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Pencils of differential operators containing the eigenvalue parameter in the boundary conditions

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The paper deals with linear pencils $N - \lambda P$ of ordinary differential operators on a finite interval with λ-dependent boundary conditions. Three different problems of this form arising in elasticity and hydrodynamics are considered. So-called linearization pairs (W, T) are constructed for the problems in question. More precisely, functional spaces W densely embedded in L_2 and linear operators T acting in W are constructed such that the eigenvalues and the eigen- and associated functions of T coincide with those of the original problems. The spectral properties of the linearized operators T are studied. In particular, it is proved that the eigen- and associated functions of all linearizations (and hence of the corresponding original problems) form Riesz bases in the spaces W and in other spaces which are obtained by interpolation between $D(T)$ and W.

1. Introduction

In this paper we study boundary eigenvalue problems for ordinary differential equations of the form

$$N(y) = \lambda P(y), \qquad y = y(x), \qquad x \in [0, 1], \tag{1.1}$$

$$U_j^0(y) = \lambda U_j^1(y), \qquad j = 1, 2, \ldots, n. \tag{1.2}$$

Here, N and P are ordinary differential expressions of order n and p, respectively, with constant leading coefficients, $n > p$, and $U_j^0(y)$, $U_j^1(y)$ are linear forms containing the variables $y^{(k)}(0)$ and $y^{(k)}(1)$ with $k = 0, 1, \ldots, n-1$. Problems of this form arise widely in mechanics, in particular, in elasticity and hydrodynamics, where usually $n = 4$ and $p = 2$.

For each fixed λ in $\mathbb{C}$, the differential expression $N(y) - \lambda P(y)$ and the boundary conditions $U_j^0(y) - \lambda U_j^1(y) = 0$, $j = 1, \dots, n$, determine an operator $\mathcal{L} = \mathcal{L}(\lambda)$ in the Hilbert space $L_2(0,1)$ with domain depending on λ. The set of values of λ for which $\mathcal{L}(\lambda)$ is not invertible is called the *spectrum* of problem (1.1), (1.2). Usually, for problems on a finite interval, the spectrum consists of isolated eigenvalues. If λ is an eigenvalue, then the problem (1.1), (1.2) has non-trivial solutions that are called the *eigenfunctions* corresponding to λ. In the usual way, the associated functions can be determined (see, for example, [11, ch. 1]).

There exists a vast literature devoted to the study of problems of the form (1.1), (1.2) and more general ones. Historical comments and references can be found in the papers [17, 21]. Here we only recall that the usual topics of investigation for problems of this kind have been the following: special asymptotics for fundamental solutions of (1.1) and estimates of Green's function; asymptotics of the eigenvalues and eigenfunctions; completeness of the eigen- and associated functions in $L_2(0,1)$ and in Sobolev spaces; and convergence of the eigenfunction expansions.

There are, however, problems that arise naturally both from mathematical and physical points of view, but these have received attention in only a few papers.

(1) *Does there exist a functional space W densely embedded in $L_2(0,1)$ such that the system of eigen- and associated functions of problem (1.1), (1.2) is complete and minimal simultaneously, or even a Riesz* (*i.e. an unconditional*) *basis in W?*

This question is closely connected with the following ones.

(2) *Do there exist a functional space W and a linear operator T acting in W such that the eigenvalues and the eigen- and associated functions of T coincide with those of problem (1.1), (1.2)?*

A pair (W, T) with this property will be called a *linearization pair* for (1.1), (1.2).

(3) *If so, how should linearization pairs for the problems in question be constructed?*

Problems of this type were posed and solved in [15] for boundary eigenvalue problems containing the eigenparameter polynomially. However, for linear problems (1.1), (1.2), the results in [15] can be applied only in the case $\deg P(y) = 0$. On the other hand, the papers [17, 18] (see also [10, 13, 16]) solve these problems for the case $\deg P(y) > 0$, but only with λ-independent boundary conditions (i.e. $U_j^1(y) = 0$, $j = 1, 2 \dots, n$).

The problem (1.1), (1.2) can be viewed as a linear pencil in the Hilbert space $L_2(0,1) \times \mathbb{C}^r$, where $r \leqslant n$ is the number of λ-dependent boundary conditions after a suitable normalization. Therefore, its natural linearizations can be also considered in $L_2(0,1) \times \mathbb{C}^r$ (we will make use of this notion in § 4). It should be mentioned that in the case $\deg P(y) = 0$ there is a theory that provides linearizations of symmetric boundary-value problems with nonlinearly λ-dependent boundary conditions in extended spaces (see, for example, [5] for an exposition of this theory, historical remarks and references). For a general class of non-symmetric boundary-value problems (which, however, does not include problems (1.1), (1.2)), linearizations in extended spaces were constructed and investigated in [15]. Problems of the form (1.1) with λ-polynomial boundary conditions (and $\deg P(y) > 0$) are treated in [19–21] via a linearization in extended spaces; properties such as completeness, minimality and Riesz basisness with finite defect are investigated for the eigen- and associated functions.

Certainly, all possible linearizations are of interest. However, those that we consider here seem to be more relevant from the physical point of view. We are not aware of publications where linearization pairs in the above-mentioned sense were constructed for problems of the form (1.1), (1.2) with $\deg P(y) > 0$ and λ-dependent boundary conditions. In this paper we shall solve this problem for some concrete problems arising in mechanics. To answer the three questions raised above for problems (1.1), (1.2) in a general setting is a difficult task (one can understand this by reading [17,18]). However, we will demonstrate a set of tools that can also be applied to other problems of this type. Naturally, we consider the problems in increasing order of complexity. We start from a buckling problem from elasticity, which is relatively easy to solve explicitly, then we proceed to a more involved problem for the well-known Orr–Sommerfeld problem of hydrodynamics. Finally, we consider a symmetric fourth-order problem arising in the stability of superposed fluids. This last problem involves a non-trivial analysis and represents the main object of the paper.

In the following we always abbreviate the space $L_2(0,1)$ by L_2 and the Sobolev spaces $W_2^k(0,1)$ by W_2^k for $k = 1, 2, \dots$. By $\|\cdot\|_k$ and $(\cdot,\cdot)_k$ we denote the norm and scalar product, respectively, in W_2^k, omitting the subscript when $k = 0$. The notation $W_{2,U}^k$ ($W_{2,\mathcal{U}}^k$) is used to denote the subspaces in W_k^2 preserving the given (the given and certain 'hidden', respectively) boundary conditions of order less than k.

2. A clamped-free elastic bar

The study of critical loads for divergence of a clamped-free elastic bar of length l and constant flexural rigidity α exposed to a non-tangential end load Q (the so-called Petterson–König rod (see [4, 19])) leads to the boundary eigenvalue problem

$$y^{(4)} = \lambda y'', \tag{2.1}$$

$$y(0) = y'(0) = 0, \tag{2.2}$$

$$y''(1) = 0, \quad y^{(3)}(1) - \lambda\gamma y'(1) = 0. \tag{2.3}$$

Here, $\lambda = Ql^2/\alpha$ and $(1-\gamma)\varphi_l$, with $\gamma \in [0,1]$, is the angle between the end load Q and the vertical axis, where φ_l denotes the angle between the tangent at the free end of the bar and the vertical axis.

The eigenfunctions and the eigenvalues of this problem can be found explicitly by standard means. However, it is by no means obvious how to find a linearization pair in this case. In the following, we shall construct such a pair explicitly.

Denote

$$y'' = z. \tag{2.4}$$

Then, due to the first two boundary conditions (2.2),

$$y(x) = \int_0^x (x-\xi)z(\xi)\,\mathrm{d}\xi =: (Kz)(x), \quad x \in [0,1]. \tag{2.5}$$

Obviously, the operator K is an isomorphism from the space L_2 onto the space

$$W_{2,U}^2 := \{y \in W_2^2 : y(0) = y'(0) = 0\}. \tag{2.6}$$

By means of the substitution (2.4), the problem (2.1)–(2.3) takes the form

$$z'' = \lambda z, \tag{2.7}$$

$$z(1) = 0, \quad z'(1) - \lambda\gamma \int_0^1 z(\xi)\,\mathrm{d}\xi = 0. \tag{2.8}$$

From the differential equation (2.7), it follows that

$$\lambda \int_0^1 z(\xi)\,\mathrm{d}\xi = \int_0^1 z''(\xi)\,\mathrm{d}\xi = z'(1) - z'(0),$$

and hence the problem (2.7), (2.8) is equivalent to

$$z'' = \lambda z, \tag{2.9}$$

$$z(1) = 0, \quad (1-\gamma)z'(1) + \gamma z'(0) = 0. \tag{2.10}$$

With the problem (2.9), (2.10), we associate the differential operator L in L_2 given by

$$\mathcal{D}(L) = \{z \in W_2^2 : z(1) = 0,\ (1-\gamma)z'(1) + \gamma z'(0) = 0\}, \tag{2.11}$$

$$Lz = z''. \tag{2.12}$$

An easy calculation shows that the eigenvalues $\{\lambda_n\}$ of the operator L are given by $\lambda_n = -\mu_n^2$, where the μ_n are the roots of the equation

$$\cos\mu = 1 - \frac{1}{\gamma},$$

and that the eigenfunctions have the form

$$z_n(x) = \sin((x-1)\mu_n), \quad x \in [0,1].$$

It follows directly from the definition that the operator L is strongly Birkhoff regular (see [11, ch. 1]). Therefore, the system $\{z_n\}$ of its eigenfunctions forms a Riesz basis in L_2 (a property shared by all strongly regular differential operators (see, for example, [6])).

Obviously, the spectra of problem (2.1)–(2.3) and of the operator L coincide and, for the eigenfunctions $\{y_n\}$ of (2.1)–(2.3), we have

$$y_n(x) = (Kz_n)(x) = \frac{1}{\mu_n^2}(\sin((x-1)\mu_n) + \sin\mu_n + x\mu_n\cos\mu_n), \quad x \in [0,1], \tag{2.13}$$

where K is the operator defined in equation (2.5). Note that the inverse operator $K^{-1} : W_{2,U}^2 \to L_2$ is given by $K^{-1}y = y''$. From this, we immediately obtain the following result.

