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Mathematical Modeling of Weak and Strong Piezoelectric Interfaces

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Abstract We study the electromechanical behavior of a thin interphase, constituted by a piezoelectric anisotropic thin layer, embedded between two generic three-dimensional piezoelectric bodies by means of the asymptotic analysis. After defining a small real dimensionless parameter ε, which will tend to zero, we characterize two different limit models and their associated limit problems, the so-called *weak* and *strong* interface models, respectively. Moreover, we identify the non classical electromechanical transmission conditions at the interface between the two three-dimensional bodies.

Keywords Asymptotic analysis · Interface models · Piezoelectric materials

Mathematics Subject Classification (2010) 74F15 · 74K30 · 74A50

1 Introduction

The conception and use of smart materials have undergone a major development over the past few decades in all fields of aeronautical, mechanical and civil engineering. Concerning with the case of the so-called *smart structures*, the strain state is constantly under control with the help of sensors and actuators, made, for instance, of piezoelectric materials, which are integrated within the structure. This kind of technology is based on the ability to allow the structural member to sense and react in a desired fashion, improving its performances. Piezoelectric materials exhibit both direct and converse piezoelectric effects. The direct effect (electric field generation as a response to mechanical strains) is used in piezoelectric sensors; the converse effect (mechanical strain is produced as a result of an electric field) is used in piezoelectric actuators.

Piezoelectric materials may be applied onto a host structure to change its shape and to enhance its mechanical properties with different configurations: for instance, a piezoelectric

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transducer can be embedded into the structure to be controlled or it can be glued on it, as in the case of piezo-patches. Moreover, the same piezoelectric actuators are often obtained by alternating different thin layers of material with highly contrasted electromechanical properties. This generates different types of complex multimaterial assemblies, in which each phase interacts with the others. An extensive list of references on the subject can be found in the following cited papers for what concerns with piezoelectric interphases/interfaces problems using classical variational tools: see, for instance, [2, 3] for curved thin interphases in conduction phenomena, [4] for anisotropic piezoelectric curved thin interphases, and [11] for piezoelectric multi-layer plate models.

The successful application of the asymptotic methods to obtain a mathematical justification of thin structure models in the field of linear and non linear elasticity (see, e.g., [9]) and in quasi-static and transient piezoelectricity, taking into account both sensor and actuator functions, (see, e.g. [10, 12, 21, 22, 24, 25] has stimulated the research toward a rational simplification of the modeling of complex structures obtained joining elements of different dimensions and/or materials of highly contrasted properties. Thin interphases represent one of the most peculiar bonded joint between two media. The treatment of the thin interphase as a separate phase of the multimaterial by a standard finite element analysis is too expensive from a computational point of view and the presence of strong contrasts in the geometry and mechanical properties causes numerical instabilities. That is why specific asymptotic expansions are used and allow to replace the original problem by a set of problems in which the thin interphase is substituted by a two-dimensional material surface, i.e., a so-called *imperfect interface*, between the two three-dimensional bodies with non classical transmission conditions. Within the theory of elasticity, the asymptotic analysis of a thin elastic interphase between two elastic materials has been deeply investigated through the years, by varying the rigidity ratios between the thin inclusion and the surrounding materials and by considering different geometry features. For instance, it is worth mentioning the pioneering work by E. Acerbi et al. [1] on the variational behavior of the elastic energy of a thin inclusion using Γ-convergence, the contributions by G. Geymonat et al. [16], F. Krasucki et al. [17], and C. Licht and G. Michaille [20] for mathematical models for linear and non linear weak bonded joints, the works by F. Lebon and R. Rizzoni [18, 19] for the case of thin interfaces with similar and hard rigidities, and, also, the works by A.-L. Bessoud et al. [6–8] in which the authors studied the case of plate-like and shell-like inclusions with high rigidity in a rigorous functional framework. It is also noteworthy the work by W. Geis et al. [13], in which the authors derive a reduced model for the problem of thin conductor plates embedded into a piezoelectric matrix with similar (undamaged electrodes) and weak (damaged electrodes) electromechanical rigidities, using functional convergence.

The goal of the present work is to identify two different interface limit models of a piezoelectric assembly constituted by a thin piezoelectric layer surrounded by two generic piezoelectric bodies by means of an asymptotic analysis. By defining a small real parameter ε, associated with the thickness and the electromechanical properties of the middle layer, we perform an asymptotic analysis by letting ε tend to zero. We analyze two different situations by varying the electromechanical stiffnesses ratios between the middle layer and the adherents: namely, the *weak* piezoelectric interface, where the electromechanical coefficients of the intermediate domain have order of magnitude ε with respect to those of the surrounding bodies; the *strong* piezoelectric interface, where the electromechanical rigidities have order of magnitude $\frac{1}{\varepsilon}$. Within the reduced models, the interphase is replaced by a material surface (*strong* case) or a constraint (*weak* case) whose energy, in both cases, is the limit of the interphase energy. This surface energy is then translated in ad hoc transmission conditions at the interface between the two piezoelectric bodies in terms of the jump of stresses, electric displacements, electric potentials and displacements.

It is important to stress that the asymptotic analysis has not yet been fully applied to generic piezoelectric multimaterials, combining the different coupled electromechanical behaviors of their constituents. The previous paper by M. Serpilli [23] is devoted to this topic: the author considers the problem of a *strong* interface model generated by inserting a transversely isotropic piezoelectric actuator plate into a three-dimensional body. In that case the author used a classical scaling of the unknowns for piezoelectric plates, i.e., $u_\alpha^\varepsilon(x^\varepsilon) = u_\alpha(\varepsilon)(x)$, $u_3^\varepsilon(x^\varepsilon) = \varepsilon^{-1}u_3(\varepsilon)(x)$ and $\varphi^\varepsilon(x^\varepsilon) = \varepsilon\varphi(\varepsilon)(x)$ (see, e.g., [24]): this scaling leads to a particular limit electromechanical state, according to which the interphase behaves as a Kirchhoff-Love plate with a quadratic electric potential with respect to x_3, depending on the values of the electric potentials at the top and bottom faces and on the transversal displacement of the plate. It is well-known that the scaling of the unknowns strongly influences the limit behavior of the solution: in particular, in [23], this peculiar scaling induces the membrane and flexural behaviors of the plate-like interphase to appear at the same order in the asymptotic expansion. That is why the author decides to follow another path for the present paper (see [5, 6, 16]), using an isotropic scaling of the unknowns, i.e., $u_\alpha^\varepsilon(x^\varepsilon) = u_\alpha(\varepsilon)(x)$, $u_3^\varepsilon(x^\varepsilon) = u_3(\varepsilon)(x)$ and $\varphi^\varepsilon(x^\varepsilon) = \varphi(\varepsilon)(x)$, without privileging any particular limit behavior of the solution. The obtained results are slightly different from those of [23], as shown in Sect. 5.

The paper is organized as follows. In Sect. 2 we define the notation and the position of the problem. In Sect. 3, we perform the asymptotic analysis of the problem. In Sects. 4 and 5, we deduce, respectively, the two limit interface models and we mathematically justify them by means of a strong convergence argument. In Sect. 6 we present some intermediate cases between the previously derived interface models. Finally, we discuss the results and propose some future developments in the concluding remarks in Sect. 7.

2 The Physical Problem

In the sequel, Greek indices range in the set $\{1,2\}$, Latin indices range in the set $\{1,2,3\}$, and the Einstein's summation convention with respect to the repeated indices is adopted. We also introduce the following notations for, respectively, the scalar and the dyadic products:

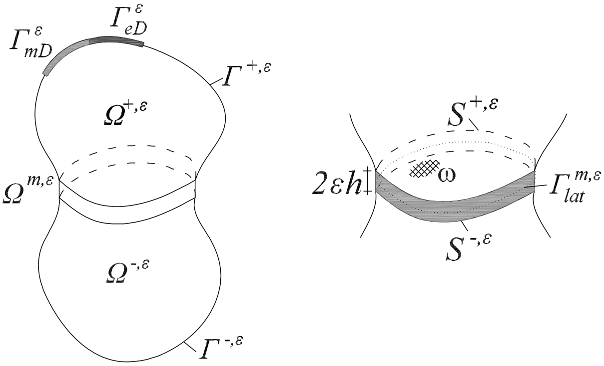
$$\mathbf{a}\cdot\mathbf{b} := a_i b_i, \quad (\mathbf{a}\otimes\mathbf{b})_{ij} := (a_i b_j), \quad \text{for all vectors } \mathbf{a} = (a_i) \text{ and } \mathbf{b} = (b_i).$$

Let us consider a three-dimensional Euclidian space identified by $\mathbb{R}^3$ and such that the three vectors $\mathbf{e}_i$ form an orthonormal basis. Let Ω^+ and Ω^- be two disjoint open domains with smooth boundaries $\partial\Omega^+$ and $\partial\Omega^-$. Let $\omega := \{\partial\Omega^+ \cap \partial\Omega^-\}^\circ$ be the interior of the common part of the boundaries which is assumed to be a non empty domain in $\mathbb{R}^2$ having a positive two-dimensional measure. We consider the assembly constituted by two solids bonded together by an intermediate thin plate-like body $\Omega^{m,\varepsilon}$ of thickness $2h^\varepsilon$, where $0 < \varepsilon < 1$ is a dimensionless small real parameter which will tend to zero. We suppose that the thickness h^ε of the middle layer depends linearly on ε, so that $h^\varepsilon = \varepsilon h$.

More precisely, we denote respectively with $\Omega^{\pm,\varepsilon} := \{x^\varepsilon := x \pm \varepsilon h\mathbf{e}_3;\ x \in \Omega^\pm\}$, the translation of Ω^+ (resp. Ω^-) along the direction $\mathbf{e}_3$ (resp. $-\mathbf{e}_3$) of the quantity εh, with $\Omega^{m,\varepsilon} := \omega \times (-\varepsilon h, \varepsilon h)$, the central plate-like domain, and with $\Omega^\varepsilon := \Omega^{+,\varepsilon} \cup \Omega^{m,\varepsilon} \cup \Omega^{-,\varepsilon}$, the reference configuration of the assembly.

Moreover, we define with $S^{\pm,\varepsilon} := \omega \times \{\pm\varepsilon h\} = \Omega^{\pm,\varepsilon} \cap \Omega^{m,\varepsilon}$, the upper and lower faces of the intermediate plate-like domain, $\Gamma^{\pm,\varepsilon} := \partial\Omega^{\pm,\varepsilon}/S^{\pm,\varepsilon}$, and $\Gamma_{lat}^{m,\varepsilon} := \partial\omega \times (-\varepsilon h, \varepsilon h)$, its lateral surface, see Fig. 1.

Fig. 1 The reference configuration of the multimaterial and the geometry of the interphase



Let $(\Gamma^{\varepsilon}_{mD}, \Gamma^{\varepsilon}_{mN})$ and $(\Gamma^{\varepsilon}_{eD}, \Gamma^{\varepsilon}_{eN})$ be two suitable partitions of $\partial\Omega^{\varepsilon} := \Gamma^{\pm,\varepsilon} \cup \Gamma^{m,\varepsilon}_{lat}$, with both $\Gamma^{\varepsilon}_{mD}$ and $\Gamma^{\varepsilon}_{eD}$ of strictly positive Haussdorff measure. The multimaterial is, on one hand, clamped along $\Gamma^{\varepsilon}_{mD}$ and at an electrical potential $\varphi^{\varepsilon}_0 = 0$ on $\Gamma^{\varepsilon}_{eD}$ and, on the other hand, subject to surface forces g^{ε}_i on $\Gamma^{\varepsilon}_{mN}$ and surface electrical charges d^{ε} on $\Gamma^{\varepsilon}_{eN}$. The assembly is also subject to body forces f^{ε}_i and electrical loadings ρ^{ε}_e acting in $\Omega^{\pm,\varepsilon}$. We suppose, without loss of generality, that $\Omega^{m,\varepsilon}$ and $\Gamma^{m,\varepsilon}_{lat}$ are both free of mechanical and electrical charges. The work of the external electromechanical loadings takes then the following form:

$$L^{\varepsilon}\big(r^{\varepsilon}\big) := \int_{\Omega^{\pm,\varepsilon}} \big(f^{\varepsilon}_i v^{\varepsilon}_i + \rho^{\varepsilon}_e \psi^{\varepsilon}\big) dx^{\varepsilon} + \int_{\Gamma^{\varepsilon}_{mN}} g^{\varepsilon}_i v^{\varepsilon}_i \, d\Gamma^{\varepsilon} + \int_{\Gamma^{\varepsilon}_{eN}} d^{\varepsilon} \psi^{\varepsilon} \, d\Gamma^{\varepsilon}.$$

