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Asymptotic Analysis of Linearly Elastic Shells: 'Generalized Membrane Shells'

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Abstract. We consider a family of linearly elastic shells indexed by their half-thickness ε, all having the same middle surface $S = \varphi(\bar{\omega})$, with $\varphi : \bar{\omega} \subset \mathbf{R}^2 \to \mathbf{R}^3$, and clamped along a portion of their lateral face whose trace on S is $\varphi(\gamma_0)$, where γ_0 is a fixed portion of $\partial\omega$ with *length* $\gamma_0 > 0$. Let $(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$ be the linearized strain tensor of S. We make an essential geometric and kinematic assumption, according to which the semi-norm $|\cdot|_\omega^M$ defined by $|\boldsymbol{\eta}|_\omega^M = \{\Sigma_{\alpha,\beta}\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{L^2(\omega)}\}^{1/2}$ is a norm over the space $\boldsymbol{V}(\omega) = \{\boldsymbol{\eta} \in \boldsymbol{H}^1(\omega); \boldsymbol{\eta} = \boldsymbol{o} \text{ on } \gamma_0\}$, excluding however the already analyzed 'membrane' shells, where $\gamma_0 = \partial\omega$ and S is elliptic. This new assumption is satisfied for instance if $\gamma_0 \neq \partial\omega$ and S is elliptic, or if S is a portion of a hyperboloid of revolution.

We then show that, as $\varepsilon \to 0$, the averages $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_i^\varepsilon \, dx_3^\varepsilon$ across the thickness of the shell of the covariant components u_i^ε of the displacement of the points of the shell strongly converge in the completion $\boldsymbol{V}_M^\sharp(\omega)$ of $\boldsymbol{V}(\omega)$ with respect to the norm $|\cdot|_\omega^M$, toward the solution of a 'generalized membrane' shell problem. This convergence result also justifies the recent formal asymptotic approach of D. Caillerie and E. Sanchez–Palencia.

The limit problem found in this fashion is 'sensitive', according to the terminology recently introduced by J.L. Lions and E. Sanchez–Palencia, in the sense that it possesses two unusual features: it is posed in a space that is not necessarily contained in a space of distributions, and its solution is 'highly sensitive' to arbitrarily small smooth perturbations of the data.

Under the same assumption, we also show that the average $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{u} \, dx_3^\varepsilon$ where $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$, and the solution $\xi^\varepsilon \in V_K(\omega)$ of Koiter's equations have the same principal part as $\varepsilon \to 0$ in the same space $\boldsymbol{V}_M^\sharp(\omega)$ as above. For such 'generalized membrane' shells, the two-dimensional shell model of W.T. Koiter is thus likewise justified.

We also treat the case where $|\cdot|_\omega^M$ is no longer a norm over $\boldsymbol{V}(\omega)$, but is a norm over the space $V_K(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\}$, thus also excluding the already analyzed 'flexural' shells. Then a convergence theorem can still be established, but only in the completion of the quotient space $\boldsymbol{V}(\omega)/\boldsymbol{V}_0(\omega)$ with repect to $|\cdot|_\omega^M$, where $\boldsymbol{V}_0(\omega) = \{\eta \in \boldsymbol{V}(\omega); \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$.

These convergence results, together with those that we already obtained for 'membrane' and 'flexural' shells, jointly with B. Miara in the second case, thus constitute an asymptotic analysis of linearly elastic shells in all possible cases.

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Introduction

Our aim here is to complete and conclude the asymptotic analysis of linearly elastic shells undertaken in Ciarlet, Lods and Miara (1996) and Ciarlet and Lods

(1996a) for 'flexural' and 'membrane' shells, together with the justification of the two-dimensional Koiter's shell equations begun in Ciarlet and Lods (1996b).

We consider a *family of linearly elastic shells of thickness* 2ε, all having the *same* middle surface $S = \varphi(\bar{\omega}) \subset \boldsymbol{R}^3$, where $\omega \subset \boldsymbol{R}^2$ is a bounded and connected open set with a Lipschitz-continuous boundary γ, and $\varphi \in \mathcal{C}^3(\bar{\omega}; \boldsymbol{R}^3)$. The shells are clamped on a portion of their lateral face, whose middle line is $\varphi(\gamma_0)$, where γ_0 is a *fixed* portion of γ with *length* $\gamma_0 > 0$.

Let

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \tfrac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3$$

denote the components of the *linearized change of metric tensor of* S, where $\Gamma^\sigma_{\alpha\beta}$ are the Christoffel symbols of S, and $b_{\alpha\beta}$ are the covariant components of the *curvature tensor* of S. In Ciarlet, Lods and Miara (1996), we made a geometrical assumption on the middle surface S and on the set γ_0, which states that *the space of inextensional displacements* (introduced by Sanchez–Palencia (1989a))

$$\begin{aligned} V_F(\omega) = \{&\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \\ &\eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0,\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\} \end{aligned}$$

contains non-zero functions. This assumptions is satisfied in particular if S is a portion of a cylinder and $\varphi(\gamma_0)$ is contained in a generatrix of S, or if S is contained in a plane, i.e., the shells are *plates.*

We then showed that, if the applied body force density is $O(1)$ with respect to ε, *the field* $\varepsilon^2\boldsymbol{u}(\varepsilon) = (\varepsilon^2 u_i(\varepsilon))$, where $u_i(\varepsilon)$ denote the three covariant components of the displacement of the points of the shell given by the equations of three-dimensional elasticity, once *'scaled'* as in (2.1) so as to be defined over the fixed domain $\Omega = \omega \times]-1, 1[$, *converges in* $\boldsymbol{H}^1(\Omega)$ *to a limit* $\boldsymbol{u}^{-2}$, *which is independent of the transverse variable. Furthermore, the average* $\boldsymbol{\xi} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}^{-2}\,\mathrm{d}x_3$, *which belongs to the space* $V_F(\omega)$, *solves the (scaled) two-dimensional equations of a 'flexural shell'*, viz.,

$$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\xi})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega \left\{\int_{-1}^{1} f^i\,\mathrm{d}x_3\right\}\eta_i\sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in V_F(\omega)$, where

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{(\lambda + 2\mu)}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})$$

are the components of the *elasticity tensor of the surface* S,

$$\begin{aligned} \rho_{\alpha\beta}(\boldsymbol{\eta}) = {}& \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 + b^\sigma_\beta(\partial_\alpha\eta_\sigma - \Gamma^\tau_{\alpha\sigma}\eta_\tau) \\ & + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\sigma_\alpha|_\beta\eta_\sigma - c_{\alpha\beta}\eta_3 \end{aligned}$$

are the components of the *linearized change of curvature tensor of* S, b_α^β are the mixed components of the curvature tensor of S, $\sqrt{a}\,\mathrm{d}y$ is the area element along S, and f^i are the scaled contravariant components of the applied body force. *If* $\boldsymbol{V}_F(\omega) \neq \{\mathbf{o}\}$, *the two-dimensional equations of a 'flexural shell' are therefore justified.*

If $\boldsymbol{V}_F(\omega) = \{\mathbf{o}\}$, the above convergence result still applies. However, the only information it provides is that $\varepsilon^2\boldsymbol{u}(\varepsilon) \to 0$ in $\boldsymbol{H}^1(\Omega)$ as $\varepsilon \to 0$. Hence a more refined asymptotic analysis of the scaled field $\boldsymbol{u}(\varepsilon)$ is needed in this case.

A first instance of such a refinement was given by Ciarlet and Lods (1996a), where we assumed that $\gamma_0 = \gamma$ *and* that the surface S is regular and *'elliptic'*, in the sense that the two principal radii of curvature are either both > 0 at all points of S, or both < 0 at all points of S. As shown in Ciarlet and Lods (1996c) and Ciarlet and Sanchez–Palencia (1996), these two conditions, together with *ad hoc* regularity assumptions, indeed imply that $\boldsymbol{V}_F(\omega) = \{\mathbf{o}\}$.

We then showed that, if the applied body force density is again $O(1)$ with respect to ε, *the field* $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$, where $u_i(\varepsilon)$ denote again the three covariant components of the displacement of the points of the shell given by the equations of three-dimensional elasticity, once 'scaled' as in (2.1) so as to be defined over the fixed domain $\Omega = \omega\times]-1[$, *converges in* $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ *as* $\varepsilon \to 0$ *to a limit* $\boldsymbol{u}$*, which is independent of the transverse variable. Furthermore, the average* $\boldsymbol{\xi} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\,\mathrm{d}x_3$*, which belongs to the space*

$$\boldsymbol{V}_M(\omega) = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega),$$

solves the (scaled) two-dimensional equations of a 'membrane shell', viz.,

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega \left\{\int_{-1}^{1} f^i\,\mathrm{d}x_3\right\}\eta_i\sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega)$ where the functions $a^{\alpha\beta\sigma\tau}, \gamma_{\alpha\beta}(\boldsymbol{\eta}), a$, and f^i have the same meanings as above. If $\gamma_0 = \gamma$ *and* S *is elliptic, the two-dimensional equations of a 'membrane shell' are therefore justified.*

It is the purpose of the present article, whose results were announced in Ciarlet and Lods (1995), *to study all the 'remaining' cases where* $\boldsymbol{V}_F(\omega) = \{\mathbf{o}\}$, e.g., when S is elliptic but *length* $\gamma_0 <$ *length* γ, or when S is a portion of a hyperboloid of revolution, etc.

Our results are perhaps best illustrated by *the important special case where the space* $\{\boldsymbol{\eta} \in \boldsymbol{H}^1(\omega); \boldsymbol{\eta} = \mathbf{o}$ on $\gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in $\omega\}$, which contains $\boldsymbol{V}_F(\omega)$, *'already' reduces to* $\{\mathbf{o}\}$, *or, equivalently, when the semi-norm*

$$|\cdot|_\omega^M : \boldsymbol{\eta} = (\eta_i) \to |\boldsymbol{\eta}|_\omega^M = \left\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2\right\}^{1/2}$$

becomes a norm over the space

$$V(\omega) = \{\eta \in H^1(\omega);\ \eta = \mathbf{o} \quad \text{on} \quad \gamma_0\}.$$

In this case, we show (Theorem 5.1 (a)) that, if the applied forces are *'admissible'* in the sense understood in Section 4 (as we shall see, this restriction on the forces seems unavoidable) and if the 'equivalent' applied body force density is again $O(1)$ with respect to ε, *the average*

$$\overline{u(\varepsilon)} = \frac{1}{2}\int_{-1}^{1} u(\varepsilon)\,\mathrm{d}x_3,$$

where the *scaled unknown* $u(\varepsilon)$ is defined as in (2.1), *converges as* $\varepsilon \to 0$ *in the space*

$$V_M^\sharp(\omega) = \text{completion of } V(\omega) \text{ with respect to } |\cdot|_\omega^M;$$

a convergence result also holds for the field $u(\varepsilon)$ itself, but it is too technical to be reproduced here (cf. Theorem 5.1(a)).

Furthermore, the limit $\xi \in V_M^\sharp(\omega)$ *solves a limit variational problem of the form*

$$B_M^\sharp(\xi, \eta) = L_M^\sharp(\eta) \quad \text{for all} \quad \eta \in V_M^\sharp(\omega),$$

where $B_M^\sharp$ is the unique extension to $V_M^\sharp(\omega)$ of the bilinear form $B_M : V(\omega) \times V(\omega) \to \mathbf{R}$ of a 'classical' membrane shell, defined by

$$B_M(\xi, \eta) = \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\xi)\gamma_{\alpha\beta}(\eta)\sqrt{a}\,\mathrm{d}y,$$

and $L_M^\sharp : V(\omega) \to \mathbf{R}$ is an *ad hoc* linear form, determined by the behavior as $\varepsilon \to 0$ of the admissible forces. In addition, this variational problem provides an instance of a *'sensitive problem'*, in the sense of Lions and Sanchez–Palencia (1994,1996).

In the 'last' case, where $V_F(\omega) = \{\mathbf{o}\}$ but $|\cdot|_\omega^M$ is a 'genuine' semi-norm over the space $V(\omega)$, a similar convergence result can be established (see Theorem 5.1(b)), but only in the completion $\dot{V}_M^\sharp(\omega)$ with respect of $|\cdot|_\omega^M$ of the *quotient* space

$$\dot{V}_M(\omega) = V(\omega)/V_0(\omega),$$

where $V_0(\omega) = \{\eta \in V(\omega); \gamma_{\alpha\beta}(\eta) = 0 \text{ in } \omega\}$.

