Mechanics of Advanced Materials and Structures
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/umcm20

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Published online: 30 Nov 2010.

To cite this article: Raúl Guinovart-Díaz , Reinaldo Rodríguez-Ramos , Julián Bravo-Castillero & Federico J. Sabina (2003) Modeling of Three-Phase Fibrous Composite Using the Asymptotic Homogenization Method, Mechanics of Advanced Materials and Structures, 10:4, 319-333, DOI: 10.1080/10759410306753

To link to this article: http://dx.doi.org/10.1080/10759410306753

Mechanics of Advanced Materials and Structures, 10: 319–333, 2003

ISSN: 1537-6494 print / 1537-6532 online
DOI: 10.1080/15376490390231791

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ABSTRACT

A three-phase concentric fiber-reinforced periodic composite is considered wherein the constituents exhibit piezoelectric properties. The cross-section of the periodic cell is a regular hexagon with two concentric circles and the periodicity is the same in two directions at an angle $\pi/3$. Simple closed-form expressions are obtained for the effective properties of this composite by means of the asymptotic homogenization method. Numerical computations have been done. The analytical solution of the required resulting plane- and antiplane-strain local problems, which turns out to be only two, makes use of potential methods of a complex variable and properties of Weierstrass elliptic and related functions of periods (1,0) and (cos $\pi/3$, sin $\pi/3$). Benveniste's universal type of relations for this composite are satisfied. Comparison with other models is shown.

§1. INTRODUCTION

A very important question in the design of piezocomposites (i.e., a composite with two or more piezoelectric phases of known properties, volume fractions, and given geometry) is the prediction of its effective or overall properties. Various techniques have been used to compute the effective physical parameters of piezoelectric composites. For instance, [1] estimated these for a transversely isotropic piezocomposite with cylindrical inclusions randomly distributed using the self-consistent method where a composite coaxial cylinder is embedded in the as-yet-unknown equivalent material; the properties of the inner cylinder are of inclusion type and the outer one has matrix properties. Reference [2] predicted the effective properties of such piezocomposites using four micromechanical models, viz., the Mori-Tanaka, dilute, self-consistent, and differential methods. An approximate model for the calculation of global behavior of fibrous piezoelectric composites for ultrasonic medical

Received 3 January 2002; accepted 12 July 2002.
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transducer applications have been developed by [3]. This simple model captures important features of the phenomena of interest. It is easily implemented and has been extended to cover other applications [4–6], which includes underwater hydrophone transducers. From the theoretical point of view, universal relations relating the effective behavior of such binary piezoelectric composites were derived by [7] based on the theory of the creation of uniform fields in heterogeneous media by appropiate boundary conditions. Over the past two decades, piezoelectric ceramic/polymer composites with different connectivities have been developed for transducer applications such as hydrophones, biomedical imaging, nondestructive testing, and air imaging. Recently, much attention has been given to a fine scale for piezoelectric ceramic rods embedded in a passive polymer. These composites allow higher operating frequencies, and thus an increased resolution in medical imaging transducers [8]. Different mathematical rigorous techniques have been adopted to estimate the effective electroelastic moduli of these kinds of heterogeneous media: the variational method called Γ-convergence was used by [9]. Bloch expansions method was applied by [10] to the dynamical equations. The double-scale expansion method developed by [11] and [12] was applied in [13] to compute macrobehavior in thermopiezoelectric solids. In all these works the same general expressions for the homogenized electroelastic coefficients were obtained. Recently, on the basis of [13], exact formulae for the layered piezocomposites and approximate formulae for some complex geometrices are given by [14]. More references can be found in [15] and [16].

The asymptotic averaging method is a mathematically rigorous method of homogenization that enables the prediction of both the local and overall averaged properties of heterogeneous media whose field equations have rapidly oscillating coefficients which characterize the physical properties of the individual phases of the composite material. It has been applied to uncoupled elastic composites [17, 24]. This mathematical framework of homogenization also allows the prediction of the overall properties of 1–3 piezocomposite materials [25]. It has been applied to deal with 2–2 piezocomposites in [26].

General expressions of the overall properties for 1–3 piezoelectric composites were recently obtained by means of the asymptotic averaging method (see [15, 27].) The composites considered here contain unidirectional cylindrical fibers periodically distributed in a matrix. The constituents of these binary composites are piezoelectric and belong to the crystalline group of hexagonal symmetry 6 mm. The results are valid when the smallest dimension of the body are much larger than the characteristic size of the fibers and the separation between them. It is interesting to note that both the linearity of the piezoelectric constitutive relations and the physical symmetry properties of the phases reduce the solution of the initial problem to that of plane uncoupled elasticity. This means that a more simpler formulation is obtained for the calculation of electromechanical effective moduli since it is reduced to similar equations that arise in the purely elastic case [16]. These formulae are a generalization of those for the purely elastic media appearing in [17]. Numerical calculations show an adequate concordance with variational bounds. The values of the properties that are predicted by the homogenization formulation are compared with the values obtained using other approximate theories and existing experimental results, found in [5] and [6]. It is shown that the agreement is quite good, and for one of the properties, the homogenization calculation is even better than the approximate one, which fails to fit the experimental data.

Numerous works, such as Theocaris's studies [18, 19], display the presence of interphase, a zone of matrix close to fiber which has different properties from the matrix and the fiber. Moreover, its influence seems to be significant on the composite's performance. Interphases, or interfacial zones, in fiber-reinforced composite materials are the thin layers between the fiber and the matrix (Figure 1). These interphases are formed due to, for

