Pergamon

Int. J. Solids Structures Vol. 35, Nos. 5–6, pp. 527–541, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00028–0

ASYMPTOTIC HOMOGENIZATION OF LAMINATED PIEZOCOMPOSITE MATERIALS

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(*Received* 20 *June* 1995; *in revised form* 12 *December* 1996)

Abstract—The objective of this paper is to apply the technique of asymptotic homogenization to determine the effective elastic, piezoelectric and dielectric moduli of a laminated piezocomposite medium with a periodic structure. Each periodic cell of the medium can possess any finite number of piezoelectric layers. The general formulae obtained are a generalization of those that appear in chapter 5 of Pobedria (Pobedria, B. E. (1984) *Mechanics of Composite Materials.* Moscow State University Press, Moscow (in Russian)) and involve both cases of Newnham's connectivity theory (Newnham, R. E., Skinner, D. P. and Cross, L. E. (1978) Connectivity and piezoelectric–pyroelectric composites. *Materials Research Bulletin* **13,** 525–536) for layered piezoelectric media. We calculate explicitly overall effective characteristics for three examples of such layered media. For the particular case of a binary layered medium, connected in parallel, with transversely isotropic constituents such formulae transform exactly to the formulae for effective constants obtained by Benveniste *et al.* (1992) in which a different method of homogenization was used. Finally, we apply these results to a piezocomposite material and obtain new piezoelectric with better global properties for hydrophone applications. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

In recent years, new piezoelectric composite materials have been intensively developed to meet the requirements of electro- and hydroacoustics for more effective transducer materials. By varying the proportions of several homogeneous constituents one can obtain effective macroscopic materials with the desired homogenized properties; e.g., a piezoelectric inactive polymer matrix—piezoelectric ceramic fibres, as in Gururaja *et al.* (1985), Chan and Unsworth (1989), Smith and Auld (1991), Smith (1993) and more recently, piezoelectric ceramic–piezoelectric polymer composites in Taunaumang *et al.* (1994).

Different techniques have been adopted to estimate the effective electroelastic properties of layered piezoelectric composites but to our knowledge no explicit solutions for all effective coefficients using asymptotic homogenization have been published. For instance, Grekov *et al.* (1987) derived the effective moduli of a binary layered medium with the individual layers possessing the 4 mm symmetry by using the hypothesis of equivalent homogeneity. Benveniste and Dvorak (1992) derived the effective behaviour of a layered medium using the theory of the creation of uniform fields in heterogeneous media by proper boundary conditions. Recently exact formulae for the layered composites and approximate formulae for some more complex geometries are given in Galka *et al.* (1996).

During the last few years, various mathematically rigorous techniques have been developed to derive the homogenized electroelastic coefficients of piezocomposite materials with periodic structure. The variational method called Γ-convergence was used to obtain the effective moduli of a piezoelectric composite with finite periodic structure, cf. Telega (1991). In Turbe and Maugin (1991) these results were extended to investigate the dynamical behavior, and the method of Bloch expansions was used. The Two-Scale Asymptotic Expansion Method developed in Bensoussan *et al.* (1978) and in Sanchez-Palencia (1980) was applied by Galka *et al.* (1992) to compute macrobehavior in thermopiezoelectric solids. In the present paper these formulae are specified for the effective moduli of laminated piezocomposite media with periodic structure.

First, in Section 2 we recall the fundamental relations of linear piezoelectric theory and the boundary value problem associated to the displacement field **u** and to the electrical potential φ. In Section 3 we seek the solution of a statical piezoelectric problem for a heterogeneous periodic medium in the form of a two-scale asymptotic expansion. Afterwards we obtain a sequence of recurrent boundary value problems with constant coefficients. In Section 4 we obtain the so called "local functions" for a general piezoelectric composite. In Section 5 the local problems are used to determine the local functions of the first order and the effective coefficients for a laminated piezoelectric medium. We determine, in Section 6, the effective coefficients for a laminated piezocomposite for which the periodic cell is composed of piezoelectric materials' layers with cubic symmetry. Moreover, overall effective characteristics are also calculated for laminated piezoelectric composites with hexagonal symmetry. In Section 7, we consider a two phase laminated piezocomposite and we show its importance in designing a typical underwater transducer (hydrophone).

2. BASIC EQUATIONS

Let $\Omega \subset R^3$ be a bounded three dimensional domain with the boundary $\Gamma = \partial\Omega$. All subscripts appearing in the text take values 1, 2 and 3. The summation convention is consequently used throughout this paper and a comma denotes partial differentiation.

The strain tensor $\underline{\varepsilon}$ and the electric field vector **E** are expressed as:

$$\varepsilon_{ij} = 1/2(u_{i,j}+u_{j,i}), \quad E_i = -\varphi_{,i}, \tag{1}$$

where **u** and φ are the displacement vector and the electric potential gradient, respectively. The stress tensor $\underline{\sigma}$ and the electric displacement vector **D** are related to the strains and electric potential gradient by the constitutive relations

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - e_{lij}\varphi_{,l}, \\ D_i &= e_{ikl}\varepsilon_{kl} + \varepsilon_{il}\varphi_{,l}, \end{aligned} \tag{2}$$

where $\underline{C}$, $\underline{e}$ and $\underline{\varepsilon}$ are the elastic (measured in a constant electric field), piezoelectric (measured at a constant strain or electric field) and the dielectric (measured at a constant strain) moduli and have the following classical properties of symmetry and positivity

$$\begin{gathered} C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \quad e_{kij} = e_{kji}, \quad \varepsilon_{ij} = \varepsilon_{ji}. \\ \exists\, \eta > 0\ \forall \varepsilon \in E_s^3\ C_{ijkl}(\mathbf{x})\varepsilon_{ij}\varepsilon_{kl} \geqslant \eta|\varepsilon|^2, \\ \exists\, \eta_1 > 0\ \forall \mathbf{a} \in R^3\ \varepsilon_{ij}(\mathbf{x})a_i a_j \geqslant \eta_1|\mathbf{a}|^2 \end{gathered}$$

for almost every $\mathbf{x} \in \Omega$. Here above E_s^3 is the space of symmetric matrices of third order.

In addition, the equations of equilibrium and Gauss' law of electrostatics in the absence of free charges can be written as

$$\sigma_{ij,j} + X_i = 0, \quad D_{i,i} = 0. \tag{3}$$

Substituting (1) and (2) into eqns (3), we obtain

$$\begin{aligned} (C_{ijkl}u_{k,l} + e_{mij}\varphi_{,m})_{,j} + X_i &= 0, \\ (e_{iml}u_{m,l} - \varepsilon_{im}\varphi_{,m})_{,i} &= 0. \end{aligned} \tag{4}$$

Equations (4) represent a system of equations for finding u_i and φ. For a complete solution, it is necessary to assign certain boundary conditions, for instance

$$u_i|_{\Gamma_0} = 0; \quad \varphi|_{\Gamma_2} = \varphi_0; \quad \sigma_{ij}n_j|_{\Gamma_1} = S_i^0; \quad D_i n_i|_{\Gamma_3} = 0 \tag{5}$$

where φ_0 and $\mathbf{S}^0$ are the electric potential on Γ_2 and the mechanical load on Γ_1. $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$, $\Gamma_0 \cap \Gamma_1 = \varnothing$, $\Gamma = \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, $\Gamma_2 \cap \Gamma_3 = \varnothing$.

Equations (4), subject to the boundary conditions (5), form a closed system of equations of the static piezoelectricity problem for an heterogeneous piezoelectric solid, see Maugin (1988).

3. HOMOGENIZATION

Let the material functions C_{ijkl}, e_{mij}, ε_{im} be Y-periodic functions. As usual, Y is the typical periodic cell, say $Y = (0, Y_1) \times (0, Y_2) \times (0, Y_3)$. We set $C_{ijkl} = C_{ijkl}(\xi)$, $e_{ijm} = e_{ijm}(\xi)$ and $\varepsilon_{im} = \varepsilon_{im}(\xi)$. Here $\xi = (\xi_1, \xi_2, \xi_3)$ is the local coordinate (or fast coordinate) and $\mathbf{x} = (x_1, x_2, x_3)$ is the global (or slow) coordinate; $\xi = x/\alpha$, and $\alpha = l/L$ is a small parameter, which represents the ratio between the characteristic length, l, of the periodic cell Y, and the characteristic length L of the whole domain.