We suppose that $f^{\varepsilon}_i \in L^2(\Omega^{\pm,\varepsilon})$, $\rho^{\varepsilon}_e \in L^2(\Omega^{\pm,\varepsilon})$, $g^{\varepsilon}_i \in L^2(\Gamma^{\varepsilon}_{mN})$ and $d^{\varepsilon} \in L^2(\Gamma^{\varepsilon}_{eN})$. We finally assume that $\Omega^{\pm,\varepsilon}$ and $\Omega^{m,\varepsilon}$ are constituted by three homogeneous linearly piezoelectric materials, whose constitutive laws are defined as follows:

$$\begin{cases} \sigma^{\varepsilon}_{ij}\big(\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}\big) = C^{\varepsilon}_{ijk\ell} e^{\varepsilon}_{k\ell}\big(\mathbf{u}^{\varepsilon}\big) - P^{\varepsilon}_{kij} E^{\varepsilon}_k\big(\varphi^{\varepsilon}\big), \\ D^{\varepsilon}_i\big(\mathbf{u}^{\varepsilon}, \varphi^{\varepsilon}\big) = P^{\varepsilon}_{ijk} e^{\varepsilon}_{jk}\big(\mathbf{u}^{\varepsilon}\big) + H^{\varepsilon}_{ij} E^{\varepsilon}_j\big(\varphi^{\varepsilon}\big), \end{cases}$$

where $(\sigma^{\varepsilon}_{ij})$ is the classical Cauchy stress tensor, $(e^{\varepsilon}_{ij}(\mathbf{u}^{\varepsilon})) := (\frac{1}{2}(\partial^{\varepsilon}_i u^{\varepsilon}_j + \partial^{\varepsilon}_j u^{\varepsilon}_i))$ is the linearized strain tensor, (D^{ε}_i) is the electrical displacement field, φ^{ε} is the electrical potential and $E^{\varepsilon}_i(\varphi^{\varepsilon}) := -\partial^{\varepsilon}_i \varphi^{\varepsilon}$ its associated electrical field. $(C^{\varepsilon}_{ijk\ell})$, (P^{ε}_{ijk}) and (H^{ε}_{ij}) represent, respectively, the classical fourth order elasticity tensor, the third order piezoelectric coupling tensor and the second order dielectric tensor related to $\Omega^{\pm,\varepsilon}$ and $\Omega^{m,\varepsilon}$.

Tensors $(C^{\varepsilon}_{ijk\ell})$ and (H^{ε}_{ij}) satisfy the following coercivity properties: for any symmetric matrix field (b_{ij}), there exists a constant $c > 0$ such that $C^{\varepsilon}_{ijk\ell} b_{k\ell} b_{ij} \geq c \sum_{i,j} |b_{ij}|^2$; for any vector field (a_i), there exists a constant $c > 0$ such that $H^{\varepsilon}_{ij} a_j a_i \geq c \sum_i |a_i|^2$. Moreover, we have the symmetries $C^{\varepsilon}_{ijk\ell} = C^{\varepsilon}_{k\ell ij} = C^{\varepsilon}_{jik\ell}$, $H^{\varepsilon}_{ij} = H^{\varepsilon}_{ji}$ and $P^{\varepsilon}_{kji} = P^{\varepsilon}_{kij}$.

Let $\Sigma^{\varepsilon} \subset \partial\Omega^{\varepsilon}$, we introduce the functional spaces

$$\mathbf{V}\big(\Omega^{\varepsilon}, \Sigma^{\varepsilon}\big) := \big\{\mathbf{v}^{\varepsilon} = \big(v^{\varepsilon}_i\big) \in H^1\big(\Omega^{\varepsilon}; \mathbb{R}^3\big);\ \mathbf{v}^{\varepsilon} = \mathbf{0} \text{ on } \Sigma^{\varepsilon}\big\},$$
$$V\big(\Omega^{\varepsilon}, \Sigma^{\varepsilon}\big) := \big\{v^{\varepsilon} \in H^1\big(\Omega^{\varepsilon}\big);\ v^{\varepsilon} = 0 \text{ on } \Sigma^{\varepsilon}\big\}.$$

The electromechanical state at the equilibrium is determined by the pair $s^\varepsilon := (\mathbf{u}^\varepsilon, \varphi^\varepsilon)$. The physical variational problem $\mathcal{P}^\varepsilon$ defined over the variable domain Ω^ε reads as follows:

$$\begin{cases} \text{Find } s^\varepsilon \in \mathbf{V}\big(\Omega^\varepsilon, \Gamma^\varepsilon_{mD}\big) \times V\big(\Omega^\varepsilon, \Gamma^\varepsilon_{eD}\big) \text{ such that} \\ A^{-,\varepsilon}\big(s^\varepsilon, r^\varepsilon\big) + A^{+,\varepsilon}\big(s^\varepsilon, r^\varepsilon\big) + A^{m,\varepsilon}\big(s^\varepsilon, r^\varepsilon\big) = L^\varepsilon\big(r^\varepsilon\big), \end{cases} \tag{1}$$

for all $r^\varepsilon \in \mathbf{V}(\Omega^\varepsilon, \Gamma^\varepsilon_{mD}) \times V(\Omega^\varepsilon, \Gamma^\varepsilon_{eD})$, where the bilinear forms $A^{\pm,\varepsilon}(\cdot,\cdot)$ and $A^{m,\varepsilon}(\cdot,\cdot)$ are defined by

$$\begin{aligned} A^{\pm,\varepsilon}\big(s^\varepsilon, r^\varepsilon\big) &:= \int_{\Omega^{\pm,\varepsilon}} \big\{C^{\pm,\varepsilon}_{ijk\ell} e^\varepsilon_{k\ell}\big(\mathbf{u}^\varepsilon\big) e^\varepsilon_{ij}\big(\mathbf{v}^\varepsilon\big) + H^{\pm,\varepsilon}_{ij} E^\varepsilon_j\big(\varphi^\varepsilon\big) E^\varepsilon_i\big(\psi^\varepsilon\big) \\ &\quad + P^{\pm,\varepsilon}_{ihk}\big(E^\varepsilon_i\big(\psi^\varepsilon\big) e^\varepsilon_{hk}\big(\mathbf{u}^\varepsilon\big) - E^\varepsilon_i\big(\varphi^\varepsilon\big) e^\varepsilon_{hk}\big(\mathbf{v}^\varepsilon\big)\big)\big\} dx^\varepsilon, \\ A^{m,\varepsilon}\big(s^\varepsilon, r^\varepsilon\big) &:= \int_{\Omega^{m,\varepsilon}} \big\{C^{m,\varepsilon}_{ijk\ell} e^\varepsilon_{k\ell}\big(\mathbf{u}^\varepsilon\big) e^\varepsilon_{ij}\big(\mathbf{v}^\varepsilon\big) + H^{m,\varepsilon}_{ij} E^\varepsilon_j\big(\varphi^\varepsilon\big) E^\varepsilon_i\big(\psi^\varepsilon\big) \\ &\quad + P^{m,\varepsilon}_{ihk}\big(E^\varepsilon_i\big(\psi^\varepsilon\big) e^\varepsilon_{hk}\big(\mathbf{u}^\varepsilon\big) - E^\varepsilon_i\big(\varphi^\varepsilon\big) e^\varepsilon_{hk}\big(\mathbf{v}^\varepsilon\big)\big)\big\} dx^\varepsilon. \end{aligned}$$

By virtue of the $\mathbf{V}(\Omega^\varepsilon, \Gamma^\varepsilon_{mD}) \times V(\Omega^\varepsilon, \Gamma^\varepsilon_{eD})$-coercivity of the bilinear forms and thanks to the Lax-Milgram lemma, problem (1) admits one and only one solution.

3 The Asymptotic Expansion Method

In order to study the asymptotic behavior of the solution of problem (1) when ε tends to zero, we rewrite the problem on a fixed domain Ω independent of ε. By using the approach of [9] we consider the bijection $\pi^\varepsilon : x \in \overline{\Omega} \mapsto x^\varepsilon \in \overline{\Omega}^\varepsilon$ given by

$$\begin{cases} \pi^\varepsilon(x_1, x_2, x_3) = \big(x_1, x_2, x_3 - h(1-\varepsilon)\big), & \text{for all } x \in \overline{\Omega}^+_{tr}, \\ \pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, \varepsilon x_3), & \text{for all } x \in \overline{\Omega}^m, \\ \pi^\varepsilon(x_1, x_2, x_3) = \big(x_1, x_2, x_3 + h(1-\varepsilon)\big), & \text{for all } x \in \overline{\Omega}^-_{tr}, \end{cases}$$

where $\Omega^\pm_{tr} := \{x \pm h\mathbf{e}_3,\ x \in \Omega^\pm\}$, $\Omega^m := \omega \times (-h, h)$ and $S^\pm := \omega \times \{\pm h\}$. In order to simplify the notation, we identify $\Omega^\pm_{tr}$ with $\Omega^\pm$, and $\overline{\Omega}$ with $\overline{\Omega}^\pm \cup \overline{\Omega}^m$. Likewise, we note $\Gamma_\pm := \partial\Omega^\pm / S^\pm$, $\Gamma^m_{lat} := \partial\omega \times (-h, h)$, $(\Gamma_{mD}, \Gamma_{mN})$ and $(\Gamma_{eD}, \Gamma_{eN})$, the partitions of $\partial\Omega := \Gamma^\pm \cup \Gamma^m_{lat}$.

Consequently,

$$\partial^\varepsilon_\alpha = \partial_\alpha \quad \text{and} \quad \partial^\varepsilon_3 = \frac{1}{\varepsilon}\partial_3 \quad \text{in } \Omega^m.$$

In the sequel, only if necessary, we will note, respectively, with $(\mathbf{v}^\pm, \psi^\pm)$ and $(\mathbf{v}^m, \psi^m)$, the restrictions of functions $(\mathbf{v}, \psi)$ to $\Omega^\pm$ and Ω^m.

With the unknown electromechanical state $s^\varepsilon = (\mathbf{u}^\varepsilon, \varphi^\varepsilon)$, we associate the scaled unknown electromechanical state $s(\varepsilon) := (\mathbf{u}(\varepsilon), \varphi(\varepsilon))$ defined by:

$$\begin{aligned} u^\varepsilon_i\big(x^\varepsilon\big) &= u_i(\varepsilon)(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon, \\ \varphi^\varepsilon\big(x^\varepsilon\big) &= \varphi(\varepsilon)(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon. \end{aligned}$$

We likewise associate with any test functions $r^\varepsilon = (\mathbf{v}^\varepsilon, \psi^\varepsilon)$, the scaled test functions $r = (\mathbf{v}, \psi)$, defined by the scalings:

$$v_i^\varepsilon\big(x^\varepsilon\big) = v_i(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,$$
$$\psi^\varepsilon\big(x^\varepsilon\big) = \psi(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon.$$

For ε sufficiently small, we associate with the constant functions $C_{ijk\ell}^{\pm,\varepsilon}$, $H_{ij}^{\pm,\varepsilon}$, $P_{ijk}^{\pm,\varepsilon} : \overline{\Omega}^{\pm,\varepsilon} \to \mathbb{R}$ the constant functions $C_{ijk\ell}^{\pm}$, $H_{ij}^{\pm}$, $P_{ijk}^{\pm} : \overline{\Omega}^{\pm} \to \mathbb{R}$ defined by

$$C_{ijk\ell}^{\pm,\varepsilon} := C_{ijk\ell}^{\pm}, \quad H_{ij}^{\pm,\varepsilon} := H_{ij}^{\pm}, \quad P_{ijk}^{\pm,\varepsilon} := P_{ijk}^{\pm} \quad \text{for all } x^\varepsilon = \pi^\varepsilon(x) \in \overline{\Omega}^{\pm,\varepsilon},$$

and we associate with the constant functions $C_{ijk\ell}^{m,\varepsilon}$, $H_{ij}^{m,\varepsilon}$, $P_{ijk}^{m,\varepsilon} : \overline{\Omega}^{m,\varepsilon} \to \mathbb{R}$ the constant functions $C_{ijk\ell}^{m}$, H_{ij}^{m}, $P_{ijk}^{m} : \overline{\Omega}^{m} \to \mathbb{R}$ defined by

$$C_{ijk\ell}^{m,\varepsilon} := \varepsilon^p C_{ijk\ell}^{m}, \quad H_{ij}^{m,\varepsilon} := \varepsilon^p H_{ij}^{m}, \quad P_{ijk}^{m,\varepsilon} := \varepsilon^p P_{ijk}^{m} \quad \text{for all } x^\varepsilon = \pi^\varepsilon(x) \in \overline{\Omega}^{m,\varepsilon},$$

with $p \in \{-1, 1\}$. Two different limit behaviors will be characterized according to the choice of the exponent p: in the case of $p = -1$, we derive a model for a strong piezoelectric interface; by choosing $p = 1$, we deduce a model for a weak piezoelectric interface.

