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Asymptotic Analysis for a Mixed Boundary-Value Contact Problem

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Abstract

The variational solution of the nonlinear Signorini contact problem determines also the active contact zone Γ_c. If the latter is known, then the elastic field is a solution of a linear mixed boundary value problem in which on Γ_c the normal displacement and tangential traction are given, while on the non-contact part the total traction is zero. Such mixed boundary conditions in general generate singularities of the solution's stress field at the points $P^{(k)}$ where the boundary conditions change. For smooth data, however, the variational solution of the Signorini contact problem actually belongs to $H^2(\Omega)^2$, which implies the disappearance of these singularities, i.e., that the corresponding stress intensity factors vanish.

This paper is devoted to the characterization of the active contact zone Γ_c by the vanishing stress intensity factors including their sensitivity with respect to varying Γ_c for two-dimensional problems provided that Γ_c consists of a finite number of intervals. We use the method of asymptotic expansions and derive an explicit formula for the sensitivity, which is rigorously justified by employing weighted Sobolev spaces with detached asymptotics. These results can be used to determine the points $P^{(k)}$ with a corresponding Newton iteration.

1. Introduction

In a plane, bounded domain Ω with smooth boundary Γ, we consider a mixed boundary value problem for the linearized elasticity equations in its natural relation to the Signorini problem in Ω. Originating from minimization problems in terms of the elastic energy functional, both problems admit a variational formulation which leads to continuous, coercive, monotone operators on $H^1(\Omega)^2$ and, therefore, provides their respective solvability and also uniqueness of their weak (energy) solutions. The Signorini problem involves unilateral constraints on Γ and, consequently, must be posed on a proper convex closed cone in $H^1(\Omega)^2$ while the active contact area is determined only after the problem is completely solved. At first sight, a possible irregular structure of the contact area seems to prevent having the Signorini solution good differentiability properties. However, due to the results in [7], [23] the solution belongs to $H^2(\Omega)^2$ if the data are sufficiently smooth. On the other hand, any smoothness restriction on the data of the mixed boundary value problem with an *a priori* given active contact area is insufficient to ensure the same property in its solution because of well-known square-root singularities at the collision points between the Neumann and the linearized contact conditions. Roughly speaking, our idea in this paper is to find the active contact area via the enforcement of additional smoothness on the solution of the mixed boundary value problem. This means that, by the proper choice of the contact area, we need to annul the so-called stress intensity factors, i.e., the coefficients at the singularities mentioned above (cf. [24]). Since those coefficients are not continuous functionals on $H^1(\Omega)^2$ and due to the complementary smoothness of the Signorini solution, we have to give up the variational formulation and treat classical solutions with singularities. For this purpose, the most suitable spaces are weighted spaces with detached asymptotics.

Now, let us describe our problem more precisely. We consider a body consisting of some homogeneous isotropic linear elastic material occupying the bounded domain $\Omega \subset \mathbb{R}^2$. The boundary Γ of this body is composed of three parts: Γ_D where boundary displacements g are given, Γ_N where boundary tractions h are prescribed, and Γ_C where the body is in active contact with a rigid obstacle. By $\boldsymbol{n} = (n_1, n_2)^\top$ we denote the exterior normal vector on Γ. The subscripts $_n$ and $_s$ denote, respectively, the normal and tangential components of vector fields on the boundary. The contact part of the boundary is assumed to be composed of $\mathcal{K}$ connected arcs $\{\Gamma(s) | s_{2k-1} \leqq s \leqq s_{2k}\}$, $k = 1, \ldots, \mathcal{K}$, with endpoints $P^{(k)} = \Gamma(s_k)$, $k = 1, \ldots, 2\mathcal{K}$. Here $\Gamma(s)$ is a parametrization of the boundary in terms of the arc-length parameter. The distance between the body and the obstacle, measured in the direction of the normal vector, will be given by the function G_n. Let $u = (u_1, u_2)^\top$ denote the displacement field, $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, 2$, the derivatives with respect to the space variables and $e_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ the components of the linearized strain tensor. In linearized elasticity, the relation between strain and stress for a homogeneous isotropic material is described by Hooke's law, i.e.,

$$\sigma_{ij}(u) = \lambda \delta_{ij} \operatorname{div}(u) + 2\mu e_{ij}(u)$$

with the Kronecker symbol δ_{ij} and the Lamé constants $\lambda \geqq 0, \mu > 0$. Let $\sigma_i^{(n)}(u) := \sigma_{ij}(u) n_j$ denote the components of the boundary traction. Here and throughout this paper, we use the Einstein summation convention.

The problem we investigate is given by the following set of equations:

$$L_i(u) := \partial_j \sigma_{ij}(u) = F_i \quad \text{in } \Omega, \tag{1}$$

$$u_i = G_i \quad \text{on } \Gamma_D, \tag{2}$$

$$\sigma_i^{(n)}(u) = H_i \quad \text{on } \Gamma_N, \ i = 1, 2\,, \tag{3}$$

$$u_n = G_n, \quad \sigma_s^{(n)}(u) = H_s \quad \text{on } \Gamma_C. \tag{4}$$

The given data F (the volume force), G and H are assumed to be sufficiently smooth functions. The differential operator is the Lamé operator, i.e.,

$$L_i(u) = (\lambda + \mu)\partial_i \partial_j u_j + \mu \partial_j \partial_j u_i. \tag{5}$$

At the endpoints $P^{(k)}$ of the active contact zone, the type of the boundary conditions changes. For a usual mixed boundary value problem (1)–(4), at these points there occur stress singularities. More precisely, the solution u admits near each of the $P^{(k)}$ a separate asymptotic expansion of the type

$$u = K_1^{(k)} r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) + \dots ,$$

where (r_k, φ_k) denote local polar coordinates with origin at $P^{(k)}$; and the dots ... symbolize terms having bounded first order derivatives. The corresponding stress intensity factor $K_1^{(k)}$ depends on the material parameters, on the given data F, G, H and on the geometry – in particular, on the position of the endpoints $P^{(k)}$. If these points are shifted along Γ to new positions $P^{(k)}(\varepsilon) = \Gamma(s_k + \varepsilon h_k)$, then the stress intensity factors change correspondingly, too. Hence, we need to derive and justify a formula describing the sensitivity of the stress intensity factors $K_1^{(k)}$ with respect to the variation of the points $P^{(k)}$ along Γ, i.e., to obtain an asymptotic representation of the type

$$K_1^{(k)}(\varepsilon) = K_1^{(k)} + \varepsilon M^{(k)}(h_1, \dots , h_{2\mathcal{K}}) + \cdots , \quad k = 1, \dots , 2\mathcal{K}.$$

Such a formula describing the sensitivity of the stress intensity factor with respect to a perturbation of the domain has been obtained in [15] for a two-dimensional straight crack and in [14] for a two-dimensional curved crack. The sensitivity of the end points of the active contact area in a Signorini problem with respect to a variation of the outer forces was studied in [17]. As was shown in [7] and [23], in the case of the Signorini problem the stress intensity factors $K_1^{(k)}$ vanish. For three-dimensional crack problems, applications of sensitivities of stress intensity factors with respect to variations of the crack front have been investigated in [22, 26, 3–5, 16, 10] (ordered according to publication dates).

For a straight boundary, the problem can be transformed to a two-dimensional crack problem. Due to some symmetry condition, the results then are the same as in [15]. However, for a curved boundary the situation is much more complicated, because the symmetry mentioned above does not lead to a crack problem but, under certain assumptions, to a problem with cuspidal boundary for which no results suitable for our purpose are available. As a consequence, the second order term in the asymptotic expansion of the solution – which is necessary to describe the sensitivity of the stress intensity factor – has a different form than in the case of crack problems. In particular we will see that there appear terms depending on the logarithm of the distance to the singular points. Our corresponding formula describing the sensitivity of the stress intensity factor is rigorously justified with the help of function spaces with detached asymptotics and overlapping cut-off functions.

The basic idea of our approach can be described as follows. The unique solution of the nonlinear Signorini contact problem coincides with the unique solution of the linear mixed boundary value problem (1)–(4) having vanishing stress intensity factors if the active contact zone Γ_C is chosen correctly. Of course, the necessary and sufficient conditions for a correct contact area are given by the compatibility conditions $u_n \leqq G_n$ and $\sigma_n = H_n$ outside Γ_C, and $\sigma_n \leqq H_n$ inside Γ_C. These relations between the normal components σ_n, u_n etc. of the fields ensure the validity of the nonlinear Signorini conditions

$$\sigma_n^{(n)} \leqq H_n,\ u_n \leqq G_n,\ \text{and } (\sigma_n^{(n)} - H_n)(u_n - G_n) = 0$$

on the whole part $\Gamma_C \cup \Gamma_N$ of the boundary, where contact is supposed to be possible. Of course, for the real physical situation considered, one needs to discuss whether or not the stress component $H_n \neq 0$ on the contact zone – e.g., superposition of stress fields as for the injection of pressurized gas, implies $H_n \neq 0$. The influence of the tangential stress in terms of friction is neglected in the Signorini problem but is determined by its solution, whereas in the mixed boundary value problem it is given as H_s in (3) and (4).

For the Signorini contact problem due to the regularity results in [7], [23], the solution has no stress singularities at the endpoints of the true contact area. Hence, the active contact zone Γ_C must be chosen in such a way that the stress intensity factors $K_1^{(k)}$ vanish. This property can already be found in SHTAERMAN's book [24] and provides us with an additional necessary condition for the determination of the true contact area. If a method for the calculation of the stress intensity factors and their sensitivity (in the sense mentioned above) is known, then one can perform the following Newton-type algorithm for the solution of contact-problems. We start with an initial guess at the active contact area. For the associated mixed boundary value problem (1)–(4) we calculate the stress intensity factors and their sensitivity. Solving the equation

$$K_1^{(k)} + \varepsilon M^{(k)}(h_1, \dots, h_{2\mathcal{K}}) = 0, \quad k = 1, \dots, 2\mathcal{K},$$

and putting $\tilde{P}^{(k)} = \Gamma(s_k + \varepsilon h_k)$, we obtain a new approximation of the contact zone. This procedure corresponds to one Newton step for solving the equation $K_1^{(k)}(P_1, \dots, P_{2\mathcal{K}}) = 0$. An iteration of this algorithm leads to a contact part Γ_C of the boundary at whose endpoints the stress intensity factors vanish. The described algorithm is related to the so-called active-set strategies for solving contact problems. Here, however, we use the results from the theory of singularities of elliptic boundary value problems which enable us to define a very efficient approximation of the contact area, and, at the same time, to improve any approximate solution of the Signorini contact problem. Clearly, during this procedure we assume that the topology of the contact zone does not change any more. Some results preventing the change of the topology can be found in [17].

2. Asymptotics of the solution near endpoints

2.1. Singular solutions for a model problem

In order to describe the behaviour of the solution near one of the endpoints $P^{(k)}$, where $k \in \{1, \dots, 2\mathcal{K}\}$ is a fixed index, we use Hadamard's natural, curvilinear coordinates: Each point in an appropriate tubular neighbourhood of $P^{(k)}$ can be represented uniquely by $(s, n) \in \mathbb{R}^2$ via $x = \Gamma(s) - n\boldsymbol{n}(s)$. Using the tubular coordinates (s, n), a new system of differential equations in the half-plane $(s, n) \in Q := \mathbb{R} \times \mathbb{R}_+$ is derived. The asymptotic expansions will later on be constructed by collecting the lower order terms on the right-hand sides of the differential equations and the boundary conditions leaving only the principal parts on the left-hand sides. The principal parts of the transformed differential equations and boundary conditions coincide with the elasticity operator and the original boundary conditions on the half-plane. Hence we obtain the following homogeneous half-plane model problem:

$$L(u) = 0 \;\text{ in } Q := \mathbb{R} \times \mathbb{R}_+, \tag{6}$$

$$\sigma_{i2}(u) = 0 \;\text{ on } \mathbb{R}_- \times \{0\}\,,\; i = 1, 2\,, \tag{7}$$

$$u_2 = 0 \;\text{ and } \sigma_{12}(u) = 0 \;\text{ on } \mathbb{R}_+ \times \{0\}\,. \tag{8}$$

In what follows, the boundary conditions (7), (8) will sometimes be denoted by $B(u)$. Problem (6)–(8) is equivalent to the model problem describing the asymptotic behaviour of a mode–I crack near the crack tip (see, e.g., [19]). To see this relation, we extend the solution u onto the lower half-plane by reflection, i.e., by the symmetry condition $u_1(x_1, -x_2) := u_1(x_1, x_2)$ and $u_2(x_1, -x_2) := -u_2(x_1, x_2)$ for $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}_+$. Due to this symmetry and the boundary conditions on $\mathbb{R}_+ \times \{0\}$, the extended solution satisfies the elasticity equations in the half-plane $\mathbb{R}^2 \setminus (\mathbb{R}_- \times \{0\})$ with a crack and the boundary conditions $\sigma_{i2}(u) = 0$ on $\mathbb{R}_- \times \{0\}$. Hence, from every solution of (6)–(8) we obtain a solution of the model problem with a crack respecting the symmetries mentioned above – these are just the solutions of a mode–I crack. Vice versa, the restriction of each solution to the mode–I crack on Q is a solution of (6)–(8). The formulae presented in this section can be found, e.g., in [19] and in [21].

Let (r, φ) denote polar coordinates in Q with the origin at $(0, 0)$. The model problem has solutions of the form

$$X^{(\nu)}(r, \varphi) = r^{\nu}\Phi^{(\nu)}(\varphi), \qquad X^{(-\nu)}(r, \varphi) = r^{-\nu}\Phi^{(-\nu)}(\varphi)$$

for $\nu \in \{0, \frac{1}{2}, 1, \dots\}$. In polar coordinates and polar components they are given by

$$\begin{aligned}
&\left(\Phi^{(\frac{m}{2})}(\varphi)\right)_r \\
&\qquad = D_{\frac{m}{2}}\left[(2 + m)\cos\left(\left(1 + \frac{m}{2}\right)\varphi\right) + (2\Theta - m)\cos\left(\left(1 - \frac{m}{2}\right)\varphi\right)\right], \\
&\left(\Phi^{(\frac{m}{2})}(\varphi)\right)_\varphi \\
&\qquad = D_{\frac{m}{2}}\left[-(2 + m)\sin\left(\left(1 + \frac{m}{2}\right)\varphi\right) - (2\Theta + m)\sin\left(\left(1 - \frac{m}{2}\right)\varphi\right)\right]
\end{aligned}$$

for even $m = 2k \neq 0$, and by

$$\begin{aligned}
&\left(\Phi^{(\frac{m}{2})}(\varphi)\right)_r \\
&\qquad = D_{\frac{m}{2}} \left[(m-2)\cos\left(\left(1+\frac{m}{2}\right)\varphi\right) + (2\Theta - m)\cos\left(\left(1-\frac{m}{2}\right)\varphi\right)\right], \\
&\left(\Phi^{(\frac{m}{2})}(\varphi)\right)_\varphi \\
&\qquad = D_{\frac{m}{2}} \left[(2-m)\sin\left(\left(1+\frac{m}{2}\right)\varphi\right) - (2\Theta + m)\sin\left(\left(1-\frac{m}{2}\right)\varphi\right)\right]
\end{aligned}$$

for odd $m = 2k+1$, where $\Theta := \frac{\lambda+3\mu}{\lambda+\mu}$. For $m = 0$, one obtains the two solutions

$$X^{(+0)}(r,\varphi) = D_0^{(1)} \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix} \tag{9}$$

and

$$\begin{aligned}
X^{(-0)}(r,\varphi) = D_0^{(2)} \Bigg(\log(r) \begin{pmatrix} 2(\Theta+1)\cos(\varphi) \\ -2(\Theta+1)\sin(\varphi) \end{pmatrix} \\
+ \begin{pmatrix} 2(\Theta-1)\varphi\sin(\varphi) \\ 2(\Theta-1)\varphi\cos(\varphi) - 4\sin(\varphi) \end{pmatrix} \Bigg).
\end{aligned} \tag{10}$$

Here, for ease of notation, we distinguish formally between index $+0$ and index -0. The so-called normalization factors $D_{\frac{m}{2}}$ are given by

$$D_{\frac{m}{2}} = \begin{cases} \dfrac{1}{2(1+\Theta)}, & m > 0 \text{ even} \\ \dfrac{1}{\sqrt{2\pi}} \dfrac{1}{4\mu m}, & m > 0 \text{ odd} \end{cases}, \quad D\frac{m}{2} = \begin{cases} \dfrac{1}{8\mu\pi m}, & m < 0 \text{ even} \\ -\dfrac{1}{\sqrt{8\pi}} \dfrac{1}{1+\Theta}, & m < 0 \text{ odd} \end{cases}$$

and $D_0^{(1)} = 1$, $D_0^{(2)} = -\frac{1}{16\mu\pi}$. For positive indices $m = \{+0, +1, \dots\}$, these normalization factors are chosen such that the conditions

$$\begin{aligned}
\sigma_{22}\left(X^{(+\frac{m}{2})}; x_1, 0\right) = \frac{1}{\sqrt{2\pi}} r^{\frac{m}{2}-1} \qquad &\text{for } x_1 > 0,\ m \geqq 1,\ m \text{ odd}, \\
X_1^{(+\frac{m}{2})}(x_1, 0) = r^{\frac{m}{2}} \qquad &\text{for } x_1 > 0,\ m \geqq 0,\ m \text{ even}
\end{aligned}$$

are valid on the contact area. For negative indices, the normalization constants follow from the orthogonality conditions

$$q\left(X^{(+\frac{m}{2})}, X^{(-\frac{\ell}{2})}\right) = \tfrac{1}{2}\delta_{\ell m}$$

with

$$\begin{aligned}
q(u,v) = \int_Q (L(\chi u)\cdot v - L(v)\cdot\chi u)\, dx \\
+ \int_{\mathbb{R}\times\{0\}} \left(\sigma^{(n)}(\chi u)\cdot v - \sigma^{(n)}(v)\cdot\chi u\right) ds_x.
\end{aligned}$$

The factor $\frac{1}{2}$ in the orthogonality condition appears because the integration is carried out over the half-plane instead of the plane with a crack. Here, $\chi \in C_0^\infty(\mathbb{R}^2, [0, 1])$ denotes an arbitrary cut-off function with $\chi \equiv 1$ near 0. The value of $q(u, v)$ is independent of the choice of χ if u and v are solutions of (6)–(8). With the help of the Green formula, the following representation of $q(u, v)$ can be obtained:

$$q(u, v) := \int_0^\pi \left\{\sigma^{(r)}(u; R, \varphi) \cdot v(R, \varphi) - \sigma^{(r)}(v; R, \varphi) \cdot u(R, \varphi)\right\} d\varphi.$$

The value of this integral is independent of the radius R if u and v are solutions of the model problem. The normalization of the power solutions is described more precisely in [20].

We remark that the solutions to the model problem satisfy the following recursive formulae of differentiation:

$$\begin{aligned} \partial_{x_1} X^{(\frac{1}{2})} &= -\frac{1+\Theta}{4\mu} X^{(-\frac{1}{2})}, \\ \partial_{x_1} X^{(\frac{m}{2})} &= \left(\frac{m}{2} - 1\right) X^{(\frac{m}{2}-1)} \text{ for } m \geqq 3, \\ \partial_{x_1} X^{(-\frac{m}{2})} &= -\frac{m}{2} X^{(-\frac{m}{2}-1)} \text{ for } m \geqq 1. \end{aligned} \tag{11}$$

These relations can be derived from the normalization conditions with the help of the identity

$$q\left(X, \frac{\partial}{\partial x_1} Y\right) = -q\left(\frac{\partial}{\partial x_1} X, Y\right)$$

which is proved in Lemma 7.4.4 of [19]. Hence, the singular functions for odd indices $m = 2k+1$ can be derived from the knowledge of $X^{(\frac{1}{2})}$ only; analogously, all functions with even indices are known if $X^{(-1)}$, $X^{(-0)}$, $X^{(+0)}$ and $X^{(1)}$ are known.