The solution of the problem (4), (5) (with periodic functions) is sought in the form of the two-scale asymptotic expansion

$$u_i(\mathbf{x}) = u_i^0(\mathbf{x}, \xi) + \alpha u_i^1(\mathbf{x}, \xi) + \alpha^2 u_i^2(\mathbf{x}, \xi) + \cdots,$$

$$\varphi(\mathbf{x}) = \varphi^0(\mathbf{x}, \xi) + \alpha\varphi^1(\mathbf{x}, \xi) + \alpha^2\varphi^2(\mathbf{x}, \xi) + \cdots.$$

Similarly, as in the linear elasticity problem, the functions u_i^0 and φ^0 do not depend on ξ. Due to the linearity of this problem and assuming both regularity of the inclusions shapes and smoothness in variation of the coefficients, we have (like in Sanchez-Palencia (1980), Bakhvalov and Panasenko (1989), Oleinik *et al.* (1992)):

$$u_i^1(\mathbf{x}, \xi) = N_{ijp}(\xi)\frac{\partial u_j^0}{\partial x_p} + \Phi_{ip}(\xi)\frac{\partial \varphi^0}{\partial x_p},$$

$$\varphi^1(\mathbf{x}, \xi) = M_{np}(\xi)\frac{\partial u_n^0}{\partial x_p} + P_p(\xi)\frac{\partial \varphi^0}{\partial x_p}.$$

This leads to seeking the asymptotic expansion of the solution in the following form

$$u_i = \sum_{q=0}^{\infty} \alpha^q [N_{ijk_1\ldots k_q}^{(q)}(\xi) V_{j,k_1\ldots k_q}(\mathbf{x}) + \Phi_{ik_1\ldots k_q}^{(q)}(\xi) S_{,k_1\ldots k_q}(\mathbf{x})],$$

$$\varphi = \sum_{q=0}^{\infty} \alpha^q [M_{nk_1\ldots k_q}^{(q)}(\xi) V_{n,k_1\ldots k_q}(\mathbf{x}) + P_{k_1\ldots k_q}^{(q)}(\xi) S_{,k_1\ldots k_q}(\mathbf{x})]. \tag{6}$$

The functions $\underline{N}^{(q)}$, $\underline{M}^{(q)}$, $\underline{\Phi}^{(q)}$ and $\underline{P}^{(q)}$ are local auxiliary Y-periodic functions, independent of $\mathbf{x}$ and satisfying the following conditions; $N_{ij}^{(0)} = \delta_{ij}$ (the Kronecker symbol), $\underline{P}^{(0)} = 1$, $\underline{M}^{(0)} = \underline{\Phi}^{(0)} = 0$. $\underline{N}^{(q)}$, $\underline{M}^{(q)}$, $\underline{\Phi}^{(q)}$, $\underline{P}^{(q)}$ are equal to zero for $q < 0$.

Moreover, to make the local auxiliary functions unique we require:

$$\langle \underline{N}^{(q)} \rangle = 0, \quad \langle \underline{M}^{(q)} \rangle = 0, \quad \langle \underline{\Phi}^{(q)} \rangle = 0, \quad \langle \underline{P}^{(q)} \rangle = 0, \quad q > 0, \tag{7}$$

where $\langle f \rangle$ stands for $(1/|Y|)\int_Y f \, \mathrm{d}Y$. The periodic conditions are

$$[\![\underline{N}^{(q)}]\!] = 0, \quad [\![\underline{M}^{(q)}]\!] = 0, \quad [\![\underline{\Phi}^{(q)}]\!] = 0, \quad [\![\underline{P}^{(q)}]\!] = 0 \tag{8}$$

and

$$[\![C_{ijml}N^{(q)}_{mnk_1\ldots k_q,l} + e_{mij}M^{(q)}_{nk_1\ldots k_q,m} + C_{ijmk_q}N^{(q-1)}_{mnk_1\ldots k_{q-1}} + e_{k_qij}M^{(q-1)}_{nk_1\ldots k_{q-1}}]\!] = 0,$$

$$[\![e_{iml}N^{(q)}_{mnk_1\ldots k_q,l} - \varepsilon_{im}M^{(q)}_{nk_1\ldots k_q,m} + e_{imk_q}N^{(q-1)}_{mnk_1\ldots k_{q-1}} - \varepsilon_{ik_q}M^{(q-1)}_{nk_1\ldots k_{q-1}}]\!] = 0,$$

$$[\![C_{ijml}\Phi^{(q)}_{mk_1\ldots k_q,l} + e_{mij}P^{(q)}_{k_1\ldots k_q,m} + C_{ijmk_q}\Phi^{(q-1)}_{mk_1\ldots k_{q-1}} + e_{k_qij}P^{(q-1)}_{k_1\ldots k_{q-1}}]\!] = 0,$$

$$[\![\varepsilon_{in}P^{(q)}_{k_1\ldots k_q,n} - e_{iml}\Phi^{(q)}_{mk_1\ldots k_q,l} + \varepsilon_{ik_q}P^{(q-1)}_{k_1\ldots k_{q-1}} - e_{imk_q}\Phi^{(q-1)}_{mk_1\ldots k_{q-1}}]\!] = 0,$$

where $[\![u]\!]$ means the difference of values of the function "u" on opposite sides of Y. $V_n(\mathbf{x}) = \langle u_n(\mathbf{x}, \xi)\rangle$ is called the averaged displacement vector and $S(\mathbf{x}) = \langle \varphi(\mathbf{x}, \xi)\rangle$ the averaged electrical potential.

We now substitute the expansions (6) into eqns (4), (5) and collect the terms of same order α^q. After some manipulations we obtain the following boundary value problems: find V_n and S being Y-periodic such that

$$\sum_{q=0}^{\infty} \alpha^{(q)}[h^{(q)}_{ijnmk_1\ldots k_q}V_{n,mk_1\ldots k_qj}(\mathbf{x}) + r^{(q)}_{mijk_1\ldots k_q}S_{,mk_1\ldots k_qj}(\mathbf{x})] + X_i = 0,$$

$$\sum_{q=0}^{\infty} \alpha^{q}[t^{(q)}_{ijmk_1\ldots k_q}V_{j,mk_1\ldots k_qi}(\mathbf{x}) - s^{(q)}_{imk_1\ldots k_q}S_{,mk_1\ldots k_qi}(\mathbf{x})] = 0. \tag{9}$$

$$\sum_{q=0}^{\infty} \alpha^{q}[N^{(q)}_{ijk_1\ldots k_q}(\xi)V_{j,k_1\ldots k_q}(\mathbf{x}) + \Phi^{(q)}_{ik_1\ldots k_q}(\xi)S_{,k_1\ldots k_q}(\mathbf{x})]|_{\Gamma_0} = 0,$$

$$\sum_{q=0}^{\infty} \alpha^{q}[M^{(q)}_{nk_1\ldots k_q}(\xi)V_{n,k_1\ldots k_q}(\mathbf{x}) + P^{(q)}_{k_1\ldots k_q}(\xi)S_{,k_1\ldots k_q}(\mathbf{x})]|_{\Gamma_2} = \varphi^0,$$

$$\sum_{q=0}^{\infty} \alpha^{q}[h^{(q)}_{ijnmk_1\ldots k_q}V_{n,mk_1\ldots k_q}(\mathbf{x}) + r^{(q)}_{mijk_1\ldots k_q}S_{,mk_1\ldots k_q}(\mathbf{x})]n_j|_{\Gamma_1} = S^0_i,$$

$$\sum_{q=0}^{\infty} \alpha^{q}[t^{(q)}_{ijmk_1\ldots k_q}V_{j,mk_1\ldots k_q}(\mathbf{x}) - s^{(q)}_{imk_1\ldots k_q}S_{,mk_1\ldots k_q}(\mathbf{x})]n_j|_{\Gamma_3} = 0, \tag{10}$$

where $\underline{h}^{(q)}$, $\underline{t}^{(q)}$, $\underline{r}^{(q)}$ and $\underline{s}^{(q)}$ are constant tensorial functions (vanishing for $q < 0$). The expressions to determine these constants will be given in the next section.

To find the functions V_n and S, we seek for the solution of (9), (10) in the asymptotic expansions form:

$$V_n = \sum_{p=0}^{\infty} \alpha^p w_n^{\{p\}}, \quad S = \sum_{p=0}^{\infty} \alpha^p y^{\{p\}}. \tag{11}$$

