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# ASYMPTOTIC ANALYSIS OF A SINGULAR PERTURBATION PROBLEM*

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**Abstract.** We study, in a rectangle $0 < x < a$ and $0 < y < b$, the Dirichlet problem for an elliptic differential equation of the form

$$-\epsilon \Delta u_\epsilon + p \frac{\partial u_\epsilon}{\partial x} + q u_\epsilon = f(x, y)$$

where $\epsilon$ is a small parameter $0 < \epsilon \ll 1$, $\Delta$ is the Laplacian operator, $p$ is a positive number, $q$ is a nonnegative number and all of the input data are smooth. We establish a constructive procedure for obtaining an asymptotic approximation of arbitrary order with respect to $\epsilon$ of this singular perturbation problem, and also give a proof of its uniform validity in the closed rectangle by the use of the maximum principle and exponential estimates of all boundary or corner layer functions. The corner singularities of parabolic boundary layer functions are removed by introducing elliptic boundary layer functions along the characteristic boundaries $y = 0$ and $y = b$. Both ordinary corner layer functions and elliptic corner layer functions are employed at the outflow corners $(a, 0)$ and $(a, b)$.

An application is made to settle a long-standing problem in the magnetohydrodynamic flow in a rectangular duct.

**Key words.** singular perturbation, outer approximation, ordinary boundary layer, elliptic boundary layer, parabolic boundary layer, ordinary corner layer, elliptic corner layer, maximum principle, magnetohydrodynamics

**AMS(MOS) subject classifications.** Primary 35B25, 35C20, 35J25; secondary 76W05

## 1. Introduction.

We study, in a rectangle $\Omega = (0, a) \times (0, b)$, the Dirichlet boundary value problem for an elliptic partial differential equation of the form

$$L_\epsilon u_\epsilon \equiv -\epsilon \Delta u_\epsilon + p \frac{\partial u_\epsilon}{\partial x} + q u_\epsilon = f(x, y) \tag{1.1}$$

with boundary conditions

$$u_\epsilon(0, y) = g_1(y), \qquad u_\epsilon(a, y) = g_2(y), \qquad 0 < y < b, \tag{1.2a,b}$$

$$u_\epsilon(x, 0) = g_3(x), \qquad u_\epsilon(x, b) = g_4(x), \qquad 0 < x < a, \tag{1.2c,d}$$

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where $\epsilon$ is a small parameter $0 < \epsilon \ll 1$, $\Delta$ is the Laplacian operator, $p$ is a positive number, $q$ is a nonnegative number, the remaining input data $f(x, y)$, $g_1(y)$, $g_2(y)$, $g_3(x)$, and $g_4(x)$ are assumed to be smooth. We also suppose that the assigned boundary functions are continuous at the corners. That is,

$$g_1(0) = g_3(0), \qquad g_1(b) = g_4(0), \tag{1.3a,b}$$

$$g_2(0) = g_3(a), \qquad g_2(b) = g_4(a). \tag{1.3c,d}$$

The distinguishing feature of a singular perturbation problem is that a small parameter multiplies some terms in the differential equation which, if absent, would change the character of the equation. Often these contain the highest derivatives in the equation and the approximation as this parameter tends to zero is therefore governed by a lower order equation which cannot satisfy all the boundary conditions prescribed. Hence the solution converges nonuniformly in the domain as the parameter tends to zero. Problems of this type frequently arise in fluid dynamics [14], [29], [34], [44], heat transfer [1], [3], theory of plates and shells [30], oil reservoir simulation [38], and magnetohydrodynamic flow [41]. The specific character of the problem (1.1), (1.2a,b,c,d) is brought about by the presence of the four corners of right angle for the domain and by the fact that the parts of the boundary, $y = 0$ and $y = b$, coincide with the characteristic curves of the reduced equation

$$p\frac{\partial u_0}{\partial x} + qu_0 = f(x, y), \tag{1.4}$$

which is obtained from (1.1) by putting $\epsilon = 0$. The boundary $x = 0$ is called the inflow boundary while the boundary $x = a$ is called the outflow boundary.

The purpose of this paper is to establish a constructive procedure for obtaining the asymptotic approximation of arbitrary order with respect to $\epsilon$ of this singular perturbation problem, and to give a proof of its uniform validity in the closed rectangle by use of the maximum principle and exponential estimates of all boundary or corner layer functions, which will be defined in Section 3. It is well known [13] that ordinary boundary layer functions appear along the outflow boundary $x = a$ while parabolic boundary layer functions occur along the characteristic boundaries $y = 0$ and $y = b$. In 1966, W. Eckhaus and E. M. De Jager [11] discovered the singularities of parabolic boundary layer functions near the inflow corners $(0, 0)$ and $(0, b)$. These corner singularities of parabolic boundary layer functions are removed by introducing elliptic boundary layer functions along the characteristic boundaries. Both ordinary corner layer functions and elliptic corner layer functions are employed at the outflow corners $(a, 0)$ and $(a, b)$.

An application is made to settle a long-standing problem in the magnetohydrodynamic flow in a rectangular duct.

We give a brief historical survey of studies on the elliptic singular perturbation problems of the type considered in this work. In providing references we have tried to give those that might be of interest for further reading, rather than presenting a comprehensive bibliography of the subject. We apologize to those who feel neglected.

## 2. Historical survey.

In 1944, W. Wasow [46] studied the problem of the following type

$$-\epsilon\Delta u + \frac{\partial u}{\partial x} = f(x,y) \tag{2.1}$$

over a finite plane domain $B$ with the smooth boundary $C$ under the prescribed boundary condition

$$u = g(x,y) \tag{2.2}$$

on $C$. Both $f$ and $g$ are smooth functions. It is shown that, as $\epsilon \to 0+$, the solution of the problem (2.1), (2.2) converges to the solution of the reduced equation

$$\frac{\partial u}{\partial x} = f(x,y),$$

assuming the prescribed boundary values along the inflow boundary of $C$, not the whole boundary.

In 1950, N. Levinson [32] considered the Dirichlet boundary value problem for the equation of the following type:

$$-\epsilon\Delta u + A(x,y)u_x + B(x,y)u_y + C(x,y)u = D(x,y) \tag{2.3}$$

over an open simply or multiply connected region $R$ whose boundary $S$ consists of a finite number of simple closed curves. Let $R \cup S$ be contained in an open connected region $R'$ and suppose all of the data of the problem are smooth in $R'$. Furthermore, we require that $A^2(x,y) + B^2(x,y) > 0$ in $R'$ and that either $R'$ is simply connected or $C(x,y) > 0$ in $R'$. Either hypothesis suffices to establish a maximum principle in $R \cup S$. Under these conditions, the Dirichlet boundary value problem for (2.3) has a unique solution in $R \cup S$ for each $\epsilon > 0$.

Let $S_1$ be a segment of one of curves of $S$ such that $(A,B)\cdot \boldsymbol{n} < 0$, where $\boldsymbol{n}$ is an outward normal vector of $S$ at point $(x,y)$. Let the characteristic curves of the reduced equation corresponding to (2.3) emanating from $S_1$ pass out of $R$ on the segment $S_2$ of a curve of $S$. The closed simply connected region in $R \cup S$ bounded by $S_1$ and $S_2$ and by the two characteristics of the reduced equation of (2.3) joining the endpoints of $S_1$ and $S_2$ is called a "regular quadrilateral". Then Levinson proved the following result.

THEOREM 2.1. *In a regular quadrilateral $Q$ in $R \cup S$ we have*

$$u(x,y) = u_0(x,y) + w(x,y;\epsilon) + O(\sqrt{\epsilon})$$

*uniformly in the quadrilateral. The function $u_0(x,y)$ is the solution of the reduced equation of* (2.3) *which takes on the given boundary value of $u$ on $S_1$. The ordinary boundary layer term $w(x,y;\epsilon)$ is defined as follows:*

$$w(x,y;\epsilon) = \begin{cases} h(x,y)\exp\left[\dfrac{-g(x,y)}{\epsilon}\right], & \textit{near } S_2, \\ \exp\left(-\dfrac{\delta}{\epsilon}\right) & \textit{for some } \delta > 0 \textit{ elsewhere in } Q, \end{cases}$$

*where $h = u - u_0$ and $g = 0$ on $S_2$ and $g$ is positive away from $S_2$. The function $g$ satisfies the nonlinear equation*

$$g_x^2 + g_y^2 + Ag_x + Bg_y = 0$$

*($g$ exists and is uniquely determined in a neighborhood of $S_2$).*

In terminology that we will establish below, Levinson treated the "ordinary boundary layer" part of the solution.

In 1957, M. I. Vishik and L. A. Lyusternik [45] studied the equation of the following type

$$-\epsilon\Delta u + \frac{\partial u}{\partial x} + u = f(x, y) \tag{2.4}$$

in the rectangle $\Omega$, $0 < x < a$ and $0 < y < b$, under the homogeneous boundary condition

$$u = 0$$

on the boundary of $\Omega$. They gave an expansion of the solution of the form for some $\delta > 0$

$$u(x,y) = u_0(x,y) + z_0(x,Y)\,\psi\Big(\frac{y}{\delta}\Big) + z_0^T(x,Y^T)\,\psi\Big(\frac{b-y}{\delta}\Big) + w_0(X_1,y)\,\psi\Big(\frac{a-x}{\delta}\Big) + R_0(x,y;\epsilon)$$

where $Y = y/\sqrt{\epsilon}$, $Y^T = (b-y)/\sqrt{\epsilon}$, $X_1 = (a-x)/\epsilon$ and the infinitely differentiable smoothing function $\psi(x)$ is identically equal to 1 for $x \le \frac{1}{3}$ and equal to zero for $x \ge \frac{2}{3}$. The function $u_0$ satisfies the reduced equation under the condition $u_0(0,y) = 0$. The function $z_0(x,Y)$ is a boundary layer near the boundary $y = 0$, which satisfies the parabolic equation

$$-\frac{\partial^2 z_0}{\partial Y^2} + \frac{\partial z_0}{\partial x} + z_0 = 0$$

over the semi-infinite region $0 < x < a$, $0 < Y < \infty$ under the conditions

$$z_0(0,Y) = 0, \quad z_0(x,0) = -u_0(x,0).$$

The function $z_0^T$ is a boundary layer near the part of the boundary given by $y = b$, $0 \le x \le a$, which has the same structure as $z_0$. The function $w_0(X_1,y)$ is a boundary layer along the boundary $x = a$, $0 \le y \le b$, which satisfies the ordinary differential equation

$$\frac{\partial^2 w_0}{\partial {X_1}^2} + \frac{\partial w_0}{\partial X_1} + \epsilon w_0 = 0$$

on the unbounded interval $0 < X_1 < \infty$ under the boundary condition

$$w_0(0,y) = -u_0(a,y) - z_0\Big(a, \frac{y}{\sqrt{\epsilon}}\Big)\,\psi\Big(\frac{y}{\delta}\Big) - z_0^T\Big(a, \frac{b-y}{\sqrt{\epsilon}}\Big)\,\psi\Big(\frac{b-y}{\delta}\Big).$$

From this Vishik and Lyusternik deduced on the basis of the maximum principle [39] that

$$R_0 = O(\epsilon)$$

everywhere except in the neighborhood of the points $(a, 0)$ and $(a, b)$, under the assumption that the smooth function $f(x, y)$ vanishes at the points $(0, 0)$ and $(0, b)$. They also asserted that on applying an iteration process, one can obtain, as above, an asymptotic formula of arbitrary order if the parameters of the problem are sufficiently smooth. In 1966, W. Eckhaus and E. M. De Jager [11] made the remark that the extension of the theory involving parabolic boundary layers to higher order approximation should not be considered a trivial matter.

In our terminology, Vishik and Lyusternik treated the "parabolic boundary layers" along the characteristic boundaries of the region.

In 1964, J. K. Knowles and R. E. Messick [30] discussed a class of singular perturbation problems arising in the theory of thin elastic plates and shells. An important feature of these problems is that the boundary of the domain coincides either partially or entirely with portions of characteristic curves of the reduced equations. In order to understand this exceptional character of a characteristic boundary, Knowles and Messick considered the equation of the following type

$$-\epsilon \Delta u + \frac{\partial u}{\partial x} = 0 \tag{2.5}$$

in the semi-infinite strip $R : 0 < x < a$ and $0 < y < \infty$, under the boundary conditions

$$u(0, y) = 0, \quad u(a, y) = 0, \quad 0 < y < \infty, \tag{2.6a, b}$$

$$u(x, 0) = g(x), \quad 0 \le x \le a, \tag{2.6c}$$

where the function $u$ and its partial derivatives are required to be bounded as $y \to \infty$. They obtained as an approximation to the solution the function

$$z_0(x, Y) + W_0(X_1, Y) + V_0(X_1, Y_1)$$

where $Y = y/\sqrt{\epsilon}$, $X_1 = (a - x)/\epsilon$, and $Y_1 = y/\epsilon$. Note that for this problem, the solution of the reduced equation under the condition $u_0(0, y) = 0$ is $u_0(x, y) = 0$, which also satisfies the boundary condition (2.6b). Therefore the ordinary boundary layer along the boundary $x = a$ is identical to zero. The function $z_0(x, Y)$ is the parabolic boundary layer function along the boundary $y = 0$, defined by the equation

$$-\frac{\partial^2 z_0}{\partial Y^2} + \frac{\partial z_0}{\partial x} = 0$$

over the domain $0 < x < a$, $0 < Y < \infty$ under the conditions

$$z_0(x, 0) = g(x), \quad z_0(0, Y) = 0,$$

and the condition that $z_0(x, Y)$ decays exponentially in $Y$. The function $W_0(X_1, Y)$ is a "corner layer" at the outflow corner $(a, 0)$, defined by the ordinary differential equation

$$\frac{\partial^2 W_0}{\partial {X_1}^2} + \frac{\partial W_0}{\partial X_1} = 0$$

on the unbounded interval $0 < X_1 < \infty$ under the conditions

$$W_0(0, Y) = -z_0(a, Y)$$

and

$$W_0(X_1, Y) \text{ has the exponential decay property in both } X_1 \text{ and } Y.$$

Then it follows that

$$W_0(X_1, Y) = -z_0(a, Y)\exp(-X_1).$$

The function $V_0(X_1, Y_1)$ is a "corner layer" at the outflow corner $(a, 0)$, defined by the elliptic equation

$$-\Big(\frac{\partial^2 V_0}{\partial {X_1}^2} + \frac{\partial^2 V_0}{\partial {Y_1}^2}\Big) + \frac{\partial V_0}{\partial X_1} = 0$$

over the quarter-plane $0 < X_1 < \infty$ and $0 < Y_1 < \infty$ under the conditions

$$\begin{aligned} &V_0(X_1, 0) = -W_0(X_1, 0) = g(a)\exp(-X_1), \\ &V_0(0, Y_1) = 0, \end{aligned}$$

and

$$V_0(X_1, Y_1) \text{ decays exponentially in } X_1 \text{ and } Y_1.$$

Knowles and Messick mentioned that using a representation of the solution to the original boundary value problem (2.5), (2.6a,b,c), which involves the expansion of the Green's function in a series of modified Bessel functions, it is possible to prove the statements listed below. First let us define some notation: Let $S$ be the boundary of $R$. $\delta$ denotes an arbitrary small positive number. The symbols ${}_\delta D$ and $D_\delta$ represent quarter-discs at $(0, 0)$ and $(a, 0)$, respectively, and are defined as follows:

$$\begin{aligned} {}_\delta D &= \{(x, y) : x \geq 0,\ y \geq 0,\ x^2 + y^2 < \delta^2\}, \\ D_\delta &= \{(x, y) : x \geq 0,\ y \leq a,\ x^2 + (y - a)^2 < \delta^2\}. \end{aligned}$$

It can be shown that

i) There exist positive constants $M(\delta)$, $c(\delta)$ such that

$$|u(x, y)| \leq M(\delta)\exp\Big(-\frac{c(\delta)}{\epsilon}\Big)$$

for all $y \geq \delta$, $0 \leq x \leq a$.

ii) There exists a positive constant $M(\delta)$ such that

$$\Big|u(x, y) - z_0\Big(x, \frac{y}{\sqrt{\epsilon}}\Big)\Big| < M(\delta)\epsilon \tag{2.7}$$

for all $(x, y) \in R \cup S - {}_\delta D - D_\delta$.

iii) If $g(x) = O(x)$ as $x \to 0+$, then there exists a constant $M(\delta) > 0$ such that (2.7) holds for $(x, y) \in R \cup S - D_\delta$.

iv) If $g(x) = O(x)$ as $x \to 0+$ and $g(x) = O(a-x)$ as $x \to a-$, then there exists a positive constant $M$ such that

$$\left|u(x,y) - z_0\Big(x, \frac{y}{\sqrt{\epsilon}}\Big) - W_0\Big(\frac{a-x}{\epsilon}, \frac{y}{\sqrt{\epsilon}}\Big)\right| \le M\epsilon$$

for all $(x,y) \in R \cup S$.