In order to describe the asymptotics of the solution to problem (1)–(4) near to the point $P^{(k)}$, the singular solutions $X^{(\frac{m}{2})}$ have to be transformed onto the domain of definition Ω. Let χ_k be a cut-off function with $\chi_k \equiv 1$ near $P^{(k)}$ such that $\operatorname{supp}(\chi_k)$ is contained in the set where the curvilinear coordinates (s, n) are defined. Then the function $\chi_k X^{(\frac{m}{2})}(s(x) - s_k, n(x))$ is defined on $\operatorname{supp}(\chi_k) \cap \Omega$ and can be extended by 0 onto the domain Ω. In what follows, the extended function will be denoted by $X^{(k, \frac{m}{2})}$. Let us define the basic function spaces with weighted norms:

Definition 2.1. *The space $V_\beta^\ell(\Omega)$, $\ell \in \mathbb{N}_0$, $\beta \in \mathbb{R}$, is the completion of $C_0^\infty(\overline{\Omega} \setminus \{P^{(1)}, \dots, P^{(2\mathcal{K})}\})$ with respect to the norm*

$$\left\| u\, ;\ V_\beta^\ell(\Omega)\right\| = \left(\left\|\left(1 - \sum_{k=1}^{2\mathcal{K}} \chi_k\right) u\, ;\ H^\ell(\Omega)^2\right\|^2 + \sum_{k=1}^{2\mathcal{K}} \sum_{|\alpha| \leqq \ell} \left\| \chi_k r_k^{\beta-\ell+|\alpha|} \partial^\alpha u\, ;\ L_2(\Omega)^2\right\|^2\right)^{\frac{1}{2}},$$

where $r_k := |x - P^{(k)}|$.

Throughout this paper, we use the notation $\|\bullet;\mathbb{B}\|$ for the norm in the Banach space $\mathbb{B}$. We also employ the introduced spaces with different smoothness and weight indices, ℓ and β respectively. As we shall see, the principal properties of the operator defined by the problem (1)–(4) with the domain $V_\beta^{\ell+1}(\Omega)^2$ are influenced by the choice of ℓ and β via the difference $\beta-\ell$ only. Note also that the relation $\gamma<\beta$ provides the inclusion $V_\gamma^{\ell+1}(\Omega)\subset V_\beta^{\ell+1}(\Omega)$ and implies faster decay of $\omega_\gamma\in V_\gamma^{\ell+1}(\Omega)$ at $P^{(k)}$ than that of $\omega_\beta\in V_\beta^{\ell+1}(\Omega)$. The following theorem on the asymptotics of solutions to the problem (1)–(4) is due to KONDRAT'EV [8].

Theorem 2.2. *Let Ω be a bounded domain with $C^{\ell+3}$-boundary Γ and let $F\in V_\gamma^{\ell-1}(\Omega)^2$, $G\in V_\gamma^{\ell+\frac12}(\Gamma)^2$ and $H\in V_\gamma^{\ell-\frac12}(\Gamma)^2$ with $\gamma-\ell\in(-1,-\frac12)$, $\ell\in\mathbb{N}$. Then any uniformly bounded solution $u\in H_{\mathrm{loc}}^{\ell+1}\big(\overline{\Omega}\setminus\{P^{(1)},\dots,P^{(2\mathcal{K})}\}\big)$ of the mixed boundary value problem* (1)–(4) *has the asymptotic expansion*

$$u(x)=\sum_{k=1}^{2\mathcal{K}}\left\{\chi_k(x)\,u(P^{(k)})+K_1^{(k)}X^{(k,\frac12)}(x)\right\}+\widetilde{u}^{(1)}(x)$$

with $\tilde{u}^{(1)}\in V_\gamma^{\ell+1}(\Omega)^2$. The terms in this asymptotic expansion satisfy the estimate

$$\begin{aligned}&\left|u\big(P^{(k)}\big)\right|+\left|K_1^{(k)}\right|+\left\|\tilde{u}^{(1)};V_\gamma^{\ell+1}(\Omega)^2\right\|\\&\quad\leqq C\left(\left\|F;V_\gamma^{\ell-1}(\Omega)^2\right\|+\left\|G;V_\gamma^{\ell+\frac12}(\Gamma)^2\right\|+\left\|H;V_\gamma^{\ell-\frac12}(\Gamma)^2\right\|\right).\end{aligned}$$

The proof of this theorem can be found in [8] and follows also from Theorems 4.2.1, 4.2.3 and Section 7.14 in [19].

2.2. Formal asymptotics of the solution near the endpoints

In order to calculate the Gateaux derivatives of the stress intensity factors with respect to a variation of the contact area it is necessary to consider one more term in the asymptotic expansion of the solution and to use local coordinates. The equations of elasticity written in the natural curvilinear coordinates (s,n) and curvilinear components have the form

$$\begin{aligned}(L(u))_s&=A\left[\partial_s\sigma_{ss}+2\kappa\sigma_{ns}\right]+\partial_n\sigma_{ns}&&=F_s,\\(L(u))_n&=A\left[\partial_s\sigma_{ns}-\kappa(\sigma_{ss}-\sigma_{nn})\right]+\partial_n\sigma_{nn}&&=F_n,\end{aligned}$$

where $\kappa=\kappa(s)$ denotes the curvature of Γ at the point $\Gamma(s)$ and $A:=(1+n\kappa)^{-1}$. The components of the stress tensor in curvilinear coordinates are given by

$$\begin{aligned}\sigma_{ss}&=A(\lambda+2\mu)\left[\partial_s u_s+\kappa u_n\right]+\lambda\partial_n u_n,\\\sigma_{ns}&=\mu\left[A(\partial_s u_n-\kappa u_s)+\partial_n u_s\right],\\\sigma_{nn}&=(\lambda+2\mu)\partial_n u_n+\lambda A(\partial_s u_s+\kappa u_n).\end{aligned}$$

Those formulae can be found in any textbook on linearized elasticity, e.g., in [9].

The boundary conditions of the corresponding model problem take the form

$$\sigma_{nn}(u) = 0,\ \sigma_{ns}(u) = 0 \text{ for } s < 0 \text{ and}$$
$$u_n = 0,\ \sigma_{ns}(u) = 0 \text{ for } s > 0.$$

For the asymptotic analysis, we introduce the following definition: The differential operator $T(n, s, \partial_n, \partial_s)$ is of *generalized order* ℓ if, for a linear transformation of coordinates $n = tn', s = ts'$, the equality $T(n', s', \partial_{n'}, \partial_{s'}) = t^\ell T(n, s, \partial_n, \partial_s)$ holds.

Using the Taylor expansions $A = 1-\kappa n+\ldots$ and $\kappa = \kappa(s_k)+\kappa'(s_k)(s-s_k)+\ldots$ near the point $P^{(k)}$, we can write the elasticity operator L and the boundary conditions B as an expansion with respect to the generalized orders,

$$\mathbf{L}(u) = \mathbf{L}^0(u) + \kappa_k \mathbf{L}^1(u) + \cdots,$$
$$\mathbf{B}(u) = \mathbf{B}^0(u) + \kappa_k \mathbf{B}^1(u) + \cdots,$$

where $\kappa_k := \kappa(s_k)$. We use bold letters to emphazise that the operators are written in curvilinear coordinates.

The terms $\mathbf{L}^0(u)$ and $\mathbf{B}^0(u)$ coincide with the original elasticity operator (5) and boundary operators (7), (8),

$$(\mathbf{L}^0(u))_s = (\lambda + 2\mu)\partial_s^2 u_s + (\lambda + \mu)\partial_s\partial_n u_n + \mu\partial_n^2 u_s,$$
$$(\mathbf{L}^0(u))_n = \mu\partial_s^2 u_n + (\lambda + \mu)\partial_s\partial_n u_s + (\lambda + 2\mu)\partial_n^2 u_n,$$

and

$$\left.\begin{aligned}(\mathbf{B}^0(u))_s &= \mu(\partial_s u_n + \partial_n u_s)\\ (\mathbf{B}^0(u))_n &= \lambda\partial_s u_s + (\lambda + 2\mu)\partial_n u_n\end{aligned}\right\} \text{ for } s < 0,$$
$$\left.\begin{aligned}(\mathbf{B}^0(u))_s &= \mu(\partial_s u_n + \partial_n u_s)\\ (\mathbf{B}^0(u))_n &= u_n\end{aligned}\right\} \text{ for } s > 0.$$

The terms $\mathbf{L}^1(u)$ and $\mathbf{B}^1(u)$ are given by

$$(\mathbf{L}^1(u))_s = (\lambda + 3\mu)\partial_s u_n + \mu\partial_n u_s - 2(\lambda + 2\mu)n\partial_s^2 u_s - (\lambda + \mu)n\partial_s\partial_n u_n,$$
$$(\mathbf{L}^1(u))_n = (\lambda + 2\mu)\partial_n u_n - (\lambda + 3\mu)\partial_s u_s - 2\mu n\partial_s^2 u_n - (\lambda + \mu)n\partial_s\partial_n u_s$$

and

$$\left.\begin{aligned}(\mathbf{B}^1(u))_s &= -\mu u_s\\ (\mathbf{B}^1(u))_n &= \lambda u_n\end{aligned}\right\} \text{ for } s < 0, \qquad \left.\begin{aligned}(\mathbf{B}^1(u))_s &= -\mu u_s\\ (\mathbf{B}^1(u))_n &= 0\end{aligned}\right\} \text{ for } s > 0.$$

The first two terms in the asymptotics of the solutions of the homogeneous model problem

$$\mathbf{L}(u) = 0 \text{ in } Q \text{ and } \mathbf{B}(u) = 0 \text{ on } \mathbb{R} \times \{0\} \tag{12}$$

are calculated as follows. In the first step, we solve the boundary value problem defined by the equations of first generalized order,

$$\mathbf{L}^0(u_0) = 0 \text{ in } Q \text{ and } \mathbf{B}^0(u_0) = 0 \text{ on } \mathbb{R} \times \{0\}. \tag{13}$$

This problem coincides with problem (6)–(8) and has the same solutions $X^{(\frac{m}{2})}$ for $m = \pm 0, \pm 1, \pm 2, \ldots$. In the next step we solve the equations of generalized second order

$$\mathbf{L}^0(u_1) = -\mathbf{L}^1\Big(X^{(\frac{m}{2})}\Big) \text{ in } Q \text{ and } \mathbf{B}^0(u_1) = -\mathbf{B}^1\Big(X^{(\frac{m}{2})}\Big) \text{ on } \mathbb{R} \times \{0\}\,. \tag{14}$$

The solution of this problem is calculated for the case $m = 1$ first. In this case, problem (14) can be significantly simplified with the help of the following observation. Since $X^{(\frac{1}{2})}$ solves the homogeneous Lamé equations, the function $n^2\partial_n X^{(\frac{1}{2})}$ satisfies the relation

$$\begin{aligned}
&\mathbf{L}^0\left(n^2\partial_n X^{(\frac{1}{2})}\right)\\
&= \begin{pmatrix} 2\mu\partial_n X_s^{(\frac{1}{2})} + 2(\lambda+\mu)n\partial_s\partial_n X_n^{(\frac{1}{2})} + 4\mu n\partial_n^2 X_s^{(\frac{1}{2})} \\ 2(\lambda+2\mu)\partial_n X_n^{(\frac{1}{2})} + 2(\lambda+\mu)n\partial_s\partial_n X_s^{(\frac{1}{2})} + 4(\lambda+2\mu)n\partial_n^2 X_n^{(\frac{1}{2})} \end{pmatrix}.
\end{aligned}$$

Expressing $\partial_n^2 X^{(\frac{1}{2})}$ in terms of $\partial_s^2 X^{(\frac{1}{2})}$ and $\partial_s\partial_n X^{(\frac{1}{2})}$ via the Lamé equations, we obtain

$$\mathbf{L}^0\left(\frac{1}{2}n^2\partial_n X^{(\frac{1}{2})}\right) = \mathbf{L}^1\Big(X^{(\frac{1}{2})}\Big) + (\lambda+3\mu)\begin{pmatrix} -\partial_s X_n^{(\frac{1}{2})} \\ \partial_s X_s^{(\frac{1}{2})} \end{pmatrix}.$$

Moreover, the homogeneous boundary conditions $\mathbf{B}^0\Big(n^2\partial_n X^{(\frac{1}{2})}\Big) = 0$ are fulfilled, too. Hence, we can use the following property:

Proposition 2.3. *The solution $u_1 \equiv Y^{(\frac{1}{2})}$ of problem* (14) *for the index $m = 1$ is given by*

$$Y^{(\frac{1}{2})} = -\frac{1}{2}n^2\partial_n X^{(\frac{1}{2})} + \tilde{u}_1,$$

where $\tilde{u}_1$ satisfies the problem

$$\mathbf{L}^0(\tilde{u}_1) = (\lambda+3\mu)\begin{pmatrix} -\partial_s X_n^{(\frac{1}{2})} \\ \partial_s X_s^{(\frac{1}{2})} \end{pmatrix} \textit{ and } \mathbf{B}^0(\tilde{u}_1) = -\mathbf{B}^1\Big(X^{(\frac{1}{2})}\Big)\,. \tag{15}$$

The solutions of problems (14) and (15) are not unique, since the homogeneous model problem has non-trivial solutions. In order to obtain a unique solution, we require the solution $Y^{(\frac{1}{2})} = -\frac{1}{2}n^2\partial_n X^{(\frac{1}{2})} + \tilde{u}_1$ of (14) to be of the form

$$\begin{aligned}
Y^{(\frac{1}{2})}(r,\varphi) &= r^{\frac{3}{2}}\Upsilon^{(\frac{3}{2})}(\log(\kappa_k r),\varphi)\\
&= r^{\frac{3}{2}}\left(\Upsilon^{(\frac{3}{2},1)}(\varphi) + \log(\kappa_k r)\Upsilon^{(\frac{3}{2},2)}(\varphi)\right).
\end{aligned} \tag{16}$$

With this condition, the function $Y^{(\frac{1}{2})}(r,\varphi)$ is defined up to a term $Cr^{\frac{3}{2}}\Phi^{(\frac{3}{2})}(\varphi)$ with a constant C. This constant is calculated from the asymptotic normalization condition

$$\sigma_{nn}\Big(r^{\frac{1}{2}}\Phi^{(\frac{1}{2})}(\varphi)+\kappa_k r^{\frac{3}{2}}\Upsilon^{(\frac{3}{2})}(\log(\kappa_k r),\varphi)\Big)\Big|_{\varphi=0} = \frac{1}{\sqrt{2\pi r}}\Big\{1+B\kappa_k r\log(\kappa_k r)+\cdots\Big\} \quad (17)$$

valid on the part of the boundary $\mathbb{R}_+\times\{0\}$ with given normal displacements. We emphasize that the curly brackets in (17) do not contain any term of the form cr; the equality $c=0$ determines the constant C uniquely since the normalization conditions in Section 2.1 imply

$$\sigma_{nn}\Big(C\tau^{\frac{3}{2}}\Phi^{\frac{3}{2}}(\varphi)\Big)\Big|_{\varphi=0}=\frac{C}{\sqrt{2\pi}}r^{\frac{1}{2}}+\mathcal{O}(r^{\frac{3}{2}})\,.$$

Note also that the product $\kappa_k r$ is dimensionless and the logarithmic term disappears for a straight boundary, i.e., for $\kappa_k r=0$. Here, σ_{nn} denotes the full normal component of the stress tensor at the boundary transformed into the curvilinear coordinates. The dots symbolize terms of higher order and B is a constant to be defined. Using the asymptotic expansion of the boundary traction and the normalization condition for the power solutions with positive, odd indices, condition (17) is simplified to

$$\sigma_{nn}^{(0)}\Big(\kappa_k r^{\frac{3}{2}}\Upsilon^{(\frac{3}{2})}(\log(\kappa_k r),\varphi)\Big)\Big|_{\varphi=0}+\sigma_{nn}^{(1)}\Big(r^{\frac{1}{2}}\Phi^{(\frac{1}{2})}(\varphi)\Big)\Big|_{\varphi=0} = \frac{1}{\sqrt{2\pi r}}B\kappa_k r\log(\kappa_k r)\ \text{ on } \mathbb{R}_+\times\{0\},$$

where $\sigma_{nn}^{(0)}(u)=(\lambda+2\mu)\partial_n u_n+\lambda\partial_s u_s$ and $\sigma_{nn}^{(1)}(u)=\lambda u_n$ denote the first two terms in the asymptotic expansion of the normal traction. Due to the fact that

$$\Big(r^{\frac{1}{2}}\Phi^{(\frac{1}{2})}(\varphi)\Big)_r\Big|_{\varphi=0}=\Big(r^{\frac{1}{2}}\Phi^{(\frac{1}{2})}(\varphi)\Big)_\varphi\Big|_{\varphi=0}=0$$

we have $\sigma_{nn}^{(1)}\Big(r^{\frac{1}{2}}\Phi^{(\frac{1}{2})}(\varphi)\Big)\Big|_{\varphi=0}=0$, and the normalization condition reads

$$\sigma_{nn}^{(0)}\Big(r^{\frac{3}{2}}\Upsilon^{(\frac{3}{2})}(\log(\kappa_k r),\varphi)\Big)\Big|_{\varphi=0}=\frac{1}{\sqrt{2\pi r}}B\kappa_k r\log(\kappa_k r)\ \text{ on } \mathbb{R}_+\times\{0\}.$$

The solution $\tilde{u}_1$ is now uniquely defined by

$$\begin{aligned}
\tilde{u}_1(r,\varphi) = {} & \frac{2\mu D_{\frac{1}{2}} r^{\frac{3}{2}}}{3\pi(\lambda+2\mu)} \\
& \cdot \left[\log(\kappa_k r) \begin{pmatrix} (2\Theta-3)\cos(\frac{\varphi}{2}) + \cos(\frac{5}{2}\varphi) \\ (2\Theta+3)\sin(\frac{\varphi}{2}) - \sin(\frac{5}{2}\varphi) \end{pmatrix} \right. \\
& \quad \left. - \varphi \begin{pmatrix} \sin(\frac{5}{2}\varphi) + (2\Theta-3)\sin(\frac{\varphi}{2}) \\ \cos(\frac{5}{2}\varphi) - (2\Theta+3)\cos(\frac{\varphi}{2}) \end{pmatrix} \right] \\
& + \Theta D_{\frac{1}{2}} r^{\frac{3}{2}} \begin{pmatrix} -\sin(\frac{\varphi}{2}) + \sin(\frac{3}{2}\varphi) \\ \cos(\frac{\varphi}{2}) + \cos(\frac{3}{2}\varphi) \end{pmatrix} - \frac{\lambda D_{\frac{1}{2}} r^{\frac{3}{2}}}{3(\lambda+2\mu)} \begin{pmatrix} \sin(\frac{5}{2}\varphi) \\ \cos(\frac{5}{2}\varphi) \end{pmatrix} \\
& + \frac{(\lambda+4\mu) D_{\frac{1}{2}} r^{\frac{3}{2}}}{3(\lambda+2\mu)} \begin{pmatrix} (2\Theta-3)\sin(\frac{\varphi}{2}) \\ -(2\Theta+3)\cos(\frac{\varphi}{2}) \end{pmatrix} \\
& - \frac{8\mu D_{\frac{1}{2}} r^{\frac{3}{2}}}{9\pi(\lambda+2\mu)} \begin{pmatrix} \Theta\cos(\frac{\varphi}{2}) - \cos(\frac{5}{2}\varphi) \\ \Theta\sin(\frac{\varphi}{2}) + \sin(\frac{5}{2}\varphi) \end{pmatrix}.
\end{aligned}$$