THEOREM 2.1. *Let $\gamma \neq 0$. Then the system of eigenfunctions $\{y_n\}$ of problem (2.1)–(2.3) forms a Riesz basis in the space $W^2_{2,U}$ defined in (2.6). Let the operator $T = KLK^{-1}$ acting in the space $\mathfrak{H} = W^2_{2,U}$ be given by*

$$\begin{aligned} \mathcal{D}(T) &= \{y \in W^2_{2,U} : y'' \in \mathcal{D}(L)\} \\ &= \{y \in W^4_2 : y(0) = y'(0) = y''(1) = (1-\gamma)y'''(1) + \gamma y'''(0) = 0\}, \\ (Ty)(x) &= y''(x) - y''(0) - xy'''(0), \quad x \in [0,1]. \end{aligned}$$

Then $(\mathfrak{H}, T)$ is a linearization pair of problem (2.1)–(2.3). As a consequence, the spectra and eigenfunctions of T and of problem (2.1)–(2.3) coincide.

In the following we investigate the properties of the eigenfunctions of (2.1)–(2.3) in other Sobolev spaces. To this end, we denote by $W^k_{2,U}$, $k = 1, 2, \ldots$, the subspace of the Sobolev space W^k_2 consisting of all functions that fulfil the given λ-independent boundary conditions (2.3) of order less than k, that is,

$$\begin{aligned} W^1_{2,U} &= \{y \in W^1_2 : y(0) = 0\}, \\ W^3_{2,U} &= \{y \in W^3_2 : y(0) = y'(0) = y''(1) = 0\}, \\ W^k_{2,U} &= W^k_2 \cap W^3_{2,U}, \quad k \geqslant 4, \end{aligned}$$

and $W^2_{2,U}$ is as in (2.6).

The domain of the linearization T in theorem 2.1 suggests that the eigenfunctions of (2.1)–(2.3) will not form a complete system in $W^4_{2,U}$ because they satisfy the 'hidden' boundary condition $(1-\gamma)y'''(1) + \gamma y'''(0) = 0$. Therefore, for $k = 4$, we also introduce

$$W^4_{2,\mathcal{U}} = \mathcal{D}(T),$$

which is a closed subspace of W^4_2. Since T is an operator with discrete spectrum and $\lambda = 0$ is not an eigenvalue, $T : W^4_{2,\mathcal{U}} \to W^2_{2,U}$ is an isomorphism. Therefore, after suitable normalization, the system $\{y_n\}$ of eigenfunctions of T also forms a Riesz basis in $W^4_{2,\mathcal{U}}$ and in all intermediate spaces $[W^4_{2,\mathcal{U}}, W^2_{2,U}]_\theta, 0 \leqslant \theta \leqslant 1$. It follows from Grisvard's theorem (see [22, p. 320]) that the spaces $[W^4_{2,\mathcal{U}}, L_2]_{k/4}$ coincide with $W^k_{2,U}$, $k = 0, 1, 2, 3$. Then the reiteration theorem (see [22, p. 105]) yields $(W^4_{2,\mathcal{U}}, W^2_{2,U})_{1/2} = W^3_{2,U}$. Hence the system $\{y_n\}$, up to normalization, forms a Riesz basis in $W^3_{2,U}$. Since T cannot be viewed as a bounded operator from $W^2_{2,U}$ to $W^1_{2,U}$, one can expect that its eigenfunctions $\{y_n\}$ may loose the basis property in $W^1_{2,U}$. In fact, this is true: the system $\{\mu_n y_n\}$ with y_n defined as in (2.13) is almost normalized in W^1_2, that is, there exist constants $C_1, C_2 > 0$ such that, for all n,

$$C_1 \leqslant \|\mu_n y_n\|_1 \leqslant C_2.$$

However, for the function defined by $f(x) = x$, we have

$$(f, \mu_n y_n)_1 = \tfrac{4}{3} \cos \mu_n + \mathrm{o}(1) \not\to 0, \quad n \to \infty;$$

a contradiction if $\{\mu_n y_n\}$ were a basis in $W^1_{2,U}$. The same arguments can be applied to show that the system $\{\mu_n^2 y_n\}$ is almost orthogonal but does not form a basis in L_2. Thus we have proved the following result.

THEOREM 2.2. *Let $\gamma \neq 0$. Then, after suitable normalization, the system of eigenfunctions $\{y_n\}$ of problem (2.1)–(2.3) forms a Riesz basis in the spaces $W^4_{2,\mathcal{U}}$, $W^3_{2,U}$ and $W^2_{2,U}$, but not in $W^1_{2,U}$ nor in L_2.*

Note that the property of completeness of the eigenfunctions $\{y_n\}$ of problem (2.1)–(2.3) in the spaces $W^4_{2,\mathcal{U}}$, $W^3_{2,U}$ and $W^2_{2,U}$ was proved in [19].

3. A model for viscous flow over an inclined plane with surface-tension gradient

In this section we consider a linearized problem for a flow of an incompressible viscous fluid over a rigid plane inclined at an angle β with respect to the horizontal (see [8, 20]). This problem leads to a boundary eigenvalue problem for the well-known Orr–Sommerfeld equation

$$((D^2 - \alpha^2)^2 - \mathrm{i}\alpha R(u(D^2 - \alpha^2) - u''))y = \lambda(D^2 - \alpha^2)y \tag{3.1}$$

on the interval $[0, 1]$. Here, R is the Reynolds number, α is the wavenumber of the perturbation, u is the velocity profile of the unperturbed basic flow and $\lambda = -\mathrm{i}\alpha Rc$ determines the exponential time dependence of the perturbation via $\psi(x, z, t) = y(x)\mathrm{e}^{\mathrm{i}\alpha(z-ct)}$.

With regard to the boundary conditions, we assume, without loss of generality, that $u(0) = 0$ (otherwise, we replace the profile u by $u - u(0)$ and introduce a new eigenvalue parameter $\mu = \lambda + \mathrm{i}\alpha Ru(0)$). If $u''(0) \neq 0$ (e.g. if the basic profile u is parabolic), then the λ-linear boundary conditions considered in [8] are equivalent to the following,

$$y(1) = 0, \quad y'(1) = 0, \tag{3.2}$$

$$\mathrm{i}\alpha Ru''(0)y(0) = \lambda(y''(0) + \alpha^2 y(0)), \tag{3.3}$$

$$y'''(0) - 3\alpha^2 y'(0) - \mathrm{i}\alpha R(\gamma(y''(0) + \alpha^2 y(0)) + u'(0)y(0)) = \lambda y'(0), \tag{3.4}$$

with

$$\gamma := \frac{2\cot\beta + \alpha^2 C}{Ru''(0)},$$

where C is the capillarity number and β is the angle of inclination of the plane.

In order to construct a linearization pair for this problem and to investigate the properties of its eigen- and associated functions, we use the substitution

$$(D^2 - \alpha^2)y = z. \tag{3.5}$$

Taking into account the boundary conditions (3.2), we find

$$y(x) = (Kz)(x) := -\frac{1}{\alpha}\int_x^1 \sinh(\alpha(x - \xi))z(\xi)\,\mathrm{d}\xi, \quad x \in [0, 1]. \tag{3.6}$$

After the substitution (3.5), the problem (3.1)–(3.4) takes the form

$$(D^2 - \alpha^2)z - \mathrm{i}\alpha R(uz - u''Kz) = \lambda z, \tag{3.7}$$

$$\mathrm{i}\alpha Ru''(0)(Kz)(0) = \lambda(z(0) + 2\alpha^2(Kz)(0)), \tag{3.8}$$

$$z'(0) - \mathrm{i}\alpha R\gamma z(0) - 2\alpha^2(Kz)'(0) - \mathrm{i}\alpha R(2\alpha^2\gamma + u'(0))(Kz)(0) = \lambda(Kz)'(0). \tag{3.9}$$

First we rewrite the boundary conditions (3.8), (3.9), so that we can apply the regularity concept from [15]. To this end, we eliminate the terms $\lambda(Kz)(0)$ and $\lambda(Kz)'(0)$ by means of the differential equation (3.7), and we obtain the equivalent boundary conditions,

$$\begin{aligned} &-2\alpha\sinh\alpha z'(1) - 2\alpha^2 z(0) + 2\alpha^2\cosh\alpha z(1) \\ &\qquad + \mathrm{i}\alpha R u''(0)(Kz)(0) - 2\alpha\int_0^1 \sinh(\alpha\xi)(K_1 z)(\xi)\,\mathrm{d}\xi = \lambda z(0), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\cosh\alpha z'(1) - \mathrm{i}\alpha R\gamma z(0) - \alpha\sinh\alpha z(1) - 2\alpha^2(Kz)'(0) \\ &\qquad - \mathrm{i}\alpha R(2\alpha^2\gamma + u'(0))(Kz)(0) - \int_0^1 \cosh(\alpha\xi)(K_1 z)(\xi)\,\mathrm{d}\xi = 0, \end{aligned} \tag{3.11}$$

where

$$K_1 z := \mathrm{i}\alpha R(uz - u''Kz).$$

In the following we are going to apply some results of [15] to the problem (3.7), (3.10), (3.11). It should be mentioned that in [15] no integral terms were present in the differential equation or in the boundary conditions, but since these terms are of lower order than the differential terms, they do not play a role for the regularity (for details, see [14]).

According to [15], one can associate a linear operator with the problem (3.7), (3.10), (3.11) either in the space $L_2 \times \mathbb{C}$ or in W_2^1 (and also in some subspaces of W_2^k for $k \geqslant 2$). To construct a linearization in W_2^1, we have to eliminate the λ-dependent term in the first boundary condition by means of (3.7). Then (3.10) is replaced by

$$z''(0) + 2\alpha\sinh\alpha z'(1) + \alpha^2 z(0) - 2\alpha^2\cosh\alpha z(1) + 2\alpha\int_0^1 \sinh(\alpha\xi)(K_1 z)(\xi)\,\mathrm{d}\xi = 0. \tag{3.12}$$

Now, with the problem (3.7), (3.11), (3.12), we can associate the operator L in W_2^1 given by

$$\mathcal{D}(L) = \{z \in W_2^3 : U_1(z) = U_2(z) = 0\}, \tag{3.13}$$

$$Lz = (D^2 - \alpha^2)z - \mathrm{i}\alpha R(uz - u''Kz), \tag{3.14}$$

where $U_1(z) = U_2(z) = 0$ are the boundary conditions given by (3.12) and (3.11). Obviously, the spectra of the operator L and of the original problem (3.1)–(3.4) coincide, and the eigen- and associated functions are connected by the relation $y_n^s(x) = Kz_n^s(x)$. Observe that K maps the space W_2^1 isomorphically onto the subspace

$$W_{2,U}^3 = \{y \in W_2^3 : y(1) = y'(1) = 0\}.$$

The inverse operator $K^{-1} = P$ is given by $\mathcal{D}(P) = W_{2,U}^3$, $P = (D^2 - \alpha^2)$. Using the representation of L according to (3.13), (3.14), (3.11) and (3.12), and observing that $(Kz)(1) = (Kz)'(1) = 0$, we obtain the following result.

