Asymptotic models for piezoelectric stack actuators with thin metal inclusions

Article · January 2004

4 authors, including:

Gennady Mishuris
Aberystwyth University

A. -M. Saendig
Universität Stuttgart

Asymptotic models for piezoelectric stack actuators with thin metal inclusions

Second International Workshop: Direct and Inverse Problems in Piezoelectricity, Hirschegg (Kleinwalsertal), Austria, July 19, 2006.

Winfried Geis[1,2], Gennady Mishuris[3], Anna-Margarete Sändig[2]

geis,saendig@ians.uni-stuttgart.de

[1]Bosch, AE/EDP 5, [2]Universität Stuttgart, IANS, [3]Rzeszów University of Technology

Outline

Part 1: The first limit problems

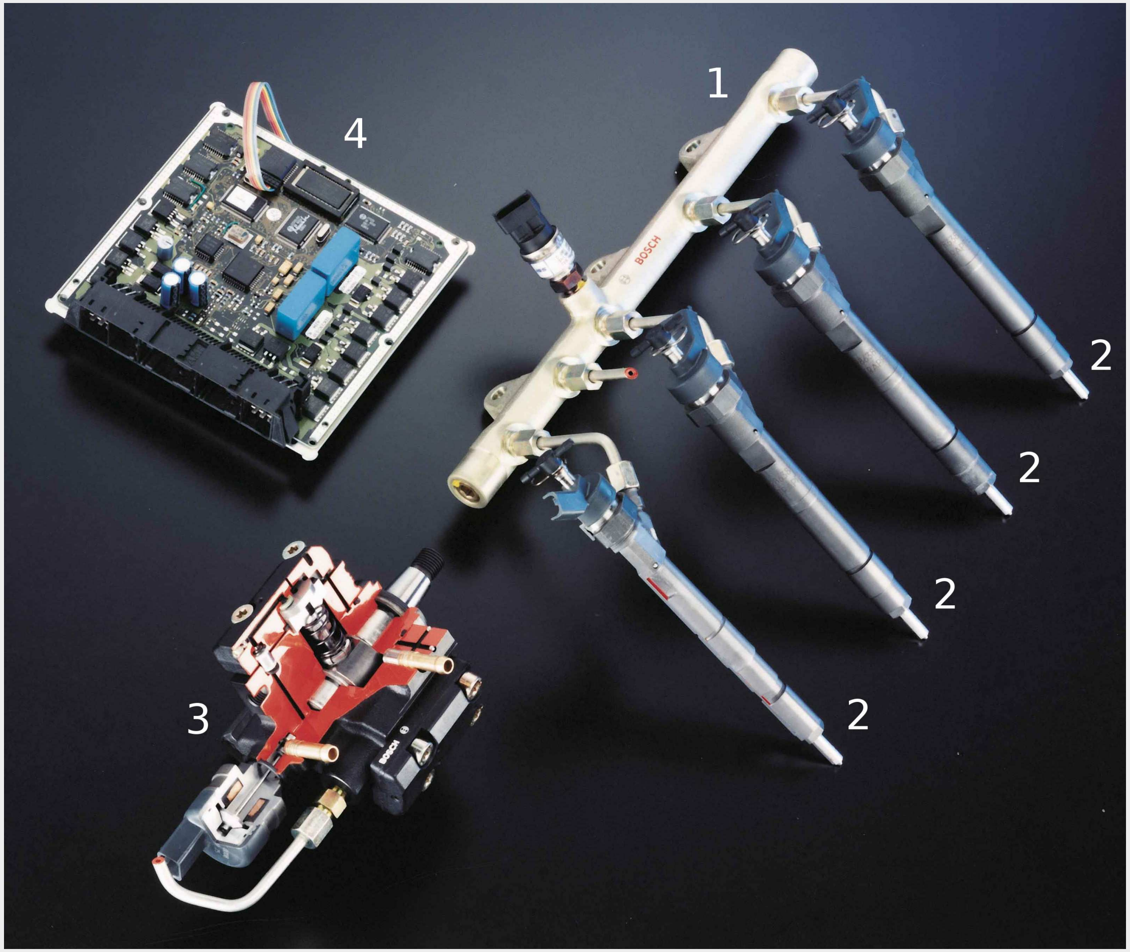
- Motivation: Engineering problem
- Goals
- Constitutive equations
  - Notation
  - Simplifications
- Mathematical model
- Asymptotic models for damaged and undamaged electrodes
  - First expansion terms, corresponding initial BVPs
  - First limit problems
  - Existence and uniqueness
- Numerical example (2D, plane strain)

Outline

Part 2: Mathematical analysis (undamaged electrodes)

- Introduction
  - Difficulties arising from the limit problems
  - Goals
- Recapitulation of the different problems
  - Exact problem P_ϵ
  - Scaled problem $P(\epsilon)$
  - First limit problem P_0
  - Equivalent first limit problem $\hat{P}_0$
- Existence of solutions
- Weak convergence
- Strong convergence
- Conclusions

Motivation: Engineering problem



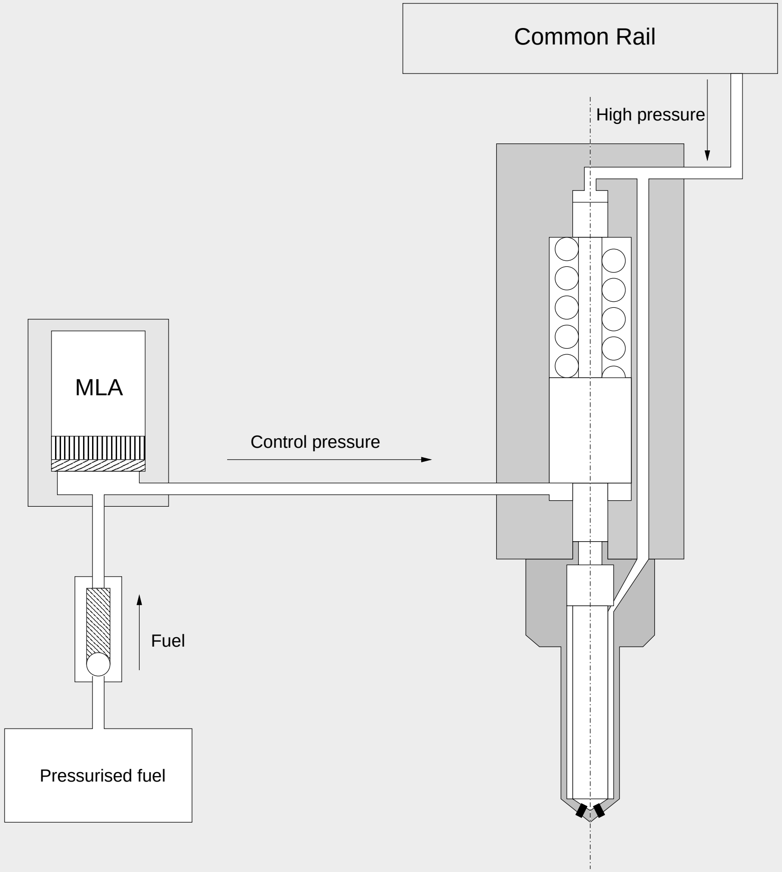
Press photo 1-DS-11654, ©Robert Bosch GmbH.

Common rail system for Diesel engines:

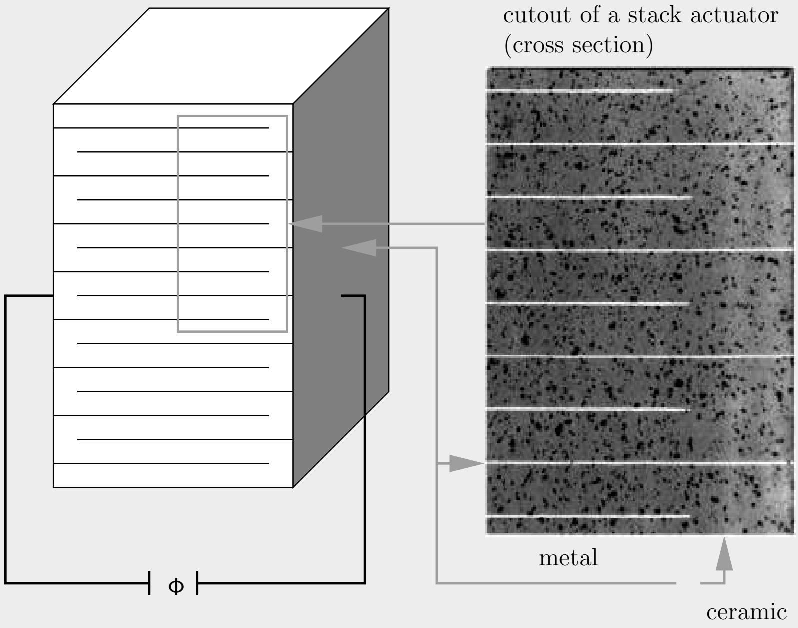
1. rail
2. piezoelectric injectors
3. high pressure fuel pump
4. electronic control unit

Motivation: Engineering problem

(a) Scheme of an injection nozzle in a common rail engine [8]. The MLA is arranged inside an earthed metal vessel.

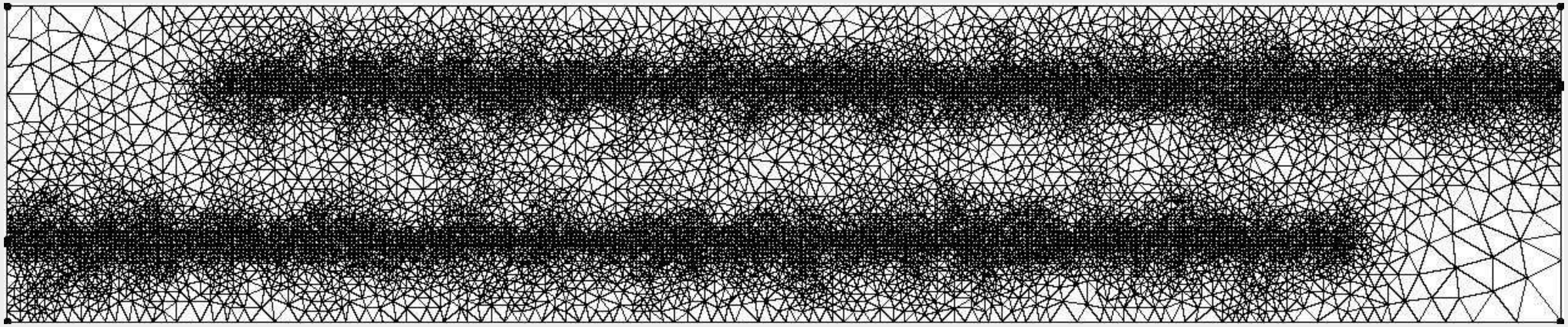


(b) Cross section of a MLC-type actuator, following [2].



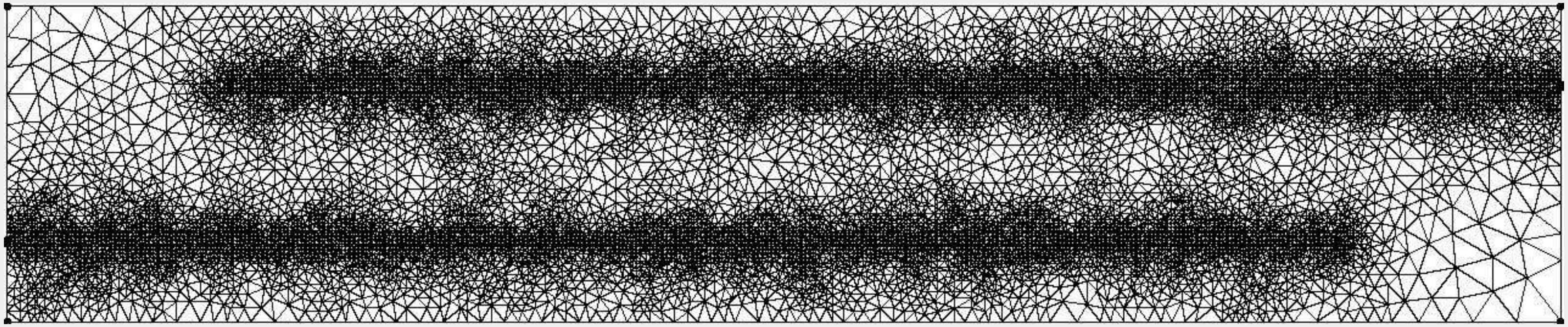
Goals

1. 2D-mesh for a cut-out of a multilayer actuator with realistic proportions (114874 elements)



Goals

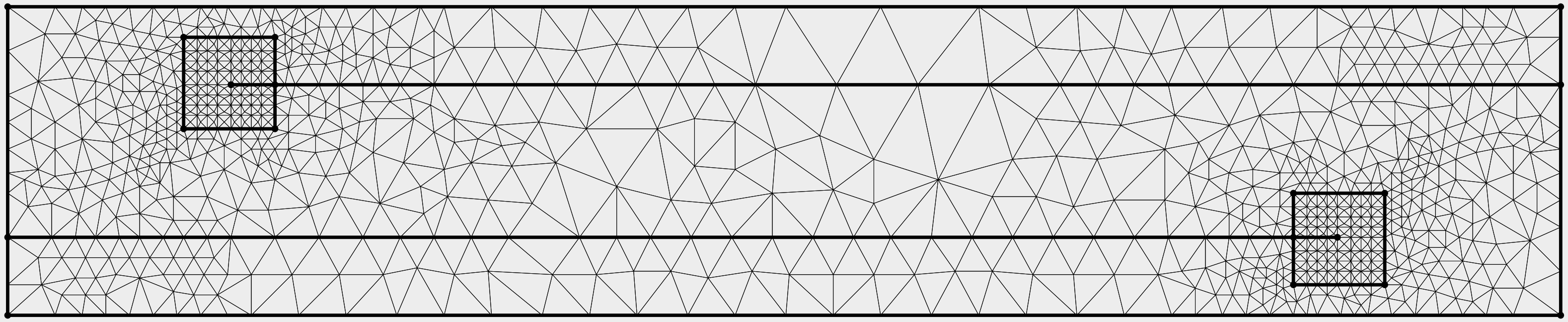
1. 2D-mesh for a cut-out of a multilayer actuator with realistic proportions (114874 elements)



Exhaust the existence of small geometrical parameters in the model, to derive simpler and easier to compute limit problems only in the ceramic domain (Ciarlet, Geymonat, et.al. [1, 3, 4, 5]).

Goals

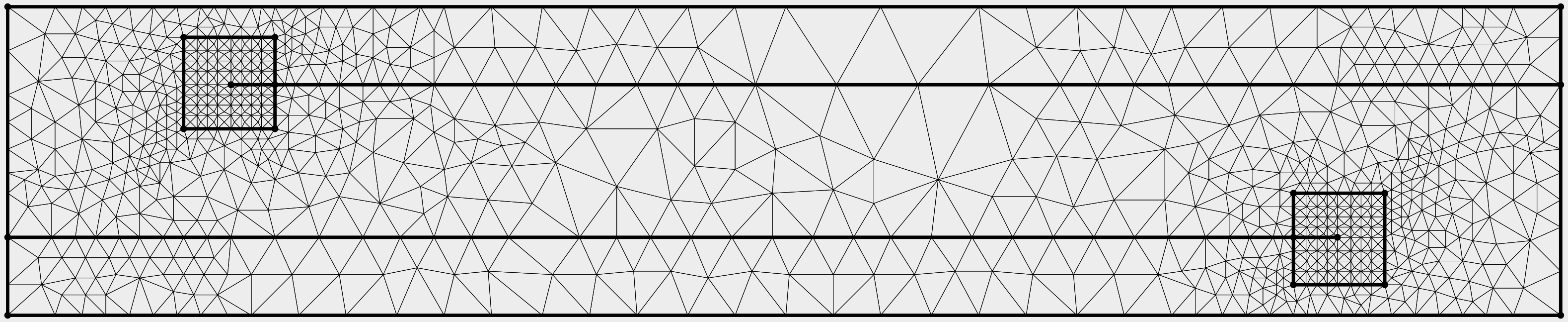
1. 2D-mesh for a cut-out of an asymptotic multilayer actuator with realistic proportions (1827 elements)



Exhaust the existence of small geometrical parameters in the model, to derive simpler and easier to compute limit problems only in the ceramic domain (Ciarlet, Geymonat, et.al. [1, 3, 4, 5]).

Goals

1. 2D-mesh for a cut-out of an asymptotic multilayer actuator with realistic proportions (1827 elements)

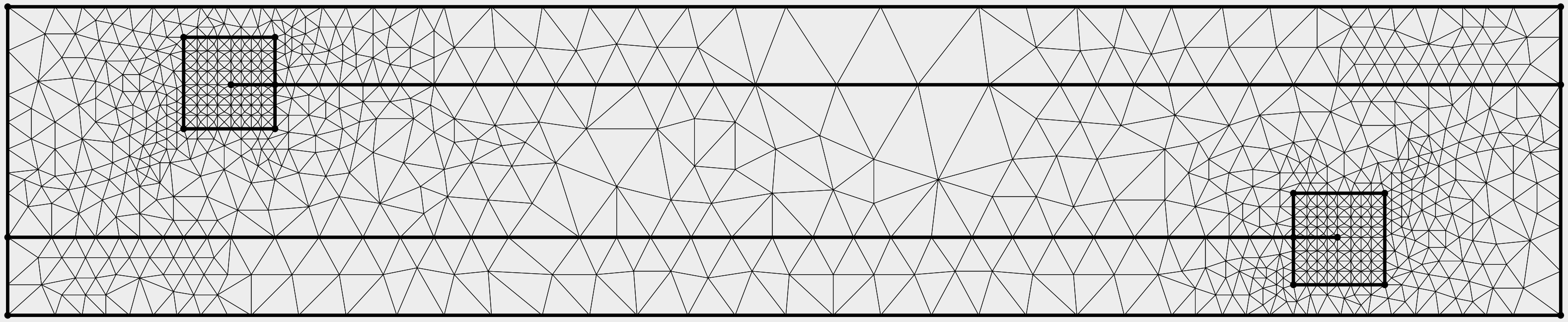
   

   Exhaust the existence of small geometrical parameters in the model, to derive simpler and easier to compute limit problems only in the ceramic domain (Ciarlet, Geymonat, et.al. [1, 3, 4, 5]).

2. Show, that unique solutions for the new asymptotic models exist in appropriate Sobolev spaces.

Goals

1. 2D-mesh for a cut-out of an asymptotic multilayer actuator with realistic proportions (1827 elements)



   Exhaust the existence of small geometrical parameters in the model, to derive simpler and easier to compute limit problems only in the ceramic domain (Ciarlet, Geymonat, et.al. [1, 3, 4, 5]).

2. Show, that unique solutions for the new asymptotic models exist in appropriate Sobolev spaces.
3. Compare the different models numerically with respect to performance and computed values.

