Revisit to the theoretical analysis of a classical piezoelectric cantilever energy harvester

Maoying Zhou

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1 Summary of the interested equations

The dynamic equations for a typical piezoelectric composite cantilever beam is

$$B_p \frac{\partial^4 w(x,t)}{\partial x^4} + m_p \frac{\partial^2 w(x,t)}{\partial t^2} = 0, \tag{1}$$

where B_p is the equivalent bending stiffness and m_p is the line mass density of the piezoelectric cantilever beam. If the piezoelectric elements attached to the cantilever beam is connected to an external electrical load R_l, we have

$$\frac{dQ_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{2}$$

For the underlying physics, we have the following constitutive equations

$$\begin{aligned} M_p(x,t) &= B_p \frac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\ q_p(x,t) &= e_p \frac{\partial^2 w(x,t)}{\partial x^2} + \varepsilon_p V_p(t), \end{aligned} \tag{3}$$

or equivalently,

$$\begin{cases} M_p(x,t) = B_p \dfrac{\partial^2 w(x,t)}{\partial x^2} - e_p V_p(t), \\ Q_p(x,t) = e_p \left[\dfrac{\partial w(x,t)}{\partial x} \right] \Bigg|_0^{l_p} + C_p V_p(t). \end{cases} \tag{4}$$

One end of the cantilever beam is fixed while the other end is free. So the boundary conditions are

$$\begin{cases} w(0,t) = w_b(t), \\ \dfrac{\partial w(0,t)}{\partial x} = 0, \end{cases} \tag{5}$$

and

$$\begin{cases} M_p(l_p,t) = B_p \dfrac{\partial^2 w(l_p,t)}{\partial x^2} - e_p V_p(t) = 0, \\ N_p(l_p,t) = \dfrac{\partial M_p(l_p,t)}{\partial x} = B_p \dfrac{\partial^3 w(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{6}$$

In the classical energy harvesting applications, the cantilever beam is subject to a periodical base excitation $w_b(t)$. Thus the dynamic response of the cantilever beam is decomposed as

$$w(x,t) = w_b(t) + w_{rel}(x,t), \tag{7}$$

where $w_{rel}(x,t)$ is the relative displacement function of the cantilever beam. In this way, the system is converted into

$$B_p \frac{\partial^4 w_{rel}(x,t)}{\partial x^4} + m_p \frac{\partial^2 w_{rel}(x,t)}{\partial t^2} = -m_p \frac{\partial^2 w_b(t)}{\partial t^2}, \tag{8}$$

$$e_p \left[\frac{\partial^2 w_{rel}(x,t)}{\partial x \partial t} \right] \Bigg|_0^{l_p} + C_p \frac{dV_p(t)}{dt} + \frac{V_p(t)}{R_l} = 0. \tag{9}$$

$$\begin{cases} w_{rel}(0,t) = 0, \\ \dfrac{\partial w_{rel}(0,t)}{\partial x} = 0, \end{cases} \tag{10}$$

and

$$\begin{cases} B_p \dfrac{\partial^2 w_{rel}(l_p,t)}{\partial x^2} - e_p V_p(t) = 0, \\ \dfrac{\partial^3 w_{rel}(l_p,t)}{\partial x^3} = 0. \end{cases} \tag{11}$$

Considering a sinusoidal base excitation

$$w_b(t) = \eta_b e^{j\sigma_b t} \tag{12}$$

where ξ_b is usually a real vibration amplitude, the steady state solution for the above system can be reasonably set as

$$w_{rel}(x,t) = \eta_{rel}(x) e^{j\sigma_b t}, \quad V_p(t) = \tilde{V}_p e^{j\sigma_b t}, \tag{13}$$

where $\eta_{rel}(x)$ and $\tilde{V}_p$ are complex amplitudes. Then the above system is again simplified as

$$B_p \frac{\partial^4 \eta_{rel}(x)}{\partial x^4} - m_p \sigma_b^2 \eta_{rel}(x) = m_p \sigma_b^2 \eta_b, \tag{14}$$

$$\begin{cases} \eta_{rel}(0) = 0, \\ \dfrac{\partial \eta_{rel}(0)}{\partial x} = 0, \end{cases} \tag{15}$$

and

$$\begin{cases} B_p \dfrac{\partial^2 \eta_{rel}(l_p)}{\partial x^2} + \dfrac{j\sigma_b R_l}{1 + j\sigma_b C_p R_l} e_p^2 \dfrac{\partial \eta_{rel}(l_p)}{\partial x} = 0, \\ \dfrac{\partial^3 \eta_{rel}(l_p)}{\partial x^3} = 0. \end{cases} \tag{16}$$

Note that here we assume a sinusoidal steady state response, which is not actually validated theoretically.