Theorem 3.1. *The operator $T = KLK^{-1}$ acting in $\mathfrak{H} = W^3_{2,U}$ is given by*

$$\begin{aligned}\mathcal{D}(T) &= \{y \in W^3_{2,U} : y'' \in \mathcal{D}(L)\}\\ &= \{y \in W^5_2 : y(1) = y'(1) = \mathcal{U}_1(y) = \mathcal{U}_2(y) = 0\},\\ (Ty)(x) &= ((D^2-\alpha^2)y)(x) + \mathrm{i}\alpha R u(x) y(x) - \frac{1}{\alpha}\sinh(\alpha(x-1))y'''(1)\\ &\quad - \cosh(\alpha(x-1))y''(1) - 2\mathrm{i}\alpha R\int_x^1 \cosh(\alpha(x-\xi))u'(\xi)y(\xi)\,\mathrm{d}\xi,\\ &\qquad x \in [0,1],\end{aligned}$$

where $\mathcal{U}_1(y) = U_1((D^2-\alpha^2)y)$ and $\mathcal{U}_2(y) = U_2((D^2-\alpha^2)y)$ have the form

$$\begin{aligned}\mathcal{U}_1(y) &= y^{(4)}(0) + 2\alpha\sinh\alpha y'''(1) + 2\alpha^2\cosh\alpha y''(1)\\ &\quad - \alpha^4 y(0) - 4\mathrm{i}\alpha^3 R\int_0^1 \cosh(\alpha\xi))u'(\xi)y(\xi)\,\mathrm{d}\xi,\\ \mathcal{U}_2(y) &= \cosh\alpha y'''(1) - \mathrm{i}\alpha R\gamma y''(0) - \alpha\sinh\alpha y''(1) - 2\alpha^2 y'(0)\\ &\quad - \mathrm{i}\alpha R(\alpha^2\gamma + 2u'(0))y(0) - 2\mathrm{i}\alpha^2 R\int_0^1 \sinh(\alpha\xi))u'(\xi)y(\xi)\,\mathrm{d}\xi.\end{aligned}$$

The pair $(W^3_{2,U}, T)$ is a linearization pair of problem (3.1)–(3.4) and hence the spectra, as well as the eigen- and associated functions of T and of problem (3.1)–(3.4), coincide.

In the following we shall study the basis properties of the eigen- and associated functions of the operator T. According to [15], the system $\{z_n^s\}$ of eigen- and associated functions of the operator L forms a Riesz basis in the space W_2^1, provided problem (3.7), (3.10), (3.11) is strongly regular. By the definition in [15], the latter problem is strongly regular if and only if the auxiliary problem consisting of its leading terms, which is given by

$$z'' = \lambda z, \qquad \lambda z(0) = 0, \qquad \cosh\alpha z'(1) = 0, \tag{3.15}$$

enjoys this property.

Again, by the definition, it is sufficient to check that the problem

$$z'' - \lambda z = 0, \qquad z(0) = z'(1) = 0$$

is strongly regular, which is immediate (see [11, ch. 1]). Hence the eigen- and associated functions $\{z_n^s\}$ of the operator L form a Riesz basis in W_2^1. Since $K : W_2^1 \mapsto W^3_{2,U}$ is an isomorphism, the system $\{y_n^s\}$ of eigen- and associated functions of the operator $T = KLK^{-1}$ is a Riesz basis in $W^3_{2,U}$. If we define the spaces

$$\begin{aligned}W^4_{2,\mathcal{U}} &:= \{y \in W_2^4 : y(1) = y'(1) = \mathcal{U}_2(y) = 0\},\\ W^5_{2,\mathcal{U}} &:= \{y \in W_2^5 : y(1) = y'(1) = \mathcal{U}_1(y) = \mathcal{U}_2(y) = 0\},\end{aligned}$$

then, obviously, $T - \lambda_0 : W^5_{2,\mathcal{U}} \to W^3_{2,U}$ is an isomorphism for some $\lambda_0 \in \mathbb{C}$ and hence the system $\{y_n^s\}$ is also a Riesz basis in $W^5_{2,\mathcal{U}}$ and in the intermediate space $W^4_{2,\mathcal{U}}$. Altogether, taking into account theorem 3.1, we have proved the following.

THEOREM 3.2. *Let $u''(0) \neq 0$. Then the system of eigenfunctions and associated functions $\{y_n^s\}$ of problem (3.1)–(3.4) forms a Riesz basis in the spaces $W^5_{2,\mathcal{U}}$, $W^4_{2,\mathcal{U}}$ and $W^3_{2,U}$.*

REMARK 3.3. By arguments analogous to those of § 2, it can be shown that $\{y_n\}$ is not a Riesz basis in $W^k_{2,U}$ for $k < 3$, where

$$W^1_{2,U} := \{y \in W^1_2 : y(1) = 0\},$$
$$W^2_{2,U} := \{y \in W^2_2 : y(1) = y'(1) = 0\}.$$

4. A fourth-order symmetric problem

4.1. Operator interpretation of the problem and plan of study

In this section, we consider a formally symmetric problem consisting of the differential equation

$$D^2(pD^2y) - D(qDy) + ry = \lambda(D(sDy) + wy) \quad \text{on } (0,1), \tag{4.1}$$

together with boundary conditions of the form

$$\left.\begin{pmatrix} -D(pD^2y) + (\lambda s + q)Dy \\ pD^2y \end{pmatrix}\right|_{x=0} = \lambda A \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}, \tag{4.2}$$

$$\left.\begin{pmatrix} -D(pD^2y) + (\lambda s + q)Dy \\ pD^2y \end{pmatrix}\right|_{x=1} = -\lambda B \begin{pmatrix} y(1) \\ y'(1) \end{pmatrix}. \tag{4.3}$$

In these expressions, the coefficients p, s, q, r and w are real valued, with p and s positive; moreover, we assume that $1/p$, $1/s$, p, s, q, r and w are all bounded (in fact, q, r and w need only lie in $L_1(0,1)$). The requirement that p and s be of the same sign is not necessary; however, it is necessary for the analysis that follows that these two functions be of one sign, and they can then be arranged to be of the same sign by making the substitution $\lambda \to -\lambda$ if necessary. The quantities A and B appearing in the boundary conditions are full rank 2×2 Hermitian matrices.

Problems of this form arise in the study of stability of superposed fluids (see, for example, problem 12 in Chandrasekhar's seminal paper [3]; our eigenparameter is $1/\sigma$ in Chandrasekhar's notation). It should be noted that our version of the differential expression, in which λ arises linearly rather than quadratically, corresponds to the particular value $\ell = 0$ for the integer parameter ℓ describing angular dependence in the model. We have included λ-dependent boundary conditions in our problem partly out of mathematical interest, and partly because such boundary conditions could arise both for mathematical and for physical reasons. Mathematically, one could obtain such conditions as a result of regularizing problems on infinite intervals and imposing boundary conditions determined by an asymptotic analysis; physically, one encounters such boundary conditions in, for example, numerous different problems of buckling under compression and torsion [4, p. 47].

We associate with this problem the Hilbert space $\mathfrak{H} = L_2(0,1) \times \mathbb{C}^4$ equipped with the scalar product

$$(\hat{f}, \hat{g}) = (f, g)_{L_2} + \sum_{j=0}^{3} f_j \bar{g}_j, \tag{4.4}$$

where

$$\hat{f} = \left\{ f, \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \right\}, \ \hat{g} = \left\{ g, \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}, \begin{pmatrix} g_2 \\ g_3 \end{pmatrix} \right\} \in \mathfrak{H}.$$

(For later convenience, we are indicating elements of $\mathbb{C}^4$ by two vectors in $\mathbb{C}^2$.) In $\mathfrak{H}$, we define operators $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ by

$$D(\mathcal{L}) = D(\mathcal{M}) = D(\mathcal{N}) = \left\{ \hat{y} = \left\{ y, \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}, \begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} \right\} \in \mathfrak{H} : y \in W_2^4 \right\}, \tag{4.5}$$

$$\mathcal{L}\hat{y} = \left\{ (D^2pD^2 - DqD + r)y, \begin{pmatrix} (DpD^2 - qD)y \\ -pD^2y \end{pmatrix} \Bigg|_{x=0}, \begin{pmatrix} (-DpD^2y + qD)y \\ pD^2y \end{pmatrix} \Bigg|_{x=1} \right\}, \tag{4.6}$$

$$\mathcal{M}\hat{y} = \left\{ (-DsD)y, \begin{pmatrix} -sy'(0) \\ 0 \end{pmatrix}, \begin{pmatrix} sy'(1) \\ 0 \end{pmatrix} \right\}, \tag{4.7}$$

$$\mathcal{N}\hat{y} = \left\{ -wy, A\begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}, B\begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} \right\}. \tag{4.8}$$

REMARK 4.1. We shall often abbreviate the notation above, for example by writing $(DpD^2 - qD)y(0)$ instead of $(DpD^2 - qD)y|_{x=0}$, or by writing $sy'(0)$ instead of $s(0)y'(0)$.

REMARK 4.2. We shall see that when one of the operators $\mathcal{L}$, $\mathcal{M} + \mathcal{N}$ is positive, the eigenvalues of the problem are all real. This should be compared with the results of Greenberg [9]: at least when the matrices A and B appearing in the boundary conditions are positive-semidefinite, his oscillation theory generalizes to the current setting and gives a counting function for the real eigenvalues of the problem.

