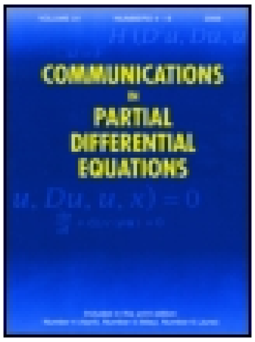

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ASYMPTOTIC BEHAVIOR OF A NONLINEAR FOURTH ORDER EIGENVALUE PROBLEM

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1 Introduction

Recently, equations involving biharmonic operator Δ^2 in R^4 have received much attention. A particular feature of biharmonic operator in R^4 is that it is conformally invariant. More precisely, let

$$P\varphi = \Delta^2\varphi + \delta(\frac{2}{3}RI - 2Ric)d\varphi$$

where δ denotes the divergence, d the differential and Ric the Ricci curvature of the metric g. Under the conformal change $g_w = e^{2w}g$, P undergoes the transformation $P_w = e^{-4w}P$. See [3] for more details.

Our purpose in this paper is to study the asymptotic behavior of the following nonlinear eigenvalue problem

$$\begin{cases} \Delta^2 u = \lambda f(u) \text{ in } \Omega \subset R^4, \\ \Delta u = 0, \text{ on } \partial\Omega, \ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in R^4, $f(u)$ is an nonnegative smooth function with exponentially dominant nonlinearity and $\lambda > 0$ is small. When

$f(u) = e^u$, equation (1.1) arises in the study of the extremal metrics of zeta function determinants on 4-manifolds (see [3]). From [3], it becomes very important to study *a priori* estimates of solutions of (1.1). Two extreme cases are $\lambda \to 0$ and $\lambda \to \infty$. It is easy to see that when λ is large, there is no solution to problem (1.1) (multiplying equation (1.1) by the first eigenfunction of $-\Delta$ and using the fact that $e^u > u$). We shall study the case when $\lambda \to 0$.

Another interesting point is that equation (1.1) can be regarded as a natural generalization of the 2-dimensional Liouville's equation

$$\begin{cases} \Delta u + \lambda e^u = 0 \text{ in } \Omega \subset R^2, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

to higher dimensions ($n = 4$). Equation (1.2) has been studied by many authors. Noteably, [2], [6], [8], [14] and the references therein. In particular in [8], Nagasaki and Suzuki studied the asymptotic bahavior of (1.2) as $\lambda \to 0$, by using the complex structure in R^2 and some elliptic estimates.

Since there is no complex structure in R^4, some new methods are needed. First, we generalize an inequality of [1] in R^2 to higher dimension by using an idea of [9]. Then we use the Pohazaev identity for the equation (1.1) to count the number of blow up points and to locate their locations. We remark that our techniques in this paper can be used to obtain a new proof of results in [8] without using analytic theory.

To introduce our main result, let $f(u)$ be a smooth function such that $f \geq 0$, $f'(u) \geq 0$ for $u \geq 0$ and $\lim_{u \to +\infty} f(u)e^{-u} = \lim_{u \to +\infty} F(u)e^{-u} = 1$ where $F(u) = \int_0^u f(s)ds$. Let u_λ be a solution of (1.1) for some $\lambda > 0$ and set

$$\Sigma_\lambda = \lambda \int_\Omega f(u_\lambda)dx \tag{1.3}$$

Then we have the following

Theorem 1.1 *Assume that Ω is a bounded convex domain. Then as $\lambda \to 0$, for any family of solutions $\{u_\lambda\}$ of (1.1), there are only three possibilities.*

Case (i), the $\{\Sigma_\lambda\}$ accumulate to 0. Then $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$.

Case (ii), the $\{\Sigma_\lambda\}$ accumulate to $2\alpha_4 m$ for some positive integer m where $\alpha_4 = 16\sigma_4$ and σ_4 is the area of the unit sphere in R^4. In this case, u_λ has

$m-$point blow up, i.e., there exists a set $S = \{x_1, ..., x_m\} \subset \Omega$ of $m-$points such that $\{u_\lambda\}$ have a limit for $x \in \overline{\Omega} \backslash S$ while $u_\lambda|_S \to \infty$.

Case (iii), the $\{\Sigma_\lambda\}$ accumulate to $+\infty$. In this case, u_λ blows up entirely, i.e. $u_\lambda(x) \to +\infty$ for all $x \in \Omega$.

Theorem 1.2 *In the second case, the limit function $u_0(x) = \lim_{\lambda \to 0} u_\lambda(x)$ has the form*

$$u_0(x) = 2\alpha_4 \sum_{i=1}^{m} G_4(x, x_j) \tag{1.4}$$

where $G_4(x, y)$ denotes the Green's function of Δ^2 under the Dirichlet condition, that is

$$\Delta^2 G_4(x, y) = \delta(x - y), G_4|_{\partial\Omega} = 0, \Delta G_4|_{\partial\Omega} = 0.$$

Furthermore, blow up points $x_j \in \Omega$ $(1 \le j \le m)$ satisfy the following relation

$$\nabla_{x_j} K_4(x_j, x_j) + \sum_{l \neq j} \nabla_{x_j} G_4(x_j, x_l) = 0 \ (1 \le j \le m) \tag{1.5}$$

where $K_4(x, y) = G_4(x, y) + \frac{\log|x-y|}{4\sigma_4}$.

