SIAM J. MATH. ANAL.
Vol. 27, No. 2, pp. 486–514, March 1996

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ASYMPTOTIC ANALYSIS OF THE BOUNDARY LAYER FOR THE REISSNER–MINDLIN PLATE MODEL*

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Abstract. We investigate the structure of the solution of the Reissner–Mindlin plate equations in its dependence on the plate thickness in the cases of soft and hard clamped, soft and hard simply supported, and traction free boundary conditions. For the transverse displacement, rotation, and shear stress, we develop asymptotic expansions in powers of the plate thickness. These expansions are uniform up to the boundary for the transverse displacement, but for the other variables there is a boundary layer, which is stronger for the soft simply supported and traction-free plate and weaker for the soft clamped plate than for the hard clamped and hard simply supported plate. We give rigorous error bounds for the errors in the expansions in Sobolev norms. As an application, we derive new regularity results for the solutions and new estimates for the difference between the Reissner–Mindlin solution and the solution to the corresponding biharmonic model.

Key words. Reissner, Mindlin, plate, boundary layer

AMS subject classifications. 73K10, 73K25

1. Introduction. The Reissner–Mindlin model for the bending of an isotropic elastic plate in equilibrium determines ω, the transverse displacement of the midplane, and $\boldsymbol{\phi}$, the rotation of fibers normal to the midplane, as the solution of the partial differential equations

$$-t^3 \,\mathbf{div}\, \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}) - \lambda t\,(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = \boldsymbol{F},$$
$$-\lambda t \,\mathrm{div}\,(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = G.$$

Here $\boldsymbol{F}$ is the applied couple per unit area, G is the applied transverse load density per unit area, t is the plate thickness, $\lambda = Ek/2(1+\nu)$ with E the Young's modulus, ν the Poisson ratio, and k the shear correction factor, $\mathcal{E}(\boldsymbol{\phi})$ is the symmetric part of the gradient of $\boldsymbol{\phi}$, and the fourth-order tensor $\mathcal{C}$ is defined by

$$\mathcal{C}\mathcal{T} = D\left[(1-\nu)\mathcal{T} + \nu\,\mathrm{tr}(\mathcal{T})\mathcal{I}\right], \quad D = \frac{E}{12(1-\nu^2)},$$

for any 2×2 matrix $\mathcal{T}$ ($\mathcal{I}$ denotes the 2×2 identity matrix). These equations are satisfied on the plane region Ω occupied by the midsection of the plate. In this paper, we investigate the dependence on the plate thickness of solutions to some boundary value problems associated to these equations.

* Received by the editors March 10, 1993; accepted for publication (in revised form) August 2, 1994.

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We consider various homogeneous boundary conditions of physical interest:

$$\phi \cdot n = \phi \cdot s = \omega = 0 \quad \text{(hard clamped)}, \tag{1.1}$$

$$\phi \cdot n = M_s(\phi) = \omega = 0 \quad \text{(soft clamped)}, \tag{1.2}$$

$$M_n(\phi) = \phi \cdot s = \omega = 0 \quad \text{(hard simply supported)}, \tag{1.3}$$

$$M_n(\phi) = M_s(\phi) = \omega = 0 \quad \text{(soft simply supported)}, \tag{1.4}$$

$$M_n(\phi) = M_s(\phi) = \partial\omega/\partial n - \phi \cdot n = 0 \quad \text{(free)}, \tag{1.5}$$

in which n and s denote the unit normal and counterclockwise tangent vectors, respectively, and $M_n(\phi) := n \cdot C\,\mathcal{E}(\phi)n$, $M_s(\phi) := s \cdot C\,\mathcal{E}(\phi)n$. Each of the first four boundary value problems admits a unique solution $\omega \in H^1(\Omega)$, $\phi \in \boldsymbol{H}^1(\Omega)$ for any $F \in \boldsymbol{L}^2(\Omega)$ and $G \in L^2(\Omega)$. The existence theory for the free plate is slightly more complicated and will be discussed in §6.

We do not treat the Reissner–Mindlin model in its full generality. In addition to the assumption of homogeneous boundary conditions, we shall assume that there is no applied couple, so $F \equiv 0$, and that the constitutive parameters E, ν, and k are independent of t. It seems clear that the techniques developed here apply to more general situations as well.

We also suppose that $G = gt^3$, where the function g does not depend on t. This is a convenient normalization, which leads to ϕ and ω having a nonzero limit as t tends to zero. Given that the first differential equation and the boundary conditions are taken to be homogeneous, this normalization is not restrictive. If G were to be proportional to some other function $h(t)$, we could make the change of dependent variables $\bar{\phi} = t^3\phi/h(t)$, $\bar{\omega} = t^3\omega/h(t)$ and the new variables would satisfy the Reissner–Mindlin equations with load proportional to t^3.

With these assumptions, the Reissner–Mindlin equations become

$$-\,\mathbf{div}\, C\,\mathcal{E}(\phi) - \lambda t^{-2}\,(\mathbf{grad}\,\omega - \phi) = 0, \tag{1.6}$$

$$-\lambda t^{-2}\,\mathrm{div}\,(\mathbf{grad}\,\omega - \phi) = g. \tag{1.7}$$

After a similar normalization of the load, the biharmonic model for plate bending may be written

$$D\,\Delta^2\,\omega_0 = g \quad \text{in } \Omega,$$

and so its solution ω_0 is independent of the plate thickness. In contrast, the solution of the Reissner–Mindlin model exhibits a complex dependence on the plate thickness, which we investigate in the present paper. In previous work [1], we gave an analysis of the boundary layer for the Reissner–Mindlin model of hard clamped and hard simply supported plates. There are many additional complications in the case of more general boundary conditions, and so the analysis of [1] is not easily extended to the soft simply supported and free plates, for example. In this paper, we analyze the boundary layer for all the boundary conditions mentioned above in a unified fashion. While the approach here is more complete, it is also simpler than that of [1] in a number of ways. Thus the present paper essentially supersedes that one. We shall show that the boundary layer is strongest for the soft simply supported and free plate, somewhat weaker for the clamped and hard simply supported plate, and weakest for the soft clamped plate. In addition, we shall demonstrate that for the soft clamped and hard simply supported plates, there is no boundary layer near a flat portion of the boundary.

We shall develop asymptotic expansions with respect to t for ω and $\boldsymbol{\phi}$ (as well as for other quantities associated with the solution such as the shear strain). The expansions take the forms

$$\omega \sim \omega_0 + t\omega_1 + t^2\omega_2 + \cdots ,$$
$$\boldsymbol{\phi} \sim \boldsymbol{\phi}_0 + \chi\boldsymbol{\Phi}_0 + t(\boldsymbol{\phi}_1 + \chi\boldsymbol{\Phi}_1) + t^2(\boldsymbol{\phi}_2 + \chi\boldsymbol{\Phi}_2) + \cdots ,$$

where the interior expansion functions ω_i and $\boldsymbol{\phi}_i$ are independent of t and the boundary correctors $\boldsymbol{\Phi}_i$ depend on t only through the quantity ρ/t, ρ being the distance of a point of Ω from the boundary. More specifically,

$$\boldsymbol{\Phi}_i = \hat{\boldsymbol{\Phi}}_i(\rho/t, \theta),$$

where θ is a coordinate which roughly gives arc length parallel to the boundary (see §2), and the function $\hat{\boldsymbol{\Phi}}_i(\eta, \theta)$ has the form of a polynomial with respect to η with coefficients depending smoothly on θ times $\exp(-\sqrt{12k}\eta)$. Thus $\boldsymbol{\Phi}_i$ represents a boundary-layer function, which essentially lives in a strip of width t around the boundary. Finally, χ is a cutoff function which is independent of t and identically equal to unity in a neighborhood of $\partial\Omega$.

After some preliminary material in §2, we construct the terms of the asymptotic expansions in §3 (for all of the boundary conditions except for those of the free plate, which are treated in §6). Then, in the following two sections, we justify the expansions rigorously in the case of the soft simply supported plate, proving a priori bounds for the terms of the expansions in §4 and performing the error analysis in §5. This analysis can be adapted easily to the cases of hard simply supported and hard and soft clamped plates and somewhat less easily to the case of the free plate. The necessary modifications are discussed in §6. To make it easier for the reader to follow some of the computations performed in the derivation and analysis of the asymptotic expansions, we have included in an appendix a summary of the main formulas we have used. In the remainder of this introduction, we summarize some of the principal results.

For each of the boundary conditions, ω_0 is the solution of the biharmonic equation

$$D\,\Delta^2\,\omega_0 = g$$

determined by appropriate boundary conditions, namely

$$\omega_0 = \frac{\partial \omega_0}{\partial n} = 0$$

for the hard and soft clamped plates,

$$\omega_0 = (1-\nu)\frac{\partial^2 \omega_0}{\partial n^2} + \nu\,\Delta\,\omega_0 = 0$$

for the hard and soft simply supported plates, and

$$(1-\nu)\frac{\partial^2 \omega_0}{\partial n^2} + \nu\,\Delta\,\omega_0 = \frac{\partial\,\Delta\,\omega_0}{\partial n} + (1-\nu)\frac{\partial}{\partial s}\left(\frac{\partial^2 \omega_0}{\partial s \partial n} - \kappa\frac{\partial \omega_0}{\partial s}\right) = 0$$

for the free plate. In the last expression, κ denotes the curvature of the boundary.

The next term in the expansion of the transverse displacement, ω_1, vanishes for the hard and soft clamped plates and the hard simply supported plate but not for the soft simply supported or free plates. In these cases, it is the solution of the homogeneous biharmonic problem

$$\Delta^2\,\omega_1 = 0 \quad \text{in } \Omega$$

with the inhomogeneous boundary conditions

$$\omega_1 = 0, \quad (1-\nu)\frac{\partial^2 \omega_1}{\partial n^2} + \nu\,\Delta\,\omega_1 = -\frac{(1-\nu)}{\sqrt{3k}}\frac{\partial^3 \omega_0}{\partial s^2 \partial n}$$

for the soft simply supported plate and

$$(1-\nu)\frac{\partial^2 \omega_1}{\partial n^2} + \nu\,\Delta\,\omega_1 = \frac{1}{\sqrt{3k}}\frac{\partial\,\Delta\,\omega_0}{\partial n},$$

$$\frac{\partial\,\Delta\,\omega_1}{\partial n} + (1-\nu)\frac{\partial}{\partial s}\left(\frac{\partial^2 \omega_1}{\partial s \partial n} - \kappa \frac{\partial \omega_1}{\partial s}\right) = -\frac{(1-\nu)}{\sqrt{3k}}\frac{\partial}{\partial s}\left(\kappa\left[\frac{\partial^2 \omega_0}{\partial s \partial n} - \kappa\frac{\partial \omega_0}{\partial s}\right]\right)$$

for the free plate.

Note that the expansions for the soft and hard simply supported plates differ already in the term ω_1. For the soft and hard clamped plates the terms ω_0, ω_1, and ω_2 all agree, but $\omega_3 = 0$ for the soft clamped plate and is generally nonzero for the hard clamped plate.

Turning to the expansion of $\boldsymbol{\phi}$, we find that in all five cases that $\boldsymbol{\phi}_0 = \mathbf{grad}\,\omega_0$ and $\boldsymbol{\phi}_1 = \mathbf{grad}\,\omega_1$ while $\boldsymbol{\phi}_2 - \mathbf{grad}\,\omega_2 = \lambda^{-1} D\,\Delta\,\omega_0$, which is never zero (except in the trivial case $g \equiv 0$). For the boundary correctors, we find that $\boldsymbol{\Phi}_0$ vanishes in all five cases. For the soft simply supported and free plates,

$$\hat{\boldsymbol{\Phi}}_1(\eta,\theta) = -\frac{1}{\sqrt{3k}}\exp(-\sqrt{12k}\eta)\left(\frac{\partial^2 \omega_0}{\partial s \partial n} - \kappa\frac{\partial \omega_0}{\partial s}\right)(0,\theta)\boldsymbol{s}.$$

For the hard clamped and hard simply supported plate, $\boldsymbol{\Phi}_1$ vanishes as well as $\boldsymbol{\Phi}_0$ and we have

$$\hat{\boldsymbol{\Phi}}_2(\eta,\theta) = -\frac{1}{6k(1-\nu)}\exp(-\sqrt{12k}\eta)\frac{\partial}{\partial s}\,\Delta\,\omega_0(0,\theta)\boldsymbol{s}$$

in both cases. For the soft clamped plate $\boldsymbol{\Phi}_0$, $\boldsymbol{\Phi}_1$, and $\boldsymbol{\Phi}_2$ all vanish. In all five cases, the first nonzero boundary corrector is purely tangential. Table 1 summarizes the terms in the asymptotic expansions of ω and $\boldsymbol{\phi}$ which vanish.

TABLE 1
Vanishing terms in the asymptotic expansions.

soft simply supported	–	–	$\boldsymbol{\Phi}_0$	$\boldsymbol{\Phi}_1 \cdot \boldsymbol{n}$
free	–	–	$\boldsymbol{\Phi}_0$	$\boldsymbol{\Phi}_1 \cdot \boldsymbol{n}$
hard clamped	ω_1	$\boldsymbol{\phi}_1$	$\boldsymbol{\Phi}_0$, $\boldsymbol{\Phi}_1$	$\boldsymbol{\Phi}_2 \cdot \boldsymbol{n}$
hard simply supported	ω_1	$\boldsymbol{\phi}_1$	$\boldsymbol{\Phi}_0$, $\boldsymbol{\Phi}_1$	$\boldsymbol{\Phi}_2 \cdot \boldsymbol{n}$
soft clamped	ω_1, ω_3	$\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_3$	$\boldsymbol{\Phi}_0$, $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$	$\boldsymbol{\Phi}_3 \cdot \boldsymbol{n}$

Using symbolic computation, we have computed exact solutions to the Reissner–Mindlin system on circular and semiinfinite plates for particular choices of the load function g, and have explicitly computed the asymptotic expansions of ω and $\boldsymbol{\phi}$ through terms of order 6. These computations verify the sharpness of the results in this paper in that no terms of the expansions vanish except those given in the table. These results have been reported in [2].

As an application of our asymptotic analysis, we can determine the asymptotic behavior of Sobolev norms of solutions of the Reissner–Mindlin system. Supposing that g is sufficiently smooth, we have the following estimates, valid for both the soft simply supported and free plate, in which the constant C depends on g, Ω, and the elastic constants but is independent of t. Here $\|\cdot\|_s$ and $|\cdot|_s$ denote the norms in the Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$ (see §2).

The transverse displacement ω and all of its derivatives are bounded uniformly in t, that is,

$$\|\omega\|_s \le C, \quad s \in \mathbb{R},$$

but the regularity of the rotation $\boldsymbol{\phi}$ is limited by the boundary layer. For example, for the soft simply supported and free plates, we have

$$\|\boldsymbol{\phi}\|_s \le Ct^{\min(0,3/2-s)}, \quad s \in \mathbb{R},$$

so derivatives of order greater than 1 will generally tend to infinity in L^2 as $t \to 0$.