In our terminology, Knowles and Messick have thus treated the "ordinary corner layer" and the "elliptic corner layer" at the outflow corner of the region.

In 1966, W. Eckhaus and E. M. De Jager [11] investigated the problem of the following type

$$L_\epsilon u \equiv -\epsilon \Delta u + \frac{\partial u}{\partial x} = 0 \tag{2.8}$$

in the domain $\Omega$, $0 < x < a$ and $0 < y < b$, with the boundary conditions

$$\begin{array}{llll} \text{(2.9a,b)} & u(0,y) = g_1(y), & u(a,y) = g_2(y), & 0 < y < b, \\ \text{(2.9c,d)} & u(x,0) = g_3(x), & u(x,b) = g_4(x), & 0 < x < a. \end{array}$$

The boundary data are assumed to be smooth and to be continuous at the corner points. The solution of the reduced problem is $g_1(y)$, which does not satisfy the given boundary conditions (2.9b,c,d) generally. The local variable $Y = y/\sqrt{\epsilon}$ along the characteristic boundary $y = 0$ transforms (2.8) into

$$-\frac{\partial^2 u}{\partial Y^2} + \frac{\partial u}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2}.$$

Define the function $z_0(x,Y)$ as the solution of the reduced equation in local coordinates, that is, we have

$$-\frac{\partial^2 z_0}{\partial Y^2} + \frac{\partial z_0}{\partial x} = 0$$

in the domain $0 < x < a$ and $0 < Y < \infty$. Let $z_0$ satisfy boundary conditions

$$\begin{aligned} &z_0(0,Y) = 0, \\ &z_0(x,0) = g_3(x) - g_1(0) \equiv \gamma(x). \end{aligned}$$

An explicit form of this function is easily obtained as follows:

$$z_0(x,Y) = \sqrt{\frac{2}{\pi}} \int_{Y/\sqrt{2x}}^{\infty} \exp\Big(-\frac{t^2}{2}\Big)\, \gamma\Big(x - \frac{Y^2}{2t^2}\Big)\, dt.$$

Furthermore, we have

$$L_\epsilon z_0 = -\epsilon \frac{\partial^2 z_0}{\partial x^2}. \tag{2.10}$$

Due to the assumed continuity of the boundary data, i.e., $\gamma(0) = 0$, we obtain

$$\frac{\partial^2 z_0(x,Y)}{\partial x^2} = \sqrt{\frac{2}{\pi}}\Big[\int_{Y/\sqrt{2x}}^{\infty} \exp\Big(-\frac{t^2}{2}\Big)\,\gamma''\Big(x-\frac{Y^2}{2t^2}\Big)\,dt + \gamma'(0)\,Y\,(2x)^{-3/2}\exp\Big(-\frac{Y^2}{4x}\Big)\Big],$$

where the primes indicate derivatives with respect to the argument. We see that $\partial^2 z_0/\partial x^2$ is uniformly bounded in the closure of $\Omega$ if and only if $\gamma'(0) = 0$, which was assumed to be true in the analysis of Vishik and Lyusternik [45]. In the case of general boundary conditions, where $\gamma'(0) \neq 0$, the right-hand side of (2.10) has a singularity at the origin $x = 0$, $y = 0$. The nature of the singularity is most clearly revealed if in (2.10) the origin is approached along any curve $Y = mx^n$, where $m$ and $n$ are constants. The presence of this "corner singularity" indicates that in attempting a proof of the asymptotic properties of the parabolic boundary layer, difficulties should be expected. Also this corner singularity gives more singular functions in the course of the construction of a high order approximation to the parabolic boundary layer.

Eckhaus and De Jager constructed a regularized parabolic boundary layer function $\bar{z}_0$ by replacing $\gamma(x)$ in $z_0$ by

$$\bar{\gamma}(x) = \gamma(x) - \gamma'(0)\,x\exp\Big(-\frac{x}{\sqrt{\epsilon}}\Big).$$

In other words,

$$\bar{z}_0(x,Y) = \sqrt{\frac{2}{\pi}}\int_{Y/\sqrt{2x}}^{\infty} \exp\Big(-\frac{t^2}{2}\Big)\,\bar{\gamma}\Big(x-\frac{Y^2}{2t^2}\Big)\,dt.$$

The important properties of this function $\bar{\gamma}(x)$ are

i) $\bar{\gamma}(0) = \bar{\gamma}'(0) = 0$, which implies that $\partial^2\bar{z}_0/\partial x^2$ is bounded uniformly in the closure of $\Omega$, and moreover,

$$\frac{\partial^2 \bar{z}_0}{\partial x^2} = O(1) + O\Big(\epsilon^{-1/2}\Big)$$

in the closure of $\Omega$.

ii) $\bar{\gamma}(x) = \gamma(x) + O(\sqrt{\epsilon})$, which implies that

$$\bar{z}_0(x,Y) = z_0(x,Y) + O\big(\sqrt{\epsilon}\big)$$

uniformly in the closure of $\Omega$.

Define another regularized parabolic boundary layer function $\bar{z}_0^T$ along the boundary $y = b$ similarly. Introduce the local coordinate $X_1 = (a-x)/\epsilon$ along the boundary $x = a$ and define the boundary layer function $\tilde{w}_0(X_1, y)$ as the solution of the ordinary differential equation

$$\frac{\partial^2 \tilde{w}_0}{\partial {X_1}^2} + \frac{\partial \tilde{w}_0}{\partial X_1} = 0$$

over the unbounded interval $0 < X_1 < \infty$ under the conditions

$$\begin{aligned}\tilde{w}_0(0, y) &= g_2(y) - g_1(y) - \bar{z}_0\Big(a, \frac{y}{\sqrt{\epsilon}}\Big) - \bar{z}_0^T\Big(a, \frac{b-y}{\sqrt{\epsilon}}\Big)\\ &\equiv \psi(y),\end{aligned}$$

and

$$\lim_{X_1 \to \infty} \tilde{w}_0(X_1, y) = 0.$$

Then it follows that

$$\tilde{w}_0(X_1, y) = \psi(y)\exp(-X_1).$$

Eckhaus and De Jager first showed by using the maximum principle [39] that the solution $u$ of the boundary value problem (2.8) with the boundary conditions (2.9a,b,c,d) has the asymptotic expansion

$$u(x,y) = g_1(y) + \bar{z}_0\Big(x, \frac{y}{\sqrt{\epsilon}}\Big) + \bar{z}_0^T\Big(x, \frac{b-y}{\sqrt{\epsilon}}\Big) + \tilde{w}_0\Big(\frac{a-x}{\epsilon}, y\Big) + R_0(x, y; \epsilon),$$

where the remainder $R_0$ satisfies

$$R_0 = O\big(\sqrt{\epsilon}\big)$$

uniformly in the closure of $\Omega$ with the exception of a neighborhood of the two corner points $(a, 0)$ and $(a, b)$. Note that at $y = 0$ and $y = b$, $\psi''(y)$ is bounded but of order of $1/\epsilon$. To get a better estimate for the remainder term, they introduced the expansion

$$u(x,y) = g_1(y) + \bar{z}_0\Big(x, \frac{y}{\sqrt{\epsilon}}\Big) + \bar{z}_0^T\Big(x, \frac{b-y}{\sqrt{\epsilon}}\Big) + \bar{w}_0\Big(\frac{a-x}{\epsilon}, y\Big) + \bar{R}_0(x, y; \epsilon),$$

with

$$\bar{w}_0\Big(\frac{a-x}{\epsilon}, y\Big) = [\psi(y) + \epsilon(a-x)\,\psi''(y)]\exp\Big(-\frac{a-x}{\epsilon}\Big)$$

and arrived at the conclusion that

$$\bar{R}_0 = O\big(\sqrt{\epsilon}\big)$$

uniformly in the closure of $\Omega$ on the basis of the maximum principle. Furthermore they showed that

$$\bar{w}_0\Big(\frac{a-x}{\epsilon}, y\Big) = \tilde{w}_0\Big(\frac{a-x}{\epsilon}, y\Big) + O(\epsilon)$$

uniformly in the closure of $\Omega$. Summarizing these results, Eckhaus and De Jager have established the following theorem.

THEOREM 2.2. *The asymptotic approximation for $u(x, y)$*

$$u(x,y) = g_1(y) + z_0\Big(x, \frac{y}{\sqrt{\epsilon}}\Big) + z_0^T\Big(x, \frac{b-y}{\sqrt{\epsilon}}\Big) + \tilde{w}_0\Big(\frac{a-x}{\epsilon}, y\Big) + O\big(\sqrt{\epsilon}\big)$$

*holds uniformly in the closure of $\Omega$, including the four corner points.*

We remark that the order of the asymptotic error in the above theorem is determined by the presence of corner singularities. If one studies the exceptional case $g_3'(0) = 0$, $g_4'(0) = 0$ in which the corner singularities are absent, one finds along the lines of the preceding analysis that the asymptotic error is no longer $O(\sqrt{\epsilon})$ but $O(\epsilon)$. J. Mauss [35], [36] claimed to improve the estimate of the above theorem and to obtain an $O(\epsilon)$ remainder by using a rather special inequality for the maximum principle when the right-hand side of the differential operator $L_\epsilon$ has the singular behavior. The result asserted by Mauss is inconsistent with higher order expansions obtained in this paper.

All of the ideas discussed here need to be modified in order to develop higher order asymptotic expansions. The difficulties involved are

i) Corner singularities appear at the inflow corners of the region in the construction of the parabolic boundary layers along the characteristic boundaries $y = 0$ and $y = b$.

ii) The parabolic boundary layer and ordinary boundary layer overlap in the vicinities of the outflow corners of the region.

The proper way to treat the item ii) is to follow the construction of the ordinary corner layer and the elliptic corner layer done by Knowles and Messick [30] after the modification of the ordinary boundary layer. More precisely, the function $\bar{w}_0(X_1, y)$ in the construction due to Eckhaus and De Jager may be decomposed as

$$\begin{aligned}\bar{w}_0(X_1, y) = w_0(X_1, y) + W_0(X_1, Y) + W_0^T(X_1, Y^T)\\ + \epsilon[W_1(X_1, Y) + W_1^T(X_1, Y^T)] + \epsilon^2 w_2(X_1, y),\end{aligned}$$

where the ordinary boundary layer functions $w_0(X_1, y)$ and $w_2(X_1, y)$ satisfy the equations

$$\frac{\partial^2 w_i}{\partial {X_1}^2} + \frac{\partial w_i}{\partial X_1} = \begin{cases} 0, & i = 0, \\ -\dfrac{\partial^2 w_{i-2}}{\partial y^2}, & i = 2, \end{cases}$$

over the unbounded interval $0 < X_1 < \infty$ under the conditions

$$w_i(0, y) = \begin{cases} g_2(y) - g_1(y), & i = 0, \\ 0, & i = 2, \end{cases}$$

and

$$w_i(X_1, y) \text{ decays exponentially as } X_1 \to \infty;$$

the ordinary corner layer functions $W_0(X_1, Y)$ and $W_1(X_1, Y)$ satisfy the equations

$$\frac{\partial^2 W_i}{\partial {X_1}^2} + \frac{\partial W_i}{\partial X_1} = \begin{cases} 0, & i = 0, \\ -\dfrac{\partial^2 W_{i-1}}{\partial Y^2}, & i = 1, \end{cases}$$

over the unbounded interval $0 < X_1 < \infty$ under the conditions

$$W_i(0, Y) = \begin{cases} -\bar{z}_0(a, Y), & i = 0, \\ 0, & i = 1, \end{cases}$$

and

$$W_i(X_1, Y) \text{ decays exponentially as } X_1 \to \infty.$$

The functions $W_0^T$ and $W_1^T$ are defined similarly. It is clear that the term $w_2$ plays no role in this improvement of estimate.

Eckhaus and De Jager investigated the singularities of parabolic boundary layers near the inflow corners, obtained a formal approximation, and proved the uniform validity of the asymptotic approximation to the solution with the estimate of $O(\sqrt{\epsilon})$. Since then, the problem of dealing with these singularities has attracted much attention from mathematicians, e.g., J. Grasman [15] – [18]; L. P. Cook, G. S. S. Ludford and J. S. Walker [9]; L. P. Cook and G. S. S. Ludford [10]; A. M. Il'in and E. F. Lelikova [24]; V. F. Butuzov [4]; and D. J. Temperley [43]. This type of difficulty also takes place in the problems of the interior layers due to the nonsmooth boundary conditions [8], [20], [36], the domains with corners and noncharacteristic boundary [20], and the nonconvex domains [20], [36].

A similar phenomenon occurs in the problem

$$-(\epsilon u_{yy} + u_{xx}) + u_y = f(x, y), \quad 0 < \epsilon \ll 1,$$

defined on the unit square $T$, $0 < x < 1$ and $0 < y < 1$, with the prescribed Dirichlet boundary conditions. In the event that the solution of this problem converges to the solution of the reduced problem, we can anticipate the ordinary boundary layer behavior in the vicinity of the upper edge of $T$, since the solution $u_0(x, y)$ of the reduced equation is uniquely determined throughout $T$ by the boundary values assumed on the other three edges. Note that the reduced equation is a partial differential equation of parabolic type. Thus when one attempts to determine an asymptotic solution, one has to treat the corner singularities of the derivative $\partial^2 u_0/\partial y^2$ first. Such singularities always exist no matter how smooth the prescribed boundary conditions may be unless certain compatibility conditions hold. Furthermore, the construction of the ordinary boundary layer requires the smoothness of the derivative $\partial^2 u_0/\partial x^2$ at point $(x, 1)$. This type of problem had been studied to obtain an asymptotic approximation of arbitrary order with respect to $\epsilon$ by G. E. Latta [31], R. E. O'Malley, Jr. [37], and Peng-Cheng Lin and Fa-Wang Liu [33]. None of these authors have treated the difficulties mentioned above.

