

Asymptotic analysis of piezoelectric energy harvester

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1 Summary of the interested equations

Here we are interested in the classical model of a piezoelectric cantilever beam energy harvester, whose model is described using the following set of equations:

$$u'''' - \lambda^2 u = 0, \quad (1)$$

and the accompanying boundary conditions:

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) + \frac{j\lambda\beta\alpha^2}{j\lambda\beta + 1}u'(1) = 0 \\ u'''(1) = 0 \end{cases}, \quad (2)$$

where λ is the eigenvalues for the problem, u denotes the displace function of the cantilever beam, β is the dimensionless externally connected resistance, and α is the dimensionless piezoelectric coefficient. They can be expressed as follows

$$\lambda = \omega \sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \alpha = e_p \sqrt{\frac{l_p}{C_p B_p}}, \quad (3)$$

where ω is angular frequency, m_p is line mass density, l_p is the length of the cantilever beam, B_p is the bending stiffness, C_p is the inherent capacitance of the piezoelectric layer, e_p is the charge accumulation number, R_l is the externally connected resistance. In practical applications, dielectric property of piezoelectric materials indicate that the parameter β is changed from a very small value, which is close to a short-circuit condition to a very large value, which corresponds to an open-circuit condition. Thus we have that $0 \leq \beta \leq \infty$.

2 Asymptotic analysis when β is small

Here we seek to find the behavior of the above system at a small value of connected resistance, i.e., $\beta \rightarrow 0$. In this case, we set β to be the parameter for asymptotic expansion, and

$$\begin{aligned} \lambda^{(k)} &= \lambda_0^{(k)} + \beta \lambda_1^{(k)} + \beta^2 \lambda_2^{(k)} + \dots \\ u^{(k)} &= u_0^{(k)} + \beta u_1^{(k)} + \beta^2 u_2^{(k)} + \dots \end{aligned} \quad (4)$$

where $\lambda^{(k)}$ and $u^{(k)}$ are the k th eigenvalue and eigenfunction respectively of the above mentioned system under perturbation. $\lambda_0^{(k)}$ and $u_0^{(k)}$ are the corresponding eigenvalue and eigenfunction of the unperturbed system at $\beta = 0$:

$$u'''' - \lambda_0^2 u = 0, \quad (5)$$

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) = 0 \\ u'''(1) = 0 \end{cases}. \quad (6)$$

Obviously, the unperturbed system is a classical eigenvalue problem with the eigenvalues determined by

$$1 + \cosh(\sqrt{\lambda_0}) \cos(\sqrt{\lambda_0}) = 0 \quad (7)$$

whose first several values are

$$\frac{\sqrt{\lambda_0^{(1)}}}{\pi} = 0.59686, \quad \frac{\sqrt{\lambda_0^{(2)}}}{\pi} = 1.49418, \quad \frac{\sqrt{\lambda_0^{(3)}}}{\pi} = 2.50025, \quad \frac{\sqrt{\lambda_0^{(4)}}}{\pi} = 3.49999, \quad \dots \quad (8)$$

Take the asymptotic expansions and substitute them into the previously derived system of equations, we have the following asymptotic expansions to different orders of β :

$O(\beta^0)$:

$$\begin{cases} u_0'''' - \lambda_0^2 u_0 = 0 \\ u_0(0) = 0 \\ u_0'(0) = 0 \\ u_0''(1) = 0 \\ u_0'''(1) = 0 \end{cases} \quad (9)$$

$O(\beta^1)$:

$$\begin{cases} u_1'''' - (\lambda_0^2 u_1 + 2\lambda_0 u_0 \lambda_1) = 0 \\ u_1(0) = 0 \\ u_1'(0) = 0 \\ u_1''(1) + j\alpha^2 \lambda_0 u_0'(1) = 0 \\ u_1'''(1) = 0 \end{cases} \quad (10)$$

$O(\beta^2)$:

$$\begin{cases} u_2'''' - (\lambda_0^2 u_2 + 2\lambda_0 u_1 \lambda_1 + \lambda_1^2 u_0 + 2\lambda_0 u_0 \lambda_2) = 0 \\ u_2(0) = 0 \\ u_2'(0) = 0 \\ u_2''(1) + \alpha^2 \lambda_0 u_0'(1) + j\alpha^2 [\lambda_0 u_1'(1) + \lambda_1 u_0'(1)] = 0 \\ u_2'''(1) = 0 \end{cases} \quad (11)$$