Constitutive equations

BOSCH

Universität Stuttgart

Thermopiezoelectricity in the ceramic

$$
\begin{aligned}
s &= \frac{\rho c}{T_0} T + \lambda_{ij}\gamma_{ij} + \chi_m E_m \\
\sigma_{ij} &= -\lambda_{ij} T + C_{ijkl}\gamma_{kl} - e_{mij} E_m \\
D_k &= \chi_k T + e_{kij}\gamma_{ij} + \varepsilon_{mk} E_m
\end{aligned}
$$

Constitutive equations

Thermopiezoelectricity in the ceramic

$$s = \frac{\rho c}{T_0} T + \lambda_{ij}\gamma_{ij} + \chi_m E_m$$

$$\sigma_{ij} = -\lambda_{ij} T + C_{ijkl}\gamma_{kl} - e_{mij} E_m$$

$$D_k = \chi_k T + e_{kij}\gamma_{ij} + \varepsilon_{mk} E_m$$

T	difference of temperature: $T_a = T_0 + T$
$\underline{\underline{\boldsymbol{\gamma}}}$	linearised strain tensor: $\gamma_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$
$\underline{\boldsymbol{E}}$	electric vector field
s	entropy density
$\underline{\underline{\boldsymbol{\sigma}}}$	stress
$\underline{\boldsymbol{D}}$	dielectric displacement
ρ	mass density
c	specific heat per unit mass
$\underline{\underline{\boldsymbol{\lambda}}}$	thermal stress coefficient
$\underline{\boldsymbol{\chi}}$	pyroelectric coefficient
$\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$	transversally isotropic (PZT-4) elasticity tensor
$\underline{\underline{\underline{\boldsymbol{e}}}}$	piezoelectric tensor (non-symmetric)
$\underline{\underline{\boldsymbol{\varepsilon}}}$	permittivity tensor (symmetric)

BOSCH

Universität Stuttgart

Thermopiezoelectricity in the ceramic

$$\sigma_{ij} = -\lambda_{ij}T + C_{ijkl}\gamma_{kl} + e_{mij}\partial_m\Phi$$
$$D_k = \chi_k T + e_{kij}\gamma_{ij} - \varepsilon_{mk}\partial_m\Phi$$

T	difference of temperature: $T_a = T_0 + T$
$\underline{\underline{\gamma}}$	linearised strain tensor: $\gamma_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$
$\underline{\boldsymbol{E}}$	electric vector field
s	entropy density
$\underline{\underline{\boldsymbol{\sigma}}}$	stress
$\underline{\boldsymbol{D}}$	dielectric displacement
ρ	mass density
c	specific heat per unit mass
$\underline{\underline{\boldsymbol{\lambda}}}$	thermal stress coefficient
$\underline{\boldsymbol{\chi}}$	pyroelectric coefficient
$\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$	transversally isotropic (PZT-4) elasticity tensor
$\underline{\underline{\underline{\boldsymbol{e}}}}$	piezoelectric tensor (non-symmetric)
$\underline{\underline{\boldsymbol{\varepsilon}}}$	permittivity tensor (symmetric)

Simplifications

- ➜ $\underline{\boldsymbol{E}}$ is curl free, $\underline{\boldsymbol{E}} = -\nabla\Phi$
- ➜ T is known

Constitutive equations

Thermoelasticity in the metal

$$s = \frac{\rho c}{T_0} T + \lambda_{ij}\gamma_{ij}$$

$$\sigma_{ij} = -\lambda_{ij} T + C_{ijkl}\gamma_{kl}$$

BOSCH

Universität Stuttgart

Thermoelasticity in the metal

$$s = \frac{\rho c}{T_0} T + \lambda_{ij}\gamma_{ij}$$

$$\sigma_{ij} = -\lambda_{ij} T + C_{ijkl}\gamma_{kl}$$

T	difference of temperature: $T_a = T_0 + T$
$\underline{\underline{\gamma}}$	linearised strain tensor: $\gamma_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$
s	entropy density
$\underline{\underline{\boldsymbol{\sigma}}}$	stress
ρ	mass density
c	specific heat per unit mass
$\underline{\underline{\boldsymbol{\lambda}}}$	thermal stress coefficient
$\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$	isotropic (AgPd alloy) elasticity tensor

BOSCH Universität Stuttgart

Thermoelasticity in the metal

$$\sigma_{ij} = -\lambda_{ij} T + C_{ijkl} \gamma_{kl}$$

T	difference of temperature: $T_a = T_0 + T$
$\underline{\underline{\gamma}}$	linearised strain tensor: $\gamma_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$
s	entropy density
$\underline{\underline{\boldsymbol{\sigma}}}$	stress
ρ	mass density
c	specific heat per unit mass
$\underline{\underline{\boldsymbol{\lambda}}}$	thermal stress coefficient
$\underline{\underline{\underline{\underline{C}}}}$	isotropic (AgPd alloy) elasticity tensor

Simplification

→ T is known

From now on, the two-index notation (Voigt mapping) is used.

Force balance equations/Gauss' law

Notation

$\underline{\mathbf{u}}_C := r|_{\Omega_C}\underline{\mathbf{u}}$
$\underline{\mathbf{u}}_M := r|_{\Omega_M}\underline{\mathbf{u}}$
$\Phi_C := \Phi_C(\underline{\boldsymbol{x}}, t)$, electric potential

Force balance equations

$$\rho_C \partial_t^2 \underline{\mathbf{u}}_C - \operatorname{Div} \boldsymbol{\sigma}_C(\underline{\mathbf{u}}_C, \Phi, T) = \underline{\mathbf{0}}$$
$$\operatorname{div} \underline{\boldsymbol{D}}_C(\underline{\mathbf{u}}_C, \Phi, T) = 0$$
$$\rho_M \partial_t^2 \underline{\mathbf{u}}_M - \operatorname{Div} \boldsymbol{\sigma}_M(\underline{\mathbf{u}}_M, T) = \underline{\mathbf{0}}$$

Simplifications

- ➜ T is known
- ➜ Two-index notation

BOSCH
Universität Stuttgart

Notation

$$\underline{\mathbf{u}}_C := r\big|_{\Omega_C}\underline{\mathbf{u}}$$
$$\underline{\mathbf{u}}_M := r\big|_{\Omega_M}\underline{\mathbf{u}}$$

$\Phi_C := \Phi_C(\underline{\boldsymbol{x}}, t)$, electric potential, Φ_M is known in $Q_M^{(0,t^*)}$

$Q_C^{(0,t^*)} := \cup_{t\in(0,t^*)}\Omega_C^t$, time-space cylinder, $Q_M^{(0,t^*)}$ analogously defined

$$\mathcal{D}^\top := \mathrm{Div} = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{pmatrix}$$

Force balance equations

$$\rho_C \ddot{\underline{\mathbf{u}}}_C - \mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_C \mathcal{D}\underline{\mathbf{u}}_C - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla\Phi_C = -\mathcal{D}^\top(\underline{\boldsymbol{\lambda}}_C T) \text{ in } Q_C^{(0,t^*)}$$

$$\mathrm{div}\,(\underline{\underline{\boldsymbol{e}}}\,\mathcal{D}\underline{\mathbf{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}}\,\nabla\Phi_C) = -\mathrm{div}\,(\chi T) \quad \text{ in } Q_C^{(0,t^*)}$$

$$\rho_M \ddot{\underline{\mathbf{u}}}_M - \mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_M \mathcal{D}\underline{\mathbf{u}}_M = -\mathcal{D}^\top(\underline{\boldsymbol{\lambda}}_M T) \text{ in } Q_M^{(0,t^*)}$$

Simplifications

- ➜ T is known
- ➜ Two-index notation

Notation

$$\underline{\mathbf{u}}_C := r|_{\Omega_C} \underline{\mathbf{u}} = \underline{\boldsymbol{u}}_C(\underline{\boldsymbol{x}})$$

$$\underline{\mathbf{u}}_M := r|_{\Omega_M} \underline{\mathbf{u}} = \underline{\boldsymbol{u}}_M(\underline{\boldsymbol{x}})$$

$\Phi_C := \Phi_C(\underline{\boldsymbol{x}}, t) = \Phi_C(\underline{\boldsymbol{x}})$, electric potential, $\Phi_M = \Phi_M(\underline{\boldsymbol{x}})$ is known in Ω_M

$$\mathcal{D}^\top := \mathrm{Div} = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{pmatrix}$$

Static force balance equations

$$-\mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\mathbf{u}}_C - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi_C = -\mathcal{D}^\top (\underline{\boldsymbol{\lambda}}_C T) \quad \text{in } \Omega_C$$

$$\mathrm{div}\,(\underline{\underline{\boldsymbol{e}}} \mathcal{D} \underline{\mathbf{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}} \nabla \Phi_C) = -\mathrm{div}\,(\chi T) \quad \text{in } \Omega_C$$

$$-\mathcal{D}^T \underline{\underline{\boldsymbol{C}}}_M \mathcal{D} \underline{\mathbf{u}}_M = -\mathcal{D}^\top (\underline{\boldsymbol{\lambda}}_M T) \text{ in } \Omega_M$$

Simplifications

- ➜ T is known
- ➜ Two-index notation
- ➜ **Ansatz:** All functions are time independent, static equations

Remark:

The restriction to the stationary model is introduced for the sake of simplicity. The following results (convergence, existence, uniqueness of weak solutions) hold also in the dynamical case.

Notation

The PDE system can be shortly written as:

$$-\mathcal{B}^\top \underline{\underline{\boldsymbol{A}}}_C \mathcal{B} \underline{\boldsymbol{U}}_C = \underline{\boldsymbol{F}}_C$$

The corresponding elastic system reads:

$$-\mathcal{B}^\top \underline{\underline{\boldsymbol{A}}}_M \mathcal{B} \underline{\boldsymbol{U}}_M = \underline{\boldsymbol{F}}_M$$

BOSCH

Universität Stuttgart

The PDE system can be shortly written as:

$$-\mathcal{B}^{\top}\underline{\underline{\boldsymbol{A}}}_{C}\mathcal{B}\underline{\boldsymbol{U}}_{C} = \underline{\boldsymbol{F}}_{C}$$

The corresponding elastic system reads:

$$-\mathcal{B}^{\top}\underline{\underline{\boldsymbol{A}}}_{M}\mathcal{B}\underline{\boldsymbol{U}}_{M} = \underline{\boldsymbol{F}}_{M}$$

Generalised material matrix $\underline{\underline{\boldsymbol{A}}}_{C}$:

$$\underline{\underline{\boldsymbol{A}}}_{C} = \left(\begin{array}{cccccc|ccc}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} \\
c_{12} & c_{11} & c_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 & 0 & 0 & -e_{33} \\
0 & 0 & 0 & c_{44} & 0 & 0 & 0 & -e_{15} & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 & -e_{15} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{c_{11}-c_{12}}{2} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & e_{15} & 0 & \varepsilon_{11} & 0 & 0 \\
0 & 0 & 0 & e_{15} & 0 & 0 & 0 & \varepsilon_{11} & 0 \\
e_{31} & e_{31} & e_{33} & 0 & 0 & 0 & 0 & 0 & \varepsilon_{33}
\end{array}\right)$$

Notation

Generalised material matrix $\underline{\underline{\boldsymbol{A}}}_M$:

$$\underline{\underline{\boldsymbol{A}}}_M = \left(\begin{array}{cccccc|c} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 & \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 & \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 & \underline{\underline{\mathbf{0}}} \\ 0 & 0 & 0 & \mu & 0 & 0 & \\ 0 & 0 & 0 & 0 & \mu & 0 & \\ 0 & 0 & 0 & 0 & 0 & \mu & \\ \hline \multicolumn{6}{c|}{\underline{\underline{\mathbf{0}}}} & \underline{\underline{\mathbf{0}}} \end{array}\right)$$

BOSCH Universität Stuttgart

Generalised material matrix $\underline{\underline{\boldsymbol{A}}}_M$:

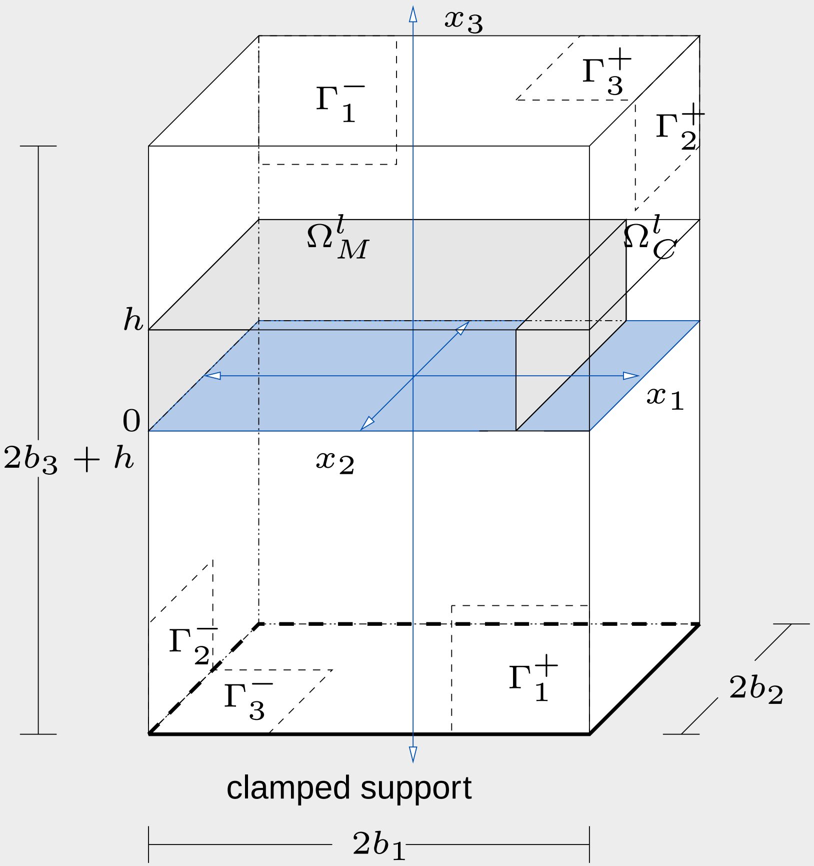
$$\underline{\underline{\boldsymbol{A}}}_M = \left(\begin{array}{cccccc|c} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 & \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 & \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 & \\ 0 & 0 & 0 & \mu & 0 & 0 & \underline{\underline{\mathbf{0}}} \\ 0 & 0 & 0 & 0 & \mu & 0 & \\ 0 & 0 & 0 & 0 & 0 & \mu & \\ \hline \multicolumn{6}{c|}{\underline{\underline{\mathbf{0}}}} & \underline{\underline{\mathbf{0}}} \end{array}\right)$$

Differential operator $\mathcal{B}$ and generalised displacement vectors $\underline{\boldsymbol{U}}$:

$$\mathcal{B} = \begin{pmatrix} \mathcal{D} & \underline{\mathbf{0}} \\ \underline{\underline{\mathbf{0}}} & -\nabla \end{pmatrix}, \quad \mathcal{D} \quad = \mathrm{Div}^{\top} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix}, \quad \underline{\boldsymbol{U}}_C = \begin{pmatrix} u_{C,1} \\ u_{C,1} \\ u_{C,3} \\ \Phi_C \end{pmatrix}, \quad \underline{\boldsymbol{U}}_M = \begin{pmatrix} u_{M,1} \\ u_{M,2} \\ u_{M,3} \\ \mp\Phi_a \end{pmatrix}$$

Additional indices always denote restrictions, e.g. to Ω_M oder Ω_C.

Simplified stack geometry with one electrode



Linear Voigt Model (ceramic) & Hooke's law (metal-electrode) for the composite

$$-\mathcal{D}^\top \underline{\underline{\boldsymbol{C}}}_C \mathcal{D} \underline{\boldsymbol{u}}_C - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla \Phi_C = \underline{\boldsymbol{F}}_C(T) \quad \text{in } \Omega_C,$$

$$\operatorname{div}\left(\underline{\underline{\boldsymbol{e}}}\, \mathcal{D} \underline{\boldsymbol{u}}_C - \underline{\underline{\boldsymbol{\varepsilon}}}\, \nabla \Phi_C\right) = f_C(T) \quad \text{in } \Omega_C,$$

$$-\mathcal{D}^\top \underline{\underline{\boldsymbol{C}}}_M \mathcal{D} \underline{\boldsymbol{u}}_M = \underline{\boldsymbol{F}}_M(T) \text{ in } \Omega_M,$$

$$\Phi_M = \pm \Phi_a \text{ known in } \Omega_M.$$

Boundary conditions

$$\boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \underline{\boldsymbol{0}} \qquad \text{on } \Gamma_2 \cup \Gamma_1 \cup \Gamma_3^+$$

$$\underline{\boldsymbol{u}}_C = \underline{\boldsymbol{0}} \qquad \text{on } \Gamma_3^-$$

$$D_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = 0 \qquad \text{on } \partial\Omega \setminus (\Gamma_2 \cup \partial\Omega_M)$$

$$\Phi_C = \pm \Phi_a \qquad \text{on } \Gamma_2 \cup \partial\Omega_M$$

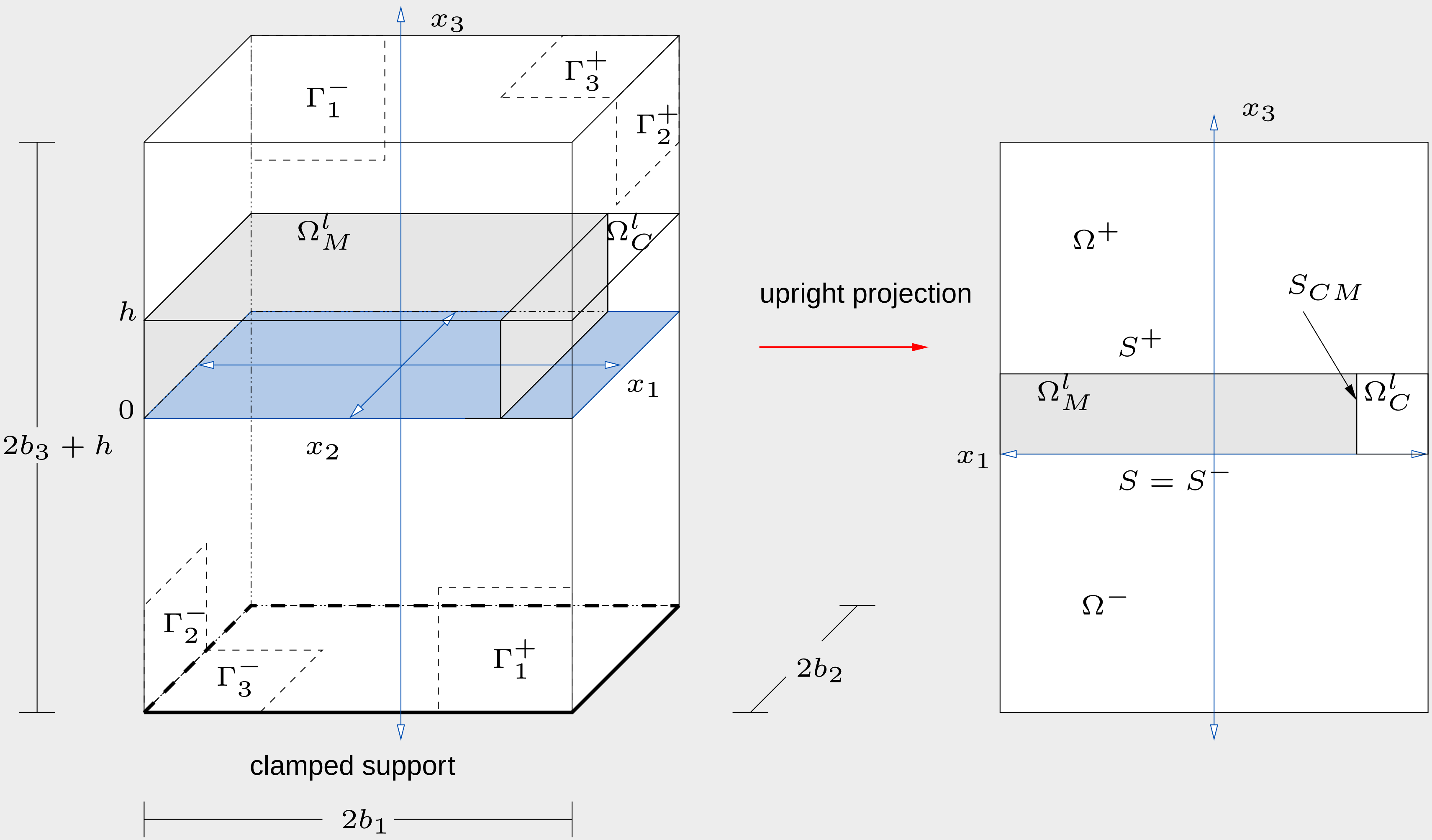
Transmission conditions on $\partial\Omega_M \cap \partial\Omega_C$:

$$\underline{\boldsymbol{u}}_C = \underline{\boldsymbol{u}}_M, \qquad \boldsymbol{\sigma}_{C_n}(\underline{\boldsymbol{u}}_C, \Phi_C) = \boldsymbol{\sigma}_{M_n}(\underline{\boldsymbol{u}}_M)$$

BOSCH

Universität Stuttgart

Simplified stack geometry with one electrode



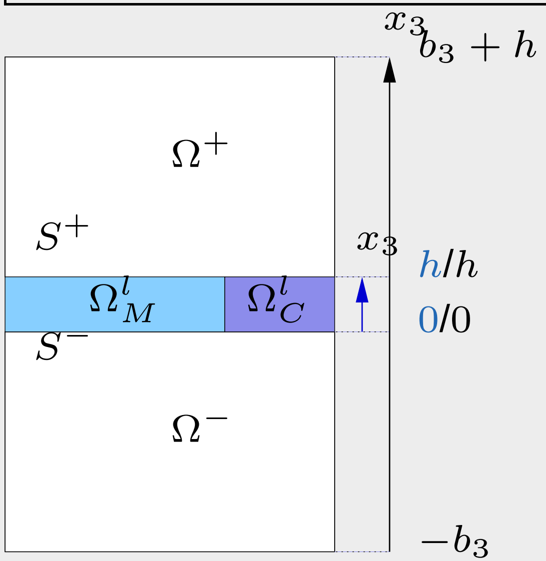
Asymptotic procedure

BOSCH
Universität Stuttgart

Final goal: Exploitation of the small geometrical quantity (electrode height h) in the original problem: reduction to a multifield problem **only** in the ceramic domain by replacing the metallic electrodes by non-standard interface conditions on the assigned planes $S = S^{\pm}$.