If $V_F(\omega) = \{\mathbf{o}\}$ (otherwise than when $\gamma_0 = \gamma$ and S is elliptic), *we have therefore justified by a convergence result the two-dimensional variational equations of a 'generalized membrane' shell*, so named because the bilinear form of the limit variational problem is an extension of the bilinear form of a 'classical' membrane shell.

This convergence result, together with those of Ciarlet, Lods and Miara (1996) and Ciarlet and Lods (1996a) thus constitute an *asymptotic analysis of linearly elastic shells in all possible cases.* All together, they provide a rigorous justification of the formal asymptotic approaches of Sanchez–Palencia (1990), Miara and Sanchez–Palencia (1996), and Caillerie and Sanchez–Palencia (1995b).

We are also able *to justify* (Theorem 6.1) *the two-dimensional shell model of Koiter in a situation that was not covered in* Ciarlet and Lods (1996b), namely when $|\cdot|_\omega^M$ is a norm over $\boldsymbol{V}(\omega)$. For, our convergence result, combined with (a slight improvement of) the asymptotic analysis of the solution $\boldsymbol{\xi}^\varepsilon$ of Koiter's model recently made by Caillerie and Sanchez–Palencia (1995a), shows that in this case, *both fields* $\overline{\boldsymbol{u}(\varepsilon)}$ and $\boldsymbol{\xi}^\varepsilon$ *have again the same principal part as* $\varepsilon \to 0$, this time in the space $\boldsymbol{V}_M^\sharp(\omega)$. This result, together with those of Ciarlet and Lods (1996b), thus constitute a *justification of Koiter's model* in all cases, save that when $\boldsymbol{V}_F(\omega) = \{\mathbf{o}\}$ but $|\cdot|_\omega^M$ is not a norm over $\boldsymbol{V}(\omega)$.

We refer to Ciarlet and Lods (1996a) for an extensive list of references about the asymptotic analysis of three-dimensional elastic shells. Among these, we mention the earlier convergence results of Destuynder (1980) for 'membrane shells', of Acerbi, Buttazzo and Percivale (1988) for shells viewed as thin inclusions in a larger surrounding elastic body, and those recently obtained by Le Dret and Raoult (1995) in the nonlinear case by means of Γ-convergence theory.

We use the following conventions and notations throught this article: *Greek* indices and exponents (except ε) belongs to the set $\{1,2\}$, *Latin* indices and exponents (except when otherwise indicated, as e.g. when they are used to index sequences) belong to the set $\{1,2,3\}$, and the *summation convention* with respect to repeated indices and exponents is systematically used. The sign := indicates that the right-hand side defines the left-hand side. Symbols such as $\delta_\beta^\alpha, \delta_i^j$, designate the Kronecker's symbol. The Euclidean scalar product and the vector product of $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^3$ are noted $\boldsymbol{a}.\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$; the Euclidean norm is noted $|\cdot|$.

A *domain* A in $\boldsymbol{R}^n$ is a bounded, open, connected subset of $\boldsymbol{R}^n$ with a Lipschitz-continuous boundary ∂A, the set A being locally on one side of ∂A. For each integer $m, H^m(A)$ and $\|\cdot\|_{m,A}$ denote the usual Sobolev spaces of real-valued functions ($H^0(A) = L^2(A)$). Boldface letters denote vector-valued or tensor-valued functions and their associated function spaces; For instance, $\boldsymbol{v} = (v_i) \in \boldsymbol{L}^2(\Omega)$ means that $v_i \in L^2(\Omega), i = 1,2,3$; $\|\boldsymbol{v}\|_{0,\Omega} = \{\Sigma_i \|v_i\|_{0,\Omega}^2\}^{1/2}$, etc.

1. The Three-Dimensional Shell Problem

All notions of differential geometry needed for shell theory may be found e.g. in Green and Zerna (1968), Niordson (1985), Ciarlet (1966). Let ω be a domain in $\boldsymbol{R}^2$ with boundary γ. Let $y = (y_\alpha)$ denote a generic point in the set $\bar{\omega}$, and let

$\partial_\alpha := \partial/\partial y_\alpha$. Let $\boldsymbol{\varphi} : \bar{\omega} \to \boldsymbol{R}^3$ be an injective mapping of class $\mathcal{C}^3$ such that the two vectors

$$\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\varphi}(y)$$

are linearly independent at all points $y \in \bar{\omega}$. They form the *covariant basis* of the tangent plane to the *surface*

$$S = \boldsymbol{\varphi}(\bar{\omega})$$

at the point $\boldsymbol{\varphi}(y)$; the two vectors $\boldsymbol{a}^\alpha(y)$ of the same tangent plane defined by the relations

$$\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$$

constitute its *contravariant basis*. We also let

$$\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y) := \frac{\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)|};$$

hence the vector $\boldsymbol{a}_3(y)$ is normal to S at the point $\boldsymbol{\varphi}(y)$, and $|\boldsymbol{a}_3(y)| = 1$.

One then defines the *first fundamental form*, also known as the *metric tensor*, $(a_{\alpha\beta})$ or $(a^{\alpha\beta})$ (in covariant or contravariant components), the *second fundamental form*, also known as the *curvature tensor*, $(b_{\alpha\beta})$ or (b^β_α) (in covariant or mixed components), and the *Christoffel symbols* $\Gamma^\sigma_{\alpha\beta}$, *of the surface* S, by letting (whenever no confusion should arise, we henceforth drop the explicit dependence on the variable $y \in \bar{\omega}$):

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \qquad a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta, \tag{1.1}$$

$$b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha, \qquad b^\beta_\alpha := a^{\beta\sigma} b_{\sigma\alpha}, \tag{1.2}$$

$$\Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha. \tag{1.3}$$

Note the symmetries:

$$a_{\alpha\beta} = a_{\beta\alpha}, \qquad a^{\alpha\beta} = a^{\beta\alpha}, \qquad b_{\alpha\beta} = b_{\beta\alpha}, \qquad \Gamma^\sigma_{\alpha\beta} = \Gamma^\sigma_{\beta\alpha}.$$

The *area element* along S is $\sqrt{a}\,\mathrm{d}y$, where

$$a := \det(a_{\alpha\beta}). \tag{1.4}$$

All the functions defined in (1.1)–(1.4) are at least continuous over the set $\bar{\omega}$. In particular, there exists a constant a_0 such that

$$0 < a_0 \leqslant a(y) \quad \text{for all} \quad y \in \bar{\omega}. \tag{1.5}$$

Let γ_0 denote a measurable subset of the boundary γ of ω, with

$$\textit{length} \quad \gamma_0 > 0.$$

For each $\varepsilon > 0$, we define the sets

$$\Omega^\varepsilon := \omega \times]-\varepsilon, \varepsilon[, \qquad \Gamma_+^\varepsilon := \omega \times \{\varepsilon\}, \qquad \Gamma_-^\varepsilon := \omega \times \{-\varepsilon\},$$
$$\Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon].$$

Let $x^\varepsilon = (x_i^\varepsilon)$ denote a generic point in the set $\bar{\Omega}^\varepsilon$, and let $\partial_i^\varepsilon := \partial/\partial x_i^\varepsilon$; hence $x_\alpha^\varepsilon = y_\alpha$ and $\partial_\alpha^\varepsilon = \partial_\alpha$.

We then define a mapping $\boldsymbol{\Phi}\colon \bar{\Omega}^\varepsilon \to \boldsymbol{R}^3$ by letting

$$\boldsymbol{\Phi}(x^\varepsilon) := \boldsymbol{\varphi}(y) + x_3^\varepsilon \boldsymbol{a}^3(y) \quad \text{for all} \quad x^\varepsilon = (y, x_3^\varepsilon) \in \bar{\Omega}^\varepsilon. \tag{1.6}$$

One can then show (cf. Ciarlet and Paumier (1986, Proposition 3.2)) that there exists $\varepsilon_0 > 0$ such that the three vectors

$$\boldsymbol{g}_i^\varepsilon(x^\varepsilon) := \partial_i^\varepsilon \boldsymbol{\Phi}(x^\varepsilon)$$

are linearly independent at all point $x^\varepsilon \in \bar{\Omega}^\varepsilon$ and the mapping $\boldsymbol{\Phi}\colon \bar{\Omega}^\varepsilon \to \boldsymbol{R}^3$ is injective for all $0 < \varepsilon \leqslant \varepsilon_0$. The injectivity of the mapping $\boldsymbol{\Phi}\colon \bar{\Omega}^\varepsilon \to \boldsymbol{R}^3$, which itself relies on the assumed injectivity of the mapping $\boldsymbol{\varphi} : \bar{\omega} \to \boldsymbol{R}^3$, insures in particular that the physical problem described below is meaningful.

The three vectors $\boldsymbol{g}_i^\varepsilon(x^\varepsilon)$ form the *covariant basis* (of the tangent space, here $\boldsymbol{R}^3$, to the manifold $\boldsymbol{\Phi}(\bar{\Omega}^\varepsilon)$) at the point $\boldsymbol{\Phi}(x^\varepsilon)$, and the three vectors $\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)$ defined by

$$\boldsymbol{g}^{j,\varepsilon}(x^\varepsilon) \cdot \boldsymbol{g}_i^\varepsilon(x^\varepsilon) = \delta_i^j,$$

form the *contravariant basis*. We then define the *metric tensor* (g_{ij}^ε) or $(g^{ij,\varepsilon})$ (in covariant or contravariant components) and the *Christoffel symbols* of the manifold $\boldsymbol{\Phi}(\bar{\Omega}^\varepsilon)$ by letting (we omit the explicit dependence on x^ε):

$$g_{ij}^\varepsilon := \boldsymbol{g}_i^\varepsilon \cdot \boldsymbol{g}_j^\varepsilon, \qquad g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}, \tag{1.7}$$

$$\Gamma_{ij}^{p,\varepsilon} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial_i^\varepsilon \boldsymbol{g}_j^\varepsilon. \tag{1.8}$$

Note the symmetries:

$$g_{ij}^\varepsilon = g_{ji}^\varepsilon, \qquad g^{ij,\varepsilon} = g^{ji,\varepsilon}, \qquad \Gamma_{ij}^{p,\varepsilon} = \Gamma_{ji}^{p,\varepsilon}. \tag{1.9}$$

The *volume element* in the set $\boldsymbol{\Phi}(\Omega^\varepsilon)$ is $\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon$, where

$$g^\varepsilon := \det(g_{ij}^\varepsilon). \tag{1.10}$$

For each $0 < \varepsilon \leqslant \varepsilon_0$, the set $\boldsymbol{\Phi}(\bar{\Omega}^\varepsilon)$ is the reference configuration of an *elastic shell*, with *middle surface* $S = \boldsymbol{\varphi}(\bar{\omega})$ and *thickness* 2ε. We assume that the material constituting the shell is *homogeneous* and *isotropic* and that $\boldsymbol{\Phi}(\Omega^\varepsilon)$ is a *natural state*, so that the material is characterized by its two *Lamé constants* $\lambda > 0$ and $\mu > 0$, assumed to be *both independent of* ε. The *unknown* of the problem is the vector field $\boldsymbol{u} = (u_i^\varepsilon)\colon \bar{\Omega}^\varepsilon \to \boldsymbol{R}^3$, where the three functions $u_i^\varepsilon : \bar{\Omega}^\varepsilon \to \boldsymbol{R}$ are the covariant components of the displacement field $u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the points of the shell; this means that $u_i^\varepsilon(x^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)$ is the displacement of the point $\boldsymbol{\Phi}(x^\varepsilon)$. Finally, we assume that the shell is *clamped* along the portion $\boldsymbol{\Phi}(\Gamma_0^\varepsilon)$ of its lateral face, i.e., that the displacement vanishes there.