Obviously we can have the following dimensionless scheme:

$$\eta_{rel} \sim u\eta_b, \quad x \sim z l_p \tag{17}$$

and therefore the following dimensionless parameters

$$\sigma = \sigma_b \sqrt{\frac{m_p l_p^4}{B_p}}, \quad \beta = R_l C_p \sqrt{\frac{B_p}{m_p l_p^4}}, \quad \delta = \frac{e_p^2 l_p}{C_p B_p}. \tag{18}$$

Now, we reach the following dimensionless system of boundary value problem

$$\begin{cases} u'''' - \sigma^2 u = \sigma^2, \\ u(0) = 0, \\ u'(0) = 0, \\ u''(1) + \dfrac{j\beta\sigma}{1 + j\beta\sigma} \delta u'(1) = 0, \\ u'''(1) = 0, \end{cases} \tag{19}$$

where the prime denotes the derivative with respect to z. The analytical solution to this problem can be formulated as

$$u(z;\delta) = A_\delta \cos\sqrt{\sigma} z + B_\delta \sin\sqrt{\sigma} z + C_\delta \cosh\sqrt{\sigma} z + D_\delta \sinh\sqrt{\sigma} z - 1 \tag{20}$$

and hence

$$\begin{aligned} u'(z;\delta) &= \sigma^{1/2} \left(-A_\delta \sin\sqrt{\sigma} z + B_\delta \cos\sqrt{\sigma} z + C_\delta \sinh\sqrt{\sigma} z + D_\delta \cosh\sqrt{\sigma} z \right), \\ u''(z;\delta) &= \sigma \left(-A_\delta \cos\sqrt{\sigma} z - B_\delta \sin\sqrt{\sigma} z + C_\delta \cosh\sqrt{\sigma} z + D_\delta \sinh\sqrt{\sigma} z \right), \\ u'''(z;\delta) &= \sigma^{3/2} \left(A_\delta \sin\sqrt{\sigma} z - B_\delta \cos\sqrt{\sigma} z + C_\delta \sinh\sqrt{\sigma} z + D_\delta \cosh\sqrt{\sigma} z \right). \end{aligned} \tag{21}$$

The coefficients A_δ, B_δ, C_δ, and D_δ are then subject to the following linear system of equations:

$$\begin{cases} A_\delta + C_\delta = 1, \\ B_\delta + D_\delta = 0, \\ \left(-A_\delta \cos\sqrt{\sigma} - B_\delta \sin\sqrt{\sigma} + C_\delta \cosh\sqrt{\sigma} + D_\delta \sinh\sqrt{\sigma}\right) + \\ \dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\delta\left(-A_\delta \sin\sqrt{\sigma} + B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma}\right) = 0, \\ A_\delta \sin\sqrt{\sigma} - B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma} = 0. \end{cases} \tag{22}$$

Analytically, we can directly obtain the solution to this problem as

$$\begin{cases} A_\delta = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ B_\delta = \dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ C_\delta = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma} + \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}, \\ D_\delta = \dfrac{-\cos\sqrt{\sigma}\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}\cosh\sqrt{\sigma} - \frac{2j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\sin\sqrt{\sigma}\sinh\sqrt{\sigma}\right)}{2\left[1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)\right]}. \end{cases} \tag{23}$$

