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AN ASYMPTOTIC THEORY FOR A CLASS OF INITIAL-BOUNDARY VALUE PROBLEMS FOR WEAKLY NONLINEAR WAVE EQUATIONS WITH AN APPLICATION TO A MODEL OF THE GALLOPING OSCILLATIONS OF OVERHEAD TRANSMISSION LINES*

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Abstract. The aim of this paper is to contribute to the foundation of the asymptotic methods for initial-boundary value problems and initial value problems for weakly nonlinear hyperbolic partial differential equations of order two. In this paper an asymptotic theory for a class of initial-boundary value problems for weakly nonlinear wave equations is presented. The theory implies the well-posedness of the problem in the classical sense and the validity of formal approximations on long timescales.

As an application of the theory, an initial-boundary value problem for a Rayleigh wave equation is studied in detail using a two-timescale perturbation method. From an aeroelastic analysis, it is shown that this initial-boundary value problem may be regarded as a model describing the growth of wind-induced oscillations of overhead transmission lines.

Key words. nonlinearly perturbed wave equations, well-posedness, asymptotic theory, two-timescale perturbation method

AMS(MOS) subject classifications. 35C20, 35L20

1. Introduction.

In this paper an asymptotic theory is presented for the following initial-boundary value problem for a nonlinearly perturbed wave equation:

$$u_{tt}-u_{xx}+\varepsilon f(x,t,u,u_t,u_x;\varepsilon)=0, \qquad 0<x<\pi, \quad t>0, \tag{1.1}$$

$$u(x,0;\varepsilon)=u_0(x;\varepsilon) \quad \text{and} \quad u_t(x,0;\varepsilon)=u_1(x;\varepsilon), \qquad 0<x<\pi, \tag{1.2}$$

$$u(0,t;\varepsilon)=u(\pi,t;\varepsilon)=0, \qquad t\geqq 0, \tag{1.3}$$

with $0<|\varepsilon|\ll 1$ and where the nonlinearity f and the initial values u_0 and u_1 have to satisfy certain smoothness properties, which are mentioned in § 2. The asymptotic theory implies the well-posedness (in the classical sense) of the initial-boundary value problem (1.1)-(1.3) and the asymptotic validity of formal approximations. In this paper formal approximations are defined to be functions that satisfy the differential equation and the initial values up to some order depending on the small parameter ε.

In [11] a similar asymptotic theory has been developed for an initial-boundary value problem for the weakly semilinear telegraph equation

$$u_{tt}-u_{xx}+u+\varepsilon f(x,t,u;\varepsilon)=0, \qquad 0<x<\pi, \quad t>0,$$

subject to the initial and boundary conditions (1.2) and (1.3). The well-posedness of that problem and the asymptotic validity of formal approximations could be established on a timescale of order $|\varepsilon|^{-1/2}$. For the initial-boundary value problem (1.1)-(1.3) it will be shown that a timescale of order $|\varepsilon|^{-1}$ can be obtained.

The asymptotic theory in [11] and the asymptotic theory presented in this paper can be regarded as an extension of the asymptotic theory for ordinary differential

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equations as for instance described in [1], [2], [8], [19]. In a number of papers (for instance, in [5], [6], [13], [16], [17]), it is suggested or assumed that an asymptotic theory for the validity of formal approximations of the solutions of initial-boundary value problems such as (1.1)-(1.3) is available. In [5], [15] it is taken for granted that in [8] a justification is given of a perturbation method introduced in [4]. An important part of the justification, namely an estimate of the difference between the exact solution and the formal approximation, is not given in [8]. Furthermore, the timescale on which the results might be valid is not specified in [8]. Some authors, (for instance [3], [8], [15]) have noticed that these validity proofs were absent or far from complete. Only recently have some asymptotic validity proofs been given in the literature. For instance in [3] a rather successful approach has been introduced to justify a number of formal perturbation methods. However, this approach is incomplete because in [3] the presumption is made that on (sufficiently large) large timescales the initial value problems under consideration are well posed in some (not specified) sense. Some other asymptotic results have been obtained in [6], [14], [21] by rewriting (1.1)-(1.3) as an initial value problem for a system of infinitely many ordinary differential equations in a Hilbert or Sobolev space.

This paper, being an attempt to contribute to the foundations of the asymptotic methods for weakly nonlinear hyperbolic partial differential equations, is organized as follows. In § 2 the well-posedness of the problem is investigated and established on a timescale of order $|\varepsilon|^{-1}$ and in § 3 the asymptotic validity of formal approximations is studied. The asymptotic theory is applied in § 5 to the initial-boundary value problem (1.1)-(1.3) with $f(x, t, u, u_t, u_x; \varepsilon) \equiv -u_t + \frac{1}{3}u_t^3$. In the early seventies this initial-boundary value problem for the Rayleigh wave equation was postulated in [16] to describe full span galloping oscillations of overhead transmission lines. In § 4 it follows from an aeroelastic analysis that this initial-boundary value problem may indeed be regarded as a model which describes the growth of wind-induced oscillations of overhead transmission lines. Using a two-timescale perturbation method, as, for instance, that successfully used in [4], [6], [11], [12], [13], an asymptotic approximation of the solution of the aforementioned initial-boundary value problem will be constructed. Finally in § 6 some remarks are made on the asymptotic theory applied to initial and initial-boundary value problems for the weakly nonlinear wave equations. Furthermore, some of the results obtained in the literature are discussed.

2. The well-posedness of the problem. In this paper the following weakly nonlinear initial-boundary value problem for a (with respect to x and t) twice continuously differentiable function $u(x, t; \varepsilon)$ is considered:

$$u_{tt} - u_{xx} + \varepsilon F(u; \varepsilon) = 0, \qquad t > 0, \quad 0 < x < \pi, \tag{2.1}$$

$$u(x, 0; \varepsilon) = u_0(x; \varepsilon), \qquad 0 < x < \pi, \tag{2.2}$$

$$u_t(x, 0; \varepsilon) = u_1(x; \varepsilon), \qquad 0 < x < \pi, \tag{2.3}$$

$$u(0, t; \varepsilon) = u(\pi, t; \varepsilon) = 0, \qquad t \geqq 0, \tag{2.4}$$

where

$$F(u; \varepsilon)(x, t) \equiv f(x, t, u(x, t; \varepsilon), u_t(x, t; \varepsilon), u_x(x, t; \varepsilon); \varepsilon), \tag{2.5}$$

$0 < |\varepsilon| \leqq \varepsilon_0 \ll 1$, and where $f(x, t, u, p, q; \varepsilon)$, $u_0(x; \varepsilon)$ and $u_1(x; \varepsilon)$ satisfy

$$f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial p}, \frac{\partial f}{\partial q} \in C([0, \pi] \times [0, \infty) \times \mathbb{R}^3 \times [-\varepsilon_0, \varepsilon_0], \mathbb{R})$$

$$\text{with } F(u;\varepsilon)(0,t)=F(u;\varepsilon)(\pi,t)=0 \quad \text{for } t\geqq 0, \tag{2.6}$$

$$u_0, \frac{\partial u_0}{\partial x}, \frac{\partial^2 u_0}{\partial x^2} \in C([0,\pi]\times[-\varepsilon_0,\varepsilon_0],\mathbb{R})$$

$$\text{with } u_0(0;\varepsilon)=u_0(\pi;\varepsilon)=u_0''(0;\varepsilon)=u_0''(\pi;\varepsilon)=0, \text{ and} \tag{2.7}$$

$$u_1, \frac{\partial u_1}{\partial x} \in C([0,\pi]\times[-\varepsilon_0,\varepsilon_0],\mathbb{R}) \quad \text{with } u_1(0;\varepsilon)=u_1(\pi;\varepsilon)=0. \tag{2.8}$$

Furthermore, $f(x, t, u, p, q; \varepsilon)$ and its partial derivatives with respect to x, u, p, and q are assumed to be uniformly bounded for those values of t under consideration.