Then it is classical (cf. e.g. Ciarlet (1996)) that the variational formulation of the corresponding *three-dimensional problem of linearized elasticity* reads as follows, when it is expressed in terms of the *curvilinear coordinates* x_i^ε of the reference configuration $\boldsymbol{\Phi}(\bar{\Omega}^\varepsilon)$: The unknown $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ satisfies

$$\boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) := \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{H}^1(\Omega^\varepsilon); \boldsymbol{v}^\varepsilon = \boldsymbol{o} \quad \text{on} \quad \Gamma_0^\varepsilon\}, \tag{1.11}$$

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{k\|l}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon = L^\varepsilon(\boldsymbol{v}^\varepsilon) \quad \text{for all} \quad \boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon), \tag{1.12}$$

where

$$A^{ijkl,\varepsilon} := \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}) \tag{1.13}$$

designate the contravariant components of the *three-dimensional elasticity tensor*,

$$e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon) := \tfrac{1}{2}(\partial_i^\varepsilon v_j^\varepsilon + \partial_j^\varepsilon v_i^\varepsilon) - \Gamma_{ij}^{p,\varepsilon} v_p^\varepsilon \tag{1.14}$$

designate the covariant components of the *linearized strain tensor* associated with an arbitrary displacement field $v_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the manifold $\boldsymbol{\Phi}(\bar{\Omega}^\varepsilon)$, and $L^\varepsilon : \boldsymbol{V}(\Omega^\varepsilon) \to \boldsymbol{R}$ is a continuous linear form that takes into account the *applied forces*. For instance,

$$L^\varepsilon(\boldsymbol{v}^\varepsilon) = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon \quad \text{for all} \quad \boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{V}(\Omega^\varepsilon), \tag{1.15}$$

if a *body force* is applied to the shell, where $f^{i,\varepsilon} \in L^\varepsilon(\Omega^\varepsilon)$ denote the contravariant components of its density (per unit volume). We record in passing the symmetries

$$A^{ijkl,\varepsilon} = A^{jikl,\varepsilon} = A^{klij,\varepsilon}, \tag{1.16}$$

and the relations (satisfied because the mapping $\boldsymbol{\Phi}$ is of the special form (1.6))

$$\Gamma_{\alpha 3}^{3,\varepsilon} = \Gamma_{33}^{p,\varepsilon} = 0 \quad \text{in} \quad \bar{\Omega}^\varepsilon, \tag{1.17}$$

$$A^{\alpha\beta\sigma 3,\varepsilon} = A^{\alpha 333,\varepsilon} = 0 \quad \text{in} \quad \bar{\Omega}^\varepsilon, \tag{1.18}$$

used in the subsequent computations.

The *three-dimensional shell problem* (1.11)–(1.12) has one and only one solution for each $\varepsilon > 0$. To see this, one may express it in Cartesian coordinates and then use the classical Korn's inequality, as in e.g. Duvaut and Lions (1972, p. 115). The $V(\Omega^\varepsilon)$-ellipticity of the bilinear form appearing in eqs. (1.12) may be also directly established in curvilinear coordinates, as in Ciarlet (1996).

2. The 'Scaled' Three-Dimensional Shell Problem over a Domain Independent of ε

Let

$$\Omega := \omega \times]-1,1[, \quad \Gamma_+ := \omega \times \{1\}, \qquad \Gamma_- := \omega \times \{-1\},$$
$$\Gamma_0 := \gamma_0 \times [-1,1],$$

let $x = (x_i)$ denote a generic point in the set $\bar{\Omega}$, and let $\partial_i := \partial/\partial x_i$. With $x^\varepsilon = (x_i^\varepsilon) \in \bar{\Omega}^\varepsilon$, we associate the point $x = (x_i) \in \bar{\Omega}$ defined by $x_\alpha = x_\alpha^\varepsilon = y_\alpha$ and $x_3 = (1/\varepsilon)x_3^\varepsilon$; we thus have $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = (1/\varepsilon)\partial_3$.

With the unknown $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) : \bar{\Omega}^\varepsilon \to \boldsymbol{R}^3$ and the vector fields $\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) : \bar{\Omega}^\varepsilon \to \boldsymbol{R}^3$ appearing in the three-dimensional problem (1.11), (1.12), we associate the *scaled unknown* $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \bar{\Omega} \to \boldsymbol{R}^3$ and the *scaled vector fields* $\boldsymbol{v} = (v_i) : \bar{\Omega} \to \boldsymbol{R}^3$ defined by

$$u_i(\varepsilon)(x) = u_i^\varepsilon(x^\varepsilon) \quad \text{and} \quad v_i(x) = v_i^\varepsilon(x^\varepsilon) \quad \text{for all} \quad v^\varepsilon \in \bar{\Omega}^\varepsilon. \tag{2.1}$$

Note that, in relations (2.1) and (2.2)–(2.4) *infra*, it is understood that x stands for the point of $\bar{\Omega}$ that is associated with the point $x^\varepsilon \in \bar{\Omega}^\varepsilon$ as indicated above.

A simple computation, which uses in particular relations (1.17), then shows that the scaled unknown $\boldsymbol{u}(\varepsilon)$ solves the *scaled three-dimensional shell problem* (2.8), (2.9), now posed over the set Ω, thus over a domain which is *independent of* ε:

THEOREM 2.1 *With the functions* $\Gamma_{ij}^{p,\varepsilon}, g^\varepsilon, A^{ijkl,\varepsilon} : \bar{\Omega}^\varepsilon \to \boldsymbol{R}$ *defined in* (1.8), (1.10), (1.13), *we associate the functions* $\Gamma_{ij}^{p,\varepsilon}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon) : \bar{\Omega} \to \boldsymbol{R}$ *defined by*

$$\Gamma_{ij}^p(\varepsilon)(x) := \Gamma_{ij}^{p,\varepsilon}(x^\varepsilon) \quad \text{for all} \quad x^\varepsilon \in \Omega^\varepsilon, \tag{2.2}$$

$$g(\varepsilon)(x) := g^\varepsilon(x^\varepsilon) \quad \text{for all} \quad x^\varepsilon \in \Omega^\varepsilon, \tag{2.3}$$

$$A^{ijkl}(\varepsilon)(x) := A^{ijkl,\varepsilon}(x^\varepsilon) \quad \text{for all} \quad x^\varepsilon \in \Omega^\varepsilon. \tag{2.4}$$

With any vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$, *we associate the symmetric tensor* $(e_{i\|j}(\varepsilon)(\boldsymbol{v})) \in \boldsymbol{L}^2(\Omega)$ *defined by*

$$e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) := \tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma_{\alpha\beta}^p(\varepsilon)v_p, \tag{2.5}$$

$$e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) := \tfrac{1}{2}\left(\partial_\alpha v_3 + \frac{1}{\varepsilon}\partial_3 v_\alpha\right) - \Gamma^\sigma_{\alpha 3}(\varepsilon)v_\sigma, \tag{2.6}$$

$$e_{3\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3. \tag{2.7}$$

Then the scaled unknown $\boldsymbol{u}(\varepsilon)$ *defined in* (2.1) *satisfies*

$$\boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega);\quad \boldsymbol{v} = \boldsymbol{o} \quad \text{on} \quad \Gamma_0\}, \tag{2.8}$$

$$\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)(\boldsymbol{u}(\varepsilon))e_{i\|j}(\varepsilon)(\boldsymbol{v})\sqrt{g(\varepsilon)}\,\mathrm{d}x = L(\varepsilon)(\boldsymbol{v})$$
$$\text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}(\Omega), \tag{2.9}$$

where the continuous linear form $L(\varepsilon)\colon \boldsymbol{V}(\Omega) \to \boldsymbol{R}$ *is defined by*

$$L^\varepsilon(\boldsymbol{v}^\varepsilon) = \varepsilon L(\varepsilon)(\boldsymbol{v}) \tag{2.10}$$

for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ *related to* $\boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega)^\varepsilon$ *as in* (2.1). □

3. Technical Preliminaries

In order to render this article as self-contained as possible, we first gather in a series of five lemmas various 'technical' results that will be needed in the proof of the convergence theorem. Save the last one, they were already used in our analysis of 'membrane' and 'flexural' shells.

From now on, whenever a symbol such as C_1, C_2, etc., or c_1, c_2 etc., appears in an inequality, it means that *there exists a constant*, denoted by this symbol, *that is* > 0 *and independent of the various variables* (e.g., the parameter ε, functions in a specific space, etc.) *involved in this inequality*. For instance, inequality (3.13) in Lemma 3.1 means that *there exists a constant* $C_2 > 0$ *independent* of $0 < \varepsilon \leqslant \varepsilon_0$, of the point $x \in \bar\Omega$, and of the symmetric tensor (t_{ij}), such that this inequality holds.

Our first result gathers all the properties needed in the sequel concerning *the behavior of the functions* $\Gamma^p_{ij}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$ as $\varepsilon \to 0$. Observe that the notational distinction between the 'three-dimensional' and 'two-dimensional' Christoffel symbols $\Gamma^\sigma_{\alpha\beta}(\varepsilon)$ and $\Gamma^\sigma_{\alpha\beta}$ is automatic, since the symbol ε appears only in the former. Also, note that the following symmetries hold (cf. (1.9) and (1.16))

$$\Gamma^p_{ij}(\varepsilon) = \Gamma^p_{ji}(\varepsilon), \qquad A^{ijkl}(\varepsilon) = A^{jikl}(\varepsilon) = A^{klij}(\varepsilon).$$

If $\omega \in C^0(\bar\Omega)$, we let

$$\|\omega\|_{0,\infty,\bar\Omega} = \sup_{x\in\bar\Omega} |w(x)|.$$

LEMMA 3.1 *The functions* $\Gamma_{ij}^p(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$ *are defined for* $\varepsilon > 0$ *as in* (2.2)–(2.4) *and the functions* $a^{\alpha\beta}, b_{\alpha\beta}, b_\beta^\alpha, \Gamma_{\alpha\beta}^\sigma, a$ *are defined as in* (1.1)–(1.3). *In addition, the covariant derivatives* $b_\beta^\sigma|_\alpha$, *and the covariant components* $c_{\alpha\beta}$ *of the third fundamental form of the surface* S *are defined by*

$$b_\beta^\sigma|_\alpha := \partial_\alpha b_\beta^\sigma + \Gamma_{\alpha\tau}^\sigma b_\beta^\tau - \Gamma_{\beta\alpha}^\tau b_\tau^\sigma, \tag{3.1}$$

$$c_{\alpha\beta} := b_\alpha^\sigma b_{\sigma\beta}. \tag{3.2}$$

All the functions $a^{\alpha\beta}, \dots, c_{\alpha\beta} \in \mathcal{C}^0(\bar{\omega})$ *are identified with functions in* $\mathcal{C}^0(\bar{\Omega})$. *Then*

$$\|\Gamma_{\alpha\beta}^\sigma(\varepsilon) - (\Gamma_{\alpha\beta}^\sigma - \varepsilon x_3 b_\beta^\sigma|_\alpha)\|_{0,\infty,\bar{\Omega}} \leqslant C_1\varepsilon^2,$$

$$\Gamma_{\alpha\beta}^3(\varepsilon) = b_{\alpha\beta} - \varepsilon x_3 c_{\alpha\beta}, \tag{3.3}$$

$$\|\partial_3\Gamma_{\alpha\beta}^p(\varepsilon)\|_{0,\infty,\bar{\Omega}} \leqslant C_1\varepsilon, \tag{3.4}$$

$$\|\Gamma_{\alpha 3}^\sigma(\varepsilon) + b_\alpha^\sigma\|_{0,\infty,\bar{\Omega}} \leqslant C_1\varepsilon, \tag{3.5}$$

$$\Gamma_{\alpha 3}^3(\varepsilon) = \Gamma_{33}^p(\varepsilon) = 0, \tag{3.6}$$

$$\|g(\varepsilon) - a\|_{0,\infty,\bar{\Omega}} \leqslant C_1\varepsilon, \tag{3.7}$$

$$\|A^{ijkl}(\varepsilon) - A^{ijkl}(0)\|_{0,\infty,\bar{\Omega}} \leqslant C_1\varepsilon, \tag{3.8}$$

$$A^{\alpha\beta\sigma 3}(\varepsilon) = A^{\alpha 333}(\varepsilon) = 0, \tag{3.9}$$

for all $0 < \varepsilon \leqslant \varepsilon_0$, *where*

$$A^{\alpha\beta\sigma\tau}(0) := \lambda a^{\alpha\beta}a^{\sigma\tau} + \mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \tag{3.10}$$

$$A^{\alpha\beta 33}(0) := \lambda a^{\alpha\beta}, \qquad A^{\alpha 3\sigma 3}(0) := \mu a^{\alpha\sigma}, \qquad A^{3333}(0) := \lambda + 2\mu, \tag{3.11}$$

$$A^{\alpha\beta\sigma 3}(0) = A^{\alpha 333}(0) := 0, \tag{3.12}$$

and finally,

$$t_{ij}t_{ij} \leqslant C_2 A^{ijkl}(\varepsilon)(x)t_{kl}t_{ij} \tag{3.13}$$

for all $0 < \varepsilon \leqslant \varepsilon_0$, *all* $x \in \bar{\Omega}$, *and all symmetric tensors* (t_{ij}).

