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# ASYMPTOTIC ANALYSIS OF VIBRATING SYSTEM CONTAINING STIFF-HEAVY AND FLEXIBLE-LIGHT PARTS

ABSTRACT. A model of strongly inhomogeneous medium with simultaneous perturbation of rigidity and mass density is studied. The medium has strongly contrasting physical characteristics in two parts with the ratio of rigidities being proportional to a small parameter  $\varepsilon$ . Additionally, the ratio of mass densities is of order  $\varepsilon^{-1}$ . We investigate the asymptotic behaviour of spectrum and eigensubspaces as  $\varepsilon \to 0$ . Complete asymptotic expansions of eigenvalues and eigenfunctions are constructed and justified.

We show that the limit operator is nonself-adjoint in general and possesses two-dimensional Jordan cells in spite of the singular perturbed problem is associated with a self-adjoint operator in appropriated Hilbert space  $\mathcal{L}_{\varepsilon}$ . This may happen if the metric in which the problem is self-adjoint depends on small parameter  $\varepsilon$  in a singular way. In particular, it leads to a loss of completeness for the eigenfunction collection. We describe how root spaces of the limit operator approximate eigenspaces of the perturbed operator.

## Introduction

We consider a model of strongly inhomogeneous medium consisting of two nearly homogeneous components. Assuming a strong contrast of the corresponding stiffness coefficients  $k_1 \ll k_2$ , we get that their ratio  $k_1/k_2$  has a small order, which we denote by  $\varepsilon$ . In general, the mass densities  $r_1$  and  $r_2$  in two parts could be quite different as well or could be the same. We model this assuming that the density ratio  $r_1/r_2$  is proportional to  $\varepsilon^{-m}$ . We investigate how the resonance vibrations of the medium change if the parameter  $\varepsilon$  tends to 0. In the one-dimensional case we consider the spectral problem

$$\frac{d}{dx}\left(k_{\varepsilon}(x)\frac{du_{\varepsilon}}{dx}\right) + \lambda^{\varepsilon} r_{\varepsilon}(x)u_{\varepsilon} = 0 \text{ in } (a,b), \quad \alpha_{1}u_{\varepsilon}'(a) + \alpha_{0}u_{\varepsilon}(a) = 0, \quad \beta_{1}u_{\varepsilon}'(b) + \beta_{0}u_{\varepsilon}(b) = 0,$$

where (a, b) is an interval in  $\mathbb{R}$  containing the origin and

$$k_{\varepsilon}(x) = \begin{cases} k(x) & \text{for } x \in (a,0) \\ \varepsilon \varkappa(x) & \text{for } x \in (0,b), \end{cases} \qquad r_{\varepsilon}(x) = \begin{cases} \varepsilon^{-m}r(x) & \text{for } x \in (a,0) \\ \rho(x) & \text{for } x \in (0,b). \end{cases}$$
(1)

Here k, r and  $\varkappa$ ,  $\rho$  are smooth positive functions in intervals [a,0] and [0,b] respectively. At point x=0 of discontinuity of the coefficients we assume that transmission conditions  $u_{\varepsilon}(-0)=u_{\varepsilon}(+0), \ (ku'_{\varepsilon})(-0)=\varepsilon(\varkappa u'_{\varepsilon})(+0)$  hold.

Of course, the limit properties of spectrum depend on the power m characterizing the density ratio. Intuitively, we expect that for large values of m the mass density perturbation has to be dominating whereas for small m the rigidity perturbation has to be leading. Then it has to be at least one critical point m separating the cases. It appears to be truth exactly for m = 1, when the mass density perturbation is strictly inverse to the stiffness one.

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This paper is devoted to the critical case m=1. We consider the Dirichlet problem

$$(k(x) u_{\varepsilon}')' + \varepsilon^{-1} \lambda^{\varepsilon} r(x) u_{\varepsilon} = 0, \qquad x \in (a, 0),$$
 (2)

$$\varepsilon \left(\varkappa(x) u_{\varepsilon}'\right)' + \lambda^{\varepsilon} \rho(x) u_{\varepsilon} = 0, \qquad x \in (0, b),$$
 (3)

$$u_{\varepsilon}(-0) = u_{\varepsilon}(+0), \quad (ku'_{\varepsilon})(-0) = \varepsilon \left(\varkappa u'_{\varepsilon}\right)(+0),$$
 (4)

$$u_{\varepsilon}(a) = 0, \qquad u_{\varepsilon}(b) = 0$$
 (5)

and investigate the asymptotic behavior of eigenvalues  $\lambda^{\varepsilon}$  and eigenfunctions  $u_{\varepsilon}$  as  $\varepsilon \to 0$ .

After a proper change of spectral parameter problem (2)-(5) can be represented as a problem with perturbation of the transmission conditions only (cf. the example with constant coefficients below). At first blush, the problem looks very simple. But the point is that the problem shows a complicated picture of the eigenspace bifurcation. In Section 2 we prove that the limit behavior of the spectrum is described in terms of a nonself-adjoint operator that has in general multiple eigenvalues and two-dimensional root spaces. At the same time, (2)-(5) is associated with a self-adjoint operator in the weighted space  $\mathcal{L}_{\varepsilon}$  with the following scalar product and norm

$$(\phi, \psi)_{\varepsilon} = \varepsilon^{-1}(r\phi, \psi)_{L_2(a,0)} + (\rho\phi, \psi)_{L_2(0,b)}, \qquad \|\phi\|_{\varepsilon} = \sqrt{(\phi, \phi)_{\varepsilon}}.$$
 (6)

It is obvious that for each fixed  $\varepsilon > 0$  the spectrum of (2)-(5) is real, discrete and simple,  $0 < \lambda_1^{\varepsilon} < \lambda_2^{\varepsilon} < \dots < \lambda_j^{\varepsilon} < \dots > \infty$  as  $j \to \infty$  and the corresponding real-valued eigenfunctions  $\{u_{\varepsilon,j}\}_{j=1}^{\infty}$  form an orthogonal basis in  $\mathcal{L}_{\varepsilon}$ . How may it happen? The metric in  $\mathcal{L}_{\varepsilon}$  for which the perturbed problem is self-adjoint, depends on small parameter  $\varepsilon$  in a singular way. In Sections 3, 4 we construct and justify the complete asymptotic expansions of eigenvalues and eigenfunctions. Therefore there exist pairs of closely adjacent eigenvalues  $\lambda_j^{\varepsilon}$  and  $\lambda_{j+1}^{\varepsilon}$  being the bifurcation of double limit eigenvalues. Although the corresponding eigenfunctions  $u_{\varepsilon,j}$  and  $u_{\varepsilon,j+1}$  remain orthogonal in  $\mathcal{L}_{\varepsilon}$  for all  $\varepsilon > 0$ , they make an infinitely small angle between them in  $L_2(a,b)$  with the standard metric and stick together at the limit. In particular, it leads to the loss of completeness in  $L_2(a,b)$  for the limit eigenfunction collection. Nevertheless both  $u_{\varepsilon,j}$  and  $u_{\varepsilon,j+1}$  converge to the same limit, a plane  $\pi(\varepsilon)$  being the linear span of these eigenfunctions has regular asymptotic behaviour as  $\varepsilon \to 0$ . In fact, a root space  $\pi$  corresponding the double eigenvalue is the limit position of plane  $\pi(\varepsilon)$  as  $\varepsilon \to 0$ , as is shown in Theorem 5. We actually prove that the completeness property of the perturbed eigenfunction collection passes into the completeness of eigenfunctions and adjoined functions of the limit nonself-adjoint operator.

This work was motivated by [1, Ch.8], where the similar problem for the Laplace operator has been considered. The authors have handled the limit operator as the direct sum of two self-adjoint operators that nevertheless does not entirely explain the bifurcation picture in perturbation theory of operators. The aim of this paper is to present more rigorous and detailed study of the case in operator framework.

Finally, let us remark that the vibrating systems with singularly perturbed stiffness and mass density have been considered in many papers. In the case of purely stiff models (with homogeneous mass density), the asymptotic behavior of spectra have been studied in [6] - [12]. Referring to problems with purely density perturbation often involving domain perturbations, we mention [13]- [18] with the latter including a broad literature overview in the area. Spectral properties of vibrating systems with mass entirely neglected in a subdomain were also studied in [19], [20]. To the best of our knowledge, the first asymptotic results for the problems with simultaneous perturbations of mass density and stiffness appear in [21], [22].

### 1. Preliminaries

We demonstrate an example where eigenvalue bifurcation is calculated explicitly. If all coefficients in (2), (3) are constant we get the eigenvalue problem

$$y_{\varepsilon}'' + \omega_{\varepsilon}^2 y_{\varepsilon} = 0, \quad x \in (a, 0) \cup (0, b), \tag{7}$$

$$y_{\varepsilon}(a) = 0, \quad y_{\varepsilon}(b) = 0, \quad y_{\varepsilon}(-0) = y_{\varepsilon}(+0), \quad y'_{\varepsilon}(-0) = \varepsilon y'_{\varepsilon}(+0),$$
 (8)

where  $\omega_{\varepsilon}^2 = \varepsilon^{-1} \lambda^{\varepsilon}$ . Then each non-zero solution can be represented by

$$y_{\varepsilon} = \begin{cases} A_{\varepsilon} \sin \omega_{\varepsilon}(x-a) & \text{for } x \in (a,0), \\ B_{\varepsilon} \sin \omega_{\varepsilon}(x-b) & \text{for } x \in (0,b), \end{cases}$$

with  $\omega_{\varepsilon} > 0$  and  $A_{\varepsilon}, B_{\varepsilon} \in \mathbb{R}$ . By virtue of (8) we have

$$A_{\varepsilon} \sin \omega_{\varepsilon} a - B_{\varepsilon} \sin \omega_{\varepsilon} b = 0$$
 and  $A_{\varepsilon} \cos \omega_{\varepsilon} a - \varepsilon B_{\varepsilon} \cos \omega_{\varepsilon} b = 0$ .