We remark that the condition that Ω is convex is only needed to show that near the boundary u_λ is bounded if $u_\lambda \in L^1_{loc}(\Omega)$. Thus our techniques can be carried out to equation (1.2) and give a new proof of results in [8].

Our paper is organized as follows. In Section 2, we derive some crucial inequalities and reduce our problem to a single case. Then we prove finite blow up in Section 3. In Section 4, we prove Theorem 1.2.

Throughout this paper, the constant C will denote various constants which is independent of λ. $B = O(A)$ means $|B| \le CA$.

2 Reduction

In this section, we reduce our problem to finite blow up problem. To this end, we first have the following boundary estimates.

Lemma 2.1 *There exists constants $\alpha > 0$ and $t_0 > 0$ such that $\xi \in R^4, |\xi| = 1, \xi \cdot n(x) \ge \alpha$ at $x \in \partial\Omega$ implies*

$$\frac{d}{dt}u(x+t\xi) < 0(\ -t_0 < t < 0)$$

for any positive solution u of (1.1).

Proof: This follows from [11], where a generalization of the *Moving Plane Method* of [4] to elliptic systems is proved.

We next state a boundary estimate lemma. The proof is standard. One combines Lemma 2.1 and elliptic regularity theory of [5].

Lemma 2.2 *Let u be a solution of equation (1.1). Suppose that there exists $C > 0$ such that*

$$\int_\Omega u\varphi_1 < C \tag{2.1}$$

where φ_1 is the positive first eigenfunction of $-\Delta$ in Ω. Then there exist a neighborhood ω of $\partial\Omega$ and a constant C depending on the geometry of Ω only such that

$$||D^\alpha u||_{L^\infty(\omega)} \leq C$$

for all $|\alpha| \leq 4$.

Next we extend the L^1 estimate of H. Brezis and F. Merle [1] to biharmonic operators. We use the level set argument as in [9].

Lemma 2.3 *Let u be a C^4 solution of*

$$\begin{cases} \Delta^2 u = f(x) \ in \ \Omega \\ \Delta u|_{\partial\Omega} = u|_{\partial\Omega} = 0 \end{cases}$$

where $f \in L^1(\Omega)$, $f \geq 0$ and $\Omega \subset R^4$. Then for every $\delta \in (0, 16\sigma_4) = (0, \alpha_4)$ we have

$$\int_\Omega \exp[\frac{(\alpha_4 - \delta)|u(x)|}{||f||_{L^1}}]dx \leq \frac{\sigma_4\alpha_4}{\delta}|\Omega|$$

where $|\Omega|$ denotes the volume of Ω.

Proof: We prove this by the symmetrization method. Consider the symmetrized problem

$$\begin{cases} \Delta^2 U = F(x) \ \text{ in } \ \Omega^* \\ \Delta U|_{\partial\Omega^*} = U|_{\partial\Omega^*} = 0 \end{cases}$$

where Ω^* is the ball centered at origin with the same volume as Ω and F

is the symmetric decreasing rearrangement of f. We refer to G. Talenti [12] and [13] for properties of the rearrangement. According to [13], we have

$$u^* \leq U$$

where u^* is the symmetric decreasing rearrangement of u. Let $\Omega^* = B_R(0)$. U clearly satisfies the following O.D.E.

$$\begin{cases} -U'' - \frac{3}{r}U' = V \\ -V'' - \frac{3}{r}V = F(r) \\ U'(0) = 0,\ U(R) = 0, V'(0) = 0, V(R) = 0 \end{cases}$$

Therefore it is easy to see that

$$-U'(r) \leq \frac{1}{4\sigma_4}\frac{1}{r}\|F\|_{L^1(\Omega^*)}.$$

Hence

$$|U(r)| \leq \frac{1}{4\sigma_4}\|F\|_{L^1(\Omega^*)}\log\frac{R}{r};$$

$$\int_{\Omega^*}\exp[(4-\epsilon)4\sigma_4\frac{U}{\|F\|_{L^1}}] \leq \int_{B(R)}\exp\log(\frac{R}{r})^{4-\epsilon}dr$$

$$= \sigma_4\int_0^R(\frac{R}{r})^{4-\epsilon}r^3dr = \epsilon^{-1}\sigma_4R^4.$$

Letting $\epsilon 4\sigma_4 = \delta$, we have

$$\int_{\Omega^*}\exp[(\alpha_4-\delta)\frac{U(r)}{\|F\|_{L^1}}] \leq \frac{\sigma_4\alpha_4|\Omega|}{\delta}.$$