Proof. All relations, except (3.4), have already been established, in Lemma 3.1 of Ciarlet and Lods (1996a) and Lemma 3.1 of Ciarlet, Lods and Miara (1996). The vectors $\boldsymbol{a}_i, \boldsymbol{a}^3, \boldsymbol{g}_\alpha^\varepsilon, \boldsymbol{g}^{p,\varepsilon}$ and the scalars $a_{\alpha\beta}, a^{\alpha\beta}, g_{\alpha\beta}^\varepsilon, g^{\alpha\beta,\varepsilon}$ are defined as in Section 1; let

$$g_{\alpha\beta}(\varepsilon)(x) := g^{\varepsilon}_{\alpha\beta}(x^{\varepsilon}), \qquad g^{\alpha\beta}(\varepsilon)(x) := g^{\alpha\beta,\varepsilon}(x^{\varepsilon}) \quad \text{for all} \quad x^{\varepsilon} \in \bar{\Omega}^{\varepsilon},$$

$$\boldsymbol{g}_{\alpha}(\varepsilon)(x) := \boldsymbol{g}^{\varepsilon}_{\alpha}(x^{\varepsilon}), \qquad \boldsymbol{g}^{p}(\varepsilon)(x) := \boldsymbol{g}^{p,\varepsilon}(x^{\varepsilon}) \quad \text{for all} \quad x^{\varepsilon} \in \bar{\Omega}^{\varepsilon},$$

where the points $x^{\varepsilon} \in \bar{\Omega}^{\varepsilon}$ and $x \in \bar{\Omega}$ are in the usual correspondence. By definition (cf. (2.2)),

$$\Gamma^{p}_{\alpha\beta}(\varepsilon) = \boldsymbol{g}^{p}(\varepsilon) \cdot \partial_{\beta}\boldsymbol{g}_{\alpha}(\varepsilon) = \boldsymbol{g}^{p}(\varepsilon) \cdot (\partial_{\beta}\boldsymbol{a}_{\alpha} + \varepsilon x_3 \partial_{\alpha\beta}\boldsymbol{a}_3);$$

whence

$$\partial_3\Gamma^{p}_{\alpha\beta}(\varepsilon) = \partial_3\boldsymbol{g}^{p}(\varepsilon) \cdot \partial_{\beta}\boldsymbol{a}_{\alpha} + O(\varepsilon),$$

where the remainder $O(\varepsilon)$ is meant with respect to $\|\cdot\|_{0,\infty,\bar{\Omega}}$. Let

$$\|w\|_{1,\infty,\bar{\Omega}} = \|w\|_{0,\infty,\bar{\Omega}} + \sum_{i} \|\partial_i w\|_{0,\infty,\bar{\Omega}}.$$

Since only second-order derivatives of φ occur in

$$g_{\alpha\beta}(\varepsilon) = (\boldsymbol{a}_{\alpha} + \varepsilon x_3 \partial_{\alpha}\boldsymbol{a}_3) \cdot (\boldsymbol{a}_{\beta} + \varepsilon x_3 \partial_{\beta}\boldsymbol{a}_3),$$

and $\varphi \in \mathcal{C}^3(\bar{\omega}; \boldsymbol{R}^3)$, we conclude that

$$g_{\alpha\beta}(\varepsilon) = a_{\alpha\beta} + \varepsilon g^{1}_{\alpha\beta}(\varepsilon) \quad \text{with} \quad \|g^{1}_{\alpha\beta}(\varepsilon)\|_{1,\infty,\bar{\Omega}} \leqslant c_1;$$

thus

$$g^{\alpha\beta}(\varepsilon) = a^{\alpha\beta} + \varepsilon g^{\alpha\beta,1}(\varepsilon) \quad \text{with} \quad \|g^{\alpha\beta,1}(\varepsilon)\|_{1,\infty,\bar{\Omega}} \leqslant c_2.$$

Relation (3.4) then follows by noting that

$$\boldsymbol{g}^{\alpha}(\varepsilon) = g^{\alpha\beta}(\varepsilon)\boldsymbol{g}_{\beta}(\varepsilon) \quad \text{and} \quad \boldsymbol{g}^{3}(\varepsilon) = \boldsymbol{a}^{3}. \qquad \square$$

Since *averages with respect to the 'transverse' variable* x_3 play a fundamental rôle in the ensuing analysis, their relevant properties are gathered in the next lemma. If v, or $\boldsymbol{v}$, are real-valued, or vector-valued, functions defined almost-everywhere over $\Omega = \omega \times]-1, 1[$, these *averages* $\bar{v}$, or $\bar{\boldsymbol{v}}$, are the real-valued, or vector-valued, functions defined almost-everywhere over ω by letting

$$\bar{v}(y) := \frac{1}{2}\int_{-1}^{1} v(y, x_3)\,\mathrm{d}x_3, \qquad \bar{\boldsymbol{v}}(y) := \frac{1}{2}\int_{-1}^{1} \boldsymbol{v}(y, x_3)\,\mathrm{d}x_3 \tag{3.14}$$

for almost all $y \in \omega$, whenever these definitions make sense (cf. Lemma 3.2 (a) for such instances).

LEMMA 3.2 *(a) Let* $v \in L^2(\Omega)$. *Then* $\bar{v}(y)$ *as given in* (3.14) *is finite for almost all* $y \in \omega$, *the function* $\bar{v}$ *defined in this fashion belongs to* $L^2(\omega)$, *and*

$$\|\bar{v}\|_{0,\omega} \leqslant \frac{1}{\sqrt{2}}\|v\|_{0,\Omega}. \tag{3.15}$$

If $\partial_3 v = 0$ *in the sense of distributions, i.e., if* $\int_\Omega v\partial_3\varphi \, dx = 0$ *for all* $\varphi \in \mathcal{D}(\Omega)$, *then* v *does not depend on* x_3, *and*

$$v(y, x_3) = \bar{v}(y) \quad \textit{for almost all} \quad (y, x_3) \in \Omega. \tag{3.16}$$

(b) Let $v \in H^1(\Omega)$. *Then* $\bar{v} \in H^1(\omega), \partial_\alpha \bar{v} = \overline{\partial_\alpha v}$, *and*

$$\|\bar{v}\|_{1,\omega} \leqslant \frac{1}{\sqrt{2}}\|v\|_{1,\Omega}. \tag{3.17}$$

Let γ_0 *be a measurable subset of* γ. *If* $v = 0$ *on* $\gamma_0 \times [-1, 1]$, *then* $\bar{v} = 0$ *on* γ_0; *in particular,* $\bar{v} \in H_0^1(\omega)$ *if* $v = 0$ *on* $\gamma \times [-1, 1]$.

Proof. See the proof of Lemma 3.2 of Ciarlet and Lods (1996a). □

The *next lemma is crucial*; it plays an essential rôle in both the proofs of a generalized Korn's inequality (Lemma 3.4) and of the convergence of the scaled unknown as $\varepsilon \to 0$ (Theorem 5.1). In the following statement, the symbols $\to$ and $\rightharpoonup$ denote strong and weak convergences, respectively, and ∂_ν denotes the outer normal derivative operator along the boundary of ω. The functions $\gamma_{\alpha\beta}(\boldsymbol{v})$ and $\rho_{\alpha\beta}(\boldsymbol{v})$ defined in (3.18), (3.19) are the three-dimensional analogs of the two-dimensional *change of metric*, and *change of curvature tensors* (the functions $b_\beta^\sigma|_\alpha$ and $c_{\alpha\beta}$ are defined in (3.1), (3.2), and the functions $\Gamma_{\alpha\beta}^\sigma, \ldots, c_{\alpha\beta} \in \mathcal{C}^0(\bar{\omega})$ are identified with functions in $\mathcal{C}^0(\bar{\Omega})$).

LEMMA 3.3 *Let* $V(\Omega)$ *be the space defined in* (2.8) *and let the functions* $e_{i\|j}(\varepsilon)(\boldsymbol{v}) \in L^2(\Omega)$, *and* $\gamma_{\alpha\beta}(\boldsymbol{v}) \in L^2(\Omega), \rho_{\alpha\beta}(\boldsymbol{v}) \in H^{-1}(\Omega)$ *be defined for any function* $\boldsymbol{v} \in V(\Omega)$ *by* (2.5)–(2.7), *and by*

$$\gamma_{\alpha\beta}(\boldsymbol{v}) := \tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma_{\alpha\beta}^\sigma v_\sigma - b_{\alpha\beta}v_3, \tag{3.18}$$

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{v}) := \ & \partial_{\alpha\beta}v_3 - \Gamma_{\alpha\beta}^\sigma \partial_\sigma v_3 + b_\beta^\sigma(\partial_\alpha v_\sigma - \Gamma_{\alpha\sigma}^\tau v_\tau) \\ & + b_\alpha^\sigma(\partial_\beta v_\sigma - \Gamma_{\beta\sigma}^\tau v_\tau) + b_\alpha^\sigma|_\beta v_\sigma - c_{\alpha\beta}v_3.\end{aligned} \tag{3.19}$$

Let $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ *be a sequence of functions* $\boldsymbol{u}(\varepsilon) \in V(\Omega)$ *such that*

$$\boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{H}^1(\Omega) \quad \text{and} \quad \boldsymbol{u}(\varepsilon) \to \boldsymbol{u} \text{ in } \boldsymbol{L}^2(\Omega), \tag{3.20}$$

$$\frac{1}{\varepsilon} e_{i\|j}(\varepsilon)(\boldsymbol{u}(\varepsilon)) \rightharpoonup e^1_{i\|j} \quad \text{in} \quad L^2(\Omega), \tag{3.21}$$

as $\varepsilon \to 0$. *Then*

$$\boldsymbol{u} = (u_i) \quad \text{is independent of the 'transverse' variable} \quad x_3, \tag{3.22}$$

$$\bar{\boldsymbol{u}} = (\bar{u}_i) := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\,\mathrm{d}x_3 \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$$

$$\text{and} \quad \bar{u}_i = \partial_\nu \bar{u}_3 = 0 \quad \text{on} \quad \gamma_0, \tag{3.23}$$

$$\gamma_{\alpha\beta}(\boldsymbol{u}) = 0, \tag{3.24}$$

$$\rho_{\alpha\beta}(\boldsymbol{u}) \in L^2(\Omega) \quad \text{and} \quad \rho_{\alpha\beta}(\boldsymbol{u}) = -\partial_3 e^1_{\alpha\|\beta}. \tag{3.25}$$

If, in addition to (3.20), (3.21), *there exist functions* $\Xi_{\alpha\beta} \in H^{-1}(\Omega)$ *such that*

$$\rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \to \Xi_{\alpha\beta} \quad \text{in} \quad H^{-1}(\Omega) \quad \text{as} \quad \varepsilon \to 0, \tag{3.26}$$

then

$$\boldsymbol{u}(\varepsilon) \to \boldsymbol{u} \quad \text{in} \quad \boldsymbol{H}^1(\Omega) \quad \text{as} \quad \varepsilon \to 0, \tag{3.27}$$

$$\rho_{\alpha\beta}(\boldsymbol{u}) = \Xi_{\alpha\beta} \quad \text{and thus} \quad \Xi_{\alpha\beta} \in L^2(\Omega). \tag{3.28}$$

Proof. See the proof of Lemma 3.3 of Ciarlet, Lods and Miara (1996). □

The key to the convergence theorem of Section 5 is the *generalized Korn's inequality* (3.29), which involves the functions $e_{i\|j}(\varepsilon)(\boldsymbol{v})$ defined in (2.5)–(2.7), instead of the 'traditional' functions $\frac{1}{2}(\partial_i v_j + \partial_j v_i)$. The 'constant' C/ε appearing in this inequality, together with an *ad hoc* assumption on the applied forces, will yield the fundamental *a priori bounds* that the family of scaled unknowns $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ satisfies (cf. Steps (i) and (ii) of the proof of Theorem 5.1).

Remarks 3.1. (1) This generalized Korn's inequality is valid for an *arbitrary* surface $S = \varphi(\bar{\omega})$ (the only requirements are that the set ω and the mapping φ satisfy the assumptions of Section 1 of Part I), irrespectively of whether the space $V_F(\omega)$ introduced in the asymptotic analysis of Section 5 reduces to $\{\mathbf{o}\}$ or not.