Looking for a non-zero solution of the algebraic system, we get the characteristic equation

$$\cos \omega_{\varepsilon} a \sin \omega_{\varepsilon} b = \varepsilon \sin \omega_{\varepsilon} a \cos \omega_{\varepsilon} b. \tag{9}$$

The latter easily gives existence of the limit  $\omega_{\varepsilon} \to \omega$  as  $\varepsilon \to 0$  such that

$$\cos \omega a \sin \omega b = 0. \tag{10}$$

Moreover, the root  $\omega$  has to be positive. Obviously, if we suppose, contrary to our claim, that  $\omega_{\varepsilon}$  goes to 0 as  $\varepsilon \to 0$ , then (9) can be written in the equivalent form

$$\frac{\cos \omega_{\varepsilon} a \sin \omega_{\varepsilon} b}{\cos \omega_{\varepsilon} b \sin \omega_{\varepsilon} a} = \varepsilon$$

for sufficiently small  $\varepsilon$ . A passage to the limit as  $\varepsilon \to 0$  and  $\omega_{\varepsilon} \to 0$  leads to a contradiction, because the left-hand side converges towards the negative number b/a.

If a and b are incommensurable number, then all roots of (10) are simple. In fact, multiple roots exist iff 2n|a|=(2l-1)b for certain natural l and n. Let us consider the case a=-1 and b=2. Then the lowest positive root  $\omega=\pi/2$  of (10) has multiplicity 2. On the other hand, equation (9) admits the factorization  $\left(\cos\omega_{\varepsilon}-\sqrt{\frac{\varepsilon}{2+2\varepsilon}}\right)\left(\cos\omega_{\varepsilon}+\sqrt{\frac{\varepsilon}{2+2\varepsilon}}\right)\sin\omega_{\varepsilon}=0$ . Hence the lowest eigenvalues  $\omega_{\varepsilon,1}=\frac{\pi}{2}-\arcsin\sqrt{\frac{\varepsilon}{2+2\varepsilon}}$ ,  $\omega_{\varepsilon,2}=\frac{\pi}{2}+\arcsin\sqrt{\frac{\varepsilon}{2+2\varepsilon}}$  are closely adjacent and converge to the same limit  $\pi/2$ . The corresponding eigenfunctions  $y_{\varepsilon,1}$  and  $y_{\varepsilon,2}$  are defined up to a constant factor as

$$y_{\varepsilon,j}(x) = \begin{cases} (-1)^j \sqrt{2\varepsilon/(1+\varepsilon)} \sin \omega_{\varepsilon,j}(x+1) & \text{for } x \in (-1,0), \\ \sin \omega_{\varepsilon,j}(x-2) & \text{for } x \in (0,2). \end{cases}$$
(11)

We see at once that the angle in  $L_2(-1,2)$  between the eigenfunctions  $y_{\varepsilon,1}$  and  $y_{\varepsilon,2}$  is infinitely small as  $\varepsilon \to 0$ , because both eigenfunctions converge towards the same function

$$y_*(x) = \begin{cases} 0 & \text{for } x \in (-1,0), \\ \sin \frac{\pi}{2}(x-2) & \text{for } x \in (0,2). \end{cases}$$

The point of the example is that the collection of eigenfunctions  $\{u_{\varepsilon,j}\}_{j=1}^{\infty}$  loses the completeness property at the limit on account of the double eigenvalues. We now turn to perturbed problem (2)-(5) in the general case. To shorten formulas below, we introduce notation  $I_a = (a, 0), I_b = (0, b)$  and

$$K(x) = \begin{cases} k(x) & \text{for } x \in I_a \\ \varkappa(x) & \text{for } x \in I_b, \end{cases} \qquad R(x) = \begin{cases} r(x) & \text{for } x \in I_a \\ \rho(x) & \text{for } x \in I_b. \end{cases}$$

**Proposition 1.** For each number  $j \in \mathbb{N}$  eigenvalue  $\lambda_j^{\varepsilon}$  of (2)-(5) is a continuous function of  $\varepsilon \in (0,1)$  and  $c \varepsilon < \lambda_j^{\varepsilon} \leq C_j \varepsilon$  with constants c,  $C_j$  being independent of  $\varepsilon$ .

*Proof.* The continuity of eigenvalues with respect to the small parameter follows immediately from the mini-max principle

$$\lambda_j^{\varepsilon} = \min_{\substack{E_j \ v \in E_j \\ v \neq 0}} \max_{\substack{v \in E_j \\ v \neq 0}} \frac{\int_a^0 kv'^2 dx + \varepsilon \int_0^b \varkappa v'^2 dx}{\varepsilon^{-1} \int_a^0 rv^2 dx + \int_0^b \rho v^2 dx},\tag{12}$$

where the minimum is taken over all the subspaces  $E_j \subset H_0^1(a,b)$  with dim  $E_j = j$ . We consider the eigenfunctions  $v_1, \ldots, v_j$  corresponding to the lowest eigenvalues  $\mu_1, \ldots, \mu_j$  of the problem

$$(\varkappa(x)v')' + \mu\rho(x)v = 0, \quad x \in I_b, \qquad v(0) = v(b) = 0.$$
 (13)

Extending each  $v_k$  by zero to (a,0) we get that the span  $\mathcal{M}$  of  $v_1,\ldots,v_j$  is an j-dimensional subspace of  $H_0^1(a,b)$ . Then

$$\lambda_{j}^{\varepsilon} \leq \max_{v \in \mathcal{M}} \frac{\int_{a}^{0} kv'^{2} dx + \varepsilon \int_{0}^{b} \varkappa v'^{2} dx}{\varepsilon^{-1} \int_{a}^{0} rv^{2} dx + \int_{0}^{b} \rho v^{2} dx} = \max_{v \in \mathcal{M}} \frac{\varepsilon \int_{0}^{b} \varkappa v'^{2} dx}{\int_{0}^{b} \rho v^{2} dx} = \varepsilon \mu_{j}, \tag{14}$$

which establishes the upper estimate. Next, by the same mini-max principle

$$\lambda_j^\varepsilon > \lambda_1^\varepsilon = \min_{H_0^1(a,b)} \frac{\int_a^0 kv'^2 \, dx + \varepsilon \int_0^b \varkappa v'^2 \, dx}{\varepsilon^{-1} \int_a^0 rv^2 \, dx + \int_0^b \rho v^2 \, dx} \geq \frac{k_*}{r_*} \min_{H_0^1(a,b)} \frac{\int_a^0 v'^2 \, dx + \varepsilon \int_0^b v'^2 \, dx}{\varepsilon^{-1} \int_a^0 v^2 \, dx + \int_0^b v^2 \, dx} = \frac{\varepsilon k_* \, \omega_{\varepsilon,1}^2}{r_*} \geq c\varepsilon,$$

where  $k_* = \min_{x \in (a,b)} K(x)$ ,  $r_* = \max_{x \in (a,b)} R(x)$  and  $\omega_{\varepsilon,1}^2$  is the first eigenvalue of problem (7)-(8) with constant coefficients. It remains to note that  $\omega_{\varepsilon,1} \to \pi/2$ .

# 2. Convergence Results and Properties of Limit Problem

Let us consider the eigenvalue problem

$$\begin{cases}
(K(x)u')' + \mu R(x)u = 0, & x \in I_a \cup I_b, \\
u(a) = 0, & u(b) = 0, & u(-0) = u(+0), & u'(-0) = 0,
\end{cases}$$
(15)

that will be referred to as the limit spectral problem. The spectrum of (15) is discrete and real (see Th. 1 below). We introduce the space  $\mathcal{H} = \{f \in H_0^1(a,b) : f_a \in H^2(a,0) \text{ and } f_b \in H^2(0,b)\}$ , where  $f_a$  and  $f_b$  are the restrictions of f to intervals  $I_a$  and  $I_b$  resp. Problem (15) admits the variational formulation: to find  $\mu \in \mathbb{C}$  and a nontrivial  $u \in \mathcal{H}$  such that

$$\int_{a}^{b} K \, u' \phi' \, dx + \varkappa(0) u'(+0) \phi(0) = \mu \int_{a}^{b} R \, u \phi \, dx \tag{16}$$

for all  $\phi \in C_0^{\infty}(a,b)$ . We first prove a conditional results.

**Proposition 2.** Given eigenvalue  $\lambda^{\varepsilon}$  and the corresponding eigenfunction  $u_{\varepsilon}$  of (2)-(5), if  $\varepsilon^{-1}\lambda^{\varepsilon} \to \mu^*$  and  $u_{\varepsilon} \to u_*$  in  $H^2$  weakly on each intervals  $I_a$ ,  $I_b$  and  $u_*$  is different from zero, then  $\mu^*$  is an eigenvalue of (15) with the eigenfunction  $u_*$ .

*Proof.* We make a change of spectral parameter  $\lambda^{\varepsilon} = \varepsilon \mu^{\varepsilon}$  in (2)-(5), whereat we can reduce equation (3) by the first order of  $\varepsilon$ . Then each pair  $(\mu^{\varepsilon}, u_{\varepsilon})$  satisfies the integral identity

$$\int_{a}^{b} K \, u_{\varepsilon}' \phi' \, dx + (1 - \varepsilon) \varkappa(0) u_{\varepsilon}'(+0) \phi(0) = \mu_{\varepsilon} \int_{a}^{b} R \, u_{\varepsilon} \phi \, dx \tag{17}$$

for all  $\phi \in C_0^{\infty}(a,b)$ . The weak convergence of  $u_{\varepsilon}$  in  $H^2(0,b)$  gives the convergence  $u_{\varepsilon} \to u_*$  in  $C^1(0,b)$ , in particular,  $u'_{\varepsilon}(+0) \to u'_*(+0)$  as well as  $u'_{\varepsilon}(-0) \to 0$ . Moreover, the limit function  $u_*$  belongs to  $\mathcal{H}$ , since each  $u_{\varepsilon}$  is a continuous function at x=0. A passage to the limit in (17) implies that pair  $(\mu^*, u_*)$  satisfies identity (16). Recall that  $u_*$  is different from zero, which completes the proof.