The angular part of the $\log(\kappa_k r)$-term is a multiple of $\Phi^{(\frac{3}{2})}(\varphi)$. Hence, the solution of the second order problem (14) for the index $m = 1$ can be represented by formula (16) with

$$\begin{aligned}
r^{\frac{3}{2}} \Upsilon^{(\frac{3}{2},1)}(\varphi) = {} & \left(-\frac{1}{2} n^2 \partial_n X^{(\frac{1}{2})} \right)(r,\varphi) \\
& + \frac{2\mu D_{\frac{1}{2}} r^{\frac{3}{2}} \varphi}{3\pi(\lambda+2\mu)} \begin{pmatrix} -(2\Theta-3)\sin(\frac{\varphi}{2}) - \sin(\frac{5}{2}\varphi) \\ (2\Theta+3)\cos(\frac{\varphi}{2}) - \cos(\frac{5}{2}\varphi) \end{pmatrix} \\
& + \Theta D_{\frac{1}{2}} r^{\frac{3}{2}} \begin{pmatrix} -\sin(\frac{\varphi}{2}) + \sin(\frac{3}{2}\varphi) \\ \cos(\frac{\varphi}{2}) + \cos(\frac{3}{2}\varphi) \end{pmatrix} - \frac{\lambda D_{\frac{1}{2}} r^{\frac{3}{2}}}{3(\lambda+2\mu)} \begin{pmatrix} \sin(\frac{5}{2}\varphi) \\ \cos(\frac{5}{2}\varphi) \end{pmatrix} \\
& + \frac{(\lambda+4\mu) D_{\frac{1}{2}} r^{\frac{3}{2}}}{3(\lambda+2\mu)} \begin{pmatrix} (2\Theta-3)\sin(\frac{\varphi}{2}) \\ -(2\Theta+3)\cos(\frac{\varphi}{2}) \end{pmatrix} \\
& - \frac{8\mu D_{\frac{1}{2}} r^{\frac{3}{2}}}{9\pi(\lambda+2\mu)} \begin{pmatrix} \Theta\cos(\frac{\varphi}{2}) - \cos(\frac{5}{2}\varphi) \\ \Theta\sin(\frac{\varphi}{2}) + \sin(\frac{5}{2}\varphi) \end{pmatrix}
\end{aligned}$$

and

$$\Upsilon^{(\frac{3}{2},2)}(\varphi) = B_{\frac{3}{2}} \Phi^{(\frac{3}{2})}(\varphi) \text{ with } B_{\frac{3}{2}} = \frac{2\mu D_{\frac{1}{2}} \left(D_{\frac{3}{2}} \right)^{-1}}{3\pi(\lambda+2\mu)} = \frac{2}{\pi} \frac{\mu}{\lambda+2\mu}.$$

The unknown factor B mentioned in the normalization condition (17) is therefore defined by $B = B_{\frac{3}{2}}$. The other solutions of the problem of generalized second order

$$\mathbf{L}^0\left(Y^{(\frac{m}{2})}\right) = -\mathbf{L}^1\left(X^{(\frac{m}{2})}\right), \quad \mathbf{B}^0\left(Y^{(\frac{m}{2})}\right) = -\mathbf{B}^1\left(X^{(\frac{m}{2})}\right), \tag{18}$$

can be obtained by the following property:

Proposition 2.4. *The solutions* $Y^{(\frac{m}{2})}$ *of problem* (18) *for* m *odd are given by*

$$Y^{(\frac{m}{2}-1)} = C_{\frac{m}{2}}^{-1} \partial_s Y^{(\frac{m}{2})}$$

with

$$C_{\frac{m}{2}} = \begin{cases} -\dfrac{1+\Theta}{4\mu} & \textit{for} \quad m = 1, \\ \dfrac{m}{2} - 1 & \textit{for} \quad m \geqq 3, \\ \dfrac{m}{2} & \textit{for} \quad m \leqq -1. \end{cases}$$

These formulae are valid, since the first two terms $\mathbf{L}^0$, $\mathbf{L}^1$ and $\mathbf{B}^0$, $\mathbf{B}^1$ of the expansions of the elasticity operator L and the boundary conditions B do not depend on s explicitly and therefore the order of the application of ∂_s and of $\mathbf{L}^0, \mathbf{L}^1$ or $\mathbf{B}^0, \mathbf{B}^1$ can be interchanged. Differentiation of equation (18) (with index $\frac{m}{2}$) with respect to s and the use of the formulae of differentiation (11) provide us with the calculation of $Y^{(\frac{m}{2}-1)}$. We then obtain the representations

$$\begin{aligned} r^{\frac{m}{2}} \Upsilon^{(\frac{m}{2},1)}(\varphi) = C_{\frac{m}{2}}^{-1} \Big[\partial_s \left(r^{\frac{m}{2}+1} \Upsilon^{(\frac{m}{2}+1,1)}(\varphi) \right) \\ + (\partial_s \log(\kappa_k r))\, r^{\frac{m}{2}+1} \Upsilon^{(\frac{m}{2}+1,2)}(\varphi) \Big] \end{aligned} \tag{19}$$

and

$$\Upsilon^{(\frac{m}{2},2)} = B_{\frac{m}{2}} \Phi^{(\frac{m}{2})}(\varphi). \tag{20}$$

The coefficients $B_{\frac{m}{2}}$ in the last formula are given by the recursive relation

$$B_{\frac{m}{2}} = B_{\frac{m}{2}+1} \frac{C_{\frac{m}{2}+1}}{C_{\frac{m}{2}}}. \tag{21}$$

Now let χ_k be again the cut-off function with $\chi_k \equiv 1$ near $P^{(k)}$ and its support contained in the set where the curvilinear coordinates (s, n) are defined. As in the previous section, the functions $\chi_k X^{(\frac{3}{2})}(s(x) - s_k, n(x))$ and $\chi_k Y^{(\frac{1}{2})}(s(x) - s_k, n(x))$ can be transformed and extended onto the domain of definition Ω. Let us denote the transformed and extended functions by $X^{(k,\frac{3}{2})}$ and $Y^{(k,\frac{1}{2})}$. Then the following theorem is valid describing the two-terms asymptotics of the solution near the points $P^{(k)}$:

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^2$ be a domain with a $C^{\ell+3}$ smooth boundary Γ and let $F \in V_\gamma^{\ell-1}(\Omega)^2$, $G \in V_\gamma^{\ell+\frac{1}{2}}(\Gamma)^2$ and $H \in V_\gamma^{\ell-\frac{1}{2}}(\Gamma)^2$ with $\gamma - \ell \in (-2, -\frac{3}{2})$. Then any uniformly bounded solution $u \in H_{\mathrm{loc}}^{\ell+1}\big(\overline{\Omega} \setminus \{P^{(1)}, \dots, P^{(2\mathcal{K})}\}\big)$ of problem* (1)–(4) *has the asymptotic expansion*

$$u(x) = \sum_{k=1}^{2\mathcal{K}} \Big\{ \chi_k(u) u(P^k) + K_1^{(k)} \Big[X^{(k,\frac{1}{2})}(x) + \kappa_k Y^{(k,\frac{1}{2})}(x) \Big] + \chi_k(x) \Lambda^{(k)}(x) + K_3^{(k)} X^{(k,\frac{3}{2})}(x) \Big\} + \widetilde{u}^{(2)}(x)\,.$$

where $\Lambda^{(k)}(x) = \Lambda_s^{(k)}(s(x) - s_k) + \Lambda_n^{(k)} n(x)$ and $\tilde{u}^{(2)} \in V_\gamma^{\ell+1}(\Omega)^2$. The terms in this expansion for $k = 1, \dots, 2\mathcal{K}$ satisfy the estimates

$$\Big|u\big(P^{(k)}\big)\Big| + \Big|K_1^{(k)}\Big| + \Big|\Lambda_s^{(k)}\Big| + \Big|\Lambda_n^{(k)}\Big| + \Big|K_3^{(k)}\Big| + \Big\|\tilde{u}^{(2)};\, V_\gamma^{\ell+1}(\Omega)^2\Big\| \leqq C \left(\Big\|F;\, V_\gamma^{\ell-1}(\Omega)^2\Big\| + \Big\|G;\, V_\gamma^{\ell+\frac{1}{2}}(\Gamma)^2\Big\| + \Big\|H;\, V_\gamma^{\ell-\frac{1}{2}}(\Gamma)^2\Big\| \right).$$

Theorem 2.5 is due to KONDRAT'EV [8]. Its proof can be also found in Section 4.2 of [19] (see Theorem 4.2.3). Note that the cut-off function χ_k has been included in the asymptotic terms $X^{(k,\frac{1}{2})}$, $X^{(k,\frac{2}{2})}$ and $Y^{(k,\frac{1}{2})}$.

We remark that the coordinates $(s(x), n(x))^\top$ and $(r(s,n), \varphi(s,n))^\top$ used in the previous theorem are not the usual Cartesian and polar coordinates. Nevertheless, the coefficients $K_1^{(k)}$ and $K_3^{(k)}$ have a clear physical meaning: Due to the normalization condition (17) for $Y^{(\frac{1}{2})}$, the normal component of the stress on the boundary has, near $P^{(k)}$ on the contact zone, the asymptotic expansion along the boundary,

$$\sigma_{nn}(u) = \frac{1}{\sqrt{2\pi r_k}} \left(K_1^{(k)} \left(1 + \frac{2}{\pi} \frac{\mu}{\lambda + 2\mu} \kappa_k r_k \log(\kappa_k r_k) \right) + K_3^{(k)} r_k + \cdots \right)$$

with $r_k := |s - s_k|$. The logarithmic term appears due to the curvature of the boundary.

3. Calculation of the stress intensity factors

In order to calculate the stress intensity factors $K_1^{(k)}$, the classical approach by BÜCKNER [1] can be employed by using appropriate weight functions. This method is described in [1] for cracks in plane isotropic media and was derived in [12] for general elliptic boundary value problems. The results can also be found in [19]. The weight function $\zeta^{(k,\frac{1}{2})}$ associated with the point $P^{(k)}$ is defined as the distributional weak solution of the homogeneous boundary value problem

$$L\Big(\zeta^{(k,\frac{1}{2})}\Big) = 0 \text{ in } \Omega, \tag{22}$$

$$\zeta^{(k,\frac{1}{2})} = 0 \text{ on } \Gamma_D, \tag{23}$$

$$\sigma^{(n)}\Big(\zeta^{(k,\frac{1}{2})}\Big) = 0 \text{ on } \Gamma_N, \tag{24}$$

$$\zeta_n^{(k,\frac{1}{2})} = 0 \text{ and } \sigma_s^{(n)}\Big(\zeta^{(k,\frac{1}{2})}\Big) = 0 \text{ on } \Gamma_C \tag{25}$$

with the prescribed singular behaviour near $P^{(k)}$,

$$\zeta^{(k,\frac{1}{2})} = r_k^{-\frac{1}{2}}\Phi^{(-\frac{1}{2})}(\varphi_k) + \cdots$$

Note that $\zeta^{(k,\frac{1}{2}} \in L^2 \setminus H^1$. In order to compute this solution, we may use the decomposition

$$\zeta^{(k,\frac{1}{2})} = X^{(k,-\frac{1}{2})}(x) + \tilde{\zeta}^{(k,\frac{1}{2})}(x), \tag{26}$$

where $X^{(k,-\frac{1}{2})}(x)$ is defined by $\chi_k(x)X^{(-\frac{1}{2})}(s(x) - s_k, n(x))$, extended by 0 onto the whole domain Ω. The remaining function $\tilde{\zeta}^{(k,\frac{1}{2})} \in H^1(\Omega)^2$ is an energy solution of the inhomogeneous boundary value problem

$$\begin{aligned}
L\Big(\tilde{\zeta}^{(k,\frac{1}{2})}\Big) &= -L\Big(X^{(k,-\frac{1}{2})}\Big) \text{ in } \Omega,\\
\tilde{\zeta}^{(k,\frac{1}{2})} &= -X^{(k,-\frac{1}{2})} \text{ on } \Gamma_D,\\
\sigma^{(n)}\Big(\tilde{\zeta}^{(k,\frac{1}{2})}\Big) &= -\sigma^{(n)}\Big(X^{(k,-\frac{1}{2})}\Big) \text{ on } \Gamma_N,\\
\tilde{\zeta}_n^{(k,\frac{1}{2})} = -X_n^{(k,-\frac{1}{2})} \text{ and } \sigma_s^{(n)}\Big(\tilde{\zeta}^{(k,\frac{1}{2})}\Big) &= -\sigma_s^{(n)}\Big(X^{(k,-\frac{1}{2})}\Big) \text{ on } \Gamma_C.
\end{aligned}$$

Using the weight function $\zeta^{(k,\frac{1}{2})}$, the stress intensity factor $K_1^{(k)}$ can be represented in terms of the given volume- and boundary-data by the formula

$$\begin{aligned}
K_1^{(k)} = &\int_\Omega F \cdot \zeta^{(k,\frac{1}{2})}\,dx + \int_{\Gamma_N} H \cdot \zeta^{(k,\frac{1}{2})}\,ds - \int_{\Gamma_D} G \cdot \sigma^{(n)}\Big(\zeta^{(k,\frac{1}{2})}\Big)\,ds\\
&+ \int_{\Gamma_C} H_s \cdot \zeta_s^{(k,\frac{1}{2})}\,ds - \int_{\Gamma_C} G_n \cdot \sigma_n^{(n)}\Big(\zeta^{(k,\frac{1}{2})}\Big)\,ds.
\end{aligned} \tag{27}$$

The estimate for the stress intensity factors in Theorem 2.2 is proved with the help of this formula.

The weight function $\zeta^{(k,\frac{1}{2})}$ has, near the point $P^{(m)}$, an expansion of the form

$$\begin{aligned}
\zeta^{(k,\frac{1}{2})} = &\, r_k^{-\frac{1}{2}}\Phi^{(-\frac{1}{2})}(\varphi_k) + c_\Omega^{kk} e_s^{(k)}\\
&+ r_k^{\frac{1}{2}}\Big(C_\Omega^{kk}\Phi^{(\frac{1}{2})}(\varphi_k) + \kappa_k \Upsilon^{(\frac{1}{2})}(\log(\kappa_k r_k), \varphi_k)\Big) + \dots
\end{aligned} \tag{28}$$

in the case $m = k$; and

$$\zeta^{(k,\frac{1}{2})} = c_\Omega^{km} e_s^{(m)} + C_\Omega^{km} r_m^{\frac{1}{2}}\Phi^{(\frac{1}{2})}(\varphi_m) + \dots \tag{29}$$

for $m \neq k$. The coefficients c_Ω^{km} are given by $c_\Omega^{km} = \zeta^{(k,\frac{1}{2})}\big(P^{(m)}\big)$, $k \neq m$, and each of the coefficients C_Ω^{km} represents the stress intensity factor at the point $P^{(m)}$ of the weight function $\zeta^{(k,\frac{1}{2})}$ with singularity at the point $P^{(k)}$. For the matrix $C_\Omega := \big(C_\Omega^{km}\big)_{k,m=1}^{2\mathcal{K}}$ of the stress intensity factors the following lemma is valid:

Lemma 3.1. *The $2\mathcal{K} \times 2\mathcal{K}$ matrix C_Ω of stress intensity factors is symmetric.*

Proof. Let us consider two indices $k \neq m$. The stress intensity factor of the function $\zeta^{(k)}$ at the point $P^{(m)}$ is given by

$$C_\Omega^{km} = \lim_{\varepsilon \to 0} \int_{\Gamma^{(m)}(\varepsilon)} \Big(\sigma^{(n)}\big(\zeta^{(m)}\big) \cdot \zeta^{(k)} - \zeta^{(m)} \cdot \sigma^{(n)}\big(\zeta^{(k)}\big)\Big)\, ds. \tag{30}$$

Here, $\Gamma^{(m)}(\varepsilon)$ denotes the intersection of the circle $\big\{x \in \mathbb{R}^2 : \big|x - P^{(m)}\big| = \varepsilon\big\}$ with Ω and $\sigma_i^{(n)} := \sigma_{ij} n_j$, where $(n_1, n_2)^\top$ denotes the normal vector on $\Gamma^{(m)}(\varepsilon)$ oriented to the point $P^{(m)}$. Relation (30) is a consequence of the orthogonality condition of the power solutions and the main part of the proof of formula (27). Let $\Omega^{km}(\varepsilon) := \big\{x \in \Omega : \big|x - P^{(m)}\big| > \varepsilon, \big|x - P^{(k)}\big| > \varepsilon\big\}$ denote the domain obtained by removing from Ω the two circular discs with common radius ε and centres $P^{(k)}$ and $P^{(m)}$. Applying the second Green formula

$$\int_{\Omega^{km}(\varepsilon)} (L(u) \cdot v - u \cdot L(v))\, dx = \int_{\partial\Omega^{km}(\varepsilon)} \Big(\sigma^{(n)}(u) \cdot v - u \cdot \sigma^{(n)}(v)\Big)\, ds$$

for $u = \zeta^{(k)}$ and $v = \zeta^{(m)}$ and using the differential equation for the weight functions and their boundary conditions we obtain

$$\int_{\Gamma^{(m)}(\varepsilon) \cup \Gamma^{(k)}(\varepsilon)} \Big(\sigma^{(n)}\big(\zeta^{(k)}\big) \cdot \zeta^{(m)} - \zeta^{(k)} \cdot \sigma^{(n)}\big(\zeta^{(m)}\big)\Big)\, ds = 0.$$

Inserting equation (30) for k and m and for their exchanged rôles proves the theorem. □

For the calculation of the second coefficients $K_3^{(k)}$ in the singular expansions, a similar method can be employed. The corresponding weight functions $\zeta^{(k,\frac{3}{2})}$ are the solutions of the homogeneous boundary value problem (22)–(25) with the singular asymptotics

$$\zeta^{(k,\frac{3}{2})}(x) \sim X^{(-\frac{3}{2})}(s(x) - s_k, n(x)).$$

Then the formula

$$\begin{aligned} K_3^{(k)} &+ K_1^{(k)}\,\Xi^{(k)} \\ &= \int_\Omega F \cdot \zeta^{(k,\frac{3}{2})}\, dx + \int_{\Gamma_N} H \cdot \zeta^{(k,\frac{3}{2})}\, ds - \int_{\Gamma_D} G \cdot \sigma^{(n)}\Big(\zeta^{(k,\frac{3}{2})}\Big)\, ds \\ &\quad + \int_{\Gamma_C} H_s \cdot \zeta_s^{(k,\frac{3}{2})}\, ds - \int_{\Gamma_C} G_n \cdot \sigma_n^{(n)}\Big(\zeta^{(k,\frac{3}{2})}\Big)\, ds \end{aligned} \tag{31}$$

is valid ([19, Section 6.2]). The factor $\Xi^{(k)}$ in this representation appears due to the perturbation term $\kappa_k r_k^{\frac{3}{2}} \Upsilon^{(\frac{3}{2})}(\log(\kappa_k r_k), \varphi_k)$ in the asymptotic expansion of the solution u from Theorem 2.5 and describes the interaction of this term with the function $X^{(\frac{1}{2})}(r_k, \varphi_k)$. Unfortunately, it is very difficult to calculate this factor. A formula better suited for the calculation of the second coefficient in the singular expansion can be derived by subtracting the principal asymptotic terms from the solution, i.e.,

$$u^{(k,3)} := u - \chi_k u\big(P^{(k)}\big) - K_1^{(k)} \chi_k \left(r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) + \kappa_k r_k^{\frac{3}{2}} \Upsilon^{(\frac{3}{2})}(\log(\kappa_k r_k), \varphi_k) \right).$$

Then there is no interaction of the terms of lower order with the weight function and we obtain the formula

$$\begin{aligned} K_3^{(k)} = & \int_\Omega L\big(u^{(k,3)}\big) \cdot \zeta^{(k,\frac{3}{2})}\, dx \\ & + \int_{\Gamma_N} \sigma^{(n)}\big(u^{(k,3)}\big) \cdot \zeta^{(k,\frac{3}{2})}\, ds - \int_{\Gamma_D} u^{(k,3)} \cdot \sigma^{(n)}\Big(\zeta^{(k,\frac{3}{2})}\Big)\, ds \\ & + \int_{\Gamma_C} \sigma_s^{(n)}\big(u^{(k,3)}\big) \cdot \zeta_s^{(k,\frac{3}{2})}\, ds - \int_{\Gamma_C} u_n^{(k,3)} \cdot \sigma_n^{(n)}\Big(\zeta^{(k,\frac{3}{2})}\Big)\, ds, \end{aligned} \tag{32}$$

where the coefficient $\Xi^{(k)}$ does not appear any more.