(2) Another Korn's inequality with a 'constant' also of the form C/ε was established in Kohn and Vogelius (1985) (see also Acerbi, Buttazzo and Percivale (1988)). It holds however over the 'variable' domain Ω^ε and besides, it involves the 'traditional' functions $\frac{1}{2}(\partial^\varepsilon_i v^\varepsilon_j + \partial^\varepsilon_j v^\varepsilon_i)$. □

LEMMA 3.4 *Let the space* $V(\Omega)$ *be defined as in* (2.8), *i.e.,*

$$V(\Omega) := \{v = (v_i) \in H^1(\Omega); v = o \quad \text{on} \quad \Gamma_0\},$$

where

$$\Gamma_0 = \gamma_0 \times [-1, 1] \quad \text{and } \textit{length} \ \ \gamma_0 > 0.$$

Then there exist $0 < \varepsilon_1 \leqslant \varepsilon_0$ *and* $C > 0$ *such that, for all* $0 < \varepsilon \leqslant \varepsilon_1$,

$$\|v\|_{1,\Omega} \leqslant \frac{C}{\varepsilon} \left\{ \sum_{i,j} \|e_{i\|j}(\varepsilon)(v)\|_{0,\Omega}^2 \right\}^{1/2} \quad \text{for all} \quad v \in V(\Omega), \tag{3.29}$$

where the symmetric tensor $(e_{i\|j}(\varepsilon)(v))$ *is defined as in* (2.5)–(2.7), *i.e.,*

$$e_{\alpha\|\beta}(\varepsilon)(v) = \tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^p_{\alpha\beta}(\varepsilon)v_p,$$

$$e_{\alpha\|3}(\varepsilon)(v) = \tfrac{1}{2}(\partial_\alpha v_3 + \frac{1}{\varepsilon}\partial_3 v_\alpha) - \Gamma^\sigma_{\alpha\beta}(\varepsilon)v_\sigma,$$

$$e_{3\|3}(\varepsilon)(v) = \frac{1}{\varepsilon}\partial_3 v_3.$$

Proof. See the proof of Theorem 4.1 of Ciarlet, Lods and Miara (1996). □

In the course of the proof of the convergence theorem of Section 5, we shall also need the following slight extension of the elegant method proposed by J.L. Lions for proving Korn's inequality in three-dimensional elasticity.

LEMMA 3.5 *Let* Ω *be a domain in* R^3. *Given* $v = (v_i) \in L^2(\Omega)$, *define the distributions*

$$e_{ij}(v) := \tfrac{1}{2}(\partial_i v_j + \partial_j v_i) \in H^{-1}(\Omega).$$

Let there be given a sequence of functions $v^k = (v_i^k) \in L^2(\Omega), k = 0, 1, \ldots,$ *such that*

$$v^k \to o \quad \text{in} \quad H^{-1}(\Omega) \quad \text{and} \quad e_{ij}(v^k) \to 0 \quad \text{in} \quad H^{-1}(\Omega). \tag{3.30}$$

Then

$$v^k \to o \quad \text{in} \quad L^2(\Omega). \tag{3.31}$$

Proof. As shown by Amrouche and Girault (1994), the implication

$$w \in H^{-2}(\Omega) \quad \text{and} \quad \partial_k w \in H^{-2}(\Omega), k = 1, 2, \ldots, n \Longrightarrow w \in H^{-1}(\Omega) \tag{3.32}$$

holds if Ω is a domain in $\boldsymbol{R}^n$. Let then $\boldsymbol{v} = (v_i)$ be such that

$$\boldsymbol{v} \in \boldsymbol{H}^{-1}(\Omega) \quad \text{and} \quad e_{ij}(\boldsymbol{v}) \in H^{-1}(\Omega).$$

Since $w \in H^{-1}(\Omega)$ implies $\partial_j w \in H^{-2}(\Omega)$, we have

$$\partial_j v_i \in H^{-2}(\Omega),$$

$$\partial_k(\partial_j v_i) = \{\partial_k e_{ij}(\boldsymbol{v}) + \partial_j e_{ik}(\boldsymbol{v}) - \partial_i e_{jk}(\boldsymbol{v})\} \in H^{-2}(\Omega),$$

so that $\partial_j v_i \in H^{-1}(\Omega)$ by (3.32). Using a classical *Lemma of J.L. Lions*, who showed that

$$w \in H^{-1}(\Omega) \quad \text{and} \quad \partial_i w \in H^{-1}(\Omega) \Longrightarrow w \in L^2(\Omega),$$

we therefore conclude that $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$. In other words,

$$\{\boldsymbol{v} = (v_i) \in \boldsymbol{H}^{-1}(\Omega); e_{ij}(\boldsymbol{v}) \in H^{-1}(\Omega)\} = \boldsymbol{L}^2(\Omega).$$

Then the isomorphism theorem of Banach shows that there exists a constant C such that

$$\|\boldsymbol{v}\|_{0,\Omega} \leqslant C\{\|\boldsymbol{v}\|_{-1,\Omega} + \sum_{i,j} \|e_{ij}(\boldsymbol{v})\|_{-1,\Omega}\}$$

for all $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, which concludes the proof. □

Remark 3.1. The *lemma of J.L. Lions* cited *supra* was first mentioned in Magenes and Stampacchia (1958), then proved in Duvaut and Lions (1972, p. 110) for domains with smooth boundaries. It was later extended to Lipschitz-continuous boundaries by Borchers and Sohr (1990), then by Amrouche and Girault (1994, Proposition 2.10) who in fact proved the more general implication (of which (3.32) is a special case)

$$w \in \mathcal{D}'(\Omega) \quad \text{and} \quad \partial_i w \in W^{m,p}(\Omega) \Longrightarrow w \in W^{m+1,p}(\Omega),$$

for *arbitrary* integers m and real numbers $1 < p < \infty$. □

4. 'Generalized Membrane' Shells and 'Admissible' Applied Forces

Let

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \tfrac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3 \tag{4.1}$$

denote the covariant components of the *linearized change of metric tensor of the surface* $S = \varphi(\bar{\omega})$ corresponding to a displacement field $\eta_i \boldsymbol{a}^i$ of the points of S (the functions $\Gamma^{\sigma}_{\alpha\beta}$ and $b_{\alpha\beta}$ are defined in (1.2), (1.3)), and let

$$\begin{aligned} \boldsymbol{V}_F(\omega) \;:=\; \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\} \end{aligned} \tag{4.2}$$

denote the *space of inextensional displacements* associated with the surface S and the set γ_0.

If the space $\boldsymbol{V}_F(\omega)$ *contains non-zero functions* (e.g. if S is a portion of cylinder and $\varphi(\gamma_0)$ is contained in a generatrix of S, or if S is a plane domain), the associated family of shells (defined as in Section 1 for all $\varepsilon > 0$) is called a 'flexural' family, for its asymptotic behavior is indeed that of a 'flexural shell', as shown in Ciarlet, Lods and Miara (1996). Our aim is to consider all the 'remaining' cases, where

$$\boldsymbol{V}_F(\omega) = \{\mathbf{o}\}. \tag{4.3}$$

Define the space

$$\begin{aligned} \boldsymbol{V}_K(\omega) \;:=\; \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\} \end{aligned} \tag{4.4}$$

(this is the natural space for studying *Koiter's model*; cf. Section 6), and the *semi-norm* $|\cdot|^M_\omega$ by

$$|\boldsymbol{\eta}|^M_\omega = \left\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega}\right\}^{1/2}, \tag{4.5}$$

for all $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$; hence *condition* (4.3) *is equivalent to stating that the semi-norm* $|\cdot|^M_\omega$ of (4.5) *is a norm on the space* $\boldsymbol{V}_K(\omega)$ *of* (4.4).

Assume that

$$\gamma_0 = \gamma, \tag{4.6}$$

and *simultaneously*, that there exists a constant c such that

$$\left\{\sum_{\alpha} \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{0,\omega}\right\}^{1/2} \leqslant c|\boldsymbol{\eta}|^M_\omega \quad \text{for all} \quad \boldsymbol{\eta} \in \boldsymbol{V}_M(\omega), \tag{4.7}$$

where the space $\boldsymbol{V}_M(\omega)$ is defined by

$$\boldsymbol{V}_M(\omega) := H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega). \tag{4.8}$$

Then in this case, the associated family of shells (defined as in Section 1 for all $\varepsilon > 0$) is called a *'membrane' family*, for its asymptotic behavior is indeed that of a *'membrane shell'*, as shown in Ciarlet and Lods (1996a).

A *'membrane family' thus provides a first instance where* $V_F(\omega) = \{\mathbf{o}\}$, since inequality (4.7) implies that the semi-norm $|\cdot|_\omega^M$ is already a norm over the space $V_M(\omega)$, whence *a fortiori* over $V_K(\omega) \subset V_M(\omega)$ (note that, by virtue of assumption (4.6), $V_K(\omega)$ is equal to $H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ in this case). Inequality (4.7) also shows that the space $V_M(\omega)$, equipped with the norm $|\cdot|_\omega^M$, is *complete* (the 'opposite' inequality clearly holds), a fact that 'sets membrane families apart' among those for which $V_F(\omega) = \{\mathbf{o}\}$, as we shall see in Section 5.

Assumption (4.7) *is in fact an assumption about the allowed 'geometries' of the surface S and the allowed sets γ_0.* To this end, we need a definition: A surface $S = \varphi(\bar{\omega})$ with $\varphi \in C^2(\bar{\omega}; \boldsymbol{R}^3)$ is *elliptic* if there exists a constant $b > 0$ such that

$$|b_{\alpha\beta}(y)\xi^\alpha\xi^\beta| \geqslant b\xi^\alpha\xi^\alpha \tag{4.9}$$

for all $y \in \bar{\omega}$ and $(\xi^\alpha) \in \boldsymbol{R}^2$; equivalently, the two principal radii of curvature are either > 0 at all points of S, or < 0 at all points of S, and their moduli lie in a compact interval of $]0, +\infty[$.

The following sufficient conditions guaranteeing that assumption (4.7) holds were proved in Ciarlet and Sanchez–Palencia (1996) and Ciarlet and Lods (1996c), respectively:

THEOREM 4.1 *Assume either that the boundary γ of ω is of class C^3 and $\varphi : \bar{\omega} \to \boldsymbol{R}^3$ is the restriction to $\bar{\omega}$ of an analytic mapping; or that γ is of class C^4 and $\varphi \in C^5(\bar{\omega}; \boldsymbol{R}^3)$. Then relation* (4.7) *is satisfied if the surface $S = \varphi(\bar{\omega})$ is elliptic.* □

Remarkably, this condition is also necessary, as recently shown by Şlicacu (1996):

THEOREM 4.2 *Assume that γ is Lipschitz-continuous, $\varphi \in C^2(\bar{\omega}; \boldsymbol{R}^3)$, and relation* (4.7) *holds. Then the surface S is elliptic.* □

Şlicaru (1996) has also shown that assumption (4.6) is likewise necessary, in the following sense:

THEOREM 4.3 *Assume that γ is Lipschitz-continuous and $\varphi \in C^2(\bar{\omega}; \boldsymbol{R}^3)$. Let γ_0 denote a measurable subset of γ and assume that inequality* (4.7) *holds for all $\eta \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ that satisfy $\eta_\alpha = 0$ on γ_0. Then $\gamma_0 = \gamma$.* □

We now give a *crucial definition*, that of the families of shells studied in this paper. A family of shells (defined as in Section 1 for all $\varepsilon > 0$), associated with a given surface $S = \varphi(\bar{\omega})$ and a given set γ_0, is a *'generalized membrane family'* if $V_F(\omega) = \{\mathbf{o}\}$, *but* the family is *not* a 'membrane family'. This is the case for instance if S is elliptic but *length* $\gamma_0 <$ *length* γ, or if S is a portion of hyperboloid; cf. Mardare (1995).