Before improving the convergent results, we first compute the spectrum of the limit problem. Let us introduce space  $\mathcal{L} = L_2(r, I_a) \oplus L_2(\rho, I_b)$ , where  $L_2(g, I)$  is a weighted  $L_2$ -space with the norm  $||v|| = (\int_I g|v|^2)^{1/2}$ . We consider two operators

$$A_{1} = -\frac{1}{r} \frac{d}{dx} k \frac{d}{dx} \text{ in } L_{2}(r, I_{a}), \qquad \mathcal{D}(A_{1}) = \left\{ u \in H^{2}(I_{a}) : u(a) = 0, u'(0) = 0 \right\},$$

$$A_{2} = -\frac{1}{\rho} \frac{d}{dx} \varkappa \frac{d}{dx} \text{ in } L_{2}(\rho, I_{b}), \qquad \mathcal{D}(A_{2}) = \left\{ u \in H^{2}(I_{b}) : u(b) = 0 \right\}.$$

For problem (15) we assign the matrix operator

$$\mathcal{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{in } \mathcal{L}, \quad \mathcal{D}(\mathcal{A}) = \{(u_1, u_2) \in \mathcal{D}(A_1) \oplus \mathcal{D}(A_2) \colon u_1(0) = u_2(0)\}.$$

The operator A is nonself-adjoint. Actually, it is easy to check that

$$\mathcal{A}^* = \begin{pmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}^*) = \{(v_1, v_2) \in \mathcal{D}(\hat{A}_1) \oplus \mathcal{D}(\hat{A}_2) : (kv_1')(0) = (\varkappa v_2')(0)\},$$

where  $\hat{A}_1$  is the extension of operator  $A_1$  to  $\mathcal{D}(\hat{A}_1) = \{u \in H^2(a,0) : u(a) = 0\}$  and  $\hat{A}_2$  is the restriction of  $A_2$  to  $\mathcal{D}(\hat{A}_2) = \{u \in \mathcal{D}(A_2) : u(0) = 0\}$ . Let  $\sigma(A)$  and  $\varrho(A)$  denote the spectrum and the resolvent set of an operator A respectively. Let  $\mathcal{R}_{\mu}(A)$  denote the resolvent  $(A - \mu \mathcal{I})^{-1}$  of an operator A, where  $\mathcal{I}$  is the identity operator in  $\mathcal{L}$ .

**Definition.** Let u be an eigenvector of  $\mathcal{A}$  with eigenvalue  $\mu$ . A solution  $u_*$  to  $(\mathcal{A} - \mu \mathcal{I})u_* = u$  is called an adjoined vector of  $\mathcal{A}$  (corresponding to the eigenvalue  $\mu$ ).

Theorem 1. (i)  $\sigma(A) = \sigma(A_1) \cup \sigma(\hat{A}_2)$ .

- (ii) If  $\mu$  belongs to  $\sigma(A) \setminus (\sigma(A_1) \cap \sigma(\hat{A}_2))$ , then  $\mu$  is a simple eigenvalue. If  $\mu \in \sigma(A_1) \cap \sigma(\hat{A}_2)$ , then  $\mu$  has multiplicity 2 and the corresponding root space is generated by an eigenvector and an adjoined vector of A.
- (iii) The set of eigenvectors and adjoined vectors of A forms a complete system in  $\mathcal{L}$ .
- Proof. (i) Let us consider the equation  $(A \mu \mathcal{I})u = f$  for fixed  $f \in \mathcal{L}$ . In the coordinate representation we have  $A_1 u_1 \mu u_1 = f_1$ ,  $A_2 u_2 \mu u_2 = f_2$ . If  $\mu \notin \sigma(A_1)$ , then  $u_1 = \mathcal{R}_{\mu}(A_1)f_1$ . In order to find  $u_2$  we introduce the bounded intertwining operator  $T_{\mu} \colon H^2(I_a) \to H^2(I_b)$  that solves the problem  $(\varkappa \psi')' + \mu \rho \psi = 0$  in  $I_b$ ,  $\psi(0) = g(0)$ ,  $\psi(b) = 0$  for each  $g \in H^2(I_a)$ . Note

that  $T_{\mu}$  is a well-defined operator for all  $\mu \in \varrho(\hat{A}_2)$ . Then  $u_2 = T_{\mu}\mathcal{R}_{\mu}(A_1)f_1 + \mathcal{R}_{\mu}(\hat{A}_2)f_2$  and the resolvent of  $\mathcal{A}$  can be written in the form

$$\mathcal{R}_{\mu}(\mathcal{A}) = \begin{pmatrix} \mathcal{R}_{\mu}(A_1) & 0 \\ T_{\mu}\mathcal{R}_{\mu}(A_1) & \mathcal{R}_{\mu}(\hat{A}_2) \end{pmatrix}. \tag{18}$$

From the explicit representation of  $\mathcal{R}_{\mu}(\mathcal{A})$  it follows that sets  $\sigma(\mathcal{A})$  and  $\sigma(A_1) \cup \sigma(\hat{A}_2)$  coincide.

(ii) We suppose that  $\mu \in \sigma(A_1) \setminus \sigma(\hat{A}_2)$ . Then there exists an eigenvector  $U_{\mu} = (u_1, T_{\mu}u_1)$ , where  $u_1$  is an eigenvector of  $A_1$  and, that is the same, one is an eigenfunction of problem  $(k\phi')' + \mu r \phi = 0$  in  $I_a$ ,  $\phi(a) = \phi'(0) = 0$ . Note that  $\mu$  is a simple eigenvalue of the problem. Indeed,  $(\mathcal{A} - \mu \mathcal{I})U_{\mu} = 0$  follows from the evident equality  $(A_2 - \mu \mathcal{I})T_{\mu} = 0$  for all  $\mu \in \varrho(\hat{A}_2)$ .

Suppose now that  $\mu \in \sigma(\hat{A}_2) \setminus \sigma(A_1)$ . Then operator  $\mathcal{A}$  has the eigenvector  $V_{\mu} = (0, u_2)$ , where  $u_2$  is an eigenvector of  $\hat{A}_2$ . In other words,  $u_2$  is an eigenfunction of the Dirichlet problem (13). Note that each point of  $\sigma(\hat{A}_2)$  is a simple eigenvalue. Furthermore, the first component  $u_1$  must be zero, since  $\mu \notin \sigma(A_1)$ .

Finally we shall show that each point of intersection  $\sigma(A_1) \cap \sigma(\hat{A}_2)$  is an eigenvalue of algebraic multiplicity 2. Obviously, vector  $V_{\mu} = (0, u_2)$ , which appears above, is an eigenvector of  $\mathcal{A}$  in this case too. Next we consider the system

$$A_1 v_1 - \mu v_1 = 0, \quad A_2 v_2 - \mu v_2 = u_2$$
 (19)

determining adjoined vectors. If  $v_1 = 0$ , then  $v_2$  must be a solution of the boundary value problem  $(\varkappa\phi')' + \mu\rho\phi = -\rho u_2$  in  $I_b$ ,  $\phi(0) = \phi(b) = 0$ , which is unsolvable. Actually, since  $\mu \in \sigma(\hat{A}_2)$ , by the Fredholm alternative the problem admits a solution iff  $\int_0^b \rho |u_2|^2 dx = 0$ . This contradicts the fact that  $u_2$  is an eigenvector of  $\hat{A}_2$ . Consequently we have to assume that  $v_1$  is an eigenvector of  $A_1$  and examine the problem  $(\varkappa v_2')' + \mu\rho v_2 = -\rho u_2$  in  $I_b$ ,  $v_2(0) = v_1(0)$ ,  $v_2(b) = 0$ . Here the Fredholm alternative gives the solvability condition

$$\varkappa(0)u_2'(0)v_1(0) = -\int_0^b \rho \, u_2^2 \, dx. \tag{20}$$

We satisfy one by normalization of  $v_1$ , because  $u_2'(0)$  is different from zero. This condition assures the existence of  $v_2$  and a solution  $V_{\mu}^* = (v_1, v_2)$  of system (19). Vector  $V_{\mu}^*$  is the adjoined vector of  $\mathcal{A}$ . Pair  $\{V_{\mu}, V_{\mu}^*\}$  forms a basis in the root space that corresponds to  $\mu$ .

The last statement of the theorem follows from the Keldysh theorem [3].

We investigate the limit behaviour of eigenfunctions  $u_{\varepsilon,n}$  normalized by conditions

$$\int_{a}^{b} R(x) u_{\varepsilon,j}^{2}(x) dx = 1, \qquad u_{\varepsilon,j}'(b) > 0.$$

$$(21)$$

Let us enumerate the eigenvalues of operator  $\mathcal{A}$  in increasing order and repeat each eigenvalue according to its multiplicity:  $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_j \leq \cdots$ . The next statement improves the conditional results of Proposition 2.

**Theorem 2.** There exists a one-to-one correspondence between the set of eigenvalues  $\{\lambda_j^{\varepsilon}\}_{j=1}^{\infty}$  of perturbed problem (2)-(5) and the spectrum of operator  $\mathcal{A}$ . Namely,  $\varepsilon^{-1}\lambda_j^{\varepsilon} \to \mu_j$  as  $\varepsilon \to 0$ , for each  $j \in \mathbb{N}$ . Furthermore, a sequence of the corresponding eigenfunctions  $u_{\varepsilon,j}$  converges in  $H^1(a,b)$  towards the eigenfunction u with eigenvalue  $\mu_j$ .

