvalue and eigenfunction respectively of the above mentioned system under perturbation. $\tilde{\lambda}_0^{(k)}$ and $\tilde{u}_0^{(k)}$ are the corresponding eigenvalue and eigenfunction of the unperturbed system at $\beta = \infty$:
$O(\frac{1}{\beta^0})$:

$$\begin{cases} \tilde{u}_0'''' - \tilde{\lambda}_0^2 \tilde{u}_0 = 0 \\ \tilde{u}_0(0) = 0 \\ \tilde{u}_0'(0) = 0 \\ \tilde{u}_0''(1) + \alpha^2 \tilde{u}_0'(1) = 0 \\ \tilde{u}_0'''(1) = 0 \end{cases} \tag{25}$$

$O(\frac{1}{\beta^1})$:

$$\begin{cases} \tilde{u}_1'''' - \left(\tilde{\lambda}_0^2 u_1 + 2\tilde{\lambda}_0 \tilde{u}_0 \tilde{\lambda}_1\right) = 0 \\ \tilde{u}_1(0) = 0 \\ \tilde{u}_1'(0) = 0 \\ \tilde{u}_1''(1) + \alpha^2 \tilde{u}_1'(1) + \dfrac{j\alpha^2}{\tilde{\lambda}_0} \tilde{u}_0'(1) = 0 \\ \tilde{u}_1'''(1) = 0 \end{cases} \tag{26}$$

$O(\frac{1}{\beta^2})$:

$$\begin{cases} \tilde{u}_2'''' - \left(\tilde{\lambda}_0^2\tilde{u}_2 + 2\tilde{\lambda}_0\tilde{u}_1\tilde{\lambda}_1 + \tilde{\lambda}_1^2\tilde{u}_0 + 2\tilde{\lambda}_0\tilde{u}_0\tilde{\lambda}_2\right) = 0 \\ \tilde{u}_2(0) = 0 \\ \tilde{u}_2'(0) = 0 \\ \tilde{u}_2''(1) + \left[\alpha^2\tilde{u}_2'(1) - \frac{\alpha^2}{\tilde{\lambda}_0^2}\tilde{u}_0'(1)\right] + j\left[\frac{\alpha^2}{\tilde{\lambda}_0}\tilde{u}_1'(1) - \frac{\alpha^2\tilde{\lambda}_1}{\tilde{\lambda}_0^2}\tilde{u}_0'(1)\right] = 0 \\ \tilde{u}_2'''(1) = 0 \end{cases} \tag{27}$$

4 Asymptotic analysis in terms of small α^2

Directly using the eigenvalue analysis method for linear boundary value problem, we arrive at the equation for the eigenvalue λ:

$$\sqrt{\lambda}\left[1 + \left(\frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{2}\right)\cos\sqrt{\lambda}\right] + \frac{j\beta\lambda\alpha^2}{1+j\beta\lambda}\left[\left(\frac{e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}}{2}\right)\cos\sqrt{\lambda} + \left(\frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{2}\right)\sin\sqrt{\lambda}\right] = 0 \tag{28}$$

or

$$\sqrt{\lambda}\left[1 + \cosh\sqrt{\lambda}\cos\sqrt{\lambda}\right] + \frac{j\beta\lambda\alpha^2}{1+j\beta\lambda}\left[\sinh\sqrt{\lambda}\cos\sqrt{\lambda} + \cosh\sqrt{\lambda}\sin\sqrt{\lambda}\right] = 0 \tag{29}$$

Taking the parameter α^2 as the small parameter ϵ and expanding the eigenvalue λ in terms of this ϵ, we have