4. Asymptotics of the stress intensity factor

Let us consider the mixed boundary value problem (1)–(4) with two slightly different contact parts of the boundary given by the endpoints $P^{(k)} = \Gamma(s_k)$, $k = 1, \dots, 2\mathcal{K}$, and the shifted endpoints $P^{(k)}(\varepsilon) = \Gamma(s_k + \varepsilon h_k)$. Here ε denotes a small parameter and $h_1, \dots, h_{2\mathcal{K}}$ are arbitrary real numbers. The stress intensity factors for the two different problems at the points $P^{(k)}$ and $P^{(k)}(\varepsilon)$ are denoted by $K_1^{(k)}$ and $K_1^{(k)}(\varepsilon)$ respectively. We are interested in an asymptotic expansion of the stress intensity factor of the form

$$K_1^{(k)}(\varepsilon) = K_1^{(k)} + \varepsilon M^{(k)}(h_1, \dots, h_{2\mathcal{K}}) + \dots .$$

In order to derive such a formula we use the method of matched asymptotic expansions described for example in [25,6], and in [19, Section 6.5]. The solution of the problem with shifted endpoints will be expanded into an outer expansion and into $2\mathcal{K}$ inner expansions, each of them related to one of the points $P^{(k)}$, $k = 1, \dots, 2\mathcal{K}$. The outer expansion is valid outside a small neighbourhood of the points $P^{(1)}, \dots, P^{(2\mathcal{K})}$. The inner expansion corresponding to the point $P^{(k)}$ is valid close to this point and will be given in terms of the stretched coordinates $\xi = (\xi_1, \xi_2)^\top$ defined by $\xi_1 = \frac{1}{\varepsilon}(s - s_k)$ and $\xi_2 = \frac{1}{\varepsilon} n$. Let (r_k, φ_k) denote polar coordinates with respect to (s, n) and with the origin at $(s_k, 0)$. Let (ρ_k, φ_k) be polar coordinates with respect to (ξ_1, ξ_2). The outer expansion consists of a formal series,

$$u(x, \varepsilon) = v^{(0)}(x) + \varepsilon v^{(1)}(x) + \dots .$$

The functions $v^{(m)}$ are solutions of the boundary value problems

$$L\big(v^{(m)}\big) = \delta_{m0} F \text{ in } \Omega, \tag{33}$$

$$v^{(m)} = \delta_{m0}\, G \text{ on } \Gamma_D, \tag{34}$$

$$\sigma^{(n)}\big(v^{(m)}\big) = \delta_{m0}\, H \text{ on } \Gamma_N, \tag{35}$$

$$v_n^{(m)} = \delta_{m0}\, G_n \text{ and } \sigma_s^{(n)}\big(v^{(m)}\big) = \delta_{m0} H_s \text{ on } \Gamma_C. \tag{36}$$

Equations (33)–(36) are obtained directly by inserting the outer expansion into (1)–(4) at some distance from $P^{(1)}, \dots, P^{(2\mathcal{K})}$ and equating the coefficients at equal powers of the small parameter ε. We emphazise that for $m \geqq 1$, the right-hand sides in (33)–(36) are zero but $v^{(1)}$ does not vanish identically since it becomes a non-energy solution, i.e. a linear combination of the weight functions $\zeta^{(k,\frac{1}{2})}$ (cf. (22)–(25)). To provide the uniqueness of $v^{(1)}$, the behaviour of $v^{(1)}(x)$ as $x \to P^{(k)}, k = 1, \dots, 2\mathcal{K}$, must be determined by matching the outer expansion with the inner expansions in the vicinities of the points $P^{(k)}$. The latter expansions are given in the stretched coordinates by the formal series

$$u(\varepsilon, x) =: w^{(k)}(\xi, \varepsilon) = w^{(k,0)}(\xi) + \varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}(\xi) + \varepsilon w^{(k,1)}(\xi) + \varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}(\xi) + \cdots$$

which satisfy the model problem equations

$$\Big(\mathbf{L}^0 + \kappa_k \varepsilon \mathbf{L}^1\Big)\, w^{(k)}(\xi, \varepsilon) = 0, \quad \xi \in \mathbb{R} \times \mathbb{R}_+, \tag{37}$$

$$\Big(\mathbf{B}_-^0 + \kappa_k \varepsilon \mathbf{B}_-^1\Big)\, w^{(k)}(\xi, \varepsilon) = \varepsilon H\big(P^{(k)}\big), \quad \xi \in (-\infty, h_k) \times \{0\}, \tag{38}$$

$$\Big(\mathbf{B}_+^0 + \kappa_k \varepsilon \mathbf{B}_+^1\Big)\, w^{(k)}(\xi, \varepsilon) = \begin{pmatrix} \varepsilon H_s\big(P^{(k)}\big) \\ G_n\big(P^{(k)}\big) + \varepsilon \partial_s G_n\big(P^{(k)}\big)\xi_1 \end{pmatrix}, \quad \xi \in (h_k, +\infty) \times \{0\}, \tag{39}$$

in an asymptotic sense, i.e., up to the order ε^2 in our case. We note that in (38) and (39) appear the values $H_s(P^{(k)})$, $G_n(P^{(k)})$, etc., and thus the right-hand sides of (2)–(4) are supposed to be continuous on the arc $\{\Gamma(s) \mid s_{2k-1} \leqq s \leqq s_{2k}\}$. An assumption on the data which provides the latter properties will be formulated in Section 5 (see (51)–(53)). Here the boundary operators $\mathbf{B}_{+/-}^{0/1}$ are defined by

$$\mathbf{B}_-^0 w := \begin{pmatrix} \sigma_{12}(w) \\ \sigma_{22}(w) \end{pmatrix}, \qquad \mathbf{B}_+^0 w := \begin{pmatrix} \sigma_{12}(w) \\ w_2 \end{pmatrix},$$

$$\mathbf{B}_-^1 w := \begin{pmatrix} -\mu w_1 \\ \lambda w_2 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_+^1 w := \begin{pmatrix} -\mu w_1 \\ 0 \end{pmatrix},$$

where $\sigma_{ij}(w, \xi) := \lambda \delta_{ij} (\partial_{\xi_1} w_1 + \partial_{\xi_2} w_2) + \mu (\partial_{\xi_i} w_j + \partial_{\xi_j} w_i)$ denote the components of the usual stress tensor. The outer and inner expansions are different representations of the same function $u(x, \varepsilon)$, therefore they coincide asymptotically in an intermediate zone, that is $\big|x - P^{(k)}\big| \sim \sqrt{\varepsilon}$ and $|\xi| \sim \frac{1}{\sqrt{\varepsilon}}$.

The first term of the outer expansion coincides with the solution of the mixed boundary value problem with non-shifted endpoints and admits near $P^{(k)}$ the asymptotic representation

$$v^{(0)}(x) = c^{(0,k)} + K_1^{(k)} r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) + \Lambda^{(k)}(s - s_k, n) \\ + r_k^{\frac{3}{2}} \left[K_3^{(k)} \Phi^{(\frac{3}{2})}(\varphi_k) + \kappa_k K_1^{(k)} \Upsilon^{(\frac{3}{2})}(\log(\kappa_k r_k), \varphi_k) \right] + \cdots .$$

Here $c^{(0,k)} = u\big(P^{(k)}\big)$ is constant and $\Lambda^{(k)}$ denotes a linear function. Written in the stretched coordinates $(\xi_1, \xi_2)^\top$, this representation reads

$$v^{(0)}(x) = c^{(0,k)} + \varepsilon^{\frac{1}{2}} K_1^{(k)} \rho_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) + \varepsilon \Lambda^{(k)}(\xi_1, \xi_2) \\ + \varepsilon^{\frac{3}{2}} \rho_k^{\frac{3}{2}} \left[K_3^{(k)} \Phi^{(\frac{3}{2})}(\varphi_k) + \kappa_k K_1^{(k)} \left(\Upsilon^{(\frac{3}{2})}(\log(\kappa_k \rho_k), \varphi_k) + \log(\varepsilon) \Upsilon^{(\frac{3}{2},2)}(\varphi_k) \right) \right] \\ + \ldots .$$

The first two terms of the inner expansion are solutions of the problems

$$\begin{aligned} \mathbf{L}^0 w^{(k,0)} &= 0, & \mathbf{L}^0 w^{(k,\frac{1}{2})} &= 0 \ \text{ in } \mathbb{R} \times \mathbb{R}_+, \\ \mathbf{B}_-^0 w^{(k,0)} &= 0, & \mathbf{B}_-^0 w^{(k,\frac{1}{2})} &= 0 \ \text{ on } (-\infty, h_k) \times \{0\}, \\ \mathbf{B}_+^0 w^{(k,0)} &= \begin{pmatrix} 0 \\ g_n\big(P^{(k)}\big) \end{pmatrix}, & \mathbf{B}_+^0 w^{(k,\frac{1}{2})} &= 0 \ \text{ on } (h_k, +\infty) \times \{0\}. \end{aligned}$$

They are chosen in such a way that their terms of lowest order (after the transformation of coordinates $(s - s_k, n)^\top = \varepsilon(\xi_1, \xi_2)^\top$) coincide with the corresponding terms of $v^{(0)}$. Thus, we obtain

$$w^{(k,0)} = c^{(0,k)} = c_s^{(0,k)} e_s^{(k)} + g_n\big(P^{(k)}\big) e_n^{(k)} \tag{40}$$

with the unit vectors $e_s^{(k)}$, $e_n^{(k)}$ of the system of curvilinear coordinates at the point $P^{(k)}$ and

$$w^{(k,\frac{1}{2})} = K_1^{(k)} (\rho_k^*)^{\frac{1}{2}} \Phi^{(\frac{1}{2})} \left(\varphi_k^*\right).$$

Here (ρ_k^*, φ_k^*) denote polar coordinates with origin at $(h_k, 0)$. The function $w^{(k,\frac{1}{2})}$ can be given in non-stretched but shifted coordinates $x^* = (x_1^*, x_2^*)^\top = (x_1 - \varepsilon h_k, x_2)^\top$ and corresponding polar coordinates $r_k^* = \varepsilon \rho_k^*$, φ_k^* by

$$\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})} = K_1^{(k)} \left(r_k^*\right)^{\frac{1}{2}} \Phi^{(\frac{1}{2})} \left(\varphi_k^*\right).$$

Applying the Taylor expansion at a distance $|x - x^*| \geqq \sqrt{\varepsilon}$ from the point $x^* = 0$ and the formula of differentiation (11) we obtain the expansion

$$
\begin{aligned}
\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}(\xi) &= K_1^{(k)} X^{(\frac{1}{2})}(x_1 - \varepsilon h_k, x_2) \\
&= K_1^{(k)} \left(X^{(\frac{1}{2})}(x) - \varepsilon h_k \partial_{x_1} X^{(\frac{1}{2})}(x) + \mathcal{O}\left(\varepsilon^2\right)\right) \qquad (41)\\
&= K_1^{(k)} \left[r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) - \varepsilon h_k C_{\frac{1}{2}} r_k^{-\frac{1}{2}} \Phi^{(-\frac{1}{2})}(\varphi_k) + \mathcal{O}\left(\varepsilon^2\right)\right],
\end{aligned}
$$

where $(\xi_1, \xi_2) = \varepsilon^{-1}(s(x) - s_k, n(x))$.

The second term in the outer expansion is defined as a solution of the homogeneous boundary value problem (33)–(36) with asymptotics

$$
v^{(1)} \sim -K_1^{(k)} C_{\frac{1}{2}} h_k r_k^{-\frac{1}{2}} \Phi^{(-\frac{1}{2})}(\varphi_k)
$$

near the point $P^{(k)}$ for $k = 1, \dots, 2\mathcal{K}$. This function is given by

$$
v^{(1)}(x) = -\sum_{m=1}^{2\mathcal{K}} K_1^{(m)} C_{\frac{1}{2}} h_m \zeta^{(m,\frac{1}{2})}(x) + c_s^{(1,k)} e_s^{(k)}
$$

with the weight functions $\zeta^{(m,\frac{1}{2})}$ defined in Section 3. Taking into account the asymptotic expansions (28), (29) of the weight functions we obtain the following asymptotic representation of $v^{(1)}$ near the point $P^{(k)}$:

$$
\begin{aligned}
v^{(1)}(x) = &-K_1^{(k)} C_{\frac{1}{2}} h_k \left(r_k^{-\frac{1}{2}} \Phi^{(-\frac{1}{2})}(\varphi_k) + \kappa_k r_k^{\frac{1}{2}} \Upsilon^{(\frac{1}{2})}(\log(\kappa_k r_k), \varphi_k)\right) \\
&- \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} C_{\frac{1}{2}} h_m \left(c_\Omega^{mk} e_s^{(k)} + C_\Omega^{mk} r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k)\right) + \cdots .
\end{aligned}
$$

In stretched coordinates this representation is given by

$$
\begin{aligned}
v^{(1)}(x) = &-\varepsilon^{-\frac{1}{2}} K_1^{(k)} C_{\frac{1}{2}} h_k \rho_k^{-\frac{1}{2}} \Phi^{(-\frac{1}{2})}(\varphi_k) - \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} C_{\frac{1}{2}} h_m c_\Omega^{mk} e_s^{(k)} \\
&- \varepsilon^{\frac{1}{2}} \Bigg[\kappa_k K_1^{(k)} C_{\frac{1}{2}} h_k \rho_k^{\frac{1}{2}} \left(\Upsilon^{(\frac{1}{2})}(\log(\kappa_k \rho_k), \varphi_k) + \log(\varepsilon) \Upsilon^{(\frac{1}{2},2)}(\varphi_k)\right) \\
&\qquad + \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} C_{\frac{1}{2}} h_m C_\Omega^{mk} \rho_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) \Bigg] + \cdots .
\end{aligned}
$$

The third term of the inner expansion is a solution of the problem

$$
\begin{aligned}
\mathbf{L}^0 w^{(k,1)} &= -\kappa_k \mathbf{L}^1 w^{(k,0)} \; = \; 0 && \text{in } \mathbb{R} \times \mathbb{R}_+, \\
\mathbf{B}^0_- w^{(k,1)} &= -\kappa_k \mathbf{B}^1_- w^{(k,0)} + H\left(P^{(k)}\right) && \text{on } (-\infty, h_k) \times \{0\}, \\
\mathbf{B}^0_+ w^{(k,1)} &= -\kappa_k \mathbf{B}^1_+ w^{(k,0)} + \begin{pmatrix} H_s\left(P^{(k)}\right) \\ \partial_s G_n\left(P^{(k)}\right)\xi_1 \end{pmatrix} && \text{on } (h_k, +\infty) \times \{0\}.
\end{aligned}
$$

The solution $w^{(k,1)}$ is a polynomial of first order. Matching with the linear part of $v^{(0)}$ and the constant part of $v^{(1)}$ gives

$$\varepsilon w^{(k,1)}(\xi) = \Lambda^{(k)}(s - s_k, n) - \varepsilon \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} C_{\frac{1}{2}} h_m c_\Omega^{mk} e_s^{(k)}. \tag{42}$$

The remaining part $w^{(k,\frac{3}{2})}$ of the inner expansion is a solution of the problem

$$\begin{aligned} \mathbf{L}^0 w^{(k,\frac{3}{2})} &= -\kappa_k \mathbf{L}^1 w^{(k,\frac{1}{2})} && \text{in } \mathbb{R} \times \mathbb{R}_+, \\ \mathbf{B}_-^0 w^{(k,\frac{3}{2})} &= -\kappa_k \mathbf{B}_-^1 w^{(k,\frac{1}{2})} && \text{on } (-\infty, h_k) \times \{0\}, \\ \mathbf{B}_+^0 w^{(k,\frac{3}{2})} &= -\kappa_k \mathbf{B}_+^1 w^{(k,\frac{1}{2})} && \text{on } (h_k, +\infty) \times \{0\}. \end{aligned}$$

This problem has been solved already for non-shifted endpoints of the contact zone in Section 2.2. The solution for shifted endpoints is

$$\begin{aligned} w^{(k,\frac{3}{2})}(\xi) &= \kappa_k K_1^{(k)} (\rho_k^*)^{\frac{3}{2}} \Upsilon^{(\frac{3}{2})}(\log(\kappa_k \rho_k^*), \varphi_k^*) \\ &\quad + T_1^{(k)} (\rho_k^*)^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k^*) + T_2^{(k)} (\rho_k^*)^{\frac{3}{2}} \Phi^{(\frac{3}{2})}(\varphi_k^*) \end{aligned}$$

with unknown coefficients $T_1^{(k)}$ and $T_2^{(k)}$. This solution can be written in non-stretched coordinates $(r_k^*, \varphi_k^*)^\top = (\varepsilon\rho_k^*, \varphi_k^*)^\top$. Due to the definition (16) of the function $Y^{(\frac{1}{2})}(\rho, \varphi) = \rho^{\frac{3}{2}} \Upsilon^{(\frac{3}{2})}(\log(\kappa_k \rho), \varphi)$ and due to relation (20) we obtain

$$\begin{aligned} \varepsilon^{\frac{3}{2}} \left(\rho_k^*\right)^{\frac{3}{2}} \Upsilon^{(\frac{3}{2})} \left(\log\left(\kappa_k \rho_k^*\right), \varphi_k^*\right) &= \left(r_k^*\right)^{\frac{3}{2}} \Upsilon^{(\frac{3}{2})} \left(\log\left(\frac{\kappa_k r_k^*}{\varepsilon}\right), \varphi_k^*\right) \\ &= \left(r_k^*\right)^{\frac{3}{2}} \Upsilon^{(\frac{3}{2})} \left(\log\left(\kappa_k r_k^*\right), \varphi_k^*\right) - \log(\varepsilon) \left(r_k^*\right)^{\frac{3}{2}} \Upsilon^{(\frac{3}{2},2)} \left(\varphi_k^*\right) \\ &= Y^{(\frac{1}{2})} \left(r_k^*, \varphi_k^*\right) - \log(\varepsilon) B_{\frac{3}{2}} X^{(\frac{3}{2})} \left(r_k^*, \varphi_k^*\right). \end{aligned}$$

The Taylor expansion, which is valid at all points bounded away from $x^* = 0$, and the application of the formulae for differentiation of the power solutions yield

$$\begin{aligned} &\varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}(\xi) \\ &= \kappa_k K_1^{(k)} \left(Y^{(\frac{1}{2})}(x_1 - \varepsilon h_k, x_2) - \log(\varepsilon) B_{\frac{3}{2}} X^{(\frac{3}{2})}(x_1 - \varepsilon h_k, x_2)\right) \\ &\quad + T_2^{(k)} X^{(\frac{3}{2})}(x_1 - \varepsilon h_k, x_2) + \varepsilon T_1^{(k)} X^{(\frac{1}{2})}(x_1 - \varepsilon h_k, x_2) \\ &= \kappa_k K_1^{(k)} \left(Y^{(\frac{1}{2})}(x) - \varepsilon h_k \partial_{x_1} Y^{(\frac{1}{2})}(x) + \mathcal{O}\big(\varepsilon^2\big)\right) \\ &\quad + \left(T_2^{(k)} - \log(\varepsilon) \kappa_k K_1^{(k)} B_{\frac{3}{2}}\right) \left(X^{(\frac{3}{2})}(x) - \varepsilon h_k \partial_{x_1} X^{(\frac{3}{2})}(x) + \mathcal{O}\big(\varepsilon^2\big)\right) \\ &\quad + \varepsilon T_1^{(k)} \left(X^{(\frac{1}{2})}(x) + \mathcal{O}(\varepsilon)\right) \\ &= \kappa_k K_1^{(k)} \left(Y^{(\frac{1}{2})}(x) - \varepsilon h_k C_{\frac{1}{2}} Y^{(-\frac{1}{2})}(x)\right) \\ &\quad + \left(T_2^{(k)} - \log(\varepsilon) \kappa_k K_1^{(k)} B_{\frac{3}{2}}\right) \left(X^{(\frac{3}{2})}(x) - \varepsilon h_k C_{\frac{3}{2}} X^{(\frac{1}{2})}(x)\right) \\ &\quad + \varepsilon T_1^{(k)} X^{(\frac{1}{2})}(x) + \mathcal{O}\big(\varepsilon^2\big) \end{aligned}$$

$$
\begin{aligned}
&= r_k^{\frac{3}{2}} \Big[\kappa_k K_1^{(k)} \Upsilon^{(\frac{3}{2})}(\log(\kappa_k r_k), \varphi_k) + \Big(T_2^{(k)} - \log(\varepsilon)\kappa_k K_1^{(k)} B_{\frac{3}{2}} \Big) \Phi^{(\frac{3}{2})}(\varphi_k) \Big] \\
&\quad + \varepsilon r_k^{\frac{1}{2}} \Big[-\kappa_k K_1^{(k)} C_{\frac{1}{2}} h_k \Upsilon^{(\frac{1}{2})}(\log(\kappa_k r_k), \varphi_k) + T_1^{(k)} \Phi^{(\frac{1}{2})}(\varphi_k) \\
&\quad - \Big(T_2^{(k)} - \log(\varepsilon)\kappa_k K_1^{(k)} B_{\frac{3}{2}} \Big) h_k C_{\frac{3}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) \Big] + \mathcal{O}\big(\varepsilon^2\big).
\end{aligned}
$$