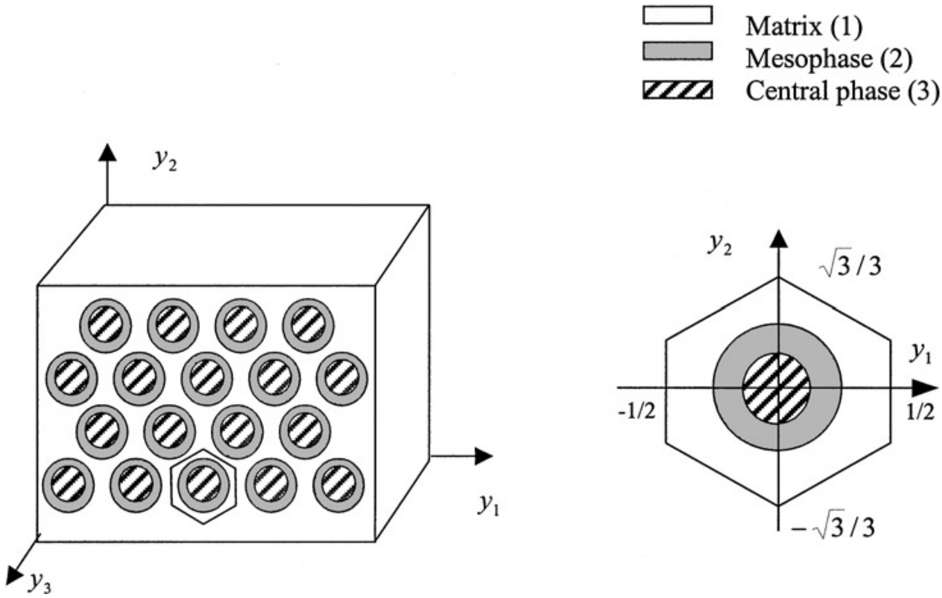

Figure 1. *Three-phase periodic fibrous piezoelectric composite and its hexagon periodic cell.*

example, chemical reaction between the matrix and fiber materials or the use of protective coatings on the fiber during manufacturing. Although small in thickness, interphases can significantly affect the overall mechanical properties of the fiber-reinforced composites, as observed in some different reports [20, 21]. It is the weakest link in the load path, and consequently most failures in fiber-reinforced composites, such as debonding, fiber pullout, and matrix cracking, occur in or near this region. Thus, it is crucial to fully understand the mechanism and effects of the interphases in fiber-reinforced composites. Numerical techniques such as the finite-element method and the boundary element method have been developed for these purposes [22, 23]. Homogenization methods could be used as a tool to evaluate the influence of such interphase on linear piezoelectric behavior.

The present paper constitutes a continuing study of the previous works [27, 33] and derives theoretical results for the overall moduli of three-phase concentric fiber-reinforced periodic composite using the double asymptotic homogenization. In section 2, a boundary value problem is posed for a piezoelectric medium with rapidly varying properties, which is reflected by the presence of a small parameter ϵ. The hybrid composite is considered. In section 3, the method of solution is described and two types of local problems, ${}_{pq}L$ and ${}_{p}L$ (p, q = 1, 2, 3), need to be solved in order to get the effective parameters. In section 4, an explanation of the two steps of homogenization is stated. The final formulae for the effective parameters is given in a relatively simpler closed form. In section 5, a discussion of the numerical calculations of the formulae is reported. Section 6 is devoted to some final remarks.

§2. Statement of the problem

A piezoelectric material is considered whose constitutive relations are given in the form

$$\begin{aligned} \sigma_{ij} &= C^{E}_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \\ D_i &= e_{ikl}\varepsilon_{kl} + \epsilon^{\varepsilon}_{ik}E_k, \end{aligned} \tag{1}$$

where the summation convention over repeated Latin indices is understood and i, j, k, and l run from 1 to 3. Here the second-order stress tensor σ_{ij} and the electric displacement vector D_i are linearly related to the second-order strain tensor ε_{ij} and the electric field vector E_i; the material properties are given by the fourth-order stiffness tensor C^E_{ijkl} measured at constant electric field C_{ijkl}, the third-order piezoelectric tensor e_{ijk}, and the second-order dielectric permittivity tensor $\epsilon^{\varepsilon}_{ij}$ measured at constant strain (see [28]). The usual superindices E, ε attached to them will not be used in what follows in order to have a less cumbersome notation. The material properties satisfy the following symmetries:

$$C_{ijkl} = C_{jikl} = C_{klij}, \quad e_{kij} = e_{kji}, \quad \epsilon_{ij} = \epsilon_{ji}. \tag{2}$$

The strain tensor components ε_{ij} and the displacement vector u_i are related through

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

A linear piezoelectric boundary value static problem for a heterogeneous medium occupying the domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ is stated. A displacement $u^{(\epsilon)}$ and electric potential $\varphi^{(\epsilon)}$ are sought in Ω such that they satisfy the field equations

$$\begin{aligned}
&\frac{\partial}{\partial x_j}\left(C_{ijkl}(\mathbf{x}/\epsilon)\frac{\partial u_k^{(\epsilon)}}{\partial x_l} + e_{kij}(\mathbf{x}/\epsilon)\frac{\partial \varphi^{(\epsilon)}}{\partial x_k}\right) + X_i = 0 \quad \text{in } \Omega,\\
&\frac{\partial}{\partial x_i}\left(e_{ikl}(\mathbf{x}/\epsilon)\frac{\partial u_k^{(\epsilon)}}{\partial x_l} - \epsilon_{ik}(\mathbf{x}/\epsilon)\frac{\partial \varphi^{(\epsilon)}}{\partial x_k}\right) = 0 \quad \text{in } \Omega,
\end{aligned} \tag{3}$$

where X_i is the body force per unit volume.

In addition, the boundary conditions on $\partial\Omega$ are prescribed as follows:

$$\begin{aligned}
&u_i^{(\epsilon)} = 0 \quad \text{on } \partial\Omega_0, \qquad \sigma_{ij}^{(\epsilon)} n_j = t_i^{(0)} \quad \text{on } \partial\Omega_1,\\
&\varphi^{(\epsilon)} = \varphi^{(0)} \quad \text{on } \partial\Omega_2, \qquad D_i^{(\epsilon)} n_i = 0 \quad \text{on } \partial\Omega_3,
\end{aligned} \tag{4}$$

where n_i is the outward unit normal vector to $\partial\Omega$, $t_i^{(0)}$ the traction vector on $\partial\Omega_1$, and $\varphi^{(0)}$ the electric potential on $\partial\Omega_2$. The boundary is partitioned in such a way that $\partial\Omega = \overline{\partial\Omega_0} \cup \overline{\partial\Omega_1} = \overline{\partial\Omega_2} \cup \overline{\partial\Omega_3}$ and $\partial\Omega_0 \cap \partial\Omega_1 = \partial\Omega_2 \cap \partial\Omega_3 = \emptyset$.