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots \tag{30}$$

and therefore:
$O(\epsilon^0)$:

$$1 + \cosh\sqrt{\lambda_0}\cos\sqrt{\lambda_0} = 0 \tag{31}$$

$O(\epsilon^1)$:

$$2j\beta\lambda_0\left(\cosh\sqrt{\lambda_0}\sin\sqrt{\lambda_0} + \sinh\sqrt{\lambda_0}\cos\sqrt{\lambda_0}\right) + (1+j\beta\lambda_0)\lambda_1\left(-\cosh\sqrt{\lambda_0}\sin\sqrt{\lambda_0} + \sinh\sqrt{\lambda_0}\cos\sqrt{\lambda_0}\right) = 0 \tag{32}$$

or equivalently

$$\lambda_1 = \frac{2j\beta\lambda_0}{(1+j\beta\lambda_0)}\frac{\left(\cosh\sqrt{\lambda_0}\sin\sqrt{\lambda_0} + \sinh\sqrt{\lambda_0}\cos\sqrt{\lambda_0}\right)}{\left(\cosh\sqrt{\lambda_0}\sin\sqrt{\lambda_0} - \sinh\sqrt{\lambda_0}\cos\sqrt{\lambda_0}\right)} \tag{33}$$

5 Asymptotic analysis in terms of small α^2

The forced vibration problem of a piezoelectric cantilever bimorph is described by

$$u'''' - \lambda^2 u = \lambda^2, \tag{34}$$

and the accompanying boundary conditions:

$$\begin{cases} u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \dfrac{j\lambda\beta}{j\lambda\beta+1}\epsilon u'(1) = 0, \\ u'''(1) = 0. \end{cases} \tag{35}$$

This problem can readily be solved using a conventional boundary value problem solver. Howevere, here we would like to develop an asymptotic expansion of the solution for the system. Using ϵ as a parameter, we have

$$u(x;\epsilon) = A_\epsilon \cos\sqrt{\lambda}x + B_\epsilon \sin\sqrt{\lambda}x + C_\epsilon \cosh\sqrt{\lambda}x + D_\epsilon \sinh\sqrt{\lambda}x - 1 \tag{36}$$

As a result, we have

$$\begin{aligned} u'(x;\epsilon) &= \sqrt{\lambda}\left(-A_\epsilon \sin\sqrt{\lambda}x + B_\epsilon \cos\sqrt{\lambda}x + C_\epsilon \sinh\sqrt{\lambda}x + D_\epsilon \cosh\sqrt{\lambda}x\right) \\ u''(x;\epsilon) &= \lambda\left(-A_\epsilon \cos\sqrt{\lambda}x - B_\epsilon \sin\sqrt{\lambda}x + C_\epsilon \cosh\sqrt{\lambda}x + D_\epsilon \sinh\sqrt{\lambda}x\right) \\ u'''(x;\epsilon) &= \lambda\sqrt{\lambda}\left(A_\epsilon \sin\sqrt{\lambda}x - B_\epsilon \cos\sqrt{\lambda}x + C_\epsilon \sinh\sqrt{\lambda}x + D_\epsilon \cosh\sqrt{\lambda}x\right) \end{aligned} \tag{37}$$

Thus the above boundary value problem is converted into the following linear equation systems:

$$\begin{cases} A_\epsilon + C_\epsilon = 1, \\ B_\epsilon + D_\epsilon = 0, \\ \left(-A_\epsilon \cos\sqrt{\lambda} - B_\epsilon \sin\sqrt{\lambda} + C_\epsilon \cosh\sqrt{\lambda} + D_\epsilon \sinh\sqrt{\lambda}\right) + \\ \dfrac{j\beta\sqrt{\lambda}}{j\lambda\beta+1}\epsilon\left(-A_\epsilon \sin\sqrt{\lambda} + B_\epsilon \cos\sqrt{\lambda}x + C_\epsilon \sinh\sqrt{\lambda}x + D_\epsilon \cosh\sqrt{\lambda}x\right) = 0, \\ A_\epsilon \sin\sqrt{\lambda} - B_\epsilon \cos\sqrt{\lambda} + C_\epsilon \sinh\sqrt{\lambda} + D_\epsilon \cosh\sqrt{\lambda} = 0. \end{cases} \tag{38}$$