*Proof.* For the perturbed problem (2)-(5) we assign the matrix operator in  $\mathcal{L}$ 

$$\mathcal{A}_{\varepsilon} = \begin{pmatrix} \hat{A}_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_{\varepsilon}) = \{(u_1, u_2) \in \mathcal{D}(\hat{A}_1) \oplus \mathcal{D}(A_2) : u_1(0) = u_2(0), \quad (ku_1')(0) = \varepsilon(\varkappa u_2')(0) \}.$$

Clearly, if  $\mu_{\varepsilon}$  belongs to  $\sigma(\mathcal{A}_{\varepsilon})$ , then  $\varepsilon \mu_{\varepsilon}$  is an eigenvalue of (2)-(5). Let us solve the equation  $(\mathcal{A}_{\varepsilon} - \mu \mathcal{I})u = f$  for  $f = (f_1, f_2) \in \mathcal{L}$  and  $\mu \in \varrho(\mathcal{A}_{\varepsilon})$ . Similarly to the previous theorem we obtain  $u_1 = \mathcal{R}_{\mu}(A_1)f_1 + \varepsilon S_{\mu}u_2$ ,  $u_2 = T_{\mu}u_1 + \mathcal{R}_{\mu}(\hat{A}_2)f_2$ , where  $S_{\mu} \colon H^2(I_b) \to H^2(I_a)$  is a bounded intertwining operator that solves the problem  $(k\psi')' + \mu r\psi = 0$  in  $I_a$ ,  $\psi(a) = 0$  and  $(k\psi')(0) = (\varkappa g')(0)$  for each  $g \in H^2(I_b)$ . This yields that

$$\begin{pmatrix} I & -\varepsilon S_{\mu} \\ -T_{\mu} & I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{\mu}(A_1) f_1 \\ \mathcal{R}_{\mu}(\hat{A}_2) f_2 \end{pmatrix}, \tag{22}$$

where the matrix operator in the left-hand side is invertible as a small perturbation of the invertible one. Letting  $\varepsilon \to 0$  we can assert that

$$\mathcal{R}_{\mu}(\mathcal{A}_{\varepsilon}) = \begin{pmatrix} I & -\varepsilon S_{\mu} \\ -T_{\mu} & I \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{R}_{\mu}(A_{1}) & 0 \\ 0 & \mathcal{R}_{\mu}(\hat{A}_{2}) \end{pmatrix} \rightarrow \begin{pmatrix} I & 0 \\ T_{\mu} & I \end{pmatrix} \begin{pmatrix} \mathcal{R}_{\mu}(A_{1}) & 0 \\ 0 & \mathcal{R}_{\mu}(\hat{A}_{2}) \end{pmatrix}.$$

Hence,  $\mathcal{R}_{\mu}(\mathcal{A}_{\varepsilon}) \to \mathcal{R}_{\mu}(\mathcal{A})$  in the uniform operator topology as  $\varepsilon \to 0$ , which establishes a number-by-number convergence of the corresponding eigenvalues [3, Th. 3.1].

Next we prove existence of the limit for the eigenfunctions under normalization condition (21). We conclude from (17) that  $\int_a^b K(x) u_\varepsilon'^2(x) \, dx + (1-\varepsilon)\varkappa(0) u_\varepsilon'(+0) u_\varepsilon(+0) = \mu_\varepsilon$ . For each  $\nu$  there exists a twice differentiable solution  $\psi(x,\nu)$  of equation  $(\varkappa v')' + \nu \rho v = 0$  in  $I_b$  that satisfies conditions v(b) = 0, v'(b) = 1. Moreover,  $\psi(x,\nu)$  is an analytic function with respect to the second argument for each fixed x [2, Th.1.5]. In particular,  $\psi(x,\mu^\varepsilon) \to \psi(x,\mu)$  in  $C^2(0,b)$  as  $\mu^\varepsilon \to \mu$ . Then there exits constant  $\beta_\varepsilon$  such that  $u_\varepsilon(x) = \beta_\varepsilon \psi(x,\mu^\varepsilon)$ . Moreover,  $\beta_\varepsilon$  is bounded as  $\varepsilon \to 0$ , which is due to condition (21). Therefore the values  $u_\varepsilon(+0)$  and  $u_\varepsilon'(+0)$  are bounded with respect to  $\varepsilon$ . Consequently we have  $\int_a^b K(x) u_\varepsilon'^2(x) \, dx \le \mu_\varepsilon + (1-\varepsilon)\varkappa(0) |u_\varepsilon'(+0)u_\varepsilon(+0)| \le M$ . Then finally the sequence  $\{u_\varepsilon\}_{\varepsilon>0}$  is precompact in the weak topology of  $H^1(a,b)$ . Let us consider a subsequence  $u_{\varepsilon'}$  such that  $u_{\varepsilon'} \to u$  in  $H^1(a,b)$  weakly. We get  $u_{\varepsilon'}(x) = \beta_{\varepsilon'}\psi(x,\mu^{\varepsilon'}) \to \beta\psi(x,\mu) = u(x)$  in  $C^2(0,b)$  for certain  $\beta$ . Note that  $\beta > 0$ , which is due to (21). Moreover,  $u_{\varepsilon'}'(+0) \to u'(+0)$  as  $\varepsilon' \to 0$ . A passage to the limit in (17) implies that partial weak limit u satisfies the identity

$$\int_{a}^{b} K(x)u'\phi' \, dx + \varkappa(0)u'(+0)\phi(0) = \mu \int_{a}^{b} R(x)u\phi \, dx$$

for all  $\phi \in C_0^{\infty}(a, b)$ . Moreover, u is different from zero, since  $\int_a^b R|u|^2 dx = 1$ . Consequently each weakly convergent subsequence of  $\{u_{\varepsilon}\}_{{\varepsilon}>0}$  tends to u, where u is an eigenfunction of (15) that corresponds to the eigenvalue  $\mu$  and satisfies conditions  $||u||_{L^2(R,(a,b))} = 1$  and u'(b) > 0. Then the same conclusion can be drawn for the entire sequence.

Remark 1. In some cases value  $\varepsilon^{-1}\lambda^{\varepsilon}$  doesn't actually depend on  $\varepsilon$ . The latter takes place if and only if the three-points problem

$$(K(x)u')' + \mu R(x)u = 0$$
 for  $x \in I_a \cup I_b$ ,  $u(a) = u(b) = u'(-0) = u'(+0) = 0$  (23)

has an eigenfunction u that is *continuous* at x=0 (for a certain eigenvalue  $\mu$ ). This situation is possible, for instance, in the case a=-b when there exists even eigenfunction of the Dirichlet problem on (-b,b). Then a trivial verification shows that  $\lambda^{\varepsilon}=\varepsilon\mu$  is an eigenvalue of (2)-(5) with the eigenfunction  $u_{\varepsilon}=u$  for all  $\varepsilon\in(0,1]$ .

Corollary 1. Restrictions of eigenfunction  $u_{\varepsilon,j}$  to the intervals  $I_a$  and  $I_b$  converge towards the corresponding restrictions of eigenfunction u in  $H^2(a,0)$  and  $H^2(0,b)$  respectively.

*Proof.* Set  $u_{\varepsilon} = u_{\varepsilon,j}$ . We consider equation (2) in the form  $u_{\varepsilon}'' = -k'k^{-1}u_{\varepsilon}' - \mu_{\varepsilon}rk^{-1}u_{\varepsilon}$  in  $I_a$ . Then from Theorem 2 we have

$$u_{\varepsilon}'' \to -k'k^{-1}u' - \mu r k^{-1}u \quad \text{in} \quad L^2(a,0),$$
 (24)

where u is an eigenfunction of (15). From (15) it follows that the limit (24) is exactly the second derivative of the limiting eigenfunction in  $I_a$ . The proof for interval  $I_b$  is the same.  $\square$ 

## 3. Formal Asymptotic Expansions of Eigenvalues and Eigenfunctions

3.1. Asymptotics of Simple Eigenvalues. In this section we construct the complete asymptotic expansions of eigenvalues  $\lambda^{\varepsilon}$  and eigenfunctions  $u_{\varepsilon}$ . We begin with the examination of eigenvalues  $\lambda^{\varepsilon}_{j}$  for which the limit  $\mu = \lim_{\varepsilon \to 0} \lambda^{\varepsilon}_{j}/\varepsilon$  is a simple eigenvalue of operator  $\mathcal{A}$ . Clearly,  $\mu$  depends on j, which we do not indicate for the sake of notation simplicity. The asymptotic expansions of the eigenvalues and the corresponding eigenfunctions are represented by

$$\lambda^{\varepsilon} \sim \varepsilon \left( \mu + \varepsilon \nu_1 + \dots + \varepsilon^n \nu_n + \dots \right), \tag{25}$$

$$u_{\varepsilon}(x) \sim \begin{cases} y_0(x) + \varepsilon y_1(x) + \dots + \varepsilon^n y_n(x) + \dots & \text{for } x \in I_a, \\ z_0(x) + \varepsilon z_1(x) + \dots + \varepsilon^n z_n(x) + \dots & \text{for } x \in I_b, \end{cases}$$
 (26)

where  $\mu$  is an arbitrary eigenvalue of limit problem (15). Then

$$u(x) = \begin{cases} y_0(x) & \text{for } x \in I_a, \\ z_0(x) & \text{for } x \in I_b \end{cases}$$
 (27)

is the corresponding eigenfunction of (15) as it follows from Th. 2. Since in this section we treat only the simple eigenvalues  $\mu$ , according to Th. 1 we only consider here two possible situations:  $\mu \in \sigma(A_1) \setminus \sigma(\hat{A}_2)$  and  $\mu \in \sigma(\hat{A}_2) \setminus \sigma(A_1)$ .

3.1.1. Case  $\mu \in \sigma(A_1) \setminus \sigma(\hat{A}_2)$ . We fix the corresponding eigenfunction  $y_0$  of operator  $A_1$  such that  $\int_a^0 ry_0^2 dx = 1$  and  $y_0(0) > 0$ . Since  $\mu$  doesn't belong to the spectrum of  $\hat{A}_2$  there exists a unique solution  $z_0$  to the problem

$$(\varkappa z_0')' + \mu \rho z_0 = 0$$
 in  $I_b$ ,  $z_0(0) = y_0(0)$ ,  $z_0(b) = 0$ . (28)

An easy computation shows that the next terms of the expansions are unique solutions to the recurrent sequence of problems

$$\begin{cases} (ky'_n)' + \mu r y_n = -\nu_n r y_0 - r \sum_{j=1}^{n-1} \nu_j \, y_{n-j} & \text{in } I_a, \\ y_n(a) = 0, \quad (ky'_n)(0) = (\varkappa z'_{n-1})(0), \quad \int_a^0 r y_n y_0 \, dx = 0, \end{cases}$$
(29)

$$\begin{cases} (\varkappa z_n')' + \mu \rho z_n = -\rho \sum_{j=1}^n \nu_j z_{n-j} & \text{in } I_b, \\ z_n(0) = y_n(0), & z_n(b) = 0 \end{cases}$$
(30)

with  $\nu_n = -(\varkappa z'_{n-1})(0)y_0(0)$  for  $n = 1, 2, \ldots$ . The last formula for  $\nu_n$  is obtained as the solvability condition of (29). Note that all solutions  $y_n, z_n$  are smooth functions.