Matching of the coefficients at $r_k^{\frac{3}{2}} \Phi^{(\frac{3}{2})}(\varphi_k)$ and $r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k)$ with the corresponding coefficients in the outer expansion gives

$$
T_2^{(k)} - \log(\varepsilon)\kappa_k K_1^{(k)} B_{\frac{3}{2}} = K_3^{(k)},
$$

$$
T_1^{(k)} - C_{\frac{3}{2}} h_k T_2^{(k)} + \log(\varepsilon)\kappa_k K_1^{(k)} h_k C_{\frac{3}{2}} B_{\frac{3}{2}} = -\sum_{m=1}^{2\mathcal{K}} K_1^{(m)} h_m C_{\frac{1}{2}} C_\Omega^{mk}.
$$

Solving this system of equations we find

$$
\begin{aligned}
T_1^{(k)} &= C_{\frac{3}{2}} h_k K_3^{(k)} - \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} h_m C_{\frac{1}{2}} C_\Omega^{mk}, \\
T_2^{(k)} &= K_3^{(k)} + \log(\varepsilon)\kappa_k K_1^{(k)} B_{\frac{3}{2}}.
\end{aligned}
$$

Inserting the expressions $C_{\frac{1}{2}}$ and $C_{\frac{3}{2}}$ we obtain

$$
\begin{aligned}
w^{(k,\frac{3}{2})}(\xi) &= \kappa_k K_1^{(k)} (\rho_k^*)^{\frac{3}{2}} \Upsilon^{(\frac{3}{2})}(\log(\varepsilon\kappa_k \rho_k^*), \varphi_k^*) \\
&\quad + \left(\frac{1}{2} h_k K_3^{(k)} + \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} h_m \frac{1+\Theta}{4\mu} C_\Omega^{mk} \right) (\rho_k^*)^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k^*) \\
&\quad + K_3^{(k)} (\rho_k^*)^{\frac{3}{2}} \Phi^{(\frac{3}{2})}(\varphi_k^*).
\end{aligned}
$$

Since the inner expansion represents the solution of the perturbed problem near the point $P^{(k)}(\varepsilon)$, the stress intensity factor of this solution is given by the coefficient of the function $(r_k^*)^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k^*)$ in the inner expansion. The functions $\varepsilon^0 w^{(0,k)}$ and $\varepsilon^1 w^{(1,k)}$ do not contain such terms, therefore we only have to consider $\varepsilon^{\frac{1}{2}} w^{(\frac{1}{2},k)}$ and $\varepsilon^{\frac{3}{2}} w^{(\frac{3}{2},k)}$. Due to the relation $r_k^* = \varepsilon \rho_k^*$ we find the following two-term asymptotics of the stress intensity factor:

$$
\begin{aligned}
K_1^{(k)}(\varepsilon) &= K_1^{(k)} + \varepsilon T_1^{(k)} + \cdots \\
&= K_1^{(k)} + \varepsilon \left[\frac{1}{2} h_k K_3^{(k)} + \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} h_m \frac{1+\Theta}{4\mu} C_\Omega^{mk} \right] + \cdots.
\end{aligned} \tag{43}
$$

This formula was obtained with a calculation of formal character only. In order to verify its validity rigorously, the justification of an estimate

$$
\left| K_1^{(k)}(\varepsilon) - K_1^{(k)} - \varepsilon T_1^{(k)} \right| \leqq C\varepsilon^{1+\delta} \tag{44}
$$

is required with constants C and $\delta > 0$ which are independent of ε. The proof of such an estimate is carried out in the next section.

5. Rigorous justification of the asymptotics

5.1. The plan of the proof

There exist two different ways to justify formula (43) characterizing the asymptotics of the stress intensity factor. In the first method, the asymptotic expansion of the weight function $\zeta^{(k,\frac{1}{2})}$ is constructed with the method of matched asymptotic expansions and inserted into the representation (27) of the stress intensity factor. This method has been carried out for a straight two-dimensional crack in [19]. However, in the case of a curved boundary as considered here this method is difficult to realize, since in the asymptotics of the stress intensity factor there appears the second coefficient $K_3^{(k)}$ of the singular expansion which is difficult to estimate due to the appearance of the unknown coefficient $\Xi^{(k)}$ in the corresponding representation (31). Hence we employ here the second method. In this approach, weighted Sobolev spaces with detached asymptotics are used, in whose norms the stress intensity factor is contained. The last condition means that the norm of the function space – say $\mathcal{V}$ – satisfies an inequality of the type

$$\sum_{m=1}^{2\mathcal{K}} |K_1^{(m)}(u)| \leqq C\|u; \mathcal{V}\|. \tag{45}$$

In other situations spaces of this type were used in [19, Section 6.2 and Section 6.3].

The left-hand side of the boundary value problem (1)–(4) defines an operator $A_\beta^\ell := \{L, B\}$, consisting of the differential operator L and the boundary conditions B, which maps the function space $V_\beta^{\ell+1}(\Omega)^2$ into the range

$$\mathcal{R}_\beta^\ell := V_\beta^{\ell-1}(\Omega)^2 \times V_\beta^{\ell+\frac{1}{2}}(\Gamma_D)^2 \times V_\beta^{\ell-\frac{1}{2}}(\Gamma_N)^2 \times V_\beta^{\ell-\frac{1}{2}}(\Gamma_C) \times V_\beta^{\ell+\frac{1}{2}}(\Gamma_C).$$

The operator

$$A_\beta^\ell : V_\beta^{\ell+1}(\Omega)^2 \to \mathcal{R}_\beta^\ell$$

is generated by an elliptic, self-adjoint boundary value problem which possesses the polynomial property (see Example 5.4.6 in [19]). Therefore, all the general results described in Section 6.1 of [19] are applicable to our problem. In what follows, we choose the indices ℓ and β and enlarge the domain of definition $V_\beta^{\ell+1}(\Omega)^2$ and the range $\mathcal{R}_\beta^\ell$ of the operator A_β^ℓ in such a way, that this operator becomes an isomorphism and the norm of the range satisfies inequality (45). There are two possible ways to accomplish this goal. One can study the weak formulation of the problem, employ the known existence and uniqueness results for such variational problems, prove certain estimates concerning the regularity of the weak solution and employ Theorem 2.2. A more elegant way, avoiding extensive calculations is the application of the general theory of Fredholm operators, where only the indices and the dimensions of the kernels and cokernels of certain operators have to be evaluated. Here we follow the second approach. For simplicity we assume that there are no collision points of Γ_D with Γ_N or Γ_C,

$$\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset \text{ and } \overline{\Gamma}_D \cap \overline{\Gamma}_C = \emptyset\,.$$

These assumptions mean that the clamped curve Γ_D is a connected component of the boundary $\partial\Omega$ and that contact can happen only along the complementary component $\partial\Omega \setminus \Gamma_D$. The violation of the first assumption does not bring a serious change to the further considerations: additional weights are to be included into the norms near to collision points of Γ_D and Γ_N which are fixed. If $\overline{\Gamma}_D \cap \overline{\Gamma}_C \neq \emptyset$, the result (22) on the higher smoothness of the solution to the Signorini problem does not hold true any longer and the behaviour of the solution at the collision points of Γ_D and Γ_C has to be taken into account. The latter is not the concern of this paper and we therefore restrict ourselves to the above situation, as do most of the publications. However, problems for which the assumption $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$ is violated, e.g., corresponding punch problems, are of practical significance and should be considered in the future.

5.2. Basic properties of the operator A^ℓ_β

The properties of our operator A^ℓ_β are strongly related to the power solutions $r_k^\nu \Phi^{(\nu)}(\log(r_k), \varphi_k)$ of the model problem which are polynomials in $\log(r_k)$, $k = 1, \dots, 2\mathcal{K}$. Let $\mathcal{E}$ denote the set of all exponents ν of these power solutions. In our case, $\mathcal{E} = \left\{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots\right\}$ (see Section 2.1). Then the following classical theorems are valid:

Theorem 5.1 (KONDRAT'EV [8]**).** *The operator A^ℓ_β is Fredholm if and only if the difference $\beta - \ell$ does not coincide with one of the eigenvalues $m \in \mathcal{E}$.*

This theorem was obtained by KONDRAT'EV in [8] and is also described in [19, Theorem 4.1.2].

Theorem 5.2 (MAZ'YA & PLAMENEVSKY [12]**).** *Let $\beta - \ell \notin \mathcal{E}$. Then the cokernel of A^ℓ_β is equivalent to the kernel of $A^\ell_{2\ell-\beta}$ in the following sense. Let T be the dual boundary operator to B, such that the Green formula*

$$\langle Lu, v\rangle_\Omega + \langle Bu, Tv\rangle_\Gamma = \langle u, Lv\rangle_\Omega + \langle Bv, Tu\rangle_\Gamma$$

holds. Then problem $Lu = F$, $Bu = G$ has a solution in $V^{\ell+1}_\beta(\Omega)^2$ if and only if

$$\langle F, v\rangle_\Omega + \langle G, Tv\rangle_\Gamma = 0 \quad \textit{for all } v \in \ker A^\ell_{2\ell-\beta}.$$

The proof of this theorem can be found in [12] and in [19, Section 4.3 and Section 8.3]. The dual operator T is given in our problem by $Tu = \sigma^{(n)}(u)$ on Γ_D, $Tu = u$ on Γ_N and $(Tu)_n = \sigma_n^{(n)}(u)$, $(Tu)_s = u_s$ on Γ_C.

Theorem 5.3 (Maz'ya, Nazarov & Plamenevsky[11]). *Let $\beta < \gamma$ and $\beta - \ell$, $\gamma - \ell \notin \mathcal{E}$. Then the Fredholm indices of the operators A^ℓ_γ and A^ℓ_β satisfy the relation*

$$\operatorname{Ind} A^\ell_\gamma = \operatorname{Ind} A^\ell_\beta + J,$$

where J is the number of linearly independent power solutions $r_k^\nu \Phi^{(\nu)}(\log(r_k), \varphi_k)$, $k = 1, \dots, 2\mathcal{K}$, which are polynomials in $\log(r_k)$ such that $\ell - \gamma < \nu < \ell - \beta$.

This assertion was proved first in [11] (see also [19, Theorem 4.3.3]). The condition $\ell - \gamma < \nu < \ell - \beta$ for the exponent means that the corresponding power solutions belong to $V^{\ell+1}_\gamma(\Omega)^2$, but not to $V^{\ell+1}_\beta(\Omega)^2$. In our case the set of exponents ν coincides with $\mathcal{E} = \left\{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots\right\}$. To any index $k = 1, \dots, 2\mathcal{K}$ there exists only one power solution with $\nu = \frac{m}{2} \neq 0$ while for $\nu = 0$ there exist two of them.

Let β_- be a real number with $\beta_- - \ell \in (-\frac{1}{2}, 0)$; let $\beta_+ = 2\ell - \beta_-$ and $\ell \geqq 0$. First we prove the relation

$$\operatorname{Ind} A^\ell_{\beta_+} = -\operatorname{Ind} A^\ell_{\beta_-} = 2\mathcal{K}. \tag{46}$$

Let $u \in V^{\ell+1}_{\beta_-}(\Omega)^2$ be an element of the kernel of $A^\ell_{\beta_-}$. Applying the Green formula

$$\langle Lu, v\rangle_\Omega = \int_\Omega \sigma_{ij}(u) e_{ij}(v)\, dx - \langle Bu, Tv\rangle_\Gamma \tag{47}$$

for $v = u$ we obtain $\int_\Omega \sigma_{ij}(u)e_{ij}(u)\, dx = 0$. Hence u describes a rigid motion. Due to the displacement condition $u = 0$ on Γ_D, we have $u \equiv 0$. Thus we have proved $\ker A^\ell_{\beta_-} = \{0\}$. Due to Theorem 5.2 there follows $\dim \operatorname{coker} A^\ell_{\beta_+} = \dim\ker A^\ell_{\beta_-} = 0$ and

$$\operatorname{Ind} A^\ell_{\beta_-} = -\dim\operatorname{coker} A^\ell_{\beta_-} = -\dim\ker A^\ell_{\beta_+} = -\operatorname{Ind} A^\ell_{\beta_+}. \tag{48}$$

Between $\ell - \beta_+ \in (-\frac{1}{2}, 0)$ and $\ell - \beta_- \in (0, \frac{1}{2})$ there is only one exponent $\nu = 0$ of all nontrivial power solutions. Such solutions are given by (9) and (10). Since the number of singular points $P^{(k)}$ equals $2\mathcal{K}$, we have $4\mathcal{K}$ independent nontrivial power solutions. According to Theorem 5.3, $\operatorname{Ind} A^\ell_{\beta_+} = \operatorname{Ind} A^\ell_{\beta_-} + 4\mathcal{K}$. Together with (48) this proves the result (46).

5.3. Attaching the unit vectors $e^{(k)}$

In order to construct an isomorphism, we extend the domain of definition of $A^\ell_{\beta_-}$ by

$$\mathcal{V}^{\ell+1}_{\beta_-}(\Omega) := V^{\ell+1}_{\beta_-}(\Omega)^2 \bigoplus_{k=1}^{2\mathcal{K}} \chi_k e_s^{(k)}.$$

The functions $\chi_k e_s^{(k)}$ (see (40)) are linearly independent and belong to $H^1(\Omega)^2$, but not to $V^{\ell+1}_{\beta_-}(\Omega)^2$. Moreover, the image $\left\{L\left(\chi_k e_s^{(k)}\right), B\left(\chi_k e_s^{(k)}\right)\right\}$ is contained in $\mathcal{R}^\ell_{\beta_-}$. Therefore the operator

$$\mathcal{A}^\ell_{\beta_-} : \mathcal{V}^{\ell+1}_{\beta_-}(\Omega) \to \mathcal{R}^\ell_{\beta_-}$$

is Fredholm with index

$$\operatorname{Ind} \mathcal{A}^{\ell}_{\beta_-} = \operatorname{Ind} A^{\ell}_{\beta_-} + \dim \left(\mathcal{V}^{\ell+1}_{\beta_-} \setminus V^{\ell+1}_{\beta_-}(\Omega)^2 \right) = -2\mathcal{K} + 2\mathcal{K} = 0.$$

Applying the Green formula (47) to an element of the kernel of $\mathcal{A}^{\ell}_{\beta_-}$, we prove with the same arguments as before that $\ker \mathcal{A}^{\ell}_{\beta_-} = \{0\}$. Hence $\mathcal{A}^{\ell}_{\beta_-}$ defines an isomorphism.

If the boundary datum g_n does not vanish at $P^{(k)}$, then $\{f, g, h\}$ do not belong to $\mathcal{R}^{\ell}_{\beta_-}$. In order to admit such data we must enlarge the spaces again. Let

$$\mathfrak{V}^{\ell+1}_{\beta_-}(\Omega) := \mathcal{V}^{\ell+1}_{\beta_-}(\Omega) \bigoplus_{k=1}^{2\mathcal{K}} \chi_k e_n^{(k)},$$
$$\mathfrak{R}^{\ell}_{\beta_-} := \mathcal{R}^{\ell}_{\beta_-} \bigoplus_{k=1}^{2\mathcal{K}} \mathcal{A}^{\ell}_{\beta_-}\left(\chi_k e_n^{(k)}\right),$$
$$\mathfrak{A}^{\ell}_{\beta_-} : \mathfrak{V}^{\ell+1}_{\beta_-} \to \mathfrak{R}^{\ell}_{\beta_-}.$$

The functions $\chi_k e_n^{(k)}$ are linearly independent and belong to $H^1(\Omega)^2$ but not to $\mathcal{V}^{\ell+1}_{\beta_-}(\Omega)$, since the integrals

$$\int_{\Omega} r_k^{2(\beta_- - (\ell+1))} \left| \chi_k e_n^{(k)} \right|^2 \, dx$$

diverge for $\beta_- - \ell < 0$ and $\chi_k e_n^{(k)}$ does not belong to $\bigoplus_{j=1}^{2\mathcal{K}} \chi_j e_s^{(j)}$. The images $\mathfrak{A}^{\ell}_{\beta_-}\left(\chi_k e_n^{(k)}\right)$ are also linearly independent, since the normal component $\left(\chi_k e_n^{(k)}\right)_n = \chi_k$ is non-zero close to the corresponding point of singularity $P^{(k)}$ and equals zero outside a close neighbourhood of $P^{(k)}$. The images do not belong to $\mathcal{R}^{\ell}_{\beta_-}$ since, for all elements $\mathcal{A}^{\ell}_{\beta_-}(u) = (L(u), u_{|\Gamma_D}, \sigma^{(n)}(u)_{|\Gamma_N}, u_{n|\Gamma_C}, \sigma_s(u)_{|\Gamma_C})^{\top}$ from this range, the fourth component u_n must vanish at $P^{(k)}$; and this is not the case for $\mathfrak{A}^{\ell}_{\beta_-}\left(\chi_k e_n^{(k)}\right)$. Consequently, the operator $\mathfrak{A}^{\ell}_{\beta_-}$ defines an isomorphism.

Now let us consider the modified image

$$\mathfrak{R}^{\ell}_{\beta_- - \frac{1}{2}} := \mathcal{R}^{\ell}_{\beta_- - \frac{1}{2}} \bigoplus_{k=1}^{2\mathcal{K}} \mathfrak{A}^{\ell}_{\beta_-}\left(\chi_k e_n^{(k)}\right)$$

and its pre-image

$$\begin{aligned} \mathfrak{V}^{\ell+1}_{\beta_- - \frac{1}{2}}(\Omega) &:= \left\{ u \in \mathfrak{V}^{\ell+1}_{\beta_-} \mid \mathfrak{A}^{\ell}_{\beta_-}(u) \in \mathfrak{R}^{\ell}_{\beta_- - \frac{1}{2}} \right\} \\ &= V^{\ell+1}_{\beta_- - \frac{1}{2}}(\Omega)^2 \bigoplus_{k=1}^{2\mathcal{K}} \chi_k e_n^{(k)} \bigoplus_{k=1}^{2\mathcal{K}} \chi_k e_s^{(k)}. \end{aligned}$$

5.4. Detaching the square-root singularities

There is only one exponent $\nu = \frac{1}{2}$ of nontrivial power solutions with $\ell - \beta_- < \nu < \ell - (\beta_- - \frac{1}{2})$. If $(f, g, h) \in \mathfrak{R}^{\ell}_{\beta_- - \frac{1}{2}}$ holds, then, according to Theorem 2.2, the unique solution $u \in \mathfrak{V}^{\ell+1}_{\beta_-}(\Omega)$ of the mixed boundary value problem has the asymptotic expansion

$$u = \sum_{k=1}^{2\mathcal{K}} \chi_k \left(c^{(k,0)} + K_1^{(k)} r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) \right) + \tilde{u}^{(1)}$$

with $\tilde{u}^{(1)} \in \mathfrak{V}^{\ell+1}_{\beta_- - \frac{1}{2}}(\Omega)$. The coefficients in this expansion and the remaining function $\tilde{u}^{(1)}$ satisfy the estimate

$$\sum_{k=1}^{2\mathcal{K}} \left(\left| c^{(k,0)} \right| + \left| K_1^{(k)} \right| \right) + \left\| \tilde{u}^{(1)} ;\, \mathfrak{V}^{\ell+1}_{\beta_- - \frac{1}{2}}(\Omega) \right\| \leqq C \left\| (F, G, H) ;\, \mathfrak{R}^{\ell}_{\beta_- - \frac{1}{2}} \right\| . \tag{49}$$

Hence the space $\mathfrak{V}^{\ell+1}_{\beta_- - \frac{1}{2}}(\Omega)$ can be equipped with the norm given by the left-hand side of (49). Then the operator

$$\mathfrak{A}^{\ell}_{\beta_- - \frac{1}{2}} : \mathfrak{V}^{\ell+1}_{\beta_- - \frac{1}{2}}(\Omega) \to \mathfrak{R}^{\ell}_{\beta_- - \frac{1}{2}}$$

defines an isomorphism.