Putting (11) in (9), (10) we obtain the following sequence of recurrent periodic boundary value problems with constant coefficients: find $w_n^{\{p\}}$, $y^{\{p\}}$ being Y-periodic such that

$$h^{(0)}_{ijmn}w^{\{p\}}_{n,mj} + r^{(0)}_{mij}y^{\{p\}}_{,mj} + X^{\{p\}}_i = 0,$$

$$t^{(0)}_{iml}w^{\{p\}}_{m,li} - s^{(0)}_{im}y^{\{p\}}_{,mi} + Y^{\{p\}} = 0, \tag{12}$$

$$w^{\{p\}}_i|_{\Gamma_0} = u^{0\{p\}}_i, \quad y^{\{p\}}|_{\Gamma_2} = \varphi^{\{p\}},$$

$$(h^{(0)}_{ijnm}w^{\{p\}}_{n,m}+r^{(0)}_{mij}y^{\{p\}}_{,m})n_j|_{\Gamma_1} = S^{0\{p\}}_i,$$

$$(t^{(0)}_{iml}w^{\{p\}}_{m,l}-s^{(0)}_{im}y^{\{p\}}_{,m})n_i|_{\Gamma_3} = q^{0\{p\}},$$

$$p = 0,1,2,\ldots \tag{13}$$

where

$$X^{\{p\}}_i = \sum_{q=1}^{p} [h^{(q)}_{ijmnk_1\ldots k_q}w^{\{p-q\}}_{m,nk_1\ldots k_qj}+r^{(q)}_{mijk_1\ldots k_q}y^{\{p-q\}}_{,mk_1\ldots k_qj}], \quad p>0$$

$$Y^{\{p\}} = \sum_{q=1}^{p} [t^{(q)}_{ijmk_1\ldots k_q}w^{\{p-q\}}_{j,mk_1\ldots k_qi}-s^{(q)}_{imk_1\ldots k_q}y^{\{p-q\}}_{,mk_1\ldots k_qi}], \quad p>0$$

$$u^{0\{p\}}_i = -\sum_{q=1}^{p} [N^{(q)}_{ijk_1\ldots k_q}w^{\{p-q\}}_{j,k_1\ldots k_q}+\Phi^{(q)}_{ik_1\ldots k_q}y^{\{p-q\}}_{,k_1\ldots k_q}]|_{\Gamma_0}, \quad p>0$$

$$\varphi^{0\{p\}} = -\sum_{q=1}^{p} [M^{(q)}_{jk_1\ldots k_q}w^{\{p-q\}}_{j,k_1\ldots k_q}+P^{(q)}_{k_1\ldots k_q}y^{\{p-q\}}_{,k_1\ldots k_q}]|_{\Gamma_2}, \quad p>0$$

$$S^{0\{p\}}_i = -\sum_{q=1}^{p} [h^{(q)}_{ijnmk_1\ldots k_q}w^{\{p-q\}}_{n,mk_1\ldots k_q}+r^{(q)}_{mijk_1\ldots k_q}y^{\{p-q\}}_{,mk_1\ldots k_q}]n_j|_{\Gamma_1}, \quad p>0$$

$$q^{0\{p\}} = -\sum_{q=1}^{p} [t^{(q)}_{ijmk_1\ldots k_q}w^{\{p-q\}}_{j,mk_1\ldots k_q}-s^{(q)}_{ijk_1\ldots k_q}y^{\{p-q\}}_{,jk_1\ldots k_q}]n_i|_{\Gamma_3}, \quad p>0$$

$$X^{\{0\}}_i \equiv X_i, \quad Y^{\{0\}} \equiv 0, \quad u^{0\{0\}}_i \equiv 0,$$

$$\varphi^{0\{0\}} \equiv \varphi^0, \quad S^{0\{0\}}_i \equiv S^0_i, \quad q^{0\{0\}} \equiv 0.$$

Finally, with the solution of the problems (12), (13) the procedure of constructing the formal asymptotic solution of problem (4), (5) is complete and we only need finding the auxiliary local functions and the tensors of effective moduli.

4. COMPUTATION OF THE EFFECTIVE MODULI TENSORS AND OF THE LOCAL AUXILIARY FUNCTIONS

To obtain the constants $\underline{h}^{(q)}$, $\underline{t}^{(q)}$, $\underline{r}^{(q)}$ and $\underline{s}^{(q)}$ and the functions $\underline{N}^{(q+1)}$, $\underline{M}^{(q+1)}$, $\underline{\Phi}^{(q+1)}$ and $\underline{P}^{(q+1)}$ we solve the following problems with periodic boundary values:

Problems $P^{(q+1,q)}_I$: find $\underline{N}^{(q)}$, $\underline{M}^{(q)}$ being Y-periodic such that

$$\begin{aligned}
&(C_{ijml}N^{(q+2)}_{mnk_1\ldots k_{q+2},l}+e_{mij}M^{(q+2)}_{nk_1\ldots k_{q+2},m})_{,j}\\
&\quad+(C_{ijmk_{q+2}}N^{(q+1)}_{mnk_1\ldots k_{q+1}}+e_{k_{q+2}ij}M^{(q+1)}_{nk_1\ldots k_{q+1}})_{,j}\\
&\quad+C_{ik_{q+2}ml}N^{(q+1)}_{mnk_1\ldots k_{q+1},l}+e_{mik_{q+2}}M^{(q+1)}_{nk_1\ldots k_{q+1},m}\\
&\quad+C_{ik_{q+2}mk_{q+1}}N^{(q)}_{mnk_1\ldots k_q}+e_{k_{q+1}ik_{q+2}}M^{(q)}_{nk_1\ldots k_q} = h^{(q)}_{ik_{q+2}nk_1\ldots k_{q+1}},\\
&(e_{iml}N^{(q+2)}_{mnk_1\ldots k_{q+2},l}-\varepsilon_{im}M^{(q+2)}_{nk_1\ldots k_{q+2},m})_{,i}\\
&\quad+(e_{imk_{q+2}}N^{(q+1)}_{mnk_1\ldots k_{q+1}}-\varepsilon_{ik_{q+2}}M^{(q+1)}_{nk_1\ldots k_{q+1}})_{,i}\\
&\quad+e_{k_{q+2}ml}N^{(q+1)}_{mnk_1\ldots k_{q+1},l}-\varepsilon_{k_{q+2}m}M^{(q+1)}_{nk_1\ldots k_{q+1},m}\\
&\quad+e_{k_{q+2}mk_{q+1}}N^{(q)}_{mnk_1\ldots k_q}-\varepsilon_{k_{q+2}k_{q+1}}M^{(q)}_{nk_1\ldots k_q} = t^{(q)}_{k_{q+2}nk_1\ldots k_{q+1}}, \qquad q=-1,0,1,\ldots
\end{aligned} \tag{14}$$

$$h^{(q)}_{ik_{q+2}nk_1\ldots k_{q+1}} = \langle C_{ik_{q+2}ml}N^{(q+1)}_{mnk_1\ldots k_{q+1},l} + C_{ik_{q+2}mk_{q+1}}N^{(q)}_{mnk_1\ldots k_q}\rangle,$$
$$+\langle e_{mik_{q+2}}M^{(q+1)}_{nk_1\ldots k_{q+1},m} + e_{k_{q+1}ik_{q+2}}M^{(q)}_{nk_1\ldots k_q}\rangle,$$
$$t^{(q)}_{k_{q+2}nk_1\ldots k_{q+1}} = \langle e_{k_{q+2}ml}N^{(q+1)}_{mnk_1\ldots k_{q+1},l} + e_{k_{q+2}mk_{q+1}}N^{(q)}_{mnk_1\ldots k_q}\rangle$$
$$-\langle \varepsilon_{k_{q+2}l}M^{(q+1)}_{nk_1\ldots k_{q+1},l} + \varepsilon_{k_{q+2}k_{q+1}}M^{(q)}_{nk_1\ldots k_q}\rangle. \qquad q = 0,1,2,\ldots. \tag{15}$$

Problems $P^{(q+1,q)}_{II}$: find the Y-periodic functions $\underline{\Phi}^{(q)}$, $\underline{P}^{(q)}$ such that

$$(C_{ijml}\Phi^{(q+2)}_{mk_1\ldots k_{q+2},l} + e_{mij}P^{(q+2)}_{k_1\ldots k_{q+2},m})_{,j}$$
$$+(C_{ijmk_{q+2}}\Phi^{(q+1)}_{mk_1\ldots k_{q+1}} + e_{k_{q+2}ij}P^{(q+1)}_{k_1\ldots k_{q+1}})_{,j}$$
$$+C_{ik_{q+2}ml}\Phi^{(q+1)}_{mk_1\ldots k_{q+1},l} + e_{mik_{q+2}}P^{(q+1)}_{k_1\ldots k_{q+1},m}$$
$$+C_{ik_{q+2}mk_{q+1}}\Phi^{(q)}_{mk_1\ldots k_q} + e_{k_{q+1}ik_{q+2}}P^{(q)}_{k_1\ldots k_q} = r^{(q)}_{k_{q+1}ik_{q+2}k_1\ldots k_q},$$
$$(\varepsilon_{in}P^{(q+2)}_{k_1\ldots k_{q+2},n} - e_{iml}\Phi^{(q+2)}_{mk_1\ldots k_{q+2},l})_{,i}$$
$$+(\varepsilon_{ik_{q+2}}P^{(q+1)}_{k_1\ldots k_{q+1}} - e_{imk_{q+2}}\Phi^{(q+1)}_{mk_1\ldots k_{q+1}})_{,i}$$
$$+\varepsilon_{k_{q+2}}P^{(q+1)}_{k_1\ldots k_{q+1},n} - e_{k_{q+2}ml}\Phi^{(q+1)}_{mk_1\ldots k_{q+1},l}$$
$$+\varepsilon_{k_{q+2}k_{q+1}}P^{(q)}_{k_1\ldots k_q} - e_{k_{q+2}mk_{q+1}}\Phi^{(q)}_{mk_1\ldots k_q} = s^{(q)}_{k_{q+2}k_1\ldots k_{q+1}}, \qquad q = -1,0,1,\ldots \tag{(16}$$

$$r^{(q)}_{k_{q+1}ik_{q+2}k_1\ldots k_q} = \langle C_{ik_{q+2}ml}\Phi^{(q+1)}_{mk_1\ldots k_{q+1},l} + C_{ij_{q+2}mk_{q+1}}\Phi^{(q)}_{mk_1\ldots k_q}\rangle$$
$$+\langle e_{mik_{q+2}}P^{(q+1)}_{kk_1\ldots k_{q+1},m} + e_{k_{q+1}ik_{q+2}}P^{(q)}_{k_1\ldots k_q}\rangle,$$
$$s^{(q)}_{k_{q+2}k_1\ldots k_{q+1}} = \langle \varepsilon_{k_{q+2}l}P^{(q+1)}_{k_1\ldots k_{q+1},l} + \varepsilon_{k_{q+2}k_{q+1}}P^{(q)}_{k_1\ldots k_q}\rangle$$
$$-\langle e_{k_{q+2}ml}\Phi^{(q+1)}_{mk_1\ldots k_{q+1},l} + e_{k_{q+2}mk_{q+1}}\Phi^{(q)}_{mk_1\ldots k_q}\rangle. \qquad q = 0,1,2,\ldots. \tag{17}$$