The resulting complex amplitudes $\tilde{V}_p$, $\tilde{I}_p$, and $\tilde{P}_p$ for output voltage $V_p(t)$, output current $I_p(t)$, and output power $P_p(t)$, respectively, can be formulated as follows

$$\begin{cases} \tilde{V}_p = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}u'(1), \\ \quad = -\dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}\sigma^{1/2}\left(-A_\delta \sin\sqrt{\sigma} + B_\delta \cos\sqrt{\sigma} + C_\delta \sinh\sqrt{\sigma} + D_\delta \cosh\sqrt{\sigma}\right) \\ \quad = \dfrac{j\sigma\beta}{j\sigma\beta+1}\dfrac{\eta_b}{l_p}\dfrac{e_p}{C_p}\dfrac{\sqrt{\sigma}\left(\sin\sqrt{\sigma} - \sinh\sqrt{\sigma}\right)}{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)} \\ \quad = \dfrac{j\sigma\beta}{j\sigma\beta+1}\left(\dfrac{\eta_b}{l_p}\right)\left(\dfrac{e_p}{C_p}\right)\chi_p, \\ \tilde{I}_p = \tilde{V}_p/R_l = \dfrac{j}{j\sigma\beta+1}\left(\dfrac{\eta_b}{l_p}\right)\left(e_p\sigma_b\right)\chi_p, \\ \tilde{P}_p = \tilde{V}_p^2/R_l = -\left(\dfrac{\eta_b}{l_p}\right)^2\left(\dfrac{e_p}{C_p}\right)\left(e_p\sigma_b\right)\dfrac{\sigma\beta}{\left(j\sigma\beta+1\right)^2}\chi_p, \end{cases} \tag{24}$$

in which we have used the notations that

$$\chi_p = \frac{\sqrt{\sigma}\left(\sin\sqrt{\sigma} - \sinh\sqrt{\sigma}\right)}{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\delta\left(\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}\right)} \tag{25}$$

In the following, we can obtain the corresponding displacement function $u(z)$ in terms of its amplitude and phase in Figure 1.

The solution function $u(z)$ is dependent upon three dimensionless parameters β, σ, and δ. β indicates the electric properties of the piezoelectric energy harvesting system, σ is the equivalent base excitation frequency and hence depicts the mechanical input part of the system, and δ reflects the electromechanical effect in the system. It is seen from the figure that the responsive displacement function $u(z)$ is tuned by the attached piezoelectric layers. Nonetheless, the tuning effect is minuscule in the way that compared to the non-piezoelectric case, amplitude of the displacement $u(z)$ is disturbed to a small degree, while the phase is little offset from a value of 0.

