

# Asymptotic analysis of piezoelectric energy harvester

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## 1 Summary of the interested equations

The dynamic equations for a typical piezoelectric composite cantilever beam is

$$B_p \frac{\partial^4 w(x, t)}{\partial x^4} + m_p \frac{\partial^2 w(x, t)}{\partial t^2} = 0, \quad (1)$$

where  $B_p$  is the equivalent bending stiffness and  $m_p$  is the line mass density of the piezoelectric cantilever beam. If the piezoelectric elements attached to the cantilever beam is connected to an external electrical load  $R_l$ , we have

$$\frac{dQ_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \quad (2)$$

For the underlying physics, we have the following constitutive equations

$$\begin{aligned} M_p(x, t) &= B_p \frac{\partial^2 w(x, t)}{\partial x^2} - e_p V_p(t), \\ q_p(x, t) &= e_p \frac{\partial^2 w(x, t)}{\partial x^2} + \varepsilon_p V_p(t), \end{aligned} \quad (3)$$

or equivalently,

$$\begin{cases} M_p(x, t) = B_p \frac{\partial^2 w(x, t)}{\partial x^2} - e_p V_p(t), \\ Q_p(x, t) = e_p \left[ \frac{\partial w(x, t)}{\partial x} \right] \Big|_0^{l_p} + C_p V_p(t). \end{cases} \quad (4)$$

One end of the cantilever beam is fixed while the other end is free. So the boundary conditions are

$$\begin{cases} w(0, t) = w_b(t), \\ \frac{\partial w(0, t)}{\partial x} = 0, \end{cases} \quad (5)$$

and

$$\begin{cases} M_p(l_p, t) = B_p \frac{\partial^2 w(l_p, t)}{\partial x^2} - e_p V_p(t) = 0, \\ Q_p(l_p, t) = B_p \frac{\partial^3 w(l_p, t)}{\partial x^3} = 0. \end{cases} \quad (6)$$

In the classical energy harvesting applications, the cantilever beam is subject to a periodical base excitation

$$w_b(t) = \xi_b e^{j\lambda t} \quad (7)$$

where  $\xi_b$  is usually a real vibration amplitude.

Here we are interested in the classical model of a piezoelectric cantilever beam energy harvester, whose model is described using the following set of equations:

$$u'''' - \lambda^2 u = 0, \quad (8)$$

and the accompanying boundary conditions:

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) + \frac{j\lambda\beta\alpha^2}{j\lambda\beta + 1} u'(1) = 0 \\ u'''(1) = 0 \end{cases}, \quad (9)$$

where  $\lambda$  is the eigenvalues for the problem,  $u$  denotes the displace function of the cantilever beam,  $\beta$  is the dimensionless externally connected resistance, and  $\alpha$  is the dimensionless piezoelectric coefficient. They can be expressed as follows

$$\lambda = \omega \sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \alpha = e_p \sqrt{\frac{l_p}{C_p B_p}}, \quad (10)$$

where  $\omega$  is angular frequency,  $m_p$  is line mass density,  $l_p$  is the length of the cantilever beam,  $B_p$  is the bending stiffness,  $C_p$  is the inherent capacitance of the piezoelectric layer,  $e_p$  is the charge accumulation number,  $R_l$  is the externally connected resistance. In practical applications, dielectric property of piezoelectric materials indicate that the parameter  $\beta$  is changed from a very small value, which is close to a short-circuit condition to a very large value, which corresponds to an open-circuit condition. Thus we have that  $0 \leq \beta \leq \infty$ .

## 2 Asymptotic analysis when $\beta$ is small

Here we seek to find the behavior of the above system at a small value of connected resistance, i.e.,  $\beta \rightarrow 0$ . In this case, we set  $\beta$  to be the parameter for asymptotic expansion, and

$$\begin{aligned} \lambda^{(k)} &= \lambda_0^{(k)} + \beta \lambda_1^{(k)} + \beta^2 \lambda_2^{(k)} + \dots \\ u^{(k)} &= u_0^{(k)} + \beta u_1^{(k)} + \beta^2 u_2^{(k)} + \dots \end{aligned} \quad (11)$$

where  $\lambda^{(k)}$  and  $u^{(k)}$  are the  $k$ th eigenvalue and eigenfunction respectively of the above mentioned system under perturbation.  $\lambda_0^{(k)}$  and  $u_0^{(k)}$  are the corresponding eigenvalue and eigenfunction of the unperturbed system at  $\beta = 0$ :

$$u'''' - \lambda_0^2 u = 0, \quad (12)$$

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) = 0 \\ u'''(1) = 0 \end{cases}. \quad (13)$$