Asymptotic procedure

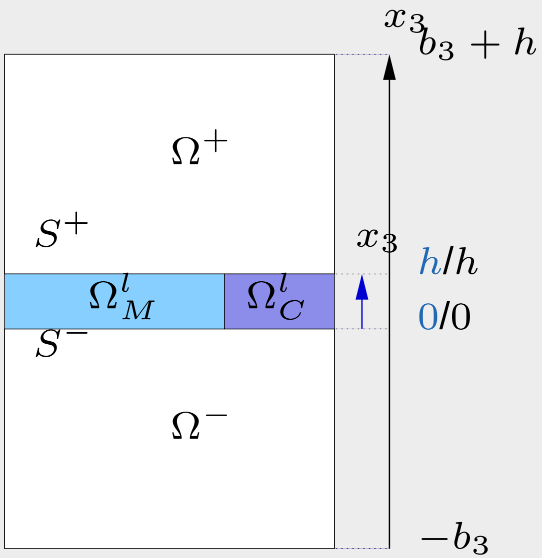
Final goal: Exploitation of the small geometrical quantity (electrode height h) in the original problem: reduction to a multifield problem **only** in the ceramic domain by replacing the metallic electrodes by non-standard interface conditions on the assigned planes $S = S^{\pm}$.

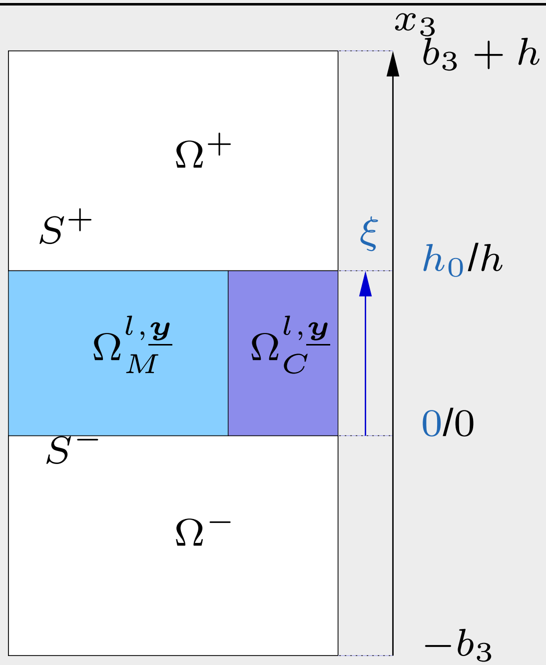


- Smooth continuation $\underline{\boldsymbol{W}}$ of Φ_a
- Weak formulation in $\Omega = \Omega^+ \cup \Omega^- \cup \Omega^l \cup S^+ \cup S^-$
- Ω^l depends on ϵ: $h = \epsilon h_0$.

Asymptotic procedure

Final goal: Exploitation of the small geometrical quantity (electrode height h) in the original problem: reduction to a multifield problem **only** in the ceramic domain by replacing the metallic electrodes by non-standard interface conditions on the assigned planes $S = S^{\pm}$.





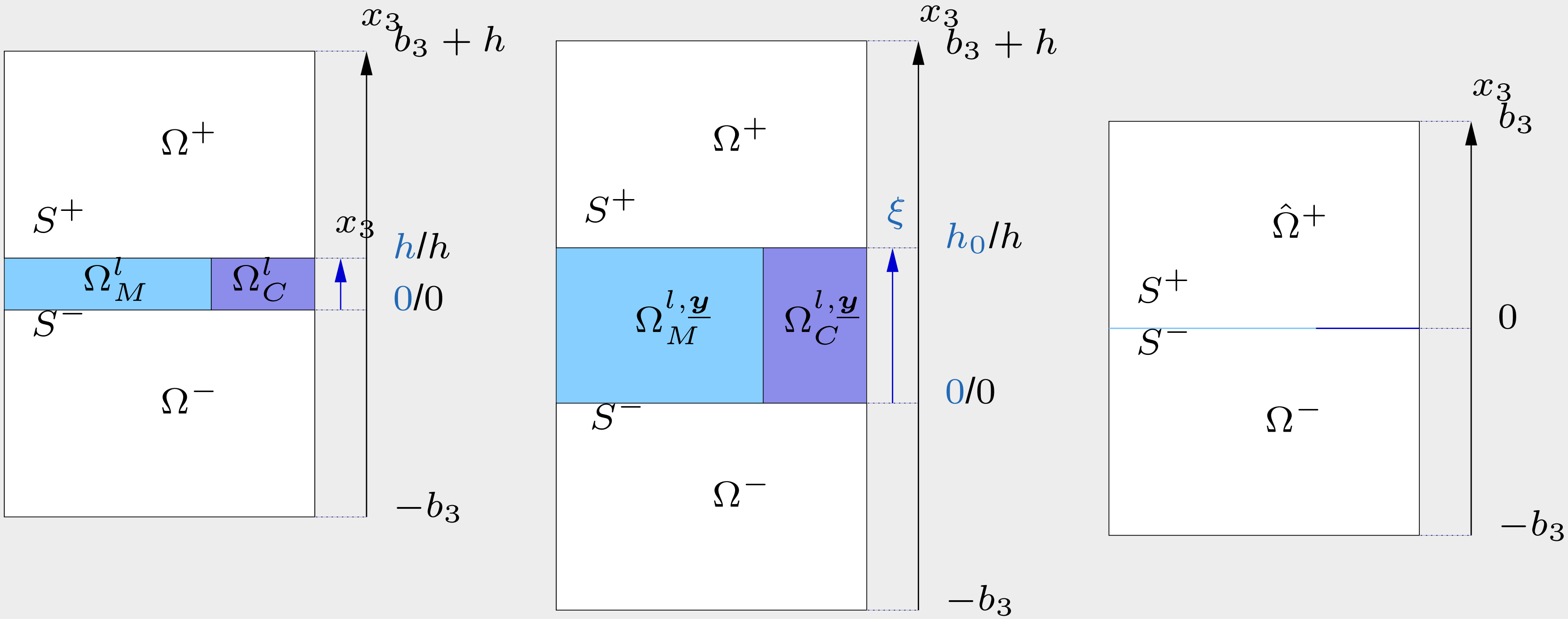
- Smooth continuation $\underline{W}$ of Φ_a
- Weak formulation in $\Omega = \Omega^+ \cup \Omega^- \cup \Omega^l \cup S^+ \cup S^- \cup S_{CM}$
- Ω^l depends on ϵ: $h = \epsilon h_0$.

- Rapid coordinates $x_3 = \epsilon\xi$
- Splitting of $\mathcal{D}$ and $\mathcal{B}$
- Power series of $\underline{U}$ in ϵ
- Hierarchical series of BVPs

BOSCH

Universität Stuttgart

Final goal: Exploitation of the small geometrical quantity (electrode height h) in the original problem: reduction to a multifield problem **only** in the ceramic domain by replacing the metallic electrodes by non-standard interface conditions on the assigned planes $S = S^{\pm}$.



- Smooth continuation $\underline{\boldsymbol{W}}$ of Φ_a
- Weak formulation in $\Omega = \Omega^+ \cup \Omega^- \cup \Omega^l \cup S^+ \cup S^- \cup S_{CM}$
- Ω^l depends on ϵ: $h = \epsilon h_0$.

- Rapid coordinates $x_3 = \epsilon\xi$
- Splitting of $\mathcal{D}$ and $\mathcal{B}$
- Power series of $\underline{\boldsymbol{U}}$ in ϵ
- Hierarchical series of BVPs

- Sort with respect to ϵ powers
- Jump conditions along $S^{\pm}$ – equivalence to $a^l(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}})$
- 1^{st} limit problem in $\hat{\Omega} = \hat{\Omega}^+ \cup \Omega^- \cup S^+ \cup S^-$

BOSCH

Universität Stuttgart

Undamaged electrodes	Damaged electrodes
The elastic constants of the thin metal layer and the ceramic matrix are comparable	In the case of isotropically damaged electrodes, the small parameter ϵ additionally occurs due to the small ratio of the elastic constants
$c_{11} \sim c_{33} \sim \lambda + 2\mu$	$\epsilon c_{11} \sim \epsilon c_{33} \sim \lambda^{\epsilon} + 2\mu^{\epsilon},$
$2c_{44} \sim c_{11} - c_{12} \sim 2\mu$	$\epsilon 2c_{44} \sim \epsilon(c_{11} - c_{12}) \sim \mu^{\epsilon}$
$c_{12} \sim c_{13} \sim \lambda$	$\epsilon c_{12} \sim \epsilon c_{13} \sim \lambda^{\epsilon}$

Undamaged electrodes	**Damaged electrodes**
The elastic constants of the thin metal layer and the ceramic matrix are comparable $c_{11} \sim c_{33} \sim \lambda + 2\mu$, $2c_{44} \sim c_{11} - c_{12} \sim 2\mu$, $c_{12} \sim c_{13} \sim \lambda$	In the case of isotropically damaged electrodes, the small parameter ϵ additionally occurs due to the small ratio of the elastic constants $\epsilon c_{11} \sim \epsilon c_{33} \sim \lambda^\epsilon + 2\mu^\epsilon,$ $\epsilon 2c_{44} \sim \epsilon(c_{11} - c_{12}) \sim \mu^\epsilon$, $\epsilon c_{12} \sim \epsilon c_{13} \sim \lambda^\epsilon$

Going over to the weak formulation in appropriate Sobolev spaces:

$$\mathcal{V}(\Omega) := \left\{ \underline{\boldsymbol{U}} \in \mathrm{H}^1(\Omega) : \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-, \Phi = 0 \text{ on } \Omega_M^l \cup \Gamma_2 \right\}.$$

Let $\underline{\boldsymbol{W}} = \begin{pmatrix} \mathbf{0} \\ \tilde{\Phi} \end{pmatrix} \in \mathrm{H}^1(\Omega)$ be a smooth continuation of the non-homogeneous electrical Dirichlet data, such that

$$\underline{\boldsymbol{U}} = \overset{\circ}{\underline{\boldsymbol{U}}} + \underline{\boldsymbol{W}},$$

where $\overset{\circ}{\underline{\boldsymbol{U}}} \in \mathcal{V}(\Omega)$, $\underline{\boldsymbol{W}} \in \mathrm{H}^1(\Omega)$ and $\underline{\boldsymbol{W}}$ is independent of x_3 in Ω^l.

BOSCH Universität Stuttgart

Undamaged electrodes	**Damaged electrodes**
The elastic constants of the thin metal layer and the ceramic matrix are comparable $$c_{11} \sim c_{33} \sim \lambda + 2\mu$$ $$2c_{44} \sim c_{11} - c_{12} \sim 2\mu$$ $$c_{12} \sim c_{13} \sim \lambda$$	In the case of isotropically damaged electrodes, the small parameter ϵ additionally occurs due to the small ratio of the elastic constants $$\epsilon c_{11} \sim \epsilon c_{33} \sim \lambda^{\epsilon} + 2\mu^{\epsilon},$$ $$\epsilon 2c_{44} \sim \epsilon(c_{11} - c_{12}) \sim \mu^{\epsilon}$$ $$\epsilon c_{12} \sim \epsilon c_{13} \sim \lambda^{\epsilon}$$

Going over to the weak formulation in appropriate Sobolev spaces:

$$\mathcal{V}(\Omega) := \left\{ \underline{\boldsymbol{U}} \in \mathrm{H}^1(\Omega) : \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-, \Phi = 0 \text{ on } \Omega_M^l \cup \Gamma_2 \right\}.$$

Let $\underline{\boldsymbol{W}} = \begin{pmatrix} \boldsymbol{0} \\ \tilde{\Phi} \end{pmatrix} \in \mathrm{H}^1(\Omega)$ be a smooth continuation of the non-homogeneous electrical Dirichlet data, such that

$$\underline{\boldsymbol{U}} = \underline{\mathring{\boldsymbol{U}}} + \underline{\boldsymbol{W}},$$

where $\underline{\mathring{\boldsymbol{U}}} \in \mathcal{V}(\Omega)$, $\underline{\boldsymbol{W}} \in \mathrm{H}^1(\Omega)$ and $\underline{\boldsymbol{W}}$ is independent of x_3 in Ω^l.

From now on, we write $\underline{\boldsymbol{U}} \in \mathcal{V}$ instead of $\underline{\mathring{\boldsymbol{U}}}$

Weak formulation of the original problem: Find $\underline{\boldsymbol{U}} \in \mathcal{V}$ such that $\forall\, \underline{\boldsymbol{V}} \in \mathcal{V}$ holds:

$$a^{+}(\underline{\boldsymbol{U}},\underline{\boldsymbol{V}}) + a^{-}(\underline{\boldsymbol{U}},\underline{\boldsymbol{V}}) + a^l_C(\underline{\boldsymbol{U}},\underline{\boldsymbol{V}}) + a^l_M(\underline{\boldsymbol{U}},\underline{\boldsymbol{V}}) = F(\underline{\boldsymbol{V}}) + F^l_M(\underline{\boldsymbol{V}}) + F^l_C(\underline{\boldsymbol{V}})$$

$$a^{+}(\underline{\boldsymbol{U}},\underline{\boldsymbol{V}}) + a^{-}(\underline{\boldsymbol{U}},\underline{\boldsymbol{V}}) + a^l_C(\underline{\boldsymbol{U}},\underline{\boldsymbol{V}}) + \epsilon a^l_M(\underline{\boldsymbol{U}},\underline{\boldsymbol{V}}) = F(\underline{\boldsymbol{V}}) + F^l_C(\underline{\boldsymbol{V}}) + \epsilon F^l_M(\underline{\boldsymbol{V}})$$

where

$$a^{\pm}(\underline{\boldsymbol{U}},\underline{\boldsymbol{V}}) := \int_{\Omega^{\pm}} \underline{\underline{\boldsymbol{A}}}_C \mathcal{B}\underline{\boldsymbol{U}} \cdot \mathcal{B}\underline{\boldsymbol{V}}\, \mathrm{d}\boldsymbol{x},$$

$$a^l_{C/M}(\underline{\boldsymbol{U}},\underline{\boldsymbol{V}},t) := \int_{\Omega^l_{C/M}} \underline{\underline{\boldsymbol{A}}}_{C/M} \mathcal{B}\underline{\boldsymbol{U}} \cdot \mathcal{B}\underline{\boldsymbol{V}}\, \mathrm{d}\boldsymbol{x},$$

$$F(\underline{\boldsymbol{V}}) = -a^{\pm}(\underline{\boldsymbol{W}},\underline{\boldsymbol{V}}) - a^{\pm}_T(T,\underline{\boldsymbol{V}}),$$

$$F^l_M(\underline{\boldsymbol{V}}) = -a^l_{T,M}(T,\underline{\boldsymbol{V}}),$$

$$F^l_C(\underline{\boldsymbol{V}}) = -a^l_C(\underline{\boldsymbol{W}},\underline{\boldsymbol{V}}) - a^l_{T,C}(T,\underline{\boldsymbol{V}}),$$

$$F(\underline{\boldsymbol{V}}) = -a^{\pm}(\underline{\boldsymbol{W}},\underline{\boldsymbol{V}}) - a^{\pm}_{T,C}(T,\underline{\boldsymbol{V}})$$

$$\epsilon F^l_M(\underline{\boldsymbol{V}}) = -\epsilon a^l_{T,M}(T,\underline{\boldsymbol{V}})$$

$$F^l_C(\underline{\boldsymbol{V}}) = -a^l_C(\underline{\boldsymbol{W}},\underline{\boldsymbol{V}}) - a^l_{T,C}(T,\underline{\boldsymbol{V}}).$$

Introduction of rapid coordinates in the layer Ω^l:

$$\underline{\boldsymbol{y}} := (x_1, x_2, \xi)^\top ,$$
$$x_3 = \epsilon\xi, \quad h = \epsilon h_0 .$$

Introduction of rapid coordinates in the layer Ω^l:

$$\underline{\boldsymbol{y}} := (x_1, x_2, \xi)^\top,$$
$$x_3 = \epsilon\xi, \quad h = \epsilon h_0.$$

Splitting of the differential operators $\mathcal{D}^{\underline{\boldsymbol{x}}} = \mathcal{D}$ and $\mathcal{B}^{\underline{\boldsymbol{x}}} = \mathcal{B}$ into (x_3) respectively (ξ) and (x_1, x_2)-dependent parts

$$\mathcal{D}^{\underline{\boldsymbol{x}}} = \underline{\underline{\boldsymbol{A}}}_{(3)}\partial_3 + \mathcal{A}_{(1,2)} = \mathcal{A}^{\underline{\boldsymbol{x}}}_{(3)} + \mathcal{A}_{(1,2)} = \frac{1}{\epsilon}\mathcal{A}^{\underline{\boldsymbol{y}}}_{(\xi)} + \mathcal{A}_{(1,2)},$$
$$\mathcal{B}^{\underline{\boldsymbol{x}}} = \underline{\underline{\boldsymbol{B}}}_{(3)}\partial_3 + \mathcal{B}_{(1,2)} = \mathcal{B}^{\underline{\boldsymbol{x}}}_{(3)} + \mathcal{B}_{(1,2)} = \frac{1}{\epsilon}\mathcal{B}^{\underline{\boldsymbol{y}}}_{(\xi)} + \mathcal{B}_{(1,2)},$$

with

$$\underline{\underline{\boldsymbol{A}}}_{(3)} := \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1\\0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, \quad \underline{\underline{\boldsymbol{B}}}_{(3)} := \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1 \end{pmatrix}, \quad \mathcal{A}_{(1,2)} := \begin{pmatrix} \partial_1&0&0\\0&\partial_2&0\\0&0&0\\0&0&\partial_2\\0&0&\partial_1\\\partial_2&\partial_1&0 \end{pmatrix}, \quad \mathcal{B}_{(1,2)} := \begin{pmatrix} \partial_1&0&0&0\\0&\partial_2&0&0\\0&0&0&0\\0&0&\partial_2&0\\0&0&\partial_1&0\\\partial_2&\partial_1&0&0\\0&0&0&-\partial_1\\0&0&0&-\partial_2\\0&0&0&0 \end{pmatrix}$$

and $\mathcal{A}_{(\xi)} = \partial_\xi \underline{\underline{\boldsymbol{A}}}_{(3)}$, $\mathcal{B}_{(\xi)} = \partial_\xi \underline{\underline{\boldsymbol{B}}}_{(3)}$.