Remark 2. It might happened that  $z_0'(0) = 0$  (cf. the proof of Th. 2). In this case function u defined by (27) is exactly an eigenfunction of the perturbed problem for each  $\varepsilon \in (0,1]$ . Then the construction of asymptotics is interrupted and we can state that there exists an eigenvalue  $\lambda^{\varepsilon} = \varepsilon \mu$  for all  $\varepsilon > 0$ . The corresponding eigenfunction

$$u_{\varepsilon}(x) = \begin{cases} y_0(x) & \text{for } x \in I_a, \\ z_0(x) & \text{for } x \in I_b \end{cases}$$

doesn't depend on  $\varepsilon$ .

3.1.2. Case  $\mu \in \sigma(\hat{A}_2) \backslash \sigma(A_1)$ . This situation immediately implies  $y_0 = 0$  (cf. the proof of Th. 1, part (ii)). We fix the corresponding eigenfunction  $z_0$  of  $\hat{A}_2$  such that  $\int_0^b \rho z_0^2 dx = 1$  and  $z_0'(0) > 0$ . A trivial verification shows that the next terms of expansions (26) are the unique smooth solutions to the problems

$$\begin{cases} (ky'_n)' + \mu r y_n = -r \sum_{j=1}^{n-1} \nu_j \, y_{n-j} & \text{in } I_a, \\ y_n(a) = 0, \quad (ky'_n)(0) = (\varkappa z'_{n-1})(0), \\ (\varkappa z'_n)' + \mu \rho z_n = -\nu_n \rho z_0 - \rho \sum_{j=1}^{n-1} \nu_j \, z_{n-j} & \text{in } I_b, \\ z_n(0) = y_n(0), \quad z_n(b) = 0, \quad \int_0^b \rho z_n z_0 \, dx = 0, \end{cases}$$
(31)

with  $\nu_n = -(\varkappa z_0')(0)y_n(0)$  for  $n = 1, 2, \ldots$ . Such choice of  $\nu_n$  assures the solvability of (31).

3.2. Asymptotics of Double Eigenvalues. In this subsection we treat the case when for two successive eigenvalues  $\lambda_j^{\varepsilon}$  and  $\lambda_{j+1}^{\varepsilon}$  the corresponding ratios  $\varepsilon^{-1}\lambda_j^{\varepsilon}$  and  $\varepsilon^{-1}\lambda_{j+1}^{\varepsilon}$  converge to the same limit  $\mu$ . It is obvious that  $\mu$  must belong to the intersection  $\sigma(A_1) \cup \sigma(\hat{A}_2)$ . Let us assume that the eigenvalues and the corresponding eigenfunctions admit expansions

$$\lambda^{\varepsilon} \sim \varepsilon \left( \mu + \sqrt{\varepsilon} \nu_1 + \varepsilon \nu_2 + \cdots \right), \tag{32}$$

$$u_{\varepsilon}(x) \sim \begin{cases} \sqrt{\varepsilon} \, w_1(x) + \varepsilon \, w_2(x) + \cdots & \text{for } x \in (a, 0), \\ v_0(x) + \sqrt{\varepsilon} \, v_1(x) + \varepsilon \, v_2(x) + \cdots & \text{for } x \in (0, b), \end{cases}$$
(33)

because the eigenvectors of operator  $\mathcal{A}$  that correspond to double eigenvalues  $\mu$  have the form  $V_{\mu} = (0, v_0)$  (see Th. 1). Substituting (32), (33) into the perturbed problem we obtain

$$(\varkappa v_0')' + \mu \rho v_0 = 0 \quad \text{in} \quad I_b, \qquad v_0(0) = v_0(b) = 0,$$
 (34)

$$(kw_1')' + \mu r w_1 = 0$$
 in  $I_a$ ,  $w_1(a) = w_1'(0) = 0$ . (35)

We fix  $\mu \in \sigma(A_1) \cup \sigma(\hat{A}_2)$  and introduce the functions

$$U(x) = \begin{cases} 0 & \text{for } x \in I_a \\ v(x) & \text{for } x \in I_b \end{cases}, \qquad U_*(x) = \begin{cases} w_*(x) & \text{for } x \in I_a \\ v_*(x) & \text{for } x \in I_b \end{cases}$$
(36)

that correspond to the eigenvector and adjoined vector of  $\mathcal{A}$  (cf. vectors  $V_{\mu}$  and  $V_{\mu}^{*}$  in Th. 1). Here v is an eigenfunction of (34) such that  $\int_{0}^{b} \rho v^{2} dx = 1$ , v'(0) > 0 and adjoined vector  $U_{*}$  is chosen such that  $(U, U_{*})_{L_{2}(R,(a,b))} = 0$ . We also introduce an eigenfunction w of (35) such that  $\int_{a}^{0} rw^{2} dx = 1$  and w(0) > 0. It follows that  $v_{0} = \alpha v$  and  $w_{1} = \beta w$  with certain constants  $\alpha$  and  $\beta$ . In addition,  $\alpha$  must be different from zero. The next problems to solve are

$$(\varkappa v_1')' + \mu \rho v_1 = -\nu_1 \alpha \rho v \quad \text{in} \quad I_b, \qquad v_1(0) = \beta w(0), \quad v_1(b) = 0,$$
 (37)

$$(kw_2')' + \mu r w_2 = -\nu_1 \beta r w \quad \text{in} \quad I_a, \qquad w_2(a) = 0, \quad k(0)w_2'(0) = \alpha \varkappa(0)v'(0).$$
 (38)

In general case both problems (37) and (38) are unsolvable, since  $\mu$  belongs to the spectra  $\sigma(A_1)$  and  $\sigma(\hat{A}_2)$  at one time. Hence we have to apply Fredholm's alternative for both the problems. After multiplying equations (38) and (37) by eigenfunctions v and w respectively and integrating by parts, one yields the common solvability condition:

$$\begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\nu_1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{39}$$

where  $\omega = (\varkappa wv')(0)$  is positive. Since the first component of vector  $\gamma = (\alpha, \beta)$  must be different from zero,  $-\nu_1$  is an eigenvalue of the matrix in (39). Therefore if either  $\nu_1 = \omega$  and  $\gamma = (1, -1)$  or  $\nu_1 = -\omega$  and  $\gamma = (1, 1)$ , then problems (37), (38) admit solutions. Moreover, functions  $\nu_1 w_*$  and  $\nu_1 v_*$  solve problems (35) and (37) respectively for both values of  $\nu_1$ . Actually these problems imply immediately  $(\mathcal{A} - \mu)U_* = \omega U$ . In other words, the first corrector is an adjoined vector of  $\mathcal{A}$  that corresponds to the eigenvector  $\omega U$ . It causes no confusion that we use the same letters U,  $U_*$  to designate a function of  $L_2(a, b)$  and a vector in  $\mathcal{L}$ .

Summarizing, we formally demonstrate that there exists a pair of closely adjacent eigenvalues  $\lambda_j^{\varepsilon}$  and  $\lambda_{j+1}^{\varepsilon}$  that admit the asymptotic expansions

$$\lambda_j^{\varepsilon} = \varepsilon \mu - \varepsilon^{3/2} \omega + O(\varepsilon^2), \qquad \lambda_{j+1}^{\varepsilon} = \varepsilon \mu + \varepsilon^{3/2} \omega + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

As of asymptotics of eigenfunctions we have

$$u_{\varepsilon,j}(x) = U(x) - \sqrt{\varepsilon} \omega U_*(x) + O(\varepsilon), \qquad u_{\varepsilon,j+1}(x) = U(x) + \sqrt{\varepsilon} \omega U_*(x) + O(\varepsilon).$$

These eigenfunctions subtend an infinitely small angle in  $L^2$ -space as  $\varepsilon \to 0$ . Hence  $u_{\varepsilon,j}$  and  $u_{\varepsilon,j+1}$  stick together at the limit. The latter gives rise to the loss of completeness of the limit eigenfunction system.

Suppose that  $\nu_1 = \omega$  and  $\gamma = (1, -1)$ . Then we will denote by  $V_1$  and  $W_2$  such solutions of the problems that  $\int_0^b \rho V_1 v \, dx = 0$  and  $\int_a^0 r W_2 w \, dx = 0$ . We see at once that  $-V_1$  and  $-W_2$  are solutions of (37), (38) for  $\nu_1 = -\omega$  and  $\gamma = (1, 1)$ .

From now on we distinct two branches of expansions (32)

$$\lambda_{-}^{\varepsilon} \sim \varepsilon(\mu - \sqrt{\varepsilon}\omega + \varepsilon\nu_{2}^{-} + \dots + \varepsilon^{n/2}\nu_{n}^{-} + \dots),$$

$$\lambda_{+}^{\varepsilon} \sim \varepsilon(\mu + \sqrt{\varepsilon}\omega + \varepsilon\nu_{2}^{+} + \dots + \varepsilon^{n/2}\nu_{n}^{+} + \dots),$$
(40)

and the corresponding branches of (33) are

$$u_{\varepsilon}^{\pm}(x) \sim \begin{cases} \mp \sqrt{\varepsilon} \, w(x) \pm \varepsilon \, w_{2}^{\pm}(x) + \dots + \varepsilon^{n/2} w_{n}^{\pm}(x) \dots, & x \in I_{a}, \\ v(x) \pm \sqrt{\varepsilon} \, v_{1}^{\pm}(x) + \varepsilon v_{2}^{\pm}(x) + \dots + \varepsilon^{n/2} v_{n}^{\pm}(x) \dots, & x \in I_{b}. \end{cases}$$
(41)