The quantity $\boldsymbol{\zeta} := t^{-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi})$, which is proportional to the shear strain, is often of interest. From the above expansions, we get

$$\boldsymbol{\zeta} \sim -t^{-1}\chi\boldsymbol{\Phi}_1 + (\mathbf{grad}\,\omega_2 - \boldsymbol{\phi}_2 - \chi\boldsymbol{\Phi}_2) + \cdots,$$

so it has a stronger boundary layer. Indeed, for the soft simply supported and free plates, $\boldsymbol{\zeta}$ is not uniformly bounded in L^2, or even in H^s for $s > -1/2$:

$$\|\boldsymbol{\zeta}\|_s \le Ct^{\min(0,-1/2-s)}, \quad s \in \mathbb{R}.$$

The corresponding estimates for the hard clamped and hard simply supported plates are

$$\|\boldsymbol{\phi}\|_s \le Ct^{\min(0,5/2-s)}, \quad s \in \mathbb{R}, \qquad \|\boldsymbol{\zeta}\|_s \le Ct^{\min(0,1/2-s)}, \quad s \in \mathbb{R},$$

and for the soft clamped plate

$$\|\boldsymbol{\phi}\|_s \le Ct^{\min(0,7/2-s)}, \quad s \in \mathbb{R}, \qquad \|\boldsymbol{\zeta}\|_s \le Ct^{\min(0,3/2-s)}, \quad s \in \mathbb{R}.$$

Of course, the boundary layer does not limit the regularity of $\boldsymbol{\phi}$ or $\boldsymbol{\zeta}$ at a positive distance from $\partial\Omega$ nor does it affect the smoothness of their restrictions to $\partial\Omega$. Thus

$$\|\boldsymbol{\phi}\|_{H^s(\Omega_c)} + |\boldsymbol{\phi}|_s + \|\boldsymbol{\zeta}\|_{H^s(\Omega_c)} + |\boldsymbol{\zeta}|_s \le C, \quad s \in \mathbb{R},$$

for any compact subdomain Ω_c of Ω.

In the limit as $t \to 0$, the variables ω and $\boldsymbol{\phi}$ tend in L^2 to the leading terms of their asymptotic expansions. The number of derivatives which converge and the rate of convergence may be determined by examining the first neglected interior and boundary terms of the expansions. For any $s \in \mathbb{R}$, we get for the soft simply supported and free plate

$$\|\omega - \omega_0\|_s \le Ct, \qquad \|\boldsymbol{\phi} - \boldsymbol{\phi}_0\|_s \le Ct^{\min(1,3/2-s)}.$$

Note that the rate of convergence for $\boldsymbol{\phi}$ depends on the Sobolev norm under consideration. For each of the variables, taking more terms from the expansion increases the rate of convergence and taking sufficiently many terms in the expansions gives approximation of any desired algebraic order of convergence in t in any desired Sobolev space (provided g is sufficiently regular). For example,

$$\|\omega - \omega_0 - t\omega_1\|_s \le Ct^2, \qquad \|\boldsymbol{\phi} - \boldsymbol{\phi}_0 - t(\boldsymbol{\phi}_1 + \chi\boldsymbol{\Phi}_1)\|_s \le Ct^{\min(2,5/2-s)}.$$

For the hard clamped and hard simply supported plates, the analogous results are

$$\|\omega - \omega_0\|_s \le Ct^2, \qquad \|\boldsymbol{\phi} - \boldsymbol{\phi}_0\|_s \le Ct^{\min(2,5/2-s)},$$

$$\|\omega - \omega_0 - t^2\omega_2\|_s \le Ct^3, \qquad \|\boldsymbol{\phi} - \boldsymbol{\phi}_0 - t^2(\boldsymbol{\phi}_2 + \chi\boldsymbol{\Phi}_2)\|_s \le Ct^{\min(3,7/2-s)}.$$

For the soft clamped plate,

$$\|\omega - \omega_0\|_s \le Ct^2, \qquad \|\phi - \phi_0\|_s \le Ct^{\min(2,7/2-s)},$$
$$\|\omega - \omega_0 - t^2\omega_2\|_s \le Ct^3, \qquad \|\phi - \phi_0 - t^2\phi_2\|_s \le Ct^{\min(3,7/2-s)}.$$

It is also possible to use our asymptotic expansions to derive estimates in function spaces other than H^s. The technique for doing this is described in [1]. Further references for the Reissner–Mindlin model and its boundary-layer behavior can also be found there. Many of the results in this paper were described without proof in [2], where explicit illustrations of the theory are constructed.

2. Notation and preliminaries. The letter C denotes a generic constant, not necessarily the same in each occurrence. We assume that Ω is a smooth, bounded, and simply connected domain in $\mathbb{R}^2$. The $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner products are denoted by $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$, respectively. We also use the usual L^2-based Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$, real $s \ge 0$, with norms denoted by $\|\cdot\|_s$ and $|\cdot|_s$. When the domain argument is omitted, L^2 and H^s refer to $L^2(\Omega)$ and $H^s(\Omega)$. The space $\mathring{H}^s = \mathring{H}^s(\Omega)$ is the closure of C_0^∞ in H^s. The interpolation inequality

$$\|g\|_{s+v}^u \le C\|g\|_s^{u-v}\|g\|_{s+u}^v, \quad s \ge 0,\ u \ge v \ge 0, \tag{2.1}$$

holds. If $g \in L^2$ and $\Delta^{-1} g$ denotes the unique function in $H^2 \cap \mathring{H}^1$ whose Laplacian is equal to g, then

$$C^{-1}\|\,\Delta^{-1} g\|_{s+2} \le \|g\|_s \le C\|\,\Delta^{-1} g\|_{s+2}, \quad s \ge 0,$$

where the constant C may depend on s and Ω but not on g. In other words, $g \mapsto \|\,\Delta^{-1} g\|_{s+2}$ defines an equivalent norm on H^s for $s \ge 0$. We also define some negatively indexed norms which maintain this equivalence:

$$\|g\|_s := \|\,\Delta^{-1} g\|_{s+2}, \quad -2 \le s < 0.$$

For $s = -1$, this is equivalent to the norm in the dual space of $\mathring{H}^1$. For $s = -2$, it is equivalent to the norm in the dual space of $H^2 \cap \mathring{H}^1$. With this definition, (2.1) holds for $s \ge -2$. We shall make frequent use of this fact to bound sums of the form $\sum_{i=0}^n t^i\|g\|_{s+i}$ by a multiple of the sum of the first and last terms.

We also require the quotient space $H^s/\mathbb{R}$. An element $p \in H^s/\mathbb{R}$ is a coset consisting of all functions in H^s differing from a fixed function by a constant. The quotient norm is given by

$$\|p\|_{s/\mathbb{R}} = \min_{q\in p}\|q\|_s.$$

(In fact, $\|p\|_{s/\mathbb{R}} = \|\bar{p}\|_s$, where $\bar{p}$ is the unique function in the coset p having mean value zero.)

We use boldface type to denote 2-vector-valued functions, operators whose values are vector-valued functions, and spaces of vector-valued functions. Script type is used in a similar way for 2×2-matrix objects. Thus, for example, $\operatorname{div}\boldsymbol{\psi} \in L^2$ for $\boldsymbol{\psi} \in \boldsymbol{H}^1$, while $\mathbf{div}\,\mathcal{T} \in \boldsymbol{L}^2$ for $\mathcal{T} \in \mathcal{H}^1$. Finally, we use various standard differential operators:

$$\mathbf{grad}\, r = \begin{pmatrix} \partial r/\partial x \\ \partial r/\partial y \end{pmatrix}, \qquad \operatorname{div}\boldsymbol{\psi} = \frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y},$$

$$\mathbf{div}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} \partial t_{11}/\partial x + \partial t_{12}/\partial y \\ \partial t_{21}/\partial x + \partial t_{22}/\partial y \end{pmatrix},$$

$$\mathbf{curl}\, p = \begin{pmatrix} -\partial p/\partial y \\ \partial p/\partial x \end{pmatrix}, \qquad \operatorname{rot} \boldsymbol{\psi} = \frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x}.$$

Note that these differential operators annihilate constants and consequently induce operators on the quotient space $H^s/\mathbb{R}$ for each s. We denote the induced operator in the same way as the original. Thus, for example, if $p \in H^1/\mathbb{R}$, $\mathbf{curl}\, p$ denotes the element of $\boldsymbol{L}^2$ obtained by applying the curl to any element in the coset p.

We record here for later reference the identity

$$\sum_{i=0}^{n} \sum_{j=0}^{i} f(i-j, j) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} f(i, j). \tag{2.2}$$

To describe the boundary layer, we define the usual boundary-fitted coordinates in a neighborhood of the boundary. Let ρ_0 be a positive number less than the minimum radius of curvature of $\partial\Omega$ and define

$$\Omega_0 = \{\, \boldsymbol{z} - \rho \boldsymbol{n}_z \mid \boldsymbol{z} \in \partial\Omega, 0 < \rho < \rho_0 \,\},$$

where $\boldsymbol{n_z}$ is the outward unit normal to Ω at $\boldsymbol{z}$. Let $\boldsymbol{z}(\theta) = (X(\theta), Y(\theta))$, $\theta \in [0, L)$, be a parametrization of $\partial\Omega$ by arclength which we extend L-periodically to $\theta \in \mathbb{R}$. The correspondence

$$(\rho, \theta) \mapsto \boldsymbol{z} - \rho \boldsymbol{n_z} = (X(\theta) - \rho Y'(\theta), Y(\theta) + \rho X'(\theta))$$

is a diffeomorphism of $(0, \rho_0) \times \mathbb{R}/L$ on Ω_0. Let $\kappa(\theta)$ denote the curvature of $\partial\Omega$ at $\boldsymbol{z}(\theta)$ and set

$$\sigma(\rho, \theta) := \frac{1}{1 - \kappa(\theta)\rho}.$$

The unit vector fields of the outward normal and the counterclockwise tangent extend from $\partial\Omega$ to Ω_0 as functions of θ, independent of ρ, and satisfy

$$\boldsymbol{n} = -\,\mathbf{grad}\, \rho = -\sigma(\rho, \theta)^{-1}\, \mathbf{curl}\, \theta, \qquad \boldsymbol{s} = \sigma(\rho, \theta)^{-1}\, \mathbf{grad}\, \theta = -\,\mathbf{curl}\, \rho.$$

We shall also use the stretched variable $\hat{\rho} = \rho/t$. When required for clarity, we use hats to denote the change of variables to $(\hat{\rho}, \theta)$ coordinates, that is,

$$\hat{f}(\hat{\rho}, \theta) := f(x, y).$$

3. An asymptotic expansion of the solution. We now develop asymptotic expansions of $\boldsymbol{\phi}$ and ω with respect to the plate thickness. Such expansions normally consist of two parts, an interior expansion and a boundary-layer expansion. Now it follows easily from (1.6) and (1.7) that the transverse displacement ω satisfies the biharmonic equation

$$D\,\Delta^2 \omega = g - \lambda^{-1} D t^2\, \Delta\, g, \tag{3.1}$$

which indicates that ω admits no boundary layer and hence can be described by an interior expansion alone. However, the rotation vector $\boldsymbol{\phi}$ satisfies the singular perturbation equation given in (1.6) and hence can be expected to include a boundary layer. Thus we shall seek expansions of the form

$$\omega \sim \sum_{i=0}^{\infty} t^i \omega_i, \qquad \boldsymbol{\phi} \sim \sum_{i=0}^{\infty} t^i \boldsymbol{\phi}_i + \sum_{i=1}^{\infty} t^i \boldsymbol{\Phi}_i,$$

where ω_i and $\boldsymbol{\phi}_i$ are smooth functions independent of t, while $\boldsymbol{\Phi}_i(x,y) = \hat{\boldsymbol{\Phi}}_i(\hat{\rho},\theta)$ with $\hat{\boldsymbol{\Phi}}$ a smooth function on $[0,\infty) \times \partial\Omega$. We have suppressed the term $\boldsymbol{\Phi}_0$ since it turns out to be zero in all cases. In order that the expansion for $\boldsymbol{\phi}$ is defined everywhere in Ω even though $\boldsymbol{\Phi}_i$ is defined only on Ω_0, we introduce a smooth cutoff function χ which is a function of ρ alone, independent of θ and t, and identically one for $0 \le \rho \le \rho_0/3$, identically zero for $\rho > 2\rho_0/3$.

In this section, we give precise definitions of all the functions $\boldsymbol{\phi}_i$, ω_i, and $\boldsymbol{\Phi}_i$. In §5 (Theorems 5.1–5.3), we shall prove the validity of the expansions. More precisely, we shall show that by choosing n large enough we can make the corresponding remainder terms

$$\omega_n^E := \omega - \sum_{i=0}^n t^i\omega_i, \qquad \boldsymbol{\phi}_n^E := \boldsymbol{\phi} - \sum_{i=0}^n t^i\boldsymbol{\phi}_i - \chi\sum_{i=1}^n t^i\boldsymbol{\Phi}_i$$

smaller than any desired power of t in any Sobolev norm.

Taking the divergence of (1.6) and using (1.7), we see that $\operatorname{div}\boldsymbol{\phi}$ satisfies Poisson's equation:

$$D\,\Delta\operatorname{div}\boldsymbol{\phi} = g.$$

This suggests an alternate form for the asymptotic expansion of $\boldsymbol{\phi}$ in which the terms of the boundary-layer expansion are divergence free and hence can be written as the curls of scalar functions. Inserting some convenient factors, the alternate expansion is

$$\boldsymbol{\phi} \sim \sum_{i=0}^\infty t^i\boldsymbol{\phi}_i - \lambda^{-1}\chi t^2\sum_{i=0}^\infty t^i\,\mathbf{curl}\,P_i$$

with $P_i(x,y) = \hat{P}_i(\hat{\rho},\theta)$ with $\hat{P}_i : [0,\infty)\times\partial\Omega \to \mathbb{R}$ smooth. Now

$$\mathbf{curl}\,P_i = \frac{\partial P_i}{\partial\rho}\,\mathbf{curl}\,\rho + \frac{\partial P_i}{\partial\theta}\,\mathbf{curl}\,\theta = -t^{-1}\frac{\partial \hat{P}_i}{\partial\hat{\rho}}\boldsymbol{s} - \sigma(\rho,\theta)\frac{\partial \hat{P}_i}{\partial\theta}\boldsymbol{n}.$$

Formally inserting the Taylor expansion

$$\sigma(\rho,\theta) = \sum_{j=0}^\infty[\kappa(\theta)\rho]^j = \sum_{j=0}^\infty[\kappa(\theta)t\hat{\rho}]^j$$

and equating the two forms of the boundary-layer expansion, we get that

$$-\lambda^{-1}t^2\sum_{i=0}^\infty t^i\,\mathbf{curl}\,P_i = \sum_{i=1}^\infty t^i\boldsymbol{\Phi}_i. \tag{3.2}$$

This gives the relation between $\boldsymbol{\Phi}_i$ and P_i:

$$\boldsymbol{\Phi}_i = \lambda^{-1}\left\{\frac{\partial P_{i-1}}{\partial\hat{\rho}}\boldsymbol{s} + \sum_{j=0}^{i-2}[\kappa(\theta)\hat{\rho}]^j\frac{\partial P_{i-j-2}}{\partial\theta}\boldsymbol{n}\right\}, \quad i \ge 1. \tag{3.3}$$

We now proceed to the definitions of w_i, $\boldsymbol{\phi}_i$, and P_i (with $\boldsymbol{\Phi}_i$ determined from P_i by (3.3)).