We also make the following assumptions on the applied mechanical and electrical forces:

$$f_i^\varepsilon\big(x^\varepsilon\big) = f_i(x) \quad \text{and} \quad g_i^\varepsilon\big(x^\varepsilon\big) = g_i(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{\pm,\varepsilon},$$
$$\rho_e^\varepsilon\big(x^\varepsilon\big) = \rho_e(x) \quad \text{and} \quad d^\varepsilon\big(x^\varepsilon\big) = d(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^{\pm,\varepsilon},$$

where functions $f_i \in L^2(\Omega^\pm)$, $\rho_e \in L^2(\Omega^\pm)$, $g_i \in L^2(\Gamma_{mN})$ and $d \in L^2(\Gamma_{eN})$ are independent of ε. Thus $L^\varepsilon(r^\varepsilon) = L(r)$.

According to the previous hypothesis, problem (1) can be reformulated on a fixed domain Ω independent of ε. Thus we obtain the following scaled problem $\mathcal{P}^p(\varepsilon)$:

$$\begin{cases} \text{Find } s(\varepsilon) \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD}) \text{ such that} \\ A^-\big(s(\varepsilon), r\big) + A^+\big(s(\varepsilon), r\big) + A^{m,p}(\varepsilon)\big(s(\varepsilon), r\big) = L(r), \end{cases} \tag{2}$$

for all $r \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD})$, $p \in \{-1, 1\}$, where

$$\begin{aligned} A^\pm\big(s(\varepsilon), r\big) := \int_{\Omega^\pm} \big\{ & C_{ijk\ell}^{\pm} e_{k\ell}\big(\mathbf{u}(\varepsilon)\big) e_{ij}(\mathbf{v}) + H_{ij}^{\pm} \partial_j\big(\varphi(\varepsilon)\big)\partial_i \psi \\ & + P_{ihk}^{\pm}\big(\partial_i \varphi(\varepsilon) e_{hk}(\mathbf{v}) - \partial_i \psi e_{hk}\big(\mathbf{u}(\varepsilon)\big)\big)\big\} dx, \end{aligned}$$
$$A^{m,p}(\varepsilon)\big(s(\varepsilon), r\big) := \varepsilon^{p-1} a^m\big(s(\varepsilon), r\big) + \varepsilon^p b^m\big(s(\varepsilon), r\big) + \varepsilon^{p+1} c^m\big(s(\varepsilon), r\big),$$

with

$$\begin{aligned} a^m\big(s(\varepsilon), r\big) := \int_{\Omega^m} \big\{ & C_{i3j3}^{m} \partial_3 u_i(\varepsilon) \partial_3 v_j + H_{33}^{m} \partial_3 \varphi(\varepsilon) \partial_3 \psi \\ & + P_{3i3}^{m}\big(\partial_3 \varphi(\varepsilon) \partial_3 v_i - \partial_3 \psi \partial_3 u_i(\varepsilon)\big)\big\} dx, \end{aligned}$$
$$b^m\big(s(\varepsilon), r\big) := \int_{\Omega^m} \big\{ C_{i3j\alpha}^{m}\big(\partial_3 u_i(\varepsilon) \partial_\alpha v_j + \partial_\alpha u_j(\varepsilon) \partial_3 v_i\big)$$

$$
\begin{aligned}
&+ H^m_{\alpha 3}\big(\partial_3\varphi(\varepsilon)\partial_\alpha\psi + \partial_3\psi\,\partial_\alpha\varphi(\varepsilon)\big)\\
&+ P^m_{3\alpha i}\big(\partial_3\varphi(\varepsilon)\partial_\alpha v_i - \partial_3\psi\,\partial_\alpha u_i(\varepsilon)\big)\\
&+ P^m_{\alpha i3}\big(\partial_\alpha\varphi(\varepsilon)\partial_3 v_i - \partial_\alpha\psi\,\partial_3 u_i(\varepsilon)\big)\big\}dx,
\end{aligned}
$$

$$
\begin{aligned}
c^m\big(s(\varepsilon), r\big) := \int_{\Omega^m} &\big\{C^m_{i\alpha j\beta}\partial_\alpha u_i(\varepsilon)\partial_\beta v_j + H^m_{\alpha\beta}\partial_\alpha\varphi(\varepsilon)\partial_\beta\psi\\
&+ P^m_{\alpha\beta i}\big(\partial_\alpha\varphi(\varepsilon)\partial_\beta v_i - \partial_\alpha\psi\,\partial_\beta u_i(\varepsilon)\big)\big\}dx.
\end{aligned}
$$

We can now perform an asymptotic analysis of the rescaled problem (2). Since the rescaled problem (2) has a polynomial structure with respect to the small parameter ε, we can look for the solution $s(\varepsilon) = (\mathbf{u}(\varepsilon), \varphi(\varepsilon))$ of the problem as a series of powers of ε:

$$
s(\varepsilon) = s^0 + \varepsilon s^1 + \varepsilon^2 s^2 + \ldots \quad \Longrightarrow \quad \begin{cases} \mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \varepsilon^2\mathbf{u}^2 + \ldots \\ \varphi(\varepsilon) = \varphi^0 + \varepsilon\varphi^1 + \varepsilon^2\varphi^2 + \ldots \end{cases}, \tag{3}
$$

with $s^q = (\mathbf{u}^q, \varphi^q) \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD})$, $q \geq 0$. By substituting (3) into the rescaled problem (2), and by identifying the terms with identical power of ε, we obtain, as customary, a set of variational problems to be solved in order to characterize the limit electromechanical state $s^0 = (\mathbf{u}^0, \varphi^0)$ and its associated limit problem, for $p \in \{-1, 1\}$.

4 The *Weak* Piezoelectric Interface: The Case of $p = 1$

In this section we characterize the limit model for a weak piezoelectric interface. By choosing $p = 1$, we obtain the following set of variational problems:

$$
\begin{aligned}
&\mathcal{P}^1_0\colon A^+\big(s^0, r\big) + A^-\big(s^0, r\big) + a^m\big(s^0, r\big) = L(r),\\
&\mathcal{P}^1_1\colon A^+\big(s^1, r\big) + A^-\big(s^1, r\big) + a^m\big(s^1, r\big) + b^m\big(s^0, r\big) = 0,\\
&\mathcal{P}^1_q\colon A^+\big(s^q, r\big) + A^-\big(s^q, r\big) + a^m\big(s^q, r\big) + b^m\big(s^{q-1}, r\big) + c^m\big(s^{q-2}, r\big) = 0, \quad q \geq 2.
\end{aligned}
\tag{4}
$$

The first problem $\mathcal{P}^1_0$ of (4) represents the so-called limit problem, which reads

$$
\begin{cases} \text{Find } s^0 = \big(\mathbf{u}^0, \varphi^0\big) \in \mathbf{W}(\Omega) \times W(\Omega) \text{ such that} \\ A^-\big(s^0, r\big) + A^+\big(s^0, r\big) + a^m\big(s^0, r\big) = L(r), \end{cases} \tag{5}
$$

for all $r \in \mathbf{W}(\Omega) \times W(\Omega)$, where

$$
\begin{aligned}
\mathbf{W}(\Omega) := \big\{&\mathbf{v} \in L^2\big(\Omega; \mathbb{R}^3\big);\ \mathbf{v}^\pm \in H^1\big(\Omega^\pm; \mathbb{R}^3\big),\\
&\partial_3\mathbf{v}^m \in L^2\big(\Omega^m; \mathbb{R}^3\big),\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{mD},\ \mathbf{v}^\pm = \mathbf{v}^m \text{ on } S^\pm\big\},\\
W(\Omega) := \big\{&v \in L^2(\Omega);\ v^\pm \in H^1\big(\Omega^\pm\big),\\
&\partial_3 v^m \in L^2\big(\Omega^m\big),\ v = 0 \text{ on } \Gamma_{eD},\ v^\pm = v^m \text{ on } S^\pm\big\}.
\end{aligned}
$$

The limit problem (5) can be simplified if one considers the structure of the bilinear form $a^m(\cdot, \cdot)$, which involves only the derivatives along the x_3-coordinates. Indeed, by choosing test functions v_i and $\psi \in \mathcal{D}(\Omega^m)$, one has

$$\int_{\Omega^m} \left\{ \left(\mathbf{C}\partial_3 \mathbf{u}^{0,m} + \mathbf{p}\partial_3 \varphi^{0,m} \right) \cdot \partial_3 \mathbf{v} + \left(H\partial_3 \varphi^{0,m} - \mathbf{p} \cdot \partial_3 \mathbf{u}^{0,m} \right) \partial_3 \psi \right\} dx = 0,$$

where $\mathbf{C} := (C^m_{i3j3})$, $\mathbf{p} := (P^m_{3i3})$ and $H := H^m_{33}$ are introduced for the compact notation of the problem. The previous variational equation implies the existence of two constant functions with respect to the x_3-coordinate, namely, $\mathbf{z} = \mathbf{z}(\tilde{x})$ and $w = w(\tilde{x})$, with $\tilde{x} = (x_\alpha)$, such that

$$\begin{cases} \mathbf{C}\partial_3 \mathbf{u}^{0,m} + \mathbf{p}\partial_3 \varphi^{0,m} = \mathbf{z}, \\ H\partial_3 \varphi^{0,m} - \mathbf{p} \cdot \partial_3 \mathbf{u}^{0,m} = w. \end{cases} \tag{6}$$

Thus we can compute explicitly $\partial_3 \mathbf{u}^0$ and $\partial_3 \varphi^0$ by solving (6):

$$\partial_3 \mathbf{u}^{0,m} = \mathbf{A}\mathbf{z} - \mathbf{b}w, \qquad \partial_3 \varphi^{0,m} = \mathbf{b} \cdot \mathbf{z} + kw, \tag{7}$$

where

$$k := \frac{1}{H + \mathbf{p} \cdot \mathbf{C}^{-1}\mathbf{p}}, \qquad \mathbf{b} := k\mathbf{C}^{-1}\mathbf{p}, \qquad \mathbf{A} := \mathbf{C}^{-1} - k\mathbf{C}^{-1}\mathbf{p} \otimes \mathbf{C}^{-1}\mathbf{p}.$$

By integrating expressions (7) along x_3 between $-h$ and h, we obtain, as customary,

$$\begin{aligned} \mathbf{u}^{0,m}\big|_{S^+} - \mathbf{u}^{0,m}\big|_{S^-} &:= [\![\mathbf{u}^0]\!] = 2h(\mathbf{A}\mathbf{z} - \mathbf{b}w), \\ \varphi^{0,m}\big|_{S^+} - \varphi^{0,m}\big|_{S^-} &:= [\![\varphi^0]\!] = 2h(\mathbf{b} \cdot \mathbf{z} + kw), \end{aligned}$$

which allows us to fully characterize $\mathbf{z}$ and w as functions of the jumps of the displacement field and electric potential at the interface between Ω^+ and Ω^-, as follows

$$\mathbf{z} = \mathbf{K}[\![\mathbf{u}^0]\!] + \mathbf{q}[\![\varphi^0]\!], \qquad w = p[\![\varphi^0]\!] - \mathbf{q} \cdot [\![\mathbf{u}^0]\!], \tag{8}$$

where

$$p := \frac{1}{2h(k + \mathbf{b} \cdot \mathbf{A}^{-1}\mathbf{b})}, \qquad \mathbf{q} := p\mathbf{A}^{-1}\mathbf{b}, \qquad \mathbf{K} := \frac{1}{2h}\mathbf{A}^{-1} - p\mathbf{A}^{-1}\mathbf{b} \otimes \mathbf{A}^{-1}\mathbf{b}.$$

By means of the Sherman-Morrison formula, we can easily invert matrix $\mathbf{A}$ and, after some computations, we can express $\mathbf{K}$, $\mathbf{q}$ and p in terms of the electromechanical coefficients $\mathbf{C}$, $\mathbf{p}$ and H. Hence,

$$p := \frac{H}{2h}, \qquad \mathbf{q} := \frac{\mathbf{b}}{2h}, \qquad \mathbf{K} := \frac{\mathbf{C}}{2h}.$$

Consequently, one has

$$\mathbf{z} = \frac{1}{2h}\left(\mathbf{C}[\![\mathbf{u}^0]\!] + \mathbf{p}[\![\varphi^0]\!] \right), \qquad w = \frac{1}{2h}\left(H[\![\varphi^0]\!] - \mathbf{p} \cdot [\![\mathbf{u}^0]\!] \right). \tag{9}$$

This implies that $\partial_3 \mathbf{u}^{0,m} = \frac{[\![\mathbf{u}^0]\!]}{2h}$ and $\partial_3 \varphi^{0,m} = \frac{[\![\varphi^0]\!]}{2h}$, and thus, $\mathbf{u}^{0,m}$ and $\varphi^{0,m}$ become two affine functions of x_3. By using the continuity conditions on S^+ and S^- of the displacement field and the electric potential and after an integration by parts on x_3, we get

$$a^m(s^0, r) = \int_{S^+} (\mathbf{z} \cdot \mathbf{v}^+ + w\psi^+) d\Gamma - \int_{S^-} (\mathbf{z} \cdot \mathbf{v}^- + w\psi^-) d\Gamma.$$