Now a three-phase composite is considered that comprises a matrix with homogeneous properties given by the following moduli tensors: elastic $C^{(1)}_{ijkl}$, piezoelectric $e^{(1)}_{ijk}$, and dielectric $\epsilon^{(1)}_{ij}$, in which two concentric parallel circular cylindrical fibers are embedded with corresponding homogeneous properties $C^{(2)}_{ijkl}$, $e^{(2)}_{ijk}$, and $\epsilon^{(2)}_{ij}$ for the intermediate fiber and $C^{(3)}_{ijkl}$, $e^{(3)}_{ijk}$, and $\epsilon^{(3)}_{ij}$ for the central fiber. The fibers are aligned in the x_3-direction and periodically distributed with period l in the direction of the Ox_1 axes and the line with slope at angle $\mu = \pi/3$. Effective properties of this piezocomposite are sought by means of the asymptotic homogenization method as outlined above. In view of this aim, a periodic hexagonal cell Y is chosen in the local coordinates $y = x/\epsilon$, where $\epsilon = l/L$ is a dimensionless parameter and L is a linear dimension of the body. The transverse section H of the periodic cell consists of a hexagon with two concentric single circular of radios R_1, R_2 ($R_2 \leq R_1$) centered at the origin on the plane (y_1, y_2), and they are denoted through H_2 and H_3, respectively. Its complement H_1 is occupied by matrix material; the interphases are defined as follows: $H_1 \cap H_2 = \Gamma_1$, $H_2 \cap H_3 = \Gamma_2$, as shown in Figure 1. The displacement, electric potential, traction, and normal electric displacement are continuous across the interfaces. In this particular case, the local problems describing the cell corresponding to the plane and antiplane formulation are decoupled [29]. The material moduli are piecewise-constant functions in H. It is assumed that they are periodic in y_1 and y_2, but not on y_3. Furthermore, the material properties C_{ijkl}, e_{kij}, and ϵ_{ij} are assumed to be Y-periodic functions in Ω.

§3. METHOD OF SOLUTION

The solution of the boundary-value problem (3)–(4) can be obtained using the well-known method of asymptotic homogenization using a two-scale asymptotic expansion [11, 12, 17, 24, 25, 30]. The solution is sought in the form of a series in powers of ϵ with coefficients depending on both the variables **x** and **y** treated as independent; they are referred to as the slow or macroscopic and fast or microscopic variables, respectively. Here, the solution is explicitly posed, as it is assumed that the solution has the form of an asymptotic expansion as $\epsilon \to 0$, which depends on both the fast **x** and the slow **y** coordinates as follows: (3), (4), with periodic functions, is sought

$$\begin{aligned} \mathbf{u}^{(\epsilon)}(\mathbf{x}) &= \mathbf{u}^{(0)}(\mathbf{x}, \mathbf{y}) + \epsilon \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) + \cdots, \\ \varphi^{(\epsilon)}(\mathbf{x}) &= \varphi^{(0)}(\mathbf{x}, \mathbf{y}) + \epsilon \varphi^{(1)}(\mathbf{x}, \mathbf{y}) + \cdots. \end{aligned} \tag{5}$$

The substitution of (5) into the Eqs. (3) and (4) and the comparison of similar powers of ϵ leads to the boundary value problems that are satisfied by $u_i^{(0)}$, $u_i^{(1)}$, $\varphi^{(0)}$, and $\varphi^{(1)}$. The relation $y = x/\epsilon$ must be considered when derivatives are computed so that the chain rule is applied. It is found, to order ϵ^{-2}, that the functions $u_i^{(0)}$ and $\varphi^{(0)}$ do not depend on y as it turns out to be the case in the corresponding elasticity problem (see [29]). To the next order of ϵ^{-1}, due to the linearity of this problem and the assumed regularity of both the inclusions shapes and the smoothness of the coefficients, a product solution type is found for $u^{(1)}$ and $\varphi^{(1)}$ as follows:

$$\begin{aligned} \mathbf{u}^{(1)}(\mathbf{x}, \mathbf{y}) &= {}_{pq}\mathbf{M}(\mathbf{y}) \frac{\partial u_p^{(0)}}{\partial x_q}(\mathbf{x}) + {}_{p}\mathbf{P}(y) \frac{\partial \varphi^{(0)}}{\partial x_q}(x), \\ \varphi^{(1)}(\mathbf{x}, \mathbf{y}) &= {}_{pq}N(\mathbf{y}) \frac{\partial u_p^{(0)}}{\partial x_q}(\mathbf{x}) + {}_{p}Q(\mathbf{y}) \frac{\partial \varphi^{(0)}}{\partial x_p}(\mathbf{x}), \end{aligned}$$

where the sets of pq-functions, ${}_{pq}M(y)$ and ${}_{pq}N(y)$, and p-functions, ${}_{p}P(y)$ and ${}_{p}Q(y)$, depend only on y. They are the unique periodic solution of the so-called pq-local and p-local problems denoted by ${}_{pq}L$ and ${}_{p}L$, respectively, over the periodic cell H (see local problems in [27] for two-phase composite).

3.1. *Local problems*

In case of three-phase composite, the solutions of the ${}_{pq}L$ problems can be reduced to find the solutions of two problems analogous to those reported in [27] for two-phase composite. It can be written in the following form:

$$ {}_{pq}\mathbf{M} = {}_{pq}^{1}\mathbf{M} + {}_{pq}^{2}\mathbf{M}, \quad {}_{pq}N = {}_{pq}^{1}N + {}_{pq}^{2}N, \tag{6}$$

where the functions ${}_{pq}^{\Upsilon}\mathbf{M}$ and ${}_{pq}^{\Upsilon}N$ are periodic of periods $\omega_1 = (1; 0)$ and $\omega_2 = (1/2; \sqrt{3}/2)$. For example, the functions ${}_{pq}^{1}\mathbf{M}$, ${}_{pq}^{1}N$ are the solutions of the following boundary-value problem (see [27]):

$$
\begin{aligned}
{}_{pq}\sigma^{(s)}_{i\delta,\delta} &= 0 \quad \text{in } H_s,\\
{}_{pq}D^{(s)}_{\delta,\delta} &= 0 \quad \text{in } H_s,\\
\left\|{}^{1}_{pq}M_i\right\|_1 &= 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad {}^{1}_{pq}M^{(2)}_i = 0 \quad \text{on } \Gamma_2,\\
\left\|{}^{1}_{pq}N\right\|_1 &= 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad {}^{1}_{pq}N^{(2)} = 0 \quad \text{on } \Gamma_2,\\
\left\|{}_{pq}\sigma_{i\delta}n_\delta\right\|_1 &= -\|C_{i\delta pq}\|_1\, n_\delta \quad \text{on } \Gamma_1,\\
{}_{pq}\sigma^{(2)}_{i\delta}n_\delta + C^{(2)}_{i\delta pq}n_\delta &= 0 \quad \text{on } \Gamma_2,\\
\|{}_{pq}D_\delta n_\delta\|_1 &= -\|e_{\delta_{pq}}\|_1 n_\delta \quad \text{on } \Gamma_1,\\
{}_{pq}D^{(2)}_\delta n_\delta + e^{(2)}_{\delta pq}n_\delta &= 0 \quad \text{on } \Gamma_2,\\
\left\langle {}^{1}_{pq}M_i\right\rangle &= \left\langle {}^{1}_{pq}N\right\rangle = 0,
\end{aligned}
\tag{7}
$$

