Asymptotic analysis of piezoelectric energy harvester

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1 Summary of the interested equations

Here we are interested in the classical model of a piezoelectric cantilever beam energy harvester, whose model is described using the following set of equations:

$$u'''' - \lambda^2 u = 0, \tag{1}$$

and the accompanying boundary conditions:

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) + \dfrac{j\lambda\beta\alpha^2}{j\lambda\beta + 1} u'(1) = 0 \\ u'''(1) = 0 \end{cases}, \tag{2}$$

where λ is the eigenvalues for the problem, u denotes the displace function of the cantilever beam, β is the dimensionless externally connected resistance, and α is the dimensionless piezoelectric coefficient. They can be expressed as follows

$$\lambda = \omega\sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \alpha = e_p \sqrt{\frac{l_p}{C_p B_p}}, \tag{3}$$

where ω is angular frequency, m_p is line mass density, l_p is the length of the cantilever beam, B_p is the bending stiffness, C_p is the inherent capacitance of the piezoelectric layer, e_p is the charge accumulation number, R_l is the externally connected resistance. In practical applications, dielectric property of piezoelectric materials indicate that the parameter β is changed from a very small value, which is close to a short-circuit condition to a very large value, which corresponds to an open-circuit condition. Thus we have that $0 \le \beta \le \infty$.

2 Asymptotic analysis when β is small

Here we seek to find the behavior of the above system at a small value of connected resistance, i.e., $\beta \sim 0$. In this case, we set that

$$\begin{aligned} \lambda_k &= \lambda_k^{(0)} + \beta\lambda_k^{(1)} + \beta^2\lambda_k^{(2)} + \cdots \\ \phi_k &= \phi_k^{(0)} + \beta\phi_k^{(1)} + \beta^2\phi_k^{(2)} + \cdots \end{aligned} \tag{4}$$

where λ_k and ϕ_k are the kth eigenvalue and eigenfunction respectively of the above mentioned system under perturbation. $\lambda_k^{(0)}$ and $\phi_k^{(0)}$ are the corresponding eigenvalue and eigenfunction of the unperturbed system at $\beta = 0$:

$$u'''' - \lambda^2 u = 0, \tag{5}$$

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \\ u''(1) = 0 \\ u'''(1) = 0 \end{cases}. \tag{6}$$

Obviously, the unperturbed system is a classical eigenvalue problem with the eigenvalues determined by

$$1 + cosh(\sqrt{\lambda})cos(\sqrt{\lambda}) = 0 \tag{7}$$

whose first several values are

$$\sqrt{\lambda_1}/\pi = 0.59686..., \sqrt{\lambda_2}/\pi = 1.49418..., \sqrt{\lambda_3}/\pi = 2.50025..., \sqrt{\lambda_4}/\pi = 3.49999..., ... \tag{8}$$