3 Asymptotic analysis when β is large

Here we seek to find the behavior of the above system at a large value of connected resistance, i.e., $\beta \rightarrow \infty$. In this case, we set $\frac{1}{\beta}$ to be the parameter for asymptotic expansion and

$$\begin{aligned} \lambda^{(k)} &= \tilde{\lambda}_0^{(k)} + \left(\frac{1}{\beta}\right) \tilde{\lambda}_1^{(k)} + \left(\frac{1}{\beta}\right)^2 \tilde{\lambda}_2^{(k)} + \dots \\ u^{(k)} &= \tilde{u}_0^{(k)} + \left(\frac{1}{\beta}\right) \tilde{u}_1^{(k)} + \left(\frac{1}{\beta}\right)^2 \tilde{u}_2^{(k)} + \dots \end{aligned} \quad (12)$$

where $\tilde{\lambda}^{(k)}$ and $\tilde{u}^{(k)}$ are the k th eigenvalue and eigenfunction respectively of the above mentioned system under perturbation. $\tilde{\lambda}_0^{(k)}$ and $\tilde{u}_0^{(k)}$ are the corresponding eigenvalue and eigenfunction of the unperturbed system at $\beta = \infty$:

$O(\frac{1}{\beta^0})$:

$$\begin{cases} \tilde{u}_0'''' - \tilde{\lambda}_0^2 \tilde{u}_0 = 0 \\ \tilde{u}_0(0) = 0 \\ \tilde{u}_0'(0) = 0 \\ \tilde{u}_0''(1) + \alpha^2 \tilde{u}_0'(1) = 0 \\ \tilde{u}_0'''(1) = 0 \end{cases} \quad (13)$$

$O(\frac{1}{\beta^1})$:

$$\begin{cases} \tilde{u}_1'''' - (\tilde{\lambda}_0^2 u_1 + 2\tilde{\lambda}_0 \tilde{u}_0 \tilde{\lambda}_1) = 0 \\ \tilde{u}_1(0) = 0 \\ \tilde{u}_1'(0) = 0 \\ \tilde{u}_1''(1) + \alpha^2 \tilde{u}_1'(1) + \frac{j\alpha^2}{\tilde{\lambda}_0} \tilde{u}_0'(1) = 0 \\ \tilde{u}_1'''(1) = 0 \end{cases} \quad (14)$$

$O(\frac{1}{\beta^2})$:

$$\begin{cases} \tilde{u}_2'''' - (\tilde{\lambda}_0^2 \tilde{u}_2 + 2\tilde{\lambda}_0 \tilde{u}_1 \tilde{\lambda}_1 + \tilde{\lambda}_1^2 \tilde{u}_0 + 2\tilde{\lambda}_0 \tilde{u}_0 \tilde{\lambda}_2) = 0 \\ \tilde{u}_2(0) = 0 \\ \tilde{u}_2'(0) = 0 \\ \tilde{u}_2''(1) + \left[\alpha^2 \tilde{u}_2'(1) - \frac{\alpha^2}{\tilde{\lambda}_0^2} \tilde{u}_0'(1) \right] + j \left[\frac{\alpha^2}{\tilde{\lambda}_0} \tilde{u}_1'(1) - \frac{\alpha^2 \tilde{\lambda}_1}{\tilde{\lambda}_0^2} \tilde{u}_0'(1) \right] = 0 \\ \tilde{u}_2'''(1) = 0 \end{cases} \quad (15)$$