where

$$
{}_{pq}\sigma^{(s)}_{i\delta} = C^{(s)}_{i\delta k\lambda}\ {}^{1}_{pq}M^{(s)}_{k,\lambda} + e^{(s)}_{\lambda i\delta}\ {}^{1}_{pq}N^{(s)}_{,\lambda}, \qquad {}_{pq}D^{(s)}_{\delta} = e^{(s)}_{\delta k\lambda}\ {}^{1}_{pq}M^{(s)}_{k,\lambda} - \epsilon^{(s)}_{\delta,\lambda}\ {}^{1}_{pq}N^{(s)}_{,\lambda}. \tag{8}
$$

The comma notation denotes a partial derivative relative to the $y\delta$ component (i.e., $U_{,\delta} \equiv \partial U/\partial y_\delta$); the summation convention is also understood for Greek indices, which run from 1 to 2; no summation is carried out over uppercase indices, whether Latin or Greek. The outward unit normal vector to the interface Γ_Υ is n. The double-bar notation denotes the jump of the function $f(y)$ across the interface Γ_Υ; i.e.,

$$
\begin{aligned}
\|f\|_1 &= f^{(1)}(\mathbf{y}) - f^{(2)}(\mathbf{y}) \quad \text{for } \mathbf{y} \in \Gamma_1,\\
\|f\|_2 &= f^{(3)}(\mathbf{y}) - f^{(2)}(\mathbf{y}) \quad \text{for } \mathbf{y} \in \Gamma_2,
\end{aligned}
$$

The angular brackets in Eq. (7) define the volume average per unit length over the cell; that is,

$$
\langle F\rangle = \frac{1}{|H|}\int_H F(\mathbf{y})\, d\mathbf{y},
$$

where the area of H is $|H|$. From Eq. (8), it is clear that ${}_{pq}\sigma_{i\delta}$ and ${}_{pq}D_\delta$ are the $i\delta$-components of the stress tensor and the δ-component of the electric displacement vector associated with the displacement ${}^{1}_{pq}M$ and the potential ${}^{1}_{pq}N$. The remaining components of stress and electric displacement follow immediately from the plane components of the displacement ${}^{1}_{pq}M_\alpha$ because it is a two-dimensional situation, as

$$
\begin{aligned}
{}_{pq}\sigma^{(s)}_{33} &= C^{(s)}_{3311}\ {}^{1}_{pq}M^{(s)}_{\alpha,\alpha},\\
{}_{pq}D^{(s)}_{3} &= e^{(s)}_{311}\ {}^{1}_{pq}M^{(s)}_{\alpha,\alpha}.
\end{aligned}
\tag{9}
$$

The contribution to the effective properties will only come from the solution of certain ${}_{pq}L$ inhomogeneous problems. The symmetry between the indices p and q shows right away that at most six problems need to be considered.

The functions ${}_{pq}^{2}\mathbf{M}$, ${}_{pq}^{2}N$ are solutions of an analogous problem to that reported in Eq. (7). The solution of the ${}_{p}L$ is found in a similar way as it was solved for the ${}_{pq}L$ problems. Therefore,

$$
{}_{p}\mathbf{P} = {}_{p}^{1}\mathbf{P} + {}_{p}^{2}\mathbf{P}, \quad {}_{p}Q = {}_{p}^{1}Q + {}_{p}^{2}Q, \tag{10}
$$

which are periodic of periods $\omega_1 = (1, 0)$ and $\omega_2 = (1/2; \sqrt{3}/2)$, that satisfy analogous boundary-value problem ${}_{p}L$ stated in [27].

Once the local problems ${}_{pq}L$ and ${}_{p}L$ are solved, the homogenized (effective) moduli are given by the following expressions:

$$
\begin{aligned}
\bar{C}_{ijpq} &= \langle C_{ijpq} + C_{ijkl}\ {}_{pq}M_{k,l} + e_{kij}\ {}_{pq}N_{,k} \rangle, \\
\bar{e}_{ipq} &= \langle e_{ipq} + e_{ikl}\ {}_{pq}M_{k,l} - \epsilon_{ik}\ {}_{pq}N_{,k} \rangle, \\
\bar{e}_{pij} &= \langle e_{pij} + C_{ijkl}\ {}_{p}P_{k,l} + e_{kij}\ {}_{p}Q_{,k} \rangle, \\
\bar{\epsilon}_{ip} &= \langle \epsilon_{ip} - e_{ikl}\ {}_{p}P_{k,l} + \epsilon_{ik}\ {}_{p}Q_{,k} \rangle,
\end{aligned} \tag{11}
$$

where the overbar refers to an overall property of the composite.

§4. Effective Coefficients

The constitutive relations (1) can also be written explicitly in terms of 10 independent parameters $k, l, n, p, m, q, r, s, t$, and u so that the nonzero terms in Eq. (1) become

$$
\begin{aligned}
\frac{1}{2}(\sigma_{11} + \sigma_{22}) &= k(\varepsilon_{11} + \varepsilon_{22}) + l\varepsilon_{33} + q\varphi_{,3}, \\
\sigma_{33} &= l(\varepsilon_{11} + \varepsilon_{22}) + n\varepsilon_{33} + r\varphi_{,3}, \\
\sigma_{11} - \sigma_{22} &= 2m(\varepsilon_{11} - \varepsilon_{22}), \\
\sigma_{23} &= 2p\varepsilon_{23} + s\varphi_{,2}, \\
\sigma_{13} &= 2p\varepsilon_{13} + s\varphi_{,1}, \\
\sigma_{12} &= 2m\varepsilon_{12}, \\
D_1 &= 2s\varepsilon_{12} - t\varphi_{,1}, \\
D_2 &= 2s\varepsilon_{23} - t\varphi_{,2}, \\
D_3 &= q(\varepsilon_{11} + \varepsilon_{22}) + r\varepsilon_{33} - u\varphi_{,3},
\end{aligned} \tag{12}
$$

where again the superindices E associated with the elastic constants and ϵ related to the dielectric parameters are dropped for the sake of clarity. No confusion will arise since no other parameters are used. The above formulae recalls that of [31] for transversely isotropic elastic media with the usual elastic constants $2k = C_{1111} + C_{1122}$, $l = C_{1133} = C_{2233}$, $n = C_{3333}$, $p = C_{1313} = C_{2323}$, $2m = C_{1111} - C_{1122} = 2C_{1212}$. The remaining coefficients are three piezoelectric $q = e_{311} = e_{322}$, $r = e_{333}$, $s = e_{113} = e_{223}$ and two dielectric $t = \epsilon_{11} = \epsilon_{22}$ and $u = \epsilon_{33}$ [32].