All coefficients are endowed with indexes + or - if they depend on the choice of the sign of the first corrector  $\nu_1 = \pm \omega$ . Note that the high order correctors in (40), (41) have to be calculated separately for both the branches. We now turn to the case  $\nu_1 = \omega$  and find coefficients  $\nu_n^+$ ,  $w_n^+$  and  $v_n^+$ . To shorten notation, we omit upper index "+" for a while. Next, we see that problems (37) and (38) admit many solutions  $v_1 = V_1 + \alpha_1 v$  and  $w_2 = W_2 + \beta_1 w$ , where  $\alpha_1, \beta_1$ are constants. These constants can be obtained from the consistency of problems

$$\begin{cases} (\varkappa v_2')' + \mu \rho v_2 = -\nu_1 \rho \left( V_1 + \alpha_1 v \right) - \nu_2 \rho v, & x \in I_b \\ v_2(0) = W_2(0) + \beta_1 w(0), & v_2(b) = 0, \end{cases}$$
(42)

$$\begin{cases}
(\varkappa v_2')' + \mu \rho v_2 = -\nu_1 \rho \left( V_1 + \alpha_1 v \right) - \nu_2 \rho v, & x \in I_b \\
v_2(0) = W_2(0) + \beta_1 w(0), & v_2(b) = 0,
\end{cases}$$

$$\begin{cases}
(kw_3')' + \mu r w_3 = -\nu_1 r \left( W_2 + \beta_1 w \right) - \nu_2 r w_1, & x \in I_a \\
w_3(a) = 0, & k(0) w_3'(0) = \varkappa(0) \left( V_1 + \alpha_1 v \right)'(0).
\end{cases}$$
(43)

The solvability conditions for problems (42) and (43), which arrive from Fredholm's alternatives, can be represented as a linear algebraic system

$$\begin{pmatrix} \nu_1 & \omega \\ \omega & \nu_1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} (\varkappa W_2 v')(0) + \nu_2 \\ (\varkappa w V_1)'(0) - \nu_2 \end{pmatrix}. \tag{44}$$

The system has solution if and only if  $\nu_2 = \frac{1}{2} \left(\varkappa w V_1' - \varkappa W_2 v'\right)$  (0). After the solvability condition is satisfied, system (44) has a partial solution  $\alpha_1 = \beta_1 = \frac{1}{2\omega} \left(\varkappa w V_1' + \varkappa W_2 v'\right)$  (0) and problems (42) and (43) admit solutions  $V_2$  and  $W_3$  such that  $\int\limits_0^b \rho V_2 v \, dx = 0$  and  $\int\limits_a^0 r W_3 w \, dx = 0$ . Therefore, all other solutions of (42) and (43) allow the representation  $v_2 \stackrel{a}{=} V_2 + \alpha_2 v$  and  $w_3 = W_3 + \beta_2 w$  with real constants  $\alpha_2$ ,  $\beta_2$ .

We construct the general terms of expansions (40) and (41) as solutions to the problems

$$\begin{cases} (\varkappa v_n')' + \mu \rho v_n = -\rho \sum_{j=1}^n \nu_j v_{n-j}, & x \in I_b, \\ v_n(0) = w_n(0), & v_n(b) = 0, \end{cases}$$
(45)

$$\begin{cases} (kw'_{n+1})' + \mu r w_{n+1} = -r \sum_{j=1}^{n} \nu_j w_{n+1-j}, & x \in I_a, \\ w_{n+1}(a) = 0, & (kw_{n+1})'(0) = (\varkappa v_{n-1})'(0), \end{cases}$$
(46)

with

$$v_{n-1} = V_{n-1} + \alpha_{n-1}v$$
 and  $w_n = W_n + \beta_{n-1}w$ , (47)

where  $V_{n-1}$  and  $W_n$  are solutions of the previous problems chosen accordingly to the orthogonality conditions  $\int_0^b \rho V_{n-1} v \, dx = 0$  and  $\int_a^b r W_n w \, dx = 0$ ,  $n \ge 2$ . Constants  $\alpha_{n-1}$  and  $\beta_{n-1}$  we find from the solvability conditions for (45) and (46) given by

$$\begin{pmatrix} \nu_{1} & \omega \\ \omega & \nu_{1} \end{pmatrix} \begin{pmatrix} \alpha_{n-1} \\ \beta_{n-1} \end{pmatrix} = \begin{pmatrix} (\varkappa W_{n} v') (0) + \sum_{j=2}^{n-1} \nu_{j} \alpha_{n-j} + \nu_{n} \\ (\varkappa w V'_{n-1}) (0) + \sum_{j=2}^{n-1} \nu_{j} \beta_{n+1-j} - \nu_{n} \end{pmatrix}. \tag{48}$$

The latter has a solution if and only if  $\nu_n = \frac{1}{2} \left( \varkappa w V'_{n-1} - \varkappa W_n v' \right) (0)$ . Then system (48) has a partial solution  $\alpha_{n-1} = \beta_{n-1} = \frac{1}{2\omega} \left( \varkappa w V'_{n-1} + \varkappa W_n v' \right) (0) + \frac{1}{\omega} \sum_{j=2}^{n-1} \nu_j \alpha_{n-j}$ . Substituting the constants into (47) we finish the general step of recurrent algorithm. Hence, after coming back our natation we obtain all coefficients  $\nu_n^+$ ,  $\nu_n^+$  and  $\nu_n^+$  of series (40) and (41).

Similarly, we can construct the coefficients  $\nu_n^-$ ,  $v_n^-$  and  $w_n^-$  of series (40) and (41). Then, by induction we get that for any natural n the coefficients satisfy relations  $\nu_n^- = (-1)^n \nu_n^+$ ,  $v_n^- = (-1)^n v_n^+$  and  $w_n^- = (-1)^n w_n^+$ .

## 4. Justification of Asymptotic Expansions

Let  $\mathcal{L}_{\varepsilon}$  be he weighted  $L_2$ -space with the scalar product and norm given by (6). We also introduce space  $\mathcal{H}_{\varepsilon}$  as the Sobolev space  $H_0^1(a,b)$  with scalar product and norm

$$\langle \phi, \psi \rangle_{\varepsilon} = \int_{a}^{0} k \phi' \, \psi' \, dx + \varepsilon \int_{0}^{b} \varkappa \phi' \, \psi' \, dx, \qquad \|\phi\|_{\mathcal{H}_{\varepsilon}} = \sqrt{\langle \phi, \phi \rangle_{\varepsilon}}. \tag{49}$$

It is easily seen that

$$c\|\phi\| \le \|\phi\|_{\varepsilon} \le C\varepsilon^{-1/2}\|\phi\|, \qquad c\varepsilon^{1/2}\|\phi\|_1 \le \|\phi\|_{\mathcal{H}_{\varepsilon}} \le C\|\phi\|_1, \tag{50}$$

where  $\|\cdot\|$  and  $\|\cdot\|_1$  are standard norms in  $L_2(a,b)$  and  $H_0^1(a,b)$  respectively.

For the sake of completeness, we introduce here below the classical result on quasimodes. Let A be a self-adjoint operator in Hilbert space H with domain  $\mathcal{D}(A)$  and  $\sigma > 0$ .

**Definition.** We will say that pair  $(\mu, u) \in \mathbb{R} \times \mathcal{D}(A)$  is a quasimode with accuracy to  $\sigma$  for operator A if  $\|(A - \mu I)u\|_H \leq \sigma$  and  $\|u\|_H = 1$ .

**Lemma 1** (Vishik and Lyusternik). Suppose that the spectrum of A is discrete. If  $(\mu, u)$  is a quasimode of A with accuracy to  $\sigma$ , then interval  $[\mu - \sigma, \mu + \sigma]$  contains an eigenvalue of A. Furthermore, if segment  $[\mu - d, \mu + d]$ , d > 0, contains one and only one eigenvalue  $\lambda$  of A, then  $\|u - v\|_H \leq 2d^{-1}\sigma$ , where v is an eigenfunction of A with eigenvalue  $\lambda$ ,  $\|v\|_H = 1$ . [4, 5]

4.1. Simple Spectrum. We will denote by  $\Lambda_{\varepsilon,n} = \varepsilon \left( \mu + \varepsilon \nu_1 + \cdots + \varepsilon^n \nu_n \right)$  and

$$U_{\varepsilon,n}(x) = \begin{cases} y_0(x) + \varepsilon y_1(x) + \dots + \varepsilon^n y_n(x) & \text{for } x \in I_a \\ z_0(x) + \varepsilon z_1(x) + \dots + \varepsilon^n z_n(x) & \text{for } x \in I_b \end{cases}$$

the partial sums of series (25), (26). The perturbed problem is associated with self-adjoint operator  $A_{\varepsilon} = -\frac{1}{r_{\varepsilon}} \frac{d}{dx} k_{\varepsilon} \frac{d}{dx}$  in  $\mathcal{L}_{\varepsilon}$  with the domain  $\mathcal{D}(A_{\varepsilon}) = \{ f \in \mathcal{H} : (kf')(-0) = \varepsilon(\varkappa f')(+0) \}$ , where coefficients  $k_{\varepsilon}$ ,  $r_{\varepsilon}$  are given by (1) for m = 1.

**Theorem 3.** If  $\mu_j \in \sigma(A_1) \setminus \sigma(\hat{A}_2)$ , then eigenfunction  $u_{\varepsilon,j}$  of (2)-(5) with eigenvalue  $\lambda_j^{\varepsilon}$  converges in  $H^1(a,b)$  towards the function

$$u(x) = \begin{cases} y(x) & \text{for } x \in I_a \\ z(x) & \text{for } x \in I_b, \end{cases}$$

where y is an eigenfunction of the problem  $(ky')' + \mu ry = 0$  in  $I_a$ , y(a) = y'(0) = 0 with eigenvalue  $\mu_j$ , and z is a unique solution of the problem  $(\varkappa z')' + \mu_j \rho z = 0$  in  $I_b$ , z(0) = y(0), z(b) = 0.

If z'(0) = 0, then  $\lambda_j^{\varepsilon} = \varepsilon \mu_j$  and  $u_{\varepsilon,j} = u$  for all  $\varepsilon > 0$ . Otherwise  $\lambda_j^{\varepsilon}$  and  $u_{\varepsilon,j}$  admit asymptotics expansions (25), (26) obtained in 3.1.1 for  $\mu = \mu_j$ . Moreover, the estimates of remainder terms hold

$$\left| \varepsilon^{-1} \lambda_j^{\varepsilon} - (\mu_j + \varepsilon \nu_1 + \dots + \varepsilon^n \nu_n) \right| \le c_n \varepsilon^{n+1}, \tag{51}$$

$$||u_{\varepsilon,j} - \vartheta_{\varepsilon} U_{\varepsilon,n}||_{H^1(a,b)} \le C_n \varepsilon^{n+1},$$
 (52)

where  $\vartheta_{\varepsilon}$  is a normalizing multiplier with strictly positive limit as  $\varepsilon \to 0$ .