Analytically, we can directly obtain the solution to this problem as

$$\begin{cases} A_\epsilon = \dfrac{1 + \cos\sqrt{\lambda}\cosh\sqrt{\lambda} - \sin\sqrt{\lambda}\sinh\sqrt{\lambda} + \epsilon\frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda}\left(\cos\sqrt{\lambda}\sinh\sqrt{\lambda}\right)}{2\left[1 + \cos\sqrt{\lambda}\cosh\sqrt{\lambda} + \epsilon\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\left(\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda}\right)\right]}, \\ B_\epsilon = \dfrac{\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda} + \epsilon\frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda}\left(\sin\sqrt{\lambda}\sinh\sqrt{\lambda}\right)}{2\left[1 + \cos\sqrt{\lambda}\cosh\sqrt{\lambda} + \epsilon\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\left(\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda}\right)\right]}, \\ C_\epsilon = \dfrac{-1 - \cos\sqrt{\lambda}\cosh\sqrt{\lambda} + \sin\sqrt{\lambda}\sinh\sqrt{\lambda} - \epsilon\frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda}\left(\cos\sqrt{\lambda}\sinh\sqrt{\lambda}\right)}{2\left[1 + \cos\sqrt{\lambda}\cosh\sqrt{\lambda} + \epsilon\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\left(\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda}\right)\right]}, \\ D_\epsilon = \dfrac{-\cos\sqrt{\lambda}\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}\cosh\sqrt{\lambda} - \epsilon\frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda}\left(\sin\sqrt{\lambda}\sinh\sqrt{\lambda}\right)}{2\left[1 + \cos\sqrt{\lambda}\cosh\sqrt{\lambda} + \epsilon\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\left(\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda}\right)\right]} \end{cases} \tag{39}$$

The resulting output voltage V_p, current I_p, and power P_p can be formulated as follows

$$\begin{cases} \tilde{V}_p = \dfrac{j\lambda\beta}{j\lambda\beta+1}\dfrac{\xi_b}{l_p}\dfrac{e_p}{C_p}u'(1), \\ \tilde{I}_p = \tilde{V}_p/R_l, \\ \tilde{P}_p = \tilde{V}_p^2/R_l. \end{cases} \tag{40}$$

Using the following regular expansion:

$$\begin{cases} A_\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots , \\ B_\epsilon = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \cdots , \\ C_\epsilon = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \cdots , \\ D_\epsilon = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots , \end{cases} \tag{41}$$

we obtain the successive expansion problem:
$O(\epsilon^0)$:

$$\begin{cases} A_0 + C_0 = 1, \\ B_0 + D_0 = 0, \\ -A_0 \cos\sqrt{\lambda} - B_0 \sin\sqrt{\lambda} + C_0 \cosh\sqrt{\lambda} + D_0 \sinh\sqrt{\lambda} = 0, \\ A_0 \sin\sqrt{\lambda} - B_0 \cos\sqrt{\lambda} + C_0 \sinh\sqrt{\lambda} + D_0 \cosh\sqrt{\lambda} = 0. \end{cases} \tag{42}$$

The solution is

$$\begin{cases} A_0 = \dfrac{1 + \cos\sqrt{\lambda}\cosh\sqrt{\lambda} - \sin\sqrt{\lambda}\sinh\sqrt{\lambda}}{2 + 2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}} \\ B_0 = \dfrac{\cosh\sqrt{\lambda}\sin\sqrt{\lambda} + \cos\sqrt{\lambda}\sinh\sqrt{\lambda}}{2 + 2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}} \\ C_0 = \dfrac{1 + \cos\sqrt{\lambda}\cosh\sqrt{\lambda} + \sin\sqrt{\lambda}\sinh\sqrt{\lambda}}{2 + 2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}} \\ D_0 = -\dfrac{\cosh\sqrt{\lambda}\sin\sqrt{\lambda} + \cos\sqrt{\lambda}\sinh\sqrt{\lambda}}{2 + 2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}} \end{cases} \tag{43}$$

Hence we have

$$-A_0 \sin\sqrt{\lambda} + B_0 \cos\sqrt{\lambda} + C_0 \sinh\sqrt{\lambda} + D_0 \cosh\sqrt{\lambda} = \frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1} \tag{44}$$