Asymptotic procedure (Rapid coordinates)

BOSCH Universität Stuttgart

The Sobolev space $\mathcal{V}$ and the bilinear forms, defined on the layer are transformed into

$$\mathcal{V}^{\underline{\boldsymbol{y}}}\left(\Omega^{\underline{\boldsymbol{y}}}\right) = \left\{ \underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} \in \mathrm{H}^1\left(\Omega^{\underline{\boldsymbol{y}}}\right) : \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-, \Phi = 0 \text{ on } \Omega_M^{l,\underline{\boldsymbol{y}}} \cup \Gamma_2 \right\}$$

The Sobolev space $\mathcal{V}$ and the bilinear forms, defined on the layer are transformed into

$$\mathcal{V}^{\underline{\boldsymbol{y}}}\left(\Omega^{\underline{\boldsymbol{y}}}\right) = \left\{\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} \in \mathsf{H}^1\left(\Omega^{\underline{\boldsymbol{y}}}\right) : \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-, \Phi = 0 \text{ on } \Omega_M^{l,\underline{\boldsymbol{y}}} \cup \Gamma_2\right\}$$

$$a^l_{C/M}\left(\underline{\boldsymbol{U}}^l, \underline{\boldsymbol{V}}\right) = \frac{1}{\epsilon} a^l_{C/M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}} \underline{\boldsymbol{V}}\right) + a^l_{C/M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right) + \epsilon a^l_{C/M,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)$$

BOSCH
Universität Stuttgart

The Sobolev space $\mathcal{V}$ and the bilinear forms, defined on the layer are transformed into

$$\mathcal{V}^{\underline{\boldsymbol{y}}}\left(\Omega^{\underline{\boldsymbol{y}}}\right)=\left\{\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\in \mathrm{H}^{1}\left(\Omega^{\underline{\boldsymbol{y}}}\right):\ \underline{\boldsymbol{u}}=\underline{\boldsymbol{0}}\text{ on }\Gamma_{3}^{-},\ \Phi=0\text{ on }\Omega_{M}^{l,\underline{\boldsymbol{y}}}\cup\Gamma_{2}\right\}$$

$$a^{l}_{C/M}\left(\underline{\boldsymbol{U}}^{l},\underline{\boldsymbol{V}}\right)=\frac{1}{\epsilon}a^{l}_{C/M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}\underline{\boldsymbol{V}}\right)+a^{l}_{C/M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}},\underline{\boldsymbol{V}}\right)+\epsilon a^{l}_{C/M,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}},\underline{\boldsymbol{V}}\right)$$

$$a^{l}_{T,M}\left(T^{l},\underline{\boldsymbol{V}}\right)=\underbrace{a^{l}_{T,M,(\xi)}\left(T^{l,\underline{\boldsymbol{y}}},\underline{\boldsymbol{V}}\right)}_{=:-F^{l}_{T,M,(\xi)}(\underline{\boldsymbol{V}})}+\underbrace{\epsilon a^{l}_{T,M,(1,2)}\left(T^{l,\underline{\boldsymbol{y}}},\underline{\boldsymbol{V}}\right)}_{=:-F^{l}_{T,M,(1,2)}(\underline{\boldsymbol{V}})},$$

The Sobolev space $\mathcal{V}$ and the bilinear forms, defined on the layer are transformed into

$$\mathcal{V}^{\underline{\boldsymbol{y}}}\left(\Omega^{\underline{\boldsymbol{y}}}\right) = \left\{ \underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} \in \mathrm{H}^1\left(\Omega^{\underline{\boldsymbol{y}}}\right) : \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-, \Phi = 0 \text{ on } \Omega_M^{l,\underline{\boldsymbol{y}}} \cup \Gamma_2 \right\}$$

$$a^l_{C/M}\left(\underline{\boldsymbol{U}}^l, \underline{\boldsymbol{V}}\right) = \frac{1}{\epsilon} a^l_{C/M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}} \underline{\boldsymbol{V}}\right) + a^l_{C/M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right) + \epsilon a^l_{C/M,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)$$

$$a^l_{T,M}\left(T^l, \underline{\boldsymbol{V}}\right) = \underbrace{a^l_{T,M,(\xi)}\left(T^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F^l_{T,M,(\xi)}(\underline{\boldsymbol{V}})} + \underbrace{\epsilon a^l_{T,M,(1,2)}\left(T^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F^l_{T,M,(1,2)}(\underline{\boldsymbol{V}})},$$

$$a^l_{T,C}\left(T^l, \underline{\boldsymbol{V}}\right) = \underbrace{a^l_{T,C,(\xi)}\left(T^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F^l_{T,C,\xi}(\underline{\boldsymbol{V}})} + \underbrace{\epsilon a^l_{T,C,(1,2)}\left(T^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F^l_{T,C,(1,2)}(\underline{\boldsymbol{V}})},$$

BOSCH
Universität Stuttgart

The Sobolev space $\mathcal{V}$ and the bilinear forms, defined on the layer are transformed into

$$\mathcal{V}^{\underline{\boldsymbol{y}}}\left(\Omega^{\underline{\boldsymbol{y}}}\right) = \left\{\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} \in \mathrm{H}^1\left(\Omega^{\underline{\boldsymbol{y}}}\right) : \underline{\boldsymbol{u}} = \underline{\mathbf{0}} \text{ on } \Gamma_3^-, \Phi = 0 \text{ on } \Omega_M^{l,\underline{\boldsymbol{y}}} \cup \Gamma_2\right\}$$

$$a^l_{C/M}\left(\underline{\boldsymbol{U}}^l, \underline{\boldsymbol{V}}\right) = \frac{1}{\epsilon} a^l_{C/M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}} \underline{\boldsymbol{V}}\right) + a^l_{C/M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right) + \epsilon a^l_{C/M,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)$$

$$a^l_{T,M}\left(T^l, \underline{\boldsymbol{V}}\right) = \underbrace{a^l_{T,M,(\xi)}\left(T^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F^l_{T,M,(\xi)}(\underline{\boldsymbol{V}})} + \underbrace{\epsilon a^l_{T,M,(1,2)}\left(T^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F^l_{T,M,(1,2)}(\underline{\boldsymbol{V}})},$$

$$a^l_{T,C}\left(T^l, \underline{\boldsymbol{V}}\right) = \underbrace{a^l_{T,C,(\xi)}\left(T^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F^l_{T,C,\xi}(\underline{\boldsymbol{V}})} + \underbrace{\epsilon a^l_{T,C,(1,2)}\left(T^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F^l_{T,C,(1,2)}(\underline{\boldsymbol{V}})},$$

$$a^l_{W,C}\left(\underline{\boldsymbol{W}}^l, \underline{\boldsymbol{V}}\right) = \frac{1}{\epsilon} \underbrace{a^l_{C,(\xi)}\left(\underline{\boldsymbol{W}}^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F_{W,C,(\xi)}} + \underbrace{a^l_{C,(1,2,\xi)}\left(\underline{\boldsymbol{W}}^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F_{W,C,(1,2,\xi)}} + \underbrace{\epsilon a^l_{C,(1,2)}\left(\underline{\boldsymbol{W}}^{l,\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right)}_{=:-F_{W,C,(1,2)}}.$$

BOSCH

Universität Stuttgart

Weak formulation: Find $\underline{\boldsymbol{U}}^{\boldsymbol{y}} \in \mathcal{V}^{\boldsymbol{y}}$ such that $\forall\, \underline{\boldsymbol{V}} \in \mathcal{V}$ holds:

$$
\begin{aligned}
& a^{+}\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + a^{-}\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \frac{1}{\epsilon} a^{l}_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + a^{l}_{C,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \epsilon a^{l}_{C,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + \frac{1}{\epsilon} a^{l}_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + a^{l}_{M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + \epsilon a^{l}_{M,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& = F\left(\underline{\boldsymbol{V}}\right) + F^{l}_{T,M,(\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon F^{l}_{T,M,(1,2)}\left(\underline{\boldsymbol{V}}\right) \\
& + F^{l}_{T,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon F^{l}_{T,C,(1,2)}\left(\underline{\boldsymbol{V}}\right) \\
& + \frac{1}{\epsilon} F_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) + F_{W,C,(1,2,\xi)}\left(\underline{\boldsymbol{V}}\right) \\
& + \epsilon F_{W,C,(1,2)}\left(\underline{\boldsymbol{V}}\right),
\end{aligned}
$$

$$
\begin{aligned}
& a^{+}\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + a^{-}\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \frac{1}{\epsilon} a^{l}_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + a^{l}_{C,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \epsilon a^{l}_{C,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + a^{l}_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \epsilon a^{l}_{M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \epsilon^{2} a^{l}_{M,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& = F\left(\underline{\boldsymbol{V}}\right) + \epsilon F^{l}_{T,M,(\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon^{2} F_{T,M,(1,2)}\left(\underline{\boldsymbol{V}}\right) \\
& + F^{l}_{T,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon F^{l}_{T,C,(1,2)}\left(\underline{\boldsymbol{V}}\right) \\
& + \frac{1}{\epsilon} F_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) + F_{W,C,(1,2,\xi)}\left(\underline{\boldsymbol{V}}\right) \\
& + F_{W,C,(1,2)}\left(\underline{\boldsymbol{V}}\right) + \frac{1}{\epsilon} F_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) \\
& + F_{W,C,(1,2,\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon F_{W,C,(1,2)}\left(\underline{\boldsymbol{V}}\right).
\end{aligned}
$$

Weak formulation: Find $\underline{\boldsymbol{U}}^{\boldsymbol{y}} \in \mathcal{V}^{\boldsymbol{y}}$ such that $\forall\, \underline{\boldsymbol{V}} \in \mathcal{V}$ holds:

$$
\begin{aligned}
& a^{+}\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + a^{-}\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \frac{1}{\epsilon} a^{l}_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + a^{l}_{C,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \epsilon a^{l}_{C,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + \frac{1}{\epsilon} a^{l}_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + a^{l}_{M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + \epsilon a^{l}_{M,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& = F\left(\underline{\boldsymbol{V}}\right) + F^{l}_{T,M,(\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon F^{l}_{T,M,(1,2)}\left(\underline{\boldsymbol{V}}\right) \\
& + F^{l}_{T,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon F^{l}_{T,C,(1,2)}\left(\underline{\boldsymbol{V}}\right) \\
& + \frac{1}{\epsilon} F_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) + F_{W,C,(1,2,\xi)}\left(\underline{\boldsymbol{V}}\right) \\
& + \epsilon F_{W,C,(1,2)}\left(\underline{\boldsymbol{V}}\right),
\end{aligned}
$$

$$
\begin{aligned}
& a^{+}\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + a^{-}\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \frac{1}{\epsilon} a^{l}_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + a^{l}_{C,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \epsilon a^{l}_{C,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) + a^{l}_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \epsilon a^{l}_{M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& + \epsilon^{2} a^{l}_{M,(1,2)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}, \underline{\boldsymbol{V}}\right) \\
& = F\left(\underline{\boldsymbol{V}}\right) + \epsilon F^{l}_{T,M,(\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon^{2} F_{T,M,(1,2)}\left(\underline{\boldsymbol{V}}\right) \\
& + F^{l}_{T,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon F^{l}_{T,C,(1,2)}\left(\underline{\boldsymbol{V}}\right) \\
& + \frac{1}{\epsilon} F_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) + F_{W,C,(1,2,\xi)}\left(\underline{\boldsymbol{V}}\right) \\
& + F_{W,C,(1,2)}\left(\underline{\boldsymbol{V}}\right) + \frac{1}{\epsilon} F_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) \\
& + F_{W,C,(1,2,\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon F_{W,C,(1,2)}\left(\underline{\boldsymbol{V}}\right).
\end{aligned}
$$

Formal Ansatz:

$$
\underline{\boldsymbol{U}}^{l,\boldsymbol{y}} := \sum_{i=0}^{\infty} \epsilon^{i} \underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_{i}(x_1, x_2, \xi) \quad \in \Omega^{l}, \qquad \underline{\boldsymbol{U}}^{\boldsymbol{y}} := \sum_{i=0}^{\infty} \epsilon^{i} \underline{\boldsymbol{U}}^{\boldsymbol{y}}_{i}(x_1, x_2, x_3) \quad \in \Omega^{+} \cup \Omega^{-}.
$$

First limit problems

Insert the formal expansion into the weak, scaled problem and sort with respect to ϵ-powers:

ϵ^{-1}-power

$$a^{l}_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)+a^{l}_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)=F_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right)$$

$$a^{l}_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)=F_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right)$$

ϵ^{0}-power

$$\begin{aligned}&a^{+}\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)+a^{-}\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)\\&+a^{l}_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{1},\underline{\boldsymbol{V}}\right)+a^{l}_{C,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)\\&+a^{l}_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{1},\underline{\boldsymbol{V}}\right)+a^{l}_{M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)\\&=F\left(\underline{\boldsymbol{V}}\right)+F^{l}_{T,M,(\xi)}\left(\underline{\boldsymbol{V}}\right)+F^{l}_{T,C,(\xi)}\left(\underline{\boldsymbol{V}}\right)\\&+F^{l}_{W,C,(1,2,\xi)}\left(\underline{\boldsymbol{V}}\right),\end{aligned}$$

$$\begin{aligned}&a^{+}\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)+a^{-}\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)\\&+a^{l}_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{1},\underline{\boldsymbol{V}}\right)+a^{l}_{C,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)\\&+a^{l}_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right)\\&=F\left(\underline{\boldsymbol{V}}\right)+F^{l}_{T,C,(\xi)}\left(\underline{\boldsymbol{V}}\right)+F^{l}_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right),\end{aligned}$$

First limit problems

Insert the formal expansion into the weak, scaled problem and sort with respect to ϵ-powers:

ϵ^{-1}-power

$$a^l_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) + a^l_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) = F_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right)$$

$$a^l_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) = F_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right)$$

ϵ^{0}-power

$$\begin{aligned} &a^+\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) + a^-\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) \\ &+a^l_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_1, \underline{\boldsymbol{V}}\right) + a^l_{C,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) \\ &+a^l_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_1, \underline{\boldsymbol{V}}\right) + a^l_{M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) \\ &= F\left(\underline{\boldsymbol{V}}\right) + F^l_{T,M,(\xi)}\left(\underline{\boldsymbol{V}}\right) + F^l_{T,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) \\ &+F^l_{W,C,(1,2,\xi)}\left(\underline{\boldsymbol{V}}\right), \end{aligned}$$

$$\begin{aligned} &a^+\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) + a^-\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) \\ &+a^l_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_1, \underline{\boldsymbol{V}}\right) + a^l_{C,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) \\ &+a^l_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_0, \underline{\boldsymbol{V}}\right) \\ &= F\left(\underline{\boldsymbol{V}}\right) + F^l_{T,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) + F^l_{W,C,(\xi)}\left(\underline{\boldsymbol{V}}\right), \end{aligned}$$

ϵ^{-1}:

$\underline{\boldsymbol{W}}$ is affine with respect to ξ in Ω^l

$\Rightarrow \underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_{C,0}$ is independent of ξ (damaged and undamaged electrodes) and $\underline{\boldsymbol{U}}^{l,\boldsymbol{y}}_{M,0}$ is independent of ξ (undamaged electrodes).

First limit problems

BOSCH
Universität Stuttgart

ϵ^0:

$\Rightarrow \underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}_{M,0}$ is independent of ξ (damaged electrodes)

ϵ^0:

$\Rightarrow \underline{\boldsymbol{U}}_{M,0}^{l,\underline{\boldsymbol{y}}}$ is independent of ξ (damaged electrodes)

Derivation of jump conditions for the displacement and the stress field outside the layer from S^+ and S^- (identification of S^+ with $S^- = S$)

- Transformation of the volume integral which defines the bilinear forms a_M^l and a_C^l into a boundary integral expression.
- Use, that $\underline{\boldsymbol{U}}_0^{l,\underline{\boldsymbol{y}}}$ and $\underline{\boldsymbol{W}}^{l,\underline{\boldsymbol{y}}}$ do not depend on ξ in Ω^l.
- Insert the transmission conditions of the original problem along S^+ and S^-.

ϵ^0:

$\Rightarrow \underline{\boldsymbol{U}}_{M,0}^{l,\underline{\boldsymbol{y}}}$ is independent of ξ (damaged electrodes)

Derivation of jump conditions for the displacement and the stress field outside the layer from S^+ and S^- (identification of S^+ with $S^- = S$)

- Transformation of the volume integral which defines the bilinear forms a_M^l and a_C^l into a boundary integral expression.
- Use, that $\underline{\boldsymbol{U}}_0^{l,\underline{\boldsymbol{y}}}$ and $\underline{\boldsymbol{W}}^{l,\underline{\boldsymbol{y}}}$ do not depend on ξ in Ω^l.
- Insert the transmission conditions of the original problem along S^+ and S^-.

Jump conditions for undamaged electrodes

$$\left[\underline{\boldsymbol{u}}^{\underline{\boldsymbol{y}}}\right]_S = \underline{\mathbf{0}}$$

$$\left[\underline{\boldsymbol{\sigma}}_n^{\underline{\boldsymbol{y}}}\left(\underline{\boldsymbol{u}}_0^{\underline{\boldsymbol{y}}}, \Phi_0^{\underline{\boldsymbol{y}}}\right)\right]_S = \underline{\mathbf{0}}$$

ϵ^0:

$\Rightarrow \underline{\boldsymbol{U}}_{M,0}^{l,\boldsymbol{y}}$ is independent of ξ (damaged electrodes)

Derivation of jump conditions for the displacement and the stress field outside the layer from S^+ and S^- (identification of S^+ with $S^- = S$)

- Transformation of the volume integral which defines the bilinear forms a_M^l and a_C^l into a boundary integral expression.
- Use, that $\underline{\boldsymbol{U}}_0^{l,\boldsymbol{y}}$ and $\underline{\boldsymbol{W}}^{l,\boldsymbol{y}}$ do not depend on ξ in Ω^l.
- Insert the transmission conditions of the original problem along S^+ and S^-.

Jump conditions for undamaged electrodes

$$\left[\underline{\boldsymbol{u}}^{\boldsymbol{y}}\right]_S = \underline{\boldsymbol{0}}$$
$$\left[\underline{\boldsymbol{\sigma}}_n^{\boldsymbol{y}}\left(\underline{\boldsymbol{u}}_0^{\boldsymbol{y}}, \Phi_0^{\boldsymbol{y}}\right)\right]_S = \underline{\boldsymbol{0}}$$

Jump conditions for damaged electrodes

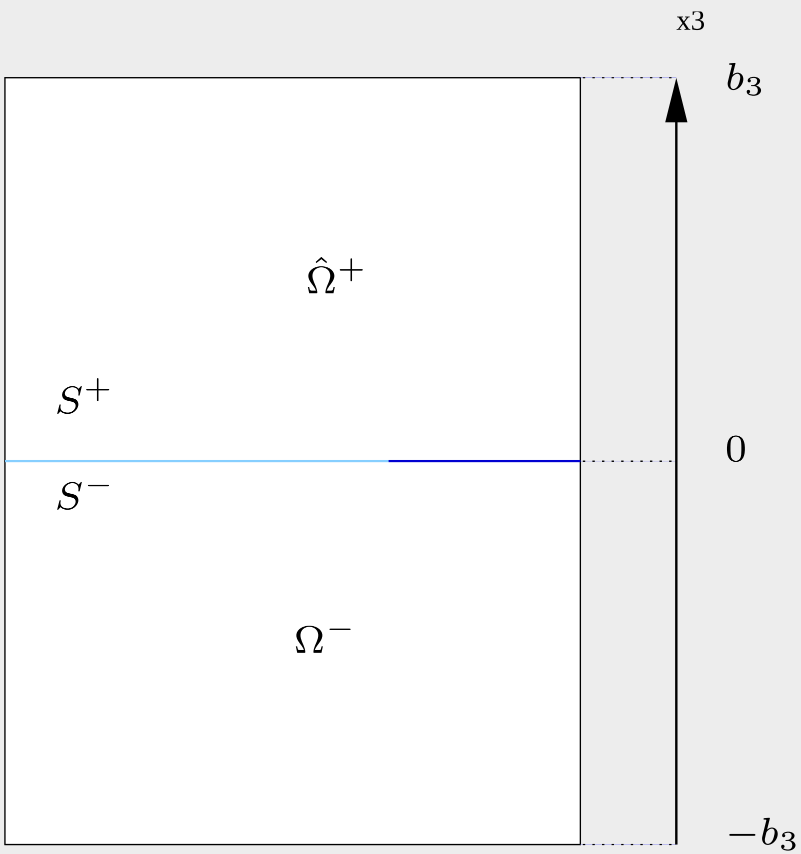
$$\left[\underline{\boldsymbol{u}}_0^{\boldsymbol{y}}\right]_S = h_0\left(\underline{\underline{\boldsymbol{A}}}_{(3)}^{\top}\underline{\underline{\boldsymbol{C}}}_M\underline{\underline{\boldsymbol{A}}}_{(3)}\right)^{-1}\underline{\boldsymbol{\sigma}}_n^{\boldsymbol{y}}\left(\underline{\boldsymbol{u}}_0^{\boldsymbol{y}}, \Phi_0^{\boldsymbol{y}}\right)$$
$$\left[\underline{\boldsymbol{\sigma}}_n^{\boldsymbol{y}}\left(\underline{\boldsymbol{u}}_0^{\boldsymbol{y}}, \Phi_0^{\boldsymbol{y}}\right)\right]_S = \underline{\boldsymbol{0}}.$$

First limit problem (undamaged electrode)