Hence, using expressions (8) and by identifying S^+ and S^- with the interface ω, the limit problem can be reformulated in the following reduced form:

$$\begin{cases} \text{Find } s^0 = \left(\mathbf{u}^0, \varphi^0\right) \in \tilde{\mathbf{W}}(\Omega) \times \tilde{W}(\Omega) \text{ such that} \\ A^-\left(s^0, r\right) + A^+\left(s^0, r\right) + \tilde{a}^m\left(s^0, r\right) = L(r), \end{cases} \tag{10}$$

for all $r \in \tilde{\mathbf{W}}(\Omega) \times \tilde{W}(\Omega)$, where

$$\tilde{\mathbf{W}}(\Omega) := \left\{\mathbf{v} \in L^2\left(\Omega; \mathbb{R}^3\right);\ \mathbf{v}^\pm \in H^1\left(\Omega^\pm; \mathbb{R}^3\right),\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{mD}\right\},$$
$$\tilde{W}(\Omega) := \left\{v \in L^2(\Omega);\ v^\pm \in H^1\left(\Omega^\pm\right),\ v = 0 \text{ on } \Gamma_{eD}\right\},$$

and

$$\tilde{a}^m\left(s^0, r\right) := \frac{1}{2h}\int_\omega \left\{\left(\mathbf{C}\left[\!\left[\mathbf{u}^0\right]\!\right] + \mathbf{p}\left[\!\left[\varphi^0\right]\!\right]\right)\cdot\left[\!\left[\mathbf{v}\right]\!\right] + \left(H\left[\!\left[\varphi^0\right]\!\right] - \mathbf{p}\cdot\left[\!\left[\mathbf{u}^0\right]\!\right]\right)\left[\!\left[\psi\right]\!\right]\right\} d\tilde{x}.$$

Remark 1 Thanks to the asymptotic analysis, we transform the limit problem into a coupled electromechanical interface problem between Ω^+ and Ω^-, with non classical transmission conditions at the interface ω. This problem represents a piezoelectric generalization of the one obtained for weak linear elastic interfaces in [16]. We rewrite problem (10) in its differential form and we obtain:

Electrostatic problems in $\Omega^\pm$

$$\begin{cases} \partial_i D_i^\pm\left(\mathbf{u}^0, \varphi^0\right) = \rho_e & \text{in } \Omega^\pm, \\ D_i^\pm\left(\mathbf{u}^0, \varphi^0\right) n_i = -d & \text{on } \Gamma_{eN}, \\ \varphi^0 = 0 & \text{on } \Gamma_{eD}, \end{cases}$$

Elasticity problems in $\Omega^\pm$

$$\begin{cases} -\partial_j \sigma_{ij}^\pm\left(\mathbf{u}^0, \varphi^0\right) = f_i & \text{in } \Omega^\pm, \\ \sigma_{ij}^\pm\left(\mathbf{u}^0, \varphi^0\right) n_j = g_i & \text{on } \Gamma_{mN}, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \Gamma_{mD}. \end{cases}$$

Transmission conditions on ω

$$\begin{cases} \sigma_{i3}^+ = -\dfrac{1}{2h}\left(C_{i3j3}^m\left[\!\left[u_j^0\right]\!\right] + P_{3i3}^m\left[\!\left[\varphi^0\right]\!\right]\right) & \text{on } \omega, \\ \sigma_{i3}^- = -\dfrac{1}{2h}\left(C_{i3j3}^m\left[\!\left[u_j^0\right]\!\right] + P_{3i3}^m\left[\!\left[\varphi^0\right]\!\right]\right) & \text{on } \omega, \\ D_3^+ = \dfrac{1}{2h}\left(H_{33}^m\left[\!\left[\varphi^0\right]\!\right] - P_{3i3}^m\left[\!\left[u_i^0\right]\!\right]\right) & \text{on } \omega, \\ D_3^- = \dfrac{1}{2h}\left(H_{33}^m\left[\!\left[\varphi^0\right]\!\right] - P_{3i3}^m\left[\!\left[u_i^0\right]\!\right]\right) & \text{on } \omega, \end{cases}$$

which can be rewritten, following [15],

$$\begin{cases} \left[\!\left[\sigma_{i3}\right]\!\right] = 0 & \text{on } \omega, \\ \left[\!\left[D_3\right]\!\right] = 0 & \text{on } \omega, \\ \sigma_{i3}^+ + \dfrac{1}{h}\left(C_{i3j3}^m u_j^{0,+} + P_{3i3}^m \varphi^{0,+}\right) = -\sigma_{i3}^- + \dfrac{1}{h}\left(C_{i3j3}^m u_j^{0,-} + P_{3i3}^m \varphi^{0,-}\right) & \text{on } \omega, \\ D_3^+ - \dfrac{1}{h}\left(H_{33}^m \varphi^{0,+} - P_{3i3}^m u_i^{0,+}\right) = -D_3^- - \dfrac{1}{h}\left(H_{33}^m \varphi^{0,-} - P_{3i3}^m u_i^{0,-}\right) & \text{on } \omega. \end{cases}$$

Remark 2 We are now in position to compute the stresses and electric displacements in Ω^m by using the constitutive law $\sigma_{ij}^{m,\varepsilon} = C_{ijk\ell}^{\varepsilon} e_{k\ell}^{m,\varepsilon}(\mathbf{u}^\varepsilon) - P_{kij}^{m,\varepsilon} \partial_k^\varepsilon \varphi^\varepsilon$ and $D_i^{m,\varepsilon} = P_{ijk}^{m,\varepsilon} e_{jk}^\varepsilon(\mathbf{u}^\varepsilon) - H_{ij}^{m,\varepsilon} \partial_j^\varepsilon \varphi^\varepsilon$. By applying the rescaling method, one has

$$\begin{cases} \sigma_{ij}^m(\varepsilon) = C_{ijk3}^m \partial_3 u_k(\varepsilon) + P_{3ij}^m \partial_3 \varphi(\varepsilon) + \varepsilon\big(C_{ijk\alpha}^m \partial_\alpha u_k(\varepsilon) + P_{\alpha ij}^m \partial_\alpha \varphi(\varepsilon)\big), \\ D_i^m(\varepsilon) = P_{ik3}^m \partial_3 u_k(\varepsilon) - H_{i3}^m \partial_3 \varphi(\varepsilon) + \varepsilon\big(P_{ik\alpha}^m \partial_\alpha u_k(\varepsilon) - H_{i\alpha}^m \partial_\alpha \varphi(\varepsilon)\big). \end{cases}$$

The asymptotic expansions method allows to look for the stresses and electric displacements as series of powers of ε, so that

$$\begin{cases} \sigma_{ij}(\varepsilon) = \sigma_{ij}^0 + \varepsilon \sigma_{ij}^1 + \varepsilon^2 \sigma_{ij}^2 + \dots, \\ D_i(\varepsilon) = D_i^0 + \varepsilon D_i^1 + \varepsilon^2 D_i^2 + \dots. \end{cases} \tag{11}$$

By using (3) and (11), and by identifying the terms with identical power, we obtain

$$\begin{cases} \sigma_{ij}^{0,m} = C_{ijk3}^m \partial_3 u_k^0 + P_{3ij}^m \partial_3 \varphi^0 = \dfrac{1}{2h}\big(C_{ijk3}^m [\![u_k^0]\!] + P_{3ij}^m [\![\varphi^0]\!]\big), \\ D_i^{0,m} = P_{ik3}^m \partial_3 u_k^0 - H_{i3}^m \partial_3 \varphi^0 = \dfrac{1}{2h}\big(P_{ik3}^m [\![u_k^0]\!] - H_{i3}^m [\![\varphi^0]\!]\big), \end{cases}$$

whereas

$$\begin{cases} \sigma_{ij}^{0,\pm} = C_{ijk\ell}^\pm e_{k\ell}(\mathbf{u}^0) + P_{kij}^\pm \partial_k \varphi^0, \\ D_i^{0,\pm} = P_{ijk}^\pm e_{jk}(\mathbf{u}^0) - H_{ij}^\pm \partial_j \varphi^0. \end{cases}$$

4.1 Strong Convergence Results

In the sequel we denote by $\|\cdot\|_{s,\Omega}$ the norm of the Sobolev space $H^s(\Omega;\mathbb{R}^d)$ for all $d \geq 1$ and $|\cdot|_{0,\Omega}$ stands for the norm in $L^2(\Omega;\mathbb{R}^d)$. Obviously, the same holds in $\Omega^\pm$, Ω^m and ω. Moreover, we introduce the norm $\|\cdot\|_W$, defined as follows

$$\|s\|_W^2 := \|\mathbf{u}^+\|_{1,\Omega^+}^2 + \|\mathbf{u}^-\|_{1,\Omega^-}^2 + |\partial_3 \mathbf{u}^m|_{0,\Omega^m}^2 + \|\varphi^+\|_{1,\Omega^+}^2 + \|\varphi^-\|_{1,\Omega^-}^2 + |\partial_3 \varphi^m|_{0,\Omega^m}^2.$$

By definition of space $\mathbf{W}(\Omega) \times W(\Omega)$, we point out that the traces on $S^\pm$ of elements of $\mathbf{W}(\Omega) \times W(\Omega)$ are well defined, which justifies the fact that the presence of $|\mathbf{u}^m|_{0,\Omega^m}$ and $|\varphi^m|_{0,\Omega^m}$ is not necessary in the definition of $\|\cdot\|_W$, (see, also, Theorem 1 of [16]). The space $\mathbf{W}(\Omega) \times W(\Omega)$, endowed with the norm $\|\cdot\|_W$, becomes an Hilbert space.

The main result of this section is stated in the following theorem:

Theorem 1 *Under the regularity assumptions*

$$\partial_\alpha u_i^{0,m} \in L^2(\Omega^m), \qquad \partial_\alpha \varphi^{0,m} \in L^2(\Omega^m), \tag{12}$$

the sequence $(s(\varepsilon))_{\varepsilon>0} = ((\mathbf{u}(\varepsilon))_{\varepsilon>0}, (\varphi(\varepsilon))_{\varepsilon>0})$ *strongly converges to* $s^0 = (\mathbf{u}^0, \varphi^0)$ *in* $\mathbf{W}(\Omega) \times W(\Omega)$, *the solution of limit problem* (5).

Proof For the sake of clarity, the proof is divided into two steps, numbered from (i) to (ii).

(i) Let $p=1$ into the rescaled problem (2) and let us choose test functions $r=s(\varepsilon)=(\mathbf{u}(\varepsilon),\varphi(\varepsilon))$, so that we have

$$\begin{aligned}&A^{-}\big(s(\varepsilon),s(\varepsilon)\big)+A^{+}\big(s(\varepsilon),s(\varepsilon)\big)+a^{m}\big(s(\varepsilon),s(\varepsilon)\big)\\&\quad+\varepsilon b^{m}\big(s(\varepsilon),s(\varepsilon)\big)+\varepsilon^{2}c^{m}\big(s(\varepsilon),s(\varepsilon)\big)=L\big(s(\varepsilon)\big).\end{aligned}$$

By virtue of the coercivity properties of tensors $(C_{ijk\ell})$ and (H_{ij}), by means of Korn's and Poincaré's inequalities and thanks to the continuity of the linear form $L(\cdot)$, there exist two positive constants c_1 and c_2 such that

$$\begin{aligned}&c_1\Big\{\big\|s(\varepsilon)\big\|_W^2+\varepsilon^2\big|\partial_\alpha\mathbf{u}^m(\varepsilon)\big|_{0,\Omega^m}^2+\varepsilon^2\big|\partial_\alpha\varphi^m(\varepsilon)\big|_{0,\Omega^m}^2\Big\}\\
&\le c_1\Bigg\{\sum_{i,j}\big|e_{ij}\big(\mathbf{u}^{\pm}(\varepsilon)\big)\big|_{0,\Omega^{\pm}}^2+\sum_i\big|\partial_i\varphi^{\pm}(\varepsilon)\big|_{0,\Omega^{\pm}}^2+\big|\partial_3\mathbf{u}^m(\varepsilon)\big|_{0,\Omega^m}^2\\
&\quad+\big|\partial_3\varphi^m(\varepsilon)\big|_{0,\Omega^m}^2+\varepsilon^2\big|\partial_\alpha\mathbf{u}^m(\varepsilon)\big|_{0,\Omega^m}^2+\varepsilon^2\big|\partial_\alpha\varphi^m(\varepsilon)\big|_{0,\Omega^m}^2\Bigg\}\\
&\le A^{-}\big(s(\varepsilon),s(\varepsilon)\big)+A^{+}\big(s(\varepsilon),s(\varepsilon)\big)+a^{m}\big(s(\varepsilon),s(\varepsilon)\big)\\
&\quad+\varepsilon b^{m}\big(s(\varepsilon),s(\varepsilon)\big)+\varepsilon^{2}c^{m}\big(s(\varepsilon),s(\varepsilon)\big)=L\big(s(\varepsilon)\big)\le c_2\big\|s(\varepsilon)\big\|_W.\end{aligned}$$