To prove existence and uniqueness in the classical sense of the solution of the initial-boundary value problem (2.1)-(2.4), an equivalent integral equation will be used. In order to derive this integral equation, the initial-boundary value problem is transformed into an initial value problem by extending the functions f, u_0, and u_1 in x to odd and 2π-periodic functions (see, for instance, [7, Chap. 5] or [22, Chap. 2]). The extensions of u, f, u_0, and u_1 are denoted, respectively, by u^*, f^*, u_0^*, and u_1^*. Then, assuming that the solution u^* of the initial value problem is twice continuously differentiable, an integral representation for the solution of the initial value problem is given by

$$\begin{aligned} u^*(x,t;\varepsilon) &= -\frac{\varepsilon}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} f^*(\xi,\tau,u^*(\xi,\tau;\varepsilon),u^*_\tau(\xi,\tau;\varepsilon),u^*_\xi(\xi,\tau;\varepsilon);\varepsilon)\,d\xi\,d\tau \\ &\quad +\frac{1}{2}u_0^*(x+t;\varepsilon)+\frac{1}{2}u_0^*(x-t;\varepsilon)+\frac{1}{2}\int_{x-t}^{x+t} u_1^*(\xi;\varepsilon)\,d\xi. \end{aligned} \tag{2.9}$$

Using reflection principles, we can rewrite (2.9) as an integral representation on the semi-infinite strip $0\leqq x\leqq \pi$, $0\leqq t<\infty$, and obtain

$$u(x,t;\varepsilon)=\frac{\varepsilon}{2}\int_0^t\int_0^\pi G(\xi,\tau;x,t)F(u;\varepsilon)(\xi,\tau)\,d\xi\,d\tau+u_l(x,t;\varepsilon), \tag{2.10}$$

where G and u_l are given by

$$\begin{aligned} G(\xi,\tau;x,t)=\sum_{k\in\mathbb{Z}}\{&H(t-\tau-\xi+2k\pi-x)H(t-\tau+\xi-2k\pi+x) \\ &-H(t-\tau+\xi+2k\pi-x)H(t-\tau-\xi-2k\pi+x)\} \end{aligned} \tag{2.11}$$

and

$$u_l(x,t;\varepsilon)=\frac{1}{2}\int_0^\pi \left\{u_0(\xi;\varepsilon)\frac{\partial G}{\partial \tau}(\xi,0;x,t)-u_1(\xi;\varepsilon)G(\xi,0;x,t)\right\}d\xi, \tag{2.12}$$

in which $H(a)$ is a function on $\mathbb{R}$ which is equal to one for $a>0$, $\frac{1}{2}$ for $a=0$, and zero otherwise. In (2.12) it is assumed that G is differentiated according to the rule

$$\frac{d}{d\tau}\{H(f(\tau))H(g(\tau))\}=\delta_0(f(\tau))\frac{df(\tau)}{d\tau}H(g(\tau))+H(f(\tau))\delta_0(g(\tau))\frac{dg(\tau)}{d\tau},$$

where δ_0 is the Dirac delta function. In fact, G as defined by (2.11) is the Green function for the differential operator $L=\partial^2/\partial t^2-\partial^2/\partial x^2$ and the boundary conditions (2.4). It is worth noting that the solution of the linear initial-boundary value problem (2.1)-(2.4) (that is with $F\equiv 0$) is given by $u_l(x,t;\varepsilon)$.

Some elementary calculations show that if $v(x,t;\varepsilon)$ is a twice continuously differentiable solution of the initial-boundary value problem (2.1)-(2.4), then $v(x,t;\varepsilon)$ is a solution of the integral equation (2.10). And if $w(x,t;\varepsilon)$ is a twice continuously differentiable solution of the integral equation (2.10), then it can easily be shown that

$w(x, t; \varepsilon)$ is a solution of the initial-boundary value problem (2.1)-(2.4). Hence, the integral equation (2.10) and the initial-boundary value problem (2.1)-(2.4) are equivalent if twice continuously differentiable solutions exist. Now it will be proved that a unique, twice continuously differentiable solution of the integral equation (2.10) exists on a region J_L of the (x, t)-plane. And so, a unique and twice continuously differentiable solution exists for the initial-boundary value problem (2.1)-(2.4) on J_L.

In order to prove existence and uniqueness in the classical sense of the solution of the nonlinear integral equation (2.10) a fixed point theorem will be used. Let J_L be given by

$$J_L = \{(x, t) \mid 0 \leqq x \leqq \pi, 0 \leqq t \leqq L|\varepsilon|^{-1}\}, \tag{2.13}$$

in which L is a sufficiently small, positive constant independent of ε. Let $C^2_M(J_L)$ be the space of all real-valued and twice continuously differentiable functions w on J_L with norm $\|\cdot\|_{J_L}$ defined by

$$\|w\|_{J_L} = \sum_{\substack{i,j=0 \\ i+j\leqq 2}}^{2} \max_{(x,t)\in J_L} \left| \frac{\partial^{i+j} w(x, t)}{\partial x^i \, \partial t^j} \right| \leqq M.$$

From the smoothness properties of u_0 and u_1 it follows that (for fixed u_0 and u_1) there exists a positive constant M_1 independent of ε such that

$$\|u_l\|_{J_L} \leqq \frac{1}{2} M_1, \tag{2.14}$$

and from the smoothness properties of $F(u; \varepsilon)(x, t)$ (as defined by (2.5) and (2.6)) it follows that there exist ε-independent constants M_2 and M_3 such that

$$\sum_{k=0}^{1} \left| \frac{d^k}{dx^k} F(v; \varepsilon)(x, t) \right| \leqq M_2, \tag{2.15}$$

$$\sum_{k=0}^{1} \left| \frac{d^k}{dx^k} (F(v; \varepsilon)(x, t) - F(w; \varepsilon)(x, t)) \right| \leqq M_3 \|v - w\|_{J_L}, \tag{2.16}$$

for all $(x, t) \in J_L$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and $v, w \in C^2_{M_1}(J_L)$. Now let the integral operator $T: C^2(J_L) \to C^2(J_L)$, which is related to the integral equation (2.10), be defined by

$$(Tw)(x, t) \equiv \frac{\varepsilon}{2} \int_0^t \int_0^\pi G(\xi, \tau; x, t) F(w; \varepsilon)(\xi, \tau) \, d\xi \, d\tau + u_l(x, t; \varepsilon), \tag{2.17}$$

where G, F, and u_l are given by (2.11), (2.5), and (2.12), respectively. According to Banach's fixed point theorem, the integral operator T has a unique fixed point in $C^2_{M_1}(J_L)$ into itself and that T is a contraction on $C^2_{M_1}(J_L)$. Banach's fixed point theorem

(i) $T: C^2_{M_1}(J_L) \to C^2_{M_1}(J_L)$, and

(ii) $\|Tv - Tw\|_{J_L} \leqq k\|v - w\|_{J_L}$ with $0 < k < 1$, for all $v, w \in C^2_{M_1}(J_L)$.

Now, it will be proved that the integral operator T satisfies these two conditions. It is not difficult to show that T maps $C^2_{M_1}(J_L)$ into the space of twice continuously differentiable functions on J_L. In order to prove that T maps $C^2_{M_1}(J_L)$ into itself, an estimate of the Green function $G(\xi, \tau; x, t)$ should be obtained for $0 \leqq \xi \leqq \pi$, $0 \leqq \tau \leqq t$ and fixed x and t. In Fig. 2.1 the characteristics from the point (x, t) and the reflected characteristics at the boundaries $\xi = 0$ and $\xi = \pi$ are drawn in the (ξ, τ)-plane. These (reflected) characteristics divide the region $V = \{(\xi, \tau) \mid 0 \leqq \xi \leqq \pi, \tau \geqq 0\}$ into a finite number of subregions. In each subregion $G(\xi, \tau; x, t)$ can be determined by evaluating (2.11). These values are given in Fig. 2.1. The following estimate of $G(\xi, \tau; x, t)$ can now be made for $0 \leqq \xi \leqq \pi$, $\tau \geqq 0$, and fixed x and t:

$$|G(\xi, \tau; x, t)| \leqq 1. \tag{2.18}$$

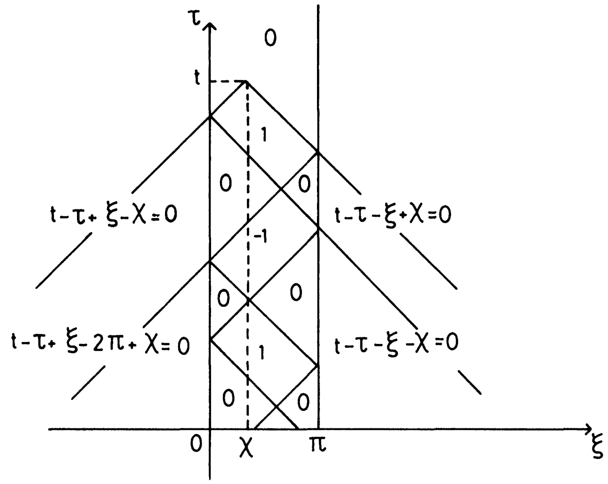

FIG. 2.1. *Subregions in V with the corresponding values of the Green's function* $G(\xi, \tau; x, t)$.