BOSCH

Universität Stuttgart

Asymptotic problem in $\hat{\Omega}$



Linear Voigt Model (ceramic) & Hooke's law (metal-electrode) for the composite

$$-\mathcal{D}^\top \underline{\underline{\boldsymbol{C}}}_C \mathcal{D}\underline{\hat{\boldsymbol{u}}} - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla\hat{\Phi} = \underline{\boldsymbol{F}}_C(\hat{T}) \text{ in } \hat{\Omega},$$

$$\operatorname{div}\left(\underline{\underline{\boldsymbol{e}}}\,\mathcal{D}\underline{\hat{\boldsymbol{u}}} - \underline{\underline{\boldsymbol{\varepsilon}}}\,\nabla\hat{\Phi}\right) = f_C(\hat{T}) \quad \text{in } \hat{\Omega},$$

Boundary conditions

$$\boldsymbol{\sigma}_{C_n}(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}) = \underline{\boldsymbol{0}} \qquad \text{on } \Gamma_2 \cup \Gamma_1 \cup \Gamma_3^+$$

$$\underline{\hat{\boldsymbol{u}}} = \underline{\boldsymbol{0}} \qquad \text{on } \Gamma_3^-$$

$$D_n(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}) = 0 \qquad \text{on } \partial\Omega \setminus (\Gamma_2 \cup S)$$

$$\hat{\Phi} = \pm\Phi_a \qquad \text{on } \Gamma_2 \cup S$$

Transmission conditions on S:

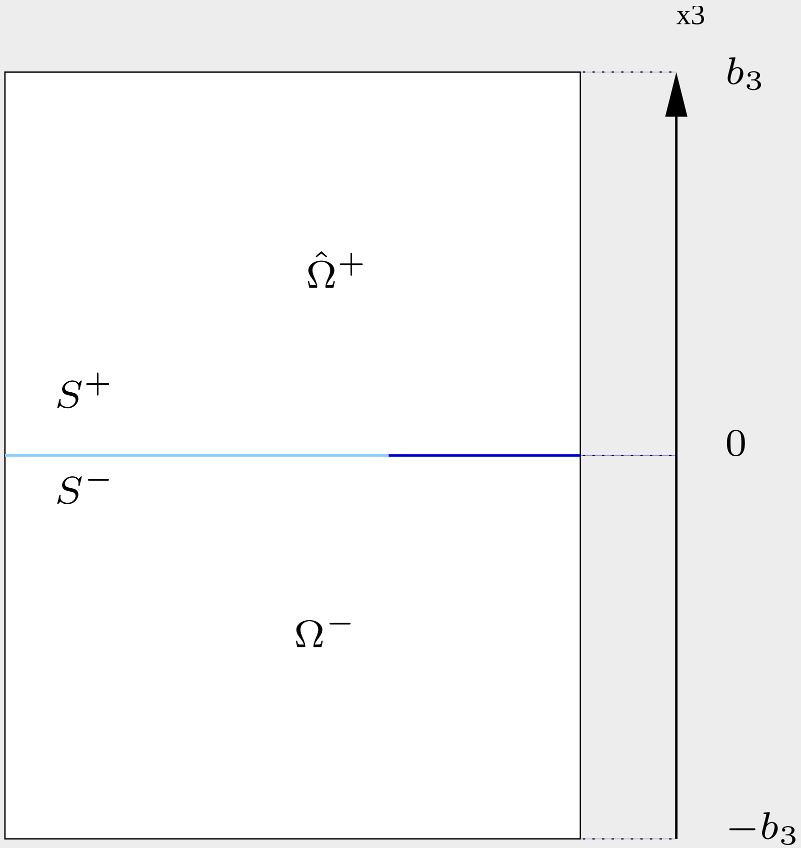
$$[\underline{\hat{\boldsymbol{u}}}]_S = \underline{\boldsymbol{0}}$$

$$\left[\underline{\boldsymbol{\sigma}}_n\left(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}\right)\right]_S = \underline{\boldsymbol{0}}$$

BOSCH

Universität Stuttgart

Asymptotic problem in $\hat{\Omega}$



Linear Voigt Model (ceramic) & Hooke's law (metal-electrode) for the composite

$$-\mathcal{D}^\top \underline{\underline{\boldsymbol{C}}}_C \mathcal{D}\underline{\hat{\boldsymbol{u}}} - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla\hat{\Phi} = \underline{\boldsymbol{F}}_C(\hat{T}) \text{ in } \hat{\Omega},$$

$$\operatorname{div}\left(\underline{\underline{\boldsymbol{e}}}\,\mathcal{D}\underline{\hat{\boldsymbol{u}}} - \underline{\underline{\boldsymbol{\varepsilon}}}\,\nabla\hat{\Phi}\right) = f_C(\hat{T}) \quad \text{in } \hat{\Omega},$$

Boundary conditions

$$\boldsymbol{\sigma}_{C_n}(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}) = \underline{\boldsymbol{0}} \qquad \text{on } \Gamma_2 \cup \Gamma_1 \cup \Gamma_3^+$$

$$\underline{\hat{\boldsymbol{u}}} = \underline{\boldsymbol{0}} \qquad \text{on } \Gamma_3^-$$

$$D_n(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}) = 0 \qquad \text{on } \partial\Omega \setminus (\Gamma_2 \cup S)$$

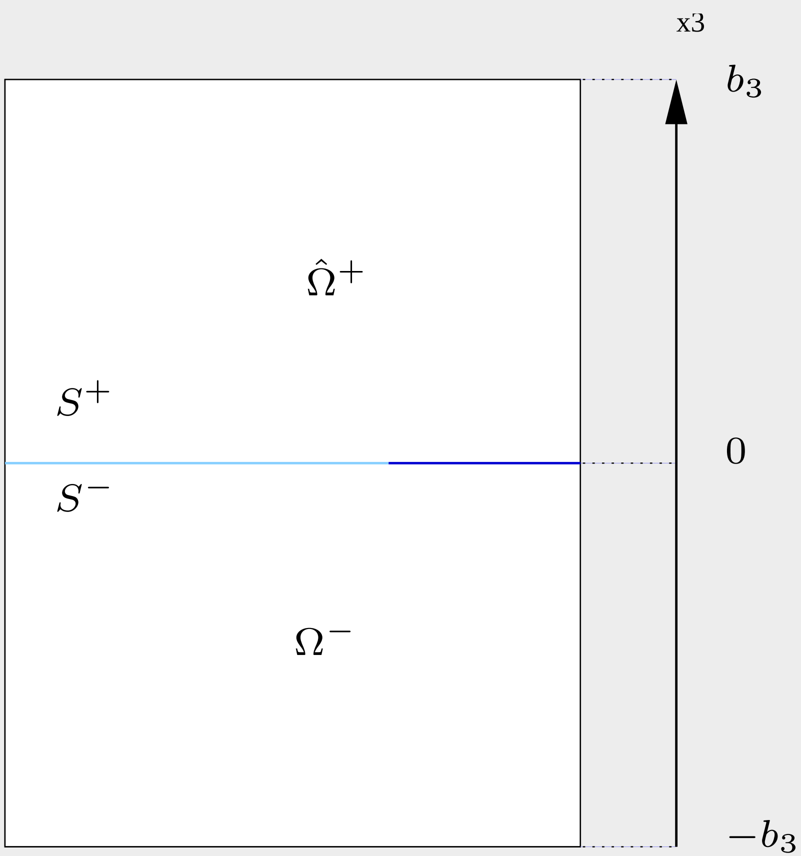
$$\hat{\Phi} = \pm\Phi_a \qquad \text{on } \Gamma_2 \cup S$$

Transmission conditions on S:

$$[\underline{\hat{\boldsymbol{u}}}]_S = h_0 \left(\underline{\underline{\boldsymbol{A}}}_{(3)}^\top \underline{\underline{\boldsymbol{C}}}_M \underline{\underline{\boldsymbol{A}}}_{(3)}\right)^{-1} \underline{\boldsymbol{\sigma}}_n \left(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}_0\right)$$

$$\left[\underline{\boldsymbol{\sigma}}_n\left(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}\right)\right]_S = \underline{\boldsymbol{0}}$$

Asymptotic problem in $\hat{\Omega}$



Linear Voigt Model (ceramic) & Hooke's law (metal-electrode) for the composite

$$-\mathcal{D}^\top \underline{\underline{\boldsymbol{C}}}_C \mathcal{D}\underline{\hat{\boldsymbol{u}}} - \mathcal{D}^\top \underline{\underline{\boldsymbol{e}}}^\top \nabla \hat{\Phi} = \underline{\boldsymbol{F}}_C(\hat{T}) \text{ in } \hat{\Omega},$$

$$\operatorname{div}\left(\underline{\underline{\boldsymbol{e}}}\,\mathcal{D}\underline{\hat{\boldsymbol{u}}} - \underline{\underline{\boldsymbol{\varepsilon}}}\,\nabla\hat{\Phi}\right) = f_C(\hat{T}) \quad \text{in } \hat{\Omega},$$

Boundary conditions

$$\boldsymbol{\sigma}_{C_n}(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}) = \underline{\mathbf{0}} \qquad \text{on } \Gamma_2 \cup \Gamma_1 \cup \Gamma_3^+$$

$$\underline{\hat{\boldsymbol{u}}} = \underline{\mathbf{0}} \qquad \text{on } \Gamma_3^-$$

$$D_n(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}) = 0 \qquad \text{on } \partial\Omega \setminus (\Gamma_2 \cup S)$$

$$\hat{\Phi} = \pm\Phi_a \qquad \text{on } \Gamma_2 \cup S$$

Transmission conditions on S:

$$[\underline{\hat{\boldsymbol{u}}}]_S = h_0 \underline{\underline{\boldsymbol{T}}}_{(3)}^{-1} \boldsymbol{\sigma}_n\left(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}_0\right)$$

$$\left[\boldsymbol{\sigma}_n\left(\underline{\hat{\boldsymbol{u}}}, \hat{\Phi}\right)\right]_S = \underline{\mathbf{0}}$$

Existence and uniqueness

Theorem.

The weak formulated first limit problem has a unique solution in $\hat{\mathcal{V}}_0$ in the undamaged and damaged case.

Existence and uniqueness

Theorem.

The weak formulated first limit problem has a unique solution in $\hat{\mathcal{V}}_0$ in the undamaged and damaged case.

Sketch of proof.

Continuity

Clear.

Ellipticity

$$a\left(\underline{\hat{\boldsymbol{U}}}, \underline{\hat{\boldsymbol{U}}}\right) = \int_{\hat{\Omega}} \begin{pmatrix} \underline{\boldsymbol{\gamma}} \\ \underline{\boldsymbol{E}} \end{pmatrix}^{\top} \begin{pmatrix} \underline{\underline{\boldsymbol{C}}} & -\underline{\underline{\boldsymbol{e}}}^{\top} \\ \underline{\underline{\boldsymbol{e}}} & \underline{\underline{\boldsymbol{\varepsilon}}} \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{\gamma}} \\ \underline{\boldsymbol{E}} \end{pmatrix} \mathrm{d}\underline{\boldsymbol{x}} + \left\langle \frac{1}{h_0} \underline{\underline{\boldsymbol{T}}}_{(3)} \left[\underline{\hat{\boldsymbol{u}}}\right], \left[\underline{\hat{\boldsymbol{u}}}\right] \right\rangle_S$$

$$= \int_{\Omega} \underline{\boldsymbol{\gamma}}\, \underline{\underline{\boldsymbol{C}}}\, \underline{\boldsymbol{\gamma}} + \underline{\boldsymbol{E}}^{\top} \underline{\underline{\boldsymbol{\epsilon}}}\, \underline{\boldsymbol{E}} \, \mathrm{d}\underline{\boldsymbol{x}} + \left\langle \frac{1}{h_0} \underline{\underline{\boldsymbol{T}}}_{(3)} \left[\underline{\hat{\boldsymbol{u}}}\right], \left[\underline{\hat{\boldsymbol{u}}}\right] \right\rangle_S$$

BOSCH

Universität Stuttgart

Theorem.

The weak formulated first limit problem has a unique solution in $\hat{\mathcal{V}}_0$ in the undamaged and damaged case.

Sketch of proof.

Continuity

Clear.

Ellipticity

$$a\left(\underline{\hat{\boldsymbol{U}}}, \underline{\hat{\boldsymbol{U}}}\right) = \int_{\hat{\Omega}} \begin{pmatrix} \underline{\gamma} \\ \underline{\boldsymbol{E}} \end{pmatrix}^{\top} \begin{pmatrix} \underline{\underline{\boldsymbol{C}}} & -\underline{\underline{\boldsymbol{e}}}^{\top} \\ \underline{\underline{\boldsymbol{e}}} & \underline{\underline{\boldsymbol{\varepsilon}}} \end{pmatrix} \begin{pmatrix} \underline{\gamma} \\ \underline{\boldsymbol{E}} \end{pmatrix} \mathrm{d}\underline{\boldsymbol{x}} + \left\langle \frac{1}{h_0} \underline{\underline{\boldsymbol{T}}}_{(3)} [\underline{\hat{\boldsymbol{u}}}], [\underline{\hat{\boldsymbol{u}}}] \right\rangle_S$$

$$= \int_{\Omega} \underline{\gamma}\, \underline{\underline{\boldsymbol{C}}}\, \underline{\gamma} + \underline{\boldsymbol{E}}^{\top} \underline{\underline{\boldsymbol{\epsilon}}}\, \underline{\boldsymbol{E}} \,\mathrm{d}\underline{\boldsymbol{x}} + \left\langle \frac{1}{h_0} \underline{\underline{\boldsymbol{T}}}_{(3)} [\underline{\hat{\boldsymbol{u}}}], [\underline{\hat{\boldsymbol{u}}}] \right\rangle_S$$

Mechanical part of the first term

$$\int_{\Omega} \gamma^{\top} \underline{\underline{\boldsymbol{C}}}\, \gamma \,\mathrm{d}\underline{\boldsymbol{x}} \geq C_0 \, \|\gamma\|^2_{[L_2(\Omega)]^3}$$

$$\overset{\text{Korn}}{\geq} C_{0,\text{Korn}}(C_0, \Omega, \Gamma_M^D) \, \|\underline{\boldsymbol{u}}\|^2_{[L_2(\Omega)]^2}$$

Existence and uniqueness

Electrical part of the first term

$$\int_\Omega \underline{\boldsymbol{E}}^\top \underline{\underline{\boldsymbol{\varepsilon}}}\, \underline{\boldsymbol{E}}\, \mathrm{d}\underline{\boldsymbol{x}} \geq \varepsilon_0 \int_\Omega \nabla\Phi\nabla\Phi\, \mathrm{d}\underline{\boldsymbol{x}}$$

$$\overset{\text{Friedrichs}}{\geq} \varepsilon_{0,\text{Friedrichs}}(\varepsilon_0, \Omega, \Gamma_e^D)\, \|\Phi\|^2_{\hat{\mathcal{V}}_0}\,.$$

Electrical part of the first term

$$\int_\Omega \underline{\boldsymbol{E}}^\top \underline{\underline{\boldsymbol{\varepsilon}}}\, \underline{\boldsymbol{E}} \,\mathrm{d}\underline{\boldsymbol{x}} \geq \varepsilon_0 \int_\Omega \nabla\Phi\nabla\Phi \,\mathrm{d}\underline{\boldsymbol{x}}$$

$$\overset{\text{Friedrichs}}{\geq} \varepsilon_{0,\text{Friedrichs}}(\varepsilon_0, \Omega, \Gamma_e^D)\, \|\Phi\|^2_{\hat{\mathcal{V}}_0}\,.$$

$\underline{\underline{\boldsymbol{T}}}$ is positive definite, and therefore

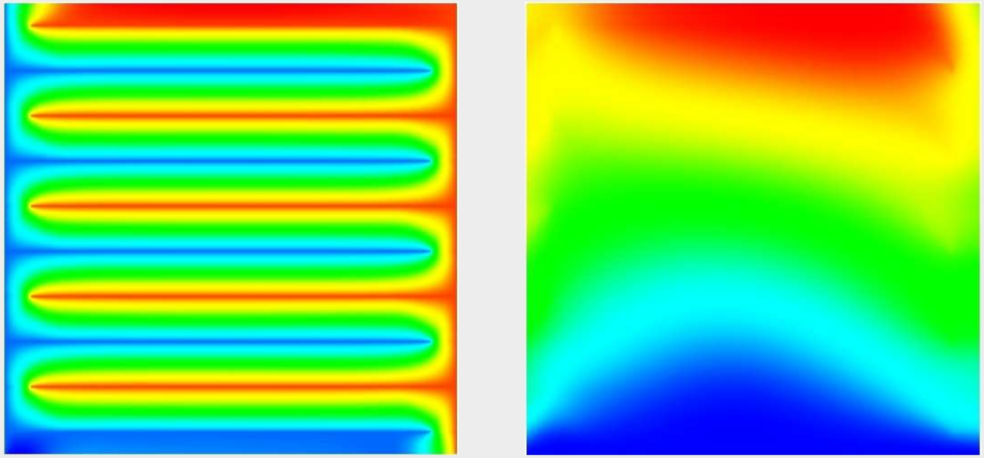
$$\left\langle \frac{1}{h_0} \underline{\underline{\boldsymbol{T}}}_{(3)} [\underline{\hat{\boldsymbol{u}}}]\,,[\underline{\hat{\boldsymbol{u}}}] \right\rangle_S \geq s_0 \left\langle [\underline{\hat{\boldsymbol{u}}}]\,,[\underline{\hat{\boldsymbol{u}}}] \right\rangle_S \geq 0$$

and therefore

$$a\left(\underline{\hat{\boldsymbol{U}}}\,,\underline{\hat{\boldsymbol{U}}}\right) \geq C \left\| \underline{\hat{\boldsymbol{U}}} \right\|^2_{\hat{\mathcal{V}}(\hat{\Omega})}\,.$$

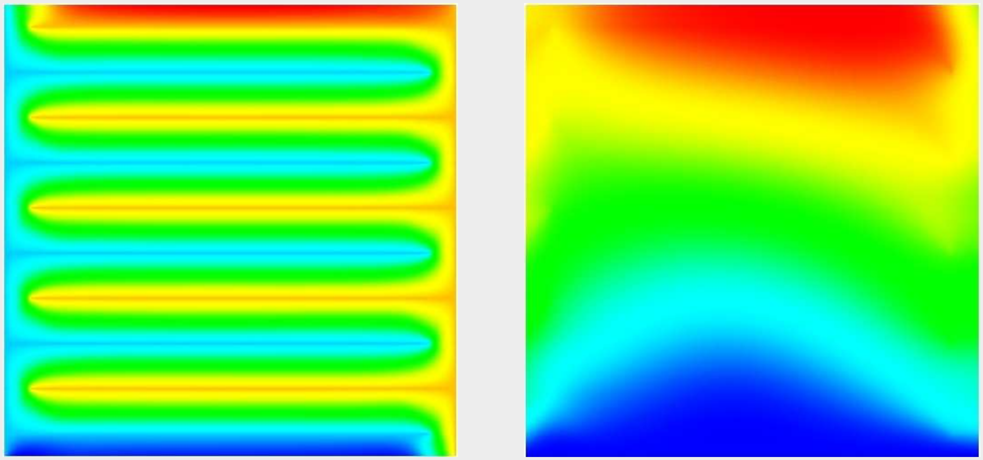
Numerical example (temperature)

Stack at 20°C



PZT 4, 20°C	
potential [kV]	[-0.256,0.243]
stroke [mm]	0.000828

Stack after heating at 30°C



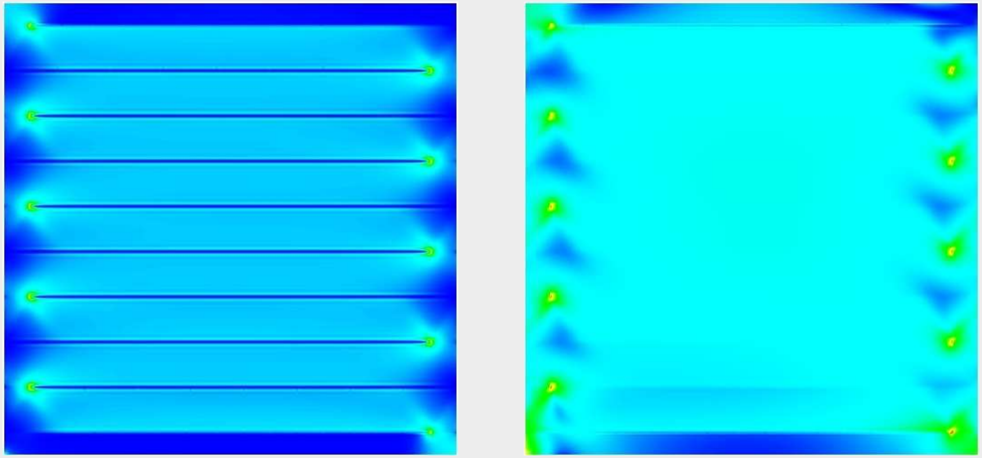
PZT 4, 30°C	
potential [kV]	[-0.311,0.303]
stroke [mm]	0.000893