To obtain $N^{(1)}_{pmn}$, $M^{(1)}_{mn}$ and $h^{(0)}_{ijmn}$, $t^{(0)}_{imn}$ we start to solve the first problem $P^{(1,0)}_{I}$, i.e., from (14), (15) we consider:

Problems $P^{(1,0)}_{I}$: find the Y-periodic functions $\underline{N}^{(1)}$, $\underline{M}^{(1)}$ such that

$$(C_{ijmn} + C_{ijpq}N^{(1)}_{pmn,q} + e_{pij}M^{(1)}_{mn,p})_{,j} = 0, \quad (e_{imn} + e_{ipq}N^{(1)}_{pmn,q} - \varepsilon_{ip}M^{(1)}_{mn,p})_{,i} = 0. \tag{18}$$

$$h^{(0)}_{ijmn} = \langle C_{ijmn} + C_{ijpq}N^{(1)}_{pmn,q} + e_{pij}M^{(1)}_{mn,p}\rangle, \tag{19$_1$}$$

$$t^{(0)}_{imn} = \langle e_{imn} + e_{ipq}N^{(1)}_{pmn,q} - \varepsilon_{ip}M^{(1)}_{mn,p}\rangle. \tag{19$_2$}$$

Analogously, $\Phi^{(1)}_{mn}$, $P^{(1)}_{n}$ and $r^{(0)}_{nij}$, $s^{(0)}_{in}$ are computed by means of the periodic problem $P^{(1,0)}_{II}$; i.e., from (16), (17) we have:

Problems $P^{(1,0)}_{II}$: find the Y-periodic function $\underline{\Phi}^{(1)}$, $\underline{P}^{(1)}$ such that

$$(e_{nij} + C_{ijpq}\Phi^{(1)}_{pn,q} + e_{pij}P^{(1)}_{n,p})_{,j} = 0,$$
$$(\varepsilon_{in} - e_{ipq}\Phi^{(1)}_{pn,q} + \varepsilon_{ip}P^{(1)}_{n,p})_{,i} = 0. \tag{20}$$

$$r^{(0)}_{nij} = \langle e_{nij} + e_{pij}P^{(1)}_{n,p} + C_{ijpq}\Phi^{(1)}_{pn,q}\rangle,$$
$$s^{(0)}_{in} = \langle \varepsilon_{in} + \varepsilon_{ip}P^{(1)}_{n,p} - e_{ipq}\Phi^{(1)}_{pn,q}\rangle. \tag{21}$$

Equations (18), (20) provide us with the system of equations for finding $N^{(1)}_{pmn}$ and $M^{(1)}_{mn}$, $\Phi^{(1)}_{pn}$ and $P^{(1)}_{n}$, taking into account (7), (8). These problems are strong formulations of the local problems, and are solvable if the periodic solutions are smooth enough. However, this regularity may be significantly relaxed if one uses a weak or variational formulation, as in Sanchez-Palencia (1980) or Oleinik *et al.* (1992).

Indeed, in the case of a laminate composite with axis of symmetry in the direction normal to the layers, the periodic local functions $\underline{N}^{(q)}$, $\underline{M}^{(q)}$, $\underline{\Phi}^{(q)}$, $\underline{P}^{(q)}$ and the material functions C_{ijmn}, e_{imn}, ε_{in} will only depend on one variable. For this kind of media we prove $t^{(0)} = r^{(0)}$. Consequently, the first problem ($p = 0$) of the recurrent sequence of boundary value problems (12), (13) is a typical boundary value problem for linear piezoelectricity in a homogeneous medium and has the form: find $w_n^{(0)}$, $y^{(0)}$ such that

$$C^h_{ijmn} w^{(0)}_{n,mj} + e^h_{mij} y^{(0)}_{,mj} + X_i = 0, \quad e^h_{iml} w^{(0)}_{m,li} - \varepsilon^h_{im} y^{(0)}_{,mi} = 0, \tag{22}$$

$$w_i^{(0)}|_{\Gamma_0} = 0, \quad y^{(0)}|_{\Gamma_2} = \varphi^{(0)}, \quad \sigma^h_{ij} n_j|_{\Gamma_1} = S_i^0, \quad D_i^h n_i|_{\Gamma_3} = 0 \tag{23}$$

where

$$\sigma^h_{ij} = C^h_{ijnm} w^{(0)}_{n,m}(\mathbf{x}) + e^h_{mij} y^{(0)}_{,m}(\mathbf{x}),$$

$$D_i^h = e^h_{ijm} w^{(0)}_{j,m}(\mathbf{x}) - \varepsilon^h_{im} y^{(0)}_{,m}(\mathbf{x})$$

$$C^h_{ijmn} = h^{(0)}_{ijmn}, \quad e^h_{mij} = t^{(0)}_{mij} = r^{(0)}_{mij}, \quad \varepsilon^h_{im} = s^{(0)}_{im},$$

and the effective moduli: C^h_{ijmn} (elastic), e^h_{mij} (piezoelectric) and ε^h_{im} (dielectric) are implicitly given by (19) and (21).

The procedure of constructing the formal asymptotic solution of linear static piezoelectric equations with typical boundary conditions for a heterogeneous and periodic medium is developed by means of Asymptotic Two-Scale Expansion. The original boundary value problem with variable coefficients is transformed into a recurrent sequence of boundary value problems with constant coefficients. Actually, this asymptotic analysis leads to the solution of two recurrent sequence of problems. The first family of these problems (problem $B(p), p = 0, 1, \ldots$) consists of the boundary value problems (12), (13). For solving the problem $B(p)$ it is necessary to solve the problems $B(r)$, $r = 0, 1, \ldots, p-1$. The solution to each of these problems determines the functions $w_n^{\{p\}}$ and $y^{\{p\}}$. Then by (11) it is possible to determine the average functions V_n and S. The solution to the original problem (4), (5) is finally obtained from V_n and S by (6). The local auxiliary periodic functions $\underline{N}^{(q)}$, $\underline{M}^{(q)}$, $\underline{\Phi}^{(q)}$, $\underline{P}^{(q)}$ are included in these formulae. Having found them we proceed to solving the second recurrent sequence of problems which is made up of $P_I^{(q+1,q)}$ (eqns (14), (15)) and $P_{II}^{(q+1,q)}$ (eqns (16), (17)). For a fixed value of q, eqns (14), (16) represent respectively, a system for finding $\underline{N}^{(q+1)}$ and $\underline{M}^{(q+1)}$, $\underline{\Phi}^{(q+1)}$ and $\underline{P}^{(q+1)}$, taking into account (7), (8). After that, by using (15), (17) the effective moduli tensor $\underline{h}^{(q)}$, $\underline{t}^{(q)}$, $\underline{r}^{(q)}$ and $\underline{s}^{(q)}$ are obtained.

5. LOCAL PROBLEMS FOR A LAMINATED MEDIUM

Let us now confine our study to a laminated piezoelectric composite, i.e., made of cells which are periodically along the axis x_3. Each cell may be made of any finite number of piezoelectric layers. The axes of symmetry of each layer are parallel to each other and the x_3-axis is perpendicular to layering. For our problem the elasticity modulus tensor $\underline{C}$, the piezoelectric modulus tensor $\underline{e}$, and the dielectric modulus tensor $\underline{\varepsilon}$ are periodic functions of the coordinate x_3 and they do not depend on x_1 and x_2.

We then introduce the fast variable in the following form,

$$\xi \equiv \xi_3 = \frac{x_3}{\alpha} \tag{24}$$

where α is the small parameter, representing the ratio between the characteristic length of the periodic cell 1 and the characteristic length of the body L. In the engineering literature these kinds of layer's distribution are known as "connectivity in series" and the case corresponding to $\xi \equiv \xi_\beta = x_\beta/\alpha$ where $\beta = 1$ or 2 is called "connectivity in parallel", see, for instance, Newnham *et al.* (1978).