Remark 4.1. The definitions given here of 'flexural', 'membrane', and 'generalized membrane' families are closely related to shells with 'non-inhibited', 'well-inhibited', and 'not well-inhibited', pure flexions, according to the terminology proposed by Sanchez-Palencia (1989a, 1989b). □

As shown in the proof of the convergence theorem of Section 5, 'generalized membrane families' need themselves to be subdivided into two different categories. To describe these, we introduce the spaces

$$V(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{H}^1(\omega); \boldsymbol{\eta} = \boldsymbol{o} \quad \text{on} \quad \gamma_0\}, \tag{4.10}$$

$$V_0(\omega) := \{\boldsymbol{\eta} \in V(\omega); \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \quad \text{in} \quad \omega\}. \tag{4.11}$$

Then a family is a '*generalized membrane family of the first kind*' if

$$V_0(\omega) = \{\boldsymbol{o}\}, \tag{4.12}$$

or equivalently, *if the semi-norm* $|\cdot|_\omega^M$ *is a norm over the space* $V(\omega)$ (whence *a fortiori* over $V_K(\omega) \subset V(\omega)$). Otherwise, i.e., if $|\cdot|_\omega^M$ is a norm over $V_K(\omega)$, but *not* over $V(\omega)$, it is a '*generalized membrane family of the second kind*'.

Remark 4.2. The space $V_F(\omega)$ plays the same rôle with respect to the space $V_K(\omega)$ as does $V_0(\omega)$ with respect to $V(\omega)$. □

To carry out our asymptotic analysis of 'sensitive membrane families', we shall also need to consider the 'three-dimensional analog' of the space $V_0(\omega)$ of (4.11), defined by

$$V_0(\Omega) := \{\boldsymbol{v} \in V(\Omega); \partial_3 \boldsymbol{v} = \boldsymbol{o} \quad \text{in} \quad \Omega, \gamma_{\alpha\beta}(\bar{\boldsymbol{v}}) = 0 \quad \text{in} \quad \omega\}, \tag{4.13}$$

where $V(\Omega)$ is the space of (2.8), and the averages $\bar{\boldsymbol{v}}$ are defined as in (3.14); we shall likewise need the 'three-dimensional analog' of the semi-norm $|\cdot|_\omega^M$ of (4.5), defined by

$$|\boldsymbol{v}|_\Omega^M := \{\|\partial_3 \boldsymbol{v}\|_{0,\Omega}^2 + (|\bar{\boldsymbol{v}}|_\omega^M)^2\}^{1/2}. \tag{4.14}$$

Observe in passing that

$$V_0(\omega) = \{\boldsymbol{o}\} \Longleftrightarrow V_0(\Omega) = \{\boldsymbol{o}\}, \tag{4.15}$$

or equivalently, that $|\cdot|_\omega^M$ *is a norm on the space* $V(\omega)$ *of* (4.10) *if and only if* $|\cdot|_\Omega^M$ *is a norm over the space* $V(\Omega)$ *of* (2.8).

In order to derive the fundamental *a priori* estimates that the family $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ of scaled unknowns satisfies, we shall need to assume that the forces applied to a '*generalized membrane family*' are of a special form, according to the following definition: Applied forces are called '*admissible*' if, for each $\varepsilon > 0$, there exist functions $\varphi^{\alpha\beta}(\varepsilon) = \varphi^{\beta\alpha}(\varepsilon) \in L^2(\omega)$ and $F^i(\varepsilon) \in L^2(\Omega)$ such that the expression $L(\varepsilon)(\boldsymbol{v})$ appearing in the right-hand side of the scaled three-dimensional shell equations (2.9) is of the form

$$L(\varepsilon)(\boldsymbol{v}) = \int_\omega \varphi^{\alpha\beta}(\varepsilon)\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\,\mathrm{d}y + \int_\Omega F^i(\varepsilon)\partial_3 v_i\,\mathrm{d}x \tag{4.16}$$

for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, *and* if there exist functions $\varphi^{\alpha\beta} = \varphi^{\beta\alpha} \in L^2(\omega)$ and $F^i \in L^2(\Omega)$ such that

$$\varphi^{\alpha\beta}(\varepsilon) \to \varphi^{\alpha\beta} \quad \text{in} \quad L^2(\omega) \quad \text{and} \quad F^i(\varepsilon) \to F^i \quad \text{in} \quad L^2(\Omega), \tag{4.17}$$

as $\varepsilon \to 0$. Thus, *if the applied forces are admissible, there exists a constant* C_3 *such that*

$$|L(\varepsilon)(\boldsymbol{v})| \leqslant C_3|\boldsymbol{v}|_\Omega^M \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}(\Omega) \quad \text{and all} \quad 0 < \varepsilon \leqslant \varepsilon_0. \tag{4.18}$$

This property is crucially needed in Step (ii) of the proof of Theorem 5.1.

Remark 4.3. The linear form $L(\varepsilon)\colon \boldsymbol{V}(\Omega) \to \boldsymbol{R}$ defined in (4.16) is also continuous with respect to the norm $\|\cdot\|_{1,\Omega}$ (to see this, use (4.18) and Lemma 3.2). This insures in particular that both three-dimensional problems (1.12) and (2.9) have a solution for each $\varepsilon > 0$. □

The expression $L(\varepsilon)(\boldsymbol{v})$ of (4.16) takes the usual form

$$L(\varepsilon)(\boldsymbol{v}) = \int_\Omega f^i(\varepsilon)v_i\sqrt{g(\varepsilon)}\,\mathrm{d}x,$$

corresponding to an applied body force density with contravariant components $f^{i,\varepsilon}$ as in (1.15) defined by $f^{i,\varepsilon}(x^\varepsilon) = f^i(\varepsilon)(x)$ for all $x \in \Omega$, if the functions $\varphi^{\alpha\beta}(\varepsilon)$ and $F^i(\varepsilon)$ satisfy the *additional* assumptions

$$\varphi^{\alpha\beta}(\varepsilon) \in H^1(\omega) \quad \text{and} \quad \varphi^{\alpha\beta}(\varepsilon)\nu_\beta = 0 \quad \text{on} \quad \gamma - \gamma_0,$$

$$F^i(\varepsilon) \in H^1(\Omega) \quad \text{and} \quad F^i(\varepsilon) = 0 \quad \text{on} \quad \omega \times (\{-1\} \cup \{1\}).$$

For it is easily verified in this case that

$$f^\alpha(\varepsilon) = -\frac{1}{2\sqrt{g(\varepsilon)}}\{\partial_\beta\varphi^{\alpha\beta}(\varepsilon) + \Gamma^\alpha_{\sigma\beta}\varphi^{\sigma\beta}(\varepsilon) + 2\partial_3F^\alpha(\varepsilon)\},$$

$$f^3(\varepsilon) = -\frac{1}{2\sqrt{g(\varepsilon)}}\{b_{\alpha\beta}\varphi^{\alpha\beta}(\varepsilon) + 2\partial_3F^\alpha(\varepsilon)\}.$$

Together with those of (4.17), these relations mean that the 'equivalent' applied body force are $O(1)$ with respect to ε, as in the case of 'membrane' shells considered in Ciarlet and Lods (1996a).

Remark 4.4. As we shall see in Ciarlet and Lods (1996d), 'admissible' forces are not restricted to 'generalized membrane families'. They can be likewise applied to a 'flexural family', in which case they give rise to 'membrane effects in flexural shells'. □

5. Asymptotic Analysis as $\varepsilon \to 0$

We now establish our main result, showing that the *averages* (cf. 3.14))

$$\overline{\boldsymbol{u}(\varepsilon)} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}(\varepsilon)\,\mathrm{d}x_3$$

of the scaled unknowns $\boldsymbol{u}(\varepsilon)$ (cf. (2.1)) corresponding to a *generalized membrane family of the first kind* converge in an appropriate *completion* of the space $\boldsymbol{V}(\omega)$ (cf. (5.1)). In addition, we show that the limit $\boldsymbol{\xi}$ of the family $\overline{\boldsymbol{u}(\varepsilon)}_{\varepsilon>0}$ solves an '*abstract*' *variational problem* (cf. (5.6)) (later identified as a 'sensitive' problem in the sense recently introduced by Lions and Sanchez–Palencia (1994); see Section 7 for further comments). We also show that a similar result holds for a *generalized membrane family of the second kind*, this time in an appropriate completion of the *quotient* space $\boldsymbol{V}(\omega)/\boldsymbol{V}_0(\omega)$ (cf. (5.7)).

The functions $a^{\alpha\beta\sigma\tau}$ defined in (5.3) are the covariant components of the *elasticity tensor* of S; the bilinear form B_M defined in (5.4) is classically that of a '*membrane shell*' (the functions a and $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ have been defined in (1.4) and (4.1)); finally, $\dot{\boldsymbol{\eta}} \in \boldsymbol{V}(\omega)/\boldsymbol{V}_0(\omega)$ denotes the equivalence class of $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$, and for $\dot{\boldsymbol{\eta}} \in \boldsymbol{V}(\omega)/\boldsymbol{V}_0(\omega)$,

$$|\dot{\boldsymbol{\eta}}|_\omega^M := |\boldsymbol{\eta}'|_\omega^M \quad \text{for any} \quad \boldsymbol{\eta}' \in \dot{\boldsymbol{\eta}}.$$

It is worth noticing that the assumption that is central to this paper, viz., that $\boldsymbol{V}_F(\omega) = \{\mathbf{o}\}$, is not needed until Step (vi) of the proof (where it is used to prove that the family $(\varepsilon\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ converges to $\mathbf{o}$).

THEOREM 5.1 *(a) Let there be given a 'generalized membrane family of the first kind' subjected to 'admissible' applied forces, according to the definitions of Section 4. The space* $\boldsymbol{V}(\omega)$ *being defined as in* (4.10) *and the semi-norm* $|\cdot|_\omega^M$ *as in* (4.5), *define the spaces*

$$\left.\begin{aligned} \boldsymbol{V}_M^\sharp(\omega) &:= \textit{completion of} \quad \boldsymbol{V}(\omega) \quad \textit{for} \quad |\cdot|_\omega^M, \\ \boldsymbol{V}_M^\sharp(\Omega) &:= \textit{completion of} \quad \boldsymbol{V}(\Omega) \quad \textit{for} \quad |\cdot|_\Omega^M. \end{aligned}\right\} \tag{5.1}$$

Let $\boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega)$ denote for each $0 < \varepsilon \leqslant \varepsilon_0$ the solution of the scaled variational problem (2.8), (2.9). *Then there exist $\boldsymbol{\xi} \in \boldsymbol{V}_M^\sharp(\omega)$ and $\boldsymbol{u} \in \boldsymbol{V}_M^\sharp(\Omega)$ such that*

$$\left.\begin{aligned} &\overline{\boldsymbol{u}(\varepsilon)} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}(\varepsilon)\,\mathrm{d}x_3 \to \boldsymbol{\xi} \quad \textit{in} \quad \boldsymbol{V}_M^\sharp(\omega) \quad \textit{as} \quad \varepsilon \to 0, \\ &\boldsymbol{u}(\varepsilon) \to \boldsymbol{u} \quad \textit{in} \quad \boldsymbol{V}_M^\sharp(\Omega) \quad \textit{as} \quad \varepsilon \to 0. \end{aligned}\right\} \tag{5.2}$$

Let

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{(\lambda+2\mu)} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \tag{5.3}$$

$$B_M(\boldsymbol{\xi},\boldsymbol{\eta}) := \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \quad \textit{for} \quad \boldsymbol{\xi},\boldsymbol{\eta} \in \boldsymbol{V}(\omega), \tag{5.4}$$

$$L(\boldsymbol{\eta}) := \int_\omega \varphi^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\,\mathrm{d}y \quad \textit{for} \quad \boldsymbol{\eta} \in \boldsymbol{V}(\omega), \tag{5.5}$$

where the functions $\varphi^{\alpha\beta} \in L^2(\omega)$ are those of (4.17), *and let $B_M^\sharp$ and $L_M^\sharp$ denote the unique continuous extensions from $\boldsymbol{V}(\omega)$ to $\boldsymbol{V}_M^\sharp(\omega)$ of the bilinear form B_M and linear form L. Then the limit $\boldsymbol{\xi}$ found in* (5.2) *is the unique solution to the equations*

$$B_M^\sharp(\boldsymbol{\xi},\boldsymbol{\eta}) = L_M^\sharp(\boldsymbol{\eta}) \quad \textit{for all} \quad \boldsymbol{\eta} \in \boldsymbol{V}_M^\sharp(\Omega), \tag{5.6}$$