J. Grasman [15] studied the problem of the following type

$$L_\epsilon u \equiv -\epsilon \Delta u + \frac{\partial u}{\partial x} = 0 \tag{2.11}$$

over the quarter-plane $0 < x < \infty$ and $0 < y < \infty$ under the boundary conditions

$$u(x, 0) = g(x), \qquad g(0) = 0, \tag{2.12a}$$

$$u(0, y) = 0, \tag{2.12b}$$

by means of the Green's theorem, which yields the exact solution. The asymptotic expansion of the integral representation of the solution is shown to be

$$
\begin{aligned}
u(x,y) = {} & \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\epsilon x}}^{\infty} \exp\Bigl(-\frac{t^2}{2}\Bigr)\, g\Bigl(x - \frac{y^2}{2\epsilon t^2}\Bigr)\, dt \\
& + \epsilon \Bigl\{ \tilde{v}_0\Bigl(\frac{x}{\epsilon}, \frac{y}{\epsilon}\Bigr) - g'(0)\,\epsilon^{-1} \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\epsilon x}}^{\infty} \exp\Bigl(-\frac{t^2}{2}\Bigr) \cdot \Bigl(x - \frac{y^2}{2\epsilon t^2}\Bigr)\, dt \\
& + \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\epsilon x}}^{\infty} \exp\Bigl(-\frac{t^2}{2}\Bigr) \frac{t^2-1}{2} \Bigl[ g'\Bigl(x - \frac{y^2}{2\epsilon t^2}\Bigr) - g'(0)\Bigr]\, dt \Bigr\} \\
& + O(\epsilon^2)
\end{aligned}
$$

uniformly for $x \geq 0$, $y \geq 0$, where the term $\epsilon \tilde{v}_0(x/\epsilon, y/\epsilon)$ represents the solution of boundary value problem (2.11), (2.12a,b) in the case that $g(x) = x g'(0)$. In order to understand this uniformly valid expansion from another viewpoint, let us define the functions $v_0(x/\epsilon, y/\epsilon; \epsilon)$, $z_0(x, y/\sqrt{\epsilon})$, and $z_1(x, y/\sqrt{\epsilon})$ by

$$
v_0\Bigl(\frac{x}{\epsilon}, \frac{y}{\epsilon}; \epsilon\Bigr) = \epsilon \tilde{v}_0\Bigl(\frac{x}{\epsilon}, \frac{y}{\epsilon}\Bigr),
$$

$$
z_0\Bigl(x, \frac{y}{\sqrt{\epsilon}}\Bigr) = \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\epsilon x}}^{\infty} \exp\Bigl(-\frac{t^2}{2}\Bigr) \Bigl[ g\Bigl(x - \frac{y^2}{2\epsilon t^2}\Bigr) - g'(0) \cdot \Bigl(x - \frac{y^2}{2\epsilon t^2}\Bigr)\Bigr]\, dt,
$$

and

$$
z_1\Bigl(x, \frac{y}{\sqrt{\epsilon}}\Bigr) = \sqrt{\frac{2}{\pi}} \int_{y/\sqrt{2\epsilon x}}^{\infty} \exp\Bigl(-\frac{t^2}{2}\Bigr) \frac{t^2-1}{2} \Bigl[ g'\Bigl(x - \frac{y^2}{2\epsilon t^2}\Bigr) - g'(0)\Bigr]\, dt,
$$

respectively. In other words, it follows that

$$
u(x,y) = z_0\Bigl(x, \frac{y}{\sqrt{\epsilon}}\Bigr) + \epsilon\, z_1\Bigl(x, \frac{y}{\sqrt{\epsilon}}\Bigr) + v_0\Bigl(\frac{x}{\epsilon}, \frac{y}{\epsilon}; \epsilon\Bigr) + O(\epsilon^2).
$$

From a direct computation, one finds that

i) The functions $z_0(x,Y)$ and $z_1(x,Y)$ are the solutions of the equations

$$
-\frac{\partial^2 z_k}{\partial Y^2} + \frac{\partial z_k}{\partial x} = \begin{cases} 0, & k = 0, \\ \dfrac{\partial^2 z_{k-1}}{\partial x^2}, & k = 1, \end{cases}
$$

over the quarter plane $0 < x < \infty$ and $0 < Y < \infty$ under the conditions

$$
z_k(0,Y) = 0,
$$

$$
z_k(x,0) = \begin{cases} g(x) - g'(0)\, x, & k = 0, \\ 0, & k = 1, \end{cases}
$$

and

$$
z_k(x,Y) \to 0 \qquad \text{as } x^2 + Y^2 \to \infty \text{ (and } x > 0 \text{ when } k = 0).
$$

ii) The function $v_0(X, Y_1; \epsilon)$ is the solution of the elliptic partial differential equation

$$-\Big(\frac{\partial^2 v_0}{\partial X^2} + \frac{\partial^2 v_0}{\partial {Y_1}^2}\Big) + \frac{\partial v_0}{\partial X} = 0$$

over the quarter-plane $0 < X < \infty$ and $0 < Y_1 < \infty$ under the conditions

$$v_0(0, Y_1; \epsilon) = 0, \quad v_0(X, 0; \epsilon) = g'(0)X\epsilon,$$

and

$$v_0(X, Y_1; \epsilon) \to 0 \qquad \text{as } X^2 + Y_1^2 \to \infty \text{ and } X > 0.$$

This means that a uniformly valid approximation with an accuracy $O(\epsilon^2)$ can be constructed by the usual perturbation method even though Grasman claimed that this cannot be done [15]. The detailed construction of such functions $v_k$, which will be called as the "elliptic boundary layer" functions, is in Section 3.3. It will be clear later that $v_k(x/\epsilon, y/\epsilon; \epsilon) \equiv 0$, for $k \geq 1$ in the problem (2.11), (2.12a,b). For the sake of convenience, let us write the function $z_0$ as

$$z_0 = z_0^1 - z_0^2$$

where $z_0^1$ denotes the integral which contains the function $g(\cdot)$ as part of integrand and $z_0^2$ is the remaining part of $z_0$. Grasman [15], [16] asserted that

i) the function $z_0^1$ is a uniformly valid approximation of the solution of problem (2.11), (2.12a,b) with a remainder term $O(\epsilon)$ for $x \geq 0$, $y \geq 0$, and

ii) the estimate $v_0 - z_0^2 = O(\epsilon)$ holds uniformly for $x \geq 0$, $y \geq 0$.

We cannot draw these conclusions because of the singularity of $z_0^2$ at the origin $x = 0$, $y = 0$. For it is clear that $v_0 - z_0^2$ vanishes when $x = 0$ or $y = 0$, but

$$L_\epsilon(v_0 - z_0^2) = \epsilon\,\frac{\partial^2 z_0}{\partial x^2} = g'(0)\sqrt{\frac{\epsilon}{4\pi}}\, y\, x^{-3/2} \exp\Big(-\frac{y^2}{4\epsilon x}\Big)$$

is singular in the neighborhood of the origin.

Cook and Ludford [10] studied the problem on a semi-infinite strip and analyzed the asymptotic approximation from an exact representation of the solution obtained by means of the Fourier sine transforms. Il'in and Lelikova [24] used the asymptotic behavior of the solution at the inflow corners to obtain the uniqueness of the parabolic boundary layer functions by matching two different asymptotic expansions. Butuzov [4] imposed certain compatibility conditions on the input data so that the corner singularities of the parabolic boundary layers disappear, and employed the corner layers at the outflow corners to obtain the asymptotic approximations of arbitrary order with respect to $\epsilon$, which is proved to be valid uniformly in the closure of the rectangular region.

**3. Main results.** The main tool used in this paper for estimating solutions of elliptic boundary value problems is furnished by the so-called maximum principle and the concept of barrier function. For the proof of the maximum principle see Eckhaus and De Jager [11]. We now repeat the formulation of the boundary layer problem:

$$L_\epsilon u_\epsilon \equiv -\epsilon \Delta u_\epsilon + p\,\frac{\partial u_\epsilon}{\partial x} + q u_\epsilon = f(x,y) \quad \text{in } \Omega, \tag{1.1}$$

with boundary conditions

$$u_\epsilon(0,y) = g_1(y), \quad u_\epsilon(a,y) = g_2(y), \quad 0 < y < b, \tag{1.2a, b}$$

$$u_\epsilon(x,0) = g_3(x), \quad u_\epsilon(x,b) = g_4(x), \quad 0 < x < a, \tag{1.2c, d}$$

where $\epsilon$ is a small parameter $0 < \epsilon \ll 1$, $\Delta$ is the Laplacian operator, $p$ is a positive number, $q$ is a nonnegative number, $\Omega$ is the rectangular region $0 < x < a$ and $0 < y < b$, and the remaining input data $f(x,y)$, $g_1(y)$, $g_2(y)$, $g_3(x)$, and $g_4(x)$ are assumed to be smooth. We also suppose that the assigned boundary functions are continuous at the corners. We are ready to state the maximum principle.

MAXIMUM PRINCIPLE. *Let $\Phi$ and $\Psi$ be twice continuously differentiable functions in $\Omega$ such that*

$$\left|L_\epsilon[\Phi]\right| \le L_\epsilon[\Psi] \quad \text{in } \Omega,$$

$$|\Phi| \le \Psi \quad \text{on } \partial\Omega.$$

*Then*

$$|\Phi| \le \Psi \quad \text{in } \bar{\Omega}.$$

*Remark* 3.1. For an elliptic differential operator of second order in an unbounded domain a maximum principle holds if the solution satisfies a certain growth condition at infinity. We will discuss this in Section 3.3.

In this work we investigate an asymptotic approximation of the solution of the elliptic boundary value problem given by (1.1), (1.2a,b,c,d). From the assumption that $q$ is nonnegative in $\Omega$, it follows that the solution is unique. Under the conditions assumed, it is well known that for a fixed value of $\epsilon$, $u_\epsilon(x,y)$ is continuous in $\bar{\Omega}$ and is smooth in $\Omega_1$ where $\Omega_1$ is any compact subregion of $\bar{\Omega}$ with positive distance from the corners. In 1979, A. Azzam [2] improved this result to obtain the following:

THEOREM 3.1. *For any fixed value of $\epsilon$, there exists a number $\nu \in (1,2)$ such that the solution $u_\epsilon$ of the Dirichlet boundary value problem* (1.1), (1.2a,b,c,d) *with the assumptions stated at the beginning of this section satisfies $u_\epsilon(x,y) \in C_\nu(\bar{\Omega})$. Moreover, in a sufficiently small neighborhood of the corner, $r^\tau D^2 u_\epsilon \in C_\mu$ for some $\tau$, $\mu \in (0,1)$, where $r$ is the distance from the corner point to $(x,y)$ and $D^2 u_\epsilon$ is any second partial derivative of $u_\epsilon$.*

*Remark* 3.2. The maximum principle implies that the solution $u_\epsilon(x,y)$ is bounded uniformly with respect to $\epsilon$ in $\bar{\Omega}$.

**3.1. Outer approximation.** In order to obtain the first rough approximation of the solution $u_\epsilon(x,y)$ for small values of the parameter $\epsilon$, we consider a function $u_0(x,y)$ which satisfies the reduced equation

$$p\,\frac{\partial u_0}{\partial x} + q u_0 = f(x,y). \tag{1.4}$$

The function $u_0(x, y)$ can satisfy only one of the prescribed boundary conditions

$$u_0(0, y) = g_1(y), \tag{1.2a}$$

and

$$u_0(a, y) = g_2(y). \tag{1.2b}$$

THEOREM 3.2. *There exists a positive constant $C$ independent of $\epsilon$ such that the inequality*

$$\left|u_\epsilon(x, y) - g_1(y)\right| \leq Cx \tag{3.1}$$

*holds uniformly in the closure of $\Omega$ for all values of $\epsilon$.*

*Proof.* The function $\Phi_\epsilon(x, y)$, defined by

$$\Phi_\epsilon(x, y) \equiv u_\epsilon(x, y) - g_1(y),$$

satisfies in $\Omega$ the differential equation

$$L_\epsilon[\Phi_\epsilon] = f(x, y) + \epsilon g_1''(y) - q g_1(y)$$

with the boundary conditions

$$\begin{aligned}
&\Phi_\epsilon(0, y) = 0, \\
&\Phi_\epsilon(a, y) = g_2(y) - g_1(y), \\
&\Phi_\epsilon(x, 0) = g_3(x) - g_1(0) = g_3(x) - g_3(0), \\
&\Phi_\epsilon(x, b) = g_4(x) - g_1(b) = g_4(x) - g_4(0).
\end{aligned}$$

We now introduce the barrier function $\Psi(x) = Cx$, where $C$ is some positive constant independent of $\epsilon$. By taking $C$ sufficiently large it follows that the inequalities

$$\left|\Phi_\epsilon(x, y)\right| \leq \Psi(x)$$

on the boundary of $\Omega$ and

$$\left|L_\epsilon[\Phi_\epsilon]\right| \leq L_\epsilon[\Psi]$$

in $\Omega$ can be satisfied for all values of $\epsilon$. Applying the maximum principle, we get the desired inequality (3.1) uniformly valid in the closure of $\Omega$ for all values of $\epsilon$. This completes the proof.

According to Theorem 3.2, as $\epsilon$ tends to zero, we are led to the inequality

$$\left|u_0(x, y) - g_1(y)\right| \leq Cx.$$

Therefore (1.2a) is the proper condition for the solution $u_0(x, y)$. Now the function $u_0$ is easily determined, and the result is

$$u_0(x, y) = g_1(y) \exp\Big(-\frac{qx}{p}\Big) + p^{-1} \int_0^x f(s, y) \exp\Big[-\frac{(x - s)q}{p}\Big]\, ds.$$

Since the remaining boundary conditions (1.2b,c,d) are not satisfied by the function $u_0(x,y)$ and the difference $u_\epsilon - u_0$ satisfies in $\Omega$ the differential equation

$$L_\epsilon[u_\epsilon - u_0] = \epsilon\Delta u_0,$$

it is quite evident that this approximation for $u_\epsilon(x,y)$ is not valid in a neighborhood of three parts of the boundary of $\Omega$, $x = a$, $y = 0$, and $y = b$, and is valid in the remaining subregion of $\Omega$ including the neighborhood of the inflow boundary $x = 0$ up to the order $O(\epsilon)$.

First of all, a better approximation of the solution $u_\epsilon(x,y)$ in this subregion is to be obtained by adding to $u_0(x,y)$ a sum $\sum_{k=1}^{n} \epsilon^k\, u_k(x,y)$ which equals zero along the inflow boundary $x = 0$ and has the property that $\sum_{k=0}^{n} \epsilon^k\, u_k(x,y)$ satisfies (1.1) up to the order $O(\epsilon^{n+1})$ for some positive integer $n$. The functions $u_k(x,y)$ are determined by iteration from the differential equations

$$p\,\frac{\partial u_k}{\partial x} + q u_k = \Delta u_{k-1}, \tag{3.2}$$

with the boundary conditions

$$u_k(0,y) = 0, \tag{3.3}$$

for $k = 1,2,\ldots,n$. Therefore it follows that the functions $u_k(x,y)$ are given by the expressions

$$u_k(x,y) = p^{-1}\int_0^x \Delta u_{k-1}(s,y)\exp\Big[-\frac{(x-s)q}{p}\Big]\,ds$$

for $k = 1,2,\ldots,n$.