Using the results obtained in the previous sections we seek the solution of this heterogeneous problem in the following form

$$u_i = \sum_{q=0}^{\infty} \alpha^q \sum_{p=0}^{q} [N^{(p)}_{ijk_1\ldots k_p}(\xi) w^{\{q-p\}}_{j,k_1\ldots k_p}(\mathbf{x}) + \Phi^{(p)}_{ik_1\ldots k_p}(\xi) y^{\{q-p\}}_{,k_1\ldots k_p}(\mathbf{x})],$$

$$\varphi = \sum_{q=0}^{\infty} \alpha^q \sum_{p=0}^{q} [M^{(p)}_{mk_1\ldots k_p}(\xi) w^{\{q-p\}}_{m,k_1\ldots k_q}(\mathbf{x}) + P^{(p)}_{k_1\ldots k_p}(\xi) y^{\{q-p\}}_{,k_1\ldots k_p}(\mathbf{x})]. \tag{25}$$

We require the periodic local auxiliary functions to verify:

$$\langle \underline{N}^{(q)} \rangle = 0, \quad \langle \underline{\underline{M}}^{(q)} \rangle = 0, \quad \langle \underline{\Phi}^{(q)} \rangle = 0, \quad \langle \underline{P}^{(q)} \rangle = 0, \quad q > 0 \tag{26}$$

where $\langle f \rangle = \int_0^1 f(\xi)\,d\xi$. Moreover, $\underline{N}^{(q)}(\xi)$, $\underline{\underline{M}}^{(q)}(\xi)$, $\underline{\Phi}^{(q)}(\xi)$ and $\underline{P}^{(q)}(\xi)$ are 1-periodic in ξ satisfying:

$$\underline{N}^{(q)}(0) = \underline{N}^{(q)}(1), \quad \underline{\underline{M}}^{(q)}(0) = \underline{\underline{M}}^{(q)}(1), \quad \underline{P}^{(q)}(0) = \underline{P}^{(q)}(1), \quad \underline{\Phi}^{(q)}(0) = \underline{\Phi}^{(q)}(1) \tag{27}$$

and

$$[\![C_{i3m3}(N^{(q)}_{mnk_1\ldots k_q})' + e_{3i3}(M^{(q)}_{nk_1\ldots k_q})' + C_{i3mk_q} N^{(q-1)}_{mnk_1\ldots k_{q-1}} + e_{k_q i3} M^{(q-1)}_{nk_1\ldots k_{q-1}}]\!] = 0,$$

$$[\![e_{3m3}(N^{(q)}_{mnk_1\ldots k_q})' - \varepsilon_{3m}(M^{(q)}_{nk_1\ldots k_q})' + e_{3mk_q} N^{(q-1)}_{mnk_1\ldots k_{q-1}} - \varepsilon_{3k_q} M^{(q-1)}_{nk_1\ldots k_{q-1}}]\!] = 0,$$

$$[\![C_{i3m3}(\Phi^{(q)}_{mk_1\ldots k_q})' + e_{3i3}(P^{(q)}_{k_1\ldots k_q})' + C_{i3mk_q} \Phi^{(q-1)}_{mk_1\ldots k_{q-1}} + e_{k_q i3} P^{(q-1)}_{k_1\ldots k_{q-1}}]\!] = 0,$$

$$[\![C_{i3m3}(\Phi^{(q)}_{mk_1\ldots k_q})' + e_{3i3}(P^{(q)}_{k_1\ldots k_q})' + C_{i3mk_q} \Phi^{(q-1)}_{mk_1\ldots k_{q-1}} + e_{k_1 i3} P^{(q-1)}_{k_1\ldots k_{q-1}}]\!] = 0,$$

where $(\cdot)'$ means $\mathrm{d}(\cdot)/\mathrm{d}\xi$ and $[\![u]\!] = 0$ means $u(0) = u(1)$.

In order to obtain the corresponding functions $N^{(1)}_{pmn}$, $M^{(1)}_{mn}$ and $h^{(0)}_{ijmn}$, $t^{(0)}_{imn}$ we have to solve the system of problems $P^{(1,0)}_I$.

Problems $P^{(1,0)}_I$: find $\underline{N}^{(1)}$, $\underline{\underline{M}}^{(1)}$ being 1-periodic such that

$$(C_{i3mn} + C_{i3p3}(N^{(1)}_{pmn})' + e_{3i3}(M^{(1)}_{mn})')' = 0,$$

$$(e_{3mn} + e_{3p3}(N^{(1)}_{pmn})' - \varepsilon_{33}(M^{(1)}_{mn})')' = 0. \tag{28}$$

$$C^h_{ijmn} = \langle C_{ijmn} + C_{ijp3}(N^{(1)}_{pmn})' + e_{3ij}(M^{(1)}_{mn})' \rangle,$$

$$e^h_{imn} = \langle e_{imn} + e_{ip3}(N^{(1)}_{pmn})' - \varepsilon_{i3}(M_{mn})' \rangle. \tag{29}$$

Analogously $\Phi^{(1)}_{mn}$, $P^{(1)}_n$ and $s^{(0)}_{in}$ are obtained from the problems $P^{(1,0)}_{II}$.

Problems $P^{(1,0)}_{II}$: find $\underline{\Phi}^{(1)}$, $\underline{P}^{(1)}$ being 1-periodic such that

$$(e_{ni3} + C_{i3p3}(\Phi^{(1)}_{pn})' + e_{3i3}(P^{(1)}_n)')' = 0,$$

$$(\varepsilon_{3n} - e_{3p3}(\Phi^{(1)}_{pn})' + \varepsilon_{33}(P^{(1)}_n)')' = 0. \tag{30}$$

$$\varepsilon^h_{in} = \langle \varepsilon_{in} + \varepsilon_{i3}(P^{(1)}_n)' - e_{ip3}(\Phi^{(1)}_{pn})' \rangle, \tag{31}$$

where, $C^h_{ijmn} = h^{(0)}_{ijmn}$ (elastic), $e^h_{mij} = t^{(0)}_{mij} = r^{(0)}_{mij}$ (piezoelectric), $\varepsilon^h_{im} = s^{(0)}_{im}$ (dielectric) are the effective moduli.

Having solved the above local problems, taking into account (26), (27), one can determine the homogenized (effective) moduli. They are given by

$$C^h_{ijnk} = \langle C_{ijnk} + C_{ijp3}\bar{C}^{-1}_{q3p3}\bar{A}_{q3nk} + e_{3ij}\bar{\varepsilon}^{-1}_{33}\bar{B}_{3nk}\rangle, \tag{32}$$

$$e^h_{ink} = \langle e_{ink} + e_{ip3}\bar{C}^{-1}_{q3p3}\bar{A}_{q3nk} - \varepsilon_{i3}\bar{\varepsilon}^{-1}_{33}\bar{B}_{3nk}\rangle, \tag{33}$$

$$\varepsilon^h_{ik} = \langle \varepsilon_{ik} - e_{im3}\bar{C}^{-1}_{p3m3}\bar{G}_{kp3} + \varepsilon_{i3}\bar{\varepsilon}^{-1}_{33}\bar{H}_{3k}\rangle, \tag{34}$$