5.5. The almost identical change of variables

Starting from this result we now justify the asymptotics of the stress intensity factor obtained in Section 4. Therefore, an inequality of the type (49) can be applied to the problem with shifted endpoints of the contact zone. However, the constant C on the right-hand side of (49) depends on the geometry of the problem, in our case in particular on the shift parameter ε. Hence it is necessary to prove that inequality (49) can still be applied to the problem with shifted endpoints with a constant independent of ε, provided ε is small enough. Therefore we construct a diffeomorphism

$$y : \overline{\Omega} \ni x \mapsto y(x) \in \overline{\Omega}$$

with $y\big(P^{(k)}(\varepsilon)\big) = P^{(k)}$ by

$$y(x) = \left(1 - \sum_{k=1}^{2\mathcal{K}} \chi_k(x) \right) x + \sum_{k=1}^{2\mathcal{K}} \chi_k(x) \left(\Gamma(s(x) - \varepsilon h_k) + n(x) n(s(x) - \varepsilon h_k) \right).$$

Here, $(s(x), n(x))$ denote the curvilinear coordinates introduced in Section 2.1 and the χ_k are the cut-off functions whose supports are contained in the set where the curvilinear coordinates are defined. Outside $\bigcup_{k=1}^{2\mathcal{K}} \operatorname{supp}(\chi_k)$ there holds $y(x) = x$. Inside $\operatorname{supp}(\chi_k)$ we have

$$
\begin{aligned}
y(x) &= x + \chi_k(x) \\
&\qquad [(\Gamma(s(x) - \varepsilon h_k) - \Gamma(s(x))) + n(x)(n(s(x) - \varepsilon h_k) - n(s(x)))] .
\end{aligned}
$$

If $\Gamma \in C^{\ell+3}$, then

$$
y(x) = x + \varepsilon J(x; \varepsilon, h), \tag{50}
$$

where the function $J(\cdot; \varepsilon, h)$ is uniformly bounded with respect to ε and $h = (h_1, \dots, h_{2\mathcal{K}})^\top$ in the space $C^{\ell+1}(\Omega)$ for bounded ε and h. If ε is small enough, then the mapping $y : \Omega \to \Omega$ is bijective. Let $(\mathcal{L}^\varepsilon, \mathcal{B}^\varepsilon)$ denote the differential operator and the boundary operator written in the new coordinates y. Using these operators, we can transform the mixed boundary value problem with shifted endpoints $P^{(k)}(\varepsilon)$ onto the domain with non-shifted endpoints $P^{(k)}$,

$$
\begin{aligned}
\mathcal{L}^\varepsilon(y, \partial_y)u &= F(x(y)) && \text{in } \Omega, \\
\mathcal{B}^\varepsilon(y, \partial_y)u &= \begin{cases} G(x(y)) & \text{on } \Gamma_D, \\ H(x(y)) & \text{on } \Gamma_N, \\ (G_n, H_s)(x(y)) & \text{on } \Gamma_C. \end{cases}
\end{aligned}
$$

Due to (50) one finds the expansions

$$
\begin{aligned}
\mathcal{L}^\varepsilon(y, \partial_y) &= \mathcal{L}^0(y, \partial_y) + \varepsilon \mathcal{L}^1(y, \partial_y, \varepsilon), \\
\mathcal{B}^\varepsilon(y, \partial_y) &= \mathcal{B}^0(y, \partial_y) + \varepsilon \mathcal{B}^1(y, \partial_y, \varepsilon),
\end{aligned}
$$

where $\mathcal{L}^0$ and $\mathcal{B}^0$ are, respectively, the differential operator and the boundary operator for the problem with non-shifted endpoints. The perturbations $\mathcal{L}^1$ and $\mathcal{B}^1$ are differential operators of the same orders as $\mathcal{L}^0$ and $\mathcal{B}^0$ with variable coefficients which are bounded uniformly with respect to y and ε in the space $C^{\ell+1}(\Omega)$. The operator $(\mathcal{L}^0, B^0) : \mathfrak{V}_{\beta_-}^{\ell+1} \to \mathfrak{R}_{\beta_-}^{\ell}$ is an isomorphism and $(\mathcal{L}^1, B^1) : \mathfrak{V}_{\beta_-}^{\ell+1} \to \mathfrak{R}_{\beta_-}^{\ell}$ is bounded. Therefore, if ε is small enough, $(\mathcal{L}^\varepsilon, B^\varepsilon) = (\mathcal{L}^0, B^0) + \varepsilon(\mathcal{L}^1, B^1) : \mathfrak{V}_{\beta_-}^{\ell+1} \to \mathfrak{R}_{\beta_-}^{\ell}$ is also an isomorphism and the corresponding operator norm is bounded uniformly with respect to ε. Hence, estimate (49) is valid for the problem with shifted endpoints of the contact zone, too, and the constant on the right-hand side of this estimate is independent of ε.

5.6. Construction of the global approximation

After these preparations we are able to justify the asymptotics of the stress intensity factors. To simplify the notation, Γ_C and Γ_N will denote the appropriate parts of the boundary with shifted endpoints. We assume $F \in H^1(\Omega)^2$, $G \in$

$H^3(\Gamma)^2$ and $H \in H^2(\Gamma)^2$. Then

$$F \in V^0_\gamma(\Omega)^2, \tag{51}$$

$$G - \sum_{k=1}^{2\mathcal{K}} \chi_k \Big(G\big(P^{(k)}\big) + (s - s_k)\partial_s G\big(P^{(k)}\big)\Big) \in V^2_{\gamma+\frac{1}{2}}(\overline{\Gamma}_C)^2 \text{ and} \tag{52}$$

$$H - \sum_{k=1}^{2\mathcal{K}} \chi_k H\big(P^{(k)}\big) \in V^1_{\gamma+\frac{1}{2}}(\overline{\Gamma}_N \cup \overline{\Gamma}_C)^2 \tag{53}$$

holds for all weight indices $\gamma > -1$. This is proved with the help of the Hardy inequalities

$$\int_0^\infty r^{2\alpha-1}|u(r)|^2\,dr \leqq \frac{1}{\alpha^2}\int_0^\infty r^{2\alpha+1}|\partial_r u(r)|^2\,dr, \tag{54}$$

$$\int_0^\infty r^{-1}|\log(r)|^{-2}|u(r)|^2\,dr \leqq 4\int_0^\infty r|\partial_r u(r)|^2\,dr. \tag{55}$$

These inequalities are valid for functions $u \in H^1(\mathbb{R}_+)$, if the corresponding right-hand sides exist. Inequality (54) is true for $\alpha \neq 0$ if the limits $\lim_{r\to 0} r^{2\alpha}|u(r)|^2$ and $\lim_{r\to\infty} r^{2\alpha}|u(r)|^2$ vanish. The second inequality (55) holds for functions satisfying $\lim_{r\to 0}\log(r)|u(r)|^2 = \lim_{r\to\infty}\log(r)|u(r)|^2 = 0$. These inequalities have been established in [2] and can be found also in [13] and [19, Section 4.5.1]. For any function $F \in H^1(\Omega)^2$, any singularity point $P^{(k)}$ and any weight $\gamma > -1$ there follows from (54) the relation

$$\begin{aligned}
\left\|r_k^\gamma \chi_k F\,;\, L_2(\Omega)\right\|^2 &= \int_\Omega r_k^{2\gamma}\,|\chi_k F|^2\,dx \\
&= \int_{r_k=0}^\infty \int_{\varphi=0}^\pi r_k^{2\gamma+1}\,|\chi_k F|^2\,|\mathcal{J}|\,d\varphi\,dr_k \\
&\leqq C\int_{r_k=0}^\infty \int_{\varphi=0}^\pi r_k^{2\gamma+3}\,|\partial_r(\chi_k F)|^2\,d\varphi\,dr_k \\
&= C\left\|r_k^{\gamma+1}\partial_r(\chi_k F)\,;\, L_2(\Omega)\right\|^2,
\end{aligned}$$

with the bounded Jacobian $\mathcal{J}$ of the transformation to the curvilinear parameters (s, n). Consequently,

$$\begin{aligned}
\left\|F\,;\, V^0_\gamma(\Omega)\right\|^2 &= \left\|\Big(1 - \sum_{k=1}^{2\mathcal{K}}\chi_k\Big)F\,;\, L_2(\Omega)\right\|^2 + \sum_{k=1}^{2\mathcal{K}}\left\|r_k^\gamma \chi_k F\,;\, L_2(\Omega)\right\|^2 \\
&\leqq C\left\|F\,;\, H^1(\Omega)\right\|^2
\end{aligned}$$

if $\gamma > -1$. This proves relation (51). Analogously, for any exponent $\gamma \neq 0$ and any function $G \in H^1(\Gamma_C)$ satisfying $G\left(P^{(k)}\right) = 0$ in the case $\gamma < 0$, we have

$$\begin{aligned}\left\| r_k^\gamma \chi_k G\,;\ L_2(\Gamma_C) \right\|^2 &= \int_{\Gamma_C} r_k^{2\gamma}\, |\chi_k G|^2\, ds_x \leqq C_1 \int_{\Gamma_C} (s - s_k)^{2\gamma}\, |\chi_k G|^2\, ds_x \\ &\leqq C_2 \int_{\Gamma_C} (s - s_k)^{2\gamma+2}\, |\partial_s (\chi_k G)|^2\, ds_x \\ &\leqq C_3 \left\| r_k^{\gamma+1} \partial_s (\chi_k G)\,;\, L_2(\Gamma_C) \right\|^2 .\end{aligned}$$

Since the function $\tilde{G} := G - \sum_{k=1}^{2\mathcal{K}} \chi_k \left(G\big(P^{(k)}\big) + (s - s_k)\partial_s G\big(P^{(k)}\big)\right)$ satisfies

$$\tilde{G}\big(P^{(k)}\big) = \partial_s \tilde{G}\big(P^{(k)}\big) = 0,$$

we obtain for all indices $\gamma > -1$, $\gamma \neq -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$, by repeating the previous arguments, the estimates

$$\left\| r_k^{\gamma+\frac{1}{2}} \partial_s^2 (\chi_k \tilde{G})\,;\ L_2(\Omega) \right\|^2 \leqq C_1 \left\| r_k^{\gamma+\frac{3}{2}} \partial_s^3 (\chi_k \tilde{G})\,;\ L_2(\Omega) \right\|^2,$$

$$\left\| r_k^{\gamma+\frac{1}{2}-1} \partial_s (\chi_k \tilde{G})\,;\ L_2(\Omega) \right\|^2 \leqq C_2 \left\| r_k^{\gamma+\frac{3}{2}} \partial_s^3 (\chi_k \tilde{G})\,;\ L_2(\Omega) \right\|^2,$$

$$\left\| r_k^{\gamma+\frac{1}{2}-2} \chi_k \tilde{G}\,;\ L_2(\Omega) \right\|^2 \leqq C_3 \left\| r_k^{\gamma+\frac{3}{2}} \partial_s^3 (\chi_k \tilde{G})\,;\ L_2(\Omega) \right\|^2.$$

The condition $\gamma \neq -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ can be removed by the application of estimate (55) in suitable cases. From this and the definition of the norm $\left\| \cdot\,;\ V^2_{\gamma+\frac{1}{2}}(\overline{\Gamma_C}) \right\|$ there follows relation (52). Relation (53) can be proved analogously.

Let us abbreviate the outer and inner expansions and the terms which have been matched in Section 4 by

$$\begin{aligned}\mathcal{V}(\varepsilon, x) &= v^{(0)}(x) + \varepsilon v^{(1)}(x), \\ \mathcal{W}^{(k)}(\varepsilon, x) &= w^{(k,0)}(\xi) + \varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}(\xi) + \varepsilon w^{(k,1)}(\xi) + \varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}(\xi), \\ \mathcal{T}^{(k)}(\varepsilon, x) &= c^{(k,0)} + K_1^{(k)} r_k^{\frac{1}{2}} \left(\Phi^{(\frac{1}{2})}(\varphi_k) + \kappa_k r_k \Upsilon^{(\frac{3}{2})}(\log(\kappa_k r_k), \varphi_k) \right) \\ &\quad + \Lambda^{(k)}(s - s_k, n) + K_3^{(k)} r_k^{\frac{3}{2}} \Phi^{(\frac{3}{2})}(\varphi_k) \\ &\quad - \varepsilon \Bigg[K_1^{(k)} C_{\frac{1}{2}} h_k r_k^{-\frac{1}{2}} \left(\Phi^{(-\frac{1}{2})}(\varphi_k) + \kappa_k r_k \Upsilon^{(\frac{1}{2})}(\log(\kappa_k r_k), \varphi_k) \right) \\ &\quad + \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} C_{\frac{1}{2}} h_m \left(c_\Omega^{mk} e_s^{(k)} + C_\Omega^{mk} r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) \right) \Bigg].\end{aligned}$$

For simplicity, the cut-off functions χ_k are defined by

$$\chi_k(x) = \begin{cases} \chi(r_k), & r_k \leqq c_2, \\ 0, & r_k > c_2 \end{cases}$$

with some function χ satisfying $\chi(r) \equiv 1$ for $r \leqq c_1$ and $\chi(r) \equiv 0$ for $r \geqq c_2$, where $0 < c_1 < c_2$. In what follows, the technique of overlapping cut-off functions is employed. The function $\chi_k(x)$ cutting off the exterior of a neighbourhood of $P^{(k)}$ is combined with the function $1 - \chi_k(\frac{x}{\varepsilon})$ which cuts off a close neighbourhood of $P^{(k)}$. If ε is small enough, then the sets $\{\chi_k = 1\}$ and $\{1 - \chi_k(\frac{\cdot}{\varepsilon}) = 1\}$ have a nontrivial intersection, and their union is the whole plane $\mathbb{R}^2$. The product of all the latter cut-off functions is given by

$$X(\varepsilon, x) = \prod_{k=1}^{2\mathcal{K}} \left(1 - \chi\left(\frac{r_k}{\varepsilon}\right)\right) = 1 - \sum_{k=1}^{2\mathcal{K}} \chi\left(\frac{r_k}{\varepsilon}\right).$$

For every $x \in \mathbb{R}^2$, either $X(\varepsilon, x) = 1$ or $\chi_k(x) = 1$ with a suitable index k. With the help of these cut-off functions we construct the vector field

$$\mathcal{U}(\varepsilon, x) = X(\varepsilon, x)\mathcal{V}(\varepsilon, x) + \sum_{k=1}^{2\mathcal{K}} \chi_k(x)\mathcal{W}^{(k)}(\varepsilon, x) - X(\varepsilon, x)\sum_{k=1}^{2\mathcal{K}} \chi_k(x)\mathcal{T}^{(k)}(\varepsilon, x).$$

There exists a nontrivial intermediate zone $\{c_2\varepsilon \leqq r_k \leqq c_1\}$ where both cut-off functions $X(\varepsilon, \cdot)$ and $\chi_k(\cdot)$ are equal to 1; hence in the first two terms on the right-hand side the terms which have been matched are considered twice. Therefore the subtraction of these terms is necessary in order to obtain a sufficiently good approximation of the solution in this intermediate zone. The function $\mathcal{U}(\varepsilon, \cdot)$ can be interpreted as a solution of the mixed boundary value problem (1)–(4) with shifted endpoints of the contact zone and the appropriately given data $(L(\mathcal{U}), \mathcal{U}, \sigma^{(n)}(\mathcal{U}))^\top$ instead of $(f, g, h)^\top$. The term $\mathcal{U}$ is equal to the outer expansion $\mathcal{V}(\varepsilon, x)$ of the solution $u(\varepsilon, x)$ outside a neighbourhood of the points $P^{(k)}$ and to the inner expansion $\mathcal{W}^{(k)}(\varepsilon, x)$ close to the point $P^{(k)}$. In particular, the stress intensity factor of $\mathcal{U}$ at the point $P^{(k)}$ coincides with the stress intensity factor of the inner expansion $\mathcal{W}^{(k)}(\varepsilon, x)$. This value is given by formula (43) and represents the asymptotics of the stress intensity factor we would like to verify. Therefore we study the difference

$$u(\varepsilon, x) - \mathcal{U}(\varepsilon, x) \tag{56}$$

of $\mathcal{U}$ and the solution $u(\varepsilon, x)$ of the problem with shifted endpoints of the contact zone. The stress intensity factor of $u(\varepsilon, x) - \mathcal{U}(\varepsilon, x)$ at the point $P^{(k)}$ is equal to the difference

$$K_1^{(k)}(\varepsilon) - \left(K_1^{(k)} + \varepsilon T_1^{(k)}\right)$$

of the stress intensity factor $K_1^{(k)}(\varepsilon)$ for the perturbed problem and its approximation $K_1^{(k)} + \varepsilon T_1^{(k)}$ derived in Section 4. In order to justify the asymptotics it is therefore sufficient to prove that the stress intensity factor of $u(\varepsilon, x) - \mathcal{U}(\varepsilon, x)$ is bounded by $C\varepsilon^{1+\delta}$ with C independent of ε and $\delta > 0$. Since estimate (49) is valid for the problem with shifted endpoints of the contact zone with a constant independent of ε, it is sufficient to show that

$$\left\| \left(F - L(\mathcal{U}), G - \mathcal{U}, H - \sigma^{(n)}(\mathcal{U}), G_n - \mathcal{U}_n, H_s - \sigma_s^{(n)}(\mathcal{U})\right)^\top ; \mathfrak{R}^1_{\beta_- - \frac{1}{2}} \right\| \leqq C\varepsilon^{1+\delta} \tag{57}$$

for a suitable value $\beta_- \in (\frac{1}{2}, 1)$. Since $u(\varepsilon, x)$ and $\mathcal{U}(\varepsilon, x)$ are equal for $x \in \Gamma_D$, this relation follows from the four estimates

$$\left\| F - L(\mathcal{U})\,;\; V^0_{\beta_- - \frac{1}{2}}(\Omega)^2 \right\| \leqq C\varepsilon^{1+\delta}, \tag{58}$$

$$\left\| G_n - \mathcal{U}_n\,;\; V^{\frac{3}{2}}_{\beta_- - \frac{1}{2}}(\Gamma_C)^2 \right\| \leqq C\varepsilon^{1+\delta}, \tag{59}$$

$$\left\| H - \sigma^{(n)}(\mathcal{U})\,;\; V^{\frac{1}{2}}_{\beta_- - \frac{1}{2}}(\Gamma_N)^2 \right\| \leqq C\varepsilon^{1+\delta}, \tag{60}$$

$$\left\| H_s - \sigma^{(n)}_s(\mathcal{U})\,;\; V^{\frac{1}{2}}_{\beta_- - \frac{1}{2}}(\Gamma_C)^2 \right\| \leqq C\varepsilon^{1+\delta}. \tag{61}$$

These estimates will be proved in what follows. There will often appear integrals over domains of the form $\Omega \cap \{r_k \leqq c\}$, $\Omega \cap \{r_k \geqq c\}$ or similar types of domains. For simplicity of notation, such domains of integration will be denoted by $r_k \leqq c$ or $r_k \geqq c$ only.