4 Asymptotic analysis in terms of small α^2

Directly using the eigenvalue analysis method for linear boundary value problem, we arrive at the equation for the eigenvalue λ :

$$\sqrt{\lambda} \left[1 + \left(\frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{2} \right) \cos \sqrt{\lambda} \right] + \frac{j\beta\lambda\alpha^2}{1+j\beta\lambda} \left[\left(\frac{e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}}{2} \right) \cos \sqrt{\lambda} + \left(\frac{e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}}{2} \right) \sin \sqrt{\lambda} \right] = 0 \quad (16)$$

or

$$\sqrt{\lambda} \left[1 + \cosh \sqrt{\lambda} \cos \sqrt{\lambda} \right] + \frac{j\beta\lambda\alpha^2}{1+j\beta\lambda} \left[\sinh \sqrt{\lambda} \cos \sqrt{\lambda} + \cosh \sqrt{\lambda} \sin \sqrt{\lambda} \right] = 0 \quad (17)$$

Taking the parameter α^2 as the small parameter ϵ and expanding the eigenvalue λ in terms of this ϵ , we have

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \dots \quad (18)$$

and therefore:

$O(\epsilon^0)$:

$$1 + \cosh \sqrt{\lambda_0} \cos \sqrt{\lambda_0} = 0 \quad (19)$$

$O(\epsilon^1)$:

$$2j\beta\lambda_0 \left(\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} + \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0} \right) + (1+j\beta\lambda_0)\lambda_1 \left(-\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} + \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0} \right) = 0 \quad (20)$$

or equivalently

$$\lambda_1 = \frac{2j\beta\lambda_0}{(1+j\beta\lambda_0)} \frac{(\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} + \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0})}{(\cosh \sqrt{\lambda_0} \sin \sqrt{\lambda_0} - \sinh \sqrt{\lambda_0} \cos \sqrt{\lambda_0})} \quad (21)$$

5 Asymptotic analysis in terms of small α^2

The forced vibration problem of a piezoelectric cantilever bimorph is described by

$$u'''' - \lambda^2 u = \lambda^2, \quad (22)$$

and the accompanying boundary conditions:

$$\begin{cases} u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \frac{j\lambda\beta}{j\lambda\beta + 1} \epsilon u'(1) = 0, \\ u'''(1) = 0. \end{cases} \quad (23)$$

This problem can readily be solved using a conventional boundary value problem solver. However, here we would like to develop an asymptotic expansion of the solution for the system. Using ϵ as a parameter, we have

$$u(x; \epsilon) = A_\epsilon \cos \sqrt{\lambda} x + B_\epsilon \sin \sqrt{\lambda} x + C_\epsilon \cosh \sqrt{\lambda} x + D_\epsilon \sinh \sqrt{\lambda} x - 1 \quad (24)$$

As a result, we have

$$\begin{aligned} u'(x; \epsilon) &= \sqrt{\lambda} \left(-A_\epsilon \sin \sqrt{\lambda} x + B_\epsilon \cos \sqrt{\lambda} x + C_\epsilon \sinh \sqrt{\lambda} x + D_\epsilon \cosh \sqrt{\lambda} x \right) \\ u''(x; \epsilon) &= \lambda \left(-A_\epsilon \cos \sqrt{\lambda} x - B_\epsilon \sin \sqrt{\lambda} x + C_\epsilon \cosh \sqrt{\lambda} x + D_\epsilon \sinh \sqrt{\lambda} x \right) \\ u'''(x; \epsilon) &= \lambda \sqrt{\lambda} \left(A_\epsilon \sin \sqrt{\lambda} x - B_\epsilon \cos \sqrt{\lambda} x + C_\epsilon \sinh \sqrt{\lambda} x + D_\epsilon \cosh \sqrt{\lambda} x \right) \end{aligned} \quad (25)$$

Thus the above boundary value problem is converted into the following linear equation systems:

$$\begin{cases} A_\epsilon + C_\epsilon = 1, \\ B_\epsilon + D_\epsilon = 0, \\ \left(-A_\epsilon \cos \sqrt{\lambda} - B_\epsilon \sin \sqrt{\lambda} + C_\epsilon \cosh \sqrt{\lambda} + D_\epsilon \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \epsilon \left(-A_\epsilon \sin \sqrt{\lambda} + B_\epsilon \cos \sqrt{\lambda} + C_\epsilon \sinh \sqrt{\lambda} + D_\epsilon \cosh \sqrt{\lambda} \right) = 0, \\ A_\epsilon \sin \sqrt{\lambda} - B_\epsilon \cos \sqrt{\lambda} + C_\epsilon \sinh \sqrt{\lambda} + D_\epsilon \cosh \sqrt{\lambda} = 0. \end{cases} \quad (26)$$