Obviously, the unperturbed system is a classical eigenvalue problem with the eigenvalues determined by

$$1 + \cosh(\sqrt{\lambda_0}) \cos(\sqrt{\lambda_0}) = 0 \quad (14)$$

whose first several values are

$$\frac{\sqrt{\lambda_0^{(1)}}}{\pi} = 0.59686, \quad \frac{\sqrt{\lambda_0^{(2)}}}{\pi} = 1.49418, \quad \frac{\sqrt{\lambda_0^{(3)}}}{\pi} = 2.50025, \quad \frac{\sqrt{\lambda_0^{(4)}}}{\pi} = 3.49999, \quad \dots \quad (15)$$

Take the asymptotic expansions and substitute them into the previously derived system of equations, we have the following asymptotic expansions to different orders of  $\beta$ :

$O(\beta^0)$ :

$$\begin{cases} u_0'''' - \lambda_0^2 u_0 = 0 \\ u_0(0) = 0 \\ u_0'(0) = 0 \\ u_0''(1) = 0 \\ u_0'''(1) = 0 \end{cases} \quad (16)$$

$O(\beta^1)$ :

$$\begin{cases} u_1'''' - (\lambda_0^2 u_1 + 2\lambda_0 u_0 \lambda_1) = 0 \\ u_1(0) = 0 \\ u_1'(0) = 0 \\ u_1''(1) + j\alpha^2 \lambda_0 u_0'(1) = 0 \\ u_1'''(1) = 0 \end{cases} \quad (17)$$

$O(\beta^2)$ :

$$\begin{cases} u_2'''' - (\lambda_0^2 u_2 + 2\lambda_0 u_1 \lambda_1 + \lambda_1^2 u_0 + 2\lambda_0 u_0 \lambda_2) = 0 \\ u_2(0) = 0 \\ u_2'(0) = 0 \\ u_2''(1) + \alpha^2 \lambda_0 u_0'(1) + j\alpha^2 [\lambda_0 u_1'(1) + \lambda_1 u_0'(1)] = 0 \\ u_2'''(1) = 0 \end{cases} \quad (18)$$

### 3 Asymptotic analysis when $\beta$ is large

Here we seek to find the behavior of the above system at a large value of connected resistance, i.e.,  $\beta \rightarrow \infty$ . In this case, we set  $\frac{1}{\beta}$  to be the parameter for asymptotic expansion and

$$\begin{aligned} \lambda^{(k)} &= \tilde{\lambda}_0^{(k)} + \left(\frac{1}{\beta}\right) \tilde{\lambda}_1^{(k)} + \left(\frac{1}{\beta}\right)^2 \tilde{\lambda}_2^{(k)} + \dots \\ u^{(k)} &= \tilde{u}_0^{(k)} + \left(\frac{1}{\beta}\right) \tilde{u}_1^{(k)} + \left(\frac{1}{\beta}\right)^2 \tilde{u}_2^{(k)} + \dots \end{aligned} \quad (19)$$

where  $\tilde{\lambda}^{(k)}$  and  $\tilde{u}^{(k)}$  are the  $k$ th eigenvalue and eigenfunction respectively of the above mentioned system under perturbation.  $\tilde{\lambda}_0^{(k)}$  and  $\tilde{u}_0^{(k)}$  are the corresponding eigenvalue and eigenfunction of the unperturbed system at  $\beta = \infty$ :

$O(\frac{1}{\beta^0})$ :

$$\begin{cases} \tilde{u}_0'''' - \tilde{\lambda}_0^2 \tilde{u}_0 = 0 \\ \tilde{u}_0(0) = 0 \\ \tilde{u}_0'(0) = 0 \\ \tilde{u}_0''(1) + \alpha^2 \tilde{u}_0'(1) = 0 \\ \tilde{u}_0'''(1) = 0 \end{cases} \quad (20)$$

$O(\frac{1}{\beta^1})$ :

$$\begin{cases} \tilde{u}_1'''' - (\tilde{\lambda}_0^2 u_1 + 2\tilde{\lambda}_0 \tilde{u}_0 \tilde{\lambda}_1) = 0 \\ \tilde{u}_1(0) = 0 \\ \tilde{u}_1'(0) = 0 \\ \tilde{u}_1''(1) + \alpha^2 \tilde{u}_1'(1) + \frac{j\alpha^2}{\tilde{\lambda}_0} \tilde{u}_0'(1) = 0 \\ \tilde{u}_1'''(1) = 0 \end{cases} \quad (21)$$