5.7. Estimating the discrepancy

Let us start with the proof of relation (58). Due to the definition of the overlapping cut-off functions, $X(\varepsilon, x) = 1$ for $c_1 \leqq r_k \leqq c_2$ if ε is small enough ($\varepsilon < \frac{c_1}{c_2}$), $\chi_k = 1$ for $r_k \leqq c_1$ and $\chi_k = 0$ for $r_k \geqq c_2$. Consequently, we obtain $X(\varepsilon, \cdot) + \chi_k - X(\varepsilon, \cdot)\chi_k = 1$ for $r_k \leqq c_2$; and the commutator of the operator L with the product of the cut-off functions reads

$$[L, X(\varepsilon, \cdot)\chi_k] = \begin{cases} [L, X(\varepsilon, \cdot)] + [L, \chi_k] & \text{for } r_k \leqq c_2, \\ 0 & \text{for } r_k \geqq c_2, \end{cases}$$
$$= \begin{cases} [L, X(\varepsilon, \cdot)] & \text{for } r_k \leqq c_1, \\ [L, \chi_k] & \text{for } c_1 \leqq r_k \leqq c_2, \\ 0 & \text{for } r_k > c_2. \end{cases}$$

Employing the relations $L\big(v^{(0)}\big) = F$ and $L\big(v^{(1)}\big) = 0$ we obtain $L\big(\mathcal{V}(\varepsilon, \cdot)\big) = F$ and

$$L(\mathcal{U}) - F = [L, X]\Bigg(\mathcal{V} - \sum_{k=1}^{2\mathcal{K}} \chi_k \mathcal{T}^{(k)}\Bigg) + (X - 1)F$$
$$+ \sum_{k=1}^{2\mathcal{K}} \Big([L, \chi_k]\Big(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\Big) + \chi_k\Big(L\big(\mathcal{W}^{(k)}\big) - XL\big(\mathcal{T}^{(k)}\big)\Big)\Big). \tag{62}$$

The norm of the second term $(X-1)F = -\sum_{k=1}^{2\mathcal{K}} \chi_k\big(\frac{\cdot}{\varepsilon}\big)F$ can be estimated with the help of relation (51) by

$$\begin{aligned}\|\chi_k\big(\tfrac{\cdot}{\varepsilon}\big)F\,;\;V^0_{\beta_- - \frac{1}{2}}(\Omega)^2\|^2 &= \int_{r_k \leqq c_2\varepsilon} r_k^{2(\beta_- - \frac{1}{2})}\left|\chi_k\Big(\frac{\cdot}{\varepsilon}\Big)F\right|^2\,dx\\ &= \int_{r_k \leqq c_2\varepsilon} r_k^{2(\beta_- - \frac{1}{2}-\gamma)} r_k^{2\gamma}\left|\chi_k\Big(\frac{\cdot}{\varepsilon}\Big)F\right|^2\,dx\\ &\leqq C_1\varepsilon^{2(\beta_- - \frac{1}{2}-\gamma)}\left\|F\,;\;V^0_\gamma(\Omega)^2\right\|^2 \;\leqq\; C_2\varepsilon^{2(\beta_- - \frac{1}{2}-\gamma)}\end{aligned}$$

with an arbitrary parameter $\gamma > -1$.

The difference of the outer expansion and the terms which have been matched can be written as

$$\mathcal{V}(\varepsilon,x) - \sum_{k=1}^{2\mathcal{K}} \chi_k(x)\mathcal{T}^{(k)}(\varepsilon,x) = \tilde{u}^{(2)}(x) - \varepsilon\sum_{k=1}^{2\mathcal{K}} K_1^{(k)} C_{\frac{1}{2}} h_k \tilde{\zeta}^{(k,\frac{1}{2})}(x). \qquad (63)$$

Here, $\tilde{u}^{(2)}$ denotes the remaining term in the asymptotic expansion of the solution u according to Theorem 2.5; and the function $\tilde{\zeta}^{(k,\frac{1}{2})}$ is the remaining part of the weight function $\zeta^{(k,\frac{1}{2})}$ in the representation

$$\begin{aligned}\zeta^{(k,\frac{1}{2})}(x) &= \chi_k\left(r_k^{-\frac{1}{2}}\Phi^{(-\frac{1}{2})}(\varphi_k) + \kappa_k r_k^{\frac{1}{2}}\Upsilon^{(\frac{1}{2})}(\log(\kappa_k r_k),\varphi_k)\right)\\ &\quad + \sum_{m=1}^{2\mathcal{K}} \chi_m\left(c_\Omega^{km} e_s^{(m)} + C_\Omega^{km} r_m^{\frac{1}{2}}\Phi^{(\frac{1}{2})}(\varphi_m)\right) + \tilde{\zeta}^{(k,\frac{1}{2})}(x)\end{aligned}$$

(see equation (28)). These terms satisfy $\tilde{u}^{(2)} \in V^2_\gamma(\Omega)^2$ and $\tilde{\zeta}^{(k,\frac{1}{2})} \in V^1_\gamma(\Omega)^2$ for every $\gamma > -1$. The commutator $[L, X] = -\sum_{k=1}^{2\mathcal{K}}\big[L, \chi_k\big(\frac{\cdot}{\varepsilon}\big)\big]$ defines a differential operator of first order having a support contained in $\bigcup_{k=1}^{2\mathcal{K}}\{r_k \leqq c_2\varepsilon\}$. Each of the operators $\big[L, \chi_k\big(\frac{\cdot}{\varepsilon}\big)\big]$ has a representation of the type

$$\big[L, \chi_k\big(\tfrac{\cdot}{\varepsilon}\big)\big] = \sum_{\substack{|\alpha+\alpha'|\leqq 2\\ |\alpha|\leqq 1}} a_{\alpha,\alpha'}(x)\partial_x^{\alpha'}\chi_k\Big(\frac{x}{\varepsilon}\Big)\partial_x^\alpha$$

with bounded coefficients $a_{\alpha,\alpha'}(x)$. Due to the relation

$$\partial_x^\alpha\chi_k\big(\tfrac{x}{\varepsilon}\big) = \varepsilon^{-|\alpha|}(\partial_x^\alpha\chi_k(y))|_{y=\frac{x}{\varepsilon}},$$

we have for any $\gamma > -1$:

$$\begin{aligned}
&\left\| \left[L, \chi_k\left(\tfrac{\cdot}{\varepsilon}\right)\right]\tilde{u}^{(2)} \,;\; V^0_{\beta_- - \frac{1}{2}}(\Omega)^2 \right\|^2 \\
&\leqq C_1 \int_{r_k \leqq c_2\varepsilon} \sum_{\substack{|\alpha+\alpha'|\leqq 2\\ |\alpha|\leqq 1}} r_k^{2(\beta_- - \frac{1}{2})} \varepsilon^{-2|\alpha'|} \left|\partial^\alpha \tilde{u}^{(2)}\right|^2 \, dx \\
&\leqq C_2 \sum_{\substack{|\alpha+\alpha'|\leqq 2\\ |\alpha|\leqq 1}} \varepsilon^{-2|\alpha'|+2(\beta_- - \frac{1}{2}+2-\gamma-|\alpha|)} \int_{r_k \leqq c_2\varepsilon} r_k^{2(\gamma-2+|\alpha|)} \left|\partial^\alpha \tilde{u}^{(2)}\right|^2 \, dx \\
&\leqq C_3\, \varepsilon^{2(\beta_- - \frac{1}{2}-\gamma)} \left\| \tilde{u}^{(2)} \,;\; V^1_{\gamma-1}(\Omega)^2 \right\|^2 \\
&\leqq C_4\, \varepsilon^{2(\beta_- - \frac{1}{2}-\gamma)} \left\| \tilde{u}^{(2)} \,;\; V^2_{\gamma}(\Omega)^2 \right\|^2 \\
&\leqq C_5\, \varepsilon^{2(\beta_- - \frac{1}{2}-\gamma)}.
\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}
&\left\| \left[L, \chi_k\left(\tfrac{\cdot}{\varepsilon}\right)\right] \varepsilon\tilde{\zeta}^{(k,\frac{1}{2})} \,;\; V^0_{\beta_- - \frac{1}{2}}(\Omega)^2 \right\|^2 \\
&\leqq C_1 \int_{r_k \leqq c_2\varepsilon} \sum_{\substack{|\alpha+\alpha'|\leqq 2\\ |\alpha|\leqq 1}} r_k^{2(\beta_- - \frac{1}{2})} \varepsilon^2 \varepsilon^{-2|\alpha'|} \left|\partial^\alpha \tilde{\zeta}^{(k,\frac{1}{2})}\right|^2 \, dx \\
&\leqq C_2\, \varepsilon^{2(\beta_- - \frac{1}{2})+2-2+2(1-\gamma)} \int_{r_k \leqq c_2\varepsilon} \sum_{|\alpha|\leqq 1} r_k^{2(\gamma-1+|\alpha|)} \left|\partial^\alpha \tilde{\zeta}^{(k,\frac{1}{2})}\right|^2 \, dx \\
&\leqq C_3\, \varepsilon^{2(\beta_- - \gamma)+1} \left\| \tilde{\zeta}^{(k,\frac{1}{2})} \,;\; V^1_{\gamma}(\Omega)^2 \right\|^2 \leqq \; C_4\, \varepsilon^{2(\beta_- - \gamma)+1}.
\end{aligned}$$

In the third term of (62) we observe $[L, \chi_k] = 0$ for $r_k \leqq c_1$ and $r_k \geqq c_2$. Hence, we can use the following Taylor expansions which are valid outside a neighbourhood of the singular point $P^{(k)}$:

$$\begin{aligned}
&\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}(\xi) \\
&\quad = K_1^{(k)} \left[r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) - \varepsilon h_k C_{\frac{1}{2}} r_k^{-\frac{1}{2}} \Phi^{(-\frac{1}{2})}(\varphi_k) \right] + \mathcal{O}\left(\varepsilon^2\right),
\end{aligned} \tag{64}$$

$$\begin{aligned}
&\varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}(\xi) \\
&\quad = \kappa_k K_1^{(k)} \left(r_k^{\frac{3}{2}} \Upsilon^{(\frac{3}{2})}(\log(\kappa_k r_k), \varphi_k) - \varepsilon h_k C_{\frac{1}{2}} r_k^{\frac{1}{2}} \Upsilon^{(\frac{1}{2})}(\log(\kappa_k r_k), \varphi_k) \right) \\
&\qquad + K_3^{(k)} r_k^{\frac{3}{2}} \Phi^{(\frac{3}{2})}(\varphi_k) - \varepsilon \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} h_m C_{\frac{1}{2}} C_\Omega^{mk} r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) + \mathcal{O}\left(\varepsilon^2\right).
\end{aligned} \tag{65}$$

Comparison of these expansions with the terms which have been matched yields

$$\mathcal{W}^{(k)}(\varepsilon, x) - \mathcal{T}^{(k)}(\varepsilon, x) = \mathcal{O}\big(\varepsilon^2\big), \tag{66}$$

$$\partial_{x_m}\big(\mathcal{W}^{(k)}(\varepsilon, x) - \mathcal{T}^{(k)}(\varepsilon, x)\big) = \mathcal{O}\big(\varepsilon^2\big) \tag{67}$$

in the set $\{c_1 \leqq r_k \leqq c_2\}$ where the commutator $[L, \chi_k]$ is non-zero. This yields

$$\begin{aligned}\Big\|[L, \chi_k]\left(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\right) \,;\, V^0_{\beta_- - \frac{1}{2}}(\Omega)^2\Big\|^2 \\ \leqq C_1 \int_{c_1 \leqq r_k \leqq c_2} \varepsilon^4 r_k^{2(\beta_- - \frac{1}{2})}\, dx \leqq C_2\, \varepsilon^4.\end{aligned}$$

In order to estimate the last term in (62) we use the decomposition

$$L\big(\mathcal{W}^{(k)}\big) - XL\big(\mathcal{T}^{(k)}\big) = XL\big(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\big) + (1 - X)L\big(\mathcal{W}^{(k)}\big).$$

The operator of elasticity, written in curvilinear coordinates, has near the point $P^{(k)}$ the representation

$$L = \mathfrak{L}^0 + \mathcal{L}^1(\kappa_k) + \mathfrak{L}^2(\kappa_k).$$

Here, $\mathfrak{L}^0 = \mathbf{L}^0$ and $\mathfrak{L}^1(\kappa_k) = \kappa_k \mathbf{L}^1$ denote the first two terms of the asymptotic expansion given in Section 2.2. The remaining operator $\mathfrak{L}^2(\kappa_k)$ is of generalized order 0 and order 2 of differentiation. Hence, $\mathfrak{L}^2(\kappa_k)$ can be written in the form

$$\mathfrak{L}^2(\kappa_k) = \sum_{i+j \leqq 2} A_{ij}(\kappa_k; s, n)(r\partial_r)^i \partial_\varphi^j$$

with bounded coefficients $A_{ij}(\kappa_k)$. Moreover, due to the construction of the inner expansion, the following relations hold:

$$\begin{gathered}L\Big(w^{(k,0)}\Big) = 0,\ \mathfrak{L}^0\Big(\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}\Big) = 0,\ \mathfrak{L}^0\Big(\varepsilon w^{(k,1)}\Big) = 0, \\ \text{and } \mathfrak{L}^0\Big(\varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}\Big) = -\mathfrak{L}^1\Big(\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}\Big).\end{gathered}$$

Therefore, we obtain

$$L\Big(\mathcal{W}^{(k)}\Big) = (\mathfrak{L}^1 + \mathfrak{L}^2)\Big(\varepsilon w^{(k,1)} + \varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}\Big) + \mathfrak{L}^2\Big(\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}\Big). \tag{68}$$

The functions $\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}$, $\varepsilon w^{(k,1)}$ and $\varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}$ satisfy near the point $P^{(k)}$ the estimates

$$\begin{aligned}\Big|\partial^\alpha \varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}\Big| &\leqq C_1 \left(r_k^*\right)^{\frac{1}{2} - |\alpha|}, \\ \Big|\partial^\alpha \varepsilon w^{(k,1)}\Big| &\leqq C_2, \\ \Big|\partial^\alpha \varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}\Big| &\leqq C_3 \varepsilon \left(r_k^*\right)^{\frac{1}{2} - |\alpha|} + C_4 \left(r_k^*\right)^{\frac{3}{2} - |\alpha|} \log(\kappa_k r_k^*).\end{aligned}$$

Inserting these estimates into (68) yields the inequality

$$\Big|L\big(\mathcal{W}^{(k)}\big)\Big| \leqq C,$$

valid for $r_k \leqq c_2\varepsilon$ with an arbitrary parameter $\delta > 0$. Hence,

$$\left\| \chi_k\left(\tfrac{\cdot}{\varepsilon}\right) L\left(\mathcal{W}^{(k)}\right);\ V^0_{\beta_- - \frac{1}{2}}(\Omega)^2 \right\| \ \leqq\ C_1 \int_{r_k \leqq c_2\varepsilon} r_k^{2(\beta_- - \frac{1}{2})}\, dx \ \leqq\ C_2\, \varepsilon^{1+2\beta_-}.$$

From the Taylor expansions (64) of $\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}$, (42) of $\varepsilon w^{(k,1)}$ and (65) of $\varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}$, we obtain

$$\begin{aligned}
&\varepsilon w^{(k,1)}(\xi) + \varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}(\xi) \ =\ \Lambda^{(k)}(s - s_k, n) - \varepsilon \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} C_{\frac{1}{2}} h_m c_\Omega^{mk} e_s^{(k)} \\
&\quad + r_k^{\frac{3}{2}} \left[\kappa_k K_1^{(k)} \Upsilon^{(\frac{3}{2})}(\log(\kappa_k r_k), \varphi_k) + K_3^{(k)} \Phi^{(\frac{3}{2})}(\varphi_k)\right] \\
&\quad - \varepsilon r_k^{\frac{1}{2}} \left[\kappa_k K_1^{(k)} C_{\frac{1}{2}} h_k \Upsilon^{(\frac{1}{2})}(\log(\kappa_k r_k), \varphi_k) + \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} h_m C_{\frac{1}{2}} C_\Omega^{mk} \Phi^{(\frac{1}{2})}(\varphi_k)\right] \\
&\quad + \tilde{\mathcal{R}}^{(k,\frac{3}{2})}(\varepsilon, x),
\end{aligned} \tag{69}$$

$$\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})} \ =\ K_1^{(k)} \left[r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) - \varepsilon h_k C_{\frac{1}{2}} r_k^{-\frac{1}{2}} \Phi^{(-\frac{1}{2})}(\varphi_k)\right] + \tilde{R}^{(k,\frac{1}{2})}(\varepsilon, x). \tag{70}$$

These expansions are valid outside a ball $B_{c_2\varepsilon}\left(P^{(k)}\right)$ containing both the non-shifted endpoint $P^{(k)}$ and the shifted endpoint $P^{(k)}(\varepsilon)$. Outside this ball the remaining terms $\tilde{\mathcal{R}}^{(k,\frac{3}{2})}$ and $\tilde{\mathcal{R}}^{(k,\frac{1}{2})}$ satisfy the estimates

$$\begin{aligned}
\left|\partial^\alpha \tilde{\mathcal{R}}^{(k,\frac{3}{2})}\right| &\leqq C\varepsilon^2 r_k^{-\frac{1}{2}-|\alpha|} \log(\kappa_k r_k) \quad \text{and} \\
\left|\partial^\alpha \tilde{\mathcal{R}}^{(k,\frac{1}{2})}\right| &\leqq C\varepsilon^2 r_k^{-\frac{3}{2}-|\alpha|} \log(\kappa_k r_k).
\end{aligned} \tag{71}$$

Bearing in mind the relations

$$\mathfrak{L}^0\left(r_k^{\frac{m}{2}} \Phi^{(\frac{m}{2})}(\varphi_k)\right) = 0 \quad \text{and}$$
$$\mathfrak{L}^0\left(\kappa_k r_k^{\frac{m}{2}} \Upsilon^{(\frac{m}{2})}(\log(\kappa_k r_k), \varphi_k)\right) = -\mathfrak{L}^1\left(r_k^{\frac{m}{2}-1} \Phi^{(\frac{m}{2}-1)}(\varphi_k)\right)$$

we can simplify the term $L\left(\mathcal{T}^{(k)}\right)$ to

$$\begin{aligned}
&(\mathfrak{L}^0 + \mathfrak{L}^1 + \mathfrak{L}^2)\left(\mathcal{T}^{(k)}\right) \\
&\quad = (\mathfrak{L}^1 + \mathfrak{L}^2)\Big\{\kappa_k K_1^{(k)} r_k^{\frac{3}{2}} \Upsilon^{(\frac{3}{2})}(\log(\kappa_k r_k), \varphi_k) \\
&\qquad + \Lambda^{(k)}(s - s_k, n) + K_3^{(k)} r_k^{\frac{3}{2}} \Phi^{(\frac{3}{2})}(\varphi_k) \\
&\qquad - \varepsilon\Big[\kappa_k K_1^{(k)} C_{\frac{1}{2}} h_k r_k^{\frac{1}{2}} \Upsilon^{(\frac{1}{2})}(\log(\kappa_k r_k), \varphi_k) \\
&\qquad + \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} C_{\frac{1}{2}} h_m \left(c_\Omega^{mk} e_s^{(k)} + C_\Omega^{mk} r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k)\right)\Big]\Big\} \\
&\qquad + \mathcal{L}^2\left(K_1^{(k)}\left(r_k^{\frac{1}{2}} \Phi^{(\frac{1}{2})}(\varphi_k) - \varepsilon h_k C_{\frac{1}{2}} r_k^{-\frac{1}{2}} \Phi^{(-\frac{1}{2})}(\varphi_k)\right)\right).
\end{aligned} \tag{72}$$

Inserting the expansions (70) and (69) into formula (68) and subtracting expression (72) from the result we obtain

$$L\big(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\big) = (\mathfrak{L}^1 + \mathfrak{L}^2)\big(\tilde{\mathcal{R}}^{(k,\frac{3}{2})}\big) + \mathcal{L}^2\big(\tilde{\mathcal{R}}^{(k,\frac{1}{2})}\big).$$

Since the operators $\mathfrak{L}^1$ and $\mathfrak{L}^2$ are of generalized order 1 and 0, respectively, and the relations (71) are valid,

$$\begin{aligned}&\Big\|\chi_k X L\big(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\big)\,;\ V^0_{\beta_- - \frac{1}{2}}(\Omega)^2\Big\| \\ &\leqq C_1 \int_{c_1\varepsilon \leqq r_k \leqq c_2} r_k^{2(\beta_- - \frac{1}{2})} \varepsilon^4 r_k^{-3} |\log(\kappa_k r_k)|^2\, dx \ \leqq\ C_2 \varepsilon^{2(1+\beta_-)} |\log(\kappa_k \varepsilon)|^2.\end{aligned}$$

The collection of all these estimates, valid for the differential operator L, yields

$$\Big\|F - L(\mathcal{U})\,;\ V^0_{\beta_- - \frac{1}{2}}(\Omega)^2\Big\|^2 \leqq C\varepsilon^{1+2\beta_- - \delta}$$

with $\beta_- \in (\frac{1}{2}, 1)$ and $\delta > 0$. This proves estimate (58).