In order to motivate the definitions of the expansion functions, we shall reason formally. Let $\boldsymbol{\phi}^I$ denote $\sum_{i=0}^\infty t^i\boldsymbol{\phi}_i$ and let p^B denote $\sum_{i=0}^\infty t^iP_i$ (these definitions are only formal, since the sums need not be convergent). We want pairs $(\boldsymbol{\phi}^I,\omega)$ to solve the Reissner–Mindlin differential equations and $-\lambda^{-1}t^2(\mathbf{curl}\,p^B, 0)$ to solve the corresponding homogeneous differential equations, so that the pair $(\boldsymbol{\phi},\omega)$, which (when $\chi\equiv 1$) is formally their sum, will satisfy the inhomogeneous equations. Inserting the expansions for $\boldsymbol{\phi}^I$ and ω into the Reissner–Mindlin equations and equating like powers of t gives the equations

$$\lambda(\boldsymbol{\phi}_i - \mathbf{grad}\, \omega_i) = \mathbf{div}\, C\, \mathcal{E}(\boldsymbol{\phi}_{i-2}), \tag{3.4}$$

$$\lambda \operatorname{div}(\boldsymbol{\phi}_i - \mathbf{grad}\, \omega_i) = \delta_{i2} g, \tag{3.5}$$

where δ_{ij} is the Kronecker symbol. These equations are to hold for $i = 0, 1, \dots$ with the convention that $\boldsymbol{\phi}_j = 0$ for $j < 0$. From (3.4) and (3.5), we easily deduce that ω_i satisfies the biharmonic problem

$$D\, \Delta^2 \omega_i = \delta_{i0} g - \delta_{i2} \lambda^{-1} D\, \Delta\, g, \tag{3.6}$$

as is to be expected in view of (3.1). It follows from (3.4) that

$$\boldsymbol{\phi}_i = \mathbf{grad} \sum_{k=0}^{[i/2]} (\lambda^{-1} D)^k\, \Delta^k\, \omega_{i-2k}$$

or, in light of (3.6),

$$\boldsymbol{\phi}_i = \mathbf{grad}\, z_i, \tag{3.7}$$

with

$$z_i = \omega_i + \lambda^{-1} D\, \Delta\, \omega_{i-2} + \delta_{i4} \lambda^{-2} D g. \tag{3.8}$$

To obtain differential equations satisfied by the boundary-layer functions, we note that $(\mathbf{curl}\, P, 0)$ solves the homogeneous Reissner–Mindlin system if and only if P solves the differential equation

$$-t^2 \lambda^{-1} D \frac{1-\nu}{2}\, \Delta\, P + P = 0. \tag{3.9}$$

In (ρ, θ) coordinates, we have (on Ω_0)

$$\begin{aligned}
\Delta\, P &= |\, \mathbf{grad}\, \rho|^2 \frac{\partial^2 P}{\partial \rho^2} + \Delta\, \rho \frac{\partial P}{\partial \rho} + |\, \mathbf{grad}\, \theta|^2 \frac{\partial^2 P}{\partial \theta^2} + \Delta\, \theta \frac{\partial P}{\partial \theta} \\
&= \frac{\partial^2 P}{\partial \rho^2} - \kappa(\theta)\sigma(\rho,\theta) \frac{\partial P}{\partial \rho} + \sigma(\rho,\theta)^2 \frac{\partial^2 P}{\partial \theta^2} + \rho \kappa'(\theta) \sigma(\rho,\theta)^3 \frac{\partial P}{\partial \theta} \\
&= \frac{\partial^2 P}{\partial \rho^2} + \sum_{j=0}^{\infty} \rho^j \left(a_1^j \frac{\partial P}{\partial \rho} + a_2^j \frac{\partial^2 P}{\partial \theta^2} + a_3^j \frac{\partial P}{\partial \theta} \right).
\end{aligned}$$

In the last step, we have (formally) replaced each coefficient with its Taylor series in ρ. It is easy to check that

$$a_1^j = -[\kappa(\theta)]^{j+1}, \qquad a_2^j = (j+1)[\kappa(\theta)]^j, \qquad a_3^j = \frac{j(j+1)}{2} [\kappa(\theta)]^{j-1} \kappa'(\theta). \tag{3.10}$$

Switching to the stretched variable $\hat{\rho}$, this becomes

$$\Delta\, P = t^{-2} \frac{\partial^2 P}{\partial \hat{\rho}^2} + \sum_{j=0}^{\infty} (t\hat{\rho})^j \left(a_1^j t^{-1} \frac{\partial P}{\partial \hat{\rho}} + a_2^j \frac{\partial^2 P}{\partial \theta^2} + a_3^j \frac{\partial P}{\partial \theta} \right).$$

Thus if we write (3.9) in $(\hat\rho, \theta)$ variables, insert $\sum_{i=0}^{\infty} t^i P_i$ for P, and equate like powers of t, we get

$$(3.11) \quad -\lambda^{-1} D \frac{1-\nu}{2} \frac{\partial^2 P_i}{\partial \hat\rho^2} + P_i = \hat F_i(\hat\rho, \theta)$$

$$:= \lambda^{-1} D \frac{1-\nu}{2} \sum_{j=0}^{i-1} \hat\rho^j \left(a_1^j \frac{\partial P_{i-j-1}}{\partial \hat\rho} + a_2^j \frac{\partial^2 P_{i-j-2}}{\partial \theta^2} + a_3^j \frac{\partial P_{i-j-2}}{\partial \theta} \right), \quad i = 0, 1, \dots,$$

where again P_j is to be interpreted as 0 for $j < 0$. Note that (3.11) is an ordinary differential equation for the function P_i in the independent variable $\hat\rho$ in which θ enters as a parameter. We shall only consider solutions which satisfy the decay condition

$$(3.12) \qquad \lim_{\hat\rho \to \infty} P_i = 0.$$

This will ensure that each P_i decays exponentially with $\hat\rho$ and is therefore negligible outside of Ω_0.

The differential equations (3.6) and (3.11), together with appropriate boundary conditions, will be used to define the functions ω_i and P_i. Then the $\boldsymbol{\phi}_i$ are given by (3.7) and the $\boldsymbol{\Phi}_i$ by (3.3).

We now derive the boundary conditions and, for each of the boundary value problems we consider, show that the ω_i and P_i are uniquely determined. The boundary conditions for ω_i and P_i will be obtained from the boundary conditions for the Reissner–Mindlin system by inserting the asymptotic expansions and equating like powers of the thickness, and then using (3.7) to eliminate the $\boldsymbol{\phi}_i$.

The hard clamped plate. The boundary condition $\omega = 0$ leads, of course, to

$$(3.13) \qquad \omega_i = 0 \quad \text{on } \partial\Omega.$$

The boundary conditions $\boldsymbol{\phi} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{\phi} \cdot \boldsymbol{s} = 0$ give $\boldsymbol{\phi}_i \cdot \boldsymbol{n} + \boldsymbol{\Phi}_i \cdot \boldsymbol{n} = 0$ and $\boldsymbol{\phi}_i \cdot \boldsymbol{s} + \boldsymbol{\Phi}_i \cdot \boldsymbol{s} = 0$. Using (3.3), these become

$$(3.14) \qquad \boldsymbol{\phi}_i \cdot \boldsymbol{n} = -\lambda^{-1} \frac{\partial P_{i-2}}{\partial \theta} \quad \text{on } \partial\Omega$$

and

$$(3.15) \qquad \boldsymbol{\phi}_i \cdot \boldsymbol{s} = -\lambda^{-1} \frac{\partial P_{i-1}}{\partial \hat\rho} \quad \text{on } \partial\Omega.$$

In view of (3.7), (3.14) can be expressed equivalently as

$$(3.16) \qquad \frac{\partial \omega_i}{\partial n} = -\lambda^{-1} \frac{\partial P_{i-2}}{\partial \theta} - \lambda^{-1} D \frac{\partial \Delta \omega_{i-2}}{\partial n} - \delta_{i4} \lambda^{-2} D \frac{\partial g}{\partial n} \quad \text{on } \partial\Omega,$$

and, using (3.13) and (3.7), we can write (3.15) as

$$(3.17) \qquad -\frac{\partial P_i}{\partial \hat\rho} = D \frac{\partial \Delta w_{i-1}}{\partial s} + \delta_{i3} \lambda^{-1} D \frac{\partial g}{\partial s} \quad \text{on } \partial\Omega.$$

We now show that all the ω_i and P_i are uniquely determined by (3.6), (3.11), (3.13)–(3.15), and (3.12). Indeed, from (3.11), (3.12), and (3.17), we immediately infer that $P_0 = 0$. We can then uniquely determine ω_i for $i = 0, 1, 2$ from (3.6), (3.13), and (3.16). These being known, P_i, $i = 1, 2, 3$ are uniquely determined, from which we can in turn compute ω_i for $i = 3, 4, 5$ and so forth. Note that ω_0 is determined from the usual boundary value problem for a clamped Kirchhoff plate. Also, (3.6), (3.13), and (3.16) all have vanishing right-hand sides for $i = 1$, so ω_1 and therefore $\boldsymbol{\phi}_1$ vanish.

The soft clamped plate. In this case, the boundary conditions (3.13) and (3.14) apply, but instead of $\boldsymbol{\phi} \cdot \boldsymbol{s} = 0$ we must enforce $M_{\boldsymbol{s}}\boldsymbol{\phi} = 0$ on $\partial\Omega$. Using (3.3) and the fact that

$$M_{\boldsymbol{s}}\boldsymbol{\Phi} = \frac{D(1-\nu)}{2}\left(-t^{-1}\frac{\partial \boldsymbol{\Phi}}{\partial \hat{\rho}} \cdot \boldsymbol{s} + \frac{\partial \boldsymbol{\Phi}}{\partial \theta} \cdot \boldsymbol{n}\right),$$

we get

(3.18)

$$\begin{aligned}\frac{\partial^2 P_i}{\partial \hat{\rho}^2} &= -\kappa\frac{\partial P_{i-1}}{\partial \hat{\rho}} + \frac{\partial^2 P_{i-2}}{\partial \theta^2} + \frac{2\lambda}{D(1-\nu)}M_{\boldsymbol{s}}\boldsymbol{\phi}_i \\ &= -\kappa\frac{\partial P_{i-1}}{\partial \hat{\rho}} + \frac{\partial^2 P_{i-2}}{\partial \theta^2} + 2\left(\frac{\partial^2}{\partial s \partial n} - \kappa\frac{\partial}{\partial s}\right)(\lambda\omega_i - D\,\Delta\,\omega_{i-2} + \delta_{i4}\lambda^{-1}Dg).\end{aligned}$$

We conclude that ω_0 is uniquely determined and that ω_1 again vanishes. Now since $\partial^2\omega_0/\partial s\partial n = \partial\omega_0/\partial s = 0$, we infer that P_0 and P_1 vanish as well and therefore that ω_3 does also. The other terms can be computed as follows: first ω_2, then P_2 and P_3, then ω_4 and ω_5, then P_4 and P_5, etc. It is interesting to note that ω_0, ω_1, ω_2, and P_0 are the same for the hard and soft clamped plates but ω_3 and P_1 are not (they vanish for the latter but not for the former).

We now show that for the soft clamped plate all the P_i vanish for any values of θ such that $\kappa(\theta) = 0$. Thus there is no boundary layer near a flat portion of the boundary. (This property holds as well for the hard simply supported plate but not for the other boundary conditions we consider.) To prove it in the case of the soft clamped plate, we note first that by (3.7), $\boldsymbol{\phi}_i$ is a gradient for all i. Using this fact, one computes that

$$M_{\boldsymbol{s}}\boldsymbol{\phi}_i = D(1-\nu)\left(\frac{\partial \boldsymbol{\phi}_i \cdot \boldsymbol{n}}{\partial s} - \kappa\boldsymbol{\phi}_i \cdot \boldsymbol{s}\right).$$

In view of (3.14), we have

$$M_{\boldsymbol{s}}\boldsymbol{\phi}_i = -\lambda^{-1}D(1-\nu)\frac{\partial^2 P_{i-2}}{\partial \theta^2}$$

wherever $\kappa = 0$. Our claim then follows from the defining equations for the P_i and induction.

The hard simply supported plate. For the hard simply supported plate, the boundary conditions are (3.13), (3.15), and, arising from the condition $M_{\boldsymbol{n}}\boldsymbol{\phi} = 0$,

$$M_{\boldsymbol{n}}\boldsymbol{\phi}_i = \lambda^{-1}D(1-\nu)\left(\kappa\frac{\partial P_{i-2}}{\partial \theta} + \frac{\partial^2 P_{i-1}}{\partial \theta \partial \hat{\rho}}\right) \quad \text{on } \partial\Omega, \tag{3.19}$$

where we have used (3.3) and the fact that

$$M_{\boldsymbol{n}}\boldsymbol{\Phi} = D\left(-t^{-1}\frac{\partial \boldsymbol{\Phi}}{\partial \hat{\rho}} \cdot \boldsymbol{n} + \nu\frac{\partial \boldsymbol{\Phi}}{\partial \theta} \cdot \boldsymbol{s}\right).$$

Using (3.7) and (3.8), we may rewrite this as

$$\begin{aligned} D\left[(1-\nu)\frac{\partial^2}{\partial n^2}+\nu\,\Delta\right]\left(\omega_i+\lambda^{-1}D\,\Delta\,\omega_{i-2}+\delta_{i4}\lambda^{-2}Dg\right) \\ =\lambda^{-1}D(1-\nu)\left(\kappa\frac{\partial P_{i-2}}{\partial\theta}+\frac{\partial^2 P_{i-1}}{\partial\theta\partial\hat\rho}\right). \end{aligned}\tag{3.20}$$

Since (3.17) holds, we again have $P_0=0$. Using (3.13) and (3.20), we see that ω_0 is determined from the usual boundary value problem for a simply supported Kirchhoff plate and that ω_1 vanishes. We can then continue by computing P_1 and P_2, then ω_2 and ω_3, etc.

Now since $\boldsymbol{\phi}_i$ is a gradient, one can verify that

$$\mathbf{div}\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_i)\cdot\boldsymbol{s}=\frac{\partial M_{\boldsymbol{n}}\boldsymbol{\phi}_i}{\partial s}+D(1-\nu)\frac{\partial}{\partial s}\left[\frac{\partial\boldsymbol{\phi}_i\cdot\boldsymbol{s}}{\partial s}-\kappa\boldsymbol{\phi}_i\cdot\boldsymbol{n}\right].$$

If we assume that κ vanishes on a nondegenerate interval, then, combining this equation with (3.19) and (3.15), we can express $\operatorname{div}\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_i)\cdot\boldsymbol{s}$ in terms of P_{i-2} and P_{i-1} for θ in this interval. Now (3.15) and (3.4) combine to give

$$\frac{\partial P_i}{\partial\hat\rho}=-\,\mathbf{div}\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_i)\cdot\boldsymbol{s}.$$

Thus, on an interval where κ vanishes, $\partial P_i/\partial\hat\rho$ may be expressed in terms of P_{i-1} and P_{i-2} on that interval. A simple induction allows us to conclude that all the P_i vanish for such θ.

The soft simply supported plate. In this case, the boundary conditions are (3.13), (3.18), and (3.19). We can compute ω_0 and $\boldsymbol{\phi}_0$ from the same equations as for the hard simply supported case. Then P_0 can be computed (it need not vanish), and then ω_1 (which also need not vanish), P_1, etc.

4. A priori estimates for the soft simply supported plate. We now consider in detail the case of the soft simply supported plate. An easy computation shows that

$$\hat P_0(\hat\rho,\theta)=D(1-\nu)\widehat{\frac{\partial^2\omega_0}{\partial s\partial n}}(0,\theta)e^{-c\hat\rho},\qquad\text{where } c=\sqrt{\frac{2\lambda}{D(1-\nu)}}=\sqrt{12k}.$$

We may show in general that $\hat P_i$ are polynomials in $\hat\rho$ times the decaying exponential $e^{-c\hat\rho}$. The specific form is given in the following theorem.