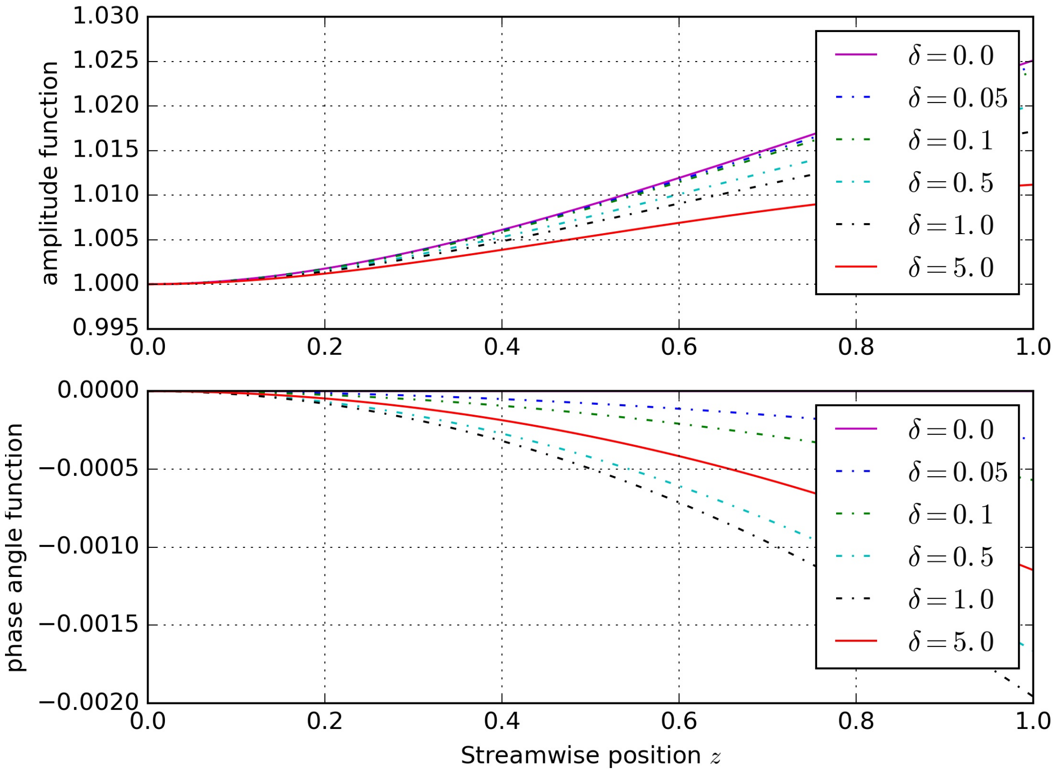

Figure 1: Amplitude and phase of the displacement function $u(z)$

Using the following regular expansion:

$$\begin{cases} A_\epsilon = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots, \\ B_\epsilon = B_0 + \epsilon B_1 + \epsilon^2 B_2 + \cdots, \\ C_\epsilon = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \cdots, \\ D_\epsilon = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \end{cases} \tag{26}$$

we obtain the successive expansion problem:
$O(\epsilon^0)$:

$$\begin{cases} A_0 + C_0 = 1, \\ B_0 + D_0 = 0, \\ -A_0 \cos\sqrt{\sigma} - B_0 \sin\sqrt{\sigma} + C_0 \cosh\sqrt{\sigma} + D_0 \sinh\sqrt{\sigma} = 0, \\ A_0 \sin\sqrt{\sigma} - B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma} = 0. \end{cases} \tag{27}$$

The solution is

$$\begin{cases} A_0 = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} - \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\ B_0 = \dfrac{\cosh\sqrt{\sigma}\sin\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\ C_0 = \dfrac{1 + \cos\sqrt{\sigma}\cosh\sqrt{\sigma} + \sin\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \\ D_0 = -\dfrac{\cosh\sqrt{\sigma}\sin\sqrt{\sigma} + \cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{2 + 2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}} \end{cases} \tag{28}$$

Hence we have

$$-A_0 \sin\sqrt{\sigma} + B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma} = \frac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma} + 1} \tag{29}$$

$O(\epsilon^1)$:

$$\begin{cases} A_1 + C_1 = 0, \\ B_1 + D_1 = 0, \\ \left(-A_1 \cos\sqrt{\sigma} - B_1 \sin\sqrt{\sigma} + C_1 \cosh\sqrt{\sigma} + D_1 \sinh\sqrt{\sigma}\right) + \\ \dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\left(-A_0 \sin\sqrt{\sigma} + B_0 \cos\sqrt{\sigma} + C_0 \sinh\sqrt{\sigma} + D_0 \cosh\sqrt{\sigma}\right) = 0, \\ A_1 \sin\sqrt{\sigma} - B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma} = 0. \end{cases} \tag{30}$$

The solution is

$$\begin{cases} A_1 = \dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ B_1 = \dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ C_1 = \dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(-\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ D_1 = \dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \end{cases} \tag{31}$$

Then we have

$$\begin{aligned} & -A_1 \sin\sqrt{\sigma} + B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma} \\ = & \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\frac{\sin\sqrt{\sigma} - \sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right) \end{aligned} \tag{32}$$

$O(\epsilon^2)$:

$$\begin{cases} A_2 + C_2 = 0, \\ B_2 + D_2 = 0, \\ \left(-A_2 \cos\sqrt{\sigma} - B_2 \sin\sqrt{\sigma} + C_2 \cosh\sqrt{\sigma} + D_2 \sinh\sqrt{\sigma}\right) + \\ \dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\left(-A_1 \sin\sqrt{\sigma} + B_1 \cos\sqrt{\sigma} + C_1 \sinh\sqrt{\sigma} + D_1 \cosh\sqrt{\sigma}\right) = 0, \\ A_2 \sin\sqrt{\sigma} - B_2 \cos\sqrt{\sigma} + C_2 \sinh\sqrt{\sigma} + D_2 \cosh\sqrt{\sigma} = 0. \end{cases} \tag{33}$$