Numerical example (temperature)

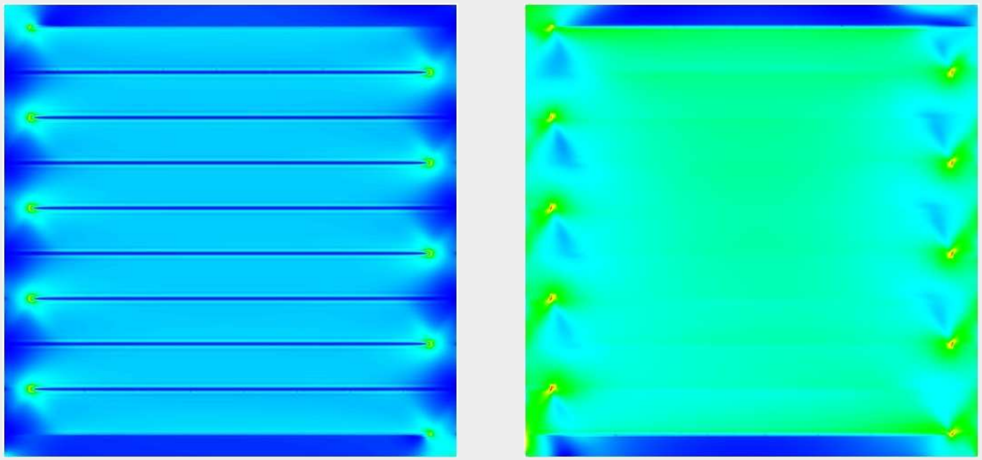
BOSCH

Universität Stuttgart

Stack at 20°C



Stack after heating at 30°C



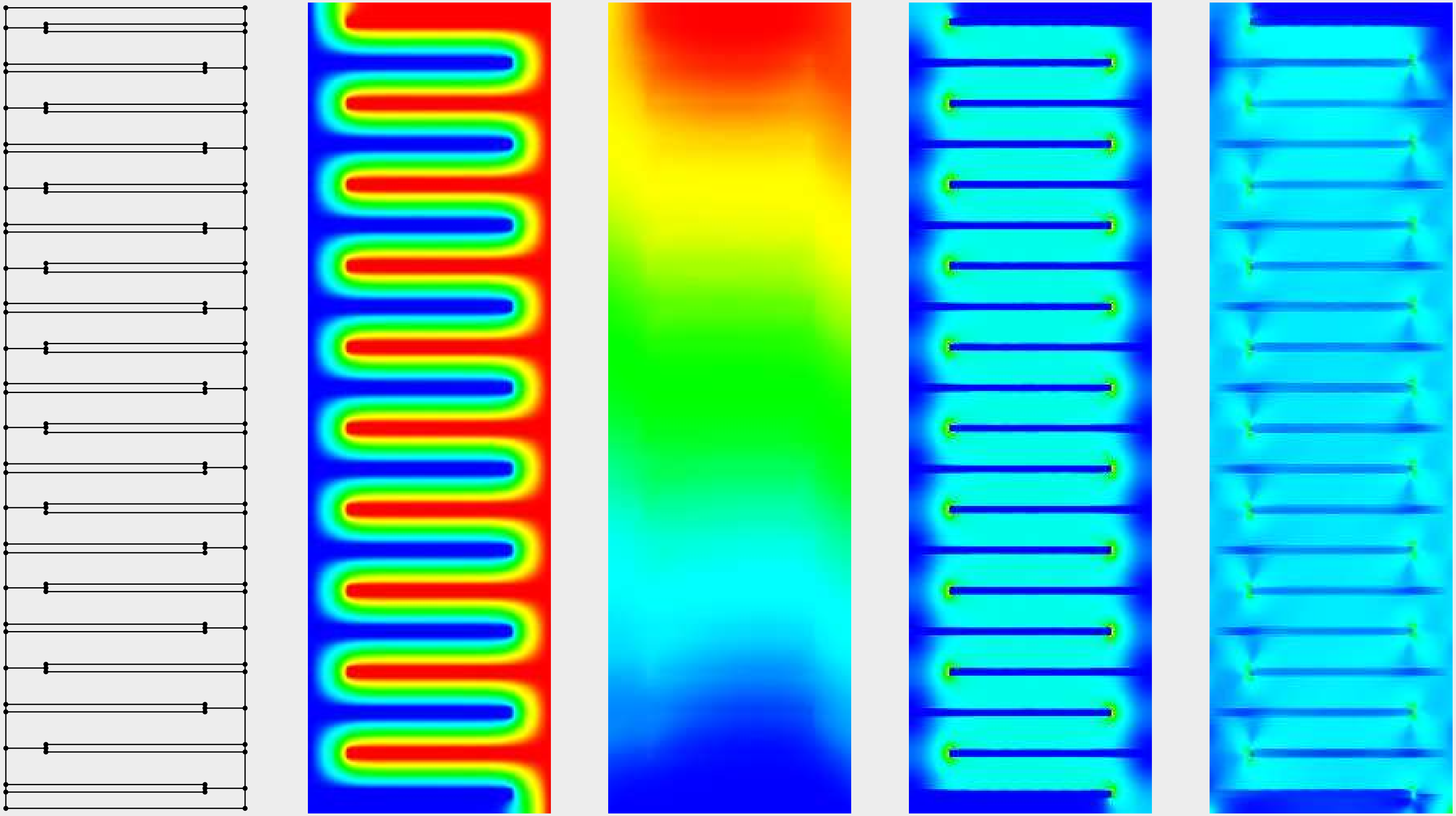
PZT 4, 20°C	
iterations	435
nodes	29857

PZT 4, 30°C	
iterations	429
nodes	29857

BOSCH

Universität Stuttgart

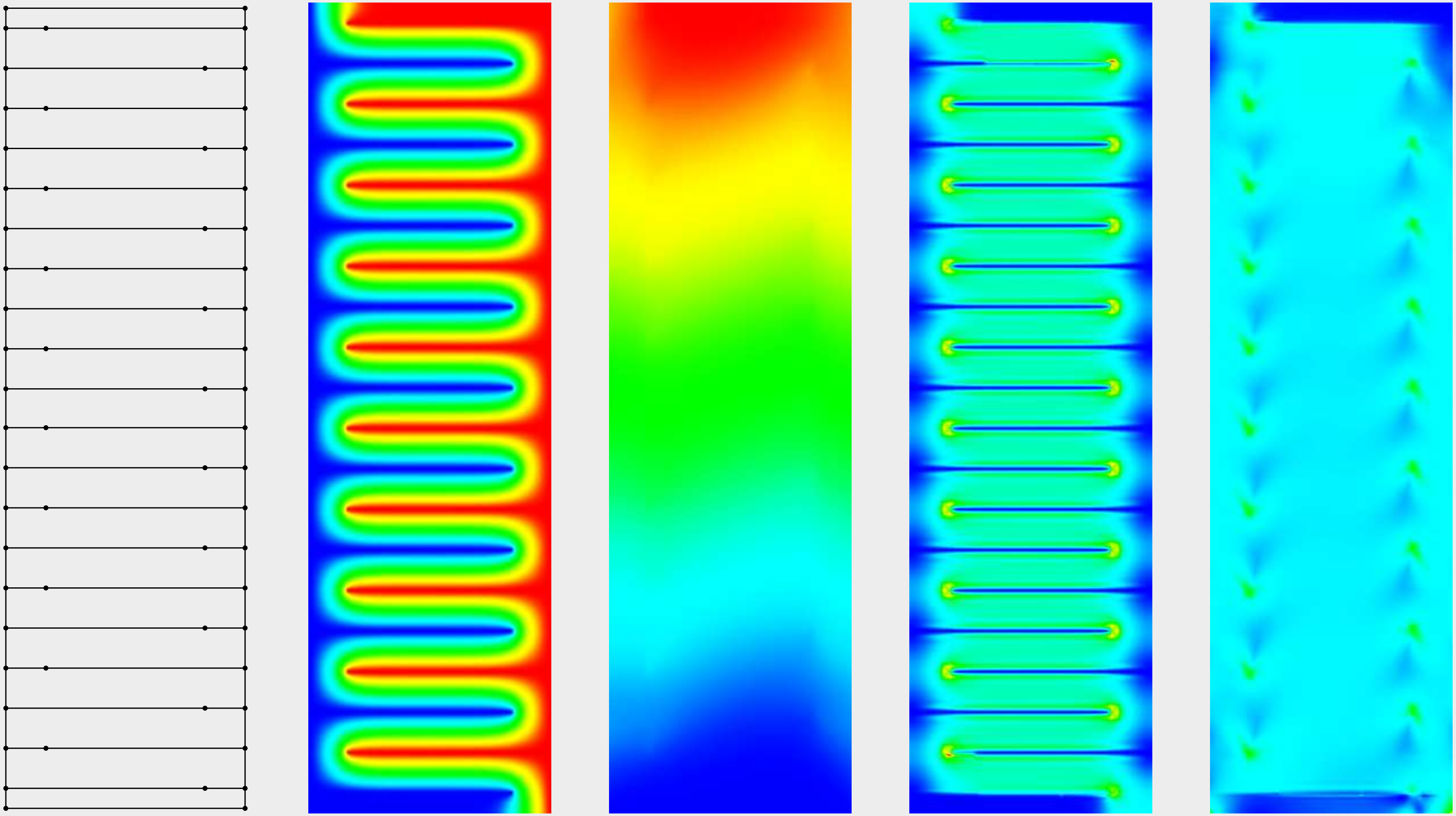
Full 2D-Problem (geometry, potential field, displacement, elektric field, stress)



	time [s]	iterations	nodes	potential [V]	stroke [mm]	CPU
BPCG	3250	1009	117713	$[-155, 161]$	0.000789	Athlon XP 2000+

Numerical example (Comparison full/asymptotic)

Full 2D-Problem (geometry, potential field, displacement, elektric field, stress)



	time [s]	iterations	nodes	potential [V]	stroke $[mm]$	CPU
BPCG	318	717	11497	$[-155, 161]$	0.000788	Athlon XP 2000+

BOSCH

Universität Stuttgart

Part 2: Mathematical analysis (undamaged electrodes)

- Introduction
  - Difficulties arising from the limit problems
  - Goals
- Recapitulation of the different problems
  - Exact problem P_ϵ
  - Scaled problem $P(\epsilon)$
  - First limit problem P_0
  - Equivalent first limit problem $\hat{P}_0$
- Existence of solutions
- Weak convergence
- Conclusions

Challenge

$P(\epsilon)$ is a typical singular perturbation problem:

Challenge

$P(\epsilon)$ is a typical singular perturbation problem:

- ➜ For $\epsilon \longrightarrow 0$, the exact solutions do not converge in the function space $\mathcal{V}$, where $\underline{\boldsymbol{U}}$ exists but in a larger space $\mathcal{V}_0$, specified by the limit problem.

BOSCH

Universität Stuttgart

Challenge

$P(\epsilon)$ is a typical singular perturbation problem:

- For $\epsilon \longrightarrow 0$, the exact solutions do not converge in the function space $\mathcal{V}$, where $\underline{\boldsymbol{U}}$ exists but in a larger space $\mathcal{V}_0$, specified by the limit problem.
- A careful mathematical analysis is necessary to provide existence and uniqueness of the
  - exact solution $\underline{\boldsymbol{U}}_{\epsilon}$
  - scaled solution $\underline{\boldsymbol{U}}(\epsilon)$
  - limit problem's solution $\underline{\boldsymbol{U}}_0$
  - limit problem's solution $\underline{\hat{\boldsymbol{U}}}_0$ in the domain $\hat{\Omega}$

  in appropriate function spaces.

BOSCH

Universität Stuttgart

Challenge

$P(\epsilon)$ is a typical singular perturbation problem:

- For $\epsilon \to 0$, the exact solutions do not converge in the function space $\mathcal{V}$, where $\underline{\boldsymbol{U}}$ exists but in a larger space $\mathcal{V}_0$, specified by the limit problem.
- A careful mathematical analysis is necessary to provide existence and uniqueness of the
  - exact solution $\underline{\boldsymbol{U}}_{\epsilon}$
  - scaled solution $\underline{\boldsymbol{U}}(\epsilon)$
  - limit problem's solution $\underline{\boldsymbol{U}}_0$
  - limit problem's solution $\underline{\hat{\boldsymbol{U}}}_0$ in the domain $\hat{\Omega}$

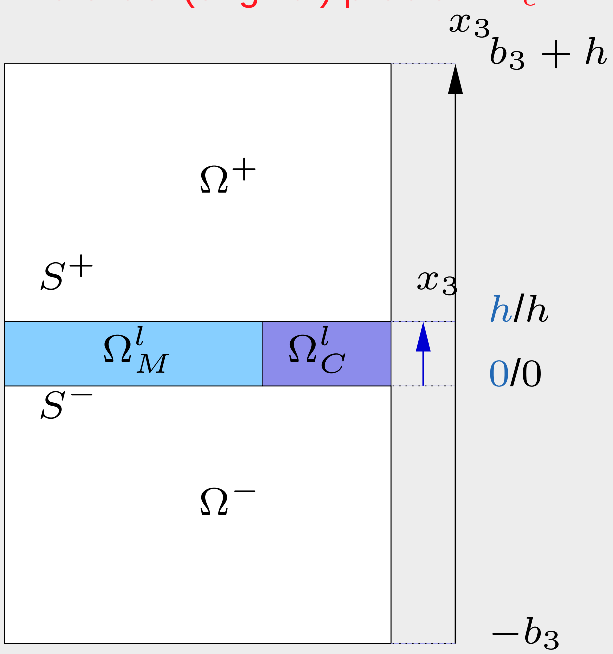
  in appropriate function spaces.

Main goal

For $\epsilon \to 0$, $\epsilon > 0$: Analysis of the convergence of $\underline{\boldsymbol{U}}(\epsilon)$ to the solution $\underline{\boldsymbol{U}}_0$ of the fi rst limit problem.

The different problems - P_ϵ

The exact (original) problem P_ϵ:



$$\Omega^l = \Omega^l_M \cup \Omega^l_C,$$

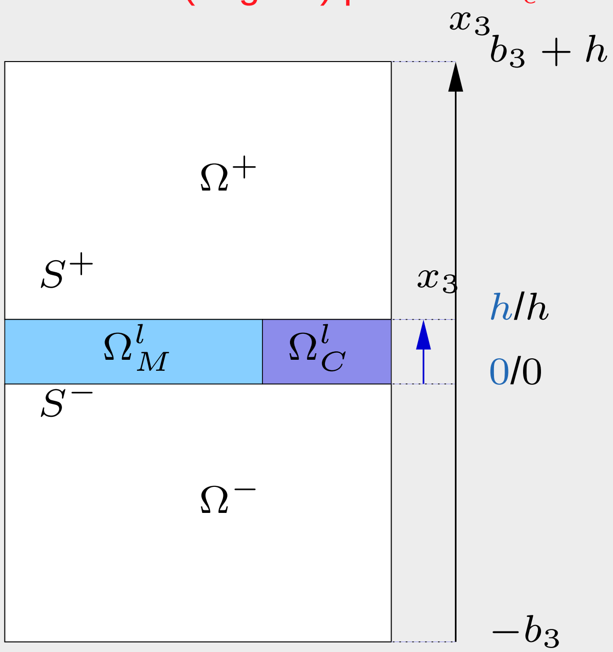
$$\Omega = \Omega^+ \cup \Omega^- \cup \Omega^l \cup S^+ \cup S^- \cup S_{CM},$$

$$\mathcal{V}(\Omega) = \Big\{ \underline{\boldsymbol{U}} \in \mathrm{H}^1(\Omega) : \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-,$$

$$\Phi = 0 \text{ on } \Omega^l_M \cup \Gamma_2 \Big\}.$$

The different problems - P_ϵ

The exact (original) problem P_ϵ:



$$\Omega^l = \Omega^l_M \cup \Omega^l_C ,$$

$$\Omega = \Omega^+ \cup \Omega^- \cup \Omega^l \cup S^+ \cup S^- \cup S_{CM} ,$$

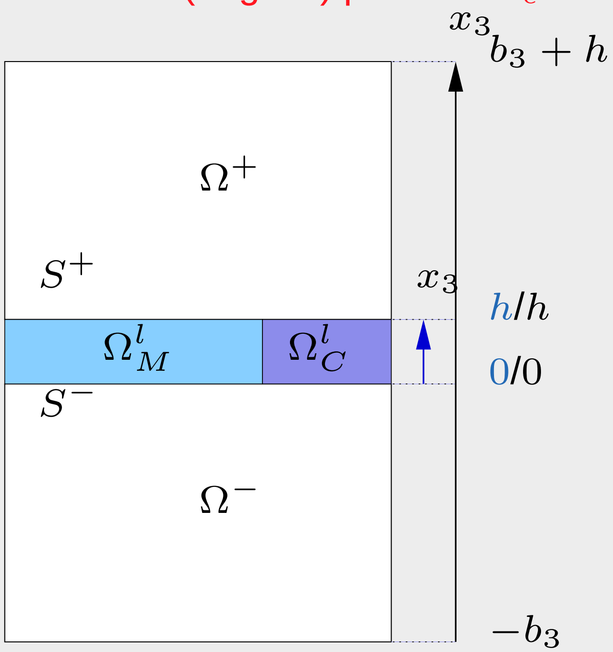
$$\mathcal{V}(\Omega) = \Big\{ \underline{\boldsymbol{U}} \in \mathrm{H}^1(\Omega) : \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^- , \quad \Phi = 0 \text{ on } \Omega^l_M \cup \Gamma_2 \Big\} .$$

Weak problem: Find $\underline{\boldsymbol{U}} \in \mathcal{V}$ such that $\forall \underline{\boldsymbol{V}} \in \mathcal{V}$ holds:

$$a^+(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}}) + a^-(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}}) + a^l_M(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}}) + a^l_C(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}}) = F(\underline{\boldsymbol{V}}) + F^l_M(\underline{\boldsymbol{V}}) + F^l_C(\underline{\boldsymbol{V}}) .$$

The different problems - P_ϵ

The exact (original) problem P_ϵ:



$$\Omega^l = \Omega^l_M \cup \Omega^l_C ,$$

$$\Omega = \Omega^+ \cup \Omega^- \cup \Omega^l \cup S^+ \cup S^- \cup S_{CM} ,$$

$$\mathcal{V}(\Omega) = \Big\{ \underline{\boldsymbol{U}} \in \mathrm{H}^1(\Omega) : \underline{\boldsymbol{u}} = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^- , \quad \Phi = 0 \text{ on } \Omega^l_M \cup \Gamma_2 \Big\} .$$

Weak problem: Find $\underline{\boldsymbol{U}} \in \mathcal{V}$ such that $\forall \underline{\boldsymbol{V}} \in \mathcal{V}$ holds:

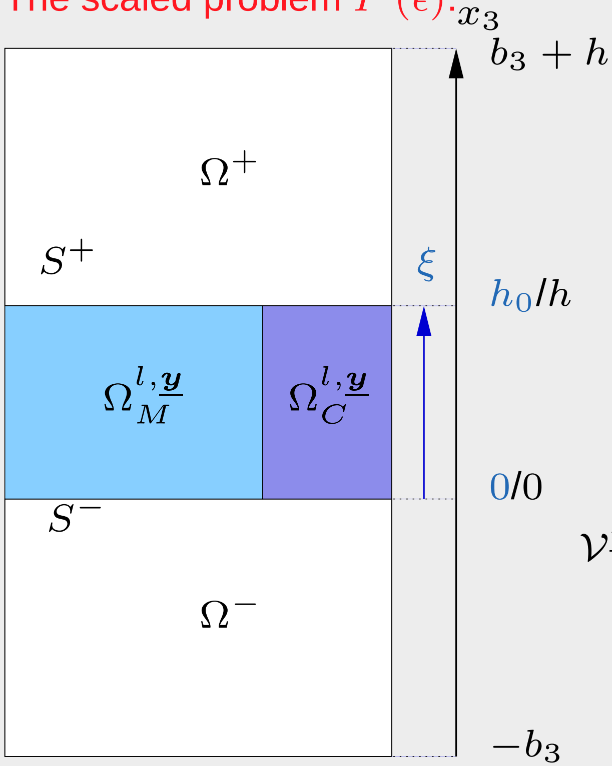
$$a^+(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}}) + a^-(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}}) + a^l_M(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}}) + a^l_C(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}}) = F(\underline{\boldsymbol{V}}) + F^l_M(\underline{\boldsymbol{V}}) + F^l_C(\underline{\boldsymbol{V}}) .$$

Scaling: Introduction of new coordinates $\underline{\boldsymbol{x}} \rightarrow \underline{\boldsymbol{y}}$ with $x_3 = \epsilon\xi$. ϵ now vanishes from the geometrical description ($h \sim \epsilon$) and appears in the differential operators (Problem $P(\epsilon)$).