As a consequence, the rescaled problem for $p=1$ is coercive in $\mathbf{V}(\Omega,\Gamma_{mD})\times V(\Omega,\Gamma_{eD})$ with respect to the norm defined on the left-hand side of the above inequality. Besides, one can get the following a priori bounds:

$$\big\|s(\varepsilon)\big\|_W\le c,\qquad \varepsilon\big|\partial_\alpha\mathbf{u}^m(\varepsilon)\big|_{0,\Omega^m}\le c,\qquad \varepsilon\big|\partial_\alpha\varphi^m(\varepsilon)\big|_{0,\Omega^m}\le c. \tag{13}$$

Besides, by definition of the $\mathbf{W}(\Omega)\times W(\Omega)$-norm, we can also claim that

$$\big\|\mathbf{u}^{\pm}(\varepsilon)\big\|_{1,\Omega^{\pm}}\le c,\qquad \big\|\varphi^{\pm}(\varepsilon)\big\|_{1,\Omega^{\pm}}\le c,\qquad \big|\partial_3\mathbf{u}^m(\varepsilon)\big|_{0,\Omega^m}\le c,\qquad \big|\partial_3\varphi^m(\varepsilon)\big|_{0,\Omega^m}\le c.$$

Therefore, we deduce that $s(\varepsilon)=(\mathbf{u}(\varepsilon),\varphi(\varepsilon))\rightharpoonup s=(\mathbf{u},\varphi)$ in $\mathbf{W}(\Omega)\times W(\Omega)$, meaning that $\mathbf{u}^{\pm}(\varepsilon)\rightharpoonup\mathbf{u}^{\pm}$ in $H^1(\Omega^{\pm};\mathbb{R}^3)$ and $\varphi^{\pm}(\varepsilon)\rightharpoonup\varphi^{\pm}$ in $H^1(\Omega^{\pm})$. Moreover, $\partial_3\mathbf{u}^m(\varepsilon)\rightharpoonup\mathbf{z}$ in $L^2(\Omega^m;\mathbb{R}^3)$ and $\partial_3\varphi^m(\varepsilon)\rightharpoonup w$ in $L^2(\Omega^m)$. Thanks to the H^1-boundedness of the sequences $(\mathbf{u}^{\pm}(\varepsilon))_{\varepsilon>0}$ and by virtue of the continuity of the trace operator, from expression

$$\mathbf{u}^m(\varepsilon)(\tilde{x},x_3)=\mathbf{u}^m(\varepsilon)|_{S^-}+\int_{-h}^{x_3}\partial_3\mathbf{u}^m(\varepsilon)(\tilde{x},\xi)d\xi,$$

it follows that there exist two positive constants, c_3 and c_4, such that $|\mathbf{u}^m(\varepsilon)|_{0,\Omega^m}\le c_3\{|\mathbf{u}^-(\varepsilon)|_{0,S^-}+|\partial_3\mathbf{u}^m(\varepsilon)|_{0,\Omega^m}\}\le c_4$. This implies that $\mathbf{u}^m(\varepsilon)\rightharpoonup\mathbf{u}^m$ in $L^2(\Omega^m;\mathbb{R}^3)$. As a consequence, $\varepsilon\mathbf{u}^m(\varepsilon)\to\mathbf{0}$ in $L^2(\Omega^m;\mathbb{R}^3)$.

Thanks to (13), we also have that $\varepsilon\partial_\alpha\mathbf{u}^m(\varepsilon)\rightharpoonup\mathbf{z}_\alpha$ in $L^2(\Omega^m;\mathbb{R}^3)$ and, by continuity of the derivative operator, we can infer that $\mathbf{z}_\alpha=\mathbf{0}$ in $L^2(\Omega^m;\mathbb{R}^3)$.

Same arguments can be used to prove the L^2-boundedness of $\varphi^m(\varepsilon)$, i.e., $\varphi^m(\varepsilon)\rightharpoonup\varphi^m$ in $L^2(\Omega^m)$, and $\varepsilon\partial_\alpha\varphi^m(\varepsilon)\to 0$ in $L^2(\Omega^m)$.

By means of the above weak convergence results, we can now compute the limit problem, by letting ε tend to zero in (2),

$$A^-(s,r)+A^+(s,r)+a^m(s,r)=L(r),$$

and, by uniqueness of the limit problem, we can identify the weak limit s with the leading term of the asymptotic expansion s^0, i.e., $\mathbf{u} = \mathbf{u}^0$ and $\varphi = \varphi^0$.

(ii) By hypothesis (12), we can use $s^0 = (\mathbf{u}^0, \varphi^0)$ as a test function in (2), and so if we set $\bar{s} := s(\varepsilon) - s^0$, namely, $\bar{\mathbf{u}} := \mathbf{u}(\varepsilon) - \mathbf{u}^0$ and $\bar{\varphi} := \varphi(\varepsilon) - \varphi^0$, we find

$$\begin{aligned}
&c\left\{\|\bar{s}\|_W^2 + \varepsilon^2\left|\partial_\alpha \bar{\mathbf{u}}^m\right|_{0,\Omega^m}^2 + \varepsilon^2\left|\partial_\alpha \bar{\varphi}^m\right|_{0,\Omega^m}^2\right\} \\
&\quad \le A^-(\bar{s}, \bar{s}) + A^+(\bar{s}, \bar{s}) + a^m(\bar{s}, \bar{s}) + \varepsilon b^m(\bar{s}, \bar{s}) + \varepsilon^2 c^m(\bar{s}, \bar{s}) \\
&\quad = L(\bar{s}) - A^-\left(s^0, \bar{s}\right) - A^+\left(s^0, \bar{s}\right) - a^m\left(s^0, \bar{s}\right) - \varepsilon b^m\left(s^0, \bar{s}\right) - \varepsilon^2 c^m\left(s^0, \bar{s}\right).
\end{aligned}$$

By taking the limit when ε tends to zero, we obtain that the right-hand side of the inequality converges to zero, and, hence, $\|\bar{s}\|_W \le c$, which implies the strong convergence of the sequence $(s(\varepsilon))_{\varepsilon>0}$ in $\mathbf{W}(\Omega) \times W(\Omega)$. This completes the proof. $\square$

5 The *Strong* Piezoelectric Interface: The Case of $p = -1$

In the sequel we identify the strong piezoelectric interface problem. By choosing $p = -1$, we obtain the following set of variational problems:

$$\begin{aligned}
&\mathcal{P}_{-2}^{-1}\colon a^m\left(s^0, r\right) = 0, \\
&\mathcal{P}_{-1}^{-1}\colon a^m\left(s^1, r\right) + b^m\left(s^0, r\right) = 0, \\
&\mathcal{P}_{0}^{-1}\colon A^+\left(s^0, r\right) + A^-\left(s^0, r\right) + a^m\left(s^2, r\right) + b^m\left(s^1, r\right) + c^m\left(s^0, r\right) = L(r), \\
&\mathcal{P}_{q}^{-1}\colon A^+\left(s^q, r\right) + A^-\left(s^q, r\right) + a^m\left(s^{q+2}, r\right) + b^m\left(s^{q+1}, r\right) + c^m\left(s^q, r\right) = 0, \quad q \ge 1.
\end{aligned}$$

Let us consider problem $\mathcal{P}_{-2}^{-1}$. By choosing test functions $r \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD})$, one has, using the compact notation,

$$\int_{\Omega^m} \left\{\left(\mathbf{C}\partial_3 \mathbf{u}^{0,m} + \mathbf{p}\partial_3 \varphi^{0,m}\right) \cdot \partial_3 \mathbf{v} + \left(H\partial_3 \varphi^{0,m} - \mathbf{p} \cdot \partial_3 \mathbf{u}^{0,m}\right)\partial_3 \psi\right\} dx = 0,$$

which is satisfied when $\mathbf{C}\partial_3 \mathbf{u}^{0,m} + \mathbf{p}\partial_3 \varphi^{0,m} = \mathbf{0}$ and $H\partial_3 \varphi^{0,m} - \mathbf{p} \cdot \partial_3 \mathbf{u}^{0,m} = 0$. Hence, $\partial_3 \mathbf{u}^{0,m} = \mathbf{0}$ and $\partial_3 \varphi^{0,m} = 0$ and so, $\mathbf{u}^{0,m}$ and $\varphi^{0,m}$ are both independent of x_3, i.e., $\mathbf{u}^{0,m} = \mathbf{u}^{0,m}(\tilde{x})$ and $\varphi^{0,m} = \varphi^{0,m}(\tilde{x})$.

Considering problem $\mathcal{P}_{-1}^{-1}$ with test functions $r \in \mathbf{V}(\Omega, \Gamma_{mD}) \times V(\Omega, \Gamma_{eD})$, we get

$$\begin{aligned}
&\int_{\Omega^m} \left\{\left(\mathbf{C}\partial_3 \mathbf{u}^{1,m} + \mathbf{p}\partial_3 \varphi^{1,m} + \left(\mathbf{C}^\alpha\right)^T \partial_\alpha \mathbf{u}^{0,m} + \mathbf{p}_3^\alpha \partial_\alpha \varphi^{0,m}\right) \cdot \partial_3 \mathbf{v}\right. \\
&\qquad \left. + \left(H\partial_3 \varphi^{1,m} - \mathbf{p} \cdot \partial_3 \mathbf{u}^{1,m} - \mathbf{p}_\alpha^3 \cdot \partial_\alpha \mathbf{u}^{0,m} + H^\alpha \partial_\alpha \varphi^{0,m}\right)\partial_3 \psi\right\} dx = 0,
\end{aligned}$$

where $\mathbf{C}^\alpha := (C_{i3j\alpha}^m)$, $\mathbf{p}_3^\alpha := (P_{\alpha i3}^m)$, $\mathbf{p}_\alpha^3 := (P_{3i\alpha}^m)$ and $H^\alpha := H_{\alpha 3}^m$. The previous variational problem is verified if

$$\begin{cases}
\mathbf{C}\partial_3 \mathbf{u}^{1,m} + \mathbf{p}\partial_3 \varphi^{1,m} = -\left(\mathbf{C}^\alpha\right)^T \partial_\alpha \mathbf{u}^{0,m} - \mathbf{p}_3^\alpha \partial_\alpha \varphi^{0,m}, \\
H\partial_3 \varphi^{1,m} - \mathbf{p} \cdot \partial_3 \mathbf{u}^{1,m} = \mathbf{p}_\alpha^3 \cdot \partial_\alpha \mathbf{u}^{0,m} - H^\alpha \partial_\alpha \varphi^{0,m}.
\end{cases} \tag{14}$$

Now we can easily compute $\partial_3 \mathbf{u}^{1,m}$ and $\partial_3 \varphi^{1,m}$, depending on $\partial_\alpha \mathbf{u}^{0,m}$ and $\partial_\alpha \varphi^{0,m}$, as follows

$$\partial_3 \mathbf{u}^{1,m} = \mathbf{K}^\alpha \partial_\alpha \mathbf{u}^{0,m} + \mathbf{q}_3^\alpha \partial_\alpha \varphi^{0,m}, \qquad \partial_3 \varphi^{1,m} = \mathbf{q}_\alpha^3 \cdot \partial_\alpha \mathbf{u}^{0,m} + p^\alpha \partial_\alpha \varphi^{0,m}, \tag{15}$$

with

$$\mathbf{K}^\alpha := -\left(\mathbf{A}\left(\mathbf{C}^\alpha\right)^T + \mathbf{b} \otimes \mathbf{p}_\alpha^3\right), \qquad \mathbf{q}_3^\alpha := -\mathbf{A}\mathbf{p}_3^\alpha + H^\alpha \mathbf{b},$$
$$\mathbf{q}_\alpha^3 := -\left(\mathbf{C}^\alpha\right)^T \mathbf{b} + k\mathbf{p}_\alpha^3, \qquad p^\alpha := -\left(\mathbf{b} \cdot \mathbf{p}_3^\alpha + kH^\alpha\right).$$

Finally we are in position to characterize the limit problem. Let us consider problem $\mathcal{P}_0^{-1}$ and let us choose test function $r \in \mathbf{X}(\Omega) \times X(\Omega)$, where

$$\begin{aligned}\mathbf{X}(\Omega) := \big\{\mathbf{v} \in L^2\left(\Omega; \mathbb{R}^3\right);\ \mathbf{v}^\pm \in H^1\left(\Omega^\pm; \mathbb{R}^3\right),\ L^2\left(\Omega^m; \mathbb{R}^3\right) \ni \partial_3 \mathbf{v}^m = \mathbf{0},\\ \mathbf{v}^m \in H^1\left(\Omega^m; \mathbb{R}^3\right),\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{mD},\ \mathbf{v}^\pm = \mathbf{v}^m \text{ on } S^\pm\big\},\\ X(\Omega) := \big\{v \in L^2(\Omega);\ v^\pm \in H^1\left(\Omega^\pm\right),\ L^2\left(\Omega^m\right) \ni \partial_3 v^m = 0,\\ v^m \in H^1\left(\Omega^m\right),\ v = 0 \text{ on } \Gamma_{eD},\ v^\pm = v^m \text{ on } S^\pm\big\}.\end{aligned}$$