*Proof.* The case z'(0) = 0 was considered in Remarks 1 and 2. Suppose that  $z'(0) \neq 0$ . We first check that the series being constructed in 3.1.1 give us the quasimodes with accuracy to an arbitrary order. It follows from (29), (30) that

$$\left| r_{\varepsilon}^{-1} (k_{\varepsilon} U_{\varepsilon,n}')' + \Lambda_{\varepsilon,n} U_{\varepsilon,n} \right| \le c_n \varepsilon^{n+2} \tag{53}$$

in [a, b] uniformly,  $U_{\varepsilon,n}(a) = U_{\varepsilon,n}(b) = 0$ ,  $U_{\varepsilon,n}(-0) = U_{\varepsilon,n}(+0)$  and

$$\beta_{\varepsilon,n} = (kU'_{\varepsilon,n})(-0) - \varepsilon(\varkappa U'_{\varepsilon,n})(+0) = O(\varepsilon^{n+1}), \quad \varepsilon \to 0.$$
 (54)

Note that  $U_{\varepsilon,n}$  doesn't belong to the domain of  $A_{\varepsilon}$  since  $\beta_{\varepsilon,n}$  is different from zero in the general case. Set  $\phi(x) = x(\frac{x}{a} - 1)$  for  $x \in (a,0)$  and  $\phi(x) = 0$  elsewhere. Then  $V_{\varepsilon,n} = U_{\varepsilon,n} + \beta_{\varepsilon,n}\phi$  belongs to  $\mathcal{D}(A_{\varepsilon})$  and a simple computation gives  $\|A_{\varepsilon}V_{\varepsilon,n} - \Lambda_{\varepsilon,n}V_{\varepsilon,n}\|_{\varepsilon} \leq c_n\varepsilon^{n+3/2}$ . Hence  $(\Lambda_{\varepsilon,n}, V_{\varepsilon,n}/\|V_{\varepsilon,n}\|_{\varepsilon})$  is a quasimode of operator  $A_{\varepsilon}$  with accuracy to  $c_n\varepsilon^{n+2}$  because  $\|V_{\varepsilon,n}\|_{\varepsilon} = O(\varepsilon^{-1/2})$ . According to the Vishik-Lyusternik Lemma there exists an eigenvalue  $\lambda^{\varepsilon}$  of  $A_{\varepsilon}$  such that  $|\lambda^{\varepsilon} - \Lambda_{\varepsilon,n}| \leq c_n\varepsilon^{n+2}$ , which establishes (51). Moreover, there exists an unique eigenvalue  $\lambda^{\varepsilon} = \lambda_j^{\varepsilon}$  with such asymptotics by Theorem 2. Next, for a certain d > 0 segment  $[\Lambda_{\varepsilon,n} - d\varepsilon, \Lambda_{\varepsilon,n} + d\varepsilon]$  contains one and only one eigenvalue of  $A_{\varepsilon}$ . Repeated application of Lemma 1 enables us to write  $\|\|u_{\varepsilon}\|_{\varepsilon}^{-1} \cdot u_{\varepsilon} - \|V_{\varepsilon,n}\|_{\varepsilon}^{-1} \cdot V_{\varepsilon,n}\|_{\varepsilon} \leq 2c_n d^{-1}\varepsilon^{n+1}$ , where  $u_{\varepsilon} = u_{\varepsilon,j}$ . Hence, by (50)

$$\left\| u_{\varepsilon} - \frac{\|u_{\varepsilon}\|_{\varepsilon}}{\|V_{\varepsilon,n}\|_{\varepsilon}} V_{\varepsilon,n} \right\|_{\varepsilon} \le \frac{2c_n}{d} \|u_{\varepsilon}\|_{\varepsilon} \varepsilon^{n+1} \le C_n \varepsilon^{n+1/2}$$

and  $\vartheta_{\varepsilon} = \frac{\|u_{\varepsilon}\|_{\varepsilon}}{\|V_{\varepsilon,n}\|_{\varepsilon}}$  converges to 1 by Theorem 2.

Pair  $(\lambda^{\varepsilon}, u_{\varepsilon})$  satisfies identity  $\langle u_{\varepsilon}, \psi \rangle_{\varepsilon} = \lambda^{\varepsilon}(u_{\varepsilon}, \psi)_{\varepsilon}$  for all  $\psi \in H_0^1(a, b)$ . Similarly,  $\langle V_{\varepsilon,n}, \psi \rangle_{\varepsilon} = \Lambda_{\varepsilon,n}(V_{\varepsilon,n}, \psi)_{\varepsilon} + \alpha_{\varepsilon}(\psi)$ , where  $|\alpha_{\varepsilon}(\psi)| \leq c\varepsilon^{n+1/2} ||\psi||_{\mathcal{H}_{\varepsilon}}$ . The latter gives

$$||u_{\varepsilon} - \vartheta_{\varepsilon} V_{\varepsilon,n}||_{\mathcal{H}_{\varepsilon}} \leq \Lambda_{\varepsilon,n} ||u_{\varepsilon} - \vartheta_{\varepsilon} V_{\varepsilon,n}||_{\varepsilon} + |\lambda^{\varepsilon} - \Lambda_{\varepsilon,n}| ||u_{\varepsilon}||_{\varepsilon} + |\alpha_{\varepsilon} (u_{\varepsilon} - \vartheta_{\varepsilon} V_{\varepsilon,n})|$$

$$\leq 2\mu_{j} C_{n} \varepsilon^{n+3/2} + c_{n} ||u_{\varepsilon}|| \varepsilon^{n+3/2} + c \varepsilon^{n+1/2} ||u_{\varepsilon} - \vartheta_{\varepsilon} V_{\varepsilon,n}||_{\mathcal{H}_{\varepsilon}}$$

and consequently  $||u_{\varepsilon} - \vartheta_{\varepsilon} V_{\varepsilon,n}||_{\mathcal{H}_{\varepsilon}} \leq C_n \varepsilon^{n+3/2}$ . From this and (50) we thus get estimate (52).  $\square$ 

The same proof works for the rest part of the simple spectrum of A.

**Theorem 4.** If  $\mu_j \in \sigma(\hat{A}_2) \backslash \sigma(A_1)$ , then eigenfunction  $u_{\varepsilon,j}$  of (2)-(5) with eigenvalue  $\lambda_j^{\varepsilon}$  converges towards function

$$u(x) = \begin{cases} 0 & \text{for } x \in I_a, \\ z(x) & \text{for } x \in I_b \end{cases}$$

in  $H^1(a,b)$ , where z is an eigenfunction of the problem  $(\varkappa z')' + \mu \rho z = 0$  in  $I_b$ , z(0) = 0, z(a) = 0 with eigenvalue  $\mu_j$ . Moreover  $\lambda_j^{\varepsilon}$  and  $u_{\varepsilon,j}$  admit asymptotic expansions (25), (26) obtained in 3.1.2 for  $\mu = \mu_j$  with the estimates of remainder terms

$$\left| \varepsilon^{-1} \lambda_j^{\varepsilon} - (\mu_j + \varepsilon \nu_1 + \dots + \varepsilon^n \nu_n) \right| \le c_n \varepsilon^{n+1}, \qquad \| u_{\varepsilon,j} - \vartheta_{\varepsilon} U_{\varepsilon,n} \|_{H^1(a,b)} \le C_n \varepsilon^{n+1}.$$

Here  $\vartheta_{\varepsilon}$  is a normalizing multiplier that converges to a positive constant as  $\varepsilon \to 0$ .

4.2. **Double Spectrum.** We introduce the partial sums of (40), (41)

$$\Lambda_{\varepsilon,n}^{\pm} = \varepsilon(\mu_j \pm \varepsilon^{1/2}\omega + \varepsilon\nu_2^{\pm} + \dots + \varepsilon^{n/2}\nu_n^{\pm}), \tag{55}$$

$$U_{\varepsilon,n}^{\pm} = \begin{cases} \mp \varepsilon^{1/2} w + \varepsilon w_2^{\pm} + \dots + \varepsilon^{n/2} w_n^{\pm} & \text{for } x \in I_a \\ v + \varepsilon^{1/2} v_1^{\pm} + \dots + \varepsilon^{n/2} v_n^{\pm} & \text{for } x \in I_b \end{cases}$$
 (56)

with all coefficients constructed in Section 3.2 for certain double eigenvalue  $\mu = \mu_j = \mu_{j+1}$ . Set  $V_{\varepsilon,n}^{\pm} = U_{\varepsilon,n}^{\pm} + \beta_{\varepsilon,n}^{\pm} \phi$ , where  $\beta_{\varepsilon,n}^{-}$  and  $\beta_{\varepsilon,n}^{+}$  are residuals in condition (4) for  $U_{\varepsilon,n}^{-}$  and  $U_{\varepsilon,n}^{+}$  respectively defined similarly as in (54). Moreover,  $\beta_{\varepsilon,n}^{\pm} = O(\varepsilon^{(n+1)/2})$  as  $\varepsilon \to 0$ .

Analysis similar to that in the proof of Theorem 3 leads to the following result.

**Proposition 3.** The pairs  $(\Lambda_{\varepsilon,n}^-, V_{\varepsilon,n}^-/\|V_{\varepsilon,n}^-\|_{\varepsilon})$  and  $(\Lambda_{\varepsilon,n}^+, V_{\varepsilon,n}^+/\|V_{\varepsilon,n}^+\|_{\varepsilon})$  are quasimodes of operator  $A_{\varepsilon}$  with accuracy to  $c_n \varepsilon^{n/2}$ .

**Proposition 4.** There exist two closely adjacent eigenvalues  $\lambda_{\varepsilon}^{-}$  and  $\lambda_{\varepsilon}^{+}$  of (2)-(5) with the asymptotics

$$\frac{\lambda_{\varepsilon}^{\pm}}{\varepsilon} = \mu_j \pm \sqrt{\varepsilon}\omega + \varepsilon\nu_2^{\pm} + \dots + \varepsilon^{n/2}\nu_n^{\pm} + O(\varepsilon^{(n+1)/2}), \tag{57}$$

where  $\mu_j$  is a double eigenvalue of operator  $\mathcal{A}$  and  $\omega$ ,  $\nu_k^{\pm}$  were defined in Sec. 3.2.

