Revisit to the theoretical analysis of a classical piezoelectric cantilever energy harvester

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1 Summary of the interested equations

The dynamic equations for a typical piezoelectric composite cantilever beam is

$$B_p \frac{\partial^4 w(x,t)}{\partial x^4} + m_p \frac{\partial^2 w(x,t)}{\partial t^2} = 0, \tag{1}$$

where B_p is the equivalent bending stiffness and m_p is the line mass density of the piezoelectric cantilever beam. If the piezoelectric elements attached to the cantilever beam is connected to an external electrical load R_l, we have

$$\frac{dQ_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{2}$$

For the underlying physics, we have the following constitutive equations

$$
\begin{aligned}
M_p(x,t) &= B_p \frac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\
q_p(x,t) &= e_p \frac{\partial^2 w(x,t)}{\partial x^2} + \varepsilon_p V_p(t),
\end{aligned}
\tag{3}
$$

or equivalently,

$$
\begin{cases}
M_p(x,t) = B_p \dfrac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\
Q_p(x,t) = e_p \left[\dfrac{\partial w(x,t)}{\partial x} \right] \Big|_0^{l_p} + C_p V_p(t).
\end{cases}
\tag{4}
$$

One end of the cantilever beam is fixed while the other end is free. So the boundary conditions are

$$
\begin{cases}
w(0,t) = w_b(t), \\
\dfrac{\partial w(0,t)}{\partial x} = 0,
\end{cases}
\tag{5}
$$

and

$$
\begin{cases}
M_p(l_p,t) = B_p \dfrac{\partial^2 w(l_p,t)}{\partial x^2} - e_p V_p(t) = 0, \\
N_p(l_p,t) = \dfrac{\partial M_p(l_p,t)}{\partial x} = B_p \dfrac{\partial^3 w(l_p,t)}{\partial x^3} = 0.
\end{cases}
\tag{6}
$$

In the classical energy harvesting applications, the cantilever beam is subject to a periodical base excitation $w_b(t)$. Thus the dynamic response of the cantilever beam is decomposed as

$$w(x,t) = w_b(t) + w_{rel}(x,t), \tag{7}$$

where $w_{rel}(x,t)$ is the relative displacement function of the cantilever beam. In this way, the system is converted into

$$B_p \frac{\partial^4 w_{rel}(x,t)}{\partial x^4} + m_p \frac{\partial^2 w_{rel}(x,t)}{\partial t^2} = -m_p \frac{\partial^2 w_b(t)}{\partial t^2}, \tag{8}$$

$$e_p \left[\frac{\partial^2 w_{rel}(x,t)}{\partial x \partial t} \right] \Big|_0^{l_p} + C_p \frac{dV_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{9}$$

$$\begin{cases} w_{rel}(0,t) = 0, \\ \dfrac{\partial w_{rel}(0,t)}{\partial x} = 0, \end{cases} \tag{10}$$

and

$$\begin{cases} B_p \dfrac{\partial^2 w_{rel}(l_p,t)}{\partial x^2} - e_p V_p(t) = 0, \\ \dfrac{\partial^3 w_{rel}(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{11}$$

Considering a sinusoidal base excitation

$$w_b(t) = \eta_b e^{j\sigma_b t} \tag{12}$$

where ξ_b is usually a real vibration amplitude, the steady state solution for the above system can be reasonably set as

$$w_{rel}(x,t) = \eta_{rel}(x) e^{j\sigma_b t}, \quad V_p(t) = \tilde{V}_p e^{j\sigma_b t}, \tag{13}$$

where $\eta_{rel}(x)$ and $\tilde{V}_p$ are complex amplitudes. Then the above system is again simplified as

$$B_p \dfrac{\partial^4 \eta_{rel}(x)}{\partial x^4} - m_p \sigma_b^2 \eta_{rel}(x) = m_p \sigma_b^2 \eta_b, \tag{14}$$

$$\begin{cases} \eta_{rel}(0) = 0, \\ \dfrac{\partial \eta_{rel}(0)}{\partial x} = 0, \end{cases} \tag{15}$$

and

$$\begin{cases} B_p \dfrac{\partial^2 \eta_{rel}(l_p)}{\partial x^2} + \dfrac{j\sigma_b R_l}{1 + j\sigma_b C_p R_l} e_p^2 \dfrac{\partial \eta_{rel}(l_p)}{\partial x} = 0, \\ \dfrac{\partial^3 \eta_{rel}(l_p)}{\partial x^3} = 0. \end{cases} \tag{16}$$

Note that here we assume a sinusoidal steady state response, which is not actually validated theoretically.