The solution is

$$\begin{cases} A_2 = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ B_2 = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{-\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ C_2 = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(-\dfrac{\cos\sqrt{\sigma} + \cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ D_2 = \left(\dfrac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^2\left(\dfrac{\sinh\sqrt{\sigma} - \sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma} + \sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\dfrac{-\sin\sqrt{\sigma} + \sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \end{cases} \tag{34}$$

To get higher order expansions, we can use the following iteration method:
$O(\epsilon^{k+1})$ $(k \geq 1)$:

$$\begin{cases} A_{k+1} + C_{k+1} = 0, \\ B_{k+1} + D_{k+1} = 0, \\ \left(-A_{k+1} \cos\sqrt{\sigma} - B_{k+1} \sin\sqrt{\sigma} + C_{k+1} \cosh\sqrt{\sigma} + D_{k+1} \sinh\sqrt{\sigma}\right) + \\ \dfrac{j\beta\sqrt{\sigma}}{j\sigma\beta+1}\left(-A_k \sin\sqrt{\sigma} + B_k \cos\sqrt{\sigma} + C_k \sinh\sqrt{\sigma} + D_k \cosh\sqrt{\sigma}\right) = 0, \\ A_{k+1} \sin\sqrt{\sigma} - B_{k+1} \cos\sqrt{\sigma} + C_{k+1} \sinh\sqrt{\sigma} + D_{k+1} \cosh\sqrt{\sigma} = 0. \end{cases} \tag{35}$$

The solution is

$$\begin{cases} A_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\frac{-\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(-\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\frac{-\sin\sqrt{\sigma}+\sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right)(Q_k) \end{cases} \tag{36}$$

where for $k \geq 2$

$$Q_k = -A_k \sin\sqrt{\sigma} + B_k \cos\sqrt{\sigma} + C_k \sinh\sqrt{\sigma} + D_k \cosh\sqrt{\sigma}, \tag{37}$$

and for $k \geq 0$

$$\begin{aligned} Q_{k+1} &= -A_{k+1}\sin\sqrt{\sigma} + B_{k+1}\cos\sqrt{\sigma} + C_{k+1}\sinh\sqrt{\sigma} + D_{k+1}\cosh\sqrt{\sigma} \\ &= -\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}+\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)Q_k, \end{aligned} \tag{38}$$

and

$$\begin{aligned} Q_1 &= -A_1\sin\sqrt{\sigma} + B_1\cos\sqrt{\sigma} + C_1\sinh\sqrt{\sigma} + D_1\cosh\sqrt{\sigma} \\ &= \frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\left(\frac{\sin\sqrt{\sigma}-\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}\cosh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right) \end{aligned} \tag{39}$$

$$Q_0 = \frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1} \tag{40}$$

Hence it is shown that for $k \geq 0$

$$\begin{aligned} Q_k &= -\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}+\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)Q_k \\ &= \left[-\left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)\left(\frac{\sin\sqrt{\sigma}\cosh\sqrt{\sigma}+\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\right]^k\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right) \end{aligned} \tag{41}$$

As a result, we obtain that for $k \geq 0$

$$\begin{cases} A_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1}\left(\frac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma}-\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^k\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{\cos\sqrt{\sigma}+\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ B_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1}\left(\frac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma}-\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^k\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{-\sinh\sqrt{\sigma}+\sin\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ C_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1}\left(\frac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma}-\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^k\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{-\cos\sqrt{\sigma}-\cosh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \\ D_{k+1} = \left(\frac{j\beta\sqrt{\sigma}}{1+j\beta\sigma}\right)^{k+1}\left(\frac{-\sin\sqrt{\sigma}\cosh\sqrt{\sigma}-\cos\sqrt{\sigma}\sinh\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)^k\left(\frac{\sinh\sqrt{\sigma}-\sin\sqrt{\sigma}}{\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+1}\right)\left(\frac{-\sin\sqrt{\sigma}+\sinh\sqrt{\sigma}}{2\cos\sqrt{\sigma}\cosh\sqrt{\sigma}+2}\right) \end{cases} \tag{42}$$