According to the properties of the symmetric decreasing function,

$$\|F\|_{L^1(\Omega^*)} = \|f\|_{L^1(\Omega)},$$

$$\int_\Omega \exp[(\alpha_4-\delta)\frac{u(x)}{\|f\|_{L^1(\Omega)}}]dx = \int_{\Omega^*}\exp[(\alpha_4-\delta)\frac{u^*(x)}{\|f\|_{L^1}}]$$

$$\leq \int_{\Omega^*}\exp[(\alpha_4-\delta)\frac{U(r)}{\|F\|_{L^1}}] \leq \frac{\sigma_4\alpha_4|\Omega|}{\delta}$$

□

An interesting consequence is

Corollary 2.4 *Let u_n be a sequence of C^4 solutions of*

$$\begin{cases} \Delta^2 u_n = V_n e^{u_n} \ in\ \Omega \subset \ R^4 \\ \Delta u_n|_{\partial\Omega} = u_n|_{\partial\Omega} = 0 \end{cases}$$

such that

$$\|V_n\|_{L^q} \leq C$$

$$\int_\Omega |V_n| e^{u_n} \leq \epsilon_0 < \frac{\alpha_4}{q'}$$

for some $1 < q < \infty$ *and* $q' = \frac{q}{q-1}$. *Then* $\{u_n\}$ *is uniformly bounded in* $L^\infty(\Omega)$.

Proof: Fix $\delta > 0$ so that $\alpha_4 - \delta > \epsilon_0(q' + \delta)$. By Lemma 2.3 we have

$$\int_\Omega \exp[(q' + \delta)|u_n|] \leq C$$

for some C independent of n. Therefore e^{u_n} is bounded in $L^{q'+\delta}(\Omega)$; hence $V_n e^{u_n}$ is bounded in $L^{1+\epsilon_0}$. Then the standard elliptic theory implies that u_n is bounded in $L^\infty(\Omega)$.

□

We now state a Pohozaev identity for equation (1.1). For the proof, see [7].

Lemma 2.5 *Let* u *be a solution of* $\Delta^2 u = f(u)$ *in* Ω. *Then we have for any* $y \in R^4$,

$$\begin{aligned} & 4\int_\Omega F(u) \\ = & \int_{\partial\Omega} \langle x + y, \nu \rangle F(u) + \frac{1}{2}\int_{\partial\Omega} v^2 \langle x + y, \nu \rangle ds + 2\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \\ & + \int_{\partial\Omega} \{ \frac{\partial v}{\partial \nu} \langle x + y, Du \rangle + \frac{\partial u}{\partial \nu} \langle x + y, Dv \rangle - \langle Dv, Du \rangle \langle x, \nu \rangle \} \end{aligned} \quad (2.2)$$

In particular, we have

$$\int_{\partial\Omega} \nu F(u) + \frac{1}{2}\int_{\partial\Omega} v^2 \nu ds + \int_{\partial\Omega} \{ \frac{\partial v}{\partial \nu} Du + \frac{\partial u}{\partial \nu} Dv - \langle Dv, Du \rangle \nu \} = 0 \quad (2.3)$$

where $F(u) = \int_0^u f(s)ds$, $-\Delta u = v$ *and* $\nu(x)$ *is the normal outward derivative of* x *on* $\partial\Omega$.

We now begin to analyze problem (1.1).

Lemma 2.6 *If* $\Sigma_\lambda \to \infty$, *then* $\{u_\lambda\}$ *make entire blow up.*

Proof: Let φ_1 be the first eigenfunction of $-\Delta$ in Ω. Then $\varphi_1 > 0, \Delta^2\varphi_1 = \lambda_1^2\varphi_1$ and we have

$$\lambda_1^2 \int_\Omega u_\lambda \varphi_1 = \lambda \int_\Omega f(u_\lambda)\varphi_1 = J.$$

Suppose that J is bounded , then by Lemma 2.4, u_λ is bounded near $\partial\Omega$. Hence

$$\int_\Omega \lambda f(u_\lambda)\varphi_1 \geq C_1\lambda \int_\Omega f(u_\lambda) - C_2 \to \infty.$$

A contradiction ! Hence $J \to \infty$.

On the other hand, by Hopf's boundary point lemma, for each $x \in \Omega$, there exists a constant $\gamma_x > 0$ such that

$$G_4(x,y) \geq \gamma_x \varphi_1(y) \quad (y \in \overline{\Omega})$$

where $G_4(x,y)$ is the Green's function of Δ^2 over Ω. Therefore

$$u_\lambda(x) = \int_\Omega G_4(x,y)\lambda f(u_\lambda) \geq \int_\Omega \gamma_x \varphi_1(y)\lambda f(u_\lambda) \geq \gamma_x J \to \infty$$

Hence $\{u_\lambda(x)\}$ makes entire blow up.