The problem (4.1)–(4.3) is now recast in the operator form,

$$[\mathcal{L} + \lambda(\mathcal{M} + \mathcal{N})]\hat{y} = \mathbf{0}. \tag{4.9}$$

Integrating by parts, one finds the quadratic forms associated with the operators $\mathcal{L}$ and $\mathcal{M}$,

$$(\mathcal{L}\hat{y}, \hat{y}) = (pD^2y, D^2y)_{L_2} + (qDy, Dy)_{L_2} + (ry, y)_{L_2}, \tag{4.10}$$

$$(\mathcal{M}\hat{y}, \hat{y}) = (sDy, Dy)_{L_2}. \tag{4.11}$$

From these expressions, we see immediately that $\mathcal{L}$ and $\mathcal{M}$ are symmetric. Because A and B are Hermitian, the operator $\mathcal{N}$ is symmetric. It is also densely defined (see proposition 4.3 below) and admits a bounded self-adjoint extension, which we also denote $\mathcal{N}$, given by

$$\mathcal{N}\left\{ f, \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \right\} = \left\{ -wf, A\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, B\begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \right\}. \tag{4.12}$$

The rest of this section is organized as follows. First, we shall show that $\mathcal{L}$ is a self-adjoint operator with discrete spectrum. Secondly, we shall find the Friedrichs extension $\mathcal{K}_F$ of the operator $\mathcal{K} = \mathcal{M} + \mathcal{N}$ (which exists because $\mathcal{M} + \mathcal{N}$ is bounded below). Then, assuming that $\mathcal{K}_F$ is boundedly invertible (later, a criterion for this property will be found), we consider the operator

$$\mathcal{T} = \mathcal{K}_F^{-1}\mathcal{L}.$$

With this operator, we associate the spaces $\mathfrak{H}_{2,\mathcal{K}}$ and $\mathfrak{H}_{1,\mathcal{K}}$, which are defined as the closures of $D(\mathcal{L})$ in the norms

$$\|\hat{y}\|_{2,\mathcal{K}} = \|\mathcal{K}_F\hat{y}\|, \qquad \|\hat{y}\|_{1,\mathcal{K}} = \||\mathcal{K}_F|^{1/2}\hat{y}\|$$

respectively, where $|\mathcal{K}_F| = (\mathcal{K}_F^2)^{1/2}$. We will show that these spaces have the representations

$$\mathfrak{H}_{2,\mathcal{K}} = \left\{\hat{y} = \left\{y, \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}, \begin{pmatrix} y(1) \\ y'(1) \end{pmatrix}\right\} : y \in W_2^2\right\}, \tag{4.13}$$

$$\mathfrak{H}_{1,\mathcal{K}} = \left\{\hat{y} = \left\{y, \begin{pmatrix} y(0) \\ \eta_0 \end{pmatrix}, \begin{pmatrix} y(1) \\ \eta_1 \end{pmatrix}\right\} : y \in W_2^1,\ \eta_0, \eta_1 \in \mathbb{C}\right\}. \tag{4.14}$$

Using results from [16], we will find that $\{\mathfrak{H}_{1,\mathcal{K}}, \mathcal{T}\}$ and $\{\mathfrak{H}_{2,\mathcal{K}}, \mathcal{T}\}$ are linearization pairs of the pencil $\mathcal{L} + \lambda\mathcal{K}$. Then we find that the space $\mathfrak{H}_{1,\mathcal{K}}$ can be equipped with an indefinite Pontryagin metric such that the operator $\mathcal{T}$ is self-adjoint in the corresponding Pontryagin space.

Assuming (without loss of generality) that $\mathcal{L}$ is invertible, we also consider the scale of Hilbert spaces associated with the operator $|\mathcal{L}|$, denoting $\mathfrak{H}_{\theta,\mathcal{L}} = D(|\mathcal{L}|^{\theta/4})$ with the norm

$$\|\hat{y}\|_{\theta,\mathcal{L}} = \||\mathcal{L}|^{\theta/4}\hat{y}\|. \tag{4.15}$$

We will show that $\mathfrak{H}_{2,\mathcal{K}} = \mathfrak{H}_{2,\mathcal{L}}$ and that there is a Pontryagin inner product such that the operator $\mathcal{L}$ is self-adjoint in $\mathfrak{H}_{2,\mathcal{L}}$ with this new inner product. From the theory of operators in spaces with a Pontryagin metric, it follows that all but finitely many eigenvalues of the operator $\mathcal{T}$ (and hence of the original problem (4.1), (4.2)) are real and the eigen- and associated functions of $\mathcal{T}$ form a Riesz basis in the spaces $\mathfrak{H}_{1,\mathcal{K}}$ and $\mathfrak{H}_{2,\mathcal{K}}$. The number of non-real eigenvalues will be estimated from above. Note that $P\hat{y} = y$ (the projection of $\hat{y}$ onto its first component) maps $\mathfrak{H}_{2,\mathcal{K}}$ onto W_2^2 isomorphically. From this we will conclude that $\{W_2^2, P\mathcal{T}P^{-1}\}$ is a linearization pair for problem (4.1), (4.2) and that the eigen- and associated functions of this problem form a Riesz basis in W_2^2. In the last section, we will find the explicit form of the expressions for $\mathcal{K}_F^{-1}$, for $\mathcal{T} = \mathcal{K}_F^{-1}\mathcal{L}$, and for the so-called 'hidden' boundary conditions that appear in the domain of $\mathcal{T}$ acting in the space $\mathfrak{H}_{2,\mathcal{K}}$.

4.2. The operators $\mathcal{L}$ and $\mathcal{K}_F = \mathcal{M}_F + \mathcal{N}$

PROPOSITION 4.3. *The operator $\mathcal{L}$ is densely defined and self-adjoint in $\mathfrak{H}$.*

Proof. For a proof that $\mathcal{L}$ is densely defined (in a general setting), see [15, lemma 1]. In our particular case, this is easily seen independently: functions $y \in W_2^4$ that vanish with their derivatives at the endpoints are dense in L_2; then, given $\epsilon > 0$ and

$$\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \quad \text{in } \mathbb{C}^2,$$

there exists a function $g \in W_2^4$ such that

$$\|g\|_{L_2} < \epsilon, \qquad \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \qquad \begin{pmatrix} g(1) \\ g'(1) \end{pmatrix} = \begin{pmatrix} f_2 \\ f_3 \end{pmatrix}.$$

Hence, given an element $\hat{f} \in \mathfrak{H}$, one can find an element $\hat{y}+\hat{g} \in D(\mathcal{L})$ approximating $\hat{f}$ with given accuracy ϵ. As $\mathcal{L}$ is symmetric, to show that $\mathcal{L}$ is self-adjoint, it is sufficient to show that $\mathcal{L} - \mu$ is invertible for some $\mu \in \mathbb{R}$; in other words, it is sufficient to show that, for some μ, the equation

$$(\mathcal{L} - \mu)\hat{y} = \hat{f} = \left\{ f, \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \right\} \tag{4.16}$$

is uniquely solvable for any $\hat{f} \in \mathfrak{H}$. To this end, let $L(\mu)$ be the operator in $L_2(0,1)$ generated by the differential expression

$$(L - \mu)y = (D^2 p D^2 - DqD + r - \mu)y$$

on the domain $D(L(\mu)) \subset W_2^4$ subject to the boundary conditions

$$\begin{aligned} U_1(y) &= (-DpD^2 + qD - \mu)y(1) = 0, \\ U_2(y) &= (pD^2 - \mu D)y(1) = 0, \\ U_3(y) &= (DpD^2 - qD - \mu)y(0) = 0, \\ U_4(y) &= (-pD^2 - \mu D)y(0) = 0. \end{aligned}$$

From the standard theory of self-adjoint differential expressions (see, for example, [11]), it is easy to see that $L(\mu)$ is self-adjoint in $L_2(0,1)$ for $\mu \in \mathbb{R}$. Thus the first component y of the solution $\hat{y}$ is given by

$$y(x) = ((L(\mu))^{-1} f)(x) + \sum_{j=1}^{4} c_j y_j(x),$$

where $\{y_1, \dots, y_4\}$ is a fundamental system of the differential equation

$$(D^2 p D^2 - DqD + r - \mu)y = 0,$$

Δ is the characteristic determinant

$$\Delta = \Delta(\mu) = \begin{vmatrix} U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) \\ U_4(y_1) & U_4(y_2) & U_4(y_3) & U_4(y_4) \end{vmatrix}$$

and $c_j = \Delta_j / \Delta$, with Δ_j being the determinant obtained by replacement of the jth column in Δ by the column vector $(f_0, f_1, f_2, f_3)^{\mathrm{T}}$. The remaining components of $\hat{y}$ are determined by the condition $\hat{y} \in D(\mathcal{L})$. It is well known that $L(\mu)$ is invertible if and only if $\Delta(\mu) \neq 0$. Thus it remains to show that $\Delta(\cdot)$ is not identically zero. Now the quadratic form of $L(\mu)$ is easily calculated to be

$$\begin{aligned} (L(\mu)y, y)_{L_2} = (pD^2 y, D^2 y)_{L_2} + (qDy, Dy)_{L_2} + ((r - \mu)y, y)_{L_2} \\ - \mu(|y(1)|^2 + |y'(1)|^2 + |y(0)|^2 + |y'(0)|^2), \quad y \in D(L(\mu)). \end{aligned}$$

Since p is uniformly positive and since q and r are bounded, we claim that it is now clear that $L(\mu)$ is a positive operator for all sufficiently large negative μ. To see this, observe first that if we define a quadratic form Q_μ on $W_2^2(0,1)$ by

$$Q_\mu(y) = (pD^2y, D^2y)_{L_2} + (qDy, Dy)_{L_2} + ((r-\mu)y, y)_{L_2},$$

then certainly $(L(\mu)y, y)_{L_2} \geqslant Q_\mu(y)$ for all $\mu < 0$. It therefore suffices to show that $Q_\mu(\cdot)$ is positive for all sufficiently large negative μ. It is well known, and easy to show by a standard calculus of variations argument, that

$$Q_\mu(y) \geqslant \kappa(\mu)(y, y)_{L_2},$$

where $\kappa(\mu)$ is the lowest eigenvalue of the differential expression $L - \mu$ subject to Neumann boundary conditions $py'' = 0 = (py'')' - qy'$ at the endpoints 0, 1. Moreover, since the Neumann boundary conditions do not depend on μ, one has

$$\kappa(\mu) = -\mu + \kappa(0).$$

Hence, for all $y \in W_2^2(0,1)$,

$$Q_\mu(y) \geqslant (-\mu + \kappa(0))(y, y)_{L_2},$$

and so $Q_\mu(\cdot)$ is positive for all sufficiently large, negative μ. This completes the proof. □

PROPOSITION 4.4. *The operator $\mathcal{M}$ is non-negative. Its Friedrichs extension $\mathcal{M}_F$ is given by*

$$D(\mathcal{M}_F) = \left\{\hat{y} = \left\{y, \begin{pmatrix} y(0) \\ \eta_0 \end{pmatrix}, \begin{pmatrix} y(1) \\ \eta_1 \end{pmatrix}\right\} : y \in W_2^2,\ \eta_0, \eta_1 \in \mathbb{C}\right\}, \tag{4.17}$$

$$\mathcal{M}_F\hat{y} = \left\{-(DsD)y, \begin{pmatrix} -sy'(0) \\ 0 \end{pmatrix}, \begin{pmatrix} sy'(1) \\ 0 \end{pmatrix}\right\}. \tag{4.18}$$

Proof. From

$$(\mathcal{M}_F\hat{y}, \hat{y}) = (sDy, Dy)_{L_2}, \quad y \in D(\mathcal{M}_F),$$

and (4.11), we see that $\mathcal{M}_F$ is symmetric and has the same lower bound as $\mathcal{M}$, namely 0. It can also be shown, using arguments similar to those in proposition 4.3 (see also proposition 4.7 below), that $\mathcal{M}_F - \mu$ is invertible in $\mathfrak{H}$ for all $\mu < 0$. Hence $\mathcal{M}_F$ is self-adjoint.