Using (2.13)-(2.15), (2.17), (2.18), we can make the following estimate:

$$\begin{aligned}\|Tv\|_{J_L} &\leqq \|Tv-u_l\|_{J_L}+\|u_l\|_{J_L}\\ &\leqq \sum_{\substack{i,j=0\\ i+j\leqq 2}}^{2} \max_{(x,t)\in J_L}\left|\frac{\partial^{i+j}}{\partial x^i\,\partial t^j}((Tv)(x,t)-u_l(x,t;\varepsilon))\right|+\frac{1}{2}M_1\\ &\leqq\left(\frac{\pi}{2}+5\right)M_2L+\varepsilon_0M_2+\frac{1}{2}M_1\end{aligned}$$

for all $v\in C^2_{M_1}(J_L)$. Now ε_0 has been assumed to be sufficiently small, and so there exists an ε-independent constant L such that $(\pi/2+5)M_2L+\varepsilon_0M_2\leqq\frac{1}{2}M_1$. Hence, $\|Tv\|_{J_L}\leqq M_1$ for all $v\in C^2_{M_1}(J_L)$. So, T maps $C^2_{M_1}$ into itself. Using (2.13), (2.16)-(2.18) we will show that T is a contraction on $C^2_{M_1}(J_L)$. Let v and $w\in C^2_{M_1}(J_L)$, then the following estimate can be obtained:

$$\|Tv-Tw\|_{J_L}\leqq\left(\left(\frac{\pi}{2}+5\right)M_3L+\varepsilon_0M_3\right)\|v-w\|_{J_L}.$$

It is obvious that there exists an ε-independent constant L such that $(\pi/2+5)M_3L+\varepsilon_0M_3\leqq k<1$. Since there always exists a constant L independent of ε such that $(\pi/2+5)M_2L+\varepsilon_0M_2<\frac{1}{2}M_1$ and $(\pi/2+5)M_3L+\varepsilon_0M_3\leqq k<1$, it follows that T maps $C^2_{M_1}(J_L)$ into itself and that T is a contraction on $C^2_{M_1}(J_L)$. Banach's fixed point theorem then implies that T has a unique fixed point in $C^2_{M_1}(J_L)$, that is, a unique and twice continuously differentiable function on J_L. Hence, the solution of the integral equation (2.10) is unique and twice continuously differentiable on J_L. And so, on J_L a unique and twice continuously differentiable solution exists for the initial-boundary value problem (2.1)-(2.4).

Next it will be shown that the solution of the initial-boundary value problem (2.1)-(2.4) depends continuously on the initial values. Let $u(x,t;\varepsilon)$ satisfy (2.1)-(2.4) and let $\bar{u}(x,t;\varepsilon)$ satisfy (2.1), (2.4), $\bar{u}(x,0;\varepsilon)=\bar{u}_0(x;\varepsilon)$, and $\bar{u}_t(x,0;\varepsilon)=\bar{u}_1(x;\varepsilon)$, where $\bar{u}_0$ and $\bar{u}_1$ satisfy (2.7) and (2.8). Let $\bar{u}_l$ be given by

$$\bar{u}_l(x,t;\varepsilon)=\frac{1}{2}\int_0^{\pi}\left\{\bar{u}_0(\xi;\varepsilon)\frac{\partial G}{\partial\tau}(\xi,0;x,t)-\bar{u}_1(\xi;\varepsilon)G(\xi,0;x,t)\right\}d\xi.$$

After subtracting the integral equations for u and $\bar{u}$, using (2.10), (2.13), (2.16), and (2.18), assuming u and $\bar{u} \in C^2_{M_1}(J_L)$, we obtain the estimate

$$\|u-\bar{u}\|_{J_L} \leqq \left(\left(\frac{\pi}{2}+5\right) M_3 L + \varepsilon_0 M_3\right) \|u-\bar{u}\|_{J_L} + \|u_l - \bar{u}_l\|_{J_L}$$

$$\leqq k\|u-\bar{u}\|_{J_L} + \|u_l-\bar{u}_l\|_{J_L} \quad \text{with } 0 \leqq k < 1.$$

This inequality implies $\|u-\bar{u}\|_{J_L} \leqq 1/(1-k)\|u_l-\bar{u}_l\|_{J_L}$ with $0 \leqq k < 1$.

So, small differences between the initial values generate small differences between the solutions u and $\bar{u}$ on J_L. In other words, the solution of the initial-boundary value problem depends continuously on the initial values. The following theorem on the well-posedness of the problem can now be formulated.

THEOREM 2.1. *Suppose that F, u_0, and u_1 satisfy the assumptions* (2.6)-(2.8). *Then for any ε satisfying $0 < |\varepsilon| \leqq \varepsilon_0 \ll 1$, the nonlinear initial-boundary value problem* (2.1)-(2.4) *and the equivalent nonlinear integral equation* (2.10) *have the same, unique and twice continuously differentiable solution for $0 \leqq x \leqq \pi$ and $0 \leqq t \leqq L|\varepsilon|^{-1}$, in which L is a sufficiently small, positive constant independent of ε. Furthermore, this unique solution depends continuously on the initial values.*

3. On the validity of formal approximations. Since the initial-boundary value problem (1.1)-(1.3) contains a small parameter ε, perturbation methods may be applied for the construction of approximations to the solution. In most perturbation methods for weakly nonlinear problems, a function is constructed that satisfies the differential equation and the initial conditions up to some order depending on the small parameter ε. Such a function is usually called a formal approximation. To show that this formal approximation is an asymptotic approximation (as $\varepsilon \to 0$) requires an additional analysis. Therefore, suppose that on J_L (given by (2.13)) a twice continuously differentiable function $v(x, t; \varepsilon)$ is constructed satisfying

$$v_{tt} - v_{xx} + \varepsilon F(v; \varepsilon) = |\varepsilon|^m c_1(x, t; \varepsilon), \qquad m > 1, \tag{3.1}$$

$$v(x, 0; \varepsilon) = u_0(x; \varepsilon) + |\varepsilon|^{m-1} c_2(x; \varepsilon) \equiv v_0(x; \varepsilon), \qquad 0 < x < \pi, \tag{3.2}$$

$$v_t(x, 0; \varepsilon) = u_1(x; \varepsilon) + |\varepsilon|^{m-1} c_3(x; \varepsilon) \equiv v_1(x; \varepsilon), \qquad 0 < x < \pi, \tag{3.3}$$

$$v(0, t; \varepsilon) = v(\pi, t; \varepsilon) = 0, \qquad 0 \leqq t \leqq L|\varepsilon|^{-1}, \tag{3.4}$$

where ε, F, u_0, and u_1 satisfy (2.5)-(2.8) and where c_1, c_2, and c_3 satisfy

$$c_1, \frac{\partial c_1}{\partial x} \in C([0, \pi] \times [0, L|\varepsilon|^{-1}] \times [-\varepsilon_0, \varepsilon_0], \mathbb{R}) \tag{3.5}$$

with $c_1(0, t; \varepsilon) = c_1(\pi, t; \varepsilon) = 0$, for $0 \leqq t \leqq L|\varepsilon|^{-1}$,

$$c_2, \frac{\partial c_2}{\partial x}, \frac{\partial^2 c_2}{\partial x^2} \in C([0, \pi] \times [-\varepsilon_0, \varepsilon_0], \mathbb{R}) \tag{3.6}$$

with $c_2(0; \varepsilon) = c_2(\pi; \varepsilon) = c_2''(0; \varepsilon) = c_2''(\pi; \varepsilon) = 0$, and

$$c_3, \frac{\partial c_3}{\partial x} \in C([0, \pi] \times [-\varepsilon_0, \varepsilon_0], \mathbb{R}) \tag{3.7}$$

with $c_3(0; \varepsilon) = c_3(\pi; \varepsilon) = 0$. Furthermore, $c_1(x, t; \varepsilon)$ and its derivative with respect to x are supposed to be uniformly bounded for those values of t and ε under consideration. From Theorem 2.1 it follows that the initial-boundary value problem (3.1)-(3.4) has a unique, twice continuously differentiable solution on a timescale of $O(|\varepsilon|^{-1})$. This initial-boundary value problem can then be transformed into the equivalent integral equation

$$v(x,t;\varepsilon)=\frac{\varepsilon}{2}\int_0^t\int_0^\pi G(\xi,\tau;x,t)\tilde{F}(v;\varepsilon)(\xi,\tau)\,d\xi\,d\tau+v_l(x,t;\varepsilon), \tag{3.8}$$

where G is given by (2.11) and where $\tilde{F}$ and v_l, respectively, are given by

$$\tilde{F}(v;\varepsilon)(x,t)\equiv F(v;\varepsilon)(x,t)-|\varepsilon|^{m-1}c_1(x,t;\varepsilon)\quad\text{and}$$

$$v_l(x,t;\varepsilon)=\frac{1}{2}\int_0^\pi\left\{v_0(\xi;\varepsilon)\frac{\partial G}{\partial\tau}(\xi,0;x,t)-v_1(\xi;\varepsilon)G(\xi,0;x,t)\right\}d\xi.$$

Now, it will be shown that the formal approximation v is an asymptotic approximation (as $\varepsilon\to0$) of the solution of the initial-boundary value problem (2.1)-(2.4) if $m>1$; that is, it will be proved that

$$\|u-v\|_{J_L}=O(\delta(\varepsilon)),\quad\text{where }\lim_{\varepsilon\to0}\delta(\varepsilon)=0.$$