THEOREM 4.1. *For $i\in\mathbb{N}$,*

$$\hat P_i(\hat\rho,\theta)=e^{-c\hat\rho}\sum_{k=0}^{i}\sum_{j=0}^{i}\sum_{l=0}^{i-j}\alpha_{ijkl}(\theta)\hat\rho^k\frac{\partial^l}{\partial\theta^l}\widehat{M_{\boldsymbol{s}}\boldsymbol{\phi}_j}(0,\theta),$$

where the α_{ijkl} are smooth functions of θ which depend only upon the domain Ω.

Proof. Let us say that a function is of type (m,n) if it is a sum of terms of the form

$$\alpha(\theta)\hat\rho^k\frac{\partial^l}{\partial\theta^l}\widehat{M_{\boldsymbol{s}}\boldsymbol{\phi}_j}(0,\theta)$$

with $k,j,l\in\mathbb{N}$ satisfying $k\le m$, $j+l\le n$, and α a smooth function of θ depending only on Ω. We wish to show that P_i is of type (i,i) for $i\in\mathbb{N}$. We shall use induction on i. The result is known for $i=0$. If we assume its validity for $0,1,\dots,i-1$, we easily check that F_i defined in (3.11) is of type $(i-1,i)$ and that the right-hand side of (3.18) is of type $(0,i)$ (there is no ρ dependence since this is on the boundary). It is then easy to see that the unique solution P_i of (3.11), (3.12), and (3.18) must be of type (i,i), as desired. □

Using this formula, we now turn to the derivation of a priori estimates for the interior expansions and boundary correctors. The following estimates are obtained immediately from the form of P_i.

THEOREM 4.2. *For any $i \in \mathbb{N}$ and $s \in \mathbb{R}$ there exists a constant C depending only on Ω, E, ν, k, s, and i, such that*

$$|P_i|_s + \left|\frac{\partial P_i}{\partial \hat\rho}\right|_s \le C \sum_{j=0}^{i} |M_s \phi_j|_{s+i-j}.$$

Using this result, we next obtain bounds for the terms in the interior expansions of ϕ and ω.

THEOREM 4.3. *For all real $s \ge 0$ and $i \in \mathbb{N}$, there exists a constant C such that*

$$\|\omega_i\|_{s+2} + \|\phi_i\|_{s+1} \le C\|g\|_{s+i-2}.$$

Proof. Let B_2 denote the boundary differential operator

$$B_2\omega = D[(1-\nu)\frac{\partial^2\omega}{\partial n^2} + \nu\,\Delta\,\omega].$$

It easily follows from (3.8), (3.6), (3.13), and (3.20) that

$$D\,\Delta^2 z_i = \delta_{i0} g \quad \text{in } \Omega, \qquad z_i = \lambda^{-1} D\,\Delta\,\omega_{i-2} + \delta_{i4}\lambda^{-2} Dg \quad \text{on } \partial\Omega,$$
$$B_2 z_i = \lambda^{-1} D(1-\nu)\left(\kappa \frac{\partial P_{i-2}}{\partial\theta} + \frac{\partial^2 P_{i-1}}{\partial\theta\partial\hat\rho}\right) \quad \text{on } \partial\Omega.$$

Applying standard estimates for the biharmonic, we obtain for $s \ge 0$ that

$$\begin{aligned}
&\|z_i\|_{s+2} \\
&\le C\left(\delta_{i0}\|g\|_{s-2} + |\,\Delta\,\omega_{i-2}|_{s+3/2} + \delta_{i4}|g|_{s+3/2} + |\frac{\partial P_{i-2}}{\partial\theta}|_{s-1/2} + \left|\frac{\partial^2 P_{i-1}}{\partial\theta\partial\hat\rho}\right|_{s-1/2}\right) \\
&\le C\left(\delta_{i0}\|g\|_{s-2} + \|\omega_{i-2}\|_{s+4} + \delta_{i4}\|g\|_{s+2} + |P_{i-2}|_{s+1/2} + \left|\frac{\partial P_{i-1}}{\partial\hat\rho}\right|_{s+1/2}\right) \\
&\le C\left(\delta_{i0}\|g\|_{s-2} + \|\omega_{i-2}\|_{s+4} + \delta_{i4}\|g\|_{s+2} + \sum_{j=0}^{i-1} |M_s\phi_j|_{s+i-j-1/2}\right) \\
&\le C\left(\delta_{i0}\|g\|_{s-2} + \|\omega_{i-2}\|_{s+4} + \delta_{i4}\|g\|_{s+2} + \sum_{j=0}^{i-1} \|\phi_j\|_{s+i-j+1}\right).
\end{aligned}$$

Now $\phi_i = \mathbf{grad}\, z_i$ and by the definition of z_i and the triangle inequality, it easily follows that

$$\|\omega_i\|_{s+2} \le C\left(\|z_i\|_{s+2} + \|\omega_{i-2}\|_{s+4} + \delta_{i4}\|g\|_{s+2}\right).$$

Combining these results, we obtain

$$\|\omega_i\|_{s+2} + \|\phi_i\|_{s+1} \le C\left(\delta_{i0}\|g\|_{s-2} + \|\omega_{i-2}\|_{s+4} + \delta_{i4}\|g\|_{s+2} + \sum_{j=0}^{i-1}\|\phi_j\|_{s+i-j+1}\right).$$

The result for $i = 0$ follows directly and the result for $i \ge 1$ is then obtained by induction. □

COROLLARY 4.4. *For real $s \ge -1/2$ and $i \in \mathbb{N}$, there exists a constant C such that*

$$|P_i|_s + \left|\frac{\partial P_i}{\partial \hat\rho}\right|_s + |M_{\boldsymbol{s}}\phi_i|_s \le C\|g\|_{s+i-3/2}.$$

Proof. Since

$$M_{\boldsymbol{s}}\phi_j = M_{\boldsymbol{s}}(\mathbf{grad}\, z_j) = D(1-\nu)\left(\frac{\partial^2}{\partial s\partial n} - \kappa\frac{\partial}{\partial s}\right)z_j$$

on $\partial\Omega$, we have

$$\begin{aligned}|M_{\boldsymbol{s}}\phi_j|_{s+i-j} &\le C\left(\left|\frac{\partial z_j}{\partial n}\right|_{s+i-j+1} + |z_j|_{s+i-j+1}\right) \le C\|z_j\|_{s+i-j+5/2}\\ &\le C\left(\|\omega_j\|_{s+i-j+5/2} + \|\omega_{j-2}\|_{s+i-j+9/2} + \delta_{j4}\|g\|_{s+i-j+5/2}\right)\\ &\le C\|g\|_{s+i-3/2}, \quad j = 0, 1, \dots, i.\end{aligned}$$

The result follows from this estimate and Theorem 4.2.

We next consider the derivation of interior norm estimates for the boundary correctors. To get these results, we make use of the following elementary lemma.

LEMMA 4.5. *Suppose $a > 0$, $b \ge 1$, and $p(x)$ is a polynomial of degree $\le n$ with positive coefficients. Then there exists a constant $K_n(a)$ depending only on n and a such that*

$$\int_b^\infty p(x)e^{-ax}\,dx \le K_n(a)e^{-ab}p(b).$$

Proof. It clearly suffices to prove the result for $p(x) = x^n$. In this case, it reduces to showing that

$$\int_0^\infty (1 + x/b)^n e^{-ax}\,dx \le K_n(a) \quad \text{for all } b \ge 1,$$

which is obvious. □

Next recall that

$$\Omega_0 = \{\, \boldsymbol{z} - \rho\boldsymbol{n_z} \mid \boldsymbol{z} \in \partial\Omega, \quad 0 < \rho < \rho_0 \,\}$$

and set

$$\Omega_1 = \{\, \boldsymbol{z} - \rho\boldsymbol{n_z} \mid \boldsymbol{z} \in \partial\Omega, \quad \rho_0/3 < \rho < \rho_0 \,\},$$

so $\chi \equiv 1$ on $\Omega_0 \setminus \Omega_1$ and $\chi \equiv 0$ on a neighborhood of $\Omega \setminus \Omega_0$. The following result is similar to results previously derived in [1] (cf. Theorems 4.1 and 4.5).

LEMMA 4.6. *Suppose $k, l, n, s \in \mathbb{N}$,*

$$\hat P(\hat\rho, \theta) = \hat\alpha(\theta)\exp(-c\hat\rho)p(\hat\rho),$$

and

$$\hat{f}(\hat{\rho},\theta) = \hat{\rho}^k \frac{\partial^{l+n}}{\partial\hat{\rho}^l \partial\theta^n} \hat{P}(\hat{\rho},\theta),$$

where α is a smooth function depending on $\partial\Omega$ and p is a polynomial. Then there exists a constant C depending only on Ω, p, k, l, n, and s such that

$$\|f\|_{s,\Omega_0} \le Ct^{1/2-s} \sum_{m=0}^{s} t^m |\alpha|_{m+n}.$$

Moreover, for any $j \ge 0$, there exists a constant C' depending on C and j such that

$$\|f\|_{s,\Omega_1} \le C' t^{1/2+j-s} \sum_{m=0}^{s} t^m |\alpha|_{m+n}.$$

We now obtain bounds on the P_i in Ω.

THEOREM 4.7. *For any $i, j, k, l, n, s \in \mathbb{N}$, there is a constant C such that*

$$\|\hat{\rho}^k \frac{\partial^{l+n}}{\partial\hat{\rho}^l \partial\theta^n} P_i\|_{s,\Omega_0} \le C(t^{1/2-s}\|g\|_{n+i-3/2} + t^{1/2}\|g\|_{n+s+i-3/2}), \tag{4.1}$$

$$\|\hat{\rho}^k \frac{\partial^{l+n}}{\partial\hat{\rho}^l \partial\theta^n} P_i\|_{s,\Omega_1} \le Ct^j(t^{1/2-s}\|g\|_{n+i-3/2} + t^{1/2}\|g\|_{n+s+i-3/2}). \tag{4.2}$$

Proof. From Lemma 4.6 and Theorem 4.1, we get

$$\|\hat{\rho}^k \frac{\partial^{l+n}}{\partial\hat{\rho}^l \partial\theta^n} P_i\|_{s,\Omega_0} \le Ct^{1/2-s} \sum_{m=0}^{s} t^m \sum_{j=0}^{i} |M_s \phi_j|_{n+m+i-j}.$$

By Corollary 4.4, this is bounded by

$$Ct^{1/2-s} \sum_{m=0}^{s} t^m \|g\|_{n+m+i-3/2} \le C(t^{1/2-s}\|g\|_{n+i-3/2} + t^{1/2}\|g\|_{n+s+i-3/2}).$$

Similarly,

$$\begin{aligned}
\|\hat{\rho}^k \frac{\partial^{l+n}}{\partial\hat{\rho}^l \partial\theta^n} P_i\|_{s,\Omega_1} &\le C_j t^{1/2+j-s} \sum_{m=0}^{s} t^m \sum_{j=0}^{i} |M_s \phi_j|_{n+m+i-j} \\
&\le C_j t^{1/2+j-s} \sum_{m=0}^{s} t^m \|g\|_{n+m+i-3/2} \\
&\le C_j t^j (t^{1/2-s}\|g\|_{n+i-3/2} + t^{1/2}\|g\|_{n+s+i-3/2}). \qquad \square
\end{aligned}$$

Using (3.3), we easily obtain the following, where $\phi_n^B = \sum_{i=0}^n t^i \Phi_i$ and $p_n^B = \sum_{i=0}^n t^i P_i$.

COROLLARY 4.8. *For any $s, n, j \in \mathbb{N}$, there is a constant C such that*

$$\|p_n^B\|_{s,\Omega_0} \le C(t^{1/2-s}\|g\|_{-3/2} + t^{n+1/2}\|g\|_{n+s-3/2}), \tag{4.3}$$

$$\|\boldsymbol{\Phi}_n\|_{s,\Omega_0} \le C(t^{1/2-s}\|g\|_{n-5/2} + t^{1/2}\|g\|_{n+s-5/2}), \tag{4.4}$$

$$\|\boldsymbol{\phi}_n^B\|_{s,\Omega_0} \le C(t^{3/2-s}\|g\|_{-3/2} + t^{n+1/2}\|g\|_{n+s-5/2}), \tag{4.5}$$

$$\|p_n^B\|_{s,\Omega_1} \le Ct^j(t^{1/2-s}\|g\|_{-3/2} + t^{n+1/2}\|g\|_{n+s-3/2}), \tag{4.6}$$

$$\|\boldsymbol{\Phi}_n\|_{s,\Omega_1} \le Ct^j(t^{1/2-s}\|g\|_{n-5/2} + t^{1/2}\|g\|_{n+s-5/2}), \tag{4.7}$$

$$\|\boldsymbol{\phi}_n^B\|_{s,\Omega_1} \le Ct^j(t^{3/2-s}\|g\|_{-3/2} + t^{n+1/2}\|g\|_{n+s-5/2}). \tag{4.8}$$

5. Error estimates for the soft simply supported plate. In this section, we shall derive estimates for differences between the solution components of the Reissner–Mindlin equations and finite sums of the asymptotic expansions. We shall not bound these differences directly but rather first bound their images under a differential operator and then apply a priori estimates for the operator. The differential operator we employ is not the Reissner–Mindlin operator but rather a singularly perturbed Stokes-like operator which arises in an equivalent formulation of the Reissner–Mindlin equations due to Brezzi and Fortin [3].

The Brezzi–Fortin formulation begins with the Helmholtz decomposition of the transverse shear stress vector

$$\lambda t^{-2}(\mathbf{grad}\,\omega - \boldsymbol{\phi}) = \mathbf{grad}\,r + \mathbf{curl}\,p, \quad r \in \mathring{H}^1, \ p \in H^1/\mathbb{R}. \tag{5.1}$$

Then it is easy to see that r may be determined by the Poisson equation

$$-\Delta r = g \tag{5.2}$$

together with the homogeneous Dirichlet boundary condition, and then $\boldsymbol{\phi}$ and p may be determined from the perturbed Stokes-like system

$$-\,\mathbf{div}\,\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}) - \mathbf{curl}\,p = \mathbf{grad}\,r, \tag{5.3}$$

$$-\operatorname{rot}\boldsymbol{\phi} + \lambda^{-1}t^2\,\Delta p = 0 \tag{5.4}$$

together with the boundary conditions

$$M_{\boldsymbol{n}}\boldsymbol{\phi} = 0, \qquad M_{\boldsymbol{s}}\boldsymbol{\phi} = 0, \qquad \boldsymbol{\phi}\cdot\boldsymbol{s} + \lambda^{-1}t^2\frac{\partial p}{\partial n} = 0. \tag{5.5}$$

Note that p is only determined modulo $\mathbb{R}$, i.e., up to an additive constant. Finally, ω satisfies

$$-\Delta\omega = -\operatorname{div}\boldsymbol{\phi} - \lambda^{-1}t^2\,\Delta r \tag{5.6}$$

and vanishes on the boundary.