□

Suppose now that $\Sigma_\lambda = O(1)$. If $\Sigma_\lambda \to 0$ as $\lambda \to 0$, then we have

Lemma 2.7 *If $\Sigma_\lambda \to 0$, then $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ as $\lambda \to 0$.*

Proof: Since $\Sigma_\lambda \to 0$, by Lemma 2.3

$$\int_\Omega e^{cu} \leq C \text{ for any } \ c > 0$$

i.e., $f(u) \in L^p(\Omega)$ for any $p > 0$. Hence $\|\lambda f(u)\|_{L^p(\Omega)} \leq C\lambda$ for any p large. Elliptic regularity arguments show that $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$.

□

In conclusion, we have reduced our problem to a single case $0 < C_1 \leq \Sigma_\lambda \leq C_2 < \infty$.

3 Finite Blow Up

In this section we suppose that $0 < C_1 \leq \Sigma_\lambda \leq C_2 < \infty$. We shall prove that $\Sigma_\lambda \to 2\alpha_4 m$ for some integer $m > 0$. Moreover, u_λ blows up at exactly m points.

To this end, we introduce the following set. Let

$$S := \left\{ x \in \overline{\Omega} : \begin{array}{c} \text{there exists } \lambda_n \to 0, \text{ solutions } u_{\lambda_n} \text{ of equation (1.1) ,} \\ x_n \in \Omega \text{ such that } u_{\lambda_n}(x_n) \to \infty, x_n \to x \end{array} \right\} \tag{3.1}$$

We denote any sequence u_{λ_n} of u_λ by u_n, Σ_{λ_n} by Σ_n, etc. Because u_n has property

$$\int_\Omega \frac{\lambda_n f(u_n)}{\Sigma_n} = 1,$$

we can extract a subsequence of u_n, still denoted by u_n, such that there is a positive finite measure μ in $M(\Omega)$, the set of all real bounded Borel measures on Ω, such that

$$\int_\Omega \frac{\lambda_n f(u_n)}{\Sigma_n} \varphi \to \int_\Omega \varphi d\mu \tag{3.2}$$

for all $\varphi \in C_0^\infty(\Omega)$. Let

$$\tilde{u}_n = u_n/\Sigma_n, g_n = \lambda_n f(u_n)/\Sigma_n$$

For any $\delta > 0$, we call x_0 is a δ-regular point if there is a function φ in $C_0(\Omega)$, $0 \leq \varphi \leq 1$, with $\varphi = 1$ in a neighborhood of x_0 such that

$$\int \varphi \mu < \frac{\alpha_4}{1 + 2\delta}. \tag{3.3}$$

We define

$$\Sigma(\delta) = \{x_0 : x_0 \text{ is not a } \delta - \text{regular point}\}. \tag{3.4}$$

We will frequently say 'regular', 'irregular' and 'Σ' not mentioning δ if no confusion exists.

Lemma 3.1 *Let S be the blow-up set defined in (3.1) of the subsequence u_n. Then S is nonempty and there is a small neighborhood ω of $\partial\Omega$ which depends on the geometry of Ω only such that S doesn't contain any point in ω.*

Proof: The second assertion follows immediately from Lemma 2.4. For the first one, observe that if $S = \phi$, then u_n is bounded. □

Lemma 3.2 *If x_0 is a δ-regular point, then $\{u_n\}$ is bounded in $L^\infty(B_{R_0}(x_0))$ for some $R_0 > 0$.*

Proof: Let x_0 be a regular point. From the definition of regular point, there exists $R_1 > 0$ such that

$$\int_{B_{R_1}(x_0)} g_n < \frac{\alpha_4}{1+\delta}.$$

Split $\tilde{u}_n$ into two parts

$$\tilde{u}_n = \tilde{u}_{1n} + \tilde{u}_{2n}$$

where $\tilde{u}_{1n}$ is the solution of

$$\begin{cases} \Delta^2 \tilde{u}_{1n} = g_n \ in \ B_{R_1}(x_0) \\ \Delta \tilde{u}_{1n}|_{\partial B_{R_1}(x_0)} = \tilde{u}_{1n}|_{\partial B_{R_1}(x_0)} = 0 \end{cases} \tag{3.5}$$

and $\tilde{u}_{2n}$ solves

$$\begin{cases} \Delta^2 \tilde{u}_{2n} = 0 \ in \ B_{R_1}(x_0) \\ \Delta \tilde{u}_{2n}|_{\partial B_{R_1}(x_0)} = \Delta \tilde{u}_n|_{\partial B_{R_1}(x_0)} \\ \tilde{u}_{2n}|_{\partial B_{R_1}(x_0)} = \tilde{u}_n|_{\partial B_{R_1}(x_0)}. \end{cases} \tag{3.6}$$

From the maximum principle, $\tilde{u}_{1n},\ \tilde{u}_{2n} > 0$. For any biharmonic equation $\Delta^2 w = 0$ in $\Omega \subset R^4$, we have the following the mean value theorem (see [10])

$$w(x) = \frac{5}{2}\frac{1}{|B_R(x)|}\int_{B_R(x)} w(y)[6 - 7\frac{|y-x|}{R}]dy, B_R(x) \subset \Omega$$