It is known (see, for example, [12, § 124]) that the Friedrichs extension P_F of a non-negative operator P is uniquely defined by the following conditions,

$$P_F^* = P_F, D(P_F) \subseteq \mathfrak{H}_P,$$

where $\mathfrak{H}_P$ is the closure of $D(P)$ with respect to the norm $\|y\|_P^2 = (Py, y) + (y, y)$. Thus it remains to be shown that the closure of $D(\mathcal{M}) = D(\mathcal{L})$ with respect to the norm

$$\begin{aligned}\|\hat{y}\|_{\mathcal{M}}^2 &= (\mathcal{M}\hat{y}, \hat{y}) + (\hat{y}, \hat{y}) \\ &= (sDy, Dy)_{L_2} + (y, y)_{L_2} + |y(0)|^2 + |y(1)|^2 + |y'(0)|^2 + |y'(1)|^2\end{aligned}$$

contains the set $D(\mathcal{M}_F)$ defined in (4.17).

To prove this, we apply similar arguments to those used in proposition 4.3. Given $\epsilon > 0$ and $\hat{y} \in D(\mathcal{M}_F)$, there exists a function $\hat{v} \in D(\mathcal{M})$ such that

$$v(0) = y(0), \qquad v'(0) = \eta_1, \qquad v(1) = y(1), \qquad v'(1) = \eta_1$$

and

$$\|\hat{y} - \hat{v}\|_{\mathcal{M}} = \sqrt{(s(y-v)', (y-v)')_{L_2}} < \epsilon.$$

Thus $\hat{y}$ can be approximated with arbitrary accuracy in the norm $\|\cdot\|_{\mathcal{M}}$ by a function in $D(\mathcal{M})$, and so $\hat{y}$ lies in the closure of $D(\mathcal{M})$ in $\|\cdot\|_{\mathcal{M}}$, which we denote by $\mathfrak{H}_{\mathcal{M}}$. This proves that $D(\mathcal{M}_F)$ lies in $\mathfrak{H}_{\mathcal{M}}$, and so $\mathcal{M}_F$ is indeed the Friedrichs extension of $\mathcal{M}$. As a byproduct, we have also proved that

$$\mathfrak{H}_{\mathcal{M}} = \left\{ \hat{y} = \left\{ y, \begin{pmatrix} y(0) \\ \eta_0 \end{pmatrix}, \begin{pmatrix} y(1) \\ \eta_1 \end{pmatrix} \right\} : y \in W_2^1,\ \eta_0, \eta_1 \in \mathbb{C} \right\}.$$

□

Let us denote $\mathcal{K} = \mathcal{M} + \mathcal{N}$. Since $\mathcal{N}$ is bounded, $\mathcal{K}$ admits a Friedrichs extension $\mathcal{K}_F$ given by $\mathcal{K}_F = \mathcal{M}_F + \mathcal{N}$. An essential assumption that we shall use in the remainder of this article is that the operator $\mathcal{K}_F$ is invertible or, equivalently, since this operator has only discrete spectrum, that $\operatorname{Ker}(\mathcal{K}_F) = \{0\}$. A criterion for the invertibility of $\mathcal{K}_F$ will be given in § 4.4.

4.3. Properties of the operator $\mathcal{T} = \mathcal{K}_F^{-1}\mathcal{L}$. Main results

Assuming that the operator $\mathcal{K}_F = \mathcal{M}_F + \mathcal{N}$ is invertible, let us consider the operator

$$\mathcal{T} = \mathcal{K}_F^{-1}\mathcal{L}, \qquad D(\mathcal{T}) = D(\mathcal{L}). \tag{4.19}$$

It cannot be guaranteed that the operator $\mathcal{T}$ is closable in the original space $\mathfrak{H}$. However, the results in [16] suggest that $\mathcal{T}$ 'behaves well' in other spaces and can be treated as a linearization of the pencil $\mathcal{L} + \lambda\mathcal{K}$. Considering $\mathcal{T}$ in a suitable space, we shall also find a linearization of the original problem (4.1), (4.2).

Let us construct some Hilbert spaces associated with the pair of operators $\mathcal{K}$ and $\mathcal{L}$.

Since the operator $\mathcal{K}_F$ is assumed to be invertible in $\mathfrak{H}$ and $\mathcal{K} \subseteq \mathcal{K}_F$, we have $\|\mathcal{K}\tilde{y}\| \geqslant \varepsilon\|\tilde{y}\|$ for $\tilde{y} \in D(\mathcal{K}) = D(\mathcal{L})$ with some $\varepsilon > 0$. Let $\mathfrak{H}_{2,K}$ be the completion of $D(\mathcal{L})$ with respect to the norm

$$\|\tilde{y}\|_{2,\mathcal{K}} := \|\mathcal{K}\tilde{y}\|, \quad \tilde{y} \in D(\mathcal{L}) \tag{4.20}$$

(which is equivalent to the usual graph norm induced by $\mathcal{K}$), and let $\mathfrak{H}_{1,K}$ be the completion of $D(\mathcal{L})$ with respect to the norm

$$\|\tilde{y}\|_{1,\mathcal{K}} := \||\mathcal{K}_F|^{1/2}\tilde{y}\|, \quad \tilde{y} \in D(\mathcal{L}), \tag{4.21}$$

where $|\mathcal{K}_F| = (\mathcal{K}_F^2)^{1/2}$. Since $\mathcal{K}_F$ is semibounded from below, we have $\mathcal{K}_F + \mu \geqslant 1$ for some $\mu > 0$ and $D(|\mathcal{K}_F|) = D(\mathcal{K}_F + \mu)$. By Heinz's inequality (see, for example, [7, ch. III, proposition 8.12]), it follows that $D(|\mathcal{K}_F|^{1/2}) = D((\mathcal{K}_F + \mu)^{1/2})$. The latter domain coincides with the domain of the closure of the quadratic form

$$((\mathcal{K}_F+\mu)\hat{y},\hat{y}) = (sDy,Dy) + ((\mu-w)y,y) \\ + \left((A+\mu)\begin{pmatrix} y(0)\\ \eta_0\end{pmatrix},\begin{pmatrix} y(0)\\ \eta_0\end{pmatrix}\right) + \left((B+\mu)\begin{pmatrix} y(1)\\ \eta_1\end{pmatrix},\begin{pmatrix} y(1)\\ \eta_1\end{pmatrix}\right).$$

Obviously, for sufficiently large μ, there exist positive constants C_1, C_2 such that

$$C_1\{(y',y')+|\eta_0|^2+|\eta_1|^2\} \leqslant ((\mathcal{K}_F+\mu)\hat{y},\hat{y}) \leqslant C_2\{(y',y')+|\eta_0|^2+|\eta_1|^2\};$$

therefore, $\mathfrak{H}_{1,\mathcal{K}}$ has the representation (4.14) and is isomorphic to $W_2^1\times\mathbb{C}^2$.

We also consider the scale of Hilbert spaces associated with the operator $\mathcal{L}$. For $0\leqslant\theta\leqslant 4$, denote by $\mathfrak{H}_{\theta,L}$ the completion of $D(\mathcal{L})$ with respect to the norm defined by (4.15). We assume here that $\mathcal{L}$ is invertible. Otherwise, in the definition (4.15), $|\mathcal{L}|^{\theta/4}$ should be replaced by $|\mathcal{L}|^{\theta/4}+\mu$ for some $\mu>0$.

Clearly, $\mathfrak{H}_{4,\mathcal{L}}=D(\mathcal{L})$ and for $\frac{3}{2}<\theta\leqslant 4$ the space $\mathfrak{H}_{\theta,\mathcal{L}}$ consists of functions (4.13) with first component $y\in W_2^\theta$ (recall the trace theorem for functions in Sobolev spaces (see, for example, [1, ch. 1])). From this, it follows that the spaces $\mathfrak{H}_{2,\mathcal{L}}$ and $\mathfrak{H}_{2,\mathcal{K}}$ coincide (both spaces consist of functions $\tilde{y}$ having the representation (4.13)). This remark will be used in an essential way in the sequel when applying results [16].

Furthermore, denote $\mathcal{I}_\mathcal{K} := |\mathcal{K}_F|\mathcal{K}_F^{-1}$. Obviously, $\mathcal{I}_\mathcal{K}$ is a symplectic operator, i.e. $\mathcal{I}_\mathcal{K}=\mathcal{I}_+-\mathcal{I}_-$, where $\mathcal{I}_\pm$ are complementary orthogonal projections ($\mathcal{I}_++\mathcal{I}_-=I$). The rank of $\mathcal{I}_-$ coincides with the number of negative eigenvalues $\nu(\mathcal{K}_F)$ of the operator $\mathcal{K}_F$.

THEOREM 4.5. *The operator $\mathcal{T}$ defined by (4.19) is closable in the space $\mathfrak{H}_{1,\mathcal{K}}$. Its closure $\mathcal{T}_\mathcal{K}$ in $\mathfrak{H}_{1,\mathcal{K}}$ has the domain $D(\mathcal{T}_\mathcal{K})=\mathfrak{H}_{3,\mathcal{L}}$. The operator $\mathcal{T}_\mathcal{K}$ is self-adjoint in $\mathfrak{H}_{1,\mathcal{K}}$, equipped with the Pontryagin inner product*

$$[\hat{y},\hat{z}] = (\mathcal{I}_\mathcal{K}\hat{y},\hat{z}).$$

It has discrete spectrum which coincides with the spectrum of the linear pencil $\mathcal{L}+\lambda\mathcal{K}$ in $\mathfrak{H}$ (counting algebraic multiplicities). All but finitely many (less than or equal to $2\nu(\mathcal{K}_F)$) eigenvalues are real. The eigen- and associated functions of $\mathcal{T}_\mathcal{K}$ form a Riesz basis in the space $\mathfrak{H}_{1,\mathcal{K}}$, and hence if $\{y_n\}$ is a system of eigen- and associated functions of problem (4.1), (4.2), (4.3), then the system $\{\{y_n,y_n'(0),y_n'(1)\}\}$ forms a Riesz basis in the space $W_2^1\times\mathbb{C}^2$.