Then, from the formulae (6), (10), and (12), the effective properties (11) can be written in the following form:

$$
\begin{aligned}
\bar{k} &= k_1V_1 + k^*(V_2+V_3) + \left\langle k\left({}_{11}^{1}M_{1,1} + {}_{11}^{1}M_{2,2}\right)\right\rangle,\\
\bar{l} &= l_1V_1 + l^*(V_2+V_3) + \left\langle l\left({}_{11}^{1}M_{1,1} + {}_{11}^{1}M_{2,2}\right)\right\rangle,\\
\bar{n} &= n_1V_1 + n^*(V_2+V_3) + \left\langle l\left({}_{33}^{1}M_{1,1} + {}_{33}^{1}M_{2,2}\right)\right\rangle,\\
\bar{m} &= m_1V_1 + m^*(V_2+V_3) + \left\langle m\left({}_{11}^{1}M_{1,1} - {}_{11}^{1}M_{2,2}\right)\right\rangle,\\
\bar{q} &= q_1V_1 + q^*(V_2+V_3) + \left\langle q\left({}_{11}^{1}M_{1,1} + {}_{11}^{1}M_{2,2}\right)\right\rangle,\\
\bar{r} &= r_1V_1 + r^*(V_2+V_3) + \left\langle q\left({}_{33}^{1}M_{1,1} + {}_{33}^{1}M_{2,2}\right)\right\rangle,\\
\bar{u} &= u_1V_1 + u^*(V_2+V_3) - \left\langle q\left({}_{3}^{1}P_{1,1} + {}_{3}^{1}P_{2,2}\right)\right\rangle,\\
\bar{p} &= p_1V_1 + p^*(V_2+V_3) + \left\langle p\ {}_{13}^{1}M_{3,1} + s\ {}_{13}^{1}N_{,1}\right\rangle,\\
\bar{s} &= s_1V_1 + s^*(V_2+V_3) + \left\langle s\ {}_{13}^{1}M_{3,1} - t\ {}_{13}^{1}N_{,1}\right\rangle,\\
\bar{t} &= t_1V_1 + t^*(V_2+V_3) - \left\langle s\ {}_{1}^{1}P_{3,1} - t\ {}_{13}^{1}Q_{,1}\right\rangle,
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
k^* &= k_2\frac{V_2}{V_2+V_3} + k_3\frac{V_3}{V_2+V_3} + \frac{1}{V_2+V_3}\left\langle k\left({}_{11}^{2}M_{1,1} + {}_{11}^{2}M_{2,2}\right)\right\rangle,\\
l^* &= l_2\frac{V_2}{V_2+V_3} + l_3\frac{V_3}{V_2+V_3} + \frac{1}{V_2+V_3}\left\langle l\left({}_{11}^{2}M_{1,1} + {}_{11}^{2}M_{2,2}\right)\right\rangle,\\
n^* &= n_2\frac{V_2}{V_2+V_3} + n_3\frac{V_3}{V_2+V_3} + \frac{1}{V_2+V_3}\left\langle l\left({}_{33}^{2}M_{1,1} + {}_{33}^{2}M_{2,2}\right)\right\rangle,\\
m^* &= m_2\frac{V_2}{V_2+V_3} + m_3\frac{V_3}{V_2+V_3} + \frac{1}{V_2+V_3}\left\langle m\left({}_{11}^{2}M_{1,1} - {}_{11}^{2}M_{2,2}\right)\right\rangle,\\
q^* &= q_2\frac{V_2}{V_2+V_3} + q_3\frac{V_3}{V_2+V_3} + \frac{1}{V_2+V_3}\left\langle q\left({}_{11}^{2}M_{1,1} + {}_{11}^{2}M_{2,2}\right)\right\rangle,\\
r^* &= r_2\frac{V_2}{V_2+V_3} + r_3\frac{V_3}{V_2+V_3} + \frac{1}{V_2+V_3}\left\langle q\left({}_{33}^{2}M_{1,1} + {}_{33}^{2}M_{2,2}\right)\right\rangle,\\
u^* &= u_2\frac{V_2}{V_2+V_3} + u_3\frac{V_3}{V_2+V_3} - \frac{1}{V_2+V_3}\left\langle q\left({}_{3}^{2}P_{1,1} + {}_{3}^{2}P_{2,2}\right)\right\rangle,\\
p^* &= p_2\frac{V_2}{V_2+V_3} + p_3\frac{V_3}{V_2+V_3} + \frac{1}{V_2+V_3}\left\langle p\ {}_{13}^{2}M_{3,1} + s\ {}_{13}^{2}N_{,1}\right\rangle,\\
s^* &= s_2\frac{V_2}{V_2+V_3} + s_3\frac{V_3}{V_2+V_3} + \frac{1}{V_2+V_3}\left\langle s\ {}_{13}^{2}M_{3,1} - t\ {}_{13}^{2}N_{,1}\right\rangle,\\
t^* &= t_2\frac{V_2}{V_2+V_3} + t_3\frac{V_3}{V_2+V_3} - \frac{1}{V_2+V_3}\left\langle s\ {}_{1}^{2}P_{3,1} - t\ {}_{13}^{2}Q_{,1}\right\rangle.
\end{aligned}
\tag{14}
$$

The area fractions occupied by the matrix, the mesophase, and the central fiber materials in H_1, H_2, and H_3 are referred to as V_1, V_2, and V_3, in that order. Note that $V_1 + V_2 + V_3 = 1$, where $V_2 = 2\pi(R_1^2 - R_2^2)/\sqrt{3}$ and $V_3 = 2\pi R_2^2/\sqrt{3}$. The subindex of $k, l, n, p, m, q, r, s, t$, and u indicates the properties of each phase.