The weak formulation of (5.3), (5.4), and the boundary conditions (5.5) seeks $\boldsymbol{\phi} \in \boldsymbol{H}^1$, $p \in H^1/\mathbb{R}$ such that

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) - (\mathbf{curl}\,p, \boldsymbol{\psi}) = (\mathbf{grad}\,r, \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \boldsymbol{H}^1, \tag{5.7}$$

$$-(\boldsymbol{\phi} + \lambda^{-1}t^2\,\mathbf{curl}\,p, \mathbf{curl}\,q) = 0 \quad \text{for all } q \in H^1/\mathbb{R}, \tag{5.8}$$

where

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) = (\mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}), \mathcal{E}(\boldsymbol{\psi})).$$

To continue, we need to define an asymptotic approximation to p. From (3.5), we see that $\lambda(\boldsymbol{\phi}_{i+2} - \mathbf{grad}\,\omega_{i+2}) + \delta_{i0}\,\mathbf{grad}\,r$ is divergence free. Therefore, we can determine a function p_i, unique modulo $\mathbb{R}$, by

$$\mathbf{curl}\,p_i = -\lambda(\boldsymbol{\phi}_{i+2} - \mathbf{grad}\,\omega_{i+2}) - \delta_{i0}\,\mathbf{grad}\,r = -\,\mathbf{div}\,C\,\mathcal{E}(\boldsymbol{\phi}_i) - \delta_{i0}\,\mathbf{grad}\,r. \tag{5.9}$$

It follows immediately from Theorem 4.3 and regularity for the Dirichlet problem that

$$\|p_n\|_{s/\mathbb{R}} \le C\|g\|_{s+n-2}, \quad s \in \mathbb{R},\ s \ge 0. \tag{5.10}$$

Note that, by (3.7), $\mathbf{curl}\,p_i$ is a gradient, so p_i is harmonic for all i. We may now write our asymptotic expansion of p:

$$p \sim \sum_{i=0}^{\infty} t^i p_i + \chi \sum_{i=0}^{\infty} t^i P_i.$$

Let us now introduce some notation for the finite interior and boundary expansion sums. Set

$$\omega_n^I = \sum_{i=0}^{n} t^i w_i, \quad \boldsymbol{\phi}_n^I = \sum_{i=0}^{n} t^i \boldsymbol{\phi}_i, \quad p_n^I = \sum_{i=0}^{n} t^i p_i, \quad \boldsymbol{\phi}_n^B = \sum_{i=0}^{n} t^i \boldsymbol{\Phi}_i, \quad p_n^B = \sum_{i=0}^{n} t^i P_i,$$

and

$$\omega_n^E = \omega - \omega_n^I, \qquad \boldsymbol{\phi}_n^E = \boldsymbol{\phi} - \boldsymbol{\phi}_n^I - \chi\boldsymbol{\phi}_n^B, \qquad p_n^E = p - p_n^I - \chi p_{n-1}^B.$$

Note that we deliberately choose one less term in the boundary-layer expansion for p than for the other terms.

The following three theorems give estimates in Sobolev norms of general index for the differences between $\boldsymbol{\phi}$, p, and ω and their finite asymptotic approximations. Note that the rates of convergence for $\boldsymbol{\phi}$ and p decline as the index of the Sobolev norm increases, but this is not true for ω. This reflects the presence of a boundary layer for the first two variables but not the third.

THEOREM 5.1. *For any $n \in \mathbb{N}$, there exists a constant C independent of t such that*

$$\|\boldsymbol{\phi}_n^E\|_1 + \|p_n^E\|_{0/\mathbb{R}} + t\|\,\mathbf{curl}\,p_n^E\|_0 \le C(t^{n+1/2}\|g\|_{n-3/2} + t^{n+3/2}\|g\|_{n-1/2}).$$

THEOREM 5.2. *For any $n, s \in \mathbb{N}$, $s \ge 2$, there exists a constant C independent of t such that*

$$\|\boldsymbol{\phi}_n^E\|_s + t\|p_n^E\|_{s/\mathbb{R}} \le C(t^{n+3/2-s}\|g\|_{n-3/2} + t^{n+1}\|g\|_{n+s-2}).$$

THEOREM 5.3. *For any $n, s \in \mathbb{N}$, $s \ge 2$, there exists a constant C independent of t such that*

$$\|\omega_n^E\|_2 \le C(t^{n+1}\|g\|_{n-1} + t^{n+5/2}\|g\|_{n+1/2}),$$
$$\|\omega_n^E\|_s \le C(t^{n+1}\|g\|_{n+s-3} + t^{n+s}\|g\|_{n+2s-4}), \quad s \ge 3.$$

The proofs depend on a number of estimates and equations which we collect here and prove at the end of the section. These results show that the formal equations (3.2) and (3.9) and the moment boundary conditions are indeed satisfied, at least to high order, by the finite boundary-layer expansions.

LEMMA 5.4. *For any $n, s \in \mathbb{N}$, there exists a constant C for which*

(5.11)
$$\|\chi\boldsymbol{\phi}_n^B + \lambda^{-1}t^2\,\mathbf{curl}(\chi p_{n-1}^B)\|_s \le C(t^{n+3/2-s}\|g\|_{n-3/2} + t^{n+3/2}\|g\|_{n+s-3/2}),$$

(5.12)
$$\|\operatorname{div}(\chi\boldsymbol{\phi}_n^B)\|_s \le C(t^{n+1/2-s}\|g\|_{n-3/2} + t^{n+1/2}\|g\|_{n+s-3/2}),$$

(5.13)
$$\|D\frac{1-\nu}{2}\operatorname{rot}(\chi\boldsymbol{\phi}_n^B) - p_{n-1}^B\|_s \le C(t^{n+1/2-s}\|g\|_{n-3/2} + t^{n+1/2}\|g\|_{s+n-3/2}),$$

(5.14)
$$D\frac{1-\nu}{2}\operatorname{rot}(\chi\boldsymbol{\phi}_n^B) - p_{n-1}^B + t^n\left(\lambda^{-1}\frac{D(1-\nu)}{2}\frac{\partial^2\hat{P}_n}{\partial\hat{\rho}^2} - \hat{P}_n\right) = 0 \quad \textit{on } \partial\Omega,$$

(5.15)
$$M_{\boldsymbol{n}}(\boldsymbol{\phi}_n^I + \boldsymbol{\phi}_n^B) = D\operatorname{div}\boldsymbol{\phi}_n^B \quad \textit{on } \partial\Omega,$$

(5.16)
$$M_{\boldsymbol{s}}(\boldsymbol{\phi}_n^I + \boldsymbol{\phi}_n^B) = \lambda^{-1}\frac{D(1-\nu)}{2}t^n\frac{\partial^2\hat{P}_n}{\partial\hat{\rho}^2} \quad \textit{on } \partial\Omega,$$

(5.17)
$$M_{\boldsymbol{s}}(\boldsymbol{\phi}_n^I + \boldsymbol{\phi}_n^B) + D\frac{1-\nu}{2}\operatorname{rot}(\chi\boldsymbol{\phi}_n^B) - p_{n-1}^B = t^n\hat{P}_n \quad \textit{on } \partial\Omega.$$

In the interest of brevity, we introduce the following notation for the quantity on the right-hand side of the estimate in Theorem 5.1:

$$\Lambda = t^{n+1/2}\|g\|_{n-3/2} + t^{n+3/2}\|g\|_{n-1/2}.$$

Proof of Theorem 5.1. It follows immediately from (5.9) that

$$-\operatorname{\mathbf{div}} \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_n^I) - \mathbf{curl}\,p_n^I = \mathbf{grad}\,r, \quad n = 0, 1, \ldots.$$

Therefore,

(5.18)
$$-a(\boldsymbol{\phi}_n^I, \boldsymbol{\psi}) + (\mathbf{curl}\,p_n^I, \boldsymbol{\psi}) = -(\mathbf{grad}\,r, \boldsymbol{\psi}) - \langle M_{\boldsymbol{n}}\boldsymbol{\phi}_n^I, \boldsymbol{\psi}\cdot\boldsymbol{n}\rangle - \langle M_{\boldsymbol{s}}\boldsymbol{\phi}_n^I, \boldsymbol{\psi}\cdot\boldsymbol{s}\rangle, \quad \boldsymbol{\psi} \in \boldsymbol{H}^1.$$

Using the identity

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) = D\frac{1-\nu}{2}(\operatorname{rot}\boldsymbol{\phi}, \operatorname{rot}\boldsymbol{\psi}) + D(\operatorname{div}\boldsymbol{\phi}, \operatorname{div}\boldsymbol{\psi}) + \langle M_{\boldsymbol{n}}\boldsymbol{\phi} - D\operatorname{div}\boldsymbol{\phi}, \boldsymbol{\psi}\cdot\boldsymbol{n}\rangle + \left\langle M_{\boldsymbol{s}} + D\frac{1-\nu}{2}\operatorname{rot}\boldsymbol{\phi}, \boldsymbol{\psi}\cdot\boldsymbol{s}\right\rangle,$$

we get

(5.19)
$$\begin{aligned}&-a(\chi\boldsymbol{\phi}_n^B, \boldsymbol{\psi}) + (\mathbf{curl}\,\chi p_{n-1}^B, \boldsymbol{\psi}) \\ &= -(D\frac{1-\nu}{2}\operatorname{rot}(\chi\boldsymbol{\phi}_n^B) - \chi p_{n-1}^B, \operatorname{rot}\boldsymbol{\psi}) - D(\operatorname{div}(\chi\boldsymbol{\phi}_n^B), \operatorname{div}\boldsymbol{\psi}) \\ &- \langle M_{\boldsymbol{n}}\boldsymbol{\phi}_n^B - D\operatorname{div}\boldsymbol{\phi}_n^B, \boldsymbol{\psi}\cdot\boldsymbol{n}\rangle - \left\langle M_{\boldsymbol{s}}\boldsymbol{\phi}_n^B + D\frac{1-\nu}{2}\operatorname{rot}\boldsymbol{\phi}_n^B - p_{n-1}^B, \boldsymbol{\psi}\cdot\boldsymbol{s}\right\rangle, \quad \boldsymbol{\psi} \in \boldsymbol{H}^1.\end{aligned}$$

Adding (5.7), (5.18), and (5.19) and using (5.15) and (5.17) gives the error equation corresponding to (5.7):

(5.20)
$$\begin{aligned}a(\boldsymbol{\phi}_n^E, \boldsymbol{\psi}) - (\mathbf{curl}\,p_n^E, \boldsymbol{\psi}) = &-\left(D\frac{1-\nu}{2}\operatorname{rot}(\chi\boldsymbol{\phi}_n^B) - \chi p_{n-1}^B, \operatorname{rot}\boldsymbol{\psi}\right) \\ &- D\left(\operatorname{div}(\chi\boldsymbol{\phi}_n^B), \operatorname{div}\boldsymbol{\psi}\right) - t^n\langle\hat{P}_n, \boldsymbol{\psi}\cdot\boldsymbol{s}\rangle, \quad \boldsymbol{\psi} \in \boldsymbol{H}^1.\end{aligned}$$

Turning to the second equation, we have from (5.9) that

$$\boldsymbol{\phi}_n^I = \mathbf{grad}\,\omega_n^I - \lambda^{-1}t^2\,\mathbf{grad}\,r - \lambda^{-1}t^2\,\mathbf{curl}\,p_{n-2}^I.$$

Combining this with (5.1), we obtain

$$\boldsymbol{\phi}_n^E = \mathbf{grad}(\omega - \omega_n^I) - \lambda^{-1}t^2\,\mathbf{curl}\,p_n^E \\ - \lambda^{-1}(t^{n+1}\,\mathbf{curl}\,p_{n-1} + t^{n+2}\,\mathbf{curl}\,p_n) - [\chi\boldsymbol{\phi}_n^B + \lambda^{-1}t^2\,\mathbf{curl}(\chi p_{n-1}^B)].$$

Multiplying by $\mathbf{curl}\,q$ for $q \in H^1$, using the orthogonality of gradients and curls, and rearranging, gives the error equation corresponding to (5.8):

$$(5.21)\quad (\boldsymbol{\phi}_n^E + \lambda^{-1}t^2\,\mathbf{curl}\,p_n^E, \mathbf{curl}\,q) = -\lambda^{-1}(t^{n+1}\,\mathbf{curl}\,p_{n-1} + t^{n+2}\,\mathbf{curl}\,p_n, \mathbf{curl}\,q) \\ - \left(\chi\boldsymbol{\phi}_n^B + \lambda^{-1}t^2\,\mathbf{curl}(\chi p_{n-1}^B), \mathbf{curl}\,q\right), \quad q \in H^1.$$

The desired error estimate will be obtained from (5.20) and (5.21) using a few choices of the test functions $\boldsymbol{\psi}$ and q. For this we will need to bound various terms arising on the right-hand sides.

Our first choice of test functions is $\boldsymbol{\psi} = \boldsymbol{\phi}_n^E$ in (5.20) and $q = p_n^E - t^n\chi P_n$ in (5.21). (The more obvious test function $q = p_n^E$ could also be used here, but not for the case of the free plate, since there we will need q to vanish on the boundary.) Adding these equations and rearranging terms, we get

$$a(\boldsymbol{\phi}_n^E, \boldsymbol{\phi}_n^E) + \lambda^{-1}t^2\|\,\mathbf{curl}\,p_n^E\|_0^2 = T_1 + T_2 + \cdots + T_9,$$

where

$$T_1 = -\left(D\frac{1-\nu}{2}\operatorname{rot}(\chi\boldsymbol{\phi}_n^B) - \chi p_{n-1}^B, \operatorname{rot}\boldsymbol{\phi}_n^E\right),$$
$$T_2 = -D\left(\operatorname{div}(\chi\boldsymbol{\phi}_n^B), \operatorname{div}\boldsymbol{\phi}_n^E\right),$$
$$T_3 = -t^n\langle \hat{P}_n, \boldsymbol{\phi}_n^E\cdot\boldsymbol{s}\rangle,$$
$$T_4 = -\lambda^{-1}\left(t^{n+1}\,\mathbf{curl}\,p_{n-1} + t^{n+2}\,\mathbf{curl}\,p_n, \mathbf{curl}\,p_n^E\right),$$
$$T_5 = \lambda^{-1}t^n\left(t^{n+1}\,\mathbf{curl}\,p_{n-1} + t^{n+2}\,\mathbf{curl}\,p_n, \mathbf{curl}(\chi\hat{P}_n)\right),$$
$$T_6 = -\left(\chi\boldsymbol{\phi}_n^B + \lambda^{-1}t^2\,\mathbf{curl}(\chi p_{n-1}^B), \mathbf{curl}\,p_n^E\right),$$
$$T_7 = t^n\left(\chi\boldsymbol{\phi}_n^B + \lambda^{-1}t^2\,\mathbf{curl}(\chi p_{n-1}^B), \mathbf{curl}(\chi\hat{P}_n)\right),$$
$$T_8 = t^n\left(\boldsymbol{\phi}_n^E, \mathbf{curl}(\chi\hat{P}_n)\right) = t^n(\operatorname{rot}\boldsymbol{\phi}_n^E, \chi\hat{P}_n) + t^n\langle\hat{P}_n, \boldsymbol{\phi}_n^E\cdot\boldsymbol{s}\rangle,$$
$$T_9 = \lambda^{-1}t^{n+2}\left(\mathbf{curl}\,p_n^E, \mathbf{curl}(\chi\hat{P}_n)\right).$$

By (5.13) and (5.12), we get

$$(5.22)\quad |T_1| + |T_2| \le C\Lambda\|\boldsymbol{\phi}_n^E\|_{1/\mathbb{R}}.$$

Next, from (4.1),

$$(5.23)\quad |T_3 + T_8| = |t^n(\operatorname{rot}\boldsymbol{\phi}_n^E, \chi\hat{P}_n)| \le C\Lambda\|\boldsymbol{\phi}_n^E\|_{1/\mathbb{R}}.$$