We will call the series

$$u(x,y;\epsilon) = \sum_{k=0}^{n} \epsilon^k\, u_k(x,y) \tag{3.4}$$

the outer asymptotic approximation (OA). (Other names that are given are the asymptotic approximation or the interior asymptotic approximation of $u_\epsilon(x,y)$ in $\Omega$.) Applying the differential operator $L_\epsilon$ to the function $u^*(x,y;\epsilon)$, defined by

$$u^* = u_\epsilon - u, \tag{3.5}$$

yields the differential equation in $\Omega$

$$L_\epsilon[u^*] = \epsilon^{n+1}\Delta u_n \tag{3.6}$$

with the boundary conditions

$$u^*(0,y;\epsilon) = 0, \tag{3.7a}$$

$$u^*(a,y;\epsilon) = g_2(y) - u_0(a,y) - \sum_{k=1}^{n} \epsilon^k\, u_k(a,y), \tag{3.7b}$$

$$u^*(x,0;\epsilon) = g_3(x) - u_0(x,0) - \sum_{k=1}^{n} \epsilon^k\, u_k(x,0), \tag{3.7c}$$

$$u^*(x,b;\epsilon) = g_4(x) - u_0(x,b) - \sum_{k=1}^{n} \epsilon^k\, u_k(x,b). \tag{3.7d}$$

Now the outer approximation $u(x, y; \epsilon)$ satisfies the boundary conditions (1.2a) and introduces discrepancies in the boundary conditions (1.2b,c,d) on the remaining parts of the boundary of $\Omega$. To obtain a "uniform" approximation of the solution $u_\epsilon$ in $\Omega$, we eliminate these discrepancies along the boundaries $x = a$, $y = 0$, and $y = b$ by introducing other functions, called boundary layer functions and corner layer functions, along the three boundaries and the two outflow corners $(a, 0)$ and $(a, b)$. These functions will have the property that when acted on by $L_\epsilon$, the result will be of order $O(\epsilon^{n+1})$ uniformly in the closure of $\Omega$. Also the boundary layer functions have the property of being asymptotically equal to zero everywhere in $\Omega$ except for a small neighborhood of one of these three boundaries while the corner layer functions have the property of being asymptotically equal to zero everywhere in $\Omega$ except for a small neighborhood of one of the outflow corners of $\Omega$. The boundary layer functions along the outflow boundary $x = a$ satisfy ordinary differential equations and define the ordinary boundary layer (OBL). The boundary layer functions along the characteristic boundaries $y = 0$ and $y = b$ are of two types. One type of function satisfies a parabolic differential equation and is the parabolic boundary layer (PBL) function. The other type of boundary layer functions along the characteristic boundaries $y = 0$ and $y = b$ satisfies an elliptic differential equation and is designed to remove the corner singularities of PBL; it is called the elliptic boundary layer (EBL). There are two types of corner layers at each of the outflow corners. One type of the corner layer function satisfies an ordinary differential equation and is employed to remove the discrepancy in the vicinity of the corner due to the PBL. This function is called an ordinary corner layer (OCL) function. The other type of corner layer function satisfies an elliptic differential equation and is used to remove the discrepancy in the vicinity of the corner due to the EBL, OBL and OCL; this function is called an elliptic corner layer (ECL) function. The detailed construction of these functions will be investigated in the subsequent sections. The location of all functions is indicated in Figure 3.1 and the order of construction is shown in Figure 3.2.

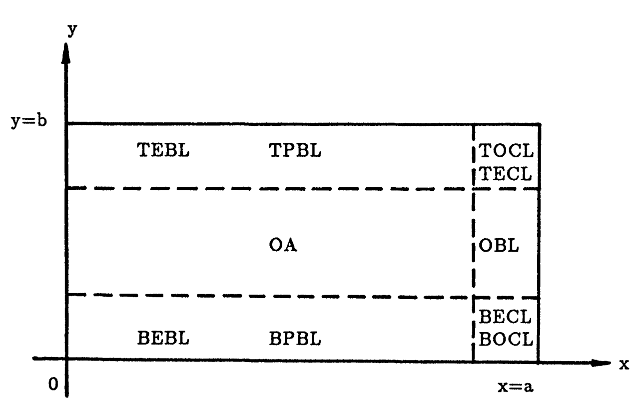


FIG. 3.1. *Location of all functions.*

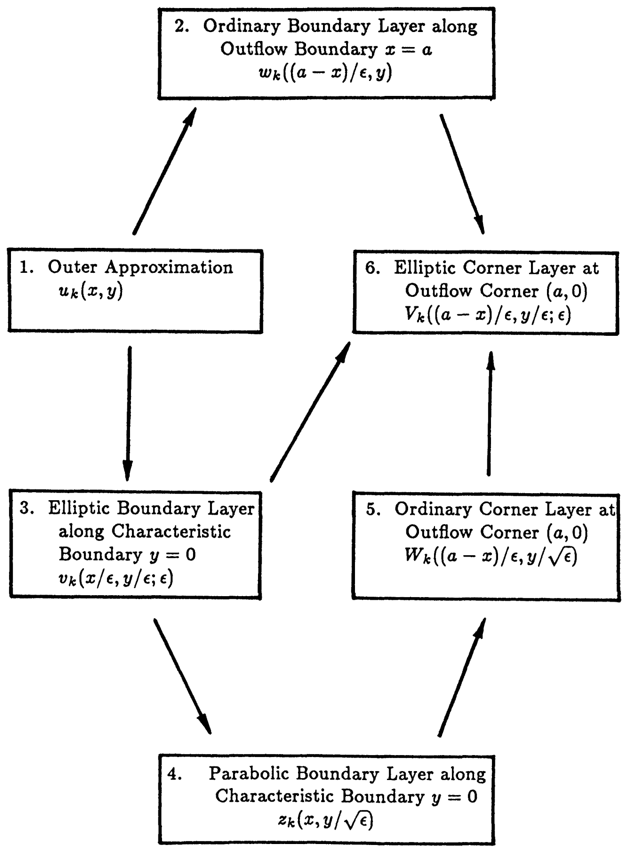


FIG. 3.2. *Order of constructions.*

The uniform asymptotic approximation of the solution $u_\epsilon(x,y)$ in the closure of $\Omega$ is expressed as follows:

$$u_\epsilon(x,y) = \mathrm{OA} + \mathrm{OBL} + \mathrm{BEBL} + \mathrm{BPBL} + \mathrm{BOCL} + \mathrm{BECL} + \mathrm{TEBL} + \mathrm{TPBL} + \mathrm{TOCL} + \mathrm{TECL} + \mathrm{REMAINDER}, \tag{3.8}$$

where BEBL and TEBL stand for the EBL along the characteristic boundaries $y = 0$ and $y = b$, respectively, etc.

In order to estimate the term REMAINDER in (3.8), we need the following result, which is a consequence of the maximum principle.

THEOREM 3.3. *If $\Phi_\epsilon(x,y)$ is the solution of the boundary value problem*

$$L_\epsilon[\Phi_\epsilon] = h_\epsilon(x,y),$$

*valid in $\Omega$ with*

$$\Phi_\epsilon(x,y)\big|_\Gamma = \Psi_\epsilon(x,y)\big|_\Gamma$$

*along the boundary* $\Gamma$ *of* $\Omega$, *and if*

$$h_\epsilon(x,y) = O(\epsilon^\mu) \ \ in\ \bar{\Omega}, \quad \mu \geq 0,$$

*and*

$$\Psi_\epsilon(x,y) = O(\epsilon^\nu) \ \ along\ \Gamma, \quad \nu \geq 0,$$

*then at most*

$$\Phi_\epsilon(x,y) = O\left(\epsilon^{\min(\mu,\nu)}\right) \ \ in\ \bar{\Omega}.$$

From this theorem, we conclude that if it is possible to have $L_\epsilon$[REMAINDER] being of order $O(\epsilon^{n+1})$ in the closure of $\Omega$ and the term REMAINDER being of order $O(\epsilon^{n+1})$ on the boundary of $\Omega$, then it follows that the estimate

$$\text{REMAINDER} = O(\epsilon^{n+1})$$

holds uniformly in the closure of $\Omega$.

**3.2. Ordinary boundary layer along the outflow boundary $x = a$.** Let the stretched variable $X_1$ along the outflow boundary $x = a$ be defined by $x = a - \epsilon X_1$. The ordinary boundary layer along the outflow boundary $x = a$ is defined by the series

$$w(X_1, y; \epsilon) = \sum_{k=0}^{n+1} \epsilon^k\, w_k(X_1, y) \tag{3.9}$$

where the functions $w_k(X_1, y)$ are defined iteratively by the ordinary differential equations

$$\frac{\partial^2 w_k}{\partial {X_1}^2} + p\,\frac{\partial w_k}{\partial X_1} = \pi_k(X_1, y) \tag{3.10}$$

over the unbounded interval $0 < X_1 < \infty$, where $y$ is a parameter $0 \leq y \leq b$, and the functions $\pi_k(X_1, y)$ have the expressions

$$\pi_0(X_1, y) = 0,$$
$$\pi_1(X_1, y) = q w_0,$$

and for $2 \leq k \leq n+1$,

$$\pi_k(X_1, y) = -\frac{\partial^2 w_{k-2}}{\partial y^2} + q w_{k-1}.$$

The boundary conditions imposed on the functions $w_k(X_1, y)$ are such that the discrepancy, due to the introduction of the outer approximation $u(x, y; \epsilon)$ in the boundary condition (1.2b) at the outflow boundary $x = a$, disappears and such that the functions $w_k(X_1, y)$ approach zero as $x \neq a$ and $\epsilon$ tends to zero. That is, we impose the boundary conditions

$$\begin{aligned} &w_0(0, y) = g_2(y) - u_0(a, y) \\ &w_k(0, y) = -u_k(a, y) \qquad \text{for } 1 \leq k \leq n, \\ &w_{n+1}(0, y) = 0 \end{aligned} \tag{3.11a}$$

and for $0 \le k \le n+1$,

$$w_k(X_1, y) \to 0 \qquad \text{as } X_1 \to \infty. \tag{3.11b}$$

It is easy to see that when $k = 0$ and $k = 1$ we obtain the solutions in the form

$$\begin{aligned} w_0(X_1, y) &= \left[g_2(y) - u_0(a, y)\right] \exp(-pX_1), \\ w_1(X_1, y) &= \left[-u_1(a, y) - \left(g_2(y) - u_0(a, y)\right)p^{-1}qX_1\right] \exp(-pX_1). \end{aligned}$$

In general, for $k \ge 2$, each of the functions $\pi_k(X_1, y)$ is the product of $\exp(-pX_1)$, the function of boundary layer type along the boundary $x = a$, and a polynomial of degree $k-1$ in $X_1$ with the coefficients depending on $y$. Hence the solutions $w_k$ can be expressed as

$$w_k(X_1, y) = \ell_k(X_1, y) \exp(-pX_1),$$

where $\ell_k(X_1, y)$ is a polynomial of degree $k$ in $X_1$ with the coefficients depending on $y$.

*Remark* 3.3. From ODE theory, if $p$ is a positive number, the integral

$$\int_0^\infty f(s) \exp(-ps)\, ds$$

converges and

$$\exp(-px) \int_0^x f(s)\, ds$$

converges to 0 as $x$ tends to $\infty$, then the solution of the ordinary differential equation

$$u''(x) + pu'(x) = f(x) \exp(-px)$$

defined over the unbounded interval $0 < x < \infty$ under the boundary conditions

$$u(0) = A \qquad \text{and} \qquad u(x) \to 0 \quad \text{as } x \to \infty,$$

has the form

$$\begin{aligned} u(x) = A \exp(-px) &- p^{-1} \exp(-px) \int_0^x f(s)\, ds \\ &- p^{-1} \int_x^\infty f(s) \exp(-ps)\, ds + p^{-1} \exp(-px) \int_0^\infty f(s) \exp(-ps)\, ds. \end{aligned} \tag{3.12}$$

THEOREM 3.4. *There exist two positive constants $C$ and $c$ independent of $\epsilon$ such that the inequalities*

$$\left| \frac{\partial^\ell}{\partial y^\ell} w_k(X_1, y) \right| \le C \exp(-cX_1) \tag{3.13}$$

*hold for $0 \le k \le n+1$ and $\ell = 0$ and 2.*

*Proof.* The above inequalities are quite clear because the functions $w_k$, and $\partial^2 w_k/\partial y^2$ are products of $\exp(-pX_1)$ and a polynomial in $X_1$ with the coefficients depending on $y$.

Applying the differential operator $L_\epsilon$ to the series $w$ gives

$$
\begin{aligned}
(3.14)\qquad L_\epsilon w &= \sum_{k=0}^{n+1} \epsilon^k\, L_\epsilon w_k \\
&= \sum_{k=0}^{n+1} \epsilon^k \Big(-\epsilon\,\frac{\partial^2 w_k}{\partial x^2} - \epsilon\,\frac{\partial^2 w_k}{\partial y^2} + p\,\frac{\partial w_k}{\partial x} + q w_k\Big).
\end{aligned}
$$

The equation (3.10) may be written as

$$
(3.15)\qquad -\epsilon\,\frac{\partial^2 w_k}{\partial x^2} + p\,\frac{\partial w_k}{\partial x} = -\epsilon^{-1}\,\pi_k\Big(\frac{a-x}{\epsilon}, y\Big).
$$

Substitution of (3.15) into (3.14) yields

$$
\begin{aligned}
(3.16)\qquad L_\epsilon w &= -\sum_{k=0}^{n+1} \epsilon^{k+1}\,\frac{\partial^2 w_k}{\partial y^2} + \sum_{k=0}^{n+1} \epsilon^k\, q w_k + \sum_{k=0}^{n+1} \epsilon^{k-1}\,\pi_k\Big(\frac{a-x}{\epsilon}, y\Big) \\
&= \epsilon^{n+1}\Big(-\epsilon\frac{\partial^2 w_{n+1}}{\partial y^2} - \frac{\partial^2 w_n}{\partial y^2} + q w_{n+1}\Big).
\end{aligned}
$$

It follows from (3.13) that the estimate

$$
L_\epsilon w = O\big(\epsilon^{n+1}\big)
$$

is valid uniformly in the closure of $\Omega$. Furthermore, let us examine this series on the boundary of $\Omega$.

i) At $x = a$, the series

$$
(3.17a)\qquad w = \sum_{k=0}^{n+1} \epsilon^k\, w_k(0, y) = g_2(y) - \sum_{k=0}^{n} \epsilon^k\, u_k(a, y)
$$

is equal to $u^*(a, y; \epsilon)$.

ii) At $x = 0$, the series

$$
(3.17b)\qquad w = \sum_{k=0}^{n+1} \epsilon^k\, w_k\Big(\frac{a}{\epsilon}, y\Big)
$$

is asymptotically exponentially small with respect to $\epsilon$ in the interval $0 \le y \le b$.

iii) At $y = 0$ and $y = b$, the series

$$
(3.17c)\qquad w = \sum_{k=0}^{n+1} \epsilon^k\, w_k\Big(\frac{a-x}{\epsilon}, y\Big)
$$

is asymptotically exponentially small with respect to $\epsilon$ for each $x$ in the interval $0 \le x < a$, but not in the closed interval $0 \le x \le a$. To circumvent this difficulty, we define two series, called the elliptic corner layers, at the outflow corners $(a, 0)$ and $(a, b)$. The construction of these elliptic corner layers is given in Section 3.6.

**3.3. Elliptic boundary layer along the characteristic boundary $y = 0$.** Usually the given problem (1.1), (1.2a,b,c,d) may not have the necessary compatibility conditions for the solution $u_\epsilon(x, y)$ at the inflow corner $(0, 0)$ to insure the smoothness of the parabolic boundary layer function $z_0(x, Y)$, which satisfies the parabolic differential equation (3.26) in a semi-strip $0 < x < a$ and $0 < Y < \infty$. The second partial derivative of $z_0$ with respect to the time-like variable $x$ is singular near the origin, and consequently the right-hand side of the differential equation for $z_1$ is singular in the vicinity of the origin. To remedy the presence of this "corner singularity," we introduce some functions $v_k(X, Y_1)$ with the stretched variables $X = x/\epsilon$ and $Y_1 = y/\epsilon$, which are defined by elliptic differential equations over the quarter plane $0 < X < \infty$ and $0 < Y_1 < \infty$ with the values zero as the boundary conditions along $X = 0$ and a suitable boundary condition along $Y_1 = 0$ such that the desired compatibility conditions for parabolic differential equations can be obtained to guarantee the boundedness of the second partial derivative of all functions $z_k(x, Y)$ with respect to $x$ in the semi-strip domain. This enables us to carry out an iteration process related to the parabolic boundary layer.