Moreover $\delta(\varepsilon)$ will be derived explicitly. This result implies that

$$\lim_{\varepsilon\to0}|u(x,t;\varepsilon)-v(x,t;\varepsilon)|=0\quad\text{for }(x,t)\in J_L.$$

Subtracting the integral equation (3.8) from the integral equation (2.10), supposing that v_l satisfies (2.14) and that $\tilde{F}$ satisfies (2.15) and (2.16), using (2.13), (2.16), (2.18), and the fact that $u, v\in C^2_{M_1}(J_L)$, we obtain the following estimate:

$$\begin{aligned}\|u-v\|_{J_L}&\leqq\left(\left(\frac{\pi}{2}+5\right)M_3L+\varepsilon_0M_3\right)\|u-v\|_{J_L}+\|c\|_{J_L}+\|u_l-v_l\|_{J_L}\\&\leqq k\|u-v\|_{J_L}+\|c\|_{J_L}+\|u_l-v_l\|_{J_L},\end{aligned}$$

with $0\leqq k<1$ and where c is given by

$$c(x,t;\varepsilon)=\frac{|\varepsilon|^m}{2}\int_0^t\int_0^\pi G(\xi,\tau;x,t)c_1(\xi,\tau;\varepsilon)\,d\xi\,d\tau,$$

and where u_l-v_l is given by

$$u_l(x,t;\varepsilon)-v_l(x,t;\varepsilon)=-\frac{|\varepsilon|^{m-1}}{2}\int_0^\pi\left\{c_2(\xi;\varepsilon)\frac{\partial G}{\partial\tau}(\xi,0;x,t)-c_3(\xi;\varepsilon)G(\xi,0;x,t)\right\}d\xi.$$

Hence,

$$\|u-v\|_{J_L}\leqq\frac{1}{1-k}\{\|c\|_{J_L}+\|u_l-v_l\|_{J_L}\}\quad\text{with }0\leqq k<1.$$

From the smoothness properties of c_1, c_2, and c_3 it follows that there exists a constant K independent of ε, such that

$$\|c\|_{J_L}\leqq\left(\left(\frac{\pi}{2}+5\right)KL+|\varepsilon|K\right)|\varepsilon|^{m-1}\quad\text{and}$$

$$\|u_l-v_l\|_{J_L}\leqq\left(\frac{\pi}{2}+11\right)K|\varepsilon|^{m-1}.$$

So,

$$\|u-v\|_{J_L}\leqq\frac{|\varepsilon|^{m-1}K}{1-k}\left\{\left(\frac{\pi}{2}+5\right)L+|\varepsilon|+\frac{\pi}{2}+11\right\}.$$

For $m>1$ this inequality implies the asymptotic validity (as $\varepsilon \to 0$) of the formal approximation v. The following theorem has now been established.

THEOREM 3.1. *Let the formal approximation v satisfy* (3.1)-(3.4), *where ε, F, u_0, and u_1 are given by* (2.5)-(2.8) *and where c_1, c_2, and c_3 satisfy* (3.5)-(3.7). *Then for $m>1$, the formal approximation v is an asymptotic approximation (as $\varepsilon \to 0$) of the solution u of the nonlinear initial-boundary value problem* (2.1)-(2.4). *The asymptotic approximation v is valid for those values of the independent variables x and t for which problem* (2.1)-(2.4) *has been proved well-posed. That is,*

$$\|u-v\|_{J_L}=O(|\varepsilon|^{m-1}), \text{ implying } |u(x,t;\varepsilon)-v(x,t;\varepsilon)|=O(|\varepsilon|^{m-1})$$

for $0 \leqq x \leqq \pi$ and $0 \leqq t \leqq L|\varepsilon|^{-1}$, in which L is a sufficiently small, positive constant independent of ε.

4. A simple model of the galloping oscillations of overhead transmission lines. In this section a simple model describing the galloping oscillations of overhead transmission lines will be derived. Galloping can be described as a low frequency, large amplitude phenomenon involving an almost purely vertical oscillation of single-conductor lines on which, for instance, ice has accreted. The frequencies involved are so low that the assumption can be made that the aerodynamic forces are as in steady flow. Another consequence of these low frequencies is that structural damping may be neglected. In severe cases galloping may give rise to conductor damage due to impact of conductor lines and due to flashover as a result of a phase-difference between conductor lines, for which the mutual distance has become too small. The usual conditions (see [20]) causing galloping are those of incipient icing in a stable atmospheric environment implying uniform (but not necessarily high velocity) airflows.

A symmetric circular conductor in a horizontal airflow cannot exhibit galloping because it cannot generate a force that lifts the conductor against gravity. On the other hand, a conductor on which ice has accreted may gallop if it adopts a suitable attitude to the wind. To describe this phenomenon, a right-handed coordinate system is set up where one of the endpoints of the conductor is the origin. Through this point three mutually perpendicular axes (the x-, y- and z-axis) are drawn, where the z-axis coincides with the direction of gravity. The three coordinate axes span the three coordinate planes in space, the (x, y)-, (x, z)- and (y, z)-planes. On each coordinate axis a unit vector is fixed: on the x-axis the vector $\mathbf{e}_x$, on the y-axis the vector $\mathbf{e}_y$, and on the z-axis the vector $\mathbf{e}_z$, which has a direction opposite to gravity. The coordinate axes are directed by these vectors, such that a right-handed coordinate system is obtained. The coordinates of the endpoints of the conductor are supposed to be $(0, 0, 0)$ and $(l, 0, 0)$, where l is the distance between the endpoints. To model galloping a cross section (perpendicular to the x-axis) of the conductor with ice ridge is considered. Assume that all cross-sectional shapes are identical and symmetric. Along the axis of symmetry of a cross section, a vector $\mathbf{e}_s$ is defined to be directing away from the ice ridge and starting in the centre of the cross section. In Fig. 4.1 the centre of the cross section is considered to be at $x=x_0$, $y=y_0$, and $z=z_0$ with $0<x_0<l \leqq l_c$, where l_c is the length of the conductor. Let $w(x_0, t)$ denote the z-coordinate of the centre of the cross section at $x=x_0$ and time t. Assume that every cross section perpendicular to the x-axis oscillates in the (y, z)-plane. Furthermore, assume that torsion of the conductor may be neglected. Let the static angle of attack α_s (assumed to be constant and identical for all cross sections) be the angle between $\mathbf{e}_s$ and the uniform airflow $\mathbf{v}_\infty$, that is, $\alpha_s := \angle(\mathbf{e}_s, \mathbf{v}_\infty)$ with $|\alpha_s| \leqq \pi$. In this uniform airflow with flow velocity $\mathbf{v}_\infty = v_\infty \mathbf{e}_y$ $(v_\infty > 0)$, the conductor may oscillate due to the lift force $L\mathbf{e}_L$ and the drag force $D\mathbf{e}_D$. It should be noted that the drag force $D\mathbf{e}_D$ has the direction of the virtual

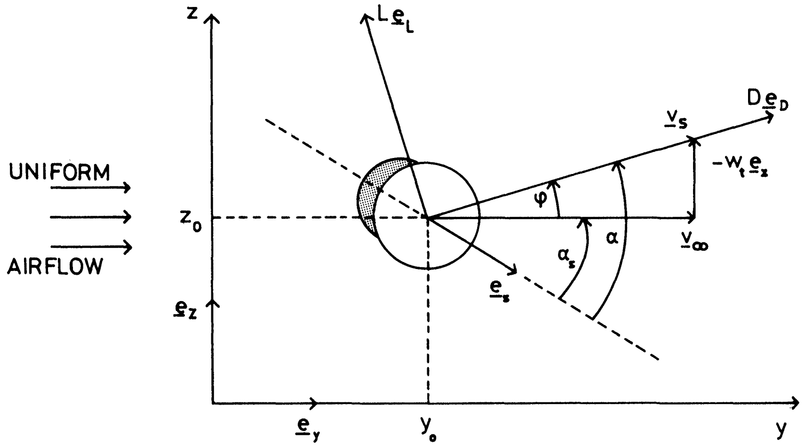

FIG. 4.1. *Cross section of the circular conductor with ice ridge.*

wind velocity $\mathbf{v}_s \equiv \mathbf{v}_\infty - (\partial w/\partial t)\mathbf{e}_z$ and that the lift force $L\mathbf{e}_L$ has a direction perpendicular to the virtual wind velocity $\mathbf{v}_s$ ($\mathbf{e}_L$ is chosen perpendicular and anti-clockwise to $\mathbf{e}_D$). In Fig. 4.1 the forces $L\mathbf{e}_L$ and $D\mathbf{e}_D$ acting on the cross section are given. Since galloping is an almost purely vertical oscillation, only vertical displacements of the conductor will be considered. Furthermore, the conductor is considered to be a one-dimensional continuum in which the only interaction between different parts is a tension T, which for all time t is assumed to be constant along the conductor. The equation describing the vertical motion of the conductor is given by

$$\rho_c A w_{tt} - TA(1+w_x^2)^{-3/2} w_{xx} = -\rho_c A g + D \sin \phi + L \cos \phi, \tag{4.1}$$