To find the functions ${}_{pq}^{\Upsilon}M_i$, ${}_{pq}^{\Upsilon}N$, ${}_{p}^{\Upsilon}P_i$, and ${}_{p}^{\Upsilon}Q$ in the formulae (13) and (14), we need to solve the problems ${}_{11}L$ and ${}_{13}L$ similarly to [27]. Both problems are solved using the methods of a complex variable and the properties of doubly periodic elliptic and related functions with periods ω_1 and ω_2. Firstly, the functions ${}_{pq}^{2}M_i$, ${}_{pq}^{2}N$, ${}_{p}^{2}P_i$, and ${}_{p}^{2}Q$ are calculated considering the two-phase composite made of mesophase (H_2) and the central fiber (H_3) with volume fraction $(\frac{V_3}{V_2+V_3})$. Then, the properties (14) are obtained. Finally, the functions ${}_{pq}^{1}M_i$, ${}_{pq}^{1}N$, ${}_{p}^{1}P_i$, and ${}_{p}^{1}Q$ are computed by the second step of homogenization between the matrix (H_1) and the equivalent medium derived of the previous homogenization step.

The analytic homogenized properties $\bar{k}, \bar{l}, \bar{n}, \bar{m}, \bar{q}, \bar{r}$, and $\bar{u}$ can be written down immediately.

$$
\begin{aligned}
\bar{k} &= k_v - (V_2 + V_3)\|k\|_1^2 K_1/m_1 - V_3\|k\|_2^2 K_2/m_2,\\
\bar{l} &= l_v - (V_2 + V_3)\|l\|_1\|k\|_1 K_1/m_1 - V_3\|l\|_2\|k\|_2 K_2/m_2,\\
\bar{n} &= n_v - (V_2 + V_3)\|l\|_1^2 K_1/m_1 - V_3\|l\|_2^2 K_2/m_2,\\
\bar{m} &= m_v - (V_2 + V_3)\|m\|_1 M_1 - V_3\|m\|_2 M_2,\\
\bar{q} &= q_v - (V_2 + V_3)\|q\|_1\|k\|_1 K_1/m_1 - V_3\|q\|_2\|k\|_2 K_2/m_2,\\
\bar{r} &= r_v - (V_2 + V_3)\|q\|_1\|l\|_1 K_1/m_1 - V_3\|q\|_2\|l\|_2 K_2/m_2,\\
\bar{u} &= u_v + (V_2 + V_3)\|q\|_1^2 K_1/m_1 + V_3\|q\|_2^2 K_2/m_2.
\end{aligned}
\tag{15}
$$

The subindex v in the first term of the right-hand side of each of the Eqs. (15) denotes the Voigt or arithmetical average (the so-called rule of mixtures) of the concerned property.

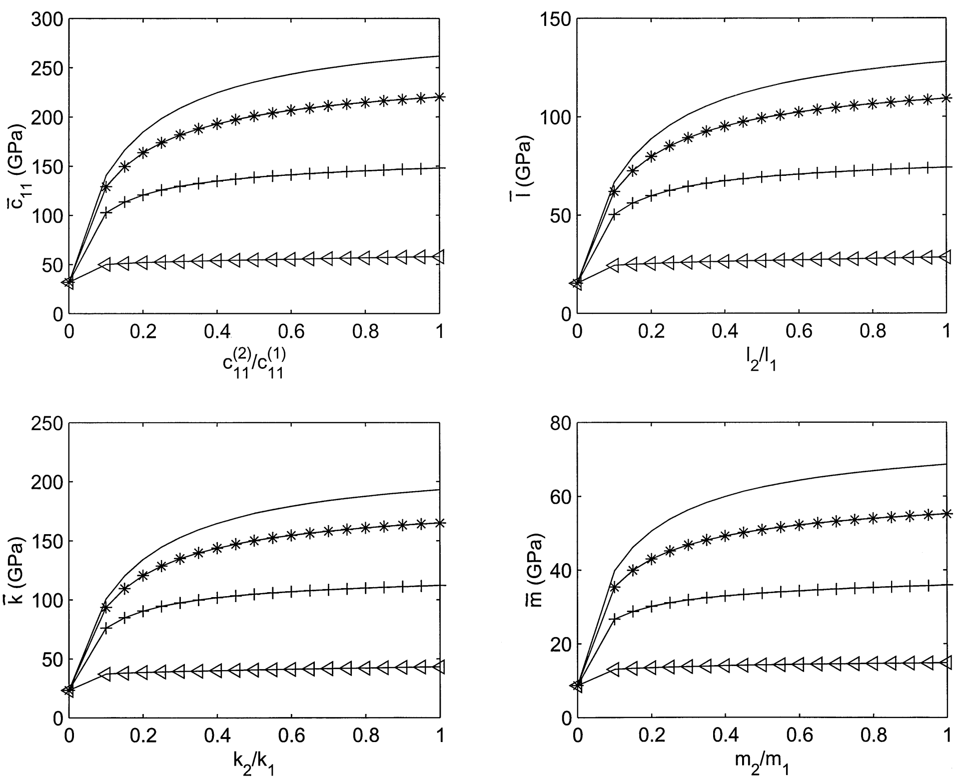

Figure 2. *Mechanic components* $\bar{C}_{11}$, $\bar{k}$, $\bar{l}$, $\bar{m}$ *of the effective coefficients for the hybrid piezoelectric composites using AHM. The matrix material used is PZT-7A. The remaining components are proportional to the matrix.*

Thus

$$k_v = V_1 k_1 + V_2 k_2 + V_3 k_3.$$

By two homogenization steps, the constants appearing in (15) can be computed. In both steps the medium is considered as a binary unidirectional fibrous composite and the formulae derived in [33] are used in the calculations. The constants K_2 and M_2 are calculated in the first step of the homogenization. In the second step of homogenization it is possible to obtain the constants K_1 and M_1. Finally, the formulae (15) can be used to compute the effective properties of three-phase composite. The effective properties $\bar{p}$, $\bar{s}$, and $\bar{t}$ related to the antiplane problem ${}_{13}L$ are calculated in a similar way, but their analytic expressions are not simple.