Obviously we can have the following dimensionless scheme:

$$\eta_{rel} \sim u\eta_b, \quad x \sim z l_p \tag{17}$$

and therefore the following dimensionless parameters

$$\sigma = \sigma_b \sqrt{\dfrac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\dfrac{B_p}{m_p l_p^4}}, \quad \delta = \dfrac{e_p^2 l_p}{C_p B_p}. \tag{18}$$

Now, we reach the following dimensionless system of boundary value problem

$$\begin{cases} u'''' - \sigma^2 u = \sigma^2, \\ u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \dfrac{j\beta\sigma}{1 + j\beta\sigma} \delta u'(1) = 0, \\ u'''(1) = 0, \end{cases} \tag{19}$$

where the prime denotes the derivative with respect to z. The analytical solution to this problem can be formulated as

$$u(z;\delta) = A_\delta \cos\sqrt{\sigma}z + B_\delta \sin\sqrt{\sigma}z + C_\delta \cosh\sqrt{\sigma}z + D_\delta \sinh\sqrt{\sigma}z - 1 \tag{20}$$

and hence

$$\begin{aligned} u'(z;\delta) &= \sigma^{1/2}\left(-A_\delta \sin\sqrt{\sigma}z + B_\delta \cos\sqrt{\sigma}z + C_\delta \sinh\sqrt{\sigma}z + D_\delta \cosh\sqrt{\sigma}z\right), \\ u''(z;\delta) &= \sigma\left(-A_\delta \cos\sqrt{\sigma}z - B_\delta \sin\sqrt{\sigma}z + C_\delta \cosh\sqrt{\sigma}z + D_\delta \sinh\sqrt{\sigma}z\right), \\ u'''(z;\delta) &= \sigma^{3/2}\left(A_\delta \sin\sqrt{\sigma}z - B_\delta \cos\sqrt{\sigma}z + C_\delta \sinh\sqrt{\sigma}z + D_\delta \cosh\sqrt{\sigma}z\right). \end{aligned} \tag{21}$$

The coefficients A_δ, B_δ, C_δ, and D_δ are then subject to the following linear system of equations:

$$
\begin{cases}
A_\delta + C_\delta = 1, \\
B_\delta + D_\delta = 0, \\
\left(-A_\delta \cos \sqrt{\sigma} - B_\delta \sin \sqrt{\sigma} + C_\delta \cosh \sqrt{\sigma} + D_\delta \sinh \sqrt{\sigma}\right) + \\
\dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1} \delta \left(-A_\delta \sin \sqrt{\sigma} + B_\delta \cos \sqrt{\sigma} + C_\delta \sinh \sqrt{\sigma} + D_\delta \cosh \sqrt{\sigma}\right) = 0, \\
A_\delta \sin \sqrt{\sigma} - B_\delta \cos \sqrt{\sigma} + C_\delta \sinh \sqrt{\sigma} + D_\delta \cosh \sqrt{\sigma} = 0.
\end{cases}
\tag{22}
$$

Analytically, we can directly obtain the solution to this problem as

$$
\begin{cases}
A_\delta = \dfrac{1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} - \sin \sqrt{\sigma} \sinh \sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta \left(\cos \sqrt{\sigma} \sinh \sqrt{\sigma}\right)}{2 \left[1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta \left(\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}\right)\right]}, \\[4mm]
B_\delta = \dfrac{\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta \left(\sin \sqrt{\sigma} \sinh \sqrt{\sigma}\right)}{2 \left[1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta \left(\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}\right)\right]}, \\[4mm]
C_\delta = \dfrac{1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \sin \sqrt{\sigma} \sinh \sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta \left(\sin \sqrt{\sigma} \cosh \sqrt{\sigma}\right)}{2 \left[1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta \left(\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}\right)\right]}, \\[4mm]
D_\delta = \dfrac{- \cos \sqrt{\sigma} \sinh \sqrt{\sigma} - \sin \sqrt{\sigma} \cosh \sqrt{\sigma} - \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta \left(\sin \sqrt{\sigma} \sinh \sqrt{\sigma}\right)}{2 \left[1 + \cos \sqrt{\sigma} \cosh \sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta \left(\cos \sqrt{\sigma} \sinh \sqrt{\sigma} + \sin \sqrt{\sigma} \cosh \sqrt{\sigma}\right)\right]}.
\end{cases}
\tag{23}
$$