where D and L, respectively, are the magnitudes of the drag and lift force's acting on the conductor per unit length of the conductor, ρ_c is the mass-density of the conductor (including the small ice ridge), A is the constant cross-sectional area of the conductor (including the small ice ridge), ϕ is the angle between $\mathbf{v}_\infty$ and $\mathbf{v}_s$ (that is, $\phi := \angle(\mathbf{v}_\infty, \mathbf{v}_s)$ with $|\phi| \leqq \pi$), and g is the gravitational acceleration. The magnitudes D and L of the aerodynamic forces may be given by

$$D = \tfrac{1}{2}\rho_a \, d c_D(\alpha) v_s^2, \tag{4.2}$$

$$L = \tfrac{1}{2}\rho_a \, d c_L(\alpha) v_s^2, \tag{4.3}$$

where ρ_a is the density of the air, d is the diameter of the cross section of the circular part of the conductor, $v_s = |\mathbf{v}_s|$, α is the angle between $\mathbf{e}_s$ and $\mathbf{v}_s$ (that is, $\alpha := \angle(\mathbf{e}_s, \mathbf{v}_s)$ with $|\alpha| \leqq \pi$), and $c_D(\alpha)$ and $c_L(\alpha)$ are the quasi-steady drag and lift coefficients, which may be obtained from wind-tunnel measurements. For a certain range of values of v_∞, some characteristic results from wind-tunnel experiments are given in Fig. 4.2 (see also [1], [18], [20]).

According to the den Hartog criterion [10] a two-dimensional section is aerodynamically unstable if

$$c_D(\alpha) + \frac{dc_L(\alpha)}{d\alpha} < 0.$$

From Fig. 4.2 it follows that this condition is likely to be satisfied for some interval in α with $\alpha_0 < \alpha < \alpha_2$, where α_0, respectively α_2, are determined by $c_D(\alpha) + (dc_L(\alpha)/d\alpha) = 0$. For these values of α the drag and lift coefficients are approximated by (see also [1])

$$c_D(\alpha) = c_{D0} \quad \text{and} \quad c_L(\alpha) = c_{L1}(\alpha - \alpha_1) + c_{L3}(\alpha - \alpha_1)^3, \tag{4.4}$$

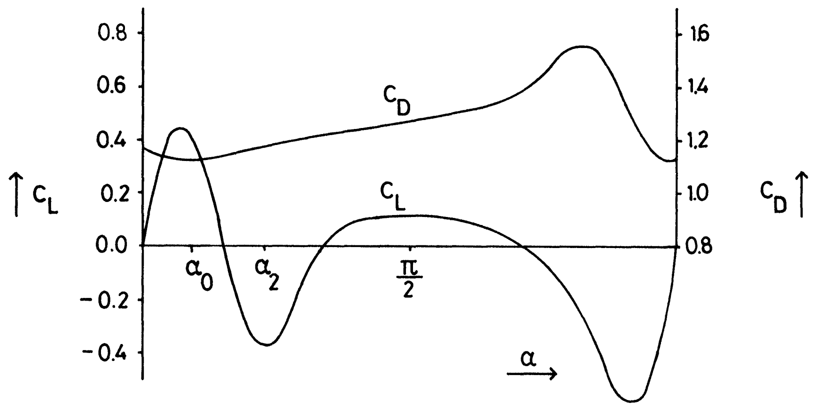

FIG. 4.2. *Typical variation of the drag and lift coefficients c_D and c_L with angle of attack for a symmetric profile with small icy nose.*

with $c_{D0} > 0$, $c_{L1} < 0$, $c_{L3} > 0$, $\alpha_0 < \alpha_1 < \alpha_2$, and $c_L(\alpha_1) = 0$. Since galloping is a low frequency oscillation, it is assumed that $|w_t| \ll v_\infty$ (so $|\phi| \ll 1$). The right-hand side of (4.1) can now be expanded near $\phi = 0$. It is also assumed that $|w_x| \ll 1$ and so, the left-hand side of (4.1) can be expanded near $w_x = 0$. Using the fact that $\phi = \arctan(-w_t/v_\infty)$ and neglecting terms of degree four and higher, we obtain, after some elementary calculations,

$$w_{tt} - c^2\left(1 - \frac{3}{2} w_x^2\right) w_{xx} = -g + \frac{\rho_a d v_\infty^2}{2\rho_c A}\left\{a_0 + \frac{a_1}{v_\infty} w_t + \frac{a_2}{v_\infty^2} w_t^2 + \frac{a_3}{v_\infty^3} w_t^3\right\} \tag{4.5}$$

where

$$\begin{aligned}
c &= (T\rho_c^{-1})^{1/2},\\
a_0 &= c_{L1}(\alpha_s - \alpha_1) + c_{L3}(\alpha_s - \alpha_1)^3,\\
a_1 &= -c_{D0} - c_{L1} - 3c_{L3}(\alpha_s - \alpha_1)^2,\\
a_2 &= (\tfrac{1}{2}c_{L1} + c_{L3})(\alpha_s - \alpha_1) + \tfrac{1}{2}c_{L3}(\alpha_s - \alpha_1)^3,\\
a_3 &= -\tfrac{1}{2}c_{D0} - \tfrac{1}{6}c_{L1} - c_{L3}(1 + (\alpha_s - \alpha_1)^2).
\end{aligned} \tag{4.6}$$

By applying the transformation $w(x, t) = \tilde{w}(x, t) + (\rho_c g/2T)x(x - l)$ and by using the dimensionless variables

$$\bar{w} = \frac{\pi c}{l v_\infty}\tilde{w}, \quad \bar{x} = \frac{\pi}{l}x, \quad \bar{t} = \frac{\pi c}{l}t,$$

(4.5) becomes

$$\begin{aligned}
&\bar{w}_{\bar{t}\bar{t}} - \left\{1 - \frac{3}{2}\left(\frac{v_\infty}{c}\right)^2\left(\bar{w}_{\bar{x}} + \frac{gl}{2\pi c v_\infty}(2\bar{x} - \pi)\right)^2\right\}\bar{w}_{\bar{x}\bar{x}}\\
&+\frac{3}{2}\left(\frac{v_\infty}{c}\right)^2\frac{gl}{2\pi c v_\infty}\left(\bar{w}_{\bar{x}} + \frac{gl}{2\pi c v_\infty}(2\bar{x} - \pi)\right)^2 = \frac{\rho_a d l}{2\pi\rho_c A}\left(\frac{v_\infty}{c}\right)\{a_0 + a_1\bar{w}_{\bar{t}} + a_2\bar{w}_{\bar{t}}^2 + a_3\bar{w}_{\bar{t}}^3\},
\end{aligned} \tag{4.7}$$

where the dimensionless constants a_0, a_1, a_2, and a_3 are given by (4.6).

Typical values of the physical quantities in a practical application are: $l=400$ m, $d=0.04$ m, $A=\pi(d/2)^2=4\pi.10^{-4}$ m^2, $\rho_c=4000$ kg/m^3, $\rho_a=1.25$ kg/m^3, $g=10$ m/s^2 and $v_\infty=10$ m/s. The tension T in the conductor is estimated by $\frac{1}{2}\rho_c g(l/2)^2 s_0^{-1}$, where s_0 (usually 2 or 3 percent of l) is the sag of the conductor. Let s_0 be 10 m, then $T=8.10^7$ kg/ms^2 and consequently $c=140$ m/s (c may be identified with the speed of propagation of transversal waves in the conductor). Then, it follows that

$$\frac{\rho_a dl}{2\pi\rho_c A}\approx\frac{5}{8},\quad \frac{gl}{2\pi c v_\infty}\approx\frac{1}{2},\quad \frac{v_\infty}{c}=\frac{1}{14}.$$