where

$$\begin{aligned}
\bar{C}_{i3mp} &= C_{i3mp} + e_{3i3}\varepsilon^{-1}_{33}e_{3mp},\\
\bar{\varepsilon}_{pq} &= \varepsilon_{pq} + e_{pm3}C^{-1}_{m3i3}e_{qi3},\\
\bar{A}_{q3nk} &= -C_{q3nk} + A_{q3nk} + e_{3q3}\varepsilon^{-1}_{33}(B_{3nk} - e_{3nk}),\\
\bar{B}_{3nk} &= e_{3nk} + e_{3p3}C^{-1}_{p3q3}(A_{q3nk} - C_{q3nk}) - B_{3nk},\\
\bar{G}_{kp3} &= -e_{kp3} + G_{kp3} + e_{3p3}\varepsilon^{-1}_{33}(\varepsilon_{3k} - H_{3k}),\\
\bar{H}_{3k} &= \varepsilon_{i3}\bar{\varepsilon}^{-1}_{33}(e_{3m3}C^{-1}_{m3p3}(-e_{kp3} + G_{kp3}) - \varepsilon_{3k} + H_{3k}),\\
A_{i3nk} &= \{\langle\bar{C}^{-1}_{i3m3}\rangle + \langle\bar{C}^{-1}_{q3m3}e_{3q3}\varepsilon^{-1}_{33}\rangle\langle\bar{\varepsilon}^{-1}_{33}\rangle^{-1}\langle C^{-1}_{p3i3}e_{3p3}\bar{\varepsilon}^{-1}_{33}\rangle\}^{-1}\\
&\quad\times\{\langle\bar{C}^{-1}_{q3m3}e_{3q3}\varepsilon^{-1}_{33}\rangle\langle\bar{\varepsilon}^{-1}_{33}\rangle^{-1}\langle\bar{\varepsilon}^{-1}_{33}(e_{3p3}C^{-1}_{p3q3}C_{q3nk} - e_{3nk})\rangle\\
&\quad+\langle\bar{C}^{-1}_{q3m3}\bar{C}_{q3nk}\rangle\},\\
B_{3nk} &= \{\langle\bar{\varepsilon}^{-1}_{33}\rangle + \langle C^{-1}_{p3q3}e_{3p3}\bar{\varepsilon}^{-1}_{33}\rangle\langle\bar{C}^{-1}_{q3m3}\rangle^{-1}\langle\bar{C}^{-1}_{q3m3}e_{3q3}\varepsilon^{-1}_{33}\rangle\}^{-1}\\
&\quad\times\{\langle\bar{\varepsilon}^{-1}_{33}(-e_{3p3}C^{-1}_{p3q3}C_{q3nk} + e_{3nk})\rangle\\
&\quad+\langle C^{-1}_{p3q3}e_{3p3}\bar{\varepsilon}^{-1}_{33}\rangle\langle\bar{C}^{-1}_{q3m3}\rangle^{-1}\langle\bar{C}^{-1}_{q3m3}\bar{C}_{q3nk}\rangle\},\\
G_{ki3} &= \{\langle\bar{C}^{-1}_{i3m3}\rangle + \langle\bar{C}^{-1}_{q3m3}e_{3q3}\varepsilon^{-1}_{33}\rangle\langle\bar{\varepsilon}^{-1}_{33}\rangle^{-1}\langle C^{-1}_{p3i3}e_{3p3}\bar{\varepsilon}^{-1}_{33}\rangle\}^{-1}\\
&\quad\times\{\langle\bar{C}^{-1}_{q3m3}(e_{kq3} - e_{3q3}\varepsilon^{-1}_{33}\varepsilon_{3k})\rangle\\
&\quad+\langle\bar{C}^{-1}_{q3m3}e_{3q3}\varepsilon^{-1}_{33}\rangle\langle\bar{\varepsilon}^{-1}_{33}\rangle^{-1}\langle\bar{\varepsilon}^{-1}_{33}\varepsilon_{3k}\rangle\},\\
H_{3k} &= \{\langle\bar{\varepsilon}^{-1}_{33}\rangle + \langle C^{-1}_{p3q3}e_{3p3}\bar{\varepsilon}^{-1}_{33}\rangle\langle\bar{C}^{-1}_{q3m3}\rangle^{-1}\langle\bar{C}^{-1}_{q3m3}e_{3q3}\varepsilon^{-1}_{33}\rangle\}^{-1}\\
&\quad\times\{\langle\bar{\varepsilon}^{-1}_{33}\varepsilon_{3k}\rangle + \langle C^{-1}_{p3q3}e_{3p3}\bar{\varepsilon}^{-1}_{33}\rangle\langle\bar{C}^{-1}_{q3m3}\rangle^{-1}\\
&\quad\times\langle\bar{C}^{-1}_{p3m3}(-e_{kp3} + e_{3p3}\varepsilon^{-1}_{33}\varepsilon_{3k})\rangle\}.
\end{aligned}$$

One can prove that $C^h_{ijkl} = C^h_{klij} = C^h_{jikl} = C^h_{ijlk}$, $e^h_{kij} = e^h_{kji}$, $\varepsilon^h_{ij} = \varepsilon^h_{ji}$.

6. EXAMPLES AND RESULTS

6.1. *Connectivity in series*

1. Let us suppose the periodic cell is composed of piezoelectric material layers with cubic symmetry ($\bar{4}3m$), see for instance Berlincourt (1964). These materials are characterized by the following independent constants: three elastic constants ($C_{1111} = C_{2222} = C_{3333}$, $C_{2323} = C_{1313} = C_{1212}$, $C_{1122} = C_{1133} = C_{2233}$), one piezoelectric constant ($e_{123} = e_{213} = e_{312}$) and one dielectric constant ($\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}$). According to eqns (32)–(34), the following formulae for overall effective properties of this kind of composite can be obtained.

Elastic effective constants

$$
\begin{aligned}
C^h_{1111} &= C^h_{2222} = \langle C_{1111}\rangle - \langle C^{-1}_{1111}C^2_{1122}\rangle + \langle C^{-1}_{1111}\rangle^{-1}\langle C_{1122}C^{-1}_{1111}\rangle^2,\\
C^h_{1122} &= \langle C_{1122}\rangle - \langle C^2_{1122}C^{-1}_{1133}\rangle + \langle C^{-1}_{1111}\rangle^{-1}\langle C_{1122}C^{-1}_{1111}\rangle^2,\\
C^h_{1212} &= \langle C_{1212}\rangle + \langle e^2_{312}\varepsilon^{-1}_{11}\rangle - \langle \varepsilon^{-1}_{11}e_{312}\rangle^2\langle \varepsilon^{-1}_{11}\rangle^{-1},\\
C^h_{3333} &= \langle C^{-1}_{1111}\rangle^{-1},\\
C^h_{1313} &= C^h_{2323} = \langle C^{-1}_{1313}\rangle^{-1},\\
C^h_{3311} &= C^h_{3322} = \langle C^{-1}_{1111}\rangle^{-1}\langle C^{-1}_{1111}C_{1122}\rangle.
\end{aligned}
\tag{35}
$$

Piezoelectric effective constants

$$
\begin{aligned}
e^h_{123} &= e^h_{213} = \langle C^{-1}_{1313}\rangle^{-1}\langle e_{123}C^{-1}_{1313}\rangle,\\
e^h_{312} &= \langle \varepsilon^{-1}_{33}\rangle^{-1}\langle e_{123}\varepsilon^{-1}_{33}\rangle.
\end{aligned}
\tag{36}
$$

Dielectric effective constants

$$
\begin{aligned}
\varepsilon^h_{11} &= \varepsilon^h_{22} = \langle \varepsilon_{11}\rangle + \langle e_{123}C^{-1}_{1313}\rangle - \langle C^{-1}_{1313}\rangle^{-1}\langle e_{123}C^{-1}_{1313}\rangle^2,\\
\varepsilon^h_{33} &= \langle \varepsilon^{-1}_{33}\rangle^{-1}.
\end{aligned}
\tag{37}
$$

As we can note, there exist six independent elastic effective constants ($C^h_{1111} = C^h_{2222}, C^h_{1122}, C^h_{1212}, C^h_{3333}, C^h_{2323} = C^h_{1313}, C^h_{3311} = C^h_{3322}$) given by (35), two piezoelectric effective constants ($e^h_{123} = e^h_{213}, e^h_{312}$), given by (36), and two dielectric effective constants ($\varepsilon^h_{11} = \varepsilon^h_{22}, \varepsilon^h_{33}$) given by (37). It can be seen that, if we have a periodic cell composed of piezoelectric material's layers with cubic symmetry ($\bar{4}3m$), the corresponding homogenized material behaves as a piezoelectric material with tetragonal symmetry ($\bar{4}2m$).

2. Let us consider a piezoelectric laminate in which the periodic cell is composed of piezoelectric layers with hexagonal symmetry (6 mm), see for instance Berlincourt (1964). These materials are characterized by the following independent constants: five elastic constants ($C_{1111} = C_{2222}, C_{1122}, C_{1133} = C_{2233}, C_{3333}, C_{2323} = C_{1313}, C_{1212} = (C_{1111} - C_{1122})/2$), three piezoelectric constants ($e_{311} = e_{322}$, e_{333}, $e_{113} = e_{223}$), and two dielectric constants ($\varepsilon_{11} = \varepsilon_{22}, \varepsilon_{33}$).

Using the expressions (32)–(34) the effective coefficients for this material are:

Elastic effective constants

$$
\begin{aligned}
C^h_{1111} &= C^h_{2222} = \langle C_{1111}\rangle + \langle \bar{C}^{-1}_{3333}(C_{1133} + e_{311}\varepsilon^{-1}_{33}e_{333})(A_{1133} - C_{1133})\rangle\\
&\quad + \langle \bar{\varepsilon}^{-1}_{33}(C_{1133}C^{-1}_{3333}e_{333} - e_{311})(B_{311} - e_{311})\rangle,\\
C^h_{1122} &= \langle C_{1122}\rangle + \langle \bar{C}^{-1}_{3333}(C_{1133} + e_{311}\varepsilon^{-1}_{33}e_{333})(A_{1133} - C_{1133})\rangle\\
&\quad + \langle \bar{\varepsilon}^{-1}_{33}(C_{1133}C^{-1}_{3333}e_{333} - e_{311})(B_{311} - e_{311})\rangle,\\
C^h_{1133} &= \langle C_{1133}\rangle + \langle \bar{C}^{-1}_{3333}(C_{1133} + e_{311}\varepsilon^{-1}_{33}e_{333})(A_{3333} - C_{3333})\rangle\\
&\quad + \langle \bar{\varepsilon}^{-1}_{33}(C_{1133}C^{-1}_{3333}e_{333} - e_{311})(B_{333} - e_{333})\rangle,\\
C^h_{3333} &= A_{3333},\\
C^h_{1313} &= C^h_{2323} = \langle C^{-1}_{2323}\rangle^{-1},\\
C^h_{1212} &= (C^h_{1111} - C^h_{1122})/2.
\end{aligned}
\tag{38}
$$