§5. Results and discussions

1. The formulae (15) is a generalization of the expressions (3.19), (3.27) for two-phase composites reported in [27]. In case of coincidence of two consecutive materials in the composite, we obtain from (15) the formulae (3.19), (3.27) mentioned above. Monolithic material can be obtained if the three materials are the same or when the radius R_1 and R_2 tend to zero.

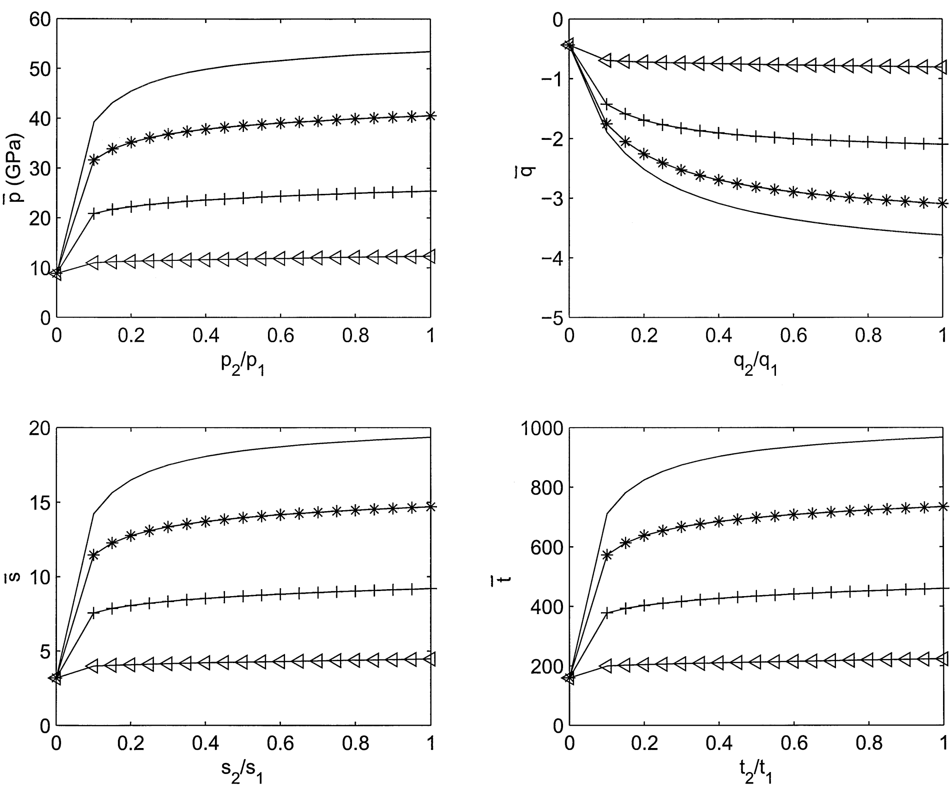

Figure 3. *Piezoelectric components* p̄*,* q̄*,* s̄*,* t̄ *of the effective coefficients for the hybrid piezoelectric composites using AHM. The matrix material used is PZT-7A. The remaining components are proportional to the matrix.*

2. The universal relations concerning the three-phase composite given by Benveniste [34] and Chen [35] are satisfied identically by the effective coefficients (15). We prove that for the calculated effective coefficients, these universal relations are constant and invariant in relation to volume fraction. This fact was not observed by [35], that is to say

$$\begin{aligned}\frac{\|k\|_1\|l\|_2 - \|k\|_2\|l\|_1}{\|k\|_1\|q\|_2 - \|k\|_2\|q\|_1} &= \frac{(\bar{k}-k_1)(\bar{l}-l_2)-(\bar{k}-k_2)(\bar{l}-l_1)}{(\bar{k}-k_1)(\bar{q}-q_2)-(\bar{k}-k_2)(\bar{q}-q_1)}\\ &= \frac{(\bar{k}-k_1)(\bar{l}-l_3)-(\bar{k}-k_3)(\bar{l}-l_1)}{(\bar{k}-k_1)(\bar{q}-q_3)-(\bar{k}-k_3)(\bar{q}-q_1)}\\ &= \frac{(\bar{k}-k_1)(\bar{n}-n_v)-(\bar{l}-l_1)(\bar{l}-l_v)}{(\bar{k}-k_1)(\bar{r}-r_v)-(\bar{q}-q_1)(\bar{l}-l_v)}\\ &= \frac{(\bar{l}-l_1)(\bar{q}-q_v)-(\bar{k}-k_1)(\bar{r}-r_v)}{(\bar{k}-k_1)(\bar{u}-u_v)+(\bar{q}-q_1)(\bar{q}-q_v)}.\end{aligned} \tag{16}$$

3. Figures 2 and 3 are shown the effective coefficients of a three-phase piezoelectric composite obtained by (15). The properties of the composite are calculated taking the coefficients of the intermediate phase and the central fiber proportional to the

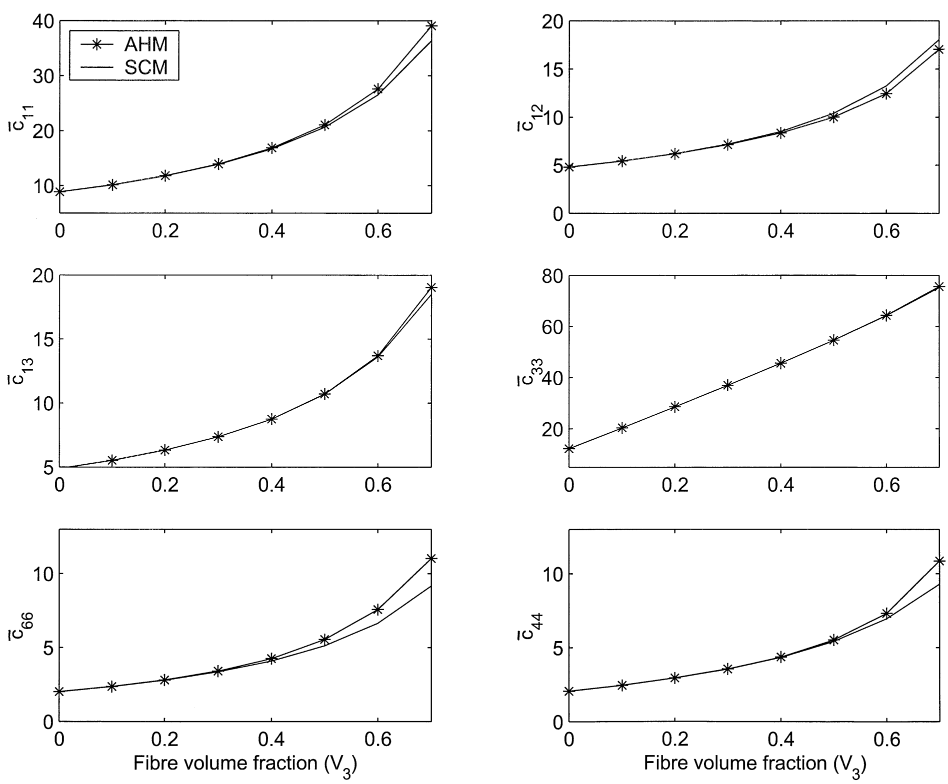

Figure 4. *Comparison of all effective mechanic properties between the SCM and AHM for a three-phase piezoelectric composite. The materials used are Araldite D (matrix), PZT-5 (mesophase), and PZT-7A (central fiber).*