$O(\frac{1}{\beta^2})$ :

$$\begin{cases} \tilde{u}_2'''' - (\tilde{\lambda}_0^2 \tilde{u}_2 + 2\tilde{\lambda}_0 \tilde{u}_1 \tilde{\lambda}_1 + \tilde{\lambda}_1^2 \tilde{u}_0 + 2\tilde{\lambda}_0 \tilde{u}_0 \tilde{\lambda}_2) = 0 \\ \tilde{u}_2(0) = 0 \\ \tilde{u}_2'(0) = 0 \\ \tilde{u}_2''(1) + \left[ \alpha^2 \tilde{u}_2'(1) - \frac{\alpha^2}{\tilde{\lambda}_0^2} \tilde{u}_0'(1) \right] + j \left[ \frac{\alpha^2}{\tilde{\lambda}_0} \tilde{u}_1'(1) - \frac{\alpha^2 \tilde{\lambda}_1}{\tilde{\lambda}_0^2} \tilde{u}_0'(1) \right] = 0 \\ \tilde{u}_2'''(1) = 0 \end{cases} \quad (22)$$

### 4 Asymptotic analysis in terms of small $\alpha^2$

Directly using the eigenvalue analysis method for linear boundary value problem, we arrive at the equation for the eigenvalue  $\lambda$ :

$$\sqrt{\lambda} \left[ 1 + \left( \frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{2} \right) \cos \sqrt{\lambda} \right] + \frac{j\beta\lambda\alpha^2}{1+j\beta\lambda} \left[ \left( \frac{e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}}{2} \right) \cos \sqrt{\lambda} + \left( \frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{2} \right) \sin \sqrt{\lambda} \right] = 0 \quad (23)$$

or

$$\sqrt{\lambda} \left[ 1 + \cosh \sqrt{\lambda} \cos \sqrt{\lambda} \right] + \frac{j\beta\lambda\alpha^2}{1+j\beta\lambda} \left[ \sinh \sqrt{\lambda} \cos \sqrt{\lambda} + \cosh \sqrt{\lambda} \sin \sqrt{\lambda} \right] = 0 \quad (24)$$

Taking the parameter  $\alpha^2$  as the small parameter  $\epsilon$  and expanding the eigenvalue  $\lambda$  in terms of this  $\epsilon$ , we have

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \dots \quad (25)$$

and therefore:

$O(\epsilon^0)$ :

$$1 + \cosh \sqrt{\lambda_0} \cos \sqrt{\lambda_0} = 0 \quad (26)$$

$O(\epsilon^1)$ :

$$2j\beta\lambda_0 \left( \cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} + \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0} \right) + (1+j\beta\lambda_0)\lambda_1 \left( -\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} + \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0} \right) = 0 \quad (27)$$

or equivalently

$$\lambda_1 = \frac{2j\beta\lambda_0}{(1+j\beta\lambda_0)} \frac{(\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} + \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0})}{(\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} - \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0})} \quad (28)$$

## 5 Asymptotic analysis in terms of small $\alpha^2$

The forced vibration problem of a piezoelectric cantilever bimorph is described by

$$u'''' - \lambda^2 u = \lambda^2, \quad (29)$$

and the accompanying boundary conditions:

$$\begin{cases} u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \frac{j\lambda\beta}{j\lambda\beta + 1} \epsilon u'(1) = 0, \\ u'''(1) = 0. \end{cases} \quad (30)$$

This problem can readily be solved using a conventional boundary value problem solver. However, here we would like to develop an asymptotic expansion of the solution for the system. Using  $\epsilon$  as a parameter, we have

$$u(x; \epsilon) = A_\epsilon \cos \sqrt{\lambda} x + B_\epsilon \sin \sqrt{\lambda} x + C_\epsilon \cosh \sqrt{\lambda} x + D_\epsilon \sinh \sqrt{\lambda} x - 1 \quad (31)$$