Piezoelectric effective constants

$$e^h_{113} = e^h_{223} = \langle C^{-1}_{1313}\rangle^{-1}\langle e_{113}C^{-1}_{1313}\rangle,$$
$$e^h_{311} = e^h_{322} = B_{311},$$
$$e^h_{333} = B_{333}. \tag{39}$$

Dielectric effective constants

$$\varepsilon^h_{11} = \varepsilon^h_{22} = \langle\varepsilon_{11}\rangle + \langle e_{113}\bar{C}^{-1}_{1313}(G_{113} - e_{113})\rangle,$$
$$\varepsilon^h_{33} = H_{33}. \tag{40}$$

As we can see, there exist five independent elastic effective constants ($C^h_{1111} = C^h_{2222}, C^h_{1122}, C^h_{1133} = C^h_{2233}, C^h_{3333}, C^h_{2323} = C^h_{1313}, C^h_{1212} = (C^h_{1111} - C^h_{1122})/2$) given by (38), three piezoelectric effective constants ($e^h_{311} = e^h_{322}, e^h_{333}, e^h_{113} = e^h_{223}$) given by (39), and two dielectric effective constants ($\varepsilon^h_{11} = \varepsilon^h_{22}, \varepsilon^h_{33}$) given by (40). Therefore, we conclude that the symmetry of piezocomposite laminated materials with periodic cells in series connection is preserved in the homogenized piezoelectric medium.

6.2. *Connectivity in parallel*

1. In this example we assume that the laminated medium possesses the same periodic properties with hexagonal symmetry (6 mm) as in the previous example, but the cells distribution is periodic along the axis x_2. The axes of symmetry of each layer are parallel to each other and the x_2-axis is perpendicular to layering. Then by using the formulae (32)–(34) (interchanging in these expressions the indices 3 and 2) we obtain the following effective coefficients:

Elastic effective constants

$$C^h_{1111} = \langle C_{1111}\rangle - \langle C^2_{1122}C^{-1}_{2222}\rangle + \langle C_{1122}C^{-1}_{2222}\rangle^2\langle C^{-1}_{2222}\rangle^{-1},$$
$$C^h_{1122} = \langle C_{1122}C^{-1}_{2222}\rangle\langle C^{-1}_{2222}\rangle^{-1},$$
$$C^h_{1133} = \langle C_{1133}\rangle - \langle C_{1122}C^{-1}_{2222}C_{2233}\rangle$$
$$+ \langle C_{1122}C^{-1}_{2222}\rangle\langle C^{-1}_{2222}\rangle^{-1}\langle C^{-1}_{2222}C_{2233}\rangle,$$
$$C^h_{2222} = \langle C^{-1}_{2222}\rangle^{-1},$$
$$C^h_{2233} = \langle C^{-1}_{2222}\rangle^{-1}\langle C_{2233}C^{-1}_{2222}\rangle,$$
$$C^h_{3333} = \langle C_{3333}\rangle - \langle C^2_{2233}C^{-1}_{2222}\rangle + \langle C_{2233}C^{-1}_{2222}\rangle^2\langle C^{-1}_{2222}\rangle^{-1},$$
$$C^h_{2323} = \langle C^{-1}_{2323}\rangle^{-1} + \langle\varepsilon^{-1}_{22}e^2_{223}\rangle - \langle\varepsilon^{-1}_{22}e^2_{223}\rangle^2\langle\varepsilon^{-1}_{22}\rangle^{-1},$$
$$C^h_{1313} = \langle C_{1313}\rangle,$$
$$C^h_{1212} = \langle C^{-1}_{1212}\rangle^{-1}. \tag{41}$$

Piezoelectric effective constants

$$e^h_{113} = \langle e_{113}\rangle,$$
$$e^h_{311} = \langle e_{311}\rangle + \langle e_{322}C^{-1}_{2222}\rangle\langle C^{-1}_{2222}\rangle^{-1}\langle C^{-1}_{2222}C_{2211}\rangle - \langle e_{322}C^{-1}_{2222}C_{2211}\rangle,$$
$$e^h_{333} = \langle e_{333}\rangle + \langle e_{322}C^{-1}_{2222}\rangle\langle C^{-1}_{2222}\rangle^{-1}\langle C^{-1}_{2222}C_{2233}\rangle - \langle e_{322}C^{-1}_{2222}C_{2233}\rangle,$$
$$e^h_{322} = \langle e_{322}C^{-1}_{2222}\rangle\langle C^{-1}_{2222}\rangle^{-1},$$
$$e^h_{223} = \langle e_{223}\varepsilon^{-1}_{22}\rangle\langle\varepsilon^{-1}_{22}\rangle^{-1}. \tag{42}$$

Dielectric effective constants

$$\varepsilon_{11}^{h} = \langle \varepsilon_{11} \rangle,$$
$$\varepsilon_{22}^{h} = \langle \varepsilon_{22}^{-1} \rangle^{-1},$$
$$\varepsilon_{33}^{h} = \langle \varepsilon_{33} \rangle + \langle e_{322}^{2} C_{2222}^{-1} \rangle - \langle e_{322} C_{2222}^{-1} \rangle^{2} \langle C_{2222}^{-1} \rangle^{-1}. \tag{43}$$

As we can observe, there exist nine independent elastic effective constants given by (41), five piezoelectric effective constants (42), and three dielectric effective constants (43). Therefore we conclude (taking into account the general classification for homogeneous piezoelectric materials; see, for instance, in Dieulesaint and Royer (1974) or Berlincourt (1964)) that if we have periodic cells composed of piezoelectric materials' layers with hexagonal symmetry (6 mm), connected in parallel, the corresponding homogenized material will like a piezoelectric material with orthorhombic symmetry (2 mm). After considerable manipulations it is verified that the effective constants in (41)–(43), for the particular case of a binary medium, are exactly the same as Benveniste and Dvorak (1992).

7. APPLICATIONS. IMPROVEMENT OF PHYSICAL CHARACTERISTICS

In order to show an application of these piezocomposite materials we will consider the case of parallel connection where each periodic cell consists only of two different homogeneous phases. The ceramic phase is a piezoelectric with hexagonal symmetry and the polymer phase is an isotropic homogeneous medium which is piezoelectrically inactive. The elastic and dielectric constants of the ceramic phase will be distinguished from those of the polymer phase by the superscripts E and S, respectively. Moreover, by utilizing the following mapping of adjacent indices:

$$(11) \to 1, \quad (22) \to 2, \quad (33) \to 3,$$
$$(23) = (32) \to 4, \quad (31) = (13) \to 5, \quad (12) = (21) \to 6,$$

we express the elastic and piezoelectric coefficients briefly as: $C_{\alpha\beta} = C_{ijkl}$, $e_{i\beta} = e_{ikl}$, where $(ij) \to \alpha$ and $(kl) \to \beta$. Taking into account the above notations we have the following expressions for the effective coefficients.

Elastic effective constants

$$\begin{aligned}
C_{11}^{h} &= \chi(C_{11}^{E} - (C_{12}^{E})^{2}(C_{11}^{E})^{-1}) + (1-\chi)(C_{11} - C_{12}(C_{11})^{-1}) \\
&\quad + (\chi C_{12}^{E}(C_{11}^{E})^{-1} + (1-\chi)C_{12}(C_{11})^{-1})^{2} \\
&\quad \times (\chi(C_{11}^{E})^{-1} + (1-\chi)(C_{11})^{-1})^{-1}, \\
C_{12}^{h} &= (\chi C_{12}^{E}(C_{11}^{E})^{-1} + (1-\chi)C_{12}(C_{11})^{-1}) \\
&\quad \times (\chi(C_{11}^{E})^{-1} + (1+\chi)(C_{11})^{-1})^{-1}, \\
C_{13}^{h} &= \chi C_{13}^{E}(1 - C_{12}^{E}(C_{11}^{E})^{-1}) + (1-\chi)C_{12}(1 - C_{12}(C_{11})^{-1}) \\
&\quad + (\chi C_{12}^{E}(C_{11}^{E})^{-1} + (1-\chi)C_{12}(C_{11})^{-1}) \\
&\quad \times (\chi(C_{11}^{E})^{-1} + (1-\chi)(C_{11})^{-1})^{-1} \\
&\quad \times (\chi C_{13}^{E}(C_{11}^{E})^{-1} + (1-\chi)C_{12}(C_{11})^{-1}), \\
C_{22}^{h} &= (\chi(C_{11}^{E})^{-1} + (1-\chi)(C_{11})^{-1})^{-1}, \\
C_{23}^{h} &= (\chi C_{13}^{E}(C_{11}^{E})^{-1} + (1-\chi)C_{12}(C_{11})^{-1}) \\
&\quad \times (\chi(C_{11}^{E})^{-1} + (1-\chi)(C_{11})^{-1})^{-1},
\end{aligned}$$

$$
\begin{aligned}
C_{33}^h &= \chi(C_{33}^E-(C_{13}^E)^2(C_{11}^E)^{-1})+(1-\chi)(C_{11}-(C_{12})^2(C_{11})^{-1}) \\
&\quad +(\chi C_{13}^E(C_{11}^E)^{-1}+(1-\chi)C_{12}(C_{11})^{-1})^2 \\
&\quad \times(\chi(C_{11}^E)^{-1}+(1-\chi)(C_{11})^{-1})^{-1}, \\
C_{44}^h &= (\chi(C_{44}^E)^{-1}+(1-\chi)(C_{44})^{-1})^{-1}+\chi(\varepsilon_{22}^S)^{-1}(e_{15})^2 \\
&\quad -\chi(\varepsilon_{22}^S)^{-1}(e_{15})^2(\chi(\varepsilon_{22}^S)^{-1}+(1-\chi)(\varepsilon_{22}^{-1})^{-1}, \\
C_{55}^h &= \chi C_{44}^E+(1-\chi)C_{44}, \\
C_{66}^h &= (\chi(C_{66}^E)^{-1}+(1-\chi)(C_{44})^{-1})^{-1}.
\end{aligned} \tag{44}
$$