Hence we obtain

$$\begin{aligned} \|\tilde{u}_{2n}\|_{L^\infty(B_{R_1/2})} &\le C\|\tilde{u}_{2n}\|_{L^1(B_{R_1})} \\ &\le C\|\tilde{u}_n\|_{L^1(\Omega)} \le C \end{aligned}$$

where the last inequality follows from Lemma 2.3. So we need only to consider $\tilde{u}_{1n}$. To estimate $\tilde{u}_{1n}$ we would like to apply Lemma 2.3.

From

$$\int_{B_{R_1}} g_n < \frac{\alpha_4}{1+\delta},$$

using Lemma 2.3, we have

$$\int_{B_{R_1}(x_0)} \exp[(1+\delta/2)|\tilde{u}_{1n}(x)|] \le C \tag{3.7}$$

if we choose ϵ in Lemma 2.3 small enough. Therefore on $B_{R_1/2}(x_0)$ since $\tilde{u}_{2n}$ is uniformly bounded, we have

$$\int_{B_{R_1/2}} g_n^{(1+\delta)} \le C \tag{3.8}$$

on $B_{R_1/2}(x_0)$. Therefore applying Corollary 2.4 to $\tilde{u}_{1n}$ on $B_{R_1/2}(x_0)$, we get uniform L^∞ bound for $\{\tilde{u}_{1n}\}$; hence uniform L^∞ bound for $\{\tilde{u}_n\}$ on $B_{R_1/4}(x_0)$. □

We immediately have

Lemma 3.3 *$S = \Sigma(\delta)$ for any $\delta > 0$.*

Proof We first claim $S = \Sigma(\delta)$ for any $\delta > 0$.

Clearly $S \subset \Sigma$. In fact, let $x_0 \notin \Sigma$; then x_0 is a regular point. Hence by Lemma 3.2 $\{v_n\}$ is bounded in $L^\infty(B_R(x_0))$ for some R, i.e. $x_0 \notin S$

Conversely suppose $x_0 \in \Sigma$. Then we have for every $R > 0$

$$\lim_{n\to\infty} \|u_n\|_{L^\infty(B_R(x_0))} = \infty. \tag{3.9}$$

Otherwise there would be some $R_0 > 0$ and a subsequence, still denoted by u_n such that

$$\|u_n\|_{L^\infty(B_{R_0}(x_0))} < C$$

for some C independent of n. Then

$$\lambda f_n \le C\lambda$$

uniformly as $n \to \infty$ on $B_{R_0}(x_0)$. Then

$$\int_{B_{R_0}(x_0)} f_n \le C\lambda \le \epsilon_0 < \frac{\alpha_4}{1+2\delta}$$

which implies that x_0 is a regular point, i.e. $x_0 \notin \Sigma$. Contradiction. (3.9) then implies by the definition (3.1) of S that $x_0 \in S$. This completes the proof of our claim.

□

Hence we arrive at the following

Lemma 3.4 *If $0 < C_1 \le \Sigma_\lambda \le C < \infty$, then $1 \le card(S) \le \frac{\overline{\lim}_{\lambda\to 0}\Sigma_\lambda}{\alpha_4}$.*

Proof: Let us go back to measure μ defined in (3.2). Clearly by (3.3)

$$1 \ge \mu(\Omega) \ge \frac{\alpha_4}{(1+2\delta)\overline{\lim}_{\lambda\to 0}\Sigma_\lambda}\#\Sigma(\delta) = \frac{\alpha_4}{(1+2\delta)\overline{\lim}_{\lambda\to 0}\Sigma_\lambda}\#S$$

Hence combining this with Lemma 3.3

$$1 \leq \#S \leq \frac{(1+2\delta)\overline{\lim}_{\lambda\to 0}\Sigma_\lambda}{\alpha_4}.$$

Applying Lemma 3.2, we finally conclude the proof of Lemma 3.4 by choosing δ small.

□

Suppose now that $S = \{x_1, ..., x_k\}$. Then we have $\|u_\lambda\|_{L^\infty_{loc}(\overline{\Omega}\setminus S)} \leq C$. Let $r > 0$ be so small that $B_r(x_i) \subset \overline{\Omega}, B_{2r}(x_i) \cap B_{2r}(x_j) = \phi, i \neq j$. Let $\Sigma^i_\lambda = \int_{B_r(x_i)} \lambda f(u)dx$. Then $\lim_{\lambda\to 0}\Sigma_\lambda = \lim_{\lambda\to 0}\Sigma^1_\lambda + ... + \lim_{\lambda\to 0}\Sigma^k_\lambda$. Moreover by Lemma 3.2, $lim_{\lambda\to 0}\Sigma_\lambda \geq \alpha_4$.