According to equations (20) and (23), the dimensionless displacement amplitude function $u(z)$ is totally determined by the three dimensionless parameters σ, β, and δ introduced before. Among the dimensionless parameters, σ is the dimensionless base excitation frequency, β is the dimensionless electrical resonant frequency, and δ is the dimensionless electromechanical coupling strength for the structure. As σ and β is determined by the base excitation and externally connected circuit respectively, only the parameter δ is fully determined by the structure itself. Hence we would like to investigate the influence of parameter δ upon the solution displacement function $u(z)$. By taking different values of δ, we calculate the displacement amplitude function $u(z)$ and plot the results in Figure 1.
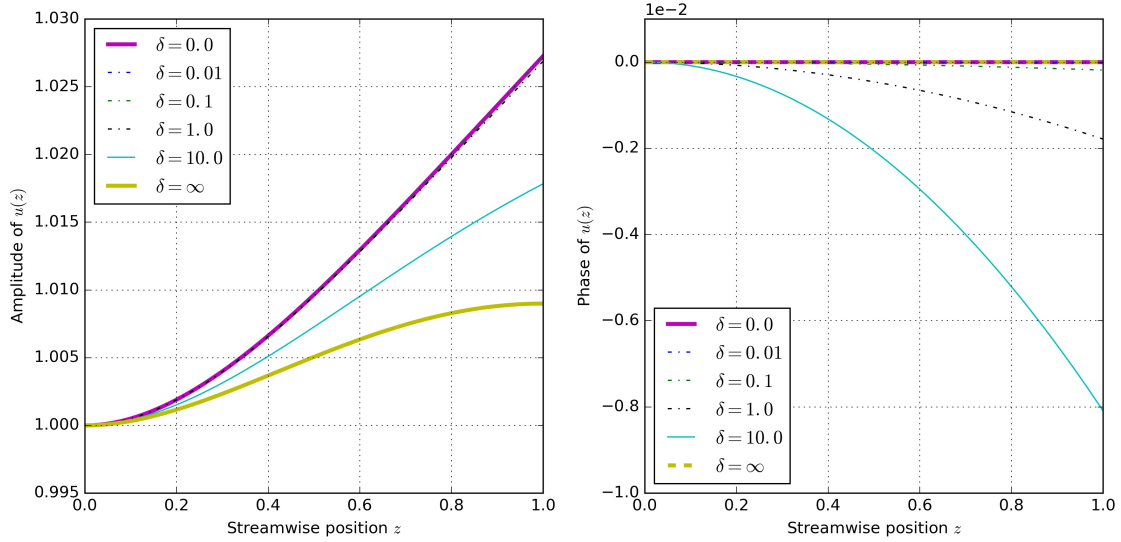

Figure 1: Amplitude and phase of the displacement function $u(z)$

It is shown in Figure 1, the parameter δ changes the function $u(z)$ through the change of the third boundary condition (to be inserted). When δ is zero, i.e., no electromechanical coupling is present, the system degenerates to the classical elastic cantilever beam problem, whose solution is

a real function. That is to say, the phase of $u(z)$ is a constant across the whole beam (in the range of $0 \leq z \leq 1$). Analytical expressions for the coefficients are

$$
\begin{cases}
A_\varnothing = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\[2mm]
B_\varnothing = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\[2mm]
C_\varnothing = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}, \\[2mm]
D_\varnothing = \dfrac{-\cos\sqrt{\sigma}\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma}\right]}.
\end{cases}
\tag{24}
$$

and the resulting dimensionless displacement function $u_\varnothing(z)$ is represented as

$$
u_\varnothing(z) = A_\varnothing \cos\sqrt{\sigma}z + B_\varnothing \sin\sqrt{\sigma}z + C_\varnothing \cosh\sqrt{\sigma}z + D_\varnothing \sinh\sqrt{\sigma}z - 1.
\tag{25}
$$