Piezoelectric effective constants

$$
\begin{aligned}
e_{15}^h &= \chi e_{15}, \\
e_{24}^h &= (\chi(\varepsilon_{22}^S)^{-1}+(1-\chi)(\varepsilon_{22})^{-1})^{-1}(\chi(\varepsilon_{22}^S)^{-1}e_{24}), \\
e_{32}^h &= (\chi(C_{22}^E)^{-1}+(1-\chi)(C_{22})^{-1})^{-1}(\chi(C_{22}^E)^{-1}e_{32}), \\
e_{31}^h &= \chi(e_{31}-e_{32}(C_{22}^E)^{-1}C_{12}^E)+(\chi(C_{22}^E)^{-1}+(1-\chi)(C_{22})^{-1})^{-1} \\
&\quad \times(\chi(C_{22}^E)^{-1}C_{21}^E+(1-\chi)(C_{22})^{-1}C_{21})(\chi(C_{22}^E)^{-1}e_{32}), \\
e_{33}^h &= \chi(e_{33}-e_{32}(C_{22}^E)^{-1}C_{23}^E)+(\chi(C_{22}^E)^{-1}+(1-\chi)(C_{22})^{-1})^{-1} \\
&\quad \times(\chi(C_{22}^E)^{-1}C_{23}^E+(1-\chi)(C_{22})^{-1}C_{23})(\chi(C_{22}^E)^{-1}e_{32}).
\end{aligned} \tag{45}
$$

Dielectric effective constants

$$
\begin{aligned}
\varepsilon_{11}^h &= \chi\varepsilon_{11}^S+(1-\chi)\varepsilon_{11}, \\
\varepsilon_{22}^h &= (\chi(\varepsilon_{22}^S)^{-1}+(1-\chi)(\varepsilon_{22})^{-1})^{-1}, \\
\varepsilon_{33}^h &= \chi(\varepsilon_{33}^S+(e_{32})^2(C_{22}^E)^{-1})+(1-\chi)\varepsilon_{33} \\
&\quad -(\chi(C_{22}^E)^{-1}e_{32})^2(\chi(C_{22}^E)^{-1}+(1-\chi)(C_{22})^{-1})^{-1},
\end{aligned} \tag{46}
$$

where χ is the ceramic's volume fraction. Moreover, the components of averaged tensors of piezomoduli $\bar{d}_{mi}$, elastic compliances $\bar{S}_{ij}^E$ and permittivity $\bar{\varepsilon}_{mn}^T$ of such piezocomposite material can be calculated by using the following formulae (see, for instance, Berlincourt (1964))

$$
\bar{S}_{ij}^E = (-1)^{i+j}\frac{\Delta_{ij}}{\Delta}, \quad \bar{d}_{mi} = \bar{e}_{mj}\bar{S}_{ji}^E, \quad \bar{\varepsilon}_{mn}^T = \bar{d}_{mp}\bar{e}_{np}+\bar{\varepsilon}_{mn}^S, \quad i,j = 1,2,\ldots,6
$$

where, Δ is the determinant of the $\bar{C}_{ij}$ matrix and Δ_{ij} is the minor obtained by excluding the ith row and jth column.

An application of piezoelectric composites is in passive detectors subjected to hydrostatic conditions (such as hydrophones, see, for instance, Haun and Newnham (1986), Gururaja *et al.* (1987) and Smith (1993)). Based on the idea of decoupling the piezoelectric $\bar{d}_{33}$ and $\bar{d}_{31}$ coefficients and lowering the permittivity $\bar{\varepsilon}_{33}^T$, these composites have produced some remarkable improvements in the hydrostatic $\bar{d}_h$ $(=\bar{d}_{33}+2\bar{d}_{31})$ and $\bar{g}_h$ $(=\bar{d}_h/\varepsilon_0\bar{\varepsilon}_{33}^T)$ coefficients, where ε_0 denotes the permittivity of free space. The principle of designing a composite material for hydrophone application is to maintain $\bar{d}_{33}$ as large as possible and to reduce $\bar{d}_{31}$ and the dielectric constant $\bar{\varepsilon}_{33}^T$, resulting in an enhanced value of $\bar{d}_h\bar{g}_h$.

Table 1. Electroelastic material properties

Parameters	C_{11}	C_{12}	C_{13}	C_{33}	C_{44}	e_{31}	e_{33}	e_{15}	$\varepsilon_{11}/\varepsilon_0$	$\varepsilon_{33}/\varepsilon_0$
Dimensions	GPa	GPa	GPa	GPa	GPa	C/m^2	C/m^2	C/m^2	—	—
PZT-5A	121	75.4	75.2	111	21.1	-5.4	15.8	12.3	916	830
Alraldite	5.46	2.94	2.94	5.46	1.26	0	0	0	7.0	7.0

$\varepsilon_0 = 8.85E-12$ (C^2/Nm^2) = permittivity of free space.

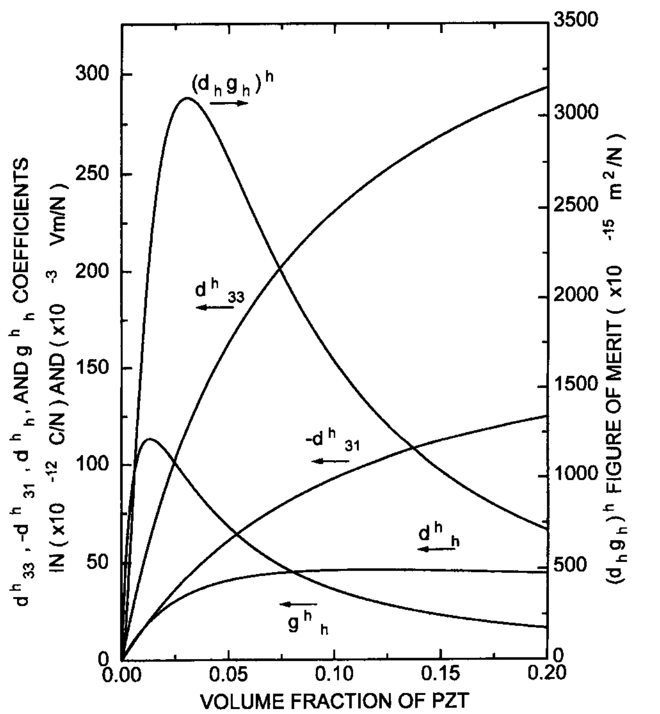

Fig. 1. Theoretical piezoelectric coefficients of a bilaminated composite plotted vs the volume fraction of PZT.

To illustrate how the composite material parameters vary with volume fraction of piezoelectric ceramic, the material parameters of piezoelectric ceramic PZT-5A and Araldite, listed in Table 1, are used in the calculus. According to the calculations for a bilayered in parallel connection shown in Fig. 1, the optimum percentage of PZT to maximise the value of $\bar{d}_h\bar{g}_h((d_h g_h)^h)$ should be about three percent of PZT. It is noted that the results for $\bar{d}_{33}$ agree with the parallel model solutions of Newnham *et al.* (1978) and Grekov *et al.* (1987).

Another important application of piezoelectric ceramic/polymer composites is in transducer for biomedical imaging applications (Chan and Unsworth (1989), Smith and Auld (1991), Taunaumang *et al.* (1994), etc.). The spatial fine-scale of these composites allow higher operating frequencies (Janas and Safari (1995)). The use of expressions (44)–(46) to design better piezoelectric composite transducers for medical imaging applications is shown in Ramos *et al.* (1996).

In this paper, the technique of asymptotic homogenization is applied to the problem of a laminated piezocomposite medium with a periodic structure. The local problems are considered and the effective elastic, piezoelectric and dielectric moduli are explicitly determined. The explicit solutions resulting from this work can be effectively used to obtain the global properties of laminated piezocomposite materials for both models of Newnham's connectivity theory and any finite number of phases.

Acknowledgements—The first author is grateful to J. J. Telega, R. Wojnar, R. E. Newnham and I. L. Guy for providing him reprints of their papers. All comments and suggestions of the reviewers are acknowledged.

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