This functional space represents the space of the admissible limit electromechanical states. Thus, $\mathcal{P}_0^{-1}$ takes the following simplified form

$$\begin{aligned}A^\pm\left(s^0, r\right) + \int_{\Omega^m} \big\{\left(\left(\mathbf{C}^{\beta\alpha}\right)^T \partial_\beta \mathbf{u}^{0,m} + \mathbf{p}^{\beta\alpha} \partial_\beta \varphi^{0,m} + \left(\mathbf{C}^\alpha\right)^T \partial_3 \mathbf{u}^{1,m} + \mathbf{p}_\alpha^3 \partial_3 \varphi^{1,m}\right) \cdot \partial_\alpha \mathbf{v}\\ + \left(H^{\beta\alpha} \partial_\beta \varphi^{0,m} - \mathbf{p}^{\alpha\beta} \cdot \partial_\beta \mathbf{u}^{0,m} - \mathbf{p}_3^\alpha \cdot \partial_3 \mathbf{u}^{1,m} + H^\alpha \partial_3 \varphi^{1,m}\right) \partial_\alpha \psi\big\} dx = 0,\end{aligned} \tag{16}$$

where $\mathbf{C}^{\alpha\beta} := (C^m_{i\alpha j\beta})$, $\mathbf{p}^{\alpha\beta} := (P^m_{\alpha\beta i})$ and $H^{\alpha\beta} := H^m_{\alpha\beta}$. By substituting expression (15) in problem (16) we obtain, as customary, the limit problem:

$$\begin{cases}\text{Find } s^0 = \left(\mathbf{u}^0, \varphi^0\right) \in \mathbf{X}(\Omega) \times X(\Omega) \text{ such that}\\ A^-\left(s^0, r\right) + A^+\left(s^0, r\right) + B^m\left(s^0, r\right) = L(r),\end{cases} \tag{17}$$

for all $r \in \mathbf{X}(\Omega) \times X(\Omega)$, where

$$\begin{aligned}B^m\left(s^0, r\right) := \int_{\Omega^m} \big\{&\left(\tilde{\mathbf{C}}^{\beta\alpha} \partial_\beta \mathbf{u}^0 + \tilde{\mathbf{p}}^{\beta\alpha} \partial_\beta \varphi^0\right) \cdot \partial_\alpha \mathbf{v}\\ &+ \left(\tilde{H}^{\beta\alpha} \partial_\beta \varphi^0 - \tilde{\mathbf{p}}^{\alpha\beta} \cdot \partial_\beta \mathbf{u}^0\right) \partial_\alpha \psi\big\} dx.\end{aligned}$$

Coefficients $\tilde{\mathbf{C}}^{\beta\alpha}$, $\tilde{\mathbf{p}}^{\beta\alpha}$ and $\tilde{H}^{\beta\alpha}$ are defined as follows

$$\begin{aligned}\tilde{\mathbf{C}}^{\beta\alpha} &:= \left(\mathbf{C}^{\beta\alpha}\right)^T + \left(\mathbf{C}^\alpha\right)^T \mathbf{K}^\beta + \mathbf{p}_\alpha^3 \otimes \mathbf{q}_\beta^3,\\ \tilde{\mathbf{p}}^{\beta\alpha} &:= \mathbf{p}^{\beta\alpha} + \left(\mathbf{C}^\alpha\right)^T \mathbf{q}_3^\beta + p^\beta \mathbf{p}_\alpha^3,\\ \tilde{H}^{\beta\alpha} &:= H^{\beta\alpha} - \mathbf{p}_3^\alpha \cdot \mathbf{q}_3^\beta + p^\beta H^\alpha.\end{aligned}$$

One can note that the space $\mathbf{X}(\Omega) \times X(\Omega)$ is isomorphic to $\tilde{\mathbf{X}}(\tilde{\Omega}) \times \tilde{X}(\tilde{\Omega})$, where

$$\tilde{\mathbf{X}}(\tilde{\Omega}) := \big\{\mathbf{v} \in H^1\left(\tilde{\Omega}; \mathbb{R}^3\right),\ \mathbf{v}|_\omega \in H^1\left(\omega; \mathbb{R}^3\right),\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_{mD}\big\},$$

$$\tilde{X}(\tilde{\Omega}) := \left\{ v \in H^1(\tilde{\Omega}),\ v|_\omega \in H^1(\omega),\ v = 0 \text{ on } \Gamma_{eD} \right\}, \quad \tilde{\Omega} := \Omega^+ \cup \omega \cup \Omega^-.$$

Thus, we can integrate $B^m(\cdot,\cdot)$ along x_3 and obtain the reduced form of the limit problem:

$$\begin{cases} \text{Find } s^0 = (\mathbf{u}^0, \varphi^0) \in \tilde{\mathbf{X}}(\tilde{\Omega}) \times \tilde{X}(\tilde{\Omega}) \text{ such that} \\ A^-(s^0, r) + A^+(s^0, r) + \tilde{B}^m(s^0, r) = L(r), \end{cases}$$

for all $r \in \tilde{\mathbf{X}}(\tilde{\Omega}) \times \tilde{X}(\tilde{\Omega})$, with

$$\tilde{B}^m(s^0, r) := 2h \int_\omega \left\{ \left(\tilde{\mathbf{C}}^{\beta\alpha} \partial_\beta \mathbf{u}^0 + \tilde{\mathbf{p}}^{\beta\alpha} \partial_\beta \varphi^0 \right) \cdot \partial_\alpha \mathbf{v} \right.$$
$$\left. + \left(\tilde{H}^{\beta\alpha} \partial_\beta \varphi^0 - \tilde{\mathbf{p}}^{\alpha\beta} \cdot \partial_\beta \mathbf{u}^0 \right) \partial_\alpha \psi \right\} d\tilde{x}.$$

Remark 3 The variational limit problem results into a non classical transmission problem between Ω^+ and Ω^- with ad hoc transmission conditions at the interface ω. This problem represents a piezoelectric generalization of the Ventcel-type transmission conditions obtained for strong elastic interfaces in [6]. After an integration by parts we can rewrite problem (17) in its differential form, so that

Electrostatic problems in $\Omega^\pm$

$$\begin{cases} \partial_i D_i^\pm(\mathbf{u}^0, \varphi^0) = \rho_e & \text{in } \Omega^\pm, \\ D_i^\pm(\mathbf{u}^0, \varphi^0) n_i = -d & \text{on } \Gamma_{eN}, \\ \varphi^0 = 0 & \text{on } \Gamma_{eD}, \\ \tilde{D}_\alpha \nu_\alpha = 0 & \text{on } \gamma_{eN}, \end{cases}$$

Elasticity problems in $\Omega^\pm$

$$\begin{cases} -\partial_j \sigma_{ij}^\pm(\mathbf{u}^0, \varphi^0) = f_i & \text{in } \Omega^\pm, \\ \sigma_{ij}^\pm(\mathbf{u}^0, \varphi^0) n_j = g_i & \text{on } \Gamma_{mN}, \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \Gamma_{mD}, \\ \tilde{\sigma}_{\alpha i} \nu_\alpha = 0 & \text{on } \gamma_{mN}, \end{cases}$$

Transmission conditions on ω

$$\begin{cases} [\![\mathbf{u}^0]\!] = \mathbf{0} & \text{on } \omega, \\ [\![\varphi^0]\!] = 0 & \text{on } \omega, \\ [\![\sigma_{i3}]\!] = -2h \partial_\alpha \tilde{\sigma}_{\alpha i} & \text{on } \omega, \\ [\![D_3]\!] = -2h \partial_\alpha \tilde{D}_\alpha & \text{on } \omega, \end{cases}$$

where $\tilde{\sigma}_{\alpha i} := \tilde{C}_{ij}^{\beta\alpha} \partial_\beta u_j^0 + \tilde{p}_i^{\beta\alpha} \partial_\beta \varphi^0$ and $\tilde{D}_\alpha := \tilde{p}_i^{\beta\alpha} \partial_\beta u_i^0 - \tilde{H}^{\beta\alpha} \partial_\beta \varphi^0$ represent, respectively, the reduced two-dimensional interface stress tensor and electric displacement defined over ω and (ν_α) denotes the unit normal vector to the uncharged electromechanical boundaries $\gamma_{eN}, \gamma_{mN} \subset \partial\omega$.

As already pointed out in the Introduction, the choice of an isotropic scaling of the unknowns produces a different limit behavior of the solution compared with [23]. In the present case, the interphase behaves as a piezoelectric membrane and, thus, the Kirchhoff-Love flexural behavior is not captured. Indeed, the jump of the shear stresses $[\![\sigma_{\beta 3}]\!]$ at the interface is related to the divergence of a membrane stress tensor $(\tilde{\sigma}_{\alpha\beta})$ as in [23], while the jump of the normal stresses $[\![\sigma_{33}]\!]$ and the jump of the normal electric displacements $[\![D_3]\!]$ have quite different expressions. This is mostly due to the particular form of the limit displacement field, satisfying the Kirchhoff-Love kinematical assumptions, and of the limit electric potential, obtained in [23].

Remark 4 Let us estimate the stresses and electric displacements in Ω^m by using the constitutive law $\sigma_{ij}^{m,\varepsilon} = C_{ijk\ell}^{\varepsilon} e_{k\ell}^{m,\varepsilon}(\mathbf{u}^{\varepsilon}) - P_{kij}^{m,\varepsilon} \partial_k^{\varepsilon} \varphi^{\varepsilon}$ and $D_i^{m,\varepsilon} = P_{ijk}^{m,\varepsilon} e_{jk}^{\varepsilon}(\mathbf{u}^{\varepsilon}) - H_{ij}^{m,\varepsilon} \partial_j^{\varepsilon} \varphi^{\varepsilon}$. By applying the rescaling method, one has

$$\begin{cases} \sigma_{ij}^m(\varepsilon) = \dfrac{1}{\varepsilon^2}\big(C_{ijk3}^m \partial_3 u_k(\varepsilon) + P_{3ij}^m \partial_3 \varphi(\varepsilon)\big) + \dfrac{1}{\varepsilon}\big(C_{ijk\alpha}^m \partial_\alpha u_k(\varepsilon) + P_{\alpha ij}^m \partial_\alpha \varphi(\varepsilon)\big), \\ D_i^m(\varepsilon) = \dfrac{1}{\varepsilon^2}\big(P_{ik3}^m \partial_3 u_k(\varepsilon) - H_{i3}^m \partial_3 \varphi(\varepsilon)\big) + \dfrac{1}{\varepsilon}\big(P_{ik\alpha}^m \partial_\alpha u_k(\varepsilon) - H_{i\alpha}^m \partial_\alpha \varphi(\varepsilon)\big). \end{cases}$$

The asymptotic expansions method allows to look for the stresses and electric displacements as series of powers of ε, so that

$$\begin{cases} \sigma_{ij}^m(\varepsilon) = \dfrac{1}{\varepsilon^2}\sigma^{-2,m} + \dfrac{1}{\varepsilon}\sigma^{-1,m} + \sigma_{ij}^{0,m} + \varepsilon\sigma_{ij}^{1,m} + \dots, \\ D_i^m(\varepsilon) = \dfrac{1}{\varepsilon^2}D^{-2,m} + \dfrac{1}{\varepsilon}D^{-1,m} + D_{ij}^{0,m} + \varepsilon D_{ij}^{1,m} + \dots. \end{cases} \tag{18}$$

By using (3) and (18), and relations (15), by identifying the terms with identical power, we obtain

$$\begin{cases} \sigma_{ij}^{-2,m} = C_{ijk3}^m \partial_3 u_k^0 + P_{3ij}^m \partial_3 \varphi^0 = 0, \\ D_i^{-2,m} = P_{ik3}^m \partial_3 u_k^0 - H_{i3}^m \partial_3 \varphi^0 = 0, \\ \sigma_{ij}^{-1,m} = C_{ijk3}^m \partial_3 u_k^1 + P_{3ij}^m \partial_3 \varphi^1 + C_{ijk\alpha}^m \partial_\alpha u_k^0 + P_{\alpha ij}^m \partial_\alpha \varphi^0 = \mathbb{C}_{ijk\alpha}^m \partial_\alpha u_k^0 + \mathbb{P}_{\alpha ij}^m \partial_\alpha \varphi^0, \\ D_i^{-1,m} = P_{ik3}^m \partial_3 u_k^1 - H_{i3}^m \partial_3 \varphi^1 + P_{ik\alpha}^m \partial_\alpha u_k^0 - H_{i\alpha}^m \partial_\alpha \varphi^0 = \mathbb{P}_{ik\alpha}^m \partial_\alpha u_k^0 - \mathbb{H}_{i\alpha}^m \partial_\alpha \varphi^0. \end{cases}$$

Expressions above are thought as a first approximation of the stress field and the electric displacement field in Ω^m: in order to have a better estimation of both stresses and electric displacements, we need to characterize the successive terms of the asymptotic expansions for the displacement field and electric potential field, such as $\mathbf{u}^2$ and φ^2. For what concerns the stresses and electric displacements in $\Omega^{\pm}$ we obtain, as customary,

$$\begin{cases} \sigma_{ij}^{0,\pm} = C_{ijk\ell}^{\pm} e_{k\ell}\big(\mathbf{u}^0\big) + P_{kij}^{\pm} \partial_k \varphi^0, \\ D_i^{0,\pm} = P_{ijk}^{\pm} e_{jk}\big(\mathbf{u}^0\big) - H_{ij}^{\pm} \partial_j \varphi^0. \end{cases}$$

5.1 Strong Convergence Results

Let us define the following norm:

$$|||s|||_{1,\Omega}^2 := \left\|\mathbf{u}^+\right\|_{1,\Omega^+}^2 + \left\|\mathbf{u}^-\right\|_{1,\Omega^-}^2 + \left\|\mathbf{u}^m\right\|_{1,\Omega^m}^2 + \left\|\varphi^+\right\|_{1,\Omega^+}^2 + \left\|\varphi^-\right\|_{1,\Omega^-}^2 + \left\|\varphi^m\right\|_{1,\Omega^m}^2,$$

which represents the natural norm for the space $\mathbf{X}(\Omega) \times X(\Omega)$. The main result of this section is claimed in the following theorem:

Theorem 2 *The sequence* $(s(\varepsilon))_{\varepsilon>0} = ((\mathbf{u}(\varepsilon))_{\varepsilon>0}, (\varphi(\varepsilon))_{\varepsilon>0})$ *strongly converges to* $s^0 = (\mathbf{u}^0, \varphi^0)$ *in* $H^1(\Omega; \mathbb{R}^3) \times H^1(\Omega)$*, the solution of limit problem* (17).