$O(\epsilon^1)$:

$$\begin{cases} A_1 + C_1 = 0, \\ B_1 + D_1 = 0, \\ \left(-A_1 \cos\sqrt{\lambda} - B_1 \sin\sqrt{\lambda} + C_1 \cosh\sqrt{\lambda} + D_1 \sinh\sqrt{\lambda}\right) + \\ \dfrac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1}\left(-A_0 \sin\sqrt{\lambda} + B_0 \cos\sqrt{\lambda} + C_0 \sinh\sqrt{\lambda} + D_0 \cosh\sqrt{\lambda}\right) = 0, \\ A_1 \sin\sqrt{\lambda} - B_1 \cos\sqrt{\lambda} + C_1 \sinh\sqrt{\lambda} + D_1 \cosh\sqrt{\lambda} = 0. \end{cases} \tag{45}$$

The solution is

$$\begin{cases} A_1 = \dfrac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\left(\dfrac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\left(\dfrac{\cos\sqrt{\lambda} + \cosh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 2}\right) \\ B_1 = \dfrac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\left(\dfrac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\left(\dfrac{-\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 2}\right) \\ C_1 = \dfrac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\left(\dfrac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\left(-\dfrac{\cos\sqrt{\lambda} + \cosh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 2}\right) \\ D_1 = \dfrac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\left(\dfrac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\left(\dfrac{-\sin\sqrt{\lambda} + \sinh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 2}\right) \end{cases} \tag{46}$$

Then we have

$$\begin{aligned} &-A_1 \sin\sqrt{\lambda} + B_1 \cos\sqrt{\lambda} + C_1 \sinh\sqrt{\lambda} + D_1 \cosh\sqrt{\lambda} \\ &= \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\left(\frac{\sin\sqrt{\lambda} - \sinh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\left(\frac{\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right) \end{aligned} \tag{47}$$

$O(\epsilon^2)$:

$$\begin{cases} A_2 + C_2 = 0, \\ B_2 + D_2 = 0, \\ \left(-A_2 \cos\sqrt{\lambda} - B_2 \sin\sqrt{\lambda} + C_2 \cosh\sqrt{\lambda} + D_2 \sinh\sqrt{\lambda}\right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1}\left(-A_1 \sin\sqrt{\lambda} + B_1 \cos\sqrt{\lambda} + C_1 \sinh\sqrt{\lambda} + D_1 \cosh\sqrt{\lambda}\right) = 0, \\ A_2 \sin\sqrt{\lambda} - B_2 \cos\sqrt{\lambda} + C_2 \sinh\sqrt{\lambda} + D_2 \cosh\sqrt{\lambda} = 0. \end{cases} \tag{48}$$

The solution is

$$\begin{cases} A_2 = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\right)^2 \left(\frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right)\left(\frac{\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right)\left(\frac{\cos\sqrt{\lambda}+\cosh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+2}\right) \\ B_2 = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\right)^2 \left(\frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right)\left(\frac{\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right)\left(\frac{-\sinh\sqrt{\lambda}+\sin\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+2}\right) \\ C_2 = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\right)^2 \left(\frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right)\left(\frac{\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right)\left(-\frac{\cos\sqrt{\lambda}+\cosh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+2}\right) \\ D_2 = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\right)^2 \left(\frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right)\left(\frac{\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right)\left(\frac{-\sin\sqrt{\lambda}+\sinh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+2}\right) \end{cases} \tag{49}$$

To get higher order expansions, we can use the following iteration method:
$O(\epsilon^{k+1})$ $(k \geq 1)$:

$$\begin{cases} A_{k+1} + C_{k+1} = 0, \\ B_{k+1} + D_{k+1} = 0, \\ \left(-A_{k+1} \cos\sqrt{\lambda} - B_{k+1} \sin\sqrt{\lambda} + C_{k+1} \cosh\sqrt{\lambda} + D_{k+1} \sinh\sqrt{\lambda}\right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1}\left(-A_k \sin\sqrt{\lambda} + B_k \cos\sqrt{\lambda} + C_k \sinh\sqrt{\lambda} + D_k \cosh\sqrt{\lambda}\right) = 0, \\ A_{k+1} \sin\sqrt{\lambda} - B_{k+1} \cos\sqrt{\lambda} + C_{k+1} \sinh\sqrt{\lambda} + D_{k+1} \cosh\sqrt{\lambda} = 0. \end{cases} \tag{50}$$