(b) Let there be given a 'generalized membrane family of the second kind' subjected to 'admissible' applied forces. The spaces $\boldsymbol{V}(\omega), \boldsymbol{V}_0(\omega)$ being defined as in (4.10), (4.11), *define the space.*

$$\left.\begin{aligned} &\dot{\boldsymbol{V}}_M^\sharp(\omega) := \textit{completion of} \quad \boldsymbol{V}(\omega)/\boldsymbol{V}_0(\omega) \quad \textit{for} \quad |\cdot|_\omega^M, \\ &\dot{\boldsymbol{V}}_M^\sharp(\Omega) := \textit{completion of} \quad \boldsymbol{V}(\Omega)/\boldsymbol{V}_0(\Omega) \quad \textit{for} \quad |\cdot|_\Omega^M. \end{aligned}\right\} \tag{5.7}$$

Then there exist $\dot{\boldsymbol{\xi}} \in \dot{\boldsymbol{V}}_M^\sharp(\omega)$ and $\dot{\boldsymbol{u}} \in \dot{\boldsymbol{V}}_M^\sharp(\Omega)$ such that

$$\left.\begin{aligned} &\dot{\overline{\boldsymbol{u}(\varepsilon)}} \to \dot{\boldsymbol{\xi}} \quad \textit{in} \quad \dot{\boldsymbol{V}}_M^\sharp(\omega) \quad \textit{as} \quad \varepsilon \to 0, \\ &\dot{\boldsymbol{u}}(\varepsilon) \to \dot{\boldsymbol{u}} \quad \textit{in} \quad \dot{\boldsymbol{V}}_M^\sharp(\Omega) \quad \textit{as} \quad \varepsilon \to 0. \end{aligned}\right\} \tag{5.8}$$

Let the forms B_M and L be defined as in (5.4), (5.5). *For $\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\eta}} \in \boldsymbol{V}(\omega)/\boldsymbol{V}_0(\omega)$, let*

$$\dot{B}_M(\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\eta}}) := B_M(\boldsymbol{\xi}',\boldsymbol{\eta}') \quad \textit{for any} \quad \boldsymbol{\xi}' \in \dot{\boldsymbol{\xi}} \quad \textit{and any} \quad \boldsymbol{\eta}' \in \dot{\boldsymbol{\eta}}, \tag{5.9}$$

$$\dot{L}(\dot{\boldsymbol{\eta}}) := L(\boldsymbol{\eta}') \quad \textit{for any} \quad \boldsymbol{\eta}' \in \dot{\boldsymbol{\eta}}, \tag{5.10}$$

and let $\dot{B}_M^\sharp$ *and* $\dot{L}_M^\sharp$ *denote the unique continuous extensions of the bilinear form* $\dot{B}_M$ *and linear form* $\dot{L}$ *form* $\boldsymbol{V}(\omega)/\boldsymbol{V}_0(\omega)$ *to* $\dot{\boldsymbol{V}}_M^\sharp(\omega)$. *Then the limit* $\dot{\boldsymbol{\xi}}$ *found in* (5.8) *is the unique solution to the equations*

$$\dot{B}_M^\sharp(\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\eta}}) = \dot{L}_M^\sharp(\dot{\boldsymbol{\eta}}) \quad \textit{for all} \quad \dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}_M^\sharp(\omega). \tag{5.11}$$

Proof. The proof is divided into ten steps, numbered (i) to (x). Throughout the proof, we let

$$e_{i\|j}(\varepsilon) := e_{i\|j}(\varepsilon)(\boldsymbol{u}(\varepsilon)),$$

and we let $\to$ and $\rightharpoonup$ denote strong and weak convergences, respectively. To save space, we only deal with generalized membrane families of the second kind, with the implicit understanding that those of the first kind are immediately recovered, simply by letting $\boldsymbol{V}_0(\omega) = \{\mathbf{o}\}$, $\dot{\boldsymbol{V}}_M^\sharp(\omega) = \boldsymbol{V}_M^\sharp(\omega)$, etc.

(i) *There exist* $0 < \varepsilon_1 \leqslant \varepsilon_0$ *and* $C > 0$ *such that, for all* $0 < \varepsilon \leqslant \varepsilon_1$,

$$|\boldsymbol{v}|_\Omega^M \leqslant C \left\{ \sum_{i,j} \|e_{i\|j}(\varepsilon)(\boldsymbol{v})\|_{0,\Omega}^2 \right\}^{1/2} \quad \text{for all} \quad \boldsymbol{v} \in \boldsymbol{V}(\Omega), \tag{5.12}$$

where $|\cdot|_\Omega^M$ *denotes the semi-norm defined in* (4.14).

Let, as in (3.18),

$$\gamma_{\alpha\beta}(\boldsymbol{v}) := \tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma_{\alpha\beta}^\sigma v_\sigma - v_{\alpha\beta} v_3$$

for $\boldsymbol{v} = (v_i) \in \boldsymbol{V}(\Omega)$. The relations

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) + (\Gamma_{\alpha\beta}^\sigma(\varepsilon) - \Gamma_{\alpha\beta}^\sigma) v_\sigma - \varepsilon x_3 c_{\alpha\beta} v_3,$$

$$\partial_3 v_\alpha = 2\varepsilon e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) - \varepsilon \partial_\alpha v_3 + 2\varepsilon \Gamma_{\alpha 3}^\sigma(\varepsilon) v_\sigma,$$

$$\partial_3 v_3 = \varepsilon e_{3\|3}(\varepsilon)(\boldsymbol{v}),$$

$$\|\Gamma_{\alpha\beta}^\sigma(\varepsilon) - \Gamma_{\alpha\beta}^\sigma\|_{0,\infty,\bar{\Omega}} = O(\varepsilon), \quad \|\Gamma_{\alpha 3}^\sigma(\varepsilon)\|_{0,\infty,\bar{\Omega}} = O(1),$$

the last ones being consequences of the estimates (3.3) and (3.5), then imply that, for all $0 < \varepsilon \leqslant \varepsilon_0$ (without loss of generality, we may assume that $\varepsilon_0 \leqslant 1$) and for all $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$,

$$\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{v})\|_{0,\Omega} + \sum_i \|\partial_3 v_i\|_{0,\Omega} \leqslant c_1 \left(\sum_{i,j} \|e_{i\|j}(\varepsilon)(\boldsymbol{v})\|_{0,\Omega} + \varepsilon \|\boldsymbol{v}\|_{1,\Omega} \right).$$

Hence inequality (5.12) is obtained by combining the inequalities

$$\|\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\|_{0,\omega} \leqslant \frac{1}{\sqrt{2}}\|\gamma_{\alpha\beta}(\boldsymbol{v})\|_{0,\Omega}$$

(which follow from Lemma 3.2 and the observation that $\gamma_{\alpha\beta}(\bar{\boldsymbol{v}}) = \overline{\gamma_{\alpha\beta}(\boldsymbol{v})}$; cf. (3.14)) with the *generalized Korn's inequality* (3.29).

(ii) *A priori bounds and extraction of weakly convergent sequences: There exists* $\varepsilon_1 > 0$ *such that the semi-norms* $|\boldsymbol{u}(\varepsilon)|_\Omega^M, |\overline{\boldsymbol{u}(\varepsilon)}|_\omega^M$, *whence the norms* $|\dot{\boldsymbol{u}}(\varepsilon)|_\Omega^M, |\dot{\overline{\boldsymbol{u}(\varepsilon)}}|_\omega^M$, *and the norms* $\|\varepsilon\boldsymbol{u}(\varepsilon)\|_{1,\Omega}, \|e_{i\|j}(\varepsilon)\|_{0,\Omega}$ *are bounded independently of* $0 < \varepsilon \leqslant \varepsilon_1$. The space $V_0(\Omega)$ being defined as in (4.13), *define the space*

$$\dot{V}_M^\sharp(\Omega) := \text{completion of} \quad V(\Omega)/V_0(\Omega) \quad \text{for} \quad \|\cdot\|_\Omega^M. \tag{5.13}$$

Then there exists a subsequence, still denoted $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ for convenience, *and there exist* $\dot{\boldsymbol{u}} \in \dot{V}_M^\sharp(\Omega), \boldsymbol{u}^{-1} = (u_i^{-1}) \in V(\Omega), e_{i\|j} \in L^2(\Omega)$, such that

$$\dot{\boldsymbol{u}}(\varepsilon) \rightharpoonup \dot{\boldsymbol{u}} \quad \text{in} \quad \dot{V}_M^\sharp(\Omega), \tag{5.14}$$

$$\varepsilon\boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u}^{-1} \quad \text{in} \quad \boldsymbol{H}^{-1}(\Omega) \quad \text{and} \quad \varepsilon\boldsymbol{u}(\varepsilon) \to \boldsymbol{u}^{-1} \quad \text{in} \quad \boldsymbol{L}^2(\Omega), \tag{5.15}$$

$$e_{i\|j}(\varepsilon) \rightharpoonup e_{i\|j} \quad \text{in} \quad L^2(\Omega), \tag{5.16}$$

$$\partial_3 u_3(\varepsilon) = \varepsilon e_{3\|3}(\varepsilon) \to 0 \quad \text{in} \quad L^2(\Omega), \tag{5.17}$$

$$\dot{\overline{\boldsymbol{u}(\varepsilon)}} \rightharpoonup \dot{\boldsymbol{\xi}} \quad \text{in} \quad \dot{V}_M^\sharp(\Omega). \tag{5.18}$$

From inequalities (1.5) and (3.7), we infer that there exist $0 < \varepsilon_1 \leqslant \varepsilon_0$ and g_0 such that

$$0 < g_0 \leqslant g(\varepsilon)(x) \quad \text{for all} \quad 0 < \varepsilon \leqslant \varepsilon_1 \quad \text{and all} \quad x \in \bar{\Omega}. \tag{5.19}$$

Combining the variational equations (2.9) with inequalities (3.13), (5.12), (5.19), and (4.18), we obtain

$$\begin{aligned}
(C^{-1}|\boldsymbol{u}(\varepsilon)|_\Omega^M)^2 &\leqslant \sum_{i,j}\|e_{i\|j}(\varepsilon)\|_{0,\Omega}^2 \\
&\leqslant C_2 g_0^{-(1/2)} \int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)e_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}\,\mathrm{d}x \\
&= C_2 g_0^{-(1/2)} L(\varepsilon)(\boldsymbol{u}(\varepsilon)) \leqslant C_2 C_3 g_0^{-(1/2)}|\boldsymbol{u}(\varepsilon)|_\Omega^M,
\end{aligned}$$

for all $0 < \varepsilon \leqslant \varepsilon_1$. The boundedness of $|\boldsymbol{u}(\varepsilon)|_\Omega^M$, whence of $|\overline{\boldsymbol{u}(\varepsilon)}|_\omega^M \leqslant |\boldsymbol{u}(\varepsilon)|_\Omega^M$, and of $\|e_{i\|j}(\varepsilon)\|_{0,\Omega}$, follows from this inequality; that of $\|\varepsilon\boldsymbol{u}(\varepsilon)\|_{1,\Omega}$ from the generalized Korn's inequality (3.29). Note that the assumption that the forces are '*admissible*' (by means of inequality (4.18)) is crucially needed here.