Since the p_n are harmonic, using (5.10) we obtain for any q that

$$\begin{aligned}|(t^{n+1}\,\mathbf{curl}p_{n-1}+t^{n+2}\,\mathbf{curl}\,p_n,\mathbf{curl}\,q)| &= \left|\left\langle t^{n+1}\frac{\partial p_{n-1}}{\partial n}+t^{n+2}\frac{\partial p_n}{\partial n},\bar q\right\rangle\right| \\ &\le |\bar q|_0\left|t^{n+1}\frac{\partial p_{n-1}}{\partial n}+t^{n+2}\frac{\partial p_n}{\partial n}\right|_0 \le |\bar q|_0\left(t^{n+1}\|p_{n-1}\|_{3/2}+t^{n+2}\|p_n\|_{3/2}\right) \\ &\le C\|\bar q\|_0^{1/2}(t\|\,\mathbf{curl}\,q\|_0)^{1/2}\Lambda \le C\Lambda^2+\delta(\|q\|^2_{0/\mathbb{R}}+t^2\|\,\mathbf{curl}\,q\|_0^2),\end{aligned}$$

where $\bar q$ is the difference between q and its mean value and δ can be any positive number and will be chosen later. Applying this twice and using (4.1), we get

$$|T_4| \le C\Lambda^2+\delta(\|p_n^E\|^2_{0/\mathbb{R}}+t^2\|\,\mathbf{curl}\,p_n^E\|_0^2), \qquad |T_5|\le C\Lambda^2. \tag{5.24}$$

Finally, by (5.11),

$$|T_6| \le Ct\Lambda\|\,\mathbf{curl}\,p_n^E\|_0, \qquad |T_7|\le C\Lambda^2, \tag{5.25}$$

and, using (4.1),

$$|T_9| \le Ct\Lambda\|\,\mathbf{curl}\,p_n^E\|_0. \tag{5.26}$$

Combining (5.22)–(5.26) gives

$$a(\boldsymbol{\phi}_n^E,\boldsymbol{\phi}_n^E)+\lambda^{-1}t^2\|\,\mathbf{curl}\,p_n^E\|_0^2 \le C_\epsilon\Lambda^2+\epsilon(\|\boldsymbol{\phi}_n^E\|^2_{1/\mathbb{R}}+\|p_n^E\|^2_{0/\mathbb{R}}+t^2\|\,\mathbf{curl}\,p_n^E\|_0^2), \tag{5.27}$$

where $\epsilon>0$ is arbitrary and $C_\epsilon>0$ depends on ϵ.

To get control over the L^2 norm of p_n^E, we use another test function in (5.20). Namely, we select $\boldsymbol{\psi}\in\mathring{\boldsymbol{H}}^1$ with $\operatorname{rot}\boldsymbol{\psi}=\bar p_n^E$ and $\|\boldsymbol{\psi}\|_1\le C\|p_n^E\|_{0/\mathbb{R}}$ (this is always possible). Then

$$\|p_n^E\|^2_{0/\mathbb{R}}=\|\bar p_n^E\|_0^2=(p_n^E,\operatorname{rot}\boldsymbol{\psi})=(\mathbf{curl}\,p_n^E,\boldsymbol{\psi})=a(\boldsymbol{\phi}_n^E,\boldsymbol{\psi})-[a(\boldsymbol{\phi}_n^E,\boldsymbol{\psi})-(\mathbf{curl}\,p_n^E,\boldsymbol{\psi})].$$

Using (5.20) and noting that $\boldsymbol{\psi}$ vanishes on $\partial\Omega$, we may write the term in brackets as

$$\left(D\frac{1-\nu}{2}\operatorname{rot}(\chi\boldsymbol{\phi}_n^B)-\chi p^B_{n-1},\mathbf{curl}\,\boldsymbol{\psi}\right)+D\left(\operatorname{div}(\chi\boldsymbol{\phi}_n^B),\operatorname{div}\boldsymbol{\psi}\right).$$

Using (5.13), (5.12), and Schwarz's inequality, we easily conclude

$$\|p_n^E\|^2_{0/\mathbb{R}}\le C\Lambda^2+C_1\|\boldsymbol{\phi}_n^E\|_1^2. \tag{5.28}$$

The above estimates give us control over $a(\boldsymbol{\phi}_n^E,\boldsymbol{\phi}_n^E)$, $\|p_n^E\|_0/\mathbb{R}$, and $t\|\,\mathbf{curl}\,p_n^E\|_0$. The theorem would follow easily were $\boldsymbol{\psi}\mapsto a(\boldsymbol{\psi},\boldsymbol{\psi})^{1/2}$ equivalent to the $\boldsymbol{H}^1$ norm. But this is not so, since $a(\boldsymbol{\psi},\boldsymbol{\psi})$ vanishes for $\boldsymbol{\psi}$ in the three-dimensional space

$$\boldsymbol{R}:=\{\,(a-by,c+bx)\mid a,b,c\in\mathbb{R}\,\}$$

of plane rigid motions. However, $\boldsymbol{\psi}\mapsto a(\boldsymbol{\psi},\boldsymbol{\psi})^{1/2}+\|\boldsymbol{P}\boldsymbol{\psi}\|_0$ is equivalent to the $\boldsymbol{H}^1$ norm, with $\boldsymbol{P}$ the $\boldsymbol{L}^2$-projection onto $\boldsymbol{R}$. Therefore, we choose q in (5.21) of mean value zero such that $\mathbf{curl}\,q=\boldsymbol{P}\boldsymbol{\phi}_n^E$, which is possible since the functions in $\boldsymbol{R}$ are divergence free. Then

$$\|\boldsymbol{P}\boldsymbol{\phi}_n^E\|_0^2=(\boldsymbol{\phi}_n^E,\mathbf{curl}\,q)=(\boldsymbol{\phi}_n^E+\lambda^{-1}t^2\,\mathbf{curl}\,p_n^E,\mathbf{curl}\,q)-\lambda^{-1}t^2(\mathbf{curl}\,p_n^E,\boldsymbol{P}\boldsymbol{\phi}_n^E).$$

Using (5.21), (5.11), (5.10), and Schwarz's inequality, we conclude

$$\|\boldsymbol{P}\boldsymbol{\phi}_n^E\|_0^2\le C\Lambda^2+C_2t^2\|\,\mathbf{curl}\,p_n^E\|_0^2. \tag{5.29}$$

It is a fairly easy matter to conclude the proof from (5.27)–(5.29). Adding $1/(2C_2)$ times (5.29) to (5.27), we get after simple manipulations

$$\|\phi_n^E\|_1^2 + t^2\|\mathbf{curl}\, p_n^E\|_0^2 \leq C\Lambda^2 + C_3\epsilon(\|\phi_n^E\|_1^2 + \|p_n^E\|_{0/\mathbb{R}}^2 + t^2\|\mathbf{curl}\, p_n^E\|_0^2),$$

for some constant C_3. Then adding $1/(2C_1)$ times (5.28) to this equation and similarly manipulating, we obtain

$$\|\phi_n^E\|_1^2 + \|p_n^E\|_{0/\mathbb{R}}^2 + t^2\|\mathbf{curl}\, p_n^E\|_0^2 \leq C\Lambda^2 + C_4\epsilon(\|\phi_n^E\|_1^2 + \|p_n^E\|_{0/\mathbb{R}}^2 + t^2\|\mathbf{curl}\, p_n^E\|_0^2).$$

Finally, choosing ϵ sufficiently small we obtain the theorem. □

Proof of Theorem 5.2. By standard regularity results for plane elasticity,

$$\|\phi_n^E\|_s \leq C\left(\|\mathbf{div}\, \mathcal{C}\,\mathcal{E}(\phi_n^E)\|_{s-2} + |M_{\boldsymbol{n}}\phi_n^E|_{s-3/2} + |M_{\boldsymbol{s}}\phi_n^E|_{s-3/2} + \|\boldsymbol{P}\phi_n^E\|_0\right).$$

From (5.20),

$$-\mathbf{div}\, \mathcal{C}\,\mathcal{E}(\phi_n^E) = \mathbf{curl}\, p_n^E - \mathbf{curl}\left[D\frac{1-\nu}{2}\operatorname{rot}(\chi\phi_n^B) - \chi p_{n-1}^B\right] + D\,\mathbf{grad}\operatorname{div}(\chi\phi_n^B).$$

Then applying (5.13), (5.12), (5.15), (5.16), (4.1), Theorem 5.1, and the trace theorem, we get

$$\|\phi_n^E\|_s \leq C\left(\|p_n^E\|_{s-1/\mathbb{R}} + t^{n+3/2-s}\|g\|_{n-3/2} + t^{n+1/2}\|g\|_{s+n-5/2}\right).$$

Next, using regularity for the Neumann problem for the Laplacian, we know that

$$\|p_n^E\|_{s/\mathbb{R}} \leq C\left(\|\Delta\, p_n^E\|_{s-2} + \left|\frac{\partial p_n^E}{\partial n}\right|_{s-3/2}\right).$$

By (5.21),

$$\Delta\, p_n^E = \lambda t^{-2}\left\{\operatorname{rot}\phi_n^E + \operatorname{rot}[\chi\phi_n^B + \lambda^{-1}t^2\,\mathbf{curl}(\chi p_{n-1}^B)]\right\}$$

and

$$\frac{\partial p_n^E}{\partial n} =$$

$$-\lambda t^{-2}\left\{\phi_n^E\cdot\boldsymbol{s} + \lambda^{-1}\left(t^{n+1}\frac{\partial p_{n-1}}{\partial n} + t^{n+2}\frac{\partial p_n}{\partial n}\right) + [\chi\phi_n^B + \lambda^{-1}t^2\,\mathbf{curl}(\chi p_{n-1}^B)]\cdot\boldsymbol{s}\right\}.$$

Applying (5.11), (5.10), the trace theorem, and (2.1), we obtain

$$\|p_n^E\|_{s/\mathbb{R}} \leq C(t^{-2}\|\phi_n^E\|_{s-1} + t^{n+1/2-s}\|g\|_{n-3/2} + t^n\|g\|_{s+n-2}).$$

Combining these bounds, we have

$$\|\phi_n^E\|_s + t\|p_n^E\|_{s/\mathbb{R}} \leq C(t^{-1}\|\phi_n^E\|_{s-1} + \|p_n^E\|_{s-1/\mathbb{R}} + t^{n+1/2-s}\|g\|_{n-3/2} + t^n\|g\|_{s+n-2}).$$

For $s = 2$, the theorem follows from this relation and Theorem 5.1, and for $s > 2$, it follows by induction on s. □

Proof of Theorem 5.3. From (5.6) and (3.5), we get

$$\Delta(\omega - \omega_n^I) = \operatorname{div}(\phi - \phi_n^I) = \operatorname{div}\left(\phi_{n+s-1}^E + \chi\phi_{n+s-1}^B + \sum_{j=n+1}^{n+s-1} t^j\phi_j\right).$$

The theorem follows by elliptic regularity for Poisson's problem, Lemma 4.3, (5.12), and Theorems 5.1 and 5.2. □

We conclude this section with the proof of Lemma 5.4.

Proof of Lemma 5.4. From (3.3) and (2.2), we can express ϕ_n^B in terms of the P_i:

$$\begin{aligned}\lambda\phi_n^B &= \sum_{i=1}^{n} t^i \frac{\partial \hat{P}_{i-1}}{\partial \hat{\rho}} \boldsymbol{s} + \sum_{i=2}^{n}\sum_{j=0}^{n-i} t^i (\kappa\hat{\rho}t)^j \frac{\partial \hat{P}_{i-2}}{\partial \theta} \boldsymbol{n} \\ &= \sum_{i=1}^{n} t^i \frac{\partial \hat{P}_{i-1}}{\partial \hat{\rho}} \boldsymbol{s} + \sum_{i=2}^{n} t^i \sigma[1-(\kappa\hat{\rho}t)^{n-i+1}] \frac{\partial \hat{P}_{i-2}}{\partial \theta} \boldsymbol{n}.\end{aligned} \tag{5.30}$$

Applying the identity

$$M_{\boldsymbol{n}}\boldsymbol{\psi} - D \operatorname{div} \boldsymbol{\psi} = -D(1-\nu)\left[\frac{\partial(\hat{\boldsymbol{\psi}}\cdot\boldsymbol{s})}{\partial\theta} + \kappa(\hat{\boldsymbol{\psi}}\cdot\boldsymbol{n})\right] \quad \text{on } \partial\Omega$$

and (3.19), we get that

$$M_{\boldsymbol{n}}\phi_n^B - D \operatorname{div} \phi_n^B = -\lambda^{-1} D(1-\nu) \sum_{i=0}^{n} t^i \left[\frac{\partial^2 \hat{P}_{i-1}}{\partial\theta\partial\hat{\rho}} + \kappa \frac{\partial \hat{P}_{i-2}}{\partial\theta}\right] = -M_{\boldsymbol{n}}\phi_n^I,$$

which proves (5.15).

Applying the identity

$$M_{\boldsymbol{s}}\boldsymbol{\psi} = \frac{D(1-\nu)}{2}\left[-t^{-1}\frac{\partial(\hat{\boldsymbol{\psi}}\cdot\boldsymbol{s})}{\partial\hat{\rho}} + \frac{\partial(\hat{\boldsymbol{\psi}}\cdot\boldsymbol{n})}{\partial\theta} - \kappa(\hat{\boldsymbol{\psi}}\cdot\boldsymbol{s})\right]$$

to (5.30) and using (3.18), we get

$$\begin{aligned}\lambda M_{\boldsymbol{s}}\phi_n^B &= \frac{D(1-\nu)}{2}\sum_{i=0}^{n} t^i \left[-t^{-1}\frac{\partial^2 \hat{P}_{i-1}}{\partial\hat{\rho}^2} + \frac{\partial^2 \hat{P}_{i-2}}{\partial\theta^2} - \kappa\frac{\partial \hat{P}_{i-1}}{\partial\hat{\rho}}\right] \\ &= \frac{D(1-\nu)}{2}\sum_{i=0}^{n} t^i \left(-t^{-1}\frac{\partial^2 \hat{P}_{i-1}}{\partial\hat{\rho}^2} + \frac{\partial^2 \hat{P}_{i}}{\partial\hat{\rho}^2}\right) - \lambda M_{\boldsymbol{s}}\phi_n^I \\ &= \frac{D(1-\nu)}{2} t^n \frac{\partial^2 \hat{P}_n}{\partial\hat{\rho}^2} - \lambda M_{\boldsymbol{s}}\phi_n^I,\end{aligned}$$

which proves (5.16).