Putting $\tilde{\varepsilon}=v_\infty/c$ and assuming that the static angle of attack α_s is such that galloping may set in according to the instability criterion of den Hartog [10], that is, by assuming that $\alpha_s=\alpha_1+O(\tilde{\varepsilon})$, (4.7) becomes up to order $\tilde{\varepsilon}$

$$\bar{w}_{\bar{t}\bar{t}}-\bar{w}_{\bar{x}\bar{x}}=\tilde{\varepsilon}\frac{\rho_a dl}{2\pi\rho_c A}(a\bar{w}_{\bar{t}}-b\bar{w}_{\bar{t}}^3), \tag{4.8}$$

where $a=-c_{D0}-c_{L1}$ and $b=\frac{1}{2}c_{D0}+\frac{1}{6}c_{L1}+c_{L3}$. For the cross-sectional shape of the conductor with small ice ridge under consideration, the aerodynamic coefficients c_{D0}, c_{L1}, and c_{L3} may be determined from wind-tunnel measurements (as, for instance, given in Fig. 4.2). From Fig. 4.2 it follows that $c_{D0}>0$, $c_{L1}<0$, $|c_{L1}|>c_{D0}$, $c_{L3}>0$, $a>0$, and $b>0$. If we consider a conductor with fixed endpoints, the boundary conditions $\bar{w}(0,\bar{t})=\bar{w}(\pi,\bar{t})=0$ are obtained. By a simple change of scale, $u(\bar{x},\bar{t})=(3b/a)^{1/2}$ $\bar{w}(\bar{x},\bar{t})$, the model equation (4.8) can be simplified to a Rayleigh wave equation

$$u_{\bar{t}\bar{t}}-u_{\bar{x}\bar{x}}=\varepsilon(u_{\bar{t}}-\tfrac{1}{3}u_{\bar{t}}^3), \tag{4.9}$$

where

$$\varepsilon=\tilde{\varepsilon}a\frac{\rho_a dl}{2\pi\rho_c A}=-\frac{v_\infty}{c}\frac{c_{D0}+c_{L1}}{2\pi}\frac{\rho_a dl}{\rho_c A}$$

is a small, positive parameter. In the next section (4.9), subject to the boundary values $u(0,\bar{t})=u(\pi,\bar{t})=0$ and the initial values $u(\bar{x},0)=w_0(\bar{x})$ and $u_{\bar{t}}(\bar{x},0)=w_1(\bar{x})$, will be studied, where $w_0(\bar{x})$ and $w_1(\bar{x})$, respectively, can be regarded as the initial displacement and the initial velocity of the conductor in vertical direction.

It is worth noting that in the early seventies [16] an equation similar to (4.9) was postulated to describe the galloping oscillations of overhead transmission lines. In that paper it was assumed that $\varepsilon u_{\bar{t}}$ and $(-\varepsilon/3)u_{\bar{t}}^3$, respectively, represent a force that tends to increase and decrease the magnitude of the oscillation. In this section it has been shown that this simple model can be derived using aerodynamical arguments.

5. An asymptotic approximation of the solution of a Rayleigh wave equation. In this section the following initial-boundary value problem for a twice continuously differentiable function $u(x,t)$ will be considered:

$$u_{tt}-u_{xx}+\varepsilon(-u_t+\tfrac{1}{3}u_t^3)=0,\qquad 0<x<\pi,\qquad t>0, \tag{5.1}$$

$$u(x,0)=u_0(x)\equiv a_n\sin nx,\qquad 0<x<\pi, \tag{5.2}$$

$$u_t(x,0)=u_1(x)\equiv b_n\sin nx,\qquad 0<x<\pi, \tag{5.3}$$

$$u(0,t)=u(\pi,t)=0,\qquad t\geqq 0, \tag{5.4}$$

where a_n and b_n are constants, n is an integer, and $0<\varepsilon\ll 1$. From Theorem 2.1 it follows that this initial-boundary value problem is well posed on J_L (given by (2.13)). In [4] a similar initial-boundary value problem was considered with $n=1$, $a_n=2$, and

$b_n = 0$. However, in that paper the asymptotic validity of the formal approximation was not given. In this section, for arbitrary n, a_n, and b_n, an asymptotic approximation (as $\varepsilon \to 0$) of the solution of (5.1)-(5.4) will be constructed. In view of computational difficulties (as has been noted in [13]) whenever we assume an infinite series representation for the solution of the nonlinear initial-boundary value problem, we may alternatively investigate the problem in the characteristic coordinates $\sigma = x - t$ and $\xi = x + t$. In this approach the initial-boundary value problem (5.1)-(5.4) is replaced by an initial value problem. This replacement requires us to extend the dependent variable $u(x, t)$ as well as the initial values $u_0(x)$ and $u_1(x)$ in x to odd and 2π-periodic functions. For simplicity the extended functions will be denoted by the same symbols. In constructing an approximation of the solution $u(x, t) = \tilde{u}(\sigma, \xi)$ of this initial value problem, a two-timescale perturbation method will be used, since the straighforward perturbation expansion $\tilde{u}_0(\sigma, \xi) + \varepsilon\tilde{u}_1(\sigma, \xi) + \cdots$ causes secular terms. When we apply the two-timescale perturbation method, $u(x, t)$ should be a function of $\sigma = x - t$, $\xi = x + t$, and $\tau = \varepsilon t$. By putting $u(x, t) \equiv v(\sigma, \xi, \tau)$, the following initial value problem for v is obtained:

$$(5.5) \qquad -4v_{\sigma\xi} + 2\varepsilon(v_{\xi\tau} - v_{\sigma\tau}) + \varepsilon^2 v_{\tau\tau} + \varepsilon(v_\sigma - v_\xi - \varepsilon v_\tau + \tfrac{1}{3}(-v_\sigma + v_\xi + \varepsilon v_\tau)^3) = 0 \quad \text{for } -\infty < \sigma < \xi < \infty,\ \tau > 0,$$

$$(5.6) \qquad v(\sigma, \xi, \tau) = u_0(\sigma) = a_n \sin n\sigma, \quad \text{for } -\infty < \sigma = \xi < \infty,\ \tau = 0,$$

$$(5.7) \qquad -v_\sigma(\sigma, \xi, \tau) + v_\xi(\sigma, \xi, \tau) + \varepsilon v_\tau(\sigma, \xi, \tau) = u_1(\sigma) = b_n \sin n\sigma, \quad \text{for } -\infty < \sigma = \xi < \infty,\ \tau = 0.$$

Furthermore, it is assumed that v may be approximated by the formal perturbation expansion $v_0(\sigma, \xi, \tau) + \varepsilon v_1(\sigma, \xi, \tau) + \varepsilon^2 v_2(\sigma, \xi, \tau) + \cdots$. By substituting this approximation into (5.5)-(5.7), and after equating the coefficients of like powers in ε, it follows from the powers zero and one of ε, respectively, that v_0 should satisfy

$$(5.8) \qquad -4v_{0_{\sigma\xi}} = 0, \qquad -\infty < \sigma < \xi < \infty, \quad \tau > 0,$$

$$(5.9) \qquad v_0(\sigma, \xi, \tau) = u_0(\sigma) = a_n \sin n\sigma, \qquad -\infty < \sigma = \xi < \infty, \quad \tau = 0,$$

$$(5.10) \qquad -v_{0_\sigma}(\sigma, \xi, \tau) + v_{0_\xi}(\sigma, \xi, \tau) = u_1(\sigma) = b_n \sin n\sigma, \qquad -\infty < \sigma = \xi < \infty, \quad \tau = 0,$$

and that v_1 should satisfy

$$(5.11) \qquad -4v_{1_{\sigma\xi}} = 2v_{0_{\sigma\tau}} - 2v_{0_{\xi\tau}} - (v_{0_\sigma} - v_{0_\xi} + \tfrac{1}{3}(-v_{0_\sigma} + v_{0_\xi})^3) \quad \text{for } -\infty < \sigma < \xi < \infty, \quad \tau > 0,$$

$$(5.12) \qquad v_1(\sigma, \xi, \tau) = 0, \qquad -\infty < \sigma = \xi < \infty, \quad \tau = 0,$$

$$(5.13) \qquad -v_{1_\sigma}(\sigma, \xi, \tau) + v_{1_\xi}(\sigma, \xi, \tau) = -v_{0_\tau}(\sigma, \xi, \tau), \qquad -\infty < \sigma = \xi < \infty, \quad \tau = 0.$$

In further analysis v_0 and v_1 will be determined, and it will be shown that on J_L $\bar{u}(x, t) \equiv v_0(x - t, x + t, \varepsilon t) + \varepsilon v_1(x - t, x + t, \varepsilon t)$ is an order ε asymptotic approximation (as $\varepsilon \to 0$) of the solution $u(x, t)$ of the initial-boundary value problem (5.1)-(5.4).