When the electromechanical coupling is extremely strong, and δ is extremely large and can be seen as ∞ in mathematical sense. In this situation, the solution $u_\infty(z)$ is again real without any phase difference in the z direction. The coefficients can be analytically expressed as

$$
\begin{cases}
A_\infty = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\[2mm]
B_\infty = \dfrac{\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\[2mm]
C_\infty = \dfrac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}, \\[2mm]
D_\infty = \dfrac{-\sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}.
\end{cases}
\tag{26}
$$

and hence the dimensionless displacement function $u_\infty(z)$ is

$$
u_\infty(z) = A_\infty \cos\sqrt{\sigma}z + B_\infty \sin\sqrt{\sigma}z + C_\infty \cosh\sqrt{\sigma}z + D_\infty \sinh\sqrt{\sigma}z - 1.
\tag{27}
$$

While a finite non-zero electromechanical coupling factor δ is present, which is expected in most applications, the resulting dimensionless displacement function $u(z)$ has varying magnitude and phase along the stream-wise direction or z direction. However, it is seen from the right panel of Figure 1 that for different values of δ, the phase change of $u(z)$ is very small in the z direction, actually in the order 10^{-2}.

The resulting complex amplitudes $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ for output voltage $V_p(t)$, output current $I_p(t)$, and output power $P_p(t)$, respectively, can be formulated as follows

$$
\begin{cases}
\tilde{V}_p = -\dfrac{j\sigma\beta}{j\sigma\beta + 1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}u'(1), \\[2mm]
\quad = -\dfrac{j\sigma\beta}{j\sigma\beta + 1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}\sigma^{1/2}\left(-A_\delta \sin\sqrt{\sigma} + B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma}\right) \\[2mm]
\quad = -\dfrac{j\sigma\beta}{j\sigma\beta + 1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}\dfrac{\sqrt{\sigma}\left(\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}\right)}{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)} \\[2mm]
\quad = -\dfrac{j\sigma\beta}{j\sigma\beta + 1}\left(\dfrac{\eta_b}{l_p}\right)\left(\dfrac{e_p}{C_p}\right)\chi_p, \\[2mm]
\tilde{I}_p = \tilde{V}_p/R_l = -\dfrac{j\sigma\beta}{j\sigma\beta + 1}\left(\dfrac{\eta_b}{l_p}\right)\left(\dfrac{e_p}{C_p R_l}\right)\chi_p, \\[2mm]
\tilde{P}_p = \tilde{V}_p^2/R_l = \left(\dfrac{\eta_b}{l_p}\right)^2\left(\dfrac{e_p}{C_p}\right)\left(\dfrac{e_p}{C_p R_l}\right)\left(\dfrac{j\sigma\beta}{j\sigma\beta + 1}\right)^2\chi_p^2,
\end{cases}
\tag{28}
$$

in which we have used the notations that

$$
\chi_p = u_1'(1) = \dfrac{\sqrt{\sigma}\left(\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}\right)}{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)}.
\tag{29}
$$

Clearly, output performance $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ of a classical piezoelectric cantilever energy harvester is heavily dependent on another dimensionless parameter $r_d = \eta_b/l_p$. To be more explicit, both $\tilde{V}_p$ and $\tilde{I}_p$ are linearly dependent on r_d and as a result, $\tilde{P}_p$ shows a quadratic dependence on r_d.

In the following, we can obtain the corresponding displacement function $u(z)$ in terms of its amplitude and phase in Figure 1.