As a result, we have

$$\begin{aligned} u'(x; \epsilon) &= \sqrt{\lambda} \left( -A_\epsilon \sin \sqrt{\lambda} x + B_\epsilon \cos \sqrt{\lambda} x + C_\epsilon \sinh \sqrt{\lambda} x + D_\epsilon \cosh \sqrt{\lambda} x \right) \\ u''(x; \epsilon) &= \lambda \left( -A_\epsilon \cos \sqrt{\lambda} x - B_\epsilon \sin \sqrt{\lambda} x + C_\epsilon \cosh \sqrt{\lambda} x + D_\epsilon \sinh \sqrt{\lambda} x \right) \\ u'''(x; \epsilon) &= \lambda \sqrt{\lambda} \left( A_\epsilon \sin \sqrt{\lambda} x - B_\epsilon \cos \sqrt{\lambda} x + C_\epsilon \sinh \sqrt{\lambda} x + D_\epsilon \cosh \sqrt{\lambda} x \right) \end{aligned} \quad (32)$$

Thus the above boundary value problem is converted into the following linear equation systems:

$$\begin{cases} A_\epsilon + C_\epsilon = 1, \\ B_\epsilon + D_\epsilon = 0, \\ \left( -A_\epsilon \cos \sqrt{\lambda} - B_\epsilon \sin \sqrt{\lambda} + C_\epsilon \cosh \sqrt{\lambda} + D_\epsilon \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \epsilon \left( -A_\epsilon \sin \sqrt{\lambda} + B_\epsilon \cos \sqrt{\lambda} + C_\epsilon \sinh \sqrt{\lambda} + D_\epsilon \cosh \sqrt{\lambda} \right) = 0, \\ A_\epsilon \sin \sqrt{\lambda} - B_\epsilon \cos \sqrt{\lambda} + C_\epsilon \sinh \sqrt{\lambda} + D_\epsilon \cosh \sqrt{\lambda} = 0. \end{cases} \quad (33)$$

Analytically, we can directly obtain the solution to this problem as

$$\begin{cases} A_\epsilon = \frac{1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} - \sin \sqrt{\lambda} \sinh \sqrt{\lambda} + \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left( \cos \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[ 1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left( \cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]}, \\ B_\epsilon = \frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left( \sin \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[ 1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left( \cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]}, \\ C_\epsilon = \frac{-1 - \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \sin \sqrt{\lambda} \sinh \sqrt{\lambda} - \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left( \cos \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[ 1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left( \cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]}, \\ D_\epsilon = \frac{-\cos \sqrt{\lambda} \sinh \sqrt{\lambda} - \sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left( \sin \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[ 1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left( \cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]} \end{cases} \quad (34)$$

The resulting output voltage  $V_p$ , current  $I_p$ , and power  $P_p$  can be formulated as follows

$$\begin{cases} \tilde{V}_p = \frac{j\lambda\beta}{j\lambda\beta + 1} \frac{\xi_b}{l_p} \frac{e_p}{C_p} u'(1), \\ \tilde{I}_p = \tilde{V}_p / R_l, \\ \tilde{P}_p = \tilde{V}_p^2 / R_l. \end{cases} \quad (35)$$

Using the following regular expansion:

$$\begin{cases} A_\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \dots, \\ B_\epsilon = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \dots, \\ C_\epsilon = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \dots, \\ D_\epsilon = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots, \end{cases} \quad (36)$$

we obtain the successive expansion problem:

$O(\epsilon^0)$ :

$$\begin{cases} A_0 + C_0 = 1, \\ B_0 + D_0 = 0, \\ -A_0 \cos \sqrt{\lambda} - B_0 \sin \sqrt{\lambda} + C_0 \cosh \sqrt{\lambda} + D_0 \sinh \sqrt{\lambda} = 0, \\ A_0 \sin \sqrt{\lambda} - B_0 \cos \sqrt{\lambda} + C_0 \sinh \sqrt{\lambda} + D_0 \cosh \sqrt{\lambda} = 0. \end{cases} \quad (37)$$

The solution is

$$\begin{cases} A_0 = \frac{1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} - \sin \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \\ B_0 = \frac{\cosh \sqrt{\lambda} \sin \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \\ C_0 = \frac{1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \sin \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \\ D_0 = -\frac{\cosh \sqrt{\lambda} \sin \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \end{cases} \quad (38)$$

Hence we have

$$-A_0 \sin \sqrt{\lambda} + B_0 \cos \sqrt{\lambda} + C_0 \sinh \sqrt{\lambda} + D_0 \cosh \sqrt{\lambda} = \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \quad (39)$$