Using (5.30) and the expansion

$$t^2 \operatorname{\mathbf{curl}} p_{n-1}^B = -\sum_{i=0}^{n-1} t^{i+2}\left(t^{-1}\frac{\partial \hat{P}_i}{\partial\hat{\rho}}\boldsymbol{s} + \sigma\frac{\partial \hat{P}_i}{\partial\theta}\boldsymbol{n}\right) = -\sum_{i=1}^{n} t^i \frac{\partial \hat{P}_{i-1}}{\partial\hat{\rho}}\boldsymbol{s} - \sum_{i=2}^{n+1} t^i \sigma \frac{\partial \hat{P}_{i-2}}{\partial\theta}\boldsymbol{n},$$

we get

$$\lambda\phi_n^B + t^2 \operatorname{\mathbf{curl}} p_{n-1}^B = -t^{n+1}\sigma\sum_{i=0}^{n-1}(\kappa\hat{\rho})^{n-i-1}\frac{\partial \hat{P}_i}{\partial\theta}\boldsymbol{n}. \tag{5.31}$$

It now follows directly from Theorem 4.7 that

$$\|\lambda\phi_n^B + t^2 \operatorname{\mathbf{curl}} p_{n-1}^B\|_{s,\Omega_0} \le C(t^{n+3/2-s}\|g\|_{n-3/2} + t^{n+3/2}\|g\|_{n+s-3/2}).$$

Finally, using (4.6), we get

$$\begin{aligned}\|\lambda\chi\phi_n^B + t^2\,\mathbf{curl}(\chi p_{n-1}^B)\|_s &\le \|\chi(\lambda\phi_n^B + t^2\,\mathbf{curl}\,p_{n-1}^B)\|_s + t^2\|p_{n-1}^B\cdot\mathbf{curl}\,\chi\|_s\\ &\le C(\|\lambda\phi_n^B + t^2\,\mathbf{curl}\,p_{n-1}^B\|_{s,\Omega_0} + t^2\|p_{n-1}^B\|_{s,\Omega_1})\\ &\le C(t^{n+3/2-s}\|g\|_{n-3/2} + t^{n+3/2}\|g\|_{n+s-3/2}),\end{aligned}$$

which proves (5.11).

Now for any ψ,

$$\operatorname{div}\psi = -\frac{\partial\psi}{\partial\rho}\cdot\boldsymbol{n} + \sigma\frac{\partial\psi}{\partial\theta}\cdot\boldsymbol{s} = \left(-t^{-1}\frac{\partial}{\partial\hat\rho} + \sigma\kappa\right)(\hat\psi\cdot\boldsymbol{n}) + \sigma\frac{\partial(\hat\psi\cdot\boldsymbol{s})}{\partial\theta}.$$

From (5.31), we then have

$$\operatorname{div}\phi_n^B = \left(-t^{-1}\frac{\partial}{\partial\hat\rho} + \sigma\kappa\right)\left[-t^{n+1}\sigma\sum_{i=0}^{n-1}(\kappa\hat\rho)^{n-i-1}\frac{\partial\hat P_i}{\partial\theta}\right].$$

It then follows easily from Theorem 4.7 that

$$\|\operatorname{div}(\chi\phi_n^B)\|_{s,\Omega_0} \le C(t^{n+1/2-s}\|g\|_{n-3/2} + t^{n+1/2}\|g\|_{n+s-3/2}).$$

To complete the proof of (5.12), we use (4.8).

Finally, we give the proof of (5.13) and (5.14). For $i = 0, 1, \dots, n$, we get by simple identities that

$$\operatorname{rot}\boldsymbol{\Phi}_i = \frac{\partial\boldsymbol{\Phi}_i}{\partial\rho}\cdot\boldsymbol{s} + \sigma\frac{\partial\boldsymbol{\Phi}_i}{\partial\theta}\cdot\boldsymbol{n} = \frac{\partial\boldsymbol{\Phi}_i}{\partial\rho}\cdot\boldsymbol{s} + \left\{\sum_{j=0}^{n-i}(\kappa\rho)^j + \sigma(\kappa\rho)^{n-i+1}\right\}\frac{\partial\boldsymbol{\Phi}_i}{\partial\theta}\cdot\boldsymbol{n}.$$

Hence,

$$\begin{aligned}\operatorname{rot}\phi_n^B &= \sum_{i=0}^n t^i\operatorname{rot}\hat{\boldsymbol{\Phi}}_i = \sum_{i=0}^n t^i\left\{t^{-1}\frac{\partial\hat{\boldsymbol{\Phi}}_i}{\partial\hat\rho}\cdot\boldsymbol{s} + \left[\sum_{j=0}^{n-i}(\kappa\rho)^j + \sigma(\kappa\rho)^{n-i+1}\right]\frac{\partial\hat{\boldsymbol{\Phi}}_i}{\partial\theta}\cdot\boldsymbol{n}\right\}\\ &= \lambda^{-1}\sum_{i=0}^{n-1}t^i\frac{\partial^2\hat P_i}{\partial\hat\rho^2} + \sum_{i=0}^n t^i\left[\sum_{j=0}^{n-i}(\kappa\rho)^j + \sigma(\kappa\rho)^{n-i+1}\right]\frac{\partial\hat{\boldsymbol{\Phi}}_i}{\partial\theta}\cdot\boldsymbol{n},\end{aligned}$$

where we used (3.3) and reindexed the first sum in the last step. Turning to the double sum on the left-hand side, we use the identity (2.2) to obtain

$$\begin{aligned}&\sum_{i=0}^n t^i\left[\sum_{j=0}^{n-i}(\kappa\hat\rho t)^j + \sigma(\kappa\hat\rho t)^{n-i+1}\right]\frac{\partial\hat{\boldsymbol{\Phi}}_i}{\partial\theta}\cdot\boldsymbol{n}\\ &\quad= \sum_{i=0}^n\sum_{j=0}^i t^{i-j}(\kappa\hat\rho t)^j\frac{\partial\hat{\boldsymbol{\Phi}}_{i-j}}{\partial\theta}\cdot\boldsymbol{n} + t^{n+1}\sum_{i=0}^n\sigma(\kappa\hat\rho)^{n-i+1}\frac{\partial\hat{\boldsymbol{\Phi}}_i}{\partial\theta}\cdot\boldsymbol{n}\\ &\quad= \sum_{i=0}^n t^i\sum_{j=0}^i(\kappa\hat\rho)^j\frac{\partial\hat{\boldsymbol{\Phi}}_{i-j}}{\partial\theta}\cdot\boldsymbol{n} + t^{n+1}\sum_{i=0}^n\sigma(\kappa\hat\rho)^{n-i+1}\frac{\partial\hat{\boldsymbol{\Phi}}_i}{\partial\theta}\cdot\boldsymbol{n}.\end{aligned}$$

Using (3.3) and (2.2), we further obtain that

$$\begin{aligned}
\sum_{j=0}^{i}(\kappa\hat\rho)^j\frac{\partial\hat{\boldsymbol{\Phi}}_{i-j}}{\partial\theta}\cdot\boldsymbol{n} &= \sum_{j=0}^{i}(\kappa\hat\rho)^j\left[\frac{\partial(\hat{\boldsymbol{\Phi}}_{i-j}\cdot\boldsymbol{n})}{\partial\theta}-\kappa\hat{\boldsymbol{\Phi}}_{i-j}\cdot\boldsymbol{s}\right]\\
&=\lambda^{-1}\sum_{j=0}^{i}(\kappa\hat\rho)^j\left\{\frac{\partial}{\partial\theta}\left[\sum_{l=0}^{i-j}(\kappa\hat\rho)^l\frac{\partial\hat P_{i-j-l-2}}{\partial\theta}\right]-\kappa\frac{\partial\hat P_{i-j-1}}{\partial\hat\rho}\right\}\\
&=\lambda^{-1}\sum_{j=0}^{i}\sum_{l=0}^{j}(\kappa\hat\rho)^{j-l}\frac{\partial}{\partial\theta}\left[(\kappa\hat\rho)^l\frac{\partial\hat P_{i-j-2}}{\partial\theta}\right]-\lambda^{-1}\sum_{j=0}^{i}\kappa(\kappa\hat\rho)^j\frac{\partial\hat P_{i-j-1}}{\partial\hat\rho}\\
&=\lambda^{-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\left[(\kappa\hat\rho)^j\frac{\partial^2\hat P_{i-j-2}}{\partial\theta^2}+l\kappa^{j-1}\kappa'\hat\rho^j\frac{\partial\hat P_{i-j-2}}{\partial\theta}\right]-\lambda^{-1}\sum_{j=0}^{i}\kappa(\kappa\hat\rho)^j\frac{\partial\hat P_{i-j-1}}{\partial\hat\rho}\\
&=\lambda^{-1}\sum_{j=0}^{i}\left[(j+1)(\kappa\hat\rho)^j\frac{\partial^2\hat P_{i-j-2}}{\partial\theta^2}+\frac{j(j+1)}{2}\kappa^{j-1}\kappa'\hat\rho^j\frac{\partial\hat P_{i-j-2}}{\partial\theta}-\kappa(\kappa\hat\rho)^j\frac{\partial\hat P_{i-j-1}}{\partial\hat\rho}\right]\\
&=\lambda^{-1}\sum_{j=0}^{i}\hat\rho^j\left[a_2^j\frac{\partial^2\hat P_{i-j-2}}{\partial\theta^2}+a_3^j\frac{\partial\hat P_{i-j-2}}{\partial\theta}+a_1^j\frac{\partial\hat P_{i-j-1}}{\partial\hat\rho}\right]\\
&=-\lambda^{-1}\frac{\partial^2\hat P_i}{\partial\hat\rho^2}+\frac{2}{D(1-\nu)}\hat P_i,
\end{aligned}$$

where the a_i^j are defined in (3.10) and we used (3.11) in the last step. Collecting these results, we have

$$\begin{aligned}
&\operatorname{rot}\boldsymbol{\phi}_n^B\\
&=\lambda^{-1}\sum_{i=0}^{n-1}t^i\frac{\partial^2\hat P_i}{\partial\hat\rho^2}-\sum_{i=0}^{n}t^i\left[\lambda^{-1}\frac{\partial^2\hat P_i}{\partial\hat\rho^2}-\frac{2}{D(1-\nu)}\hat P_i\right]+t^{n+1}\sum_{i=0}^{n}\sigma(\kappa\hat\rho)^{n-i+1}\frac{\partial\hat{\boldsymbol{\Phi}}_i}{\partial\theta}\cdot\boldsymbol{n}\\
&=-\lambda^{-1}t^n\frac{\partial^2\hat P_n}{\partial\hat\rho^2}+\frac{2}{D(1-\nu)}p_n^B+t^{n+1}\sum_{i=0}^{n}\sigma(\kappa\hat\rho)^{n-i+1}\frac{\partial\hat{\boldsymbol{\Phi}}_i}{\partial\theta}\cdot\boldsymbol{n},
\end{aligned}$$

and so

$$D\frac{1-\nu}{2}\operatorname{rot}\boldsymbol{\phi}_n^B-p_{n-1}^B=t^n\left[-D\frac{1-\nu}{2}\lambda^{-1}\frac{\partial^2\hat P_n}{\partial\hat\rho^2}+\hat P_n\right]+t^{n+1}\sum_{i=0}^{n}\sigma(\kappa\hat\rho)^{n-i+1}\frac{\partial\hat{\boldsymbol{\Phi}}_i}{\partial\theta}.$$

Equation (5.14) follows directly. Using Theorem 4.7 and (4.4), we then obtain

$$\|D\frac{1-\nu}{2}\operatorname{rot}\boldsymbol{\phi}_n^B-p_{n-1}^B\|_{s,\Omega_0}\le C(t^{n+1/2-s}\|g\|_{n-3/2}+t^{n+1/2}\|g\|_{s+n-3/2}).$$

Finally, using (4.8), we obtain

$$\begin{aligned}
&\|D\frac{1-\nu}{2}\operatorname{rot}(\chi\boldsymbol{\phi}_n^B)-\chi p_{n-1}^B\|_s\\
&\qquad\le\|\chi\left(D\frac{1-\nu}{2}\operatorname{rot}\boldsymbol{\phi}_n^B-p_{n-1}^B\right)\|_s+\|D\frac{1-\nu}{2}\boldsymbol{\phi}_n^B\cdot\mathbf{curl}\,\chi\|_s\\
&\qquad\le C(\|D\frac{1-\nu}{2}\operatorname{rot}\boldsymbol{\phi}_n^B-p_{n-1}^B\|_{s,\Omega_0}+\|\boldsymbol{\phi}_n^B\|_{s,\Omega_1})\\
&\qquad\le C(t^{n+1/2-s}\|g\|_{n-3/2}+t^{n+1/2}\|g\|_{s+n-3/2}).
\end{aligned}$$

This completes the verification of (5.13). $\square$

6. Other boundary conditions. In this section, we discuss the modifications to the foregoing analysis necessary to handle the remaining four other boundary conditions discussed in the introduction: the hard clamped plate, the soft clamped plate, the hard simply supported plate, and the free plate. We shall see that Theorems 5.1–5.3 remain true as stated in all cases.

For the *hard clamped* and *hard simply supported* plates, these were proved in [1]. (The method of proof was somewhat different and required slightly more regularity to obtain the estimates for ω. However, the present method of proof can easily be adapted to correct this.) Since $\phi_1 = 0$ for these boundary conditions, it follows from (5.9) that $p_1 = 0$ as well. Exploiting this, one may slightly improve the regularity requirements for the estimates of $\boldsymbol{\phi}_1^E$ and p_1^E. See [1] for the precise result.

The analysis for the soft clamped plate is very close to that presented here. The space $\boldsymbol{H}^1$ in which $\boldsymbol{\phi}$ is sought is replaced by the subspace of $\boldsymbol{H}^1$ consisting of functions whose normal component vanishes on the boundary. Because of this, a few terms which we estimated in §5 are zero, so the analysis is slightly simpler. A more essential difference between the soft clamped and soft simply supported plates is that the boundary layer for the former is much weaker. In fact, the boundary layer for the soft clamped plate is weaker than for any of the other four boundary conditions we consider. Specifically, as shown in §3, the boundary-layer expansion functions P_0 and P_1 and consequently $\boldsymbol{\Phi}_0$, $\boldsymbol{\Phi}_1$, and $\boldsymbol{\Phi}_2$ all vanish. Moreover, the interior expansion functions ω_i, ϕ_i, and p_i, $i = 1$ and 3, vanish as well. Consequently, $\boldsymbol{\phi}_0^E = \boldsymbol{\phi}_1^E = \boldsymbol{\phi}_2^E + t^2\phi_2$ and $p_0^E = p_1^E = p_2^E + t^2 p_2$. Thus, for example, we see from Theorem 5.1 that $\phi - \phi_0$ is $O(t^2)$ in H^1 and $p - p_0$ is $O(t^2)$ in L^2. (These quantities are only order $O(t^{1/2})$ for the soft simply supported plate and the free plate and $O(t^{3/2})$ for the hard clamped and hard simply supported plates.)

It remains to consider the case of the free plate. First we summarize some basic existence results for the biharmonic and Reissner–Mindlin plate models with traction boundary conditions. Given functions $g \in L^2(\Omega)$, $f, h \in L^2(\partial\Omega)$, the variational problem to find $\omega \in H^2(\Omega)$ satisfying

$$(\mathcal{C}\,\mathcal{E}(\mathbf{grad}\,\omega), \mathcal{E}(\mathbf{grad}\,\mu)) = (g, \mu) - \langle f, \mu\rangle + \left\langle h, \frac{\partial \mu}{\partial n}\right\rangle \quad \text{for all } \mu \in H^2(\Omega)$$

has a solution if and only if the given data is compatible in the sense that

$$(g, \mu) - \langle f, \mu\rangle + \left\langle h, \frac{\partial \mu}{\partial n}\right\rangle = 0 \quad \text{for all } \mu \in \mathbb{L},$$

where $\mathbb{L}$ denotes the three-dimensional space of linear polynomial functions on Ω. In this case, the solution is determined up to the addition of an arbitrary element of $\mathbb{L}$. Performing integration by parts, one obtains the identity

$$(\mathcal{C}\,\mathcal{E}(\mathbf{grad}\,\omega), \mathcal{E}(\mathbf{grad}\,\mu)) = (D\,\Delta^2\,\omega, \mu) - \langle B_3\omega, \mu\rangle + \left\langle B_2\omega, \frac{\partial \mu}{\partial n}\right\rangle, \quad \omega, \mu \in H^2,$$

where

$$B_2\omega := M_{\boldsymbol{n}}\,\mathbf{grad}\,\omega, \qquad B_3\omega := \frac{\partial}{\partial s} M_{\boldsymbol{s}}\,\mathbf{grad}\,\omega + [\mathbf{div}\,\mathcal{C}\,\mathcal{E}(\mathbf{grad}\,\omega)] \cdot \boldsymbol{n}.$$

From this we deduce the boundary value problem corresponding to the weak formulation just discussed:

$$D\,\Delta^2\,\omega = g \quad \text{in } \Omega, \qquad B_2\omega = h, \quad B_3\omega = f \quad \text{on } \partial\Omega.$$

Note that the traction-free biharmonic plate problem, i.e., the case when $f = h = 0$, has a solution if and only if the load function g is orthogonal to $\mathbb{L}$.