Proof For convenience, the proof is divided into three steps, numbered from (i) to (iii).

(i) Let $p=-1$ in the rescaled problem (2) and let us choose test functions $r=s(\varepsilon)=(\mathbf{u}(\varepsilon),\varphi(\varepsilon))$, so that we have

$$A^-\big(s(\varepsilon),s(\varepsilon)\big)+A^+\big(s(\varepsilon),s(\varepsilon)\big)+\frac{1}{\varepsilon^2}a^m\big(s(\varepsilon),s(\varepsilon)\big)$$
$$+\frac{1}{\varepsilon}b^m\big(s(\varepsilon),s(\varepsilon)\big)+c^m\big(s(\varepsilon),s(\varepsilon)\big)=L\big(s(\varepsilon)\big). \tag{19}$$

By virtue of the coercivity properties of tensors $(C_{ijk\ell})$ and (H_{ij}), by means of Korn's and Poincaré's inequalities and thanks to the continuity of the linear form $L(\cdot)$, there exist two positive constants c_1 and c_2 such that

$$\begin{aligned}
c_1|||s(\varepsilon)|||^2_{1,\Omega} &\le c_1\Big\{\big\|\mathbf{u}^+(\varepsilon)\big\|^2_{1,\Omega^+}+\big\|\mathbf{u}^-(\varepsilon)\big\|^2_{1,\Omega^-}+\big\|\varphi^+(\varepsilon)\big\|^2_{1,\Omega^+}+\big\|\varphi^-(\varepsilon)\big\|^2_{1,\Omega^-}\\
&\quad+\frac{1}{\varepsilon^2}\big|\partial_3\mathbf{u}^m(\varepsilon)\big|^2_{0,\Omega^m}+\frac{1}{\varepsilon^2}\big|\partial_3\varphi^m(\varepsilon)\big|^2_{0,\Omega^m}+\big|\partial_\alpha\mathbf{u}^m(\varepsilon)\big|^2_{0,\Omega^m}+\big|\partial_\alpha\varphi^m(\varepsilon)\big|^2_{0,\Omega^m}\Big\}\\
&\le c_1\Big\{\sum_{i,j}\big|e_{ij}\big(\mathbf{u}^\pm(\varepsilon)\big)\big|^2_{0,\Omega^\pm}+\sum_i\big|\partial_i\varphi^\pm(\varepsilon)\big|^2_{0,\Omega^\pm}+\frac{1}{\varepsilon^2}\big|\partial_3\mathbf{u}^m(\varepsilon)\big|^2_{0,\Omega^m}\\
&\quad+\frac{1}{\varepsilon^2}\big|\partial_3\varphi^m(\varepsilon)\big|^2_{0,\Omega^m}+\big|\partial_\alpha\mathbf{u}^m(\varepsilon)\big|^2_{0,\Omega^m}+\big|\partial_\alpha\varphi^m(\varepsilon)\big|^2_{0,\Omega^m}\Big\}\\
&\le A^-\big(s(\varepsilon),s(\varepsilon)\big)+A^+\big(s(\varepsilon),s(\varepsilon)\big)+\frac{1}{\varepsilon^2}a^m\big(s(\varepsilon),s(\varepsilon)\big)\\
&\quad+\frac{1}{\varepsilon}b^m\big(s(\varepsilon),s(\varepsilon)\big)+c^m\big(s(\varepsilon),s(\varepsilon)\big)=L\big(s(\varepsilon)\big)\le c_2|||s(\varepsilon)|||_{1,\Omega}.
\end{aligned}$$

Thus, the rescaled problem for $p=-1$ is coercive in $\mathbf{V}(\Omega,\Gamma_{mD})\times V(\Omega,\Gamma_{eD})$ with respect to the norm $|||\cdot|||_{1,\Omega}$. The inequalities above imply that $|||s(\varepsilon)|||_{1,\Omega}\le c$, which means that

$$\big\|\mathbf{u}^\pm(\varepsilon)\big\|_{1,\Omega^\pm}\le c,\qquad \big\|\varphi^\pm(\varepsilon)\big\|_{1,\Omega^\pm}\le c,\qquad \big\|\mathbf{u}^m(\varepsilon)\big\|_{1,\Omega^m}\le c,\qquad \big\|\varphi^m(\varepsilon)\big\|_{1,\Omega^m}\le c,$$

and, thus, $s(\varepsilon)=(\mathbf{u}(\varepsilon),\varphi(\varepsilon))\rightharpoonup s=(\mathbf{u},\varphi)$ in $H^1(\Omega;\mathbb{R}^3)\times H^1(\Omega)$, i.e., $\mathbf{u}^\pm(\varepsilon)\rightharpoonup\mathbf{u}^\pm$ in $H^1(\Omega^\pm;\mathbb{R}^3)$, $\varphi^\pm(\varepsilon)\rightharpoonup\varphi^\pm$ in $H^1(\Omega^\pm)$, $\mathbf{u}^m(\varepsilon)\rightharpoonup\mathbf{u}^m$ in $H^1(\Omega^m;\mathbb{R}^3)$ and $\varphi^m(\varepsilon)\rightharpoonup\varphi^m$ in $H^1(\Omega^m)$. Moreover, $e_{ij}(\mathbf{u}^\pm(\varepsilon))\rightharpoonup e_{ij}(\mathbf{u}^\pm)$ in $L^2(\Omega^\pm)$.

We also obtain the following a priori bounds:

$$\frac{1}{\varepsilon}\big|\partial_3\mathbf{u}^m(\varepsilon)\big|_{0,\Omega^m}\le c,\qquad \big|\partial_\alpha\mathbf{u}^m(\varepsilon)\big|_{0,\Omega^m}\le c,$$
$$\frac{1}{\varepsilon}\big|\partial_3\varphi^m(\varepsilon)\big|_{0,\Omega^m}\le c,\qquad \big|\partial_\alpha\varphi^m(\varepsilon)\big|_{0,\Omega^m}\le c,$$

which allows us to characterize the weak limit in Ω^m. Hence, one has that $\partial_3\mathbf{u}^m(\varepsilon)\to\mathbf{0}$ in $L^2(\Omega^m;\mathbb{R}^3)$, so $\partial_3\mathbf{u}^m=\mathbf{0}$, thanks to the H^1-bound of $\mathbf{u}^m$, meaning that $\mathbf{u}^m=\mathbf{u}^m(\tilde{x})$ is independent of x_3. Besides, $\partial_\alpha\mathbf{u}^m(\varepsilon)\rightharpoonup\partial_\alpha\mathbf{u}^m$ in $L^2(\Omega^m;\mathbb{R}^3)$ and $\frac{1}{\varepsilon}\partial_3\mathbf{u}^m(\varepsilon)\rightharpoonup\mathbf{w}$ in $L^2(\Omega^m;\mathbb{R}^3)$. Same arguments can be used for the weak convergence of the sequence $(\varphi(\varepsilon))_{\varepsilon>0}$: thus, $\partial_3\varphi^m(\varepsilon)\to 0$ in $L^2(\Omega^m)$, so $\partial_3\varphi^m=0$. Moreover, $\partial_\alpha\varphi^m(\varepsilon)\rightharpoonup\partial_\alpha\varphi^m$ in $L^2(\Omega^m)$ and $\frac{1}{\varepsilon}\partial_3\varphi^m(\varepsilon)\rightharpoonup\phi$ in $L^2(\Omega^m)$.

We can notice that, by virtue of the a priori bounds, we prove that the weak limit $s=(\mathbf{u},\varphi)$ belongs to the space $\mathbf{X}(\Omega)\times X(\Omega)$.

(ii) Now we can characterize the expression of the limit problem. Let us multiply the rescaled problem (19) by ε and let ε tends to zero, we find that

$$\int_{\Omega^m} \big\{ \big(\mathbf{C}\mathbf{w} + \mathbf{p}\phi + \big(\mathbf{C}^\alpha\big)^T \partial_\alpha \mathbf{u}^m + \mathbf{p}_3^\alpha \partial_\alpha \varphi^m \big) \cdot \partial_3 \mathbf{v} \\ + \big(H\phi - \mathbf{p}\cdot\mathbf{w} - \mathbf{p}_\alpha^3 \cdot \partial_\alpha \mathbf{u}^m + H^\alpha \partial_\alpha \varphi^m \big) \partial_3 \psi \big\} dx = 0.$$

Following the same procedure adopted in Sect. 4, by solving this variational problem, we can explicitly write $\mathbf{w}$ and ϕ in terms of $\partial_\alpha \mathbf{u}^m$ and $\partial_\alpha \varphi^m$, so that

$$\mathbf{w} = \mathbf{K}^\alpha \partial_\alpha \mathbf{u}^m + \mathbf{q}_3^\alpha \partial_\alpha \varphi^m, \qquad \phi = \mathbf{q}_\alpha^3 \cdot \partial_\alpha \mathbf{u}^m + p^\alpha \partial_\alpha \varphi^m. \tag{20}$$

Le us choose test functions $r = (\mathbf{v}, \psi) \in \mathbf{X}(\Omega) \times X(\Omega)$ in problem (19). By letting ε tend to zero and using relations (20), we obtain, after some mathematical technicalities, that the weak limit verifies the limit problem

$$\begin{cases} \text{Find } s = (\mathbf{u}, \varphi) \in \mathbf{X}(\Omega) \times X(\Omega) \text{ such that} \\ A^-(s,r) + A^+(s,r) + B^m(s,r) = L(r), \end{cases}$$

for all $r \in \mathbf{X}(\Omega) \times X(\Omega)$, having the same expression of problem (17). Hence, we can identify the weak limit $s = (\mathbf{u}, \varphi)$ with the leading term of the asymptotic expansion $s^0 = (\mathbf{u}^0, \varphi^0)$.