Analytically, we can directly obtain the solution to this problem as

$$\begin{cases} A_\epsilon = \frac{1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} - \sin \sqrt{\lambda} \sinh \sqrt{\lambda} + \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]}, \\ B_\epsilon = \frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\sin \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]}, \\ C_\epsilon = \frac{-1 - \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \sin \sqrt{\lambda} \sinh \sqrt{\lambda} - \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]}, \\ D_\epsilon = \frac{-\cos \sqrt{\lambda} \sinh \sqrt{\lambda} - \sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \epsilon \frac{2j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\sin \sqrt{\lambda} \sinh \sqrt{\lambda} \right)}{2 \left[1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \epsilon \frac{j\beta\sqrt{\lambda}}{1+j\beta\lambda} \left(\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda} \right) \right]} \end{cases} \quad (27)$$

The resulting output voltage V_p , current I_p , and power P_p can be formulated as follows

$$\begin{cases} \tilde{V}_p = \frac{j\lambda\beta}{j\lambda\beta + 1} \frac{\xi_b}{l_p} \frac{e_p}{C_p} u'(1), \\ \tilde{I}_p = \tilde{V}_p / R_l, \\ \tilde{P}_p = \tilde{V}_p^2 / R_l. \end{cases} \quad (28)$$

Using the following regular expansion:

$$\begin{cases} A_\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \dots, \\ B_\epsilon = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \dots, \\ C_\epsilon = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \dots, \\ D_\epsilon = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots, \end{cases} \quad (29)$$

we obtain the successive expansion problem:

$O(\epsilon^0)$:

$$\begin{cases} A_0 + C_0 = 1, \\ B_0 + D_0 = 0, \\ -A_0 \cos \sqrt{\lambda} - B_0 \sin \sqrt{\lambda} + C_0 \cosh \sqrt{\lambda} + D_0 \sinh \sqrt{\lambda} = 0, \\ A_0 \sin \sqrt{\lambda} - B_0 \cos \sqrt{\lambda} + C_0 \sinh \sqrt{\lambda} + D_0 \cosh \sqrt{\lambda} = 0. \end{cases} \quad (30)$$

The solution is

$$\begin{cases} A_0 = \frac{1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} - \sin \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \\ B_0 = \frac{\cosh \sqrt{\lambda} \sin \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \\ C_0 = \frac{1 + \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + \sin \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \\ D_0 = -\frac{\cosh \sqrt{\lambda} \sin \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{2 + 2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda}} \end{cases} \quad (31)$$

Hence we have

$$-A_0 \sin \sqrt{\lambda} + B_0 \cos \sqrt{\lambda} + C_0 \sinh \sqrt{\lambda} + D_0 \cosh \sqrt{\lambda} = \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \quad (32)$$

$O(\epsilon^1)$:

$$\begin{cases} A_1 + C_1 = 0, \\ B_1 + D_1 = 0, \\ \left(-A_1 \cos \sqrt{\lambda} - B_1 \sin \sqrt{\lambda} + C_1 \cosh \sqrt{\lambda} + D_1 \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \left(-A_0 \sin \sqrt{\lambda} + B_0 \cos \sqrt{\lambda} + C_0 \sinh \sqrt{\lambda} + D_0 \cosh \sqrt{\lambda} \right) = 0, \\ A_1 \sin \sqrt{\lambda} - B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} = 0. \end{cases} \quad (33)$$

The solution is

$$\begin{cases} A_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ B_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ C_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ D_1 = \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \end{cases} \quad (34)$$

Then we have

$$\begin{aligned} & -A_1 \sin \sqrt{\lambda} + B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} \\ &= \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sin \sqrt{\lambda} - \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \end{aligned} \quad (35)$$

$$O(\epsilon^2): \quad \left\{ \begin{array}{l} A_2 + C_2 = 0, \\ B_2 + D_2 = 0, \\ \left(-A_2 \cos \sqrt{\lambda} - B_2 \sin \sqrt{\lambda} + C_2 \cosh \sqrt{\lambda} + D_2 \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \left(-A_1 \sin \sqrt{\lambda} + B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} \right) = 0, \\ A_2 \sin \sqrt{\lambda} - B_2 \cos \sqrt{\lambda} + C_2 \sinh \sqrt{\lambda} + D_2 \cosh \sqrt{\lambda} = 0. \end{array} \right. \quad (36)$$