The elliptic boundary layer along the characteristic boundary $y = 0$ is defined by the series

$$v(X, Y_1; \epsilon) = \sum_{k=0}^{n+1} \epsilon^k \, v_k(X, Y_1; \epsilon), \tag{3.18}$$

where the functions $v_k(X, Y_1; \epsilon)$ are defined iteratively by the elliptic differential equations

$$-\Big(\frac{\partial^2 v_k}{\partial X^2} + \frac{\partial^2 v_k}{\partial {Y_1}^2}\Big) + p\,\frac{\partial v_k}{\partial X} + \epsilon q v_k = 0 \tag{3.19}$$

over the quarter plane $0 < X < \infty$ and $0 < Y_1 < \infty$. We impose the boundary conditions for $v_k$ in the following way:

$$v_k(0, Y_1; \epsilon) = 0, \quad 0 \le k \le n + 1, \tag{3.20a}$$

and

$$v_k(X, 0; \epsilon) = \omega_k(X; \epsilon), \quad 0 \le k \le n + 1, \tag{3.20b}$$

where the functions $\omega_k(X; \epsilon)$ have the expressions

$$\omega_0(X; \epsilon) = \sum_{i=1}^{N} \frac{X^i}{i!}\Big(g_3^{(i)}(0) - \frac{\partial^i}{\partial x^i}\, u_0(0, 0)\Big)\epsilon^i,$$

and for $1 \le k \le n$,

$$\omega_k(X;\epsilon) = -\sum_{i=1}^{N-2k} \frac{X^i}{i!} \frac{\partial^i}{\partial x^i} u_k(0,0)\, \epsilon^i,$$

and

$$\omega_{n+1}(X;\epsilon) = 0.$$

Note that $\omega_k(0;\epsilon) = 0$. In addition to the Dirichlet boundary conditions (3.20a,b), we impose the following conditions at infinity:

$$\text{(3.20c)} \qquad v_k(X, Y_1;\epsilon) \to 0 \qquad \text{as } X^2 + Y_1^2 \to \infty \text{ and } Y_1 > 0,\ 0 \le k \le n+1.$$

We remark that an equivalent condition is

$$v_k(X, Y_1;\epsilon) \text{ does not grow exponentially as } X^2 + Y_1^2 \to \infty.$$

The elliptic equation (3.19) with the conditions (3.20a,b,c) has a unique solution, and the maximum principle is valid for this problem [12], [39]. The parameter $\epsilon$ appears in this problem as a regular perturbation parameter. Therefore, $v_k(X, Y_1;\epsilon)$ could itself be written as a finite series in $\epsilon$ plus a remainder term that is $O(\epsilon^{n+1})$. The particular form of the function $v_k(X, Y_1;\epsilon)$ was chosen to make subsequent computations more tractable.

THEOREM 3.5. *The solutions $v_k(X, Y_1;\epsilon)$ have the integral representations*

$$v_k(X, Y_1;\epsilon) = \frac{\tau Y_1}{\pi} \int_0^\infty \left[r_5^{-1} K_1(\tau r_5) - r_6^{-1} K_1(\tau r_6)\right] \omega_k(s;\epsilon) \exp\left[-\frac{p(s-X)}{2}\right] ds,$$

*where*

$$\tau = \left(\frac{p^2}{4} + \epsilon q\right)^{1/2},$$

$$r_5 = \left[(X-s)^2 + Y_1^2\right]^{1/2},$$

$$r_6 = \left[(X+s)^2 + Y_1^2\right]^{1/2},$$

*and $K_1$ is the modified Bessel function of the second kind of the first order.*

*Proof.* The transformation

$$v_k(X, Y_1;\epsilon) = v_k^*(X, Y_1;\epsilon) \exp\left(\frac{pX}{2}\right)$$

yields the differential equations for $v_k^*$

$$\text{(3.21)} \qquad -\left(\frac{\partial^2 v_k^*}{\partial X^2} + \frac{\partial^2 v_k^*}{\partial {Y_1}^2}\right) + \left(\frac{p^2}{4} + \epsilon q\right) v_k^* = 0$$

over the quarter plane $0 < X < \infty$ and $0 < Y_1 < \infty$ under the boundary conditions

$$v_k^*(X,0;\epsilon) = \omega_k(X;\epsilon)\exp\Big(-\frac{pX}{2}\Big),$$
$$v_k^*(0,Y_1;\epsilon) = 0,$$

and

$$v_k^*(X,Y_1;\epsilon) \to 0 \quad \text{as } X^2+Y_1^2 \to \infty.$$

Since the fundamental solution for this differential operator (3.21) is

$$\frac{1}{2\pi}K_0\Big(\tau\sqrt{X^2+Y_1^2}\Big),$$

the Green's function for this differential operator over the quarter plane is given by

$$G(X,Y_1;s,t) = \frac{1}{2\pi}\big[K_0(\tau r_1) - K_0(\tau r_2) + K_0(\tau r_3) - K_0(\tau r_4)\big]$$

where $K_0$ is the modified Bessel function of the second kind of the zeroth order, and

$$r_1 = \big[(X-s)^2+(Y_1-t)^2\big]^{1/2}, \qquad r_2 = \big[(X+s)^2+(Y_1-t)^2\big]^{1/2},$$
$$r_3 = \big[(X+s)^2+(Y_1+t)^2\big]^{1/2}, \qquad r_4 = \big[(X-s)^2+(Y_1+t)^2\big]^{1/2}.$$

It follows that the functions $v_k^*$ have the expressions

$$v_k^*(X,Y_1;\epsilon) = \int_0^\infty \omega_k(s;\epsilon)\exp\Big(-\frac{ps}{2}\Big)\frac{\partial G}{\partial t}(X,Y_1;s,0)\,ds.$$

A computation gives, by using $K_0'(x) = -K_1(x)$,

$$\frac{\partial G}{\partial t}(X,Y_1;s,0) = \frac{\tau Y_1}{\pi}\big[r_5^{-1}K_1(\tau r_5) - r_6^{-1}K_1(\tau r_6)\big].$$

Therefore we have the desired integral representations for $v_k^*(X,Y_1;\epsilon)$. This completes the proof.

The value of the positive integer $N$ should be at least $2n+1$ in order to guarantee the smoothness of the parabolic boundary layer function $(\partial^2/\partial x^2)z_n(x,Y)$ and the ordinary corner layer function $(\partial^2/\partial Y^2)W_{n+1}(X_1,Y)$, which will be discussed shortly.

In order to estimate the elliptic boundary layer, the following exponential estimates for $v_k(X, Y_1; \epsilon)$ are needed.

THEOREM 3.6. *There exist two positive constants $C$ and $c$ independent of $\epsilon$ such that the inequalities*

$$|v_k(X, Y_1; \epsilon)| \le C \exp\left[-c\left(\sqrt{X^2 + Y_1^2} - X\right)\right] \tag{3.22}$$

*hold for $0 \le k \le n + 1$.*

*Proof.* The condition at infinity (3.20c) enables us to have the maximum principle for the boundary value problem over the unbounded domain. By the linearity, it suffices to prove that the solutions $v_k(X, Y_1; \epsilon)$ for the differential equation (3.19) over the quarter plane $0 < X < \infty$ and $0 < Y_1 < \infty$ under the boundary conditions (3.20a) and

$$v_k(X, 0; \epsilon) = X^i \epsilon^i,$$

and (3.20c) satisfy the estimate (3.22). Let the barrier function $U(X, Y_1; \epsilon)$ be defined by

$$U(X, Y_1; \epsilon) = \epsilon^i \sum_{j=0}^{i} c_j \,(X^2 + Y_1^2)^{j/2} \exp\left[-\frac{p}{2}\left(\sqrt{X^2 + Y_1^2} - X\right)\right],$$

where the values of positive constants $c_j$ will be determined in the course of this proof. Note that the function $U$ satisfies the restricted growth condition at infinity (3.20c). It is clear that

$$U(X, 0; \epsilon) = \epsilon^i \sum_{j=0}^{i} c_j \, X^j \ge v_k(X, 0; \epsilon)$$

for all $k$ if $c_j = 1$ and $c_j \ge 0$ for $0 \le j \le i - 1$, and

$$U(0, Y_1; \epsilon) = \epsilon^i \sum_{j=0}^{i} c_j \, Y_1^j \exp\left(-\frac{p}{2} Y_1\right) > v_k(0, Y_1; \epsilon)$$

for all $k$. A computation yields

$$
\begin{aligned}
&-\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial {Y_1}^2}\right)+p\,\frac{\partial U}{\partial X}+\epsilon q U\\
&=\epsilon^i \exp\left[-\frac{p}{2}\left(\sqrt{X^2+Y_1^2}-X\right)\right]\left(\sum_{j=0}^{i} c_j p(j+\tfrac12)(X^2+Y_1^2)^{(j-1)/2}\right.\\
&\qquad\left.-\sum_{j=0}^{i} c_j j^2 (X^2+Y_1^2)^{(j-2)/2}+\sum_{j=0}^{i}\epsilon q c_j (X^2+Y_1^2)^{j/2}\right)\\
&=\epsilon^i \exp\left[-\frac{p}{2}\left(\sqrt{X^2+Y_1^2}-X\right)\right]\\
&\quad\cdot\left(\epsilon q c_i (X^2+Y_1^2)^{i/2}+\left[\epsilon q c_{i-1}+c_i p(i+\tfrac12)\right](X^2+Y_1^2)^{(i-1)/2}\right.\\
&\qquad+\sum_{j=1}^{i-1}\left[\epsilon q c_{j-1}+c_j p(j+\tfrac12)-c_{j+1}(j+1)^2\right]\\
&\qquad\quad\left.\cdot(X^2+Y_1^2)^{(j-1)/2}\left(c_0\frac{p}{2}-c_1\right)(X^2+Y_1^2)^{-1/2}\right)\\
&\geq \epsilon^i \exp\left[-\frac{p}{2}\left(\sqrt{X^2+Y_1^2}-X\right)\right]\left(c_i p(i+\tfrac12)(X^2+Y_1^2)^{(i-1)/2}\right.\\
&\qquad\left.+\sum_{j=0}^{i-1}\left[c_j p(j+\tfrac12)-c_{j+1}(j+1)^2\right](X^2+Y_1^2)^{(j-1)/2}\right),
\end{aligned}
$$

which is greater than or equal to zero if the numbers $c_j$ are chosen so that the inequalities

$$c_j p(j+\tfrac12)\geq c_{j+1}(j+1)^2$$

hold for $j=i-1,\ i-2,\ \ldots,\ 2,\ 1,\ 0$, then it follows from the maximum principle that the inequalities

$$\left|v_k(X,Y_1;\epsilon)\right|\leq \epsilon^i\sum_{j=0}^{i} c_j (X^2+Y_1^2)^{j/2}\exp\left[-\frac{p}{2}\left(\sqrt{X^2+Y_1^2}-X\right)\right]$$

hold for $0\leq x<\infty$ and $0\leq Y_1<\infty$. By the definition of the stretched variables $X$ and $Y_1$, we obtain

$$\left|v_k\left(\frac{x}{\epsilon},\frac{y}{\epsilon};\epsilon\right)\right|\leq \sum_{j=0}^{i} c_j (a^2+b^2)^{j/2}\exp\left[-\frac{p}{2\epsilon}\left(\sqrt{x^2+y^2}-x\right)\right].$$

Hence over the closure of $\Omega$, we are led to the estimate

$$\left|v_k\left(\frac{x}{\epsilon},\frac{y}{\epsilon};\epsilon\right)\right|\leq C\exp\left[-\frac{p}{2\epsilon}\left(\sqrt{x^2+y^2}-x\right)\right],$$

where $C$ is some constant independent of $\epsilon$. This completes the proof.

Now applying the differential operator $L_\epsilon$ to the series $v$ yields

$$\begin{aligned} L_\epsilon v &= \sum_{k=0}^{n+1} \epsilon^k L_\epsilon v_k \\ &= \sum_{k=0}^{n+1} \epsilon^k \Big[ -\epsilon\Big(\frac{\partial^2 v_k}{\partial x^2} + \frac{\partial^2 v_k}{\partial y^2}\Big) + p\,\frac{\partial v_k}{\partial x} + q v_k \Big]. \end{aligned}$$

Equation (3.19) can be written as

$$-\epsilon\Big(\frac{\partial^2 v_k}{\partial x^2} + \frac{\partial^2 v_k}{\partial y^2}\Big) + p\,\frac{\partial v_k}{\partial x} + q v_k = 0.$$

Hence it follows that

$$L_\epsilon v = 0 \tag{3.23}$$

is valid uniformly in the closure of $\Omega$. Moreover, let us examine this series on the boundary of $\Omega$.

i) At $y = 0$, the series becomes

$$v = \sum_{i=1}^{N} \frac{x^i}{i!}\Big[g_3^{(i)}(0) - \frac{\partial^i}{\partial x^i}\, u_0(0,0)\Big] - \sum_{k=1}^{n} \epsilon^k \left[\sum_{i=1}^{N-2k} \frac{x^i}{i!}\,\frac{\partial^i}{\partial x^i}\, u_k(0,0)\right]. \tag{3.24a}$$

ii) At $x = 0$,

$$v = 0. \tag{3.24b}$$

iii) At $y = b$, the series

$$v = \sum_{k=0}^{n} \epsilon^k\, v_k\Big(\frac{x}{\epsilon}, \frac{b}{\epsilon}; \epsilon\Big) \tag{3.24c}$$

is asymptotically exponentially small with respect to $\epsilon$ in the closed interval $0 \le x \le a$.

iv) At $x = a$, the series

$$v = \sum_{k=0}^{n} \epsilon^k\, v_k\Big(\frac{a}{\epsilon}, \frac{y}{\epsilon}; \epsilon\Big) \tag{3.24d}$$

is asymptotically exponentially small with respect to $\epsilon$ in the interval $0 < y \le b$, but not in the closed interval $0 \le y \le b$. To overcome this difficulty, we will define a series, called an elliptic corner layer (ECL), at the outflow corner $(a, 0)$. The ECL is constructed in Section 3.6.