(iii) In order to prove the strong convergence, we adapt the procedure used in [5, 6]. We introduce a sequence $(q^\eta)_{\eta>0} = ((\mathbf{w}^\eta)_{\eta>0}, (\varphi^\eta)_{\eta>0}) \subset \mathcal{D}(\omega;\mathbb{R}^3) \times \mathcal{D}(\omega)$ which strongly converges in $L^2(\omega;\mathbb{R}^3) \times L^2(\omega)$ to $q = (\mathbf{w}, \phi)$, whose expressions are defined in (20). Namely, $\mathbf{w}^\eta \to \mathbf{w}$ in $L^2(\omega;\mathbb{R}^3)$ and $\varphi^\eta \to \phi$ in $L^2(\omega)$, as η tends to zero. Moreover, let us define a sequence $(\varsigma^\eta)_{\eta>0} = ((\mathbf{z}^\eta)_{\eta>0}, (\zeta^\eta)_{\eta>0}) \subset \mathcal{D}(\Omega;\mathbb{R}^3) \times \mathcal{D}(\Omega)$, such that $\mathbf{z}^\eta := x_3 \mathbf{w}^\eta(\tilde{x})$ and $\zeta^\eta := x_3 \varphi^\eta(\tilde{x})$ for all $x \in \Omega^m$. Being $s(\varepsilon) - s - \varepsilon\varsigma^\eta \in \mathbf{X}(\Omega) \times X(\Omega)$, one has

$$\begin{aligned} \mathcal{A}\big(s(\varepsilon) - s - \varepsilon\varsigma^\eta, s(\varepsilon) - s - \varepsilon\varsigma^\eta\big) := {} & A^\pm\big(s(\varepsilon) - s - \varepsilon\varsigma^\eta, s(\varepsilon) - s - \varepsilon\varsigma^\eta\big) \\ & + \frac{1}{\varepsilon^2} a^m\big(s(\varepsilon) - s - \varepsilon\varsigma^\eta, s(\varepsilon) - s - \varepsilon\varsigma^\eta\big) \\ & + \frac{1}{\varepsilon} b^m\big(s(\varepsilon) - s - \varepsilon\varsigma^\eta, s(\varepsilon) - s - \varepsilon\varsigma^\eta\big) \\ & + c^m\big(s(\varepsilon) - s - \varepsilon\varsigma^\eta, s(\varepsilon) - s - \varepsilon\varsigma^\eta\big). \end{aligned}$$

The coercivity implies that

$$\mathcal{A}\big(s(\varepsilon) - s - \varepsilon\varsigma^\eta, s(\varepsilon) - s - \varepsilon\varsigma^\eta\big) \geq c \big\|\big| s(\varepsilon) - s - \varepsilon\varsigma^\eta \big|\big\|^2_{1,\Omega}.$$

Besides, by applying the limit $\varepsilon \to 0$, we get, as customary,

$$\begin{aligned} & \lim_{\varepsilon\to 0} \mathcal{A}\big(s(\varepsilon) - s - \varepsilon\varsigma^\eta, s(\varepsilon) - s - \varepsilon\varsigma^\eta\big) \\ & \quad = L(s) - A^\pm(s,s) - c^m(s,s) \\ & \qquad + \int_{\Omega^m} \big\{ \mathbf{C}\mathbf{w}^\eta \cdot \big(\mathbf{w}^\eta - 2\mathbf{w}\big) + H\varphi^\eta\big(\varphi^\eta - 2\phi\big) - 2\big(\big(\mathbf{C}^\alpha\big)^T \mathbf{w} \cdot \partial_\alpha \mathbf{u} + H^\alpha \phi \partial_\alpha \varphi\big) \big\} dx. \end{aligned} \tag{21}$$

By taking the limit for $\eta \to 0$, by means of the strong convergence of $\mathbf{w}^\eta \to \mathbf{w}$ in $L^2(\omega; \mathbb{R}^3)$ and $\varphi^\eta \to \phi$ in $L^2(\omega)$, we can reconstruct, after some mathematical computations, the expression of the limit problem on the right-hand side of (21), namely, $L(s) - A^{\pm}(s,s) - B^m(s,s)$, and finally prove that it tends to zero. Then, by virtue of a diagonalization lemma, we deduce the existence of an application $\varepsilon \mapsto \eta(\varepsilon)$, verifying $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\lim_{\varepsilon \to 0} \mathcal{A}\big(s(\varepsilon) - s - \varepsilon \varsigma^{\eta(\varepsilon)}, s(\varepsilon) - s - \varepsilon \varsigma^{\eta(\varepsilon)}\big) = 0.$$

Thus the announced strong convergence result holds. □

6 Intermediate Cases

In this section we present some examples of intermediate cases between the formerly derived interface models. More precisely, we study the case of a rigid highly conducting interface with small piezoelectric coupling and the case of a rigid weakly conducting interface with small piezoelectric coupling. In the sequel we decide to skip the details of the asymptotic analysis, which is slightly analogous to those already presented in the previous sections, and proceed directly to the results.

Let us consider the first case and suppose the following scalings for the electromechanical coefficients of $\Omega^{m,\varepsilon}$:

$$C^{m,\varepsilon}_{ijk\ell} := \frac{1}{\varepsilon} C^m_{ijk\ell}, \qquad H^{m,\varepsilon}_{ij} := \frac{1}{\varepsilon} H^m_{ij}, \qquad P^{m,\varepsilon}_{ijk} := \varepsilon P^m_{ijk}.$$

The resulting limit model reads as follows:

$$\begin{cases} \text{Find } s^0 = (\mathbf{u}^0, \varphi^0) \in \tilde{\mathbf{X}}(\tilde{\Omega}) \times \tilde{X}(\tilde{\Omega}) \text{ such that} \\ A^-(s^0, r) + A^+(s^0, r) + \tilde{\tilde{B}}^m(s^0, r) = L(r), \end{cases}$$

for all $r \in \tilde{\mathbf{X}}(\tilde{\Omega}) \times \tilde{X}(\tilde{\Omega})$, with

$$\tilde{\tilde{B}}^m(s^0, r) := 2h \int_\omega \{\tilde{\tilde{\mathbf{C}}}^{\beta\alpha} \partial_\beta \mathbf{u}^0 \cdot \partial_\alpha \mathbf{v} + \tilde{\tilde{H}}^{\beta\alpha} \partial_\beta \varphi^0 \partial_\alpha \psi\} d\tilde{x},$$

where $\tilde{\tilde{\mathbf{C}}}^{\beta\alpha} := (\mathbf{C}^{\beta\alpha})^T - (\mathbf{C}^\beta)^T \mathbf{C}^{-1} (\mathbf{C}^\alpha)^T$ and $\tilde{\tilde{H}}^{\beta\alpha} := H^{\beta\alpha} - \frac{H^\alpha H^\beta}{H}$. As the reader can immediately notice, even though the material of the intermediate layer is piezoelectric, the resulting interface asymptotic model does not present any kind of coupled behavior between electricity and elasticity.

For what concerns with the second case, we choose this kind of scalings for the electromechanical coefficients of $\Omega^{m,\varepsilon}$, which correspond to a rigid interphase from a mechanical point of view, poorly conducting from the electric point of view and with small coupling:

$$C^{m,\varepsilon}_{ijk\ell} := \frac{1}{\varepsilon} C^m_{ijk\ell}, \qquad H^{m,\varepsilon}_{ij} := \varepsilon H^m_{ij}, \qquad P^{m,\varepsilon}_{ijk} := \varepsilon P^m_{ijk}.$$

The associated limit model takes the following form:

$$\begin{cases} \text{Find } s^0 = (\mathbf{u}^0, \varphi^0) \in \tilde{\mathbf{X}}(\tilde{\Omega}) \times \tilde{W}(\Omega) \text{ such that} \\ A^-(s^0, r) + A^+(s^0, r) + \tilde{\tilde{a}}^m(s^0, r) = L(r), \end{cases}$$

for all $r \in \tilde{\mathbf{X}}(\tilde{\Omega}) \times \tilde{W}(\Omega)$, with

$$\tilde{\tilde{a}}^m(s^0, r) := \int_\omega \left\{ 2h\tilde{\tilde{\mathbf{C}}}^{\beta\alpha} \partial_\beta \mathbf{u}^0 \cdot \partial_\alpha \mathbf{v} + \frac{1}{2hH} [\![\varphi^0]\!] [\![\psi]\!] \right\} d\tilde{x}.$$

Even in this case the interface model does not exhibit the initial piezoelectric coupling. Moreover, we can notice that the bilinear form, associated with the interface energy, is a precise combination of, respectively, the weak and strong interface models. The strong convergence of the solution of the rescaled problem towards the solution of the limit problem can be easily proved for the intermediate cases as well.

7 Concluding Remarks

In the present work we derive two limit interface models corresponding to a generic piezoelectric assembly with a piezoelectric interphase through an asymptotic analysis. We analyze two particular cases: the first case, for $p = 1$, corresponding from a mechanical point of view to a soft weakly conducting piezoelectric interphase, leads to a *weak* interface model; the latter, for $p = -1$, corresponding to a rigid highly conducting interphase into two piezoelectric media, leads to an *strong* interface model. In both cases, the interphase is replaced by a particular surface energy which is associated with ad hoc transmission conditions at the interface of the two bodies.

It is worth mentioning that these limit models are extremely versatile because they can capture the electromechanical behavior of different assemblies, just by varying the nature of the constituent materials. Here, we propose the more general situation, in which the multimaterial is constituted by three different anisotropic piezoelectric materials. However, we can adapt the model by using other material combinations: for instance, we can choose two elastic and conductor bodies separated by an intermediate piezoelectric layer, which could describe the behavior of a piezoelectric actuator embedded within a certain structural member.

As future developments, we would like to study more complex interface problems taking into account thermo-electromagnetoelastic couplings and time-dependent phenomena. Moreover we are numerically implementing the model by adapting the domain decomposition algorithm, presented in [14], to the piezoelectric case.

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References

1. Acerbi, E., Buttazzo, G., Percivale, D.: Thin inclusions in linear elasticity: a variational approach. J. Reine Angew. Math. **386**, 99–115 (1988)
2. Benveniste, Y.: An $O(h^N)$ interface model for a three-dimensional curved interphase in conduction phenomena. Proc. R. Soc. A **462**, 1619–1627 (2006)
3. Benveniste, Y.: Two models of three-dimensional thin interphases with variable conductivity and their fulfillment of the reciprocal theorem. J. Mech. Phys. Solids **60**(10), 1740–1752 (2012)
4. Benveniste, Y.: An interface model for a three-dimensional curved thin piezoelectric interphase between two piezoelectric media. Math. Mech. Solids **14**, 102–122 (2009)
5. Bessoud, A.-L.: Modélisations mathématiques d'un multi-matériau. Ph.D. Thesis, Université Montpellier II (2009)

6. Bessoud, A.-L., Krasucki, F., Michaille, G.: Multi-materials with strong interface: variational modelings. Asymptot. Anal. **1**, 1–19 (2009)
7. Bessoud, A.-L., Krasucki, F., Serpilli, M.: Asymptotic analysis of shell-like inclusions with high rigidity. J. Elast. **103**, 153–172 (2011)
8. Bessoud, A.-L., Krasucki, F., Serpilli, M.: Plate-like and shell-like inclusions with high rigidity. C. R. Acad. Sci. Paris Ser. I **346**, 697–702 (2008)
9. Ciarlet, P.G.: Mathematical Elasticity, vol. II: Theory of Plates. North-Holland, Amsterdam (1997)
10. Costa, L., Figueiredo, I., Leal, R., Oliveira, P., Stadler, G.: Modeling and numerical study of actuator and sensor effects for laminated piezoelectric plate. Comput. Struct. **85**, 385–403 (2007)
11. Fernandes, A., Pouget, J.: An accurate modelling of piezoelectric multi-layer plates. Eur. J. Mech. A, Solids **2**, 629–651 (2002)
12. Narra Figueiredo, I.M., Franco Leal, C.M.: A piezoelectric anisotropic plate model. Asymptot. Anal. **44**(3–4), 327–346 (2005)
13. Geis, W., Mishuris, G., Sändig, A.M.: Asymptotic models for piezoelectric stack actuators with thin metal inclusions. Berichte aus dem Institut für Angewandte Analysis und Numerische Simulation 2004/001, Preprint IANS, Univ. Stuttgart, Germany (2004)
14. Geymonat, G., Hendili, S., Krasucki, F., Serpilli, M., Vidrascu, M.: Asymptotic expansions and domain decomposition in Domain Decomposition Methods XXI. Lecture Notes in Computational Science and Engineering, vol. 98. Springer, Berlin (2014)
15. Geymonat, G., Krasucki, F., Marini, D., Vidrascu, M.: A domain decomposition method for a bonded structure. Math. Models Methods Appl. Sci. **8**, 1387–1402 (1998)
16. Geymonat, G., Krasucki, F., Lenci, S.: Mathematical analysis of a bonded joint with a soft thin adhesive. Math. Mech. Solids **4**, 201–225 (1999)
17. Krasucki, F., Münch, A., Ousset, Y.: Mathematical analysis of nonlinear bonded joint models. Math. Models Methods Appl. Sci. **14**, 1–22 (2004)
18. Lebon, F., Rizzoni, R.: Asymptotic analysis of a thin interface: the case involving similar rigidity. Int. J. Eng. Sci. **48**, 473–486 (2010)
19. Lebon, F., Rizzoni, R.: Asymptotic behavior of a hard thin linear interphase: An energy approach. Int. J. Solids Struct. **48**, 441–449 (2011)
20. Licht, C., Michaille, G.: A modelling of elastic adhesive bonded joints. Adv. Math. Sci. Appl. **7**(2), 711–740 (1997)
21. Raoult, A., Sène, A.: Modelling of piezoelectric plates including magnetic effects. Asymptot. Anal. **34**, 1–40 (2003)
22. Sène, A.: Modelling of piezoelectric static thin plates. Asymptot. Anal. **25**, 1–20 (2001)
23. Serpilli, M.: Asymptotic analysis of a multimaterial with a thin piezoelectric interphase. Meccanica **49**, 1641–1652 (2014)
24. Weller, T., Licht, C.: Asymptotic modeling of thin piezoelectric plates. Ann. Solid Struct. Mech. **1**, 173–188 (2010)
25. Weller, T., Licht, C.: Mathematical modeling of piezomagnetoelectric thin plates. Eur. J. Mech. A, Solids **29**, 928–937 (2010)