$O(\epsilon^1)$ :

$$\begin{cases} A_1 + C_1 = 0, \\ B_1 + D_1 = 0, \\ \left( -A_1 \cos \sqrt{\lambda} - B_1 \sin \sqrt{\lambda} + C_1 \cosh \sqrt{\lambda} + D_1 \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \left( -A_0 \sin \sqrt{\lambda} + B_0 \cos \sqrt{\lambda} + C_0 \sinh \sqrt{\lambda} + D_0 \cosh \sqrt{\lambda} \right) = 0, \\ A_1 \sin \sqrt{\lambda} - B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} = 0. \end{cases} \quad (40)$$

The solution is

$$\begin{cases} A_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ B_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ C_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( -\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ D_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \end{cases} \quad (41)$$

Then we have

$$\begin{aligned} & -A_1 \sin \sqrt{\lambda} + B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} \\ &= \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left( \frac{\sin \sqrt{\lambda} - \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \end{aligned} \quad (42)$$

$$O(\epsilon^2): \quad \left\{ \begin{array}{l} A_2 + C_2 = 0, \\ B_2 + D_2 = 0, \\ \left( -A_2 \cos \sqrt{\lambda} - B_2 \sin \sqrt{\lambda} + C_2 \cosh \sqrt{\lambda} + D_2 \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \left( -A_1 \sin \sqrt{\lambda} + B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} \right) = 0, \\ A_2 \sin \sqrt{\lambda} - B_2 \cos \sqrt{\lambda} + C_2 \sinh \sqrt{\lambda} + D_2 \cosh \sqrt{\lambda} = 0. \end{array} \right. \quad (43)$$

The solution is

$$\left\{ \begin{array}{l} A_2 = \left( \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right)^2 \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ B_2 = \left( \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right)^2 \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ C_2 = \left( \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right)^2 \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( -\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ D_2 = \left( \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right)^2 \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \end{array} \right. \quad (44)$$

To get higher order expansions, we can use the following iteration method:  
 $O(\epsilon^{k+1})$  ( $k \geq 1$ ):

$$\left\{ \begin{array}{l} A_{k+1} + C_{k+1} = 0, \\ B_{k+1} + D_{k+1} = 0, \\ \left( -A_{k+1} \cos \sqrt{\lambda} - B_{k+1} \sin \sqrt{\lambda} + C_{k+1} \cosh \sqrt{\lambda} + D_{k+1} \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \left( -A_k \sin \sqrt{\lambda} + B_k \cos \sqrt{\lambda} + C_k \sinh \sqrt{\lambda} + D_k \cosh \sqrt{\lambda} \right) = 0, \\ A_{k+1} \sin \sqrt{\lambda} - B_{k+1} \cos \sqrt{\lambda} + C_{k+1} \sinh \sqrt{\lambda} + D_{k+1} \cosh \sqrt{\lambda} = 0. \end{array} \right. \quad (45)$$

The solution is

$$\left\{ \begin{array}{l} A_{k+1} = \left( \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right) \left( \frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \\ B_{k+1} = \left( \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right) \left( \frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \\ C_{k+1} = \left( \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right) \left( -\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \\ D_{k+1} = \left( \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right) \left( \frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \end{array} \right. \quad (46)$$

where for  $k \geq 2$

$$Q_k = -A_k \sin \sqrt{\lambda} + B_k \cos \sqrt{\lambda} + C_k \sinh \sqrt{\lambda} + D_k \cosh \sqrt{\lambda}, \quad (47)$$

and for  $k \geq 0$

$$\begin{aligned} Q_{k+1} &= -A_{k+1} \sin \sqrt{\lambda} + B_{k+1} \cos \sqrt{\lambda} + C_{k+1} \sinh \sqrt{\lambda} + D_{k+1} \cosh \sqrt{\lambda} \\ &= -\left( \frac{\sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \right) Q_k, \end{aligned} \quad (48)$$

and

$$\begin{aligned} Q_1 &= -A_1 \sin \sqrt{\lambda} + B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} \\ &= \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left( \frac{\sin \sqrt{\lambda} - \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \end{aligned} \quad (49)$$

$$Q_0 = \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \quad (50)$$

Hence it is shown that for  $k \geq 0$

$$\begin{aligned} Q_k &= - \left( \frac{\sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) Q_k \\ &= \left[ - \left( \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) \left( \frac{\sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \right]^k \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \end{aligned} \quad (51)$$

As a result, we obtain that for  $k \geq 0$

$$\begin{cases} A_{k+1} = \left( \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left( \frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ B_{k+1} = \left( \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left( \frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ C_{k+1} = \left( \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left( \frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{-\cos \sqrt{\lambda} - \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ D_{k+1} = \left( \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left( \frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left( \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left( \frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \end{cases} \quad (52)$$