The verification of estimate (59) is much simpler. Due to the relations $\mathcal{V}_n(\varepsilon, x) = G_n(x)$ and $\mathcal{W}_n^{(k)}(\varepsilon, x) = \mathcal{T}_n^{(k)}(\varepsilon, x) = G_n\big(P^{(k)}\big) + \partial_s G_n\big(P^{(k)}\big)(s - s_k)$, we have

$$\begin{aligned}\mathcal{U}_n(\varepsilon, x) &= X(\varepsilon, x) G_n(x) \\ &\quad + \sum_{k=1}^{2\mathcal{K}} (1 - X(\varepsilon, x))\chi_k(x) \Big(G_n\big(P^{(k)}\big) + \partial_s G_n\big(P^{(k)}\big)(s - s_k)\Big).\end{aligned}$$

The overlapping cut-off functions satisfy the relation $(1 - X(\varepsilon, x))\chi_k(x) = \chi_k\big(\frac{x}{\varepsilon}\big)\chi_k(x) = \chi_k\big(\frac{x}{\varepsilon}\big)$, since $\chi_k(x)$ is equal to 1 in the support of $\chi_k(\frac{x}{\varepsilon})$. Hence, we have

$$\mathcal{U}_n(\varepsilon, x) = G_n(x) + \sum_{k=1}^{2\mathcal{K}} \chi_k(\tfrac{x}{\varepsilon}) \Big(G_n\big(P^{(k)}\big) + \partial_s G_n\big(P^{(k)}\big)(s - s_k) - G_n(x)\Big).$$

Let us use the abbreviation $\tilde{G}_n^{(k)}(x) = G_n(x) - G_n\big(P^{(k)}\big) - \partial_s G_n\big(P^{(k)}\big)(s - s_k)$. The cut-off function $\chi_k\big(\frac{x}{\varepsilon}\big)$ satisfies the relation $\Big|\partial_x^j \chi_k\big(\frac{x}{\varepsilon}\big)\Big| \leqq C r_k^{-j}$, since $\Big|\partial_x^j \chi_k\big(\frac{x}{\varepsilon}\big)\Big| \leqq C\varepsilon^{-j}$ and $\partial_x^j \chi_k\big(\frac{x}{\varepsilon}\big) = 0$ for $r_k < c_1\varepsilon$, $j \geqq 1$. Using this relation we obtain

$$\begin{aligned}\Big\|\mathcal{U}_n - G_n;\, V^{\frac{3}{2}}_{\beta_- - \frac{1}{2}}(\Gamma_C)\Big\|^2 &\leqq C_1 \Big\|\mathcal{U}_n - G_n\,;\ V^2_{\beta_-}(\Gamma_C)\Big\|^2 \\ &\leqq C_2 \sum_{k=1}^{2\mathcal{K}} \sum_{m=0}^{2} \Big\|r_k^{\beta_- - 2 + m} \partial_s^m \Big(\chi_k(\frac{\cdot}{\varepsilon})\tilde{G}^{(k)}\Big)\,;\ L_2(\Gamma_C)\Big\|^2 \\ &\leqq C_3 \sum_{k=1}^{2\mathcal{K}} \sum_{m=0}^{2} \Big\|r_k^{\beta_- - 2 + m} \partial_s^m \tilde{G}^{(k)}\,;\ L_2(\Gamma_C)^2\Big\|^2 \\ &\leqq C_4 \int_{|s - s_k| \leqq c_2\varepsilon} \Big|\partial_s^3 G(s)\Big|^2 |s - s_k|^{2(\beta_- + 1)}\, ds \\ &\leqq C_5 \varepsilon^{2(\beta_- + 1)} \Big\|G\,;\ H^3(\Gamma_C)\Big\|^2.\end{aligned}$$

Hence, estimate (59) is proved.

The verification of the estimates (60) and (61) is similar to the proof of (58). We present here only the proof of (60). The boundary stress of $\mathcal{U}$ can be represented by

$$\sigma^{(n)}(\mathcal{U}) = [\sigma^{(n)}, X]\left(\mathcal{V} - \sum_{k=1}^{2\mathcal{K}} \chi_k \mathcal{T}^{(k)}\right) + XH$$
$$+\sum_{k=1}^{2\mathcal{K}} \Big[[\sigma^{(n)}, \chi_k]\big(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\big) + \chi_k \Big[\sigma^{(n)}\big(\mathcal{W}^{(k)}\big) - X\sigma^{(n)}\big(\mathcal{T}^{(k)}\big)\Big)\Big].$$

Recalling the definition of X and the equation

$$\chi_k\sigma^{(n)}\big(\mathcal{W}^{(k)}\big) = (\chi_k X + \chi_k(1-X))\sigma^{(n)}\big(\mathcal{W}^{(k)}\big) = \big(\chi_k X + \chi_k\big(\tfrac{\cdot}{\varepsilon}\big)\big)\sigma^{(n)}\big(\mathcal{W}^{(k)}\big),$$

the difference of $\sigma^{(n)}(\mathcal{U})$ and H can be written as

$$\sigma^{(n)}(\mathcal{U}) - H = [\sigma^{(n)}, X]\left(\mathcal{V} - \sum_{k=1}^{2\mathcal{K}} \chi_k \mathcal{T}^{(k)}\right) + \sum_{k=1}^{2\mathcal{K}} \chi_k\big(\tfrac{\cdot}{\varepsilon}\big)\big(H\big(P^{(k)}\big) - H\big)$$
$$+ \sum_{k=1}^{2\mathcal{K}} \Big[[\sigma^{(n)}, \chi_k]\big(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\big) + \chi_k X\sigma^{(n)}\big(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\big)$$
$$+ \chi_k\big(\tfrac{\cdot}{\varepsilon}\big)\Big(\sigma^{(n)}\big(\mathcal{W}^{(k)}\big) - H\big(P^{(k)}\big)\Big)\Big]. \tag{73}$$

The definition of the boundary stress $\sigma^{(n)}$ can be extended in a natural way from the boundary Γ to the tubular neighbourhood where the curvilinear coordinates (s, n) are defined. The extended operator will be denoted by $\sigma^{(n)}$ again. Since $\sigma^{(n)}$ is a differential operator of first order, the commutator $\big[\sigma^{(n)}, X\big]$ is a matrix-valued function. Due to the definition of X there holds $\big[\sigma^{(n)}, X\big] = 0$ for $x \notin \bigcup_{k=1}^{2\mathcal{K}}\{c_1\varepsilon \leqq r_k \leqq c_2\varepsilon\}$ and $\big|\partial_s^m\big[\sigma^{(n)}, X\big]\big| \leqq C_1\varepsilon^{-1-m} \leqq C_2\varepsilon^{-1}r_k^{-m}$ for $x \in \{c_1\varepsilon \leqq r_k \leqq c_2\varepsilon\}$. In order to estimate the first term in (73) we use the representation (63) of the difference $\mathcal{V} - \sum_{k=1}^{2\mathcal{K}} \mathcal{T}^{(k)}$ in terms of the functions $\tilde{u}^{(2)}$ and $\tilde{\zeta}^{(k,\frac{1}{2})}$. Using the trace theorem for weighted Sobolev spaces we obtain

$$\left\|\big[\sigma^{(n)}, X\big]\tilde{u}^{(2)}\,;\ V^{\frac{1}{2}}_{\beta_- - \frac{1}{2}}(\Gamma_N)^2\right\|^2 \leqq C_1 \left\|\big[\sigma^{(n)}, X\big]\tilde{u}^{(2)}\,;\ V^{1}_{\beta_-}(\Gamma_N)^2\right\|^2$$
$$\leqq C_2\,\varepsilon^{-2}\sum_{k=1}^{2\mathcal{K}} \int_{\Gamma_N\cap\{c_1\varepsilon\leqq r_k\leqq c_2\varepsilon\}} \sum_{m=0}^{1} r_k^{2(\beta_- -1+m)}\left|\partial_s^m\tilde{u}^{(2)}\right|^2 ds$$
$$\leqq C_3\,\varepsilon^{2(\beta_- -\gamma-\frac{1}{2})}\sum_{k=1}^{2\mathcal{K}} \int_{\Gamma_N\cap\{c_1\varepsilon\leqq r_k\leqq c_2\varepsilon\}} \sum_{m=0}^{1} r_k^{2(\gamma-\frac{3}{2}+m)}\left|\partial_s^m\tilde{u}^{(2)}\right|^2 ds$$
$$\leqq C_4\,\varepsilon^{2(\beta_- -\gamma)-1}\left\|\tilde{u}^{(2)}\,;\ V^{1}_{\gamma-\frac{1}{2}}(\Gamma_N)^2\right\|^2 \leqq C_5\,\varepsilon^{2(\beta_- -\gamma)-1}\left\|\tilde{u}^{(2)}\,;\ V^{2}_{\gamma}(\Omega)^2\right\|^2.$$

For $\tilde\zeta^{(k,\frac12)}$ we derive analogously

$$\begin{aligned}
&\Big\| \big[\sigma^{(n)}, X\big]\,\varepsilon\tilde\zeta^{(k,\frac12)}\,;\ V^{\frac12}_{\beta_- -\frac12}(\Gamma_N)^2\Big\|^2 \ \leqq\ C_1 \Big\|\varepsilon\big[\sigma^{(n)}, X\big]\tilde\zeta^{(k,\frac12)}\,;\ V^1_{\beta_- -\frac12}(\Omega)^2\Big\|^2\\
&\qquad \leqq C_2 \sum_{m=1}^{2\mathcal{K}} \int_{r_m \leqq c_2\varepsilon} \sum_{|\alpha|\leqq 1} r_m^{2(\beta_- -\frac32+|\alpha|)} \Big|\partial^\alpha \tilde\zeta^{(k,\frac12)}\Big|^2\, dx\\
&\qquad \leqq C_3\,\varepsilon^{2(\beta_- -\frac12-\gamma)} \sum_{m=1}^{2\mathcal{K}} \int_{r_m \leqq c_2\varepsilon} \sum_{|\alpha|\leqq 1} r_m^{2(\gamma-1+|\alpha|)} \Big|\partial^\alpha \tilde\zeta^{(k,\frac12)}\Big|^2\, dx\\
&\qquad \leqq C_4\,\varepsilon^{2(\beta_- -\gamma)-1} \Big\|\tilde\zeta^{(k,\frac12)}\,;\ V^1_\gamma(\Omega)^2\Big\|^2 \ \leqq\ C_5\,\varepsilon^{2(\beta_- -\gamma)-1}
\end{aligned}$$

with an arbitrary parameter $\gamma > -1$. In the third term of equation (73) there appears the commutator $\big[\sigma^{(n)}, \chi_k\big]$. This is a bounded function with bounded derivatives and a support contained in the set $\{c_1 \leqq r_k \leqq c_2\}$. With the estimates (66) and (67) it follows that

$$\Big\|\big[\sigma^{(n)}, \chi_k\big]\big(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\big)\,;\ V^{\frac12}_{\beta_- -\frac12}(\Gamma_N)^2\Big\|^2 \ \leqq\ C_2\varepsilon^4.$$

The norm of the second expression in (73) can be estimated with the help of the inequality $\big|\partial_s^m \chi_k\big(\frac{\cdot}{\varepsilon}\big)\big| \leqq C r_k^{-m}$ by

$$\begin{aligned}
&\Big\|\chi_k\big(\tfrac{\cdot}{\varepsilon}\big)\Big(H\big(P^{(k)}\big) - H\Big)\,;\ V^1_{\beta_-}(\Gamma_N)^2\Big\|^2\\
&= \int_{|s-s_k|\leqq c_2\varepsilon} \sum_{m=0}^{1} |s-s_k|^{2(\beta_- -1+m)} \Big|\partial_s^m \Big(\chi_k\big(\tfrac{\cdot}{\varepsilon}\big)\Big(H\big(P^{(k)}\big) - H\Big)\Big)\Big|^2\, ds\\
&\leqq C_1 \int_{|s-s_k|\leqq c_2\varepsilon} \sum_{m=0}^{1} |s-s_k|^{2(\beta_- -1+m)} \Big|\partial_s^m \Big(H\big(P^{(k)}\big) - H\Big)\Big|^2\, ds\\
&\leqq C_2\,\varepsilon^{2(\beta_- -\gamma)-1} \int_{|s-s_k|\leqq c_2\varepsilon} \sum_{m=0}^{1} |s-s_k|^{2(\gamma-\frac12+m)} \Big|\partial_s^m \Big(H\big(P^{(k)}\big) - H\Big)\Big|^2\, ds\\
&\leqq C_3\,\varepsilon^{2(\beta_- -\gamma)-1} \Big\|H\big(P^{(k)}\big) - H\,;\ V^1_{\gamma+\frac12}(\Gamma_N)^2\Big\|^2\\
&\leqq C_4\,\varepsilon^{2(\beta_- -\gamma)-1} \Big\|h\,;\ H^2(\Gamma_N)^2\Big\|^2.
\end{aligned}$$

For the estimate of the remaining fourth and fifth term in (73) we need the boundary stress operator $\sigma^{(n)}$ written in curvilinear coordinates. This operator has the representation

$$\sigma^{(n)} = \sigma^{(n,0)} + \sigma^{(n,1)}(\kappa_k) + \sigma^{(n,2)}(\kappa_k),$$

where $\sigma^{(n,0)}$ and $\sigma^{(n,1)}(\kappa_k)$ correspond to the boundary operators $\mathbf{B}^0$ and $\kappa_k\mathbf{B}^1$ from the asymptotic expansion in Section 2.2. The operators $\sigma^{(n,1)}$ and $\sigma^{(n,2)}$ are of generalized order 0 and -1, respectively. Due to the construction of the inner expansion $\mathcal{W}^{(k)}$, $\sigma^{(n)}\big(w^{(k,0)}\big) = 0$, $\sigma^{(n,0)}\big(\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}\big) = 0$, $\sigma^{(n,0)}\big(\varepsilon w^{(k,1)}\big) = h\big(P^{(k)}\big)$, and $\sigma^{(n,0)}\big(\varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}\big) = -\sigma^{(n,1)}\big(\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}\big)$. Consequently, we obtain

$$\sigma^{(n)}\big(\mathcal{W}^{(k)}\big) = H\big(P^{(k)}\big) + \big(\sigma^{(n,1)} \\ + \sigma^{(n,2)}\big)\Big(\varepsilon w^{(k,1)} + \varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}\Big) + \sigma^{(n,2)}\Big(\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}\Big).$$

From the definition of the terms $\mathcal{T}^{(k)}$ which have been matched and the relations

$$\sigma^{(n,0)}\Big(r_k^{\frac{m}{2}}\,\Phi^{(\frac{m}{2})}(\varphi_k)\Big) = 0, \quad \text{and}$$
$$\sigma^{(n,0)}\Big(\kappa_k r_k^{\frac{m}{2}}\,\Upsilon^{(\frac{m}{2})}(\log(\kappa_k r_k), \varphi_k)\Big) = -\sigma^{(n,1)}\Big(r_k^{\frac{m}{2}-1}\,\Phi^{(\frac{m}{2}-1)}(\varphi_k)\Big),$$

it follows that

$$\begin{aligned}\sigma^{(n)}\big(\mathcal{T}^{(k)}\big) &= H\big(P^{(k)}\big) \\ &\quad + \sigma^{(n,1)}\Big(\kappa_k K_1^{(k)} r_k^{\frac{3}{2}}\,\Upsilon^{(\frac{3}{2})}(\log(\kappa_k r_k), \varphi_k) + \Lambda^{(k)}(s - s_k, n) \\ &\quad + K_3^{(k)} r_k^{\frac{3}{2}}\,\Phi^{(\frac{3}{2})}(\varphi_k) - \varepsilon\Big[\kappa_k K_1^{(k)} C_{\frac{1}{2}} h_k r_k^{\frac{1}{2}}\,\Upsilon^{(\frac{1}{2})}(\log(\kappa_k r_k), \varphi_k) \\ &\quad + \sum_{m=1}^{2\mathcal{K}} K_1^{(m)} C_{\frac{1}{2}} h_m C_\Omega^{mk} r_k^{\frac{1}{2}}\,\Phi^{(\frac{1}{2})}(\varphi_k)\Big]\Big) + \sigma^{(n,2)}\big(\mathcal{T}^{(k)}\big).\end{aligned}$$

The difference between these values can be represented as

$$\sigma^{(n)}\big(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\big) = \big(\sigma^{(n,1)} + \sigma^{(n,2)}\big)\Big(\tilde{\mathcal{R}}^{(k,\frac{3}{2})}\Big) + \sigma^{(n,2)}\Big(\tilde{\mathcal{R}}^{(k,\frac{1}{2})}\Big)$$

with remainders $\tilde{\mathcal{R}}^{(k,\frac{3}{2})}$ and $\tilde{\mathcal{R}}^{(k,\frac{1}{2})}$ satisfying the estimates

$$\Big|\partial_s^j \tilde{\mathcal{R}}^{(k,\frac{3}{2})}\Big| \leqq C\varepsilon^2 r_k^{-\frac{1}{2}-j} \log(\kappa_k r_k) \quad \text{and}$$
$$\Big|\partial_s^j \tilde{\mathcal{R}}^{(k,\frac{1}{2})}\Big| \leqq C\varepsilon^2 r_k^{-\frac{3}{2}-j} \log(\kappa_k r_k).$$

Hence we get, for any $\delta > 0$,

$$\Big\|\chi_k X \sigma^{(n)}\big(\mathcal{W}^{(k)} - \mathcal{T}^{(k)}\big);\ V_{\beta_-}^1(\Gamma_N)^2\Big\|^2 \leqq C_1\varepsilon^4 \int_{c_1\varepsilon \leqq r_k \leqq c_2\varepsilon} r_k^{2(\beta_- -1)-1-\delta}\, ds \\ \leqq C_2\, \varepsilon^{2(\beta_- +1)-\delta}.$$

Finally,

$$\sigma^{(n)}\Big(\mathcal{W}^{(k)} - H\big(P^{(k)}\big)\Big) \\ = \big(\sigma^{(n,1)} + \sigma^{(n,2)}\big)\Big(\varepsilon w^{(k,1)} + \varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}\Big) + \sigma^{(n,2)}\Big(\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}\Big).$$

Since the terms $\varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}$ and $\varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}$ satisfy the estimates

$$\left|\partial^\alpha \varepsilon^{\frac{1}{2}} w^{(k,\frac{1}{2})}\right| \leqq C_1 r_k^{\frac{1}{2}-|\alpha|} \quad \text{and}$$
$$\left|\partial^\alpha \varepsilon^{\frac{3}{2}} w^{(k,\frac{3}{2})}\right| \leqq C_2 r_k^{\frac{3}{2}-|\alpha|} |\log(\kappa_k r_k)| \text{ for } r_k \leqq c_2\varepsilon,$$

we have

$$\begin{aligned}\left\| \chi_k\left(\tfrac{\cdot}{\varepsilon}\right)\sigma^{(n)}\left(\mathcal{W}^{(k)} - H\left(P^{(k)}\right)\right) \; ; \; V^1_{\beta_-}(\Gamma_N)^2 \right\|^2 \\ \leqq C \int_{r_k \leqq c_2\varepsilon} r_k^{2(\beta_- -1)+2} |\log(\kappa_k r_k)|^2 \, ds \\ \leqq C\varepsilon^{2\beta_- +1}\left(|\log(\varepsilon)|^2 + 1\right).\end{aligned}$$

All these estimates involving the boundary stress operator $\sigma^{(n)}$ prove inequality (60). Estimate (61) is shown in the same way. The inequalities (58)–(61) and (49) provide in particular the estimate

$$\left|K_1^{(k)}(u(\varepsilon,\cdot) - \mathcal{U}(\varepsilon,\cdot))\right| \leqq C\varepsilon^{\frac{3}{2}-\delta}$$

for the stress intensity factor of $u(\varepsilon,\cdot) - \mathcal{U}(\varepsilon,\cdot)$ with any $\delta > 0$. Since the function $\mathcal{U}(\varepsilon,\cdot)$ was constructed such that its stress intensity factor is given by (43), the formula (43) for the sensitivity of the stress intensity factor is rigorously justified. □

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