BOSCH

Universität Stuttgart

The scaled problem $P(\epsilon)$:



$$\Omega^{l,\underline{\boldsymbol{y}}} = \Omega_M^{l,\underline{\boldsymbol{y}}} \cup \Omega_C^{l,\underline{\boldsymbol{y}}},$$

$$= \left\{ \underline{\boldsymbol{y}} = \underline{\boldsymbol{x}} + (1-\epsilon)x_3 \underline{\boldsymbol{e}}_3, \quad \underline{\boldsymbol{x}} \in \Omega^l \right\},$$

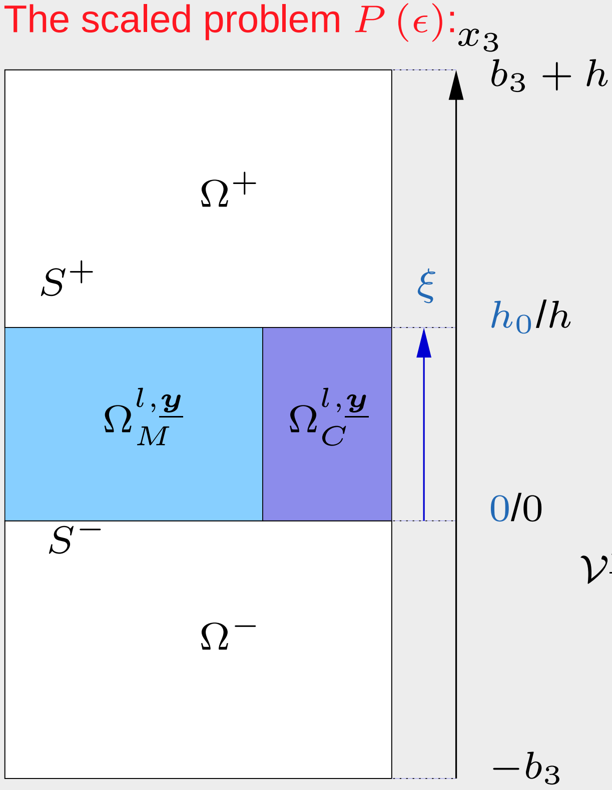
$$\Omega^{\underline{\boldsymbol{y}}} = \Omega^+ \cup \Omega^- \cup \Omega^{l,\underline{\boldsymbol{y}}} \cup S^+ \cup S^- \cup S_{CM},$$

$$\underline{\boldsymbol{U}} \mapsto \underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} = \left(\underline{\boldsymbol{U}}^+, \underline{\boldsymbol{U}}^-, \underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}} \right), \text{ and } \underline{\boldsymbol{U}}^{\pm} = r_{|\Omega^{\pm}} \underline{\boldsymbol{U}},$$

$$\mathcal{V}^{\underline{\boldsymbol{y}}}\left(\Omega^{\underline{\boldsymbol{y}}}\right) = \left\{ \underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} \in \mathrm{H}^1\left(\Omega^{\underline{\boldsymbol{y}}}\right) : \right.$$

$$\left. \underline{\boldsymbol{u}}^- = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-, \ \Phi = 0 \text{ on } \Omega_M^{l,\underline{\boldsymbol{y}}} \cup \Gamma_2 \right\}.$$

The scaled problem $P\left(\epsilon\right)$:



$$\Omega^{l,\underline{\boldsymbol{y}}} = \Omega_M^{l,\underline{\boldsymbol{y}}} \cup \Omega_C^{l,\underline{\boldsymbol{y}}}, \\ = \left\{\underline{\boldsymbol{y}} = \underline{\boldsymbol{x}} + (1-\epsilon)x_3\underline{\boldsymbol{e}}_3, \quad \underline{\boldsymbol{x}} \in \Omega^l\right\},$$

$$\Omega^{\underline{\boldsymbol{y}}} = \Omega^+ \cup \Omega^- \cup \Omega^{l,\underline{\boldsymbol{y}}} \cup S^+ \cup S^- \cup S_{CM},$$

$$\underline{\boldsymbol{U}} \mapsto \underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} = \left(\underline{\boldsymbol{U}}^+, \underline{\boldsymbol{U}}^-, \underline{\boldsymbol{U}}^{l,\underline{\boldsymbol{y}}}\right), \text{ and } \underline{\boldsymbol{U}}^{\pm} = r\big|_{\Omega^\pm}\underline{\boldsymbol{U}},$$

$$\mathcal{V}^{\underline{\boldsymbol{y}}}\left(\Omega^{\underline{\boldsymbol{y}}}\right) = \Big\{\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} \in \mathrm{H}^1\left(\Omega^{\underline{\boldsymbol{y}}}\right): \\ \underline{\boldsymbol{u}}^- = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-, \ \Phi = 0 \text{ on } \Omega_M^{l,\underline{\boldsymbol{y}}} \cup \Gamma_2\Big\}.$$

Weak problem: Find $\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} \in \mathcal{V}^{\underline{\boldsymbol{y}}}$ such that $\forall\, \underline{\boldsymbol{V}} \in \mathcal{V}^{\underline{\boldsymbol{y}}}$ holds:

$$a^+\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right) + a^-\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right) + a^l_{(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right) + \frac{1}{\epsilon}a^l_{(\xi)}\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right) + \epsilon a^l_{(1,2)}\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right) = \\ F\left(\underline{\boldsymbol{V}}\right) + F^l_{(\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon F^l_{(1,2)}\left(\underline{\boldsymbol{V}}\right) + \frac{1}{\epsilon}F_{\underline{\boldsymbol{W}},C,(\xi)}\left(\underline{\boldsymbol{V}}\right) + F_{\underline{\boldsymbol{W}},C,(1,2,\xi)}\left(\underline{\boldsymbol{V}}\right) + \epsilon F_{\underline{\boldsymbol{W}},C,(1,2)}\left(\underline{\boldsymbol{V}}\right). \tag{1}$$

BOSCH Universität Stuttgart

The auxiliary problem P_{-1}:

Insertion of the asymptotic expansion for $\underline{\boldsymbol{U}}^{\,\underline{\boldsymbol{y}}}$ into equation (1) provides:

$$\left.\begin{aligned} a^{l}_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{\,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right) &= 0 \\ a^{l}_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{\,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right) &= 0 \end{aligned}\right\} \Rightarrow \underline{\boldsymbol{U}}^{\,l,\underline{\boldsymbol{y}}}_{0} \text{ is affine with respect to } \xi \tag{2}$$

From (2) follows for test functions $\underline{\boldsymbol{V}}$ which are affine with respect to ξ that :

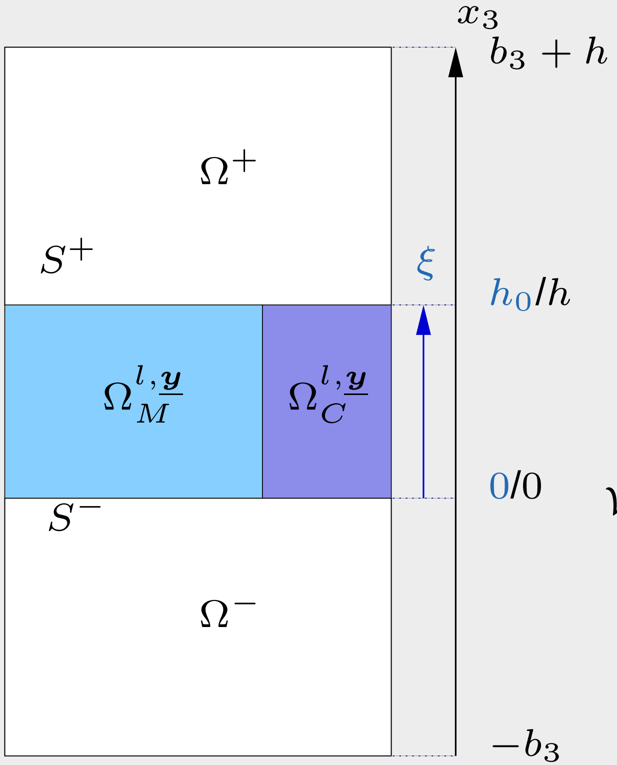
$$\begin{aligned} a^{l}_{M,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{\,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right) &= 0, \\ a^{l}_{C,(1,2,\xi)}\left(\underline{\boldsymbol{U}}^{\,\underline{\boldsymbol{y}}}_{0},\underline{\boldsymbol{V}}\right) &= 0, \\ F_{T,M,(\xi)}\left(\underline{\boldsymbol{V}}\right) = F_{T,C,(\xi)}\left(\underline{\boldsymbol{V}}\right) = F^{l}_{\underline{\boldsymbol{W}},C,(1,2,\xi)}\left(\underline{\boldsymbol{V}}\right) &= 0 \end{aligned} \tag{3}$$

and thus with (3):

$$\begin{aligned} a_{C,(\xi)}\left(\underline{\boldsymbol{U}}^{\,\underline{\boldsymbol{y}}}_{1},\underline{\boldsymbol{V}}\right) &= 0 \\ a_{M,(\xi)}\left(\underline{\boldsymbol{U}}^{\,\underline{\boldsymbol{y}}}_{1},\underline{\boldsymbol{V}}\right) &= 0. \end{aligned} \tag{4}$$

The different problems - P_0

The first limit problem P_0:



$$\begin{aligned}
\Omega^{l,\underline{y}} &= \Omega_M^{l,\underline{y}} \cup \Omega_C^{l,\underline{y}}\,, \\
&= \left\{ \underline{y} = \underline{x} + (1-\epsilon)x_3 \underline{e}_3, \quad \underline{x} \in \Omega^l \right\}, \\
\Omega^{\underline{y}} &= \Omega^+ \cup \Omega^- \cup \Omega^{l,\underline{y}} \cup S^+ \cup S^- \cup S_{CM}\,,
\end{aligned}$$

$$\underline{U}_0 \mapsto \underline{U}_0^{\underline{y}} = \left(\underline{U}_0^+, \underline{U}_0^-, \underline{U}_0^{l,\underline{y}} \right), \quad \underline{U}_0^{\pm} = r_{|\Omega^\pm} \underline{U}_0\,,$$

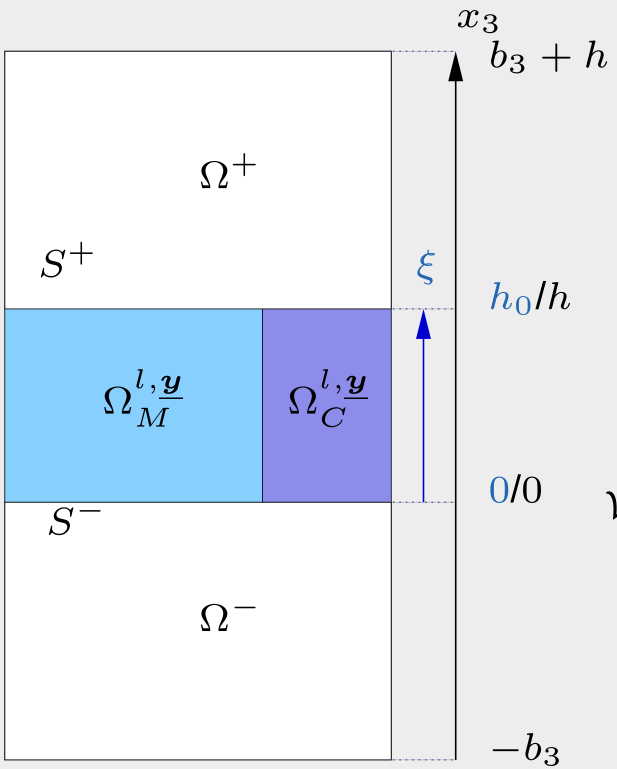
$$\begin{aligned}
\mathcal{V}_0^{\underline{y}}\left(\Omega^{\underline{y}}\right) = \Big\{ \underline{V} \in \mathcal{V}^{\underline{y}}\left(\Omega^{\underline{y}}\right) :& \\
&\underline{u}_0^- = \underline{0} \text{ on } \Gamma_3^-,\ \Phi_0 = 0 \text{ on } \Omega_M^{l,\underline{y}} \cup \Gamma_2\,, \\
&\partial_\xi \underline{V} = \underline{0} \text{ on } \Omega^l \Big\}.
\end{aligned}$$

Formal problem : Find $\underline{U}_0^{\underline{y}}$ such that $\forall\, \underline{V} \in \mathcal{V}_0^{\underline{y}}$ holds:

$$\begin{aligned}
a^+\left(\underline{U}_0^{\underline{y}}, \underline{V}\right) + a^-\left(\underline{U}_0^{\underline{y}}, \underline{V}\right) + a^l_{(1,2,\xi)}\left(\underline{U}_0^{\underline{y}}, \underline{V}\right) + a^l_{(\xi)}\left(\underline{U}_1^{\underline{y}}, \underline{V}\right) = \\
F\left(\underline{V}\right) + F^l_{T,(\xi)}\left(\underline{V}\right) + F^l_{\underline{W},C,(1,2,\xi)}\left(\underline{V}\right).
\end{aligned} \qquad (5)$$

Insert (2)-(4) into (5)

The first limit problem P_0:



$$\Omega^{l,\underline{\boldsymbol{y}}} = \Omega_M^{l,\underline{\boldsymbol{y}}} \cup \Omega_C^{l,\underline{\boldsymbol{y}}},$$

$$= \left\{ \underline{\boldsymbol{y}} = \underline{\boldsymbol{x}} + (1-\epsilon)x_3 \underline{\boldsymbol{e}}_3, \quad \underline{\boldsymbol{x}} \in \Omega^l \right\},$$

$$\Omega^{\underline{\boldsymbol{y}}} = \Omega^+ \cup \Omega^- \cup \Omega^{l,\underline{\boldsymbol{y}}} \cup S^+ \cup S^- \cup S_{CM},$$

$$\underline{\boldsymbol{U}}_0 \mapsto \underline{\boldsymbol{U}}_0^{\underline{\boldsymbol{y}}} = \left(\underline{\boldsymbol{U}}_0^+, \underline{\boldsymbol{U}}_0^-, \underline{\boldsymbol{U}}_0^{l,\underline{\boldsymbol{y}}} \right), \quad \underline{\boldsymbol{U}}_0^{\pm} = r_{|\Omega^{\pm}} \underline{\boldsymbol{U}}_0,$$

$$\mathcal{V}_0^{\underline{\boldsymbol{y}}}\left(\Omega^{\underline{\boldsymbol{y}}}\right) = \Big\{ \underline{\boldsymbol{V}} \in \mathcal{V}^{\underline{\boldsymbol{y}}}\left(\Omega^{\underline{\boldsymbol{y}}}\right) :$$

$$\underline{\boldsymbol{u}}_0^- = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-, \ \Phi_0 = 0 \text{ on } \Omega_M^{l,\underline{\boldsymbol{y}}} \cup \Gamma_2,$$

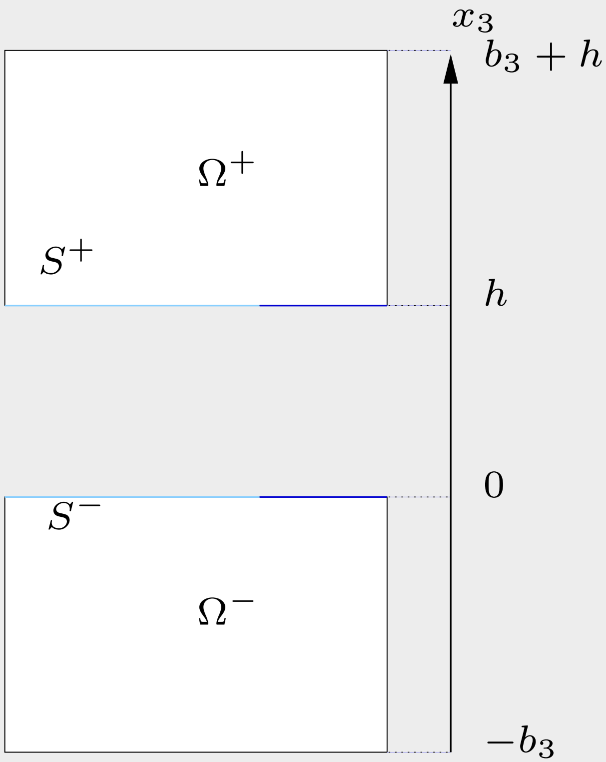
$$\partial_\xi \underline{\boldsymbol{V}} = \underline{\boldsymbol{0}} \text{ on } \Omega^l \Big\}.$$

Weak problem: Find $\underline{\boldsymbol{U}}_0^{\underline{\boldsymbol{y}}} \in \mathcal{V}_0^{\underline{\boldsymbol{y}}}$ such that $\forall\, \underline{\boldsymbol{V}} \in \mathcal{V}_0^{\underline{\boldsymbol{y}}}$ holds:

$$a^+\left(\underline{\boldsymbol{U}}_0^{\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right) + a^-\left(\underline{\boldsymbol{U}}_0^{\underline{\boldsymbol{y}}}, \underline{\boldsymbol{V}}\right) + = F\left(\underline{\boldsymbol{V}}\right) \qquad (5)$$

BOSCH Universität Stuttgart

The reduced first limit problem $P_0^{\pm}$:



$$\Omega^{\pm} = \Omega^{+} \cup \Omega^{-},$$

$$\underline{\boldsymbol{U}}_0 \mapsto \underline{\boldsymbol{U}}_0^{\pm} = \left(\underline{\boldsymbol{U}}_0^{+}, \underline{\boldsymbol{U}}_0^{-}\right) \text{ and } \underline{\boldsymbol{U}}^{\pm} = r_{|\Omega^{\pm}} \underline{\boldsymbol{U}}_0,$$

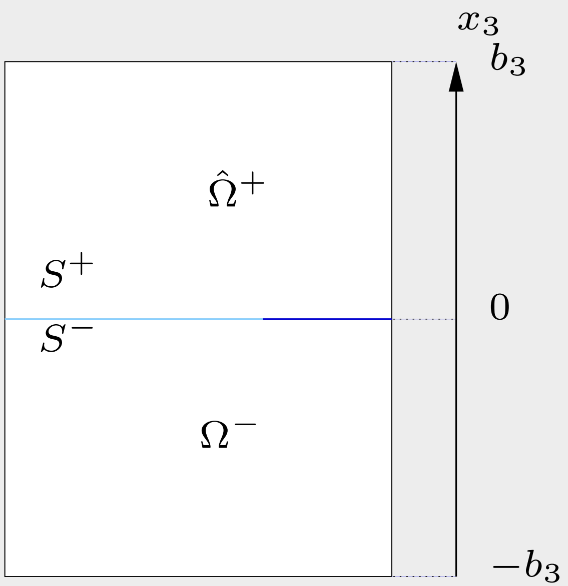
$$\mathcal{V}_0^{\pm}\left(\Omega^{\pm}\right) = \Big\{ \underline{\boldsymbol{U}}_0^{\pm} \in \mathrm{H}^1\left(\Omega^{+}\right) \times \mathrm{H}^1\left(\Omega^{-}\right) : \\ \underline{\boldsymbol{u}}_0^{-} = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^{-},\ \Phi = 0 \text{ on } S_M^{\pm} \cup \Gamma_2, \\ \underline{\boldsymbol{U}}_0^{+}\Big|_{S^+} = \underline{\boldsymbol{U}}_0^{-}\Big|_{S^-}, \sigma_{C_n}^{+} = \sigma_{C_n}^{-} \Big\}.$$