Proof. Consider the operator

$$\mathcal{T}_1 := |\mathcal{K}_F|^{-1}\mathcal{L}, \qquad D(\mathcal{T}_1)=D(\mathcal{L})$$

in $\mathfrak{H}_{1,\mathcal{K}}$. It follows from [16, theorem 1] that $\mathcal{T}_1$ is essentially self-adjoint in $\mathfrak{H}_{1,\mathcal{K}}$, i.e. its closure $\bar{\mathcal{T}}_1$ is self-adjoint. Recalling $\mathfrak{H}_{2,\mathcal{L}}=\mathfrak{H}_{2,\mathcal{K}}$ and applying [16, theorem 2], we obtain that $D(\bar{\mathcal{T}}_1)=\mathfrak{H}_{3,\mathcal{L}}$.

From (4.21), it follows that $\mathcal{I}_\mathcal{K}$ is not only symplectic in $\mathfrak{H}$, but also in $\mathfrak{H}_{1,\mathcal{K}}$ (since $\mathcal{I}_\mathcal{K}$ commutes with $\mathcal{K}$). Hence the closure of $\mathcal{T}_1$ in $\mathfrak{H}_{1,\mathcal{K}}$ coincides with the operator $\mathcal{T}_\mathcal{K}=\mathcal{I}_\mathcal{K}\bar{\mathcal{T}}_1$ (which also has the domain $\mathfrak{H}_{3,\mathcal{L}}$). Hence $\mathcal{T}_\mathcal{K}$ is self-adjoint in the inner product generated by $\mathcal{I}_\mathcal{K}$, i.e. $\mathcal{I}_\mathcal{K}\mathcal{T}_\mathcal{K}$ is self-adjoint in $\mathfrak{H}_{1,\mathcal{K}}$.

Since the embedding $\mathfrak{H}_{4,\mathcal{L}} \to \mathfrak{H}$ is compact, it follows from [16, theorem 2] that $\mathcal{T}_{\mathcal{K}}$ has discrete spectrum and its eigenvalues as well as its eigen- and associated functions coincide with those of the pencil $\mathcal{L} + \lambda\mathcal{K}$.

By Pontryagin's theorem (see [2, ch. 1]), the number of non-real eigenvalues of $\mathcal{T}_{\mathcal{K}}$ does not exceed $2 \cdot \operatorname{rank} \mathcal{I}_{\mathcal{K}} = 2\nu(\mathcal{K}_F)$, and the non-real spectrum is symmetric with respect to the real axis. Finally, a theorem of Azizov and Iokhvidov (see [2, ch. 5]) implies that the system of eigen- and associated functions of $\mathcal{T}_{\mathcal{K}}$ forms a Riesz basis in $\mathfrak{H}_{1,\mathcal{K}}$. Since the projection

$$P : \left\{ y, \begin{pmatrix} y(0) \\ \eta_0 \end{pmatrix}, \begin{pmatrix} y(1) \\ \eta_1 \end{pmatrix} \right\} \mapsto \{y, \eta_0, \eta_1\}$$

maps $\mathfrak{H}_{1,\mathcal{K}}$ isomorphically onto $W_2^1 \times \mathbb{C}^2$, we therefore deduce that the system $\{y_n, y_n'(0), y_n'(1)\}$ is a Riesz basis in $W_2^1 \times \mathbb{C}^2$. □

Let us consider the operator $\mathcal{T} = \mathcal{K}_F^{-1}\mathcal{L}$ in the space $\mathfrak{H}_{2,\mathcal{K}} = \mathfrak{H}_{2,\mathcal{L}}$, but with a different domain as in (4.19). If we were to use the same domain $D(\mathcal{T}) = D(\mathcal{L})$, $\mathcal{T}$ would map $D(\mathcal{T})$ into a larger space than $\mathfrak{H}_{2,\mathcal{L}}$. According to [16], to overcome this difficulty, the domain of $\mathcal{T}$ must be restricted to the following:

$$D(\mathcal{T}) = \{\hat{y} \in D(\mathcal{L}) : (\mathcal{L}\hat{y}, \tilde{y}_0) = 0 \text{ for all } \tilde{y}_0 \in \operatorname{Ker}(\bar{\mathcal{K}})\}.$$

In the next section, we show that $\operatorname{Ker}(\bar{\mathcal{K}})$ is two dimensional, so that

$$D(\mathcal{T}) = \{\hat{y} \in D(\mathcal{L}) : (\mathcal{L}\hat{y}, \tilde{y}_j) = 0 \text{ for } \tilde{y}_j \in \operatorname{Ker}(\bar{\mathcal{K}}),\ j = 1, 2\}, \tag{4.22}$$

where $\{\tilde{y}_1, \tilde{y}_2\}$ is a basis of $\operatorname{Ker}(\bar{\mathcal{K}})$.

Theorem 4.6. *Consider $\mathcal{T} = \mathcal{K}_F^{-1}\mathcal{L}$ as an operator in $\mathfrak{H}_{2,\mathcal{L}}$ with domain (4.22). Then $(\mathfrak{H}_{2,\mathcal{L}}, \mathcal{T})$ is a linearization pair of the pencil $\mathcal{L} + \lambda\mathcal{K}$. In particular, the eigenvalues, eigen- and associated functions of $\mathcal{T}$ and of the pencil $\mathcal{L} + \lambda\mathcal{K}$ coincide. The operator $\mathcal{T}$ is self-adjoint in the Pontryagin inner product generated by $\mathcal{I}_{\mathcal{L}} := |\mathcal{L}|\mathcal{L}^{-1}$ (which is a symplectic operator in $\mathfrak{H}_{2,\mathcal{L}}$). The number of non-real eigenvalues of $\mathcal{T}$ does not exceed $2\nu(\mathcal{L})$, where $\nu(\mathcal{L})$ is the number of negative eigenvalues of $\mathcal{L}$. The eigen- and associated functions of $\mathcal{T}$ (and hence of the pencil $\mathcal{L} + \lambda\mathcal{K}$) form a Riesz basis in $\mathfrak{H}_{2,\mathcal{L}}$. As a consequence, the eigen- and associated functions of problem (4.1), (4.2), (4.3) form a Riesz basis in W_2^2.*

Proof. The statement that $(\mathfrak{H}_{2,\mathcal{L}}, \mathcal{T})$ is a linearization pair of the pencil $\mathcal{L} + \lambda\mathcal{K}$ has been proved in [16, theorem 3]. Next we show that $\mathcal{I}_{\mathcal{L}}\mathcal{T}$ is a symmetric operator in $\mathfrak{H}_{2,\mathcal{L}}$. Indeed, for $\hat{y} \in D(\mathcal{T})$ defined by (4.22), we have

$$(\mathcal{I}_{\mathcal{L}}\mathcal{T}\hat{y}, \hat{y})_{2,\mathcal{L}} = (|\mathcal{L}|^{1/2}\mathcal{I}_{\mathcal{L}}\mathcal{K}_F^{-1}\mathcal{L}\hat{y}, |\mathcal{L}|^{1/2}\hat{y}) = (\mathcal{I}_{\mathcal{L}}\mathcal{K}_F^{-1}\mathcal{L}\hat{y}, |\mathcal{L}|\hat{y}) = (\mathcal{K}_F^{-1}\mathcal{L}\hat{y}, \mathcal{L}\hat{y}) \in \mathbb{R}$$

as $\mathcal{K}_F^{-1}$ is self-adjoint in $\mathfrak{H}$. But $\mathcal{I}_{\mathcal{L}}\mathcal{T}$ is invertible in $\mathfrak{H}_{2,\mathcal{L}}$, since both $\mathcal{I}_{\mathcal{L}}$ and $\mathcal{T}$ are invertible in $\mathfrak{H}_{2,\mathcal{L}}$. This proves that $\mathcal{T}$ is self-adjoint in $\mathfrak{H}_{2,\mathcal{L}}$, with the inner product $[\hat{y}, \hat{z}] = (\mathcal{I}_{\mathcal{L}}\hat{y}, \hat{z})$. All other statements are obtained in the same way as in theorem 2.1. Recall that the map $P : \mathfrak{H}_{2,\mathcal{L}} \to W_2^2$ defined by $P\hat{y} = y$ is an isomorphism. Therefore, $(W_2^2, P\mathcal{T}P^{-1})$ is a linearization pair for the problem (4.1), (4.2). □

4.4. Expressions for the operators $\mathcal{K}_F^{-1}$ and $\mathcal{T}$. The domain of $\mathcal{T}$ acting in $\mathfrak{H}_{2,\mathcal{K}}$ and the linearization T

It is not an easy task to find an explicit expression for the operator $\mathcal{T}$ and its compression $T = P\mathcal{T}P^{-1}$ which realizes the linearization of the problem (4.1), (4.2). In the following, we shall show how these operators may be calculated.

PROPOSITION 4.7. *Let*

$$A^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} b_{11} & b_{12} \\ \bar{b}_{12} & b_{22} \end{pmatrix}.$$

Let K be the operator on $L_2(0,1)$ defined by

$$D(K) = \{y \in W_2^2 : V_0(y) := y(0) - a_{11}sy'(0) = 0,\ V_1(y) := y(1) + b_{11}sy'(1) = 0\}, \tag{4.23}$$

$$Ky = (-DsD - w)y. \tag{4.24}$$

Then $\mathcal{K}_F = \mathcal{M}_F + \mathcal{N}$ is invertible if and only if K is invertible; in this case, if $\mathfrak{H}$ is identified with the direct product $L_2 \times \mathbb{C}^2 \times \mathbb{C}^2$, then the inverse operator $\mathcal{K}_F^{-1}$ is represented by the block operator matrix

$$\mathcal{K}_F^{-1} = \begin{pmatrix} K^{-1} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}, \tag{4.25}$$

whose entries are given by

$$\left.\begin{aligned} K_{12}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} &= (a_{11}f_0 + a_{12}f_1)y_2, \\ K_{13}\begin{pmatrix} f_2 \\ f_3 \end{pmatrix} &= (b_{11}f_2 + b_{12}f_3)y_1, \end{aligned}\right\} \quad \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \in \mathbb{C}^2, \tag{4.26}$$

$$K_{21}f = \begin{pmatrix} a_{11} \\ \bar{a}_{12} \end{pmatrix}(f, y_2), \quad K_{31}f = \begin{pmatrix} b_{11} \\ \bar{b}_{12} \end{pmatrix}(f, y_1), \quad f \in L_2, \tag{4.27}$$

$$\left.\begin{aligned} K_{22} &= \begin{pmatrix} a_{11}y_2(0) & a_{12}y_2(0) \\ \bar{a}_{12}y_2(0) & a_{22} + |a_{12}|^2 sy_2'(0) \end{pmatrix}, \\ K_{23} &= \begin{pmatrix} b_{11}y_1(0) & b_{12}y_1(0) \\ \bar{a}_{12}b_{11}sy_1'(0) & \bar{a}_{12}b_{12}sy_1'(0)| \end{pmatrix}, \end{aligned}\right\} \tag{4.28}$$

$$\left.\begin{aligned} K_{32} &= \begin{pmatrix} a_{11}y_2(1) & a_{12}y_2(1) \\ -a_{11}\bar{b}_{12}sy_2'(1) & -a_{12}\bar{b}_{12}sy_2'(1) \end{pmatrix}, \\ K_{33} &= \begin{pmatrix} b_{11}y_1(1) & b_{12}y_1(1) \\ \bar{b}_{12}y_1(1) & b_{22} - |b_{12}|^2 sy_1'(1) \end{pmatrix}, \end{aligned}\right\} \tag{4.29}$$

where y_1 and y_2 are linearly independent solutions of the differential equation

$$(-DsD - w)y = 0 \tag{4.30}$$

satisfying the boundary conditions $V_0(y_1) = 0$, $V_0(y_2) = 1$, $V_1(y_1) = 1$, $V_1(y_2) = 0$ defined in (4.23).