The solution is

$$\left\{ \begin{array}{l} A_2 = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^2 \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ B_2 = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^2 \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ C_2 = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^2 \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ D_2 = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^2 \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \end{array} \right. \quad (37)$$

To get higher order expansions, we can use the following iteration method:
 $O(\epsilon^{k+1})$ ($k \geq 1$):

$$\left\{ \begin{array}{l} A_{k+1} + C_{k+1} = 0, \\ B_{k+1} + D_{k+1} = 0, \\ \left(-A_{k+1} \cos \sqrt{\lambda} - B_{k+1} \sin \sqrt{\lambda} + C_{k+1} \cosh \sqrt{\lambda} + D_{k+1} \sinh \sqrt{\lambda} \right) + \\ \frac{j\beta\sqrt{\lambda}}{j\lambda\beta + 1} \left(-A_k \sin \sqrt{\lambda} + B_k \cos \sqrt{\lambda} + C_k \sinh \sqrt{\lambda} + D_k \cosh \sqrt{\lambda} \right) = 0, \\ A_{k+1} \sin \sqrt{\lambda} - B_{k+1} \cos \sqrt{\lambda} + C_{k+1} \sinh \sqrt{\lambda} + D_{k+1} \cosh \sqrt{\lambda} = 0. \end{array} \right. \quad (38)$$

The solution is

$$\left\{ \begin{array}{l} A_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) \left(\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) \left(\frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) \left(\frac{-\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) \left(\frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) (Q_k) \end{array} \right. \quad (39)$$

where for $k \geq 2$

$$Q_k = -A_k \sin \sqrt{\lambda} + B_k \cos \sqrt{\lambda} + C_k \sinh \sqrt{\lambda} + D_k \cosh \sqrt{\lambda}, \quad (40)$$

and for $k \geq 0$

$$\begin{aligned} Q_{k+1} &= -A_{k+1} \sin \sqrt{\lambda} + B_{k+1} \cos \sqrt{\lambda} + C_{k+1} \sinh \sqrt{\lambda} + D_{k+1} \cosh \sqrt{\lambda} \\ &= - \left(\frac{\sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) Q_k, \end{aligned} \quad (41)$$

and

$$\begin{aligned} Q_1 &= -A_1 \sin \sqrt{\lambda} + B_1 \cos \sqrt{\lambda} + C_1 \sinh \sqrt{\lambda} + D_1 \cosh \sqrt{\lambda} \\ &= \frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \left(\frac{\sin \sqrt{\lambda} - \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} \sinh \sqrt{\lambda} + \sin \sqrt{\lambda} \cosh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \end{aligned} \quad (42)$$

$$Q_0 = \frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \quad (43)$$

Hence it is shown that for $k \geq 0$

$$\begin{aligned} Q_k &= - \left(\frac{\sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) Q_k \\ &= \left[- \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right) \left(\frac{\sin \sqrt{\lambda} \cosh \sqrt{\lambda} + \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \right]^k \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \end{aligned} \quad (44)$$

As a result, we obtain that for $k \geq 0$

$$\begin{cases} A_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left(\frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{\cos \sqrt{\lambda} + \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left(\frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sinh \sqrt{\lambda} + \sin \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left(\frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\cos \sqrt{\lambda} - \cosh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\lambda}}{1 + j\beta\lambda} \right)^{k+1} \left(\frac{-\sin \sqrt{\lambda} \cosh \sqrt{\lambda} - \cos \sqrt{\lambda} \sinh \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right)^k \left(\frac{\sinh \sqrt{\lambda} - \sin \sqrt{\lambda}}{\cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 1} \right) \left(\frac{-\sin \sqrt{\lambda} + \sinh \sqrt{\lambda}}{2 \cos \sqrt{\lambda} \cosh \sqrt{\lambda} + 2} \right) \end{cases} \quad (45)$$