The general solution of the partial differential equation (5.8) is given by $v_0(\sigma, \xi, \tau) = f_0(\sigma, \tau) + g_0(\xi, \tau)$. The initial values (5.9) and (5.10) imply that f_0 and g_0 have to satisfy $f_0(\sigma, 0) + g_0(\sigma, 0) = u_0(\sigma)$ and $-f_0'(\sigma, 0) + g_0'(\sigma, 0) = u_1(\sigma)$, where the prime denotes differentiation with respect to the first argument. From the odd and 2π-periodic extension of the dependent variable of problem (5.1)-(5.4), it follows that f_0 and g_0 also have to satisfy $g_0(\sigma, \tau) = -f_0(-\sigma, \tau)$ and $f_0(\sigma, \tau) = f_0(\sigma + 2\pi, \tau)$ for $-\infty < \sigma < \infty$ and $\tau \geqq 0$. The undetermined behaviour of f_0 with respect to τ will be used to avoid secular terms in v_1. From the well-posedness theorem it follows that u, u_t, and u_x are

$O(1)$ on J_L. So, v and its first derivatives must remain $O(1)$ on $-\infty<x<\infty$ and $0\leqq t\leqq L|\varepsilon|^{-1}$. Furthermore, it should be noted that the equations for v_0 and v_1 have been derived under the assumption that v_0, v_1, and their derivatives up to order two are $O(1)$. These boundedness conditions on v_0 and v_1 determine the behaviour of f_0 with respect to τ. From (5.11)-(5.13) v_{1_σ} and v_{1_ξ} may be obtained easily. For instance,

(5.14)

$$\begin{aligned}-4v_{1_\sigma}(\sigma,\xi,\tau)=&-4v_{1_\sigma}(\sigma,\sigma,\tau)+(\xi-\sigma)\left(2f_{0_{\sigma\tau}}(\sigma,\tau)-f_{0_\sigma}(\sigma,\tau)+\frac{1}{3}f_{0_\sigma}^3(\sigma,\tau)\right)\\&+f_{0_\sigma}(\sigma,\tau)\int_\sigma^\xi g_{0_\theta}^2(\theta,\tau)\,d\theta\\&+\int_\sigma^\xi\left\{-2g_{0_{\theta\tau}}(\theta,\tau)+g_{0_\theta}(\theta,\tau)-f_{0_\sigma}^2(\sigma,\tau)g_{0_\theta}(\theta,\tau)-\frac{1}{3}g_{0_\theta}^3(\theta,\tau)\right\}d\theta\\&+h(\sigma,\tau),\end{aligned}$$

where h will be determined later. Since the first integral in (5.14) contains a nonnegative and 2π-periodic integrand, it follows that this integral will grow with the length $\xi-\sigma$ of the integration interval. It turns out that this integral can be written in a part which is $O(1)$ for all values of σ and ξ and in a part which is linear in $\xi-\sigma$:

$$\int_\sigma^\xi g_{0_\theta}^2(\theta,\tau)\,d\theta=\int_\sigma^\xi\left\{g_{0_\theta}^2(\theta,\tau)-\frac{1}{2\pi}\int_0^{2\pi}g_{0_\psi}^2(\psi,\tau)\,d\psi\right\}d\theta+\frac{\xi-\sigma}{2\pi}\int_0^{2\pi}g_{0_\psi}^2(\psi,\tau)\,d\psi.$$

Noting that $\xi-\sigma=2t$, it follows that $\xi-\sigma$ is of $O(|\varepsilon|^{-1})$ on a timescale of $O(|\varepsilon|^{-1})$. So, v_{1_σ} will be of $O(|\varepsilon|^{-1})$ unless f_0 and g_0 are such that in (5.14) the terms of $O(|\varepsilon|^{-1})$ (that is, terms linear in $\xi-\sigma$) disappear. It turns out that both v_{1_σ} and v_{1_ξ} are $O(1)$ on a timescale of $O(|\varepsilon|^{-1})$ if f_0 and g_0 satisfy the following two conditions:

$$2f_{0_{\sigma\tau}}-f_{0_\sigma}+\frac{1}{3}f_{0_\sigma}^3+f_{0_\sigma}\frac{1}{2\pi}\int_0^{2\pi}g_{0_\theta}^2(\theta,\tau)\,d\theta=0,$$

$$-2g_{0_{\xi\tau}}+g_{0_\xi}-\frac{1}{3}g_{0_\xi}^3-g_{0_\xi}\frac{1}{2\pi}\int_0^{2\pi}f_{0_\theta}^2(\theta,\tau)\,d\theta=0.$$

From $g_0(\theta,\tau)=-f_0(-\theta,\tau)$, it follows that these two conditions are equivalent. So, v_{1_σ} and v_{1_ξ} are both $O(1)$ on a timescale of $O(|\varepsilon|^{-1})$ if f_0 satisfies

$$2f_{0_{\sigma\tau}}-f_{0_\sigma}+\frac{1}{3}f_{0_\sigma}^3+f_{0_\sigma}\frac{1}{2\pi}\int_0^{2\pi}f_{0_\theta}^2(\theta,\tau)\,d\theta=0. \tag{5.15}$$

In [4] an equation similar to (5.15) has been solved. If the method introduced in [4] is applied to (5.15), we obtain after some calculations $f_0(\sigma,\tau)$, and so $v_0(\sigma,\xi,\tau)=f_0(\sigma,\tau)-f_0(-\xi,\tau)$. It turns out that f_0 and v_0 are given by

$$f_0(\sigma,\tau)=\frac{\lambda(\tau)}{n\phi^{1/2}(\tau)}\arcsin\left[\left[\frac{c_n\phi(\tau)}{1+c_n\phi(\tau)}\right]^{1/2}\sin(\alpha+n\sigma)\right]+k(\tau), \tag{5.10}$$

$$\begin{aligned}v_0(\sigma,\xi,\tau)=\frac{\lambda(\tau)}{n\phi^{1/2}(\tau)}&\left\{\arcsin\left[\left[\frac{c_n\phi(\tau)}{1+c_n\phi(\tau)}\right]^{1/2}\sin(\alpha+n\sigma)\right]\right.\\&\left.-\arcsin\left[\left[\frac{c_n\phi(\tau)}{1+c_n\phi(\tau)}\right]^{1/2}\sin(\alpha-n\xi)\right]\right\},\end{aligned} \tag{5.17}$$

where $k(\tau)$ is an arbitrary function in τ with $k(0)=0$, $\sigma=x-t$, $\xi=x+t$, $\tau=\varepsilon t$, $c_n=n^2a_n^2+b_n^2$, α is given by $\cos\alpha=na_nc_n^{-1/2}$ and $\sin\alpha=b_nc_n^{-1/2}$, where $\lambda(\tau)$ and $\phi(\tau)$ are implicitly given by $\lambda(\tau)=4e^{\tau/2}m^{-3}(\tau)$ and $\phi(\tau)=(m(\tau)/c_n)(m(\tau)-2)$ with $m(\tau)$ determined by $m^8(\tau)-(8/7)m^7(\tau)=(2^6c_n/3)(e^\tau-1)+(3.2^8/7)$.

Now the linear initial value problem (5.11)-(5.13) can be solved, and it turns out that v_1 is given by

$$v_1(\sigma, \xi, \tau) = \frac{1}{4}(f_0(\sigma, \tau) - f_0(-\xi, \tau)) \int_\sigma^\xi \left\{ f_{0_\theta}^2(\theta, \tau) - \frac{1}{2\pi}\int_0^{2\pi} f_{0_\psi}^2(\psi, \tau)\, d\psi \right\} d\theta$$

$$(5.18) \qquad -\frac{1}{4}\int_\sigma^\xi \left\{ f_{0_\theta}^2(\theta, \tau) - \frac{1}{2\pi}\int_0^{2\pi} f_{0_\psi}^2(\psi, \tau)\, d\psi \right\} (f_0(\theta, 0) - f_0(-\theta, 0))\, d\theta$$

$$+ f_1(\sigma, \tau) + g_1(\xi, \tau),$$

where f_0 is given by (5.16) and where (for $\sigma = \xi$ and $\tau = 0$) $f_1 + g_1$ is determined by the initial values (5.12) and (5.13). The undetermined behaviour of f_1 and g_1 with respect to τ can be used to avoid secular terms in v_2. However, in this analysis v_2 will not be determined. For that reason it may be assumed that $f_1 = f_1(\sigma)$ and $g_1 = g_1(\xi)$, and then

$$f_1(\sigma) + g_1(\xi) = -\frac{1}{2}\int_\sigma^\xi (f_{0_\tau}(\theta, 0) + g_{0_\tau}(\theta, 0))\, d\theta$$

$$= \frac{b_n}{n^2}\left\{ \frac{(3c_n - 2^4)}{2^5}(\sin(n\xi) - \sin(n\sigma)) + \frac{(3n^2 a_n^2 - b_n^2)}{3^3.2^5}(\sin(3n\xi) - \sin(3n\sigma)) \right\}.$$

It can be shown from (5.17) and (5.18) that v_0, v_1 and their derivatives up to order two are of $O(1)$ on J_L. So, the assumptions under which the equations for v_0 and v_1 have been derived are justified. So far a function $v_0(\sigma, \xi, \tau) + \varepsilon v_1(\sigma, \xi, \tau) \equiv \bar{v}(\sigma, \xi, \tau) = \bar{v}(x - t, x + t, \varepsilon t) \equiv \bar{u}(x, t)$ has been constructed. It can easily be seen that $\bar{u}(x, t)$ satisfies (5.2) and (5.4) exactly, and (5.3) up to order ε^2 in the sense of Theorem 3.1. After rather lengthy but elementary calculations it can also be shown that $\bar{u}(x, t)$ satisfies (5.1) up to $\varepsilon^2 c_1(x, t; \varepsilon)$, where c_1, $\partial c_1/\partial x \in C([0, \pi] \times [0, L|\varepsilon|^{-1}] \times [-\varepsilon_0, \varepsilon_0], \mathbb{R})$ with $c_1(0, t; \varepsilon) = c_1(\pi, t; \varepsilon) = 0$ for $0 \leqq t \leqq L|\epsilon|^{-1}$. Furthermore, c_1 and $\partial c_1/\partial x$ are uniformly bounded in ε. Then it follows from Theorem 3.1 that $\bar{u}(x, t)$ is an order ε asymptotic approximation (as $\varepsilon \to 0$) of the solution of the initial-boundary value problem (5.1)-(5.4) for $(x, t) \in J_L$, that is $\|u - \bar{u}\|_{J_L} = O(\varepsilon)$. From this estimate the following estimate can be obtained:

$$\|u - v_0\|_{J_L} = \|u - \bar{u} + \bar{u} - v_0\|_{J_L} \leqq \|u - \bar{u}\|_{J_L} + \|\varepsilon v_1\|_{J_L} = O(\varepsilon).$$

Hence, $v_0(x - t, x + t, \varepsilon t)$ given by (5.17) is also an order ε asymptotic approximation (as $\varepsilon \to 0$) of the solution $u(x, t)$ of problem (5.1)-(5.4) for $0 \leqq x \leqq \pi$ and $0 \leqq t \leqq L\varepsilon^{-1}$, in which L is an ε-independent, positive constant.