Proof. For arbitrary $\hat f \in L_2 \times \mathbb{C}^2 \times \mathbb{C}^2$, we have to solve the equation

$$(\mathcal{M}_F + \mathcal{N})\hat y = \hat f$$

with some $\hat y \in L_2 \times \mathbb{C}^2 \times \mathbb{C}^2$, or, in component form,

$$(\mathcal{M}_F + \mathcal{N})\left\{y, \begin{pmatrix} y(0) \\ \eta_0 \end{pmatrix}, \begin{pmatrix} y(1) \\ \eta_1 \end{pmatrix}\right\} = \left\{f, \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ f_3 \end{pmatrix}\right\}.$$

The first component consists of the differential equation

$$q(-DsD - w)y = f, \tag{4.31}$$

while the 'boundary conditions' are given by the remaining components, and are

$$\begin{pmatrix} -sy'(0) \\ 0 \end{pmatrix} + A\begin{pmatrix} y(0) \\ \eta_0 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \qquad \begin{pmatrix} sy'(1) \\ 0 \end{pmatrix} + B\begin{pmatrix} y(1) \\ \eta_1 \end{pmatrix} = \begin{pmatrix} f_2 \\ f_3 \end{pmatrix}.$$

These boundary conditions can be rearranged as

$$\begin{pmatrix} V_0(y) \\ \eta_0 - \bar a_{12} s y'(0) \end{pmatrix} = A^{-1}\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \qquad \begin{pmatrix} V_1(y) \\ \eta_1 + \bar b_{12} s y'(1) \end{pmatrix} = B^{-1}\begin{pmatrix} f_2 \\ f_3 \end{pmatrix}. \tag{4.32}$$

Thus it suffices to solve (4.31) subject to the boundary conditions

$$V_0(y) = a_{11} f_0 + a_{12} f_1, \qquad V_1(y) = b_{11} f_2 + b_{12} f_3, \tag{4.33}$$

and then recover the values of η_0 and η_1 from the formulae

$$\eta_0 = \bar a_{12} s y'(0) + \bar a_{12} f_0 + a_{22} f_1, \qquad \eta_1 = -\bar b_{12} s y'(1) + \bar b_{12} f_2 + b_{22} f_3. \tag{4.34}$$

Now let y_1 and y_2 be linearly independent solutions of the differential equation (4.30) as above (such linearly independent solutions exist if and only if K is invertible) and define Green's function,

$$G(x,t) := \begin{cases} \dfrac{-1}{\Delta} y_1(t) y_2(x), & 0 \leqslant t \leqslant x \leqslant 1, \\ \dfrac{-1}{\Delta} y_1(x) y_2(t), & 0 \leqslant x \leqslant t \leqslant 1, \end{cases} \tag{4.35}$$

where Δ is the constant

$$\Delta := \det\begin{pmatrix} y_1 & y_2 \\ sy_1' & sy_2' \end{pmatrix}. \tag{4.36}$$

Then it is easy to verify that $(K^{-1}f)(x) = \int_0^1 f(t)G(x,t)\,\mathrm{d}t$ and the solution of (4.31) subject to the boundary conditions (4.33) is given by

$$y(x) = \int_0^1 f(t)G(x,t)\,\mathrm{d}t + (a_{11} y_2(x)) f_0 + (a_{12} y_2(x)) f_1 + (b_{11} y_1(x)) f_2 + (b_{12} y_1(x)) f_3 \tag{4.37}$$

for $x \in [0,1]$. From (4.34) and (4.37), we therefore obtain the remaining components of the solution,

$$\begin{aligned}\eta_0 = \bar{a}_{12}[sy_1'(0)(b_{11}f_2 + b_{12}f_3 - \Delta^{-1}(f, y_2)) \\ + sy_2'(0)(a_{11}f_0 + a_{12}f_1)] + \bar{a}_{12}f_0 + a_{22}f_1,\end{aligned} \tag{4.38}$$

$$\begin{aligned}\eta_1 = -\bar{b}_{12}[sy_2'(1)(a_{11}f_0 + a_{12}f_1 - \Delta^{-1}(f, y_1)) \\ + sy_1'(1)(b_{11}f_2 + b_{12}f_3)] + \bar{b}_{12}f_2 + b_{22}f_3.\end{aligned} \tag{4.39}$$

Using the boundary conditions for y_1, y_2, we find

$$\begin{aligned}\Delta &= \begin{vmatrix} y_1(0) & y_2(0) \\ sy_1'(0) & sy_2'(0) \end{vmatrix} \\ &= \begin{vmatrix} y_1(0) - a_{11}sy_1'(0) & y_2(0) - a_{11}sy_2'(0) \\ sy_1'(0) & sy_2'(0) \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 \\ sy_1'(0) & sy_2'(0) \end{vmatrix} \\ &= -sy_1'(0),\end{aligned} \tag{4.40}$$

$$\begin{aligned}\Delta &= \begin{vmatrix} y_1(1) & y_2(1) \\ sy_1'(1) & sy_2'(1) \end{vmatrix} \\ &= \begin{vmatrix} y_1(1) + b_{11}sy_1'(1) & y_2(1) + b_{11}sy_2'(1) \\ sy_1'(1) & sy_2'(1) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ sy_1'(1) & sy_2'(1) \end{vmatrix} \\ &= sy_2'(1),\end{aligned} \tag{4.41}$$

and, as a consequence,

$$y_1(0) = -a_{11}\Delta, y_2(1) = -b_{11}\Delta. \tag{4.42}$$

The formula (4.25) with the blocks given by (4.26)–(4.29) now follows from algebraic manipulations. □

REMARK 4.8. The block operator matrix $\mathcal{K}$ is symmetric, but this is not immediately clear from the expressions we have given for its elements. We shall indicate how this symmetry can be checked by showing that, for example,

$$K_{32}^* = K_{23}.$$

Checking the $(1,1)$ terms of this matrix equation, we must prove that

$$a_{11}y_2(1) = b_{11}y_1(0).$$

This is immediate from (4.42). The $(2,2)$ terms give

$$-sy_2'(1) = sy_1'(0),$$

which follows from (4.40) and (4.41). The $(1,2)$ term requires

$$y_1(0) = -a_{11}sy_2'(1).$$

This follows from (4.41) and (4.42). Finally, the $(2,1)$ term requires

$$y_2(1) = b_{11} s y_1'(0),$$

which follows from (4.40) and (4.42).

Remark 4.9. One can guarantee the existence of K^{-1} (and hence of $\mathcal{K}_F^{-1}$) if $s > 0$, $w < 0$ and A and B are positive matrices. In this case, it is not difficult to show that $(Ky, y)_{L_2} > 0$ for non-zero y. See also remark 4.2 above.

Our next objective is to find the operator

$$\mathcal{T} = (\mathcal{M}_F + \mathcal{N})^{-1}\mathcal{L} = \mathcal{K}_F^{-1}\mathcal{L}.$$

First we must determine the domain $D(\mathcal{T})$ given by (4.22).

Proposition 4.10. *The adjoint of the operator $\mathcal{K} = \mathcal{M} + \mathcal{N}$ is given by the formulae*

$$D(\mathcal{K}^*) = \left\{ \hat{f} = \left\{ f, \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \right\} : f \in W_2^2,\ f_j \in \mathbb{C},\ j = 0,1,2,3 \right\}, \tag{4.43}$$

$$\mathcal{K}^* \hat{f} = \left\{ (-DsD - w)f, \begin{pmatrix} -sf'(0) \\ s(0)(f(0) - f_0) \end{pmatrix} + A \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} sf'(1) \\ s(1)(f_2 - f(1)) \end{pmatrix} + B \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \right\}. \tag{4.44}$$

Proof. The operator $\bar{\mathcal{K}}$ (the closure of $\mathcal{K}$) has the domain

$$D(\bar{\mathcal{K}}) = \left\{ \hat{f} = \left\{ f, \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}, \begin{pmatrix} f(1) \\ f'(1) \end{pmatrix} \right\} : f \in W_2^2 \right\}.$$

Also, $\bar{\mathcal{K}}$ has deficiency indices $(2,2)$, because $\dim(D(\mathcal{K}_F)/D(\bar{\mathcal{K}})) = 2$ and $\mathcal{K}_F$ is self-adjoint. Let $\hat{y} \in D(\mathcal{L})$ be of the form

$$\hat{y} = \left\{ y, \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}, \begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} \right\}$$

and let

$$\hat{f} = \left\{ f, \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \right\}.$$

Then integration by parts shows that

$$(\mathcal{K}\hat{y}, \hat{f}) = (\hat{y}, \hat{g}),$$

in which $\hat{g} \in \mathfrak{H}$ is given by

$$\hat{g} = \left\{ Kf, A \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} + \begin{pmatrix} -sf'(0) \\ s(0)(f(0) - f_0) \end{pmatrix}, B \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} + \begin{pmatrix} sf'(1) \\ s(1)(f_2 - sf_2) \end{pmatrix} \right\}.$$

Thus, for the operator $\mathcal{K}^*$ defined by (4.44), we have the identity

$$(\mathcal{K}\hat{y}, \hat{f}) = (\hat{y}, \mathcal{K}^* \hat{f}), \quad \hat{y} \in D(\mathcal{L}), \quad \hat{f} \in D(\mathcal{K}^*),$$

and $\dim(D(\mathcal{K}^*)/D(\bar{\mathcal{K}})) = 4$. The result then follows from von Neumann's formula for the domain $D(\mathcal{K}^*)$ (see, for example, [11]). $\square$