For a typical piezoelectric cantilever energy harvester in the literature [1, 2], this parameter δ is rather small
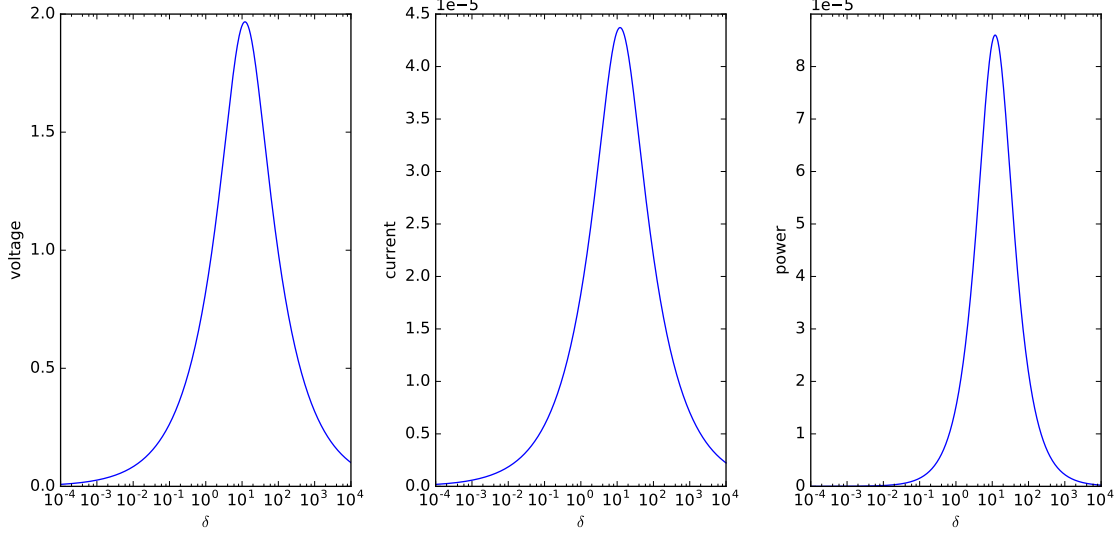

Figure 2: Voltage, current and power output for the piezoelectric cantilever energy harvester

Using the following regular expansion:

$$
\begin{cases}
A_\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots, \\
B_\epsilon = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \cdots, \\
C_\epsilon = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \cdots, \\
D_\epsilon = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots,
\end{cases}
\tag{30}
$$

we obtain the successive expansion problem:
$O(\epsilon^0)$:

$$
\begin{cases}
A_0 + C_0 = 1, \\
B_0 + D_0 = 0, \\
-A_0 \cos\sqrt{\sigma} - B_0 \sin\sqrt{\sigma} + C_0 \cosh\sqrt{\sigma} + D_0 \sinh\sqrt{\sigma} = 0, \\
A_0 \sin\sqrt{\sigma} - B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma} = 0.
\end{cases}
\tag{31}
$$

The solution is

$$
\begin{cases}
A_0 = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\[2ex]
B_0 = \dfrac{\cosh\sqrt{\sigma}\sin\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\[2ex]
C_0 = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\[2ex]
D_0 = -\dfrac{\cosh\sqrt{\sigma}\sin\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}}
\end{cases}
\tag{32}
$$

Hence we have

$$
-A_0 \sin\sqrt{\sigma} + B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma} = \frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}
\tag{33}
$$

$O(\epsilon^1)$:

$$
\begin{cases}
A_1 + C_1 = 0, \\
B_1 + D_1 = 0, \\
\left(-A_1 \cos\sqrt{\sigma} - B_1 \sin\sqrt{\sigma} + C_1 \cosh\sqrt{\sigma} + D_1 \sinh\sqrt{\sigma}\right) + \\
\dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1}\left(-A_0 \sin\sqrt{\sigma} + B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma}\right) = 0, \\
A_1 \sin\sqrt{\sigma} - B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma} = 0.
\end{cases}
\tag{34}
$$

The solution is

$$
\begin{cases}
A_1 = \dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\[2mm]
B_1 = \dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\[2mm]
C_1 = \dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(-\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\[2mm]
D_1 = \dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right)
\end{cases}
\tag{35}
$$

Then we have

$$
-A_1 \sin\sqrt{\sigma} + B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma}
$$
$$
= \dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\left(\dfrac{\sin\sqrt{\sigma} - \sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)
\tag{36}
$$