The solution is

$$\begin{cases} A_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\right)\left(\frac{\cos\sqrt{\lambda}+\cosh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+2}\right)(Q_k) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\right)\left(\frac{-\sinh\sqrt{\lambda}+\sin\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+2}\right)(Q_k) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\right)\left(-\frac{\cos\sqrt{\lambda}+\cosh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+2}\right)(Q_k) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\right)\left(\frac{-\sin\sqrt{\lambda}+\sinh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+2}\right)(Q_k) \end{cases} \tag{51}$$

where for $k \geq 2$

$$Q_k = -A_k \sin\sqrt{\lambda} + B_k \cos\sqrt{\lambda} + C_k \sinh\sqrt{\lambda} + D_k \cosh\sqrt{\lambda}, \tag{52}$$

and for $k \geq 0$

$$\begin{aligned} Q_{k+1} &= -A_{k+1} \sin\sqrt{\lambda} + B_{k+1} \cos\sqrt{\lambda} + C_{k+1} \sinh\sqrt{\lambda} + D_{k+1} \cosh\sqrt{\lambda} \\ &= -\left(\frac{\sin\sqrt{\lambda}\cosh\sqrt{\lambda} + \cos\sqrt{\lambda}\sinh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right)\left(\frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\right) Q_k, \end{aligned} \tag{53}$$

and

$$\begin{aligned} Q_1 &= -A_1 \sin\sqrt{\lambda} + B_1 \cos\sqrt{\lambda} + C_1 \sinh\sqrt{\lambda} + D_1 \cosh\sqrt{\lambda} \\ &= \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda}\left(\frac{\sin\sqrt{\lambda} - \sinh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right)\left(\frac{\cos\sqrt{\lambda}\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}\cosh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda}+1}\right) \end{aligned} \tag{54}$$

$$Q_0 = \frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1} \tag{55}$$

Hence it is shown that for $k \geq 0$

$$\begin{aligned} Q_k &= -\left(\frac{\sin\sqrt{\lambda}\cosh\sqrt{\lambda} + \cos\sqrt{\lambda}\sinh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\right)Q_k \\ &= \left[-\left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\right)\left(\frac{\sin\sqrt{\lambda}\cosh\sqrt{\lambda} + \cos\sqrt{\lambda}\sinh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\right]^k\left(\frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right) \end{aligned} \tag{56}$$

As a result, we obtain that for $k \geq 0$

$$\begin{cases} A_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\right)^{k+1}\left(\frac{-\sin\sqrt{\lambda}\cosh\sqrt{\lambda} - \cos\sqrt{\lambda}\sinh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)^k\left(\frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\left(\frac{\cos\sqrt{\lambda} + \cosh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 2}\right) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\right)^{k+1}\left(\frac{-\sin\sqrt{\lambda}\cosh\sqrt{\lambda} - \cos\sqrt{\lambda}\sinh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)^k\left(\frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\left(\frac{-\sinh\sqrt{\lambda} + \sin\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 2}\right) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\right)^{k+1}\left(\frac{-\sin\sqrt{\lambda}\cosh\sqrt{\lambda} - \cos\sqrt{\lambda}\sinh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)^k\left(\frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\left(\frac{-\cos\sqrt{\lambda} - \cosh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 2}\right) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda}\right)^{k+1}\left(\frac{-\sin\sqrt{\lambda}\cosh\sqrt{\lambda} - \cos\sqrt{\lambda}\sinh\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)^k\left(\frac{\sinh\sqrt{\lambda} - \sin\sqrt{\lambda}}{\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 1}\right)\left(\frac{-\sin\sqrt{\lambda} + \sinh\sqrt{\lambda}}{2\cos\sqrt{\lambda}\cosh\sqrt{\lambda} + 2}\right) \end{cases} \tag{57}$$