**3.4. Parabolic boundary layer along the characteristic boundary $y = 0$.** The stretched variable $Y$ along the characteristic boundary $y = 0$ is defined by $y = \sqrt{\epsilon}\, Y$. The parabolic boundary layer along the characteristic boundary $y = 0$ is defined by the series

$$z(x, Y; \epsilon) = \sum_{k=0}^{n} \epsilon^k z_k(x, Y), \tag{3.25}$$

where the functions $z_k(x, Y)$ are defined iteratively by the parabolic differential equations

$$-\frac{\partial^2 z_k}{\partial Y^2} + p \frac{\partial z_k}{\partial x} + q z_k = \mu_k(x, Y) \tag{3.26}$$

over the semi-strip $0 < x < a$ and $0 < Y < \infty$. The functions $\mu_k$ are given by

$$\mu_0(x, Y) = 0$$

and for $1 \le k \le n$,

$$\mu_k(x, Y) = \frac{\partial^2 z_{k-1}}{\partial x^2}.$$

We impose boundary conditions on the functions $z_k$ to eliminate the discrepancy along the boundary $y = 0$ introduced by both the outer approximation $u(x, y; \epsilon)$ and the elliptic boundary layer $v(X, Y_1; \epsilon)$, and to arrange that the functions $z_k(x, Y)$ approach zero for $Y \neq 0$ and as $\epsilon \to 0$. For this, we impose the following initial-boundary conditions

$$z_k(0, Y) = 0, \tag{3.27a}$$

$$z_k(x, 0) = \gamma_k(x), \tag{3.27b}$$

and

$$z_k(x, Y) \to 0 \quad \text{as } Y \to \infty. \tag{3.27c}$$

The functions $\gamma_k(x)$ are defined by

$$\gamma_0(x) = g_3(x) - u_0(x, 0) - \sum_{i=1}^{N} \frac{x^i}{i!} \Big[ g_3^{(i)}(0) - \frac{\partial^i}{\partial x^i} u_0(0, 0) \Big],$$

and for $1 \le k \le n$,

$$\gamma_k(x) = -\Big[ u_k(x, 0) - \sum_{i=1}^{N-2k} \frac{x^i}{i!} \frac{\partial^i}{\partial x^i} u_k(0, 0) \Big].$$

Note that $\gamma_k(0) = 0$ for each $k$.

THEOREM 3.7. *The solutions $z_k(x,Y)$ for the initial-boundary value problems* (3.26), (3.27a,b,c) *have the integral representations*

$$z_0(x,Y) = \sqrt{\frac{2}{\pi}} \int_{Y/\sqrt{2x/p}}^{\infty} \exp\Big(-\frac{t^2}{2}\Big)\, \gamma_0\Big(x - \frac{pY^2}{2t^2}\Big) \exp\Big(-\frac{qY^2}{2t^2}\Big)\, dt, \tag{3.28}$$

*and for* $1 \le k \le n$,

$$\begin{aligned} z_k(x,Y) = {} & \sqrt{\frac{2}{\pi}} \int_{Y/\sqrt{2x/p}}^{\infty} \exp\Big(-\frac{t^2}{2}\Big)\, \gamma_k\Big(x - \frac{pY^2}{2t^2}\Big) \exp\Big(-\frac{qY^2}{2t^2}\Big)\, dt \\ & + \frac{1}{2\sqrt{\pi}} \int_0^{x/p} \int_0^{\infty} \frac{1}{\sqrt{s}} \Big\{ \exp\Big[-\frac{(Y-t)^2}{4s}\Big] - \exp\Big[-\frac{(Y+t)^2}{4s}\Big] \Big\} \\ & \cdot \frac{\partial^2}{\partial x^2}\, z_{k-1}(x - ps, t) \exp(-qs)\, dt\, ds. \end{aligned} \tag{3.29}$$

*Proof.* The transformation

$$z_k(x,Y) = z_k^*\Big(\frac{x}{p}, Y\Big) \exp\Big(-\frac{qx}{p}\Big)$$

yields the heat equation for $z_k^*(x,Y)$

$$-\frac{\partial^2 z_k^*}{\partial Y^2} + \frac{\partial z_k^*}{\partial x} = \mu_k(px, Y) \exp(qx) \tag{3.30}$$

over the semi-strip $0 < x < a/p$ and $0 < Y < \infty$ under the initial-boundary conditions

$$z_k^*(0,Y) = 0, \quad z_k^*(x,0) = \gamma_k(px) \exp(qx),$$

and

$$z_k^*(x,Y) \to 0 \quad \text{as } Y \to \infty.$$

The fundamental solution for the differential operator of (3.30) is

$$K(x,Y) = \frac{1}{\sqrt{4\pi x}} \exp\Big(-\frac{Y^2}{4x}\Big),$$

and hence Green's function for the differential operator over the semi-strip $0 < x < a/p$ and $0 < Y < \infty$ is given by

$$G(x,Y;s,t) = K(x-s, Y-t) - K(x-s, Y+t).$$

Therefore the solutions $z_k^*(x,Y)$ can be expressed as [5]

$$\begin{aligned} z_k^*(x,Y) = {} & \int_0^x \frac{\partial G}{\partial t}(x,Y;s,0)\, \gamma_k(ps) \exp(qs)\, ds \\ & + \int_0^x \int_0^{\infty} G(x,Y;s,t)\, \mu_k(ps,t) \exp(qs)\, dt\, ds. \end{aligned}$$

A computation shows that

$$\frac{\partial G}{\partial t}(x, Y; s, 0) = -2\,\frac{\partial}{\partial Y}\,K(x-s, Y)$$
$$= \frac{Y}{2\sqrt{\pi}\,(x-s)^{3/2}}\exp\Big[-\frac{Y^2}{4(x-s)}\Big],$$

and then we obtain

$$z_k^*(x,Y) = \frac{Y}{2\sqrt{\pi}}\int_0^x \frac{1}{(x-s)^{3/2}}\exp\Big[-\frac{Y^2}{4(x-s)}\Big]\,\gamma_k(ps)\exp(qs)\,ds$$
$$+\int_0^x\int_0^\infty G(x,Y;s,t)\,\mu_k(ps,t)\exp(qs)\,dt\,ds.$$

Making the changes of integrators as follows: let $Y/\sqrt{2(x-t)} = t$ in the first integral, and let $x - s = s'$ and then replace $s'$ by $s$ in the second integral of $z_k^*$, we get

$$z_k^*(x,Y) = \sqrt{\frac{2}{\pi}}\int_{Y/\sqrt{2x}}^\infty \exp\Big(-\frac{t^2}{2}\Big)\,\gamma_k\Big(px-\frac{pY^2}{2t^2}\Big)\exp\Big(qx-\frac{qY^2}{2t^2}\Big)\,dt$$
$$+\frac{1}{2\sqrt{\pi}}\int_0^x\int_0^\infty \frac{1}{\sqrt{s}}\Big\{\exp\Big[-\frac{(Y-t)^2}{4s}\Big]-\exp\Big[-\frac{(Y+t)^2}{4s}\Big]\Big\}$$
$$\cdot\,\mu_k(px-ps,t)\exp[q(x-s)]\,dt\,ds,$$

which gives the desired expressions for $z_k(x,Y)$. This completes the proof.

THEOREM 3.8. *There exist two positive constants $C$ and $c$ independent of $\epsilon$ such that the inequalities*

$$\Big|\frac{\partial^i}{\partial x^i}\,z_k(x,Y)\Big| \le C\exp(-cY), \tag{3.31}$$

*and*

$$\Big|\frac{\partial^{2i}}{\partial Y^{2i}}\,z_k(x,Y)\Big| \le C\exp(-cY), \tag{3.32}$$

*hold for $0 \le i \le 2n+2-2k$. The inequalities* (3.32) *will be used in Section* 3.5 *to obtain estimates for the ordinary corner layer functions $W_k(X_1,Y)$.*

*Proof.* First of all, let us prove the inequality (3.31) when $k = 0$. Thanks to the construction of $\gamma_0(x)$, it is easy to see that

$$\gamma_0^{(i)}(0) = 0,$$

for $i = 0, 1, 2, \ldots, 2n+1$ (the value of $N$ should be at least $2n+1$), and hence we have

$$\frac{\partial^i}{\partial x^i}\,z_0(x,Y) = \sqrt{\frac{2}{\pi}}\int_{Y/\sqrt{2x/p}}^\infty \exp\Big(-\frac{t^2}{2}\Big)\,\gamma_0^{(i)}\Big(x-\frac{pY^2}{2t^2}\Big)\,dt,$$

for $0 \le i \le 2n+2$. Since

$$\left|\gamma_0^{(i)}\Big(x - \frac{pY^2}{2t^2}\Big)\right| \le C$$

and for arbitrary constant $c > 0$,

$$\begin{aligned}\exp\Big(-\frac{t^2}{2}\Big) &\le \exp\Big(\frac{c^2}{2} - ct\Big) = \exp\Big(\frac{c^2}{2}\Big)\cdot\exp(-ct)\\ &= C\exp(-ct),\end{aligned}$$

we have

$$\begin{aligned}\left|\frac{\partial^i}{\partial x^i} z_0(x,Y)\right| &\le C\int_{Y/\sqrt{2x/p}}^{\infty} \exp(-ct)\,dt = C\exp\Big(-c\frac{Y}{\sqrt{2x/p}}\Big)\\ &\le C\exp\Big(-c\frac{Y}{\sqrt{2a/p}}\Big) = C\exp(-cY),\end{aligned}$$

for $0 \le Y < \infty$. It follows from (3.26) that $\partial^{2i}z_0/\partial Y^{2i}$ is a linear combination of $\partial^j z_0/\partial x^j$ for $0 \le j \le i$. Consequently, the inequality (3.32) is obtained for $k = 0$. Next consider the case $k = 1$, we have from (3.29)

$$\begin{aligned}\frac{\partial^i}{\partial x^i} z_1(x,Y) &= \sqrt{\frac{2}{\pi}}\int_{Y/\sqrt{2x/p}}^{\infty} \exp\Big(-\frac{t^2}{2}\Big)\gamma_1^{(i)}\Big(x - \frac{pY^2}{2t^2}\Big)\exp\Big(-\frac{qY^2}{2t^2}\Big)\,dt\\ &\quad + \frac{1}{2\sqrt{\pi}}\int_0^{x/p}\int_0^{\infty}\frac{1}{\sqrt{s}}\Big[\exp\Big(-\frac{(Y-t)^2}{4s}\Big) - \exp\Big(-\frac{(Y+t)^2}{4s}\Big)\Big]\\ &\qquad\qquad \cdot\frac{\partial^{2+i}}{\partial x^{2+i}} z_0(x-ps,t)\,dt\,ds\end{aligned}$$

for $0 \le i \le 2n$, because of

$$\gamma_1^{(i)}(0) = 0$$

for $0 \le i \le 2n-1$ and

$$\frac{\partial^j}{\partial x^j} z_0(0,y) = 0$$

for $2 \le j \le 2n{+}1$. (Note that $(\partial^j/\partial x^j)z_0(0,Y)$ is a linear combination of $(\partial^{2i}/\partial Y^{2i})z_0(0,Y)$ for $0 \le i \le j$.) As in the case $k = 0$, the first integral of $(\partial^i/\partial x^i)z_1(x,Y)$ has the desired estimate, and therefore we consider the second integral only. Now the estimates

$$\left|\frac{\partial^{2+i}}{\partial x^{2+i}} z_0(x-ps,t)\right| \le C\exp(-ct)$$

and

$$\begin{aligned}&\left|\exp\Big(-\frac{(Y-t)^2}{4s}\Big) - \exp\Big(-\frac{(Y+t)^2}{4s}\Big)\right|\\ &\quad\le \exp\Big(-\frac{(Y-t)^2}{4s}\Big) \le C\exp\Big(-c\frac{|Y-t|}{2\sqrt{s}}\Big)\\ &\quad\le C\exp\Big(-c\frac{|Y-t|}{2\sqrt{x/p}}\Big) \le C\exp(-c|Y-t|)\end{aligned}$$

imply

$$\begin{aligned}
&\left|\text{second integral of } \frac{\partial^i}{\partial x^i}\, z_1(x,Y)\right| \\
&\quad \le C\int_0^{x/p}\int_0^{\infty} \frac{1}{\sqrt{s}}\big[\exp(-c|Y-t|)\exp(-ct)\big]\,dt\,ds \\
&\quad = C\int_0^{x/p} \frac{1}{\sqrt{s}}\,ds \int_0^{\infty} \exp\big[-c(|Y-t|+t)\big]\,dt \\
&\quad = C\exp(-cY).
\end{aligned}$$

This completes the case $k = 1$ for inequality (3.31). From (3.26), one can see that $\partial^{2i} z_1/\partial Y^{2i}$ depends linearly on $\partial^j z_1/\partial x^j$ and $\partial^{\ell+1} z_0/\partial x^{\ell+1}$ for $0 \le i \le 2n$, $0 \le j \le i$ and $1 \le \ell \le i$.

Continuing in this manner, one can show the inequalities (3.31) and (3.32) for $k = 2, 3, \ldots, n$. This completes the proof.

Now applying the differential operator $L_\epsilon$ to the series $z$ yields

$$\begin{aligned}
L_\epsilon z &= \sum_{k=0}^{n} \epsilon^k\, L_\epsilon z_k \\
&= \sum_{k=0}^{n} \epsilon^k \Big(-\epsilon\,\frac{\partial^2 z_k}{\partial x^2} - \epsilon\,\frac{\partial^2 z_k}{\partial y^2} + p\,\frac{\partial z_k}{\partial x} + q z_k\Big).
\end{aligned} \tag{3.33}$$

Equation (3.26) can be written as

$$-\epsilon\,\frac{\partial^2 z_k}{\partial y^2} + p\,\frac{\partial z_k}{\partial x} + q z_k = \mu_k\Big(x, \frac{y}{\sqrt{\epsilon}}\Big). \tag{3.34}$$

Substitution of (3.34) into (3.33) gives

$$\begin{aligned}
L_\epsilon z &= -\sum_{k=0}^{n} \epsilon^{k+1}\,\frac{\partial^2 z_k}{\partial x^2} + \sum_{k=0}^{n} \epsilon^k\,\mu_k\Big(x, \frac{y}{\sqrt{\epsilon}}\Big) \\
&= -\epsilon\,\frac{\partial^2 z_n}{\partial x^2}\,.
\end{aligned} \tag{3.35}$$

It follows from (3.31) that the estimate

$$L_\epsilon z = O\big(\epsilon^{n+1}\big)$$

is valid uniformly in the closure of $\Omega$. Moreover, let us examine the parabolic boundary layer $z$ along the boundary of $\Omega$.

i) At $x = 0$,

$$z = 0. \tag{3.36a}$$

ii) At $y = 0$,

$$z = u^*(x,0;\epsilon) - v\Big(\frac{x}{\epsilon}, 0; \epsilon\Big). \tag{3.36b}$$

iii) At $y = b$, the series

$$z = \sum_{k=0}^{n} \epsilon^k z_k\left(x, \frac{b}{\sqrt{\epsilon}}\right) \tag{3.36c}$$

is asymptotically exponentially small with respect to $\epsilon$ in the closed interval $0 \leq x \leq a$.

iv) At $x = a$, the series

$$z = \sum_{k=0}^{n} \epsilon^k z_k\left(a, \frac{y}{\sqrt{\epsilon}}\right) \tag{3.36d}$$

is asymptotically exponentially small with respect to $\epsilon$ in the interval $0 < y \leq b$, but not $0 \leq y \leq b$. In the next section we define a series, called ordinary corner layer, at the outflow corner $(a, 0)$ to overcome this difficulty.