Next we would like to claim that

Lemma 3.5 $lim_{\lambda\to 0}\Sigma^i_\lambda = 2\alpha_4, i = 1, .., k.$

Proof: Without loss of generality, we assume that $i = 1, x_1 = x_0$. Consider the function w which is a solution of the following problem

$$\begin{cases} \Delta^2 w = 0 \ in \ B_r(x_0) \\ \Delta w|_{\partial B_{R_1}(x_0)} = \Delta u_\lambda|_{\partial B_r(x_0)} \\ w|_{\partial B_r(x_0)} = u_\lambda|_{\partial B_r(x_0)}. \end{cases} \tag{3.10}$$

Then $|w| \leq C$ in $B_r(x_0)$ and $h_\lambda = u_\lambda - w$ satisfies

$$\begin{cases} \Delta^2 h_\lambda = \lambda f(u_\lambda) \ in \ B_r(x_0) \\ \Delta h_\lambda|_{\partial B_r(x_0)} = 0 \\ h_\lambda|_{\partial B_r(x_0)} = 0. \end{cases} \tag{3.11}$$

Put $\tilde{h}_\lambda = h_\lambda/\Sigma^1_\lambda$. Since $\int_{B_r(x_0)} \frac{\lambda f(u_\lambda)}{\Sigma_\lambda} = 1$ and $\lambda f(u_\lambda) \to 0$ in $C^4_{loc}(B_r(x_0)\setminus\{x_0\})$, then we have that $\tilde{h}_\lambda \to G^r_4(\cdot, x_0)$ in $C^4_{loc}(B_r(x_0)\setminus\{x_0\})$, where $G^r_4(\cdot, x_0)$ is the Green's function of Δ^2 on $B_r(x_0)$. It is easy to see that $G^r_4(\cdot, x_0) = -\frac{1}{4\sigma_4}\log\frac{|x-x_0|}{r}$. We can assume that $x_0 = 0$. Let $0 < \epsilon < r$ be a fixed small number. Since $\tilde{h}_\lambda \to G^r_4(\cdot, 0)$ in C^4_{loc}, we have

$$u_\lambda = \Sigma^1_\lambda(G^r_4(\cdot, 0) + g_\lambda), v_\lambda = -\Delta u_\lambda = \Sigma^1_\lambda(\frac{1}{2\sigma_4|x|^2} - \Delta g_\lambda) \text{ on } \partial B_\epsilon$$

where $|D^\alpha g_\lambda| \leq C$ for $|\alpha| \leq 3$. By Lemma 2.2, we have

$$4\int_{B_\epsilon} \lambda F(u_\lambda)$$

$$= \int_{\partial B_\epsilon} \langle x, \nu \rangle \lambda F(u_\lambda) + \frac{1}{2} \int_{\partial B_\epsilon} (v_\lambda^2 \langle x, \nu \rangle) ds$$
$$+ \int_{\partial B_\epsilon} (\frac{\partial v_\lambda}{\partial \nu} \langle x, Du_\lambda \rangle + \frac{\partial u_\lambda}{\partial \nu} \langle x, Dv_\lambda \rangle - \langle Du_\lambda, Dv_\lambda \rangle \langle x, Du_\lambda \rangle) ds + 2 \int_{\partial B_\epsilon} \frac{\partial u_\lambda}{\partial \nu} v_\lambda$$

where $v_\lambda = -\Delta u_\lambda, F(u_\lambda) = \int_0^{u_\lambda} f(s) ds$.

Fix ϵ, we have

$$\int_{\partial B_\epsilon} \langle x, \nu \rangle \lambda F(u_\lambda) \to 0$$
$$\frac{1}{2} \int_{\partial B_\epsilon} v_\lambda^2 \langle x, \nu \rangle ds = \frac{(\Sigma_\lambda^1)^2}{8\sigma_4} + O(\epsilon)$$
$$\int_{\partial B_\epsilon} \frac{\partial v_\lambda}{\partial \nu} \langle x, Du_\lambda \rangle = \int_{\partial B_\epsilon} \epsilon \frac{\partial v_\lambda}{\partial r} \frac{\partial u_\lambda}{\partial r} = \frac{(\Sigma_\lambda^1)^2}{4\sigma_4} + O(\epsilon)$$
$$\int_{\partial B_\epsilon} \frac{\partial u_\lambda}{\partial \nu} \langle x, Dv_\lambda \rangle = \int_{\partial B_\epsilon} \epsilon \frac{\partial v_\lambda}{\partial r} \frac{\partial u_\lambda}{\partial r} = \frac{(\Sigma_\lambda^1)^2}{4\sigma_4} + O(\epsilon)$$
$$\int_{\partial B_\epsilon} \langle Du_\lambda, Dv_\lambda \rangle \langle x, \nu \rangle = \frac{(\Sigma_\lambda^1)^2}{4\sigma_4} + O(\epsilon)$$
$$2 \int_{\partial B_\epsilon} \frac{\partial u_\lambda}{\partial \nu} v_\lambda = -\frac{(\Sigma_\lambda^1)^2}{4\sigma_4} + O(\epsilon)$$