Weak problem: Find $\underline{\boldsymbol{U}}_0^{\pm} \in \mathcal{V}_0^{\pm}$ such that $\forall\, \underline{\boldsymbol{V}} \in \mathcal{V}_0^{\pm}$ holds:

$$a^{+}\left(\underline{\boldsymbol{U}}_0^{+}, \underline{\boldsymbol{V}}\right) + a^{-}\left(\underline{\boldsymbol{U}}_0^{-}, \underline{\boldsymbol{V}}\right) = F\left(\underline{\boldsymbol{V}}\right).$$

The different problems - $\hat{P}_0$

Equivalent reduced first limit problem $\hat{P}_0$:



$$\hat{\Omega}^+ = \left\{ \underline{\boldsymbol{z}} = \underline{\boldsymbol{x}} - h x_3 \underline{\boldsymbol{e}}_3, \quad \underline{\boldsymbol{x}} \in \Omega^+ \right\},$$

$$\hat{\Omega} = \hat{\Omega}^+ \cup \Omega^- \cup S^+ \cup S^-,$$

$$\underline{\boldsymbol{U}}_0 \mapsto \underline{\hat{\boldsymbol{U}}}_0 = \left(\underline{\boldsymbol{U}}_0^+, \underline{\boldsymbol{U}}_0^- \right) \text{ and } \underline{\boldsymbol{U}}_0^\pm = r_{|\Omega^\pm} \underline{\boldsymbol{U}}_0,$$

$$\hat{\mathcal{V}}_0 \left(\hat{\Omega} \right) = \left\{ \underline{\hat{\boldsymbol{U}}}_0 \in \mathrm{H}^1 \left(\hat{\Omega} \right) : \right.$$

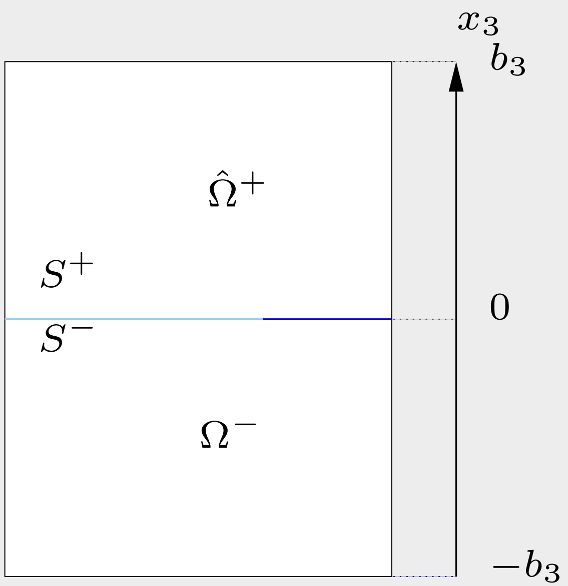
$$\left. \underline{\hat{\boldsymbol{u}}}_0^- = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-, \ \Phi_0 = 0 \text{ on } S_M^\pm \cup \Gamma_2 \right\}.$$

The different problems - $\hat{P}_0$

BOSCH

Universität Stuttgart

Equivalent reduced first limit problem $\hat{P}_0$:



$$\hat{\Omega}^+ = \left\{ \underline{\boldsymbol{z}} = \underline{\boldsymbol{x}} - hx_3\underline{\boldsymbol{e}}_3, \quad \underline{\boldsymbol{x}} \in \Omega^+ \right\},$$

$$\hat{\Omega} = \hat{\Omega}^+ \cup \Omega^- \cup S^+ \cup S^-,$$

$$\underline{\boldsymbol{U}}_0 \mapsto \underline{\hat{\boldsymbol{U}}}_0 = \left(\underline{\boldsymbol{U}}_0^+, \underline{\boldsymbol{U}}_0^- \right) \text{ and } \underline{\boldsymbol{U}}_0^\pm = r_{|\Omega^\pm} \underline{\boldsymbol{U}}_0,$$

$$\hat{\mathcal{V}}_0 \left(\hat{\Omega} \right) = \left\{ \underline{\hat{\boldsymbol{U}}}_0 \in \mathrm{H}^1 \left(\hat{\Omega} \right) : \right.$$

$$\left. \underline{\hat{\boldsymbol{u}}}_0^- = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^-, \ \Phi_0 = 0 \text{ on } S_M^\pm \cup \Gamma_2 \right\}.$$

Weak problem: Find $\underline{\hat{\boldsymbol{U}}}_0 \in \hat{\mathcal{V}}_0$ such that $\forall\, \underline{\boldsymbol{V}} \in \hat{\mathcal{V}}_0$ holds:

$$\hat{a}^+ \left(\underline{\hat{\boldsymbol{U}}}_0, \underline{\boldsymbol{V}} \right) + a^- \left(\underline{\hat{\boldsymbol{U}}}_0, \underline{\boldsymbol{V}} \right) = F \left(\underline{\boldsymbol{V}} \right).$$

Existence of solutions

Due to the continuity and ellipticity of $a\left(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}}\right)$, the Lax-Milgram lemma assures the existence and uniqueness of solutions for

P_ϵ: $\exists!\, \underline{\boldsymbol{U}} \in \mathcal{V}\left(\Omega\right)$ [6]

$P\left(\epsilon\right)$: $\exists!\, \underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} \in \mathcal{V}^{\underline{\boldsymbol{y}}}\left(\Omega^{\underline{\boldsymbol{y}}}\right)$, due to the equivalence to P_ϵ

$\hat{P}_0$: $\exists!\, \hat{\underline{\boldsymbol{U}}}_0 \in \hat{\mathcal{V}}_0\left(\hat{\Omega}\right)$, [7]

P_0: $\exists!\, \underline{\boldsymbol{U}}_0 \in \mathcal{V}_0^{\pm}\left(\Omega^{\pm}\right)$, due to the equivalence to $\hat{P}_0$

Due to the continuity and ellipticity of $a(\underline{\boldsymbol{U}}, \underline{\boldsymbol{V}})$, the Lax-Milgram lemma assures the existence and uniqueness of solutions for

P_ϵ: $\exists!\, \underline{\boldsymbol{U}} \in \mathcal{V}(\Omega)$ [6]
$P(\epsilon)$: $\exists!\, \underline{\boldsymbol{U}}^{\boldsymbol{y}} \in \mathcal{V}^{\boldsymbol{y}}(\Omega^{\boldsymbol{y}})$, due to the equivalence to P_ϵ
$\hat{P}_0$: $\exists!\, \underline{\hat{\boldsymbol{U}}}_0 \in \hat{\mathcal{V}}_0\left(\hat{\Omega}\right)$, [7]
P_0: $\exists!\, \underline{\boldsymbol{U}}_0 \in \mathcal{V}_0^{\pm}(\Omega^{\pm})$, due to the equivalence to $\hat{P}_0$

Problem:

$\underline{\boldsymbol{U}}_0$ and $\underline{\boldsymbol{U}}^{\boldsymbol{y}}$ live in different spaces

$$\underline{\boldsymbol{U}}^{\boldsymbol{y}} \in \mathcal{V}^{\boldsymbol{y}}(\Omega^{\boldsymbol{y}}), \qquad \underline{\boldsymbol{U}}_0 \in \mathcal{V}_0^{\pm}\left(\Omega^{\pm}\right).$$

Therefore we introduce the space

$$\mathcal{V}^{\pm}\left(\Omega^{\pm}\right) = \left\{\underline{\boldsymbol{U}}_0^{\pm} \in \mathrm{H}^1\left(\Omega^{+}\right) \times \mathrm{H}^1\left(\Omega^{-}\right) : \underline{\boldsymbol{u}}_0^{-} = \underline{\boldsymbol{0}} \text{ on } \Gamma_3^{-},\ \Phi = 0 \text{ on } S_M^{\pm} \cup \Gamma_2.\right\}.$$

Note that $\mathcal{V}^{\boldsymbol{y}} \subset \mathcal{V}^{\pm}, \mathcal{V}_0^{\pm} \subset \mathcal{V}^{\pm}$.
To show:

$$\underline{\boldsymbol{U}}^{\boldsymbol{y}}(\epsilon) \rightarrow \underline{\boldsymbol{U}}_0^{\pm} \text{ for } \epsilon \rightarrow 0 \text{ in } \mathcal{V}^{\pm}$$

.

Weak convergence

BOSCH
Universität Stuttgart

Lemma:
For $\epsilon \longrightarrow 0$ holds, that $\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}(\epsilon) \rightharpoonup \underline{\boldsymbol{U}}_0^{\pm}$ in $\mathcal{V}^{\pm}$.

BOSCH Universität Stuttgart

Lemma:
For $\epsilon \longrightarrow 0$ holds, that $\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}(\epsilon) \rightharpoonup \underline{\boldsymbol{U}}_0^{\pm}$ in $\mathcal{V}^{\pm}$.

Proof.
First step: Uniform boundedness

$$
\begin{aligned}
\|\underline{\boldsymbol{U}}\|^2_{\mathrm{H}^1(\Omega^{\underline{\boldsymbol{y}}})} &= \|\underline{\boldsymbol{U}}\|^2_{\mathrm{H}^1(\Omega^+)} + \|\underline{\boldsymbol{U}}\|^2_{\mathrm{H}^1(\Omega^-)} + \|\underline{\boldsymbol{U}}\|^2_{\mathrm{H}^1(\Omega^l)} \\
&\overset{\text{transf.}}{=} \|\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\|^2_{\mathrm{H}^1(\Omega^+)} + \|\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\|^2_{\mathrm{H}^1(\Omega^-)} + \\
&\quad \int_{\Omega^{l,\underline{\boldsymbol{y}}}} \left| \frac{1}{\epsilon}\left(\mathcal{B}^{\underline{\boldsymbol{y}}}_{(\xi)}\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\right)^2 + \epsilon\left(\mathcal{B}_{(1,2)}\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\right)^2 + 2\mathcal{B}_{(1,2)}\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\mathcal{B}_{(\xi)}\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} \right| \mathrm{d}\underline{\boldsymbol{y}} \\
&\overset{\text{ell.}}{\leq} c_1 \Bigg(\|\underline{\boldsymbol{F}}_r\|^2_{L^2(\Omega^+)} + \|\underline{\boldsymbol{F}}_r\|^2_{L^2(\Omega^-)} + \frac{1}{\epsilon}\left\|\underline{\boldsymbol{F}}_{r,(\xi)}\right\|_{L^2\left(\Omega^{l,\underline{\boldsymbol{y}}}\right)} \\
&\quad + \left\|\underline{\boldsymbol{F}}_{r,(1,2,\xi)}\right\|_{L^2\left(\Omega^{l,\underline{\boldsymbol{y}}}\right)} + \epsilon\left\|\underline{\boldsymbol{F}}_{(1,2)}\right\|^2_{L^2\left(\Omega^{l,\underline{\boldsymbol{y}}}\right)} \Bigg),
\end{aligned}
$$

where $\underline{\boldsymbol{F}}_r$ denotes the right hand sides of the problem (electrical loads, temperature load)

Weak convergence

BOSCH
Universität Stuttgart

Lemma:
For $\epsilon \longrightarrow 0$ holds, that $\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}(\epsilon) \rightharpoonup \underline{\boldsymbol{U}}_0^{\pm}$ in $\mathcal{V}^{\pm}$.

Proof.
First step: Uniform boundedness

$$
\begin{aligned}
\|\underline{\boldsymbol{U}}\|^2_{\mathrm{H}^1(\Omega^{\underline{\boldsymbol{y}}})} &= \|\underline{\boldsymbol{U}}\|^2_{\mathrm{H}^1(\Omega+)} + \|\underline{\boldsymbol{U}}\|^2_{\mathrm{H}^1(\Omega-)} + \|\underline{\boldsymbol{U}}\|^2_{\mathrm{H}^1(\Omega^l)} \\
&\overset{\text{transf.}}{=} \|\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\|^2_{\mathrm{H}^1(\Omega+)} + \|\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\|^2_{\mathrm{H}^1(\Omega-)} + \\
&\quad \int_{\Omega^{l,\underline{\boldsymbol{y}}}} \left| \frac{1}{\epsilon}\left(\mathcal{B}^{\boldsymbol{y}}_{(\xi)}\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\right)^2 + \epsilon\left(\mathcal{B}_{(1,2)}\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\right)^2 + 2\mathcal{B}_{(1,2)}\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\mathcal{B}_{(\xi)}\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}} \right| \mathrm{d}\underline{\boldsymbol{y}} \\
&\overset{\text{ell.}}{\leq} c_1 \Bigg(\|\underline{\boldsymbol{F}}_r\|^2_{L^2(\Omega+)} + \|\underline{\boldsymbol{F}}_r\|^2_{L^2(\Omega-)} + \frac{1}{\epsilon}\left\|\underline{\boldsymbol{F}}_{r,(\xi)}\right\|_{L^2\left(\Omega^{l,\underline{\boldsymbol{y}}}\right)} \\
&\quad + \left\|\underline{\boldsymbol{F}}_{r,(1,2,\xi)}\right\|_{L^2\left(\Omega^{l,\underline{\boldsymbol{y}}}\right)} + \epsilon\left\|\underline{\boldsymbol{F}}_{(1,2)}\right\|^2_{L^2\left(\Omega^{l,\underline{\boldsymbol{y}}}\right)} \Bigg),
\end{aligned}
$$

where $\underline{\boldsymbol{F}}_r$ denotes the right hand sides of the problem (electrical loads, temperature load)

And thus

$$
\left\|\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}\right\|_{\mathcal{V}^{\pm}} \leq c_2 .
$$

Second step: Convergence

It follows, that there exists a subsequence

$$\underline{\boldsymbol{U}}^{\,\underline{\boldsymbol{y}}}\,(\epsilon_n) \rightharpoonup \underline{\boldsymbol{U}}^{\,*} \quad \text{in } \mathcal{V}^{\pm},$$

that means, that

$$\left\langle \underline{\boldsymbol{f}}\,,\underline{\boldsymbol{U}}^{\,\underline{\boldsymbol{y}}}\,(\epsilon_n)\right\rangle \to \left\langle \underline{\boldsymbol{f}}\,,\underline{\boldsymbol{U}}^{\,*}\right\rangle \quad \forall\,\underline{\boldsymbol{f}} \in \mathcal{V}^{\pm\,\prime} \subset \mathcal{V}^{\underline{\boldsymbol{y}}\,\prime}.$$

Second step: Convergence

It follows, that there exists a subsequence

$$\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}(\epsilon_n) \rightharpoonup \underline{\boldsymbol{U}}^{*} \quad \text{in } \mathcal{V}^{\pm},$$

that means, that

$$\left\langle \underline{\boldsymbol{f}}, \underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}(\epsilon_n) \right\rangle \to \left\langle \underline{\boldsymbol{f}}, \underline{\boldsymbol{U}}^{*} \right\rangle \quad \forall\, \underline{\boldsymbol{f}} \in \mathcal{V}^{\pm'} \subset \mathcal{V}^{\underline{\boldsymbol{y}}'}.$$

Now let $\underline{\boldsymbol{Z}} \in \mathcal{V}_0^{\pm}$. $\underline{\boldsymbol{Z}}$ generates a special functional from $\mathcal{V}_0^{\pm'}$:

$$\langle \underline{\boldsymbol{Z}}, \underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}(\epsilon_n) \rangle_{\Omega+} := a^{+}\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}(\epsilon_n), \underline{\boldsymbol{Z}}\right) \to a^{+}\left(\underline{\boldsymbol{U}}^{*}, \underline{\boldsymbol{Z}}\right)$$

$$\langle \underline{\boldsymbol{Z}}, \underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}(\epsilon_n) \rangle_{\Omega-} := a^{-}\left(\underline{\boldsymbol{U}}^{\underline{\boldsymbol{y}}}(\epsilon_n), \underline{\boldsymbol{Z}}\right) \to a^{-}\left(\underline{\boldsymbol{U}}^{*}, \underline{\boldsymbol{Z}}\right)$$

$\underline{\boldsymbol{Z}}$ can be extended to an element from $\mathcal{V}_0^{\underline{\boldsymbol{y}}}(\Omega^{\underline{\boldsymbol{y}}})$

Weak convergence

Due to (1) for the extended elements $\underline{\boldsymbol{Z}}$ it holds

$$a^{+}\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}(\epsilon_n), \underline{\boldsymbol{Z}}\right) + a^{-}\left(\underline{\boldsymbol{U}}^{\boldsymbol{y}}(\epsilon_n), \underline{\boldsymbol{Z}}\right) \overset{\epsilon_n \to 0}{\longrightarrow} F(\underline{\boldsymbol{Z}}).$$

And therefore

$$\begin{aligned} a^{+}\left(\underline{\boldsymbol{U}}^{*}, \underline{\boldsymbol{Z}}\right) + a^{-}\left(\underline{\boldsymbol{U}}^{*}, \underline{\boldsymbol{Z}}\right) &= F(\underline{\boldsymbol{Z}}) \\ &= a^{+}\left(\underline{\boldsymbol{U}}_0^{+}, \underline{\boldsymbol{Z}}\right) + a^{-}\left(\underline{\boldsymbol{U}}_0^{-}, \underline{\boldsymbol{Z}}\right) \quad \forall \underline{\boldsymbol{Z}} \in \mathcal{V}_0^{\pm}. \end{aligned}$$

Since $\underline{\boldsymbol{U}}_0^{\pm}$ is unique, it follows:

$$\underline{\boldsymbol{U}}^{*} = \underline{\boldsymbol{U}}_0^{\pm} \quad \text{in} \quad \mathcal{V}_0^{\pm}. \tag{5}$$

Third step: Weak convergence for all subsequences.

Conclusions

Universität Stuttgart

➜ The linear multistructure-multifield model has been reduced to an asymptotic model (first limit problem).

Conclusions

Universität Stuttgart

- The linear multistructure-multifield model has been reduced to an asymptotic model (first limit problem).
- The first limit problem has a uniquely defined weak solution in $\hat{\mathcal{V}} \subset \mathrm{H}^1\left(\hat{\Omega}\right)$.

Conclusions

- The linear multistructure-multifield model has been reduced to an asymptotic model (first limit problem).
- The first limit problem has a uniquely defined weak solution in $\hat{\mathcal{V}} \subset \mathrm{H}^1\left(\hat{\Omega}\right)$.
- Numerical experiments confirm the efficiency of the asmyptotic model.

Conclusions

Bosch
Universität Stuttgart

- The linear multistructure-multifield model has been reduced to an asymptotic model (first limit problem).
- The first limit problem has a uniquely defined weak solution in $\hat{\mathcal{V}} \subset \mathrm{H}^1\left(\hat{\Omega}\right)$.
- Numerical experiments confirm the efficiency of the asmyptotic model.
- For $\epsilon \longrightarrow 0$, the solution of the full multistructure-multifield problem converges weakly towards the solution of the first limit problem.

BOSCH

Universität Stuttgart

References

[1] P. G. Ciarlet. *Plates and Junctions in Elastic Multi-Structures, an Asymptotic Analysis*. Masson, Springer Verlag, Paris, Berlin, Heidelberg, 1990.

[2] Informationszentrum Technische Keramik Verband d. Keramischen Industrie e.V., editor. *Brevier Technische Keramik*. Fahner, Nürnberg, 1999.

[3] F. Krasucki G. Geymonat. Analyse asymptotique du comportement en flexion de deux plaques colliées. *Comptes Rendus de l' Académie des Sciences, Paris, Série IIb*, 325:307–314, 1997.

[4] S. Lenci G. Geymonat, F. Krasucki. Analyse asymptotique du comportement d'un assemblage collé. *Comptes Rendus de l' Académie des Sciences, Paris, Série 1*, 322:1107–1112, 1996.

[5] S. Lenci G. Geymonat, F. Krasucki. Mathematical analysis of a bonded joint with a soft thin adhesive. *Mathematics and Mechanics of Solids*, 4:201–225, 1999.

[6] W. Geis, G. Mishuris, and A.-M. Sändig. Piezoelectricity in multilayer actuators. Modelling and analysis in two and three dimensions. Preprint 2003/023, University of Stuttgart, http://preprints.ians.uni-stuttgart.de, 2003.

[7] W. Geis, G. Mishuris, and A.-M. Sändig. Asymptotic models for piezoelectric stack actuators with thin metal inclusions. Preprint 2004/001, University of Stuttgart, http://preprints.ians.uni-stuttgart.de, 2004.

[8] S. Keppeler. Common-Rail-Einspritzsystem für den direkteinspritzenden Dieselmotor. Dissertationsschrift, Technische Hochschule, Aachen, 1997.