**3.5. Ordinary corner layer at the outflow corner $(a, 0)$.** As defined above, the stretched variables $X_1$ and $Y$ are expressed by $x = a - \epsilon X_1$ and $y = \sqrt{\epsilon}\, Y$. The ordinary corner layer at the outflow corner $(a, 0)$ is defined by the series

$$W(X_1, Y; \epsilon) = \sum_{k=0}^{n+1} \epsilon^k W_k(X_1, Y), \tag{3.37}$$

where the functions $W_k(X_1, Y)$ are defined iteratively by the ordinary differential equations

$$\frac{\partial^2 W_k}{\partial {X_1}^2} + p\, \frac{\partial W_k}{\partial X_1} = \tau_k(X_1, Y) \tag{3.38}$$

over the unbounded interval $0 < X_1 < \infty$. In these equations, $Y$ may be regarded as a parameter $0 \leq Y < \infty$ and the functions $\tau_k$ are defined as

$$\tau_0(X_1, Y) = 0,$$

and for $1 \leq k \leq n + 1$,

$$\tau_k(X_1, Y) = -\frac{\partial^2 W_{k-1}}{\partial Y^2} + qW_{k-1}.$$

The boundary conditions imposed on the functions $W_k$ remove the discrepancy along the outflow boundary $x = a$ introduced by the parabolic boundary layer $z$ and are such that the functions $W_k(X_1, Y)$ become boundary layer functions with respect to $X_1$. That is, we define

$$\begin{aligned} &W_k(0, Y) = -z_k(a, Y) \quad \text{for } 0 \leq k \leq n, \\ &W_{n+1}(0, Y) = 0, \end{aligned} \tag{3.39a}$$

and for $0 \le k \le n+1$,

$$W_k(X_1, Y) \to 0 \quad \text{as } X_1 \to \infty. \tag{3.39b}$$

It is easy to see that the function $W_0(X_1, Y)$ has the following representation:

$$W_0(X_1, Y) = -z_0(a, Y)\exp(-pX_1).$$

In general, for $k \ge 1$, the functions $\tau_k(X_1, Y)$ are the products of a boundary layer function $\exp(-pX_1)$ and a polynomial of degree $k-1$ in $X_1$ with the coefficients depending on the parabolic boundary layer functions $z_i(a, Y)$, $0 \le i \le k-1$, and their even-order partial derivatives with respect to $Y$. Therefore, it follows from (3.12) that the solutions $W_k(X_1, Y)$ can be expressed as

$$W_k(X_1, Y) = m_k(X_1, Y)\exp(-pX_1),$$

where the function $m_k$ is a polynomial of degree $k$ in $X_1$ with the coefficients depending on the functions $z_i(a, Y)$, $0 \le i \le k$ ($0 \le i \le n$ when $k = n+1$), and their even-order partial derivatives with respect to $Y$, $(\partial^{2j}/\partial Y^{2j})z_i(a, Y)$, for $0 \le i \le k-1$, and $1 \le j \le k-i$.

Note that the functions $W_k(X_1, Y)$ and their second partial derivatives with respect to $Y$ (the latter is required for the estimate of $L_\epsilon W$) determine how smooth the parabolic boundary layer functions $z_k(x, Y)$ should be in the domain $0 \le x \le a$ and $0 \le Y < \infty$.

THEOREM 3.9. *There exist two positive constants $C$ and $c$ independent of $\epsilon$ such that the inequalities*

$$\left|\frac{\partial^i}{\partial Y^i} W_k(X_1, Y)\right| \le C\exp\left[-c(X_1 + Y)\right] \tag{3.40}$$

*hold for $i = 0$ and $i = 2$.*

*Proof.* The functions $W_k$ and $\partial^2 W_k/\partial Y^2$ are products of $\exp(-pX_1)$ and a polynomial in $X_1$ with the coefficients depending on the functions $z_i$ and $\partial^{2j} z_i/\partial Y^{2j}$. It follows from (3.32) that we have the desired inequalities.

Now applying the differential operator $L_\epsilon$ to the ordinary corner layer $W$ yields

$$\begin{aligned} L_\epsilon W &= \sum_{k=0}^{n+1} \epsilon^k L_\epsilon W_k \\ &= \sum_{k=0}^{N+1} \epsilon^k \Big(-\epsilon \frac{\partial^2 W_k}{\partial x^2} - \epsilon \frac{\partial^2 W_k}{\partial y^2} + p\frac{\partial W_k}{\partial x} + qW_k\Big). \end{aligned} \tag{3.41}$$

Equation (3.38) can be written as

$$-\epsilon\frac{\partial^2 W}{\partial x^2} + p\frac{\partial W_k}{\partial x} = -\epsilon^{-1}\tau_k\Big(\frac{a-x}{\epsilon}, \frac{y}{\sqrt{\epsilon}}\Big). \tag{3.42}$$

Substitution of (3.42) into (3.41) gives

$$L_\epsilon W = \epsilon^{n+1}\Big(-\frac{\partial^2 W_{n+1}}{\partial Y^2} + qW_{n+1}\Big). \tag{3.43}$$

It follows from (3.40) that the estimate

$$L_\epsilon W = O(\epsilon^{n+1})$$

holds uniformly in the closure of $\Omega$. Moreover, let us examine the ordinary corner layer $W$ along the boundary of $\Omega$.

i) At $x = a$,

$$W = -z(a, Y; \epsilon). \tag{3.44a}$$

ii) At $x = 0$, the series

$$W = \sum_{k=0}^{n+1} \epsilon^k W_k\Big(\frac{a}{\epsilon}, \frac{y}{\sqrt{\epsilon}}\Big) \tag{3.44b}$$

is asymptotically exponentially small with respect to $\epsilon$ in the closed interval $0 \le y \le b$.

iii) At $y = b$, the series

$$W = \sum_{k=0}^{n+1} \epsilon^k W_k\Big(\frac{a-x}{\epsilon}, \frac{b}{\sqrt{\epsilon}}\Big) \tag{3.44c}$$

is asymptotically exponentially small with respect to $\epsilon$ in the closed interval $0 \le x \le a$.

iv) At $y = 0$, the series

$$W = \sum_{k=0}^{n+1} \epsilon^k W_k\Big(\frac{a-x}{\epsilon}, 0\Big) \tag{3.44d}$$

is asymptotically exponentially small with respect to $\epsilon$ in the interval $0 \le x < a$, but not the closed interval $0 \le x \le a$. We shall define a series, called the elliptic corner layer, at the outflow corner $(a, 0)$ to overcome this difficulty.

**3.6. Elliptic corner layer at the outflow corner $(a,0)$.** As defined above, the stretched variables $X_1$ and $Y_1$ are expressed by $x = a - \epsilon X_1$ and $y = \epsilon Y_1$. The elliptic corner layer at the outflow corner $(a, 0)$ is defined by the series

$$V(X_1, Y_1; \epsilon) = \sum_{k=0}^{n+1} \epsilon^k V_k(X_1, Y_1; \epsilon), \tag{3.45}$$

where the functions $V_k(X_1, Y_1; \epsilon)$ are defined by the elliptic differential equations

$$-\Big(\frac{\partial^2 V_k}{\partial {X_1}^2} + \frac{\partial^2 V_k}{\partial {Y_1}^2}\Big) - p\frac{\partial V_k}{\partial X_1} + \epsilon q V_k = 0 \tag{3.46}$$

over the quarter plane $0 < X_1 < \infty$ and $0 < Y_1 < \infty$. The boundary conditions for $V_k(X_1, Y_1; \epsilon)$ are specified so that

i) the discrepancy introduced by both the ordinary boundary layer $w(X_1, y; \epsilon)$ and the ordinary corner layer $W(X_1, Y; \epsilon)$ in the boundary condition along $y = 0$ is eliminated;

ii) the discrepancy introduced by the elliptic boundary layer $v(X, Y_1; \epsilon)$ in the boundary condition along $x = a$ is eliminated; and

iii) the functions $V_k(X_1, Y_1; \epsilon)$ are corner layer functions with respect to $X_1$ and $Y_1$.

Thus, we impose the following conditions

$$\begin{aligned} V_k(X_1, 0; \epsilon) &= \xi_k(X_1) \qquad\qquad 0 \le k \le n+1, \\ &\equiv -w_k(X_1, 0) - W_k(X_1, 0), \end{aligned} \tag{3.47a}$$

$$\begin{aligned} V_k(0, Y_1; \epsilon) &= \varsigma_k(Y_1; \epsilon) \\ &\equiv \begin{cases} -v_k\left(\dfrac{a}{\epsilon}, Y_1; \epsilon\right), & 0 \le k \le n, \\ 0, & k = n+1, \end{cases} \end{aligned} \tag{3.47b}$$

and

$$V_k(X_1, Y_1; \epsilon) \to 0 \qquad \text{as } X_1^2 + Y_1^2 \to \infty,\ 0 \le k \le n+1. \tag{3.47c}$$

Note that $\xi_k(0) = \varsigma_k(0; \epsilon)$ for $0 \le k \le n+1$. The elliptic problem (3.46), (3.47a,b,c) has a unique solution, and the maximum principle is valid for this problem. Note that the parameter $\epsilon$ appears in the equation (3.46) as a regular perturbation problem. The particular form of the functions $V_k(X_1, Y_1; \epsilon)$ was chosen to make the computations more tractable.

THEOREM 3.10. *The solutions* $V_k(X_1, Y_1; \epsilon)$ *of the boundary value problem* (3.46), (3.47a,b,c) *have the integral representations*

$$\begin{aligned} V_k(X_1, Y_1; \epsilon) = {} & \frac{\tau Y_1}{\pi} \int_0^\infty \left( \frac{K_1(\tau\rho_5)}{\rho_5} - \frac{K_1(\tau\rho_6)}{\rho_6} \right) \xi_k(s) \exp\left( \frac{p(s - X_1)}{2} \right) ds \\ & + \frac{\tau X_1}{\pi} \int_0^\infty \left( \frac{K_1(\tau\rho_7)}{\rho_7} - \frac{K_1(\tau\rho_8)}{\rho_8} \right) \varsigma_k(t; \epsilon) \exp\left( -\frac{pX_1}{2} \right) dt, \end{aligned}$$

*where*

$$\begin{aligned} \rho_5 &= \left[ (X_1 - s)^2 + Y_1^2 \right]^{1/2}, & \rho_6 &= \left[ (X_1 + s)^2 + Y_1^2 \right]^{1/2}, \\ \rho_7 &= \left[ X_1^2 + (Y_1 - t)^2 \right]^{1/2}, & \rho_8 &= \left[ X_1^2 + (Y_1 + t)^2 \right]^{1/2}, \end{aligned}$$

*and*

$$\tau = \left[ \frac{p^2}{4} + \epsilon q \right]^{1/2}.$$

*Proof.* The transformation

$$V_k(X_1, Y_1; \epsilon) = V_k^*(X_1, Y_1; \epsilon) \exp\left( -\frac{pX_1}{2} \right)$$

yields the differential equation for $V_k^*$

$$-\left( \frac{\partial^2 V_k^*}{\partial {X_1}^2} + \frac{\partial^2 V_k^*}{\partial {Y_1}^2} \right) + \left( \frac{p^2}{4} + \epsilon q \right) V_k^* = 0$$

over the quarter plane $0 < X_1 < \infty$ and $0 < Y_1 < \infty$ under the boundary conditions

$$V_k^*(X_1, 0; \epsilon) = \xi_k(X_1) \exp\Big(\frac{pX_1}{2}\Big),$$

$$V_k^*(0, Y_1; \epsilon) = \varsigma_k(Y_1; \epsilon)$$

and

$$V_k^*(X_1, Y_1; \epsilon) \to 0 \qquad \text{as } X_1^2 + Y_1^2 \to \infty,\ Y_1 > 0.$$

As in the case of the elliptic boundary layer, the solution $V_k^*$ has the expression of the form

$$V_k^*(X_1, Y_1; \epsilon) = \int_0^\infty \xi_k(s) \exp\Big(\frac{ps}{2}\Big) \frac{\partial G}{\partial t}(X_1, Y_1; s, 0)\, ds + \int_0^\infty \varsigma_k(t; \epsilon) \frac{\partial G}{\partial s}(X_1, Y_1; 0, t)\, dt,$$

where the Green's function for this problem is given by

$$G(X_1, Y_1; s, t) = \frac{1}{2\pi}\big[K_0(\tau\rho_1) - K_0(\tau\rho_2) + K_0(\tau\rho_3) - K_0(\tau\rho_4)\big],$$

with

$$\rho_1 = \big[(X_1 - s)^2 + (Y_1 - t)^2\big]^{1/2}, \qquad \rho_2 = \big[(X_1 + s)^2 + (Y_1 - t)^2\big]^{1/2},$$
$$\rho_3 = \big[(X_1 + s)^2 + (Y_1 + t)^2\big]^{1/2}, \qquad \rho_4 = \big[(X_1 - s)^2 + (Y_1 + t)^2\big]^{1/2}.$$

Computations give

$$\frac{\partial G}{\partial t}(X_1, Y_1; s, 0) = \frac{\tau Y_1}{\pi}\Big[\frac{K_1(\tau\rho_5)}{\rho_5} - \frac{K_1(\tau\rho_6)}{\rho_6}\Big],$$

and

$$\frac{\partial G}{\partial s}(X_1, Y_1; 0, t) = \frac{\tau X_1}{\pi}\Big[\frac{K_1(\tau\rho_7)}{\rho_7} - \frac{K_1(\tau\rho_8)}{\rho_8}\Big],$$

and hence we obtain the desired integral representations. This completes the proof.

THEOREM 3.11. *There exist two positive constants $C$ and $c$ independent of $\epsilon$ such that the inequalities*

$$|V_k(X_1, Y_1; \epsilon)| \le C \exp\Big[-c\Big(X_1 + \sqrt{\Big(\frac{a}{\epsilon}\Big)^2 + Y_1^2} - \frac{a}{\epsilon}\Big)\Big] \tag{3.48}$$

*hold.*

*Proof.* From (3.22), we have

$$|V_k(0, Y_1; \epsilon)| \le C \exp\Big[-\frac{p}{2}\Big(\sqrt{\Big(\frac{a}{\epsilon}\Big)^2 + Y_1^2} - \frac{a}{\epsilon}\Big)\Big].$$

The estimates (3.13) and (3.40) give

$$|V_k(X_1, 0; \epsilon)| \le C \exp(-cX_1)$$

for some constant $c$ between $p/2$ and $p$. Let the barrier function $V^*(X_1, Y_1; \epsilon)$ be defined by

$$V^*(X_1, Y_1; \epsilon) = C \exp\Big[-\frac{p}{2}\Big(X_1 + \sqrt{\Big(\frac{a}{\epsilon}\Big)^2 + Y_1^2} - \frac{a}{\epsilon}\Big)\Big].$$

Then a computation yields

$$\begin{aligned} &-\Big(\frac{\partial^2 V^*}{\partial X_1{}^2} + \frac{\partial^2 V^*}{\partial Y_1{}^2}\Big) - p\,\frac{\partial V^*}{\partial X_1} + \epsilon q V^* \\ &\quad = V^*\Big(\frac{p^2}{4}\,\frac{(a/\epsilon)^2}{(a/\epsilon)^2 + Y_1^2} + \frac{p}{2}\,\frac{(a/\epsilon)^2}{\big[(a/\epsilon)^2 + Y_1^2\big]^{3/2}} + \epsilon q\Big) > 0. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \big|V_k(0, Y_1; \epsilon)\big| &\le V^*(0, Y_1; \epsilon), \\ \big|V_k(X_1, 0; \epsilon)\big| &\le V^*(X_1, 0; \epsilon), \end{aligned}$$

for sufficiently large values of $C$ in the definition of $V^*$. By the maximum principle, we are led to have the desired inequalities for $0 \le k \le n+1$, $0 \le X_1 < \infty$, and $0 \le Y_1 < \infty$. This completes the proof.