Analogously, the Reissner–Mindlin boundary value problem for a traction-free plate, given by equations (1.6) and (1.7) and the boundary conditions (1.5), has a solution if and only if the load g is compatible with the traction-free conditions, i.e., it is L^2-orthogonal to $\mathbb{L}$. The solution pair $(\omega, \boldsymbol{\phi})$ is then determined up to the addition of a pair in

$$\mathbb{L}_\nabla := \{\, (l, \mathbf{grad}\, l) \,|\, l \in \mathbb{L} \,\}\,.$$

We henceforth assume that g is compatible. We now proceed to the construction of the expansion functions ω_i, $\boldsymbol{\phi}_i$, P_i, and p_i in the case of the free plate. The boundary conditions we use are (3.18), (3.19), and, from the last equality in (1.5),

$$(\boldsymbol{\phi}_i - \mathbf{grad}\,\omega_i)\cdot\boldsymbol{n} = -\lambda^{-1}\frac{\partial \hat{P}_{i-2}}{\partial\theta} \tag{6.1}$$

or, in view of (3.4),

$$\mathbf{div}\, C\,\mathcal{E}(\boldsymbol{\phi}_i)\cdot\boldsymbol{n} = -\frac{\partial \hat{P}_i}{\partial\theta}. \tag{6.2}$$

Now, from (3.18) and (6.2), we have

$$\begin{aligned}\frac{\partial}{\partial s}M_{\boldsymbol{s}}\boldsymbol{\phi}_i + \mathbf{div}\, C\,\mathcal{E}(\boldsymbol{\phi}_i)\cdot\boldsymbol{n} &= \lambda^{-1}D\frac{1-\nu}{2}\frac{\partial}{\partial\theta}\left(\frac{\partial^2\hat{P}_i}{\partial\hat{\rho}^2} + \kappa\frac{\partial\hat{P}_{i-1}}{\partial\hat{\rho}} - \frac{\partial^2\hat{P}_{i-2}}{\partial\theta^2}\right) - \frac{\partial\hat{P}_i}{\partial\theta}\\ &= \lambda^{-1}D(1-\nu)\frac{\partial}{\partial\theta}\left(\kappa\frac{\partial\hat{P}_{i-1}}{\partial\hat{\rho}} - \frac{\partial^2\hat{P}_{i-2}}{\partial\theta^2}\right),\end{aligned}$$

where we used (3.11) with $\hat{\rho} = 0$ in the last step. Using (3.7), we convert this to a boundary condition on ω_i:

(6.3)

$$B_3\omega_i = -B_3(\lambda^{-1}D\,\Delta\,\omega_{i-2} + \delta_{i4}\lambda^{-2}Dg) + \lambda^{-1}D(1-\nu)\frac{\partial}{\partial\theta}\left(\kappa\frac{\partial\hat{P}_{i-1}}{\partial\hat{\rho}} - \frac{\partial^2\hat{P}_{i-2}}{\partial\theta^2}\right).$$

The construction of expansion functions satisfying (3.6), (3.7), (3.11), (5.9), (3.18), (3.19), (6.1), and (3.12) proceeds as follows. First we define $\omega_i \in H^2$ from the biharmonic equation (3.6) together with the boundary conditions (3.20), which we may write as

$$B_2\omega_i = -B_2(\lambda^{-1}D\,\Delta\,\omega_{i-2} + \delta_{i4}\lambda^{-2}Dg) + \lambda^{-1}D(1-\nu)\left(\kappa\frac{\partial\hat{P}_{i-2}}{\partial\theta} + \frac{\partial^2\hat{P}_{i-1}}{\partial\theta\partial\hat{\rho}}\right), \tag{6.4}$$

and (6.3). Note that for $i = 0$ this is simply the biharmonic problem for a traction-free plate with load g, so ω_0 is determined up to addition of a linear function. As we shall show shortly, this problem always admits a solution, so that once P_j is known for $j < i$, ω_i is determined up to addition of a linear function. Then $\boldsymbol{\phi}_i$ is given by (3.7) and (3.8) as before, so the pair $(\omega_i, \boldsymbol{\phi}_i)$ is determined up to addition of an element of $\mathbb{L}_\nabla$. Note that $M_{\boldsymbol{s}}\boldsymbol{\phi}_i$ is determined completely, and so we can uniquely determine P_i by the differential equation (3.11), the boundary condition (3.18), and the decay condition (3.12). Thus we compute, in order, ω_0, $\boldsymbol{\phi}_0$, P_0, ω_1, $\boldsymbol{\phi}_1$, P_1, ..., always with $(\omega_i, \boldsymbol{\phi}_i)$ determined up to addition of an element of $\mathbb{L}_\nabla$, and P_i determined completely.

To see that the biharmonic problems for the ω_i admit solutions, we must show that

$$(6.5) \qquad (\delta_{i0}g - \delta_{i2}\lambda^{-1}D\,\Delta\, g, \mu) - \langle f, \mu\rangle + \left\langle h, \frac{\partial \mu}{\partial n}\right\rangle = 0 \quad \text{for all } \mu \in \mathbb{L},$$

when f is given by the right-hand side of (6.3) and h by the right-hand side of (6.4). Setting $u = \lambda^{-1}D\,\Delta\,\omega_{i-2} + \delta_{i4}\lambda^{-2}Dg$ and using the biharmonic equation satisfied by ω_{i-2} (which we can assume by induction), we get $D\,\Delta^2\, u = \delta_{i2}\lambda^{-1}Dg$. Hence if $\mu \in \mathbb{L}$,

$$(\delta_{i0}g - \delta_{i2}\lambda^{-1}Dg, \mu) = -\langle B_3 u, \mu\rangle + \left\langle B_2 u, \frac{\partial \mu}{\partial n}\right\rangle.$$

Thus, to complete the verification of (6.5), it suffices to show

$$\left\langle \frac{\partial}{\partial \theta}\left(\kappa \frac{\partial \hat{P}_{i-1}}{\partial \hat{\rho}}\right), \mu \right\rangle = \left\langle \frac{\partial^2 \hat{P}_{i-1}}{\partial \theta \partial \hat{\rho}}, \frac{\partial \mu}{\partial n}\right\rangle$$

and

$$\left\langle \frac{\partial \hat{P}_{i-2}}{\partial \theta^3}, \mu \right\rangle = -\left\langle \kappa \frac{\partial \hat{P}_{i-2}}{\partial \theta}, \frac{\partial \mu}{\partial n}\right\rangle,$$

for all $\mu \in \mathbb{L}$. These may be verified with elementary calculus, independent of the particular functions P_{i-1} and P_{i-2}.

We now define functions p_i and r, as was done in the beginning of §5 for the soft simply supported plate. From (3.7), (3.8), and (3.6), we see that $\operatorname{div} \mathbf{div}\, \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_i) = \delta_{i0}g$. Hence, defining $r \in H^1/\mathbb{R}$ by

$$-\Delta\, r = g \quad \text{in } \Omega, \quad \frac{\partial r}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

we see that $\mathbf{div}\, \mathcal{C}\,\mathcal{E}(\boldsymbol{\phi}_i) + \delta_{i0}\,\mathbf{grad}\, r$ is divergence free. Hence we may again define a function $p_i \in H^1$, unique modulo $\mathbb{R}$, by (5.9). Now from (5.9) and (6.2), we see that $\partial(p_i + P_i)/\partial s = 0$. Therefore, we may normalize p_i so that

$$(6.6) \qquad p_i + P_i = 0 \quad \text{on } \partial\Omega.$$

This completes the construction of the expansion functions.

In §§4 and 5, we presented the analysis of the asymptotic expansions in such a way that they adapt with a minimum of effort to the case of the free plate. Due to the different boundary conditions, we need to use different negatively indexed Sobolev norms. Instead of the definition given in §2, we define $\|\cdot\|_s$ to be the norm in the dual space H^s. With this understanding, all of the results of §4 hold with essentially the same proofs. Of course, in the proof of Theorem 4.3, we use the traction problem for the biharmonic rather than the simply supported plate problem.

Turning to the error analysis in §5, we again use the Helmholtz decomposition as in (5.1), except that now $r \in H^1/\mathbb{R}$ and $p \in \mathring{H}^1$. We then recover the differential equations (5.2)–(5.4), and (5.6), now with the boundary conditions

$$\frac{\partial r}{\partial n} = M_{\boldsymbol{n}}\boldsymbol{\phi} = M_{\boldsymbol{s}}\boldsymbol{\phi} = p = \frac{\partial \omega}{\partial n} - \boldsymbol{\phi}\cdot\boldsymbol{n} = 0.$$

These determine $(r, \boldsymbol{\phi}, p, \omega)$ up to an additive constant in r and addition of an element of $\mathbb{L}_\nabla$ to $(\omega, \boldsymbol{\phi})$.

The norms on the left-hand sides of the estimates in Theorems 5.1–5.3 need to be modified in the obvious ways because of the indeterminancy in (ϕ, ω) and the determinancy of p. That is, the norms of ϕ_n^E are in the Sobolev spaces modulo $\mathbb{R}$, those on ω_n^E in the Sobolev spaces modulo $\mathbb{L}$, and those on p_n^E in the full Sobolev spaces. The proofs of these theorems carry over easily. In particular, Lemma 5.4 holds without change.

The main part of the proof of Theorem 5.1 involved the choice of test functions $\psi = \phi_n^E$ in (5.20) and $q = p_n^E - t^n \chi P_n$ in (5.21). Notice that this choice of q vanishes on the boundary because of (6.6) and so is an allowable test function. This part of the proof carries over to the free case without problem.

Two more choices of test functions complete the proof of the theorem. For the second one, we take $\psi \in \boldsymbol{H}^1$ with $\operatorname{rot} \psi = p_n^E$, which allows us to get control over the full L^2 norm of p_n^E. Finally, to control the infinitesimal rotation in ϕ_n^E, we choose a test function $q \in \mathring{H}^1$ in (5.21) with nonvanishing integral and use the fact that

$$\psi \to a(\psi, \psi)^{1/2} + \left| \int_\Omega q \operatorname{rot} \psi \right|$$

defines a norm equivalent to the usual norm in $\boldsymbol{H}^1/\mathbb{R}^2$.

The proof of Theorem 5.2 adapts easily. Naturally, we use a Dirichlet rather than a Neumann problem to obtain bounds on p_n^E, using that fact that $p_n^E = t^n P_n$ on $\partial\Omega$. Analogously, to prove Theorem 5.3, we use a Neumann problem for ω_n^E and the fact that

$$\frac{\partial \omega_n^E}{\partial n} = \phi_n^E \cdot \boldsymbol{n} = \phi_{n+s-1}^E \cdot \boldsymbol{n} + \sum_{n+1}^{n+s-1} t^j (\phi_j + \boldsymbol{\Phi}_j) \cdot \boldsymbol{n}.$$

Appendix. In this appendix, we collect some elementary formulas for the convenience of the reader.

It follows immediately from the definitions of rot and **curl** that

$$\operatorname{rot} \mathbf{curl}\, q = -\Delta q, \qquad \mathbf{curl}\, q \cdot \boldsymbol{n} = -\frac{\partial q}{\partial s}, \qquad \mathbf{curl}\, q \cdot \boldsymbol{s} = \frac{\partial q}{\partial n},$$

and

$$(\operatorname{rot} \psi, q) = (\psi, \mathbf{curl}\, q) - \langle \psi \cdot \boldsymbol{s}, q \rangle.$$

Simple computations show that

$$\begin{aligned} \mathbf{div}\, C\,\mathcal{E}(\mathbf{grad}\, v) &= D\, \mathbf{grad}\, \Delta v, & \mathbf{div}\, C\,\mathcal{E}(\mathbf{curl}\, p) &= D \frac{1-\nu}{2} \mathbf{curl}\, \Delta p, \\ \operatorname{div} \mathbf{div}\, C\,\mathcal{E}(\phi) &= D\, \Delta \operatorname{div} \phi, & \Delta \phi &= \mathbf{grad} \operatorname{div} \phi - \mathbf{curl} \operatorname{rot} \phi, \end{aligned}$$

and on $\partial\Omega$,

$$M_{\boldsymbol{n}}\boldsymbol{\phi} := \boldsymbol{n}\cdot C\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n} = D\left(\frac{\partial\boldsymbol{\phi}}{\partial n}\cdot\boldsymbol{n} + \nu\frac{\partial\boldsymbol{\phi}}{\partial s}\cdot\boldsymbol{s}\right),$$

$$M_{\boldsymbol{s}}\boldsymbol{\phi} := \boldsymbol{s}\cdot C\,\mathcal{E}(\boldsymbol{\phi})\boldsymbol{n} = \frac{D(1-\nu)}{2}\left(\frac{\partial\boldsymbol{\phi}}{\partial n}\cdot\boldsymbol{s} + \frac{\partial\boldsymbol{\phi}}{\partial s}\cdot\boldsymbol{n}\right),$$

$$M_{\boldsymbol{n}}(\mathbf{grad}\,v) = D\left[(1-\nu)\frac{\partial^2 v}{\partial n^2} + \nu\,\Delta\,v\right],$$

$$M_{\boldsymbol{s}}(\mathbf{grad}\,v) = D(1-\nu)\left[\frac{\partial^2 v}{\partial s\partial n} - \kappa\frac{\partial v}{\partial s}\right],$$

$$M_{\boldsymbol{n}}(\mathbf{curl}\,p) = D(1-\nu)\left[-\frac{\partial^2 p}{\partial s\partial n} + \kappa\frac{\partial p}{\partial s}\right],$$

$$M_{\boldsymbol{s}}(\mathbf{curl}\,p) = D\frac{(1-\nu)}{2}\left[\frac{\partial^2 p}{\partial n^2} - \frac{\partial^2 p}{\partial s^2} - \kappa\frac{\partial p}{\partial n}\right],$$

and

$$\frac{\partial\boldsymbol{n}}{\partial s} = \kappa\boldsymbol{s}, \qquad \frac{\partial\boldsymbol{s}}{\partial s} = -\kappa\boldsymbol{n},$$

$$\boldsymbol{n}\cdot\mathcal{H}(v)\boldsymbol{n} = \frac{\partial^2 v}{\partial n^2}, \qquad \boldsymbol{s}\cdot\mathcal{H}(v)\boldsymbol{n} = \frac{\partial^2 v}{\partial s\partial n} - \kappa\frac{\partial v}{\partial s}, \qquad \boldsymbol{s}\cdot\mathcal{H}(v)\boldsymbol{s} = \frac{\partial^2 v}{\partial s^2} + \kappa\frac{\partial v}{\partial n},$$

where $\mathcal{H}(v)$ denotes the Hessian matrix of second partial derivatives of v.

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