PROPOSITION 4.11. *Let y_1 and y_2 be the solutions of the differential equation (4.30) defined in proposition 4.7 above. Then* $\dim \operatorname{Ker}(\mathcal{K}^*) = 2$. *Moreover,* $\operatorname{Ker}(\mathcal{K}^*) = \operatorname{span}\{\tilde{y}_1, \tilde{y}_2\}$, *where*

$$\tilde{y}_1 := \begin{cases} \left\{y_1, \begin{pmatrix} y_1(0) \\ \bar{a}_{12} s y_1'(0) \end{pmatrix}, \begin{pmatrix} y_1(1) - (1 + b_{12}s(1))^{-1} \\ b_{22}s(1)(1 + b_{12}s(1))^{-1} - \bar{b}_{12} s y_1'(1) \end{pmatrix}\right\} \\ & \textit{if } b_{12}s(1) \neq -1, \\ \left\{0, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -b_{22}s(1) \end{pmatrix}\right\} & \textit{if } b_{12}s(1) = -1, \end{cases}$$

$$\tilde{y}_2 := \begin{cases} \left\{y_2, \begin{pmatrix} y_2(0) - (1 - a_{12}s(0))^{-1} \\ -a_{22}s(0)(1 - a_{12}s(0))^{-1} + \bar{a}_{12} s y_2'(0) \end{pmatrix}, \begin{pmatrix} y_2(1) \\ -\bar{b}_{12} s y_2'(1) \end{pmatrix}\right\} \\ & \textit{if } a_{12}s(0) \neq 1, \\ \left\{0, \begin{pmatrix} 1 \\ a_{22}s(0) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} & \textit{if } a_{12}s(0) = 1. \end{cases}$$

Proof. Note that $\dim \operatorname{Ker}(\mathcal{K}^*) \leqslant 2$, since $\operatorname{Ker}(\mathcal{K}_F) = \{0\}$ and the deficiency indices of $\mathcal{K}^*$ are $(2, 2)$. It now follows by direct substitution of the expressions for $\tilde{y}_1$ and $\tilde{y}_2$ into (4.44) that $\tilde{y}_1, \tilde{y}_2 \in \operatorname{Ker}(\mathcal{K}^*)$. It is also clear that $\tilde{y}_1$ and $\tilde{y}_2$ are linearly independent. $\square$

PROPOSITION 4.12. *The boundary conditions*

$$(\mathcal{L}\hat{y}, \tilde{y}_j) = 0, \quad j = 1, 2,$$

given in (4.22) have the explicit form

$$\begin{aligned} U_j(y) = {} & u_{j,0}((py'')' - qy')(0) + u_{j,1}((py'')' - qy')(1) \\ & + u_{j,2}py''(0) + u_{j,3}py''(1) + u_{j,4}y'(0) + u_{j,5}y'(1) \\ & + u_{j,6}y(0) + u_{j,7}y(1) + \int_0^1 y(D^2pD^2 - DqD + r)y_j \, dx, \quad j = 1, 2, \end{aligned} \tag{4.45}$$

where, for $j = 1, 2$,

$$\left.\begin{aligned} u_{j,4} &= -py_j''(0), \\ u_{j,5} &= py_j''(1), \\ u_{j,6} &= ((py_j'')' - qy_j')(0), \\ u_{j,7} &= -((py_j'')' - qy_j')(1), \end{aligned}\right\} \tag{4.46}$$

and the remaining coefficients $u_{j,k}$ are as follows. If $b_{12}s(1) \neq -1$, then

$$\begin{aligned} u_{1,0} &= 0, \\ u_{1,1} &= (1 + b_{12}s(1))^{-1}, \\ u_{1,2} &= (1 - \bar{a}_{12}s(0))y_1'(0), \\ u_{1,3} &= b_{22}s(1)(1 + b_{12}s(1))^{-1} - (1 + \bar{b}_{12}s(1))y_1'(1); \end{aligned}$$

otherwise,

$$\begin{aligned} u_{1,0} &= -y_1(0), \\ u_{1,1} &= y_1(1) - 1, \\ u_{1,2} &= y_1'(0), \\ u_{1,3} &= -y_1'(1) - b_{22}s(1). \end{aligned}$$

If $a_{12}s(0) \neq 1$, *then*

$$\begin{aligned} u_{2,0} &= (a_{12}s(0) - 1)^{-1}, \\ u_{2,1} &= 0, \\ u_{2,2} &= a_{22}s(0)(1 - a_{12}s(0))^{-1} + (1 - \bar{a}_{12}s(0))y_2'(0), \\ u_{2,3} &= -(1 + \bar{b}_{12}s(1))y_2'(1); \end{aligned}$$

otherwise,

$$\begin{aligned} u_{2,0} &= 1 - y_2(0), \\ u_{2,1} &= y_2(1), \\ u_{2,2} &= y_2'(0) - a_{22}s(0), \\ u_{2,3} &= -y_2'(1). \end{aligned}$$

Proof. Corresponding to the functions y_j, $j = 1, 2$, define

$$\hat{y}_j = \left\{ y_j, \begin{pmatrix} y_j(0) \\ y_j'(0) \end{pmatrix}, \begin{pmatrix} y_j(1) \\ y_j'(1) \end{pmatrix} \right\}, \quad j = 1, 2,$$

and observe that

$$(\mathcal{L}\hat{y}, \tilde{y}_j) = (\mathcal{L}\hat{y}, \hat{y}_j) + (\mathcal{L}\hat{y}, \tilde{y}_j - \hat{y}_j) = (\hat{y}, \mathcal{L}\hat{y}_j) + (\mathcal{L}\hat{y}, \tilde{y}_j - \hat{y}_j), \quad j = 1, 2.$$

The term $(\hat{y}, \mathcal{L}\hat{y}_j)$ is easily calculated from the expression for $\mathcal{L}$ to be equal to

$$\int_0^1 y(D^2pD^2 - DqD + r)y_j \, \mathrm{d}x \\ + \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} \cdot \begin{pmatrix} ((py_j'')' - qy_j')(0) \\ -py_j''(0) \end{pmatrix} + \begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} \cdot \begin{pmatrix} (-(py_j'')' + qy_j')(1) \\ py_j''(1) \end{pmatrix},$$

which yields the integral term in (4.45) and the coefficients $u_{j,4}$, $u_{j,5}$ and $u_{j,6}$ in (4.46). The remaining coefficients come from the terms $(\mathcal{L}\hat{y}, \tilde{y}_j - \hat{y}_j)$; these do not involve integrals as the first component of $\tilde{y}_j - \hat{y}_j$ is zero, and can be calculated using the expressions in proposition 4.11. □

PROPOSITION 4.13. *If the coefficients* p, q *and* r *are sufficiently smooth, then*

$$D^2pD^2 - DqD + r = (-DsD - w)(-Dz_0D + z_1D + z_2) - w_1D - w_2,$$

where

$$\begin{aligned}
z_0 &= \frac{p}{s},\\
z_1 &= z_0' - \frac{s'}{s} z_0 = -\frac{p'}{s} - 2p\left(\frac{1}{s}\right)',\\
z_2 &= \frac{q + w z_0 - s z_1' + s z_0' + (s z_0')'}{s},\\
w_1 &= -s z_2' - w' z_0 - w z_1,\\
w_2 &= -s z_2'' - s' z_2' - w z_2 - r.
\end{aligned}$$

Proof. The proof is by direct verification. □

THEOREM 4.14. *Let y_1 and y_2 be the solutions of the differential equation (4.30) defined in proposition 4.7 and let a_{ij} and b_{ij} be the entries of the matrices A^{-1} and B^{-1}, respectively. Define the operator T acting in the space W_2^2 by*

$$\begin{aligned}
Ty(x) = (-D z_0 D + z_1 D + z_2) y(x) &+ K^{-1}(w_1 D + w_2) y(x)\\
&+ y_1(x)[b_{12} p y'' + b_{11}(w z_0 + s z_0' + s' z_0) y'\\
&\qquad - b_{11} s z_2' y - (z_0 y')' + z_1 y' + z_2 y](1)\\
&- y_2(x)[a_{12} p y'' + a_{11}(w z_0 + s z_0' + s' z_0) y'\\
&\qquad - a_{11} s z_2' y - (z_0 y')' + z_1 y' + z_2 y](0),
\end{aligned} \tag{4.47}$$

where the functions z_j and w_j are defined in proposition 4.13, $K^{-1}(w_1 D + w_2) y$ is the integral term

$$K^{-1}(w_1 D + w_2) y(x) = \int_0^1 G(x,t)[w_1(t) y'(t) + w_2(t) y(t)]\,\mathrm{d}t,$$

with $G(x,t)$ given by (4.35) and

$$D(T) = \{y \in W_2^4 : U_j(y) = 0,\ j = 1,2\},$$

where the linear forms U_j are defined as in proposition 4.12. Then (W_2^2, T) is a linearization pair of the problem (4.1), (4.2).

Proof. Recalling the definition of the operator $\mathcal{L}$ and the representation of the operator $\mathcal{K}_F^{-1}$ given in proposition 4.7, we obtain

$$\mathcal{T}\hat{y} = \mathcal{K}_F^{-1}\mathcal{L}\hat{y} = \begin{pmatrix} Ty \\ \star \\ \star \end{pmatrix},$$

where

$$\begin{aligned}
Ty(x) = (K^{-1}(D^2 p D^2 - D q D + r) y)(x)&\\
+ \{a_{11}((D p D^2 - q D) y)(0) - a_{12}(p D^2 y)(0)\} y_2(x)&\\
+ \{b_{11}((-D p D^2 + q D) y)(1) + b_{12}(p D^2 y)(1)\} y_1(x).&
\end{aligned} \tag{4.48}$$

From proposition 4.13, we have

$$\begin{aligned} Sy &:= K^{-1}(D^2pD^2 - DqD + r)y \\ &= (-Dz_0D + z_1D + z_2)y - K^{-1}(w_1D + w_2)y + c_1y_1 + c_2y_2, \end{aligned}$$

where the constants c_1 and c_2 may be found from the conditions

$$V_j(Sy) = 0, \quad j = 0, 1,$$

the V_j being as in (4.23). These conditions yield

$$\begin{aligned} c_1 &= [a_{11}s(-D^2z_0D + Dz_1D + Dz_2)y - (-Dz_0D + z_1D + z_2)y](0), \\ c_2 &= [b_{11}s(D^2z_0D - Dz_1D - Dz_2)y + (-Dz_0D + z_1D + z_2)y](1). \end{aligned}$$

Combining c_1y_1 and c_2y_2 with the multiples of y_1 and y_2 already appearing on the right-hand side of (4.48), and using the identities in proposition 4.13, a substantial amount of cancellation takes place; in particular, all third-order derivatives cancel out, leaving (4.47). The description of the domain $D(T)$ follows from the description of the domain $D(\mathcal{T})$ in proposition 4.12. This completes the proof. □

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