Now applying the differential operator $L_\epsilon$ to the elliptic corner layer $V$ yields

$$L_\epsilon V = \sum_{k=0}^{n+1} \epsilon^k \Big(-\epsilon\,\frac{\partial^2 V_k}{\partial x^2} - \epsilon\,\frac{\partial^2 V_k}{\partial y^2} + p\,\frac{\partial V_k}{\partial x} + q V_k\Big). \tag{3.49}$$

Equation (3.46) can be written as

$$-\epsilon\Big(\frac{\partial^2 V_k}{\partial x^2} + \frac{\partial^2 V_k}{\partial y^2}\Big) + p\,\frac{\partial V_k}{\partial x} + q V_k = 0, \tag{3.50}$$

for each $k$. Therefore substitution of (3.50) into (3.49) gives

$$L_\epsilon V = 0, \tag{3.51}$$

uniformly in the closure of $\Omega$. Moreover, let us examine this function $V$ along the boundary of $\Omega$.

i) At $y = 0$,

$$V = -w\Big(\frac{a-x}{\epsilon}, 0; \epsilon\Big) - W\Big(\frac{a-x}{\epsilon}, 0; \epsilon\Big). \tag{3.52a}$$

ii) At $x = a$,

$$V = -v\Big(\frac{a}{\epsilon}, \frac{y}{\epsilon}; \epsilon\Big). \tag{3.52b}$$

iii) At $y = b$, the series

$$V = \sum_{k=0}^{n+1} \epsilon^k V_k\Big(\frac{a-x}{\epsilon}, \frac{b}{\epsilon}; \epsilon\Big) \tag{3.52c}$$

is asymptotically exponentially small with respect to $\epsilon$ in the closed interval $0 \leq x \leq a$.

iv) At $x = 0$, the series

$$V = \sum_{k=0}^{n+1} \epsilon^k V_k\Big(\frac{a}{\epsilon}, \frac{y}{\epsilon}; \epsilon\Big) \tag{3.52d}$$

is asymptotically exponentially small with respect to $\epsilon$ in the closed interval $0 \leq y \leq b$.

By symmetry, the remaining functions $v_k^T$, $z_k^T$, $W_k^T$, and $V_k^T$ can be constructed similarly along the upper characteristic boundary $y = b$ for the terms TEBL, TPBL, TOCL, and TECL, respectively.

**3.7. Asymptotic representation of the solution.** Consider the remainder term $R_{n+1}(x, y; \epsilon)$, which is defined by

$$\begin{aligned}R_{n+1} &= u_\epsilon(x,y) - u(x,y;\epsilon) - w(X_1, y; \epsilon)\\ &\quad - v(X, Y_1; \epsilon) - z(x, Y; \epsilon) - W(X_1, Y; \epsilon) - V(X_1, Y_1; \epsilon)\\ &\quad - v^T(X, Y_1^T; \epsilon) - z^T(x, Y^T; \epsilon) - W^T(X_1, Y^T; \epsilon) - V^T(X_1, Y_1^T; \epsilon),\end{aligned}$$

where $u_\epsilon$ is the solution of the boundary value problem (1.1), (1.2a,b,c,d); the functions $u$, $w$, $v$, $z$, $W$, $V$ are defined in (3.4), (3.9), (3.18), (3.25), (3.37), (3.45), respectively; and analogously for the other terms. Here we have used the stretched variables: $X_1 = (a - x)/\epsilon$, $X = x/\epsilon$, $Y_1 = y/\epsilon$, $Y = y/\sqrt{\epsilon}$, $Y_1^T = (b - y)/\epsilon$, $Y^T = (b - y)/\sqrt{\epsilon}$. Then it follows from the analysis of the preceding sections that the function $R_{n+1}$ satisfies the elliptic differential equation

$$\begin{aligned}L_\epsilon R_{n+1} = \epsilon^{n+1}\Big(&\Delta u_n + \epsilon \frac{\partial^2 w_{n+1}}{\partial y^2} + \frac{\partial^2 w_n}{\partial y^2} - q w_{n+1}\\ &+ \frac{\partial^2 z_n}{\partial x^2} + \frac{\partial^2 W_{n+1}}{\partial Y^2} - q W_{n+1} + \frac{\partial^2 z_n^T}{\partial x^2} + \frac{\partial^2 W_{n+1}^T}{\partial Y^{T2}} - q W_{n+1}^T\Big),\end{aligned}$$

which is of order $O(\epsilon^{n+1})$ in the closure of $\Omega$, and we have the following conditions for $R_{n+1}$ along the boundary of $\Omega$.

i) At $x = 0$, the remainder

$$\begin{aligned}R_{n+1} = &-w\Big(\frac{a}{\epsilon}, y; \epsilon\Big) - W\Big(\frac{a}{\epsilon}, \frac{y}{\sqrt{\epsilon}}; \epsilon\Big) - V\Big(\frac{a}{\epsilon}, \frac{y}{\epsilon}; \epsilon\Big)\\ &- W^T\Big(\frac{a}{\epsilon}, \frac{b-y}{\sqrt{\epsilon}}; \epsilon\Big) - V^T\Big(\frac{a}{\epsilon}, \frac{b-y}{\epsilon}; \epsilon\Big)\end{aligned}$$

is asymptotically exponentially small for $0 \leq y \leq b$.

ii) At $x = a$,

$$R_{n+1} = 0 \quad \text{for } 0 \le y \le b.$$

iii) At $y = 0$, the remainder

$$R_{n+1} = -v^T\Big(\frac{x}{\epsilon}, \frac{b}{\epsilon}; \epsilon\Big) - z^T\Big(x, \frac{b}{\sqrt{\epsilon}}; \epsilon\Big) - W^T\Big(\frac{a-x}{\epsilon}, \frac{b}{\sqrt{\epsilon}}; \epsilon\Big) - V^T\Big(\frac{a-x}{\epsilon}, \frac{b}{\epsilon}; \epsilon\Big)$$

is asymptotically exponentially small for $0 \le x \le a$.

iv) At $y = b$, the remainder

$$R_{n+1} = -v\Big(\frac{x}{\epsilon}, \frac{b}{\epsilon}; \epsilon\Big) - z\Big(x, \frac{b}{\sqrt{\epsilon}}; \epsilon\Big) - W\Big(\frac{a-x}{\epsilon}, \frac{b}{\sqrt{\epsilon}}; \epsilon\Big) - V\Big(\frac{a-x}{\epsilon}, \frac{b}{\epsilon}; \epsilon\Big)$$

is asymptotically exponentially small for $0 \le x \le a$.

By using Theorem 3.3., we have the following estimate:

$$R_{n+1} = O\big(\epsilon^{n+1}\big),$$

which holds uniformly in the closure of $\Omega$. Furthermore, the functions $w_{n+1}$, $v_{n+1}$, $W_{n+1}$, and $V_{n+1}$ are uniformly bounded in the closure of $\Omega$. Consequently, we obtain finally the following:

THEOREM 3.12. *The solution $u_\epsilon(x, y)$ of the boundary value problem* (1.1), (1.2a,b,c,d) *has a uniform approximation $U(x, y; \epsilon)$ in the closure of the rectangular region $\Omega$ with error $O\big(\epsilon^{n+1}\big)$, where $U$ is defined by the series.*

$$\begin{aligned} U(x, y; \epsilon) = \sum_{k=0}^{n} \epsilon^k \big[&u_k(x, y) + w_k(X_1, y) + v_k(X, Y_1) + z_k(x, Y) \\ &+ W_k(X_1, Y) + V_k(X_1, Y_1) + v_k^T(X, Y_1^T) + z_k^T(x, Y^T) \\ &+ W_k^T(X_1, Y^T) + V_k^T(X_1, Y_1^T)\big]. \end{aligned}$$

**4. Magnetohydrodynamic flow in a rectangular duct with nonconducting walls.** The design of magnetohydrodynamic generators, flow-meters, pumps and accelerators requires an understanding of the flows of conducting fluids in rectangular ducts under transverse magnetic fields. These flows have received much attention from theoreticians because the governing equations are linear but the phenomena are neither trivially simple nor physically attainable in the laboratory.

In 1937, Jul. Hartmann [19] solved the one-dimensional problem where the flow was between two parallel walls, the fluid being virtually infinite in directions perpendicular to the imposed transverse magnetic field. Coordinates are then dependent on the transverse coordinate only.

We are concerned with the flow of a steady, incompressible, electrically conducting fluid through a rectangular duct with a uniform, external magnetic field applied transverse to the flow and parallel to two of the walls. Various forms of this problem with different combinations of conducting and nonconducting bounding walls have been considered, see J. A. Shercliff [41], C. C. Chang and T. S. Lundgren [6], W. E. Williams [47], J. C. R. Hunt [21], J. C. R. Hunt and K. Stewartson [22], D. Chiang and T. Lundgren [7], J. C. R. Hunt and J. A. Shercliff [23], D. J. Temperley and L. Todd [42], and L. A. Kalyakin [25] – [28].

As is known from the work of Shercliff [41] and Chang and Lundgren [6], the problem of magnetohydrodynamic flow in a rectangular duct with an applied magnetic field transverse to the axis of the duct is described by two second-order elliptic partial differential equations for the fluid velocity $V$ and the axial component of the magnetic field $B$, namely, the $z$-component of the momentum equation

$$\rho\nu\Delta V + \frac{B_0}{\mu_0}\frac{\partial B}{\partial x} = \frac{\partial p}{\partial z},$$

and the $z$-component of the curl of Ohm's law

$$\Delta B + \sigma\mu_0 B_0 \frac{\partial V}{\partial x} = 0,$$

where $\rho$ is the mass density; $\nu$ is the kinematic viscosity; $B_0$ is the magnitude of the transverse magnetic field which is applied in the $x$-direction; $\mu_0$ is the permeability in a vacuum; $\partial p/\partial z$ is the pressure gradient, which is a constant; and $\sigma$ is the electrical

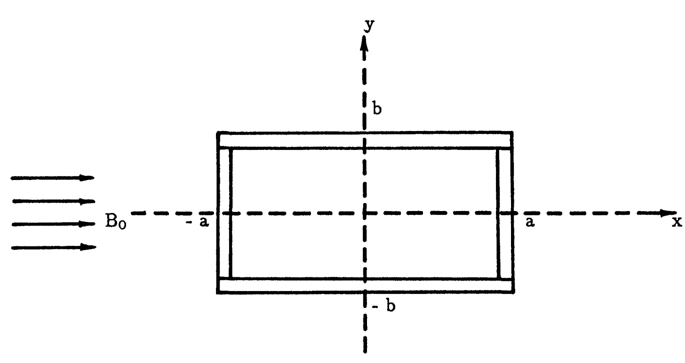


FIG. 4.1. *Rectangular duct.*

conductivity of the incompressible fluid medium. Let the origin be the centerline of the duct and let $2a$ and $2b$ be the lengths of the sides of the duct (see Figure 4.1).

In the case of nonconducting walls, the boundary conditions are

$$V = B = 0 \quad \text{at } x = \pm a,$$
$$V = B = 0 \quad \text{at } y = \pm b.$$

We shall examine the flow for large Hartmann number $M$, which is defined by

$$M = B_0 a\sqrt{\frac{\sigma}{\rho\nu}},$$

or the small value of $\epsilon = 1/M$. With these dimensionless variables

$$\xi = \frac{x}{a}, \qquad \nu = \frac{y}{a}, \qquad V^* = 2\mu_0\sqrt{\rho\nu\sigma}\, B_0^{-1}V,$$
$$B^* = 2B_0^{-1}B, \qquad P = -2\mu_0 a B_0^{-2}\frac{\partial p}{\partial z},$$

the problem is to solve

$$\epsilon\Big(\frac{\partial^2 V^*}{\partial \xi^2} + \frac{\partial^2 V^*}{\partial \eta^2}\Big) + \frac{\partial B^*}{\partial \xi} = -P,$$
$$\epsilon\Big(\frac{\partial^2 B^*}{\partial \xi^2} + \frac{\partial^2 B^*}{\partial \eta^2}\Big) + \frac{\partial V^*}{\partial \xi} = 0,$$

with the boundary conditions

$$V^* = B^* = 0 \quad \text{at } \xi = \pm 1,$$
$$V^* = B^* = 0 \quad \text{at } \eta = \pm\ell, \ \ell = b/a.$$

The equations can be decoupled by the change of variables

$$u = P^{-1}(V^* - B^*),$$

and

$$v = P^{-1}(V^* - B^*).$$

The equations for $u$ and $v$ are

$$-\epsilon\Delta u + \frac{\partial u}{\partial \xi} = 1,$$

and

$$-\epsilon\Delta v - \frac{\partial v}{\partial \xi} = 1,$$

with the boundary conditions

$$u = v = 0 \quad \text{at } \xi = \pm 1,$$
$$u = v = 0 \quad \text{at } \eta = \pm\ell.$$

It is obvious that having determined the functions $u$, we can find $v$ by the relationship

$$v(\xi, \eta) = u(-\xi, \eta).$$

Hence the function $u$ alone needs to be investigated. The asymptotic approximation to $u$, being uniformly valid in the closure of the rectangular duct for large Hartmann number $M$, is still an open question in the literature, see Shercliff [41], Roberts [40, pp. 186–190], Cook, Ludford and Walker [9], and Temperley [43].

With an application of the preceding analysis, we conclude that the solution $u$ can be written as

$$u(\xi, \eta) = (1 + \xi) - 2\exp(-\xi_2) + v_1(\xi_1, \eta_1; \epsilon) + V_1(\xi_2, \eta_1; \epsilon)$$
$$+ v_2(\xi_2, \eta_1; \epsilon) + V_2(\xi_2, \eta_2; \epsilon) + \text{AES},$$

where the stretched variables are

$$\xi_1 = \frac{\xi+1}{\epsilon}, \qquad \xi_2 = \frac{1-\xi}{\epsilon}, \qquad \eta_1 = \frac{\ell-\eta}{\epsilon} \qquad \text{and} \qquad \eta_2 = \frac{\eta+\ell}{\epsilon};$$

the function $1+\xi$ is known as a "core flow"; the function $-2\exp(-\xi_2)$ is called a "Hartmann boundary layer"; the functions $v_1(\xi_1, \eta_1; \epsilon)$ and $v_2(\xi_1, \eta_2; \epsilon)$ satisfy the elliptic partial differential equation of the form

$$-\Big(\frac{\partial^2 v_i}{\partial {\xi_1}^2} + \frac{\partial^2 v_i}{\partial {\eta_i}^2}\Big) + \frac{\partial v_i}{\partial \xi_1} = 0 \quad \text{for } i = 1, 2,$$

over the quarter plane $0 < \xi_1 < \infty$ and $0 < \eta_i < \infty$ under the boundary conditions

$$v_i(\xi_1, 0; \epsilon) = -\epsilon\xi_1, \quad v_i(0, \eta_1; \epsilon) = 0,$$

and

$$v_i(\xi_1, \eta_i; \epsilon) \to 0 \quad \text{as } \xi_1^2 + \eta_i^2 \to \infty \text{ and } \eta_i > 0;$$

and the functions $V_1(\xi_2, \eta_1; \epsilon)$ and $V_2(\xi_2, \eta_2; \epsilon)$ satisfy the elliptic partial differential equation of the form

$$\Big(\frac{\partial^2 V_i}{\partial {\xi_2}^2} + \frac{\partial^2 V_i}{\partial {\eta_i}^2}\Big) + \frac{\partial V_i}{\partial \xi_2} = 0 \quad \text{for } i = 1, 2,$$

over the quarter plane $0 < \xi_2 < \infty$ and $0 < \eta_i < \infty$ under the boundary conditions

$$V_i(\xi_2, 0; \epsilon) = 2\exp(-\xi_2),$$
$$V_i(0, \eta_i; \epsilon) = -v_i\Big(\frac{2}{\epsilon}, \eta_i; \epsilon\Big),$$

and

$$V_i(\xi_2, \eta_i; \epsilon) \to 0 \quad \text{as } \xi_2^2 + \eta_i^2 \to \infty;$$

and AES denotes asymptotically exponentially small terms with respect to $\epsilon$ in the closure of the rectangular duct.

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