$O(\epsilon^2)$:

$$
\begin{cases}
A_2 + C_2 = 0, \\
B_2 + D_2 = 0, \\
\left(-A_2 \cos\sqrt{\sigma} - B_2 \sin\sqrt{\sigma} + C_2 \cosh\sqrt{\sigma} + D_2 \sinh\sqrt{\sigma}\right) + \\
\dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1}\left(-A_1 \sin\sqrt{\sigma} + B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma}\right) = 0, \\
A_2 \sin\sqrt{\sigma} - B_2 \cos\sqrt{\sigma} + C_2 \sinh\sqrt{\sigma} + D_2 \cosh\sqrt{\sigma} = 0.
\end{cases}
\tag{37}
$$

The solution is

$$
\begin{cases}
A_2 = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\right)^2\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\[2mm]
B_2 = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\right)^2\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\[2mm]
C_2 = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\right)^2\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(-\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right) \\[2mm]
D_2 = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma}\right)^2\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}\right)\left(\dfrac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2}\right)
\end{cases}
\tag{38}
$$

To get higher order expansions, we can use the following iteration method:
$O(\epsilon^{k+1})$ $(k \geq 1)$:

$$
\begin{cases}
A_{k+1} + C_{k+1} = 0, \\
B_{k+1} + D_{k+1} = 0, \\
\left(-A_{k+1} \cos\sqrt{\sigma} - B_{k+1} \sin\sqrt{\sigma} + C_{k+1} \cosh\sqrt{\sigma} + D_{k+1} \sinh\sqrt{\sigma}\right) + \\
\dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta + 1}\left(-A_k \sin\sqrt{\sigma} + B_k \cos\sqrt{\sigma} + C_k \sinh\sqrt{\sigma} + D_k \cosh\sqrt{\sigma}\right) = 0, \\
A_{k+1} \sin\sqrt{\sigma} - B_{k+1} \cos\sqrt{\sigma} + C_{k+1} \sinh\sqrt{\sigma} + D_{k+1} \cosh\sqrt{\sigma} = 0.
\end{cases}
\tag{39}
$$

The solution is

$$
\begin{cases}
A_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right) \left(\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2} \right) (Q_k) \\[3mm]
B_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right) \left(\dfrac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2} \right) (Q_k) \\[3mm]
C_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right) \left(-\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2} \right) (Q_k) \\[3mm]
D_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right) \left(\dfrac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2} \right) (Q_k)
\end{cases}
\tag{40}
$$

where for $k \geq 2$

$$
Q_k = -A_k\sin\sqrt{\sigma} + B_k\cos\sqrt{\sigma} + C_k\sinh\sqrt{\sigma} + D_k\cosh\sqrt{\sigma},
\tag{41}
$$

and for $k \geq 0$

$$
\begin{aligned}
Q_{k+1} &= -A_{k+1}\sin\sqrt{\sigma} + B_{k+1}\cos\sqrt{\sigma} + C_{k+1}\sinh\sqrt{\sigma} + D_{k+1}\cosh\sqrt{\sigma} \\
&= -\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right) \left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right) Q_k,
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
Q_1 &= -A_1\sin\sqrt{\sigma} + B_1\cos\sqrt{\sigma} + C_1\sinh\sqrt{\sigma} + D_1\cosh\sqrt{\sigma} \\
&= \frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \left(\frac{\sin\sqrt{\sigma} - \sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right) \left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right)
\end{aligned}
\tag{43}
$$

$$
Q_0 = \frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1}
\tag{44}
$$

Hence it is shown that for $k \geq 0$

$$
\begin{aligned}
Q_k &= -\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right) \left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right) Q_k \\
&= \left[-\left(\frac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right) \left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right) \right]^k \left(\frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right)
\end{aligned}
\tag{45}
$$

As a result, we obtain that for $k \geq 0$

$$
\begin{cases}
A_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right)^{k+1} \left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right)^k \left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right) \left(\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2} \right) \\[3mm]
B_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right)^{k+1} \left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right)^k \left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right) \left(\dfrac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2} \right) \\[3mm]
C_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right)^{k+1} \left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right)^k \left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right) \left(\dfrac{-\cos\sqrt{\sigma} - \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2} \right) \\[3mm]
D_{k+1} = \left(\dfrac{j\beta\sqrt{\sigma}}{1 + j\beta\sigma} \right)^{k+1} \left(\dfrac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right)^k \left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \right) \left(\dfrac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 2} \right)
\end{cases}
\tag{46}
$$

Reference

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