matrix coefficient. The solid lines correspond to a proportion or ratio of 10 between the properties of the fiber and the matrix, the "asterisk" lines have a proportion of 3.5, the lines with "plus" correspond to a proportion of 1, and the lines with "triangle" has a proportion of 0.1. The material taken as matrix is PZT-7A, the volume fraction of the interphase and the fiber are $V_2 = 0.08$ and $V_3 = 0.4$, respectively. It can be observed in the figures that all the curves have a common start point and it is related to the effective property of a two-phase composite with empty fibers. As the properties of the interphase tends to infinity, all of the curves finish at the same point for each effective property and it is related to two-phase composite with rigid fibers. The behavior of the coefficients are qualitatively the same as the curves reported by [36] for elastic composite of ceramic matrix.

4. Figures 4 and 5 display effective coefficients of a three-phase piezoelectric composite calculated by two different models (i.e., the asymptotic homogenization method (AHM) and the self-consistent model (SCM) proposed by [37]). An extension of the results reported in [37] is used for the hybrid composite (see Figure 1). Double iteration has been applied to the self-consistent method, using first the properties of the central and intermediate fibers for searching the coefficients C^*, e^*, ϵ^*, and thereafter the self-consistent method is again applied between C^*, e^*, ϵ^*, and the properties of the matrix. The configuration of the composite is matrix Araldite D,

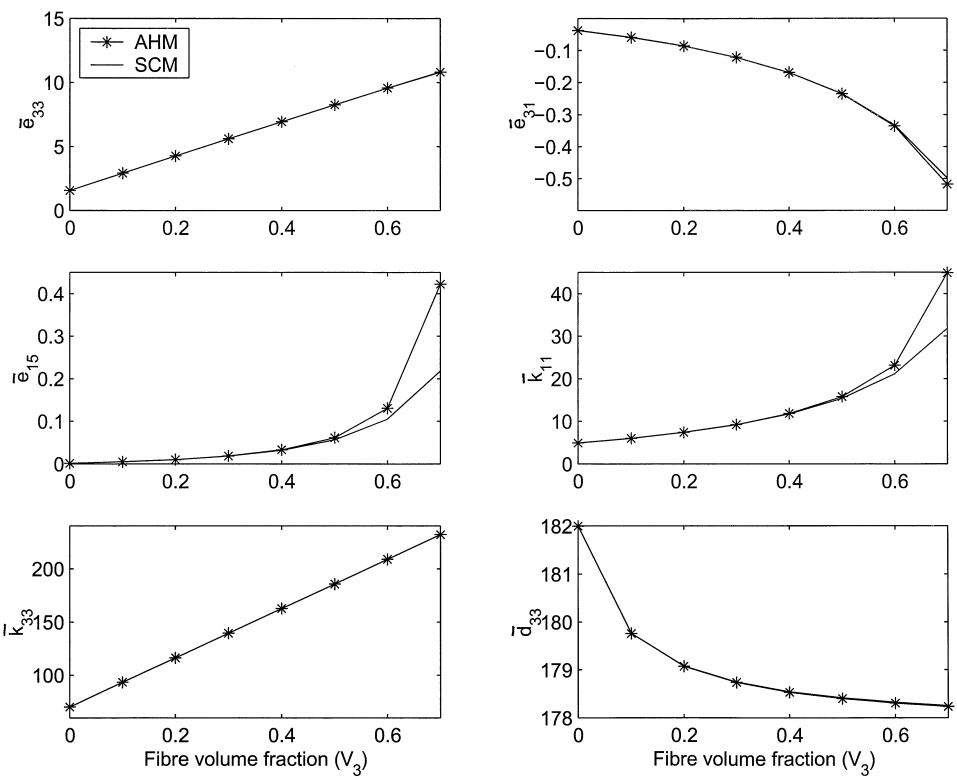

Figure 5. *Comparison of all effective piezoelectric and dielectric properties between the SCM and AHM for a three-phase piezoelectric composite. The materials used are Araldite D (matrix), PZT-5 (mesophase), and PZT-7A (central fiber).*

the intermediate phase PZT-5, and the central fiber PZT-7A. The mesophase volume fraction is $V_2 = 0.08$. In the figures are observed a good coincidence between the two models in all variation range of volume fraction of the central fiber.

§6. Concluding remarks

A fiber-reinforced three-phase composite having hexagonal symmetry has been studied here. The concentric fibers have a circular cross-section and are periodically distributed in the matrix. The matrix and fibers are occupied by homogeneous piezoelectric material with transversely isotropic properties. They are in welded contact. The asymptotic homogenization method is applied to find the overall properties of the composite. It was shown that the overall properties, $\bar{k}$, $\bar{l}$, $\bar{n}$, $\bar{q}$, $\bar{r}$, $\bar{u}$, $\bar{p}$, $\bar{s}$, and $\bar{t}$ (15), satisfy the known universal relations [34, 35]. The influence of the intermediate phase in the composite is studied. Analytical expressions for all effective coefficients of the piezocomposite were obtained by double iteration of the closed-form given in [27]. Double iteration for the self-consistent model (SCM) proposed by [37] was done as well.

It should be mentioned that the universal relations are only necessary conditions. These exact formulae (15) of the effective coefficients can be useful, besides its theoretical importance, for checking numerical codes.

Acknowledgments

This work was sponsored by the project PAPIIT, DGAPA, UNAM project number IN 103301. The German Research Foundation (DFG) has partially supported the work under project No. INK-25/B1. The authors are also grateful to Dr. Ulrich Gabbert for useful discussions and suggestions.

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