6. Some general remarks. The asymptotic theory presented in this paper may be directly applied to initial value problems for weakly nonlinear wave equations. The well-posedness of these problems on the infinite domain $-\infty < x < \infty$ and the asymptotic validity of formal approximations may be established on a timescale of order $|\varepsilon|^{-1/2}$. This timescale follows from the integration over the characteristic triangle (with an area of $O(t^2)$) in the integral equation, which is equivalent to the initial value problem. In some special cases (for instance, if (5.1)-(5.3) is considered as an initial value problem on $-\infty < x < \infty$) a timescale of $O(|\varepsilon|^{-1})$ can be obtained. However, the question remains open if for general initial value problems (that is, problems like (2.1)-(2.3) on $-\infty < x < \infty$) the well-posedness in the classical sense can be established on a timescale of $O(|\varepsilon|^{-1})$. To obtain such a timescale most likely we will have to use a different function space, perhaps a suitable Sobolev space.

In [4], [13] formal approximations of the solutions of a number of initial value and initial-boundary value problems for weakly nonlinear wave equations were constructed. In those references the asymptotic validity of the formal approximations has not been investigated. However, the asymptotic theory presented in this paper can be used successfully to justify those results; that is, estimates of the differences between the exact solutions and the formal approximations can be given on ε-dependent timescales. It is also interesting to mention that only smoothness conditions are required (see (2.6)-(2.8)) and that no other assumptions are made about the nonlinear perturbation term F. Thus, the asymptotic theory presented in this paper is applicable to those initial-boundary value problems whose solutions, while being bounded at times of $O(|\varepsilon|^{-1})$, could eventually become unbounded. Such, for example, is the case for the initial-boundary value problem (2.1)-(2.4) with $F\equiv -u_t^3$ and $0<\varepsilon\ll 1$.

In § 3 the condition $m>1$ is introduced. It should be noted that this condition for the asymptotic validity of formal approximations on a timescale of $O(|\varepsilon|^{-1})$ is a sufficient, but not a necessary one as can be seen from § 5. The asymptotic approximation v_0 (which is valid on a timescale of $O(|\varepsilon|^{-1})$) satisfies the partial differential equation and the initial values up to order ε, that is $m=1$. It may be remarked that $u_l(x, t)=(a_n \cos nt+(b_n/n)\sin nt)\sin nx$, which is the solution of the linear initial-boundary value problem (5.1)-(5.4) (that is, (5.1) with $\varepsilon=0$), also satisfies the weakly nonlinear partial differential equation and the initial values up to order ε. In general u_l will not approximate the exact solution of the nonlinear initial-boundary value problem on a timescale of $O(|\varepsilon|^{-1})$. However, on a smaller timescale the asymptotic validity of u_l can easily be established; that is, it can be shown using the methods discussed in §§ 2 and 3 that $|u(x, t)-u_l(x, t)|\leqq|\varepsilon|Mt$, where M is a constant independent of ε. This inequality implies $u(x, t)=u_l(x, t)+O(|\varepsilon|^{1-\alpha})$ for $0\leqq x\leqq\pi$ and $0\leqq t\leqq L|\varepsilon|^{-\alpha}$ with $0\leqq\alpha<1$. From the asymptotic validity of u_l on a timescale of order $|\varepsilon|^{-\alpha}$ with $0\leqq\alpha<1$, it follows that whenever we want to study the effect of the small (ε-dependent) and nonlinear terms in the partial differential equation, we must construct approximations with a validity on a timescale of order $|\varepsilon|^{-1}$.

In a number of papers [4], [15], [16], [17] initial value and initial-boundary value problems for the Rayleigh wave equation have been studied by constructing formal approximations of the solutions or by deriving some properties of the approximations for large times. An interesting result (without an asymptotic justification) has been found in [15]. For a rather general class of initial values it has been shown in [15] that the first-order approximation tends to a superposition of standing triangular waves as $\varepsilon t\to\infty$. How the solution tends to these standing triangular waves can be determined by solving a nonlinear integro-differential equation. It is not made clear in [15] how to solve the integro-differential equation, but it is the author's opinion based upon the results in this paper and in [4] that only for a restricted class of initial values, such as (5.2) and (5.3), this equation may be solved analytically.

As can be seen from (5.17) v_0 also tends to a standing triangular wave (with amplitude $\pi/2n\sqrt{3}$ and period $2\pi/n$) as $\varepsilon t\to\infty$. However, it should be emphasized that nothing can be said about the asymptotic validity of v_0 as $\varepsilon t\to\infty$, since the asymptotic validity of v_0 could only be established for finite εt (that is, $0\leqq|\varepsilon t|\leqq L<\infty$).

In [9] it was concluded from the behaviour of the first-order approximation as $\varepsilon t\to\infty$ that the Rayleigh wave equation (postulated in [16]) is not a good model for galloping oscillations since the approximation allows at least one, and possibly infinitely many, sharp bends. From a mathematical point of view, the validity of this conclusion is rather doubtful since the asymptotic validity of the results so far has only been obtained for finite εt, that is for $|\varepsilon t| \leqq L < \infty$. And on this finite timescale the solution and the asymptotic approximations are at least two times continuously differentiable with respect to the independent variables if the initial values are sufficiently smooth.

In § 5 monochromatic initial values have been considered which apply to the description of galloping oscillations because these oscillations often affect only a single mode of vibration. To obtain some information about the maximum oscillation-amplitudes, the following formula may be used:

$$w(x, t) = \frac{\rho_c g}{2T} x(x-l) + \left(\frac{a}{3b}\right)^{1/2} \frac{l v_\infty}{\pi c} u\left(\frac{\pi}{l} x, \frac{\pi c}{l} t\right),$$

where $w(x, t)$, ρ_c, g, T, a, b, l, v_∞, c, and $u((\pi/l)x, (\pi c/l)t)$ are defined as in § 4. The first term in this formula may be considered as the position of the conductor at rest, whereas the second term represents the change of the position of the conductor due to galloping. From (5.39) it follows that the maximum amplitude of $u((\pi/l)x, (\pi c/l)t)$ for $\varepsilon t \to \infty$ is $\pi/2n\sqrt{3}$. So, the maximum oscillation-amplitude of $w(x, t)$ may be approximated by

$$\left(\frac{-c_{D0} - c_{L1}}{\frac{1}{2}c_{D0} + \frac{1}{6}c_{L1} + c_{L3}}\right)^{1/2} \frac{\pi v_\infty}{2c} \frac{l}{n\pi},$$

where $l/n\pi$ is the frequency of the monochromatic initial values and where c_{D0}, c_{L1}, and c_{L3} are the aerodynamic coefficients, which may be obtained from wind-tunnel measurements.

Finally, it should be noted that the two-timescale perturbation method is applicable to perturbations not solely depending on derivatives of the dependent variable, and is applicable to perturbations depending in a special way on the dependent variable and its derivatives. Consider, for instance, the initial-boundary value problem (2.1)-(2.4) with $f(x, t, u, u_t, u_x; \varepsilon) \equiv (-1+u^2)u_t$. The partial differential equation (2.1) can then be considered as a generalized Van der Pol equation. As is well known this equation is related to the Rayleigh wave equation, which has been introduced in § 4 and treated in § 5. Again a two-timescale perturbation method can be used to construct an asymptotic approximation of the solution. The equation for $f_0(\sigma, \tau)$ now becomes

$$2f_{0_{\sigma\tau}} - f_{0_\sigma} + \frac{1}{3} f_0^2 f_{0_\sigma} + f_{0_\sigma} \frac{1}{2\pi} \int_0^{2\pi} f_0^2(\theta, \tau)\, d\theta = 0,$$

which can be integrated with respect to σ. As in § 5 an order ε asymptotic approximation can be constructed on a timescale of order $|\varepsilon|^{-1}$.

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