*Proof.* From Proposition 3 and the Vishik-Lyusternik Lemma it follows that there exists at least one eigenvalue of  $A_{\varepsilon}$  in each  $\varepsilon^{n/2}$ -vicinity of  $\Lambda_{\varepsilon,n}^-$  and  $\Lambda_{\varepsilon,n}^+$ . Moreover,  $|\lambda_{\varepsilon}^{\pm} - \Lambda_{\varepsilon,n}^{\pm}| \leq c_n \varepsilon^{n/2}$ . Evidently, eigenvalues  $\lambda_{\varepsilon}^-$ ,  $\lambda_{\varepsilon}^+$  are different, because  $\Lambda_{\varepsilon,n}^+ - \Lambda_{\varepsilon,n}^- \geq \omega \varepsilon^{3/2}$  and  $\varepsilon^{n/2}$ -vicinities of  $\Lambda_{\varepsilon,n}^-$  and  $\Lambda_{\varepsilon,n}^+$  don't intersect for n > 3 and sufficient small  $\varepsilon$ . In fact,  $|\lambda_{\varepsilon}^+ - \lambda_{\varepsilon}^-| \geq c\varepsilon^{3/2}$  for certain positive c. We conclude from  $|\lambda_{\varepsilon}^{\pm} - \Lambda_{\varepsilon,n+3}^{\pm}| \leq c_{n+3}\varepsilon^{(n+3)/2}$  that

$$\left| \frac{\lambda_{\varepsilon}^{\pm}}{\varepsilon} - (\mu_j \pm \sqrt{\varepsilon}\omega + \dots + \varepsilon^{\frac{n}{2}}\nu_n^{\pm}) \right| \le c_{n+3}\varepsilon^{\frac{n+1}{2}} + \sum_{k=1}^3 \varepsilon^{\frac{n+k}{2}} |\nu_{n+k}^{\pm}| \le C_n\varepsilon^{\frac{n+1}{2}},$$

which establishes (57).

We consider two planes in  $L_2(a,b)$ . Let  $\pi$  be the root subspace that corresponds to double eigenvalue  $\mu_i$  and  $\pi(\varepsilon)$  be the linear span of two eigenfunctions  $u_{\varepsilon}^-$  and  $u_{\varepsilon}^+$  that correspond to eigenvalues  $\lambda_{\varepsilon}^-$  and  $\lambda_{\varepsilon}^+$ . These eigenfunctions as above are normalized by (21).

**Theorem 5.** The root subspace  $\pi$  is the limit position of plane  $\pi(\varepsilon)$  as  $\varepsilon \to 0$ , namely  $||P_{\pi(\varepsilon)} - P_{\pi}|| \to 0$ , where  $P_{\pi(\varepsilon)}$  and  $P_{\pi}$  are the orthogonal projectors onto planes  $\pi(\varepsilon)$  and  $\pi$  respectively.

*Proof.* Nevertheless both eigenfunction  $u_{\varepsilon}^-$  and  $u_{\varepsilon}^+$  converge to the same limit being the eigenfunction of  $\mathcal{A}$  with eigenvalue  $\mu_j$ , the  $\pi_{\varepsilon}$  has regular asymptotic behaviour as  $\varepsilon \to 0$ . We choose new  $L_2(R,(a,b))$ -orthogonal basis in  $\pi(\varepsilon)$ :  $f_{\varepsilon} = \frac{1}{2}(u_{\varepsilon}^+ + u_{\varepsilon}^-)$ ,  $g_{\varepsilon} = \frac{1}{2\omega\sqrt{\varepsilon}}(u_{\varepsilon}^+ - u_{\varepsilon}^-)$ .

By Theorem 2 the first vector  $f_{\varepsilon}$  converges in  $L_2$  towards eigenfunction  $U \in \pi$  given by (36). Next, function  $g_{\varepsilon}$  solves the problem

$$\begin{cases} (kg_{\varepsilon}')' + \frac{\lambda_{\varepsilon}^{+}}{\varepsilon} rg_{\varepsilon} = \frac{\lambda_{\varepsilon}^{-} - \lambda_{\varepsilon}^{+}}{2\omega\varepsilon\sqrt{\varepsilon}} ru_{\varepsilon}^{-} & \text{in } I_{a}, \\ g_{\varepsilon}(a) = 0, \quad g_{\varepsilon}(b) = 0, \quad g_{\varepsilon}(-0) = g_{\varepsilon}(+0), \quad (kg_{\varepsilon}')(-0) = \varepsilon(\varkappa g_{\varepsilon}')(+0). \end{cases}$$

Since  $\varepsilon^{-1}\lambda_{\varepsilon}^{+} \to \mu_{j}$ ,  $\varepsilon^{-3/2}(\lambda_{\varepsilon}^{+} - \lambda_{\varepsilon}^{-}) \to 2\omega$  by (57) and the right-hand side is orthogonal to the eigenfunction  $u_{\varepsilon}^{+}$  in  $\mathcal{L}_{\varepsilon}$ , one obtains that norms  $\|g_{\varepsilon}\|_{H^{2}(a,0)}$  and  $\|g_{\varepsilon}\|_{H^{2}(0,b)}$  are bounded as  $\varepsilon \to 0$ . Taking into account Corollary 1 we can assert that each converging subsequence  $g_{\varepsilon'}$  converges as  $\varepsilon \to 0$  towards a solution of the problem

$$\begin{cases} (kg')' + \mu_j \, rg = 0 & \text{in } I_a, \\ g(a) = 0, \quad g(b) = 0, \quad g(-0) = g(+0), \quad g'(-0) = 0, \end{cases}$$

because  $u_{\varepsilon}^-$  converges to eigenfunction U, which equals v in  $I_b$  and vanishes in  $I_a$ . Hence, all partial limits of the second basis vector  $g_{\varepsilon}$  have to be the adjoined vectors corresponding to the eigenvalue  $\mu_j$ . In fact, by orthogonality of  $f_{\varepsilon}$  and  $g_{\varepsilon}$  these limits belong to the line  $\{\alpha U_* \mid \alpha \in \mathbb{R}\} \subset \pi$ , which is orthogonal to U (see (36) for definition of  $U_*$ ).

Indeed, in previous statements  $\lambda_{\varepsilon}^{-} = \lambda_{j}^{\varepsilon}$ ,  $\lambda_{\varepsilon}^{+} = \lambda_{j+1}^{\varepsilon}$  and  $u_{\varepsilon}^{-} = u_{\varepsilon,j}$ ,  $u_{\varepsilon}^{+} = u_{\varepsilon,j+1}$ , by Theorem 2. Next theorem summarizes all information on bifurcation of the double spectrum.

**Theorem 6.** Let  $\mu_j \in \sigma(A_1) \cap \sigma(A_2)$  be a double eigenvalue with eigenfunction U and adjoined function  $U_*$  given by (36),  $\mu_j = \mu_{j+1}$ . Then both eigenfunction  $u_{\varepsilon,j}$  and  $u_{\varepsilon,j+1}$  converge to the same eigenfunction U and the difference  $\frac{1}{\sqrt{\varepsilon}}(u_{\varepsilon,j+1} - u_{\varepsilon,j})$  converges to adjoined function  $\gamma U_*$  for certain  $\gamma \neq 0$ . Besides,  $\lambda_{\varepsilon}^- = \lambda_j^{\varepsilon}$ ,  $\lambda_{\varepsilon}^+ = \lambda_{j+1}^{\varepsilon}$  and  $u_{\varepsilon,j}$ ,  $u_{\varepsilon,j+1}$  admit asymptotic expansions (40), (41) derived in Section 3.2 for  $\mu = \mu_j$ . The estimates of remainder terms hold

$$\left|\varepsilon^{-1}\lambda_{\varepsilon}^{\pm} - \left(\mu_{i} \pm \sqrt{\varepsilon}\omega + \varepsilon\nu_{2}^{\pm} + \dots + \varepsilon^{n/2}\nu_{n}^{\pm}\right)\right| \le c_{n}^{\pm}\varepsilon^{(n+1)/2},\tag{58}$$

$$||u_{\varepsilon,j} - \vartheta_{\varepsilon}^{-} U_{\varepsilon,n}^{-}||_{H^{1}(a,b)} \le C_{n}^{-} \varepsilon^{\frac{n+1}{2}}, \quad ||u_{\varepsilon,j+1} - \vartheta_{\varepsilon}^{+} U_{\varepsilon,n}^{+}||_{H^{1}(a,b)} \le C_{n}^{+} \varepsilon^{\frac{n+1}{2}}, \tag{59}$$

where  $\vartheta_{\varepsilon}^{\pm}$  are normalizing multipliers with strictly positive limit as  $\varepsilon \to 0$ .

*Proof.* It remains to prove estimates (59). From (58) and Theorem 2 it may be concluded that for certain d>0 and  $n\geq 2$  interval  $[\Lambda_{\varepsilon,n}^--d\varepsilon^2,\Lambda_{\varepsilon,n}^-+d\varepsilon^2]$  contains eigenvalue  $\lambda_j^\varepsilon$  only. In view of Prop. 3 and the Vishik-Lyusternik Lemma, we have

$$\left\| u_{\varepsilon,j} - \frac{\|u_{\varepsilon,j}\|_{\varepsilon}}{\|V_{\varepsilon,n}^{-}\|_{\varepsilon}} V_{\varepsilon,n}^{-} \right\|_{\varepsilon} \le \frac{2c_n}{d\varepsilon^2} \|u_{\varepsilon}\|_{\varepsilon} \varepsilon^{n/2} \le C_n \varepsilon^{\frac{n-5}{2}}.$$

As in the proof of Theorem 3 we can obtain  $||u_{\varepsilon,j} - \vartheta_{\varepsilon}^{-} U_{\varepsilon,n}^{-}||_{H^{1}(a,b)} \leq C_{n} \varepsilon^{\frac{n-4}{2}}$ . Since all the coefficients of sum  $U_{\varepsilon,n}^{-}$  are bounded in  $H^{1}(a,b)$ , the first estimate (59) follows from the last inequality with n replaced by n+5. The same proof works for  $u_{\varepsilon,j+1}$ .

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