Combining all these estimates we have

$$4\lambda \int_{B_\epsilon} F(u) = \frac{(\Sigma_\lambda^1)^2}{8\sigma_4} + O(\epsilon) \tag{3.12}$$

since

$$4 \int_{B_\epsilon} \lambda F(u) - 4 \int_{B_\epsilon} \lambda f(u) \to 0,$$

we have by letting $\epsilon \to 0$,

$$\lim_{\lambda \to 0} \Sigma_\lambda^1 = 32\sigma_4 = 2\alpha_4 \tag{3.13}$$

which proves Lemma 3.4. □

In conclusion, we have

Lemma 3.6 $\lim_{\lambda \to 0} \Sigma_\lambda = 2k\alpha_4$. *Moreover,* $\lim_{\lambda \to 0} \Sigma_\lambda^i = 2\alpha_4, i = 1, ..., k$ *and* $S = \{x_1, ..., x_k\}$.

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

First we have the following lemma.

Lemma 4.1 *$u_\lambda \to u_0$ in $C^4_{loc}(\overline{\Omega}\backslash S)$ where u_0 is a solution of*

$$\begin{cases} \Delta^2 u_0 = 0 \text{ in } \overline{\Omega}\backslash S, \\ \Delta_0|_{\partial\Omega} = u_0|_{\partial\Omega} = 0. \end{cases}$$

Moreover, u_0 is given by

$$u_0 = 2\alpha_4 \sum_{j=1}^{k} G_4(\cdot, x_j). \tag{4.1}$$

Proof: From Lemma 3.2 and Lemma 3.6, we have that

$$u_\lambda \leq C \tag{4.2}$$

on any compact $K \subset \overline{\Omega}\backslash S$, which implies

$$\lambda f(u_\lambda) \to 0$$

uniformly on any compact $K \subset \overline{\Omega}\backslash S$. By Lemma 3.6, $\int_{B_r(x_i)} \lambda f(u_\lambda) \to 2\alpha_4$ for $i = 1, 2, ..., k$ and r very small.

Take $\varphi \in C_0(\Omega)$. We have for δ small

$$|\int_\Omega \lambda f(u_\lambda)\varphi dx - 2\alpha_4(\sum_{i=1}^{k} \varphi(x_i))|$$

$$\leq \sum_{i=1}^{k}[|\int_{B_\delta(x_i)}(\lambda f(u_\lambda)\varphi(x) - 2\alpha_4 \int_{B_\delta(x_i)} \varphi(x)|]$$

$$+[\sum_{i=1}^{k} \int_{B_\delta(x_i)} \lambda f(u_\lambda)|\varphi(x) - \varphi(x_0)|dx] + O(\lambda)$$

$$\to 0$$

if we first choose δ small enough then choose λ small enough.

On any compact $K \subset \overline{\Omega}\backslash S$, because $u_\lambda, v_\lambda = -\Delta u_\lambda$ are bounded and $\lambda f(u_\lambda) \to 0$ uniformly, we have by the elliptic regularity theory a subsequence of $u_\lambda, v_\lambda = -\Delta u_\lambda$, still denoted by u_λ, v_λ that approaches functions, say $G', -\Delta G'$ in $C^{4,\alpha}(K), C^{2,\alpha}(K)$ and $L^1(\Omega)$, respectively while the second convergence comes from the $W^{1,q}(\Omega)$ boundedness of u_λ, v_λ for $1 < q < \frac{4}{3}$ and the compactness of the embedding

$$W^{1,q}(\Omega) \hookrightarrow L^1(\Omega).$$

Then for any $\varphi \in C_0(\Omega)$

$$\int_\Omega G' \Delta^2 \varphi dx = \lim_{n\to\infty} \int_\Omega u_\lambda \Delta\varphi$$

$$= \lim_{n\to\infty} \int_\Omega \Delta^2 u_\lambda \varphi = \lim_{n\to\infty} \int_\Omega \lambda f(u_\lambda)\varphi$$

$$= 2\alpha_4 \sum_{i=1}^k \varphi(x_i)$$

Therefore $G' = 2\alpha_4 \sum_{i=1}^k G(\ ,x_i)$. Lemma is proved.

□

Let us now prove Theorem 1.2. Let r be a number so small that $B_{2r}(x_i) \subset \overline{\Omega}, B_{2r}(x_i) \cap B_{2r}(x_j) = \phi$ for $i \neq j$. Apply the Pohozaev identity to u_λ on $\Omega \backslash B_r(x_1)$, we have

$$\int_{\partial(\Omega\backslash B_r(x_1))} \nu F(u) + \frac{1}{2}\int_{\partial\Omega} v^2\nu + \int_{\partial(\Omega\backslash B_r(x_1))} \{\frac{\partial v}{\partial n} Du + \frac{\partial u}{\partial n} Dv - \langle Du, Dv\rangle \nu\} ds = 0.$$

Hence we have

$$\begin{aligned} 0 &= \int_{\partial B_r(x_1)} \nu F(u) + \frac{1}{2}\int_{\partial B_r(x_1)} v^2 \nu + \int_{\partial B_r(x_1)} \{\frac{\partial v}{\partial n} Du + \frac{\partial u}{\partial n} Dv - \langle Du, Dv\rangle\nu\} \\ O(\lambda) &= \frac{1}{2}\int_{\partial B_r(x_1)} v^2\nu + \int_{\partial B_r(x_1)} \{\frac{\partial v}{\partial n} Du + \frac{\partial u}{\partial n} Dv - \langle Du, Dv\rangle\nu\} \\ 0 &= \frac{1}{2}\int_{\partial B_r(x_1)} v_0^2 \nu + \int_{\partial B_r(x_1)} \{\frac{\partial v_0}{\partial n} Du_0 + \frac{\partial u}{\partial n} Dv_0 - \langle Du_0, Dv_0\rangle\nu\} \end{aligned}$$

By Lemma 3.4, $u_0 = 2\alpha_4 \sum_{j=1}^k G_4(x, x_j), v_0 = 2\alpha_4 \sum_{j=1}^k G_2(x, x_j)$, where G_4 is the Green's function for Δ^2 and G_2 is the Green's function for $-\Delta$.

Write u_0, v_0 as

$$\frac{1}{2\alpha_4} u_0 = K_4(x, x_1) + \sum_{j=2}^k G_4(x, x_j) - \frac{1}{4\sigma_4} \log\ |x - x_1| \tag{4.3}$$

$$\frac{1}{2\alpha_4} v_0 = K_2(x, x_1) + \sum_{j=2}^k G_2(x, x_j) + \frac{1}{2\sigma_4 |x - x_1|^2} \tag{4.4}$$

where $K_2(x,x_1) = -\Delta K_4(x,x_1), G_2(x,x_j) = -\Delta G(x,x_j)$.

Hence

$$\frac{1}{(2\alpha_4)^2}\int_{\partial B_r(x_1)} v_0^2\nu$$
$$= \int_{\partial B_r(x_1)} (\frac{1}{2\sigma_4 r^2} + O(1))^2\nu = O(r^2)$$
$$\frac{1}{(2\alpha_4)^2}\int_{\partial B_{r_1}(x_1)} \frac{\partial v_0}{\partial\nu} Du_0$$
$$= \int_{\partial B_r(x_1)} (-\frac{1}{\sigma_4 r^3} + O(1))(D_x(K_4(x,x_1) + \sum_{j=2}^k G_4(x,x_j)) + \frac{x}{4\sigma_4 r^2})$$
$$= -\frac{1}{\sigma_4 r^3}\int_{\partial B_r(x_1)} D_x(K_4(x,x_1) + \sum_{j=2}^k G_4(x,x_j) + O(r)$$
$$= -D_x(K_4(x_*^1,x_1) + \sum_{j=2}^k G_4(x_*^1,x_j) + O(r)$$
$$\frac{1}{(2\alpha_4)^2}\int_{\partial B_r(x_1)} \frac{\partial u_0}{\partial\nu} Dv_0$$
$$= \int_{\partial B_r(x_1)} (O(1) - \frac{x}{\sigma_4 r^4})(D_r(K_4(x,x_1) + \sum_{j=2}^k G_4(x,x_j)) - \frac{1}{4\sigma_4 r})$$
$$= -\frac{1}{\sigma_4 r^3}\int_{\partial B_r(x_1)} (D_x(K_4(x,x_1) + \sum_{j=2}^k G_4(x,x_j))) + O(r)$$
$$= -D_x(K_4(x_*^2,x_1) + \sum_{j=2}^k G_4(x_*^2,x_j)) + O(r)$$
$$\frac{1}{(2\alpha_4)^2}\int_{\partial B_r(x_1)} \langle Du_0, Dv_0\rangle\nu$$
$$= \int_{\partial B_r(x_1)} \langle O(1) - \frac{x}{\sigma_4 r^4}, D(K_4(x,x_1) + \sum_{j=2}^k G_4(x,x_j)) + \frac{x}{4\sigma_4 r^2}\rangle\nu$$
$$= -D_x(K_4(x_*^3,x_1) + \sum_{j=2}^k G_4(x_*^3,x_j)) + O(r)$$

where $x_*^1, x_*^2, x_*^3 \in B_r(x_1)$.

All together, we have as $r \to 0, x_*^j \to x_1, j = 1,2,3$ and

$$\nabla_{x_1} K_4(x_1,x_1) + \sum_{j=2}^k \nabla_{x_1} G(x,x_j) = 0. \tag{4.5}$$

Similarly, we have the other identities. Theorem 1.2 is proved.

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