(iii) *The limit functions* $e_{i\|j}$ *found in* (5.16) *satisfy*

$$e_{\alpha\|3} = 0, \quad e_{3\|3} = -\frac{\lambda}{\lambda+2\mu} a^{\alpha\beta} e_{\alpha\|\beta}. \tag{5.20}$$

Let $\boldsymbol{v} = (v_i)$ be an arbitrary function in the space $\boldsymbol{V}(\Omega)$. Then

$$\varepsilon e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) \to 0 \quad \text{in} \quad L^2(\Omega), \tag{5.21}$$

$$\varepsilon e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) \to \tfrac{1}{2}\partial_3 v_\alpha \quad \text{in} \quad L^2(\Omega), \tag{5.22}$$

$$\varepsilon e_{3\|3}(\varepsilon)(\boldsymbol{v}) = \partial_3 v_3 \quad \text{for all} \quad \varepsilon > 0, \tag{5.23}$$

by definition of the functions $e_{i\|j}(\varepsilon)(\boldsymbol{v})$. The variational equations (2.9) satisfied by $\boldsymbol{u}(\varepsilon)$ may be also written as (note that $A^{\alpha\beta\sigma 3}(\varepsilon) = A^{\alpha 333}(\varepsilon) = 0$; cf. (1.18) and (2.4)):

$$\begin{aligned}
&\int_\Omega \{A^{\alpha\beta\sigma\tau}(\varepsilon) e_{\sigma\|\tau}(\varepsilon) + A^{\alpha\beta 33}(\varepsilon) e_{3\|3}(\varepsilon)\}\{\varepsilon e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v})\}\sqrt{g(\varepsilon)}\,\mathrm{d}x \\
&\quad + \int_\Omega \{4A^{\alpha 3\sigma 3}(\varepsilon) e_{\sigma\|3}(\varepsilon)\}\{\varepsilon e_{\alpha\|3}(\varepsilon)(\boldsymbol{v})\}\sqrt{g(\varepsilon)}\,\mathrm{d}x \\
&\quad + \int_\Omega \{A^{33\sigma\tau}(\varepsilon) e_{\sigma\|\tau}(\varepsilon) + A^{3333}(\varepsilon) e_{3\|3}(\varepsilon)\}\partial_3 v_3 \sqrt{g(\varepsilon)}\,\mathrm{d}x = \varepsilon L(\varepsilon)(\boldsymbol{v}).
\end{aligned}$$

Keep $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ *fixed* and let $\varepsilon \to 0$. Using relations (3.8), (3.10), (5.21)–(5.23), and the weak convergences (5.16), we obtain

$$\int_\Omega \{2\mu a^{\alpha\sigma} e_{\sigma\|3}\partial_3 v_3 + [\lambda a^{\sigma\tau} e_{\sigma\|\tau} + (\lambda+2\mu) e_{3\|3}]\partial_3 v_3\}\sqrt{a}\,\mathrm{d}x = 0.$$

Letting $\boldsymbol{v}$ vary in $\boldsymbol{V}(\Omega)$ then yields relations (5.20) (if $w \in L^2(\Omega)$ and $\int_\Omega w\partial_3 v\,\mathrm{d}x = 0$ for all $v \in H^1(\Omega)$ that vanish on Γ_0, then $w = 0$; cf. e.g. Ciarlet (1990, p. 19)).

(iv) *For the whole family* $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$,

$$\{\overline{e_{\alpha\|\beta}(\varepsilon)} - \gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)})\} \to 0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \varepsilon \to 0; \tag{5.24}$$

consequently, for the subsequence considered in Step (ii),

$$\gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)}) \rightharpoonup \overline{e_{\alpha\|\beta}} \quad \text{in} \quad L^2(\omega). \tag{5.25}$$

The definitions of the functions $e_{\alpha\|\beta}(\varepsilon)$ and $\gamma_{\alpha\beta}(\boldsymbol{u}(\varepsilon))$, together with inequalities (3.3), imply that

$$\overline{e_{\alpha\|\beta}(\varepsilon)} - \gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)}) = \overline{\Gamma^{\sigma,2}_{\alpha\beta}(\varepsilon)u_\sigma(\varepsilon)} + \overline{\varepsilon x_3 b^\sigma_\beta|_\alpha u_\sigma(\varepsilon)} + \overline{\varepsilon x_3 c_{\alpha\beta} u_3(\varepsilon)},$$

where the functions

$$\Gamma^{\sigma,2}_{\alpha\beta}(\varepsilon) := \Gamma^\sigma_{\alpha\beta}(\varepsilon) - \{\Gamma^\sigma_{\alpha\beta} - \varepsilon x_3 b^\sigma_\beta|_\alpha\} : \bar{\Omega} \to \boldsymbol{R}$$

satisfy

$$\|\Gamma^{\sigma,2}_{\alpha\beta}(\varepsilon)\|_{0,\infty,\bar{\Omega}} \leqslant c_2\varepsilon^2$$

for all $0 < \varepsilon \leqslant \varepsilon_0$. Since $\|\bar{v}\|_{0,\omega} \leqslant \frac{1}{\sqrt{2}}\|v\|_{0,\Omega}$ (cf. Lemma 3.2 (i)) and $\overline{x_3 v} = \overline{\frac{1}{2}(1-x_3^2)\partial_3 v}$ (this relation, which is easily seen to hold if $v \in C^1(\bar{\Omega})$, is then extended to $v \in H^1(\Omega)$ by means of Lemma 3.2),

$$\|\overline{x_3 v}\|_{0,\omega} = \|\overline{\tfrac{1}{2}(1-x_3^2)\partial_3 v}\|_{0,\omega} \leqslant c_4\|\partial_3 v\|_{0,\Omega},$$

and thus

$$\|\overline{e_{\alpha\|\beta}(\varepsilon)} - \gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)})\|_{0,\omega} \leqslant c_3\varepsilon\left\{\sum_\alpha \|\varepsilon u_\alpha(\varepsilon)\|_{0,\Omega} + \sum_i \|\partial_3 u_i(\varepsilon)\|_{0,\Omega}\right\}.$$

Hence relation (5.24) follows from the boundedness of the norms $\|\varepsilon\boldsymbol{u}(\varepsilon)\|_{1,\Omega}$ and semi-norms $|\boldsymbol{u}(\varepsilon)|^M_\Omega$ established in Step (ii).

(v) *The limit functions* $e_{\alpha\|\beta}$ *found in Step (ii) satisfy* (the space $V(\omega)$ is defined in (4.10))

$$\int_\omega a^{\alpha\beta\sigma\tau}\overline{e_{\sigma\|\tau}}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y$$

$$= \int_\omega \varphi^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\,\mathrm{d}y \quad \text{for all} \quad \boldsymbol{\eta} \in V(\omega), \tag{5.26}$$

where the functions $\varphi^{\alpha\beta}$ *are those appearing in relations* (4.17).

With each $\boldsymbol{\eta} \in V(\omega)$, associate the function $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{\eta})$ in the space $V(\Omega)$ that satisfies $\partial_3\boldsymbol{v} = \mathbf{o}$ and $\bar{\boldsymbol{v}} = \boldsymbol{\eta}$. For such a $\boldsymbol{v}$,

$$e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) \to \gamma_{\alpha\beta}(\boldsymbol{v}) \quad \text{in} \quad L^2(\Omega), \tag{5.27}$$

$$e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) \to \{\tfrac{1}{2}\partial_\alpha v_3 + b^\sigma_\alpha v_\sigma\} \quad \text{in} \quad L^2(\Omega), \tag{5.28}$$

$$e_{3\|3}(\varepsilon)(\boldsymbol{v}) = 0. \tag{5.29}$$

Keep $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ fixed, let $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{\eta})$ in the variational equations (2.9), and let $\varepsilon \to 0$. Relations (3.7)–(3.12), the strong convergences (5.27), (5.28), relation (5.29), and the weak convergences (5.16) to the limits $e_{i\|j}$ satisfying (5.20), together imply that

$$\lim_{\varepsilon\to 0}\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)e_{i\|j}(\varepsilon)(\boldsymbol{v})\sqrt{g(\varepsilon)}\,\mathrm{d}x$$
$$= \frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}e_{\sigma\|\tau}\gamma_{\alpha\beta}(\boldsymbol{v})\sqrt{a}\,\mathrm{d}x = \int_\omega a^{\alpha\beta\sigma\tau}\overline{e_{\sigma\|\tau}}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y. \tag{5.30}$$

Since $\partial_3 v_i = 0$ and the forces are '*admissible*',

$$L(\varepsilon)(\boldsymbol{v}) = \int_\omega \varphi^{\alpha\beta}(\varepsilon)\gamma_{\alpha\beta}(\bar{\boldsymbol{v}})\,\mathrm{d}y \to \int_\omega \varphi^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\,\mathrm{d}y \tag{5.31}$$

as $\varepsilon \to 0$, by (4.17). The conjunction of (5.30), (5.31) then imply (5.26).

(vi) *The subsequence $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ found in Step (ii) is such that*

$$\varepsilon\boldsymbol{u}(\varepsilon) \rightharpoonup \mathbf{o} \quad \text{in} \quad \boldsymbol{H}^1(\Omega) \quad \text{and} \quad \varepsilon\boldsymbol{u}(\varepsilon) \to \mathbf{o} \quad \text{in} \quad \boldsymbol{L}^2(\Omega), \tag{5.32}$$

$$\partial_3 u_\alpha(\varepsilon) \rightharpoonup 0 \quad \text{in} \quad L^2(\Omega), \tag{5.33}$$

as $\varepsilon \to 0$. Furthermore,

$$e_{\alpha\|\beta} \quad \text{is independent of the 'transverse' variable} \quad x_3. \tag{5.34}$$

The functions

$$\boldsymbol{u}^{-1}(\varepsilon) := \varepsilon\boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega) \tag{5.35}$$

associated with the subsequence found in (ii) satisfy

$$\boldsymbol{u}^{-1}(\varepsilon) \rightharpoonup \boldsymbol{u}^{-1} \quad \text{in} \quad \boldsymbol{H}^1(\Omega) \quad \text{and} \quad \boldsymbol{u}^{-1}(\varepsilon) \to \boldsymbol{u}^{-1} \quad \text{in} \quad \boldsymbol{L}^2(\Omega),$$
$$\frac{1}{\varepsilon}e_{i\|j}(\varepsilon)(\boldsymbol{u}^{-1}(\varepsilon)) \rightharpoonup e_{i\|j} \quad \text{in} \quad L^2(\Omega).$$

Hence, by Lemma 3.3, $\overline{\boldsymbol{u}^{-1}} \in \boldsymbol{V}_F(\omega)$, and consequently $\overline{\boldsymbol{u}^{-1}} = \mathbf{o}$ *since* $\boldsymbol{V}_F(\omega) = \{\mathbf{o}\}$ *by assumption*. By the same lemma, $\boldsymbol{u}^{-1}$ is independent of x_3; hence

$$\boldsymbol{u}^{-1} = \mathbf{o}, \tag{5.36}$$

and the convergences (5.32) are established. These, combined with the relations (cf. definitions (2.6))

$$\partial_3 u_\alpha(\varepsilon) = 2\varepsilon e_{\alpha\|3}(\varepsilon) - \varepsilon\partial_\alpha u_3(\varepsilon) + 2\varepsilon\Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)$$

and the boundedness of the sequence $(\Gamma^\sigma_{\alpha 3}(\varepsilon))_{\varepsilon>0}$ in $\mathcal{C}^0(\bar\Omega)$ (cf. (3.5)), in turn yield the weak convergences (5.33). Again by Lemma 3.3,

$$\partial_3 e_{\alpha\|\beta} = -\rho_{\alpha\beta}(\boldsymbol{u}^{-1}) = 0,$$

by (5.36). The functions $e_{\alpha\|\beta}$ are thus independent of x_3.

(vii) *The weak convergences* (5.16), (5.18), (5.25), *and* (5.32), *are in fact strong, i.e.,*

$$e_{i\|j}(\varepsilon) \to e_{i\|j}(\varepsilon) \quad \text{in} \quad L^2(\Omega), \tag{5.37}$$

$$\varepsilon \boldsymbol{u}(\varepsilon) \to \boldsymbol{o} \quad \text{in} \quad \boldsymbol{H}^1(\Omega), \tag{5.38}$$

$$\gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)}) \to \overline{e_{\alpha\|\beta}} \quad \text{in} \quad L^2(\omega), \tag{5.39}$$

$$\overline{\dot{\boldsymbol{u}}(\varepsilon)} \to \dot{\boldsymbol{\xi}} \quad \text{in} \quad \dot{\boldsymbol{V}}^\sharp_M(\omega), \tag{5.40}$$

as $\varepsilon \to 0$.

Letting $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in the variational equations (2.9), and using inequalities (3.13) and (5.19), we obtain

$$\sum_{i,j} \|e_{i\|j}(\varepsilon) - e_{i\|j}\|^2_{0,\Omega} \leqslant c_4 \Lambda(\varepsilon), \tag{5.41}$$

where

$$\Lambda(\varepsilon) := L(\varepsilon)(\boldsymbol{u}(\varepsilon)) + \int_\Omega A^{ijkl}(\varepsilon)(e_{k\|l} - 2e_{k\|l}(\varepsilon)) e_{i\|j} \sqrt{g(\varepsilon)}\, \mathrm{d}x.$$

The strong convergences (4.17), (5.17) and the weak convergences (5.25), (5.33) together imply that

$$\begin{aligned} L(\varepsilon)(\boldsymbol{u}(\varepsilon)) &= \int_\omega \varphi^{\alpha\beta}(\varepsilon)\gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)})\, \mathrm{d}y \\ &\quad + \int_\Omega F^i(\varepsilon)\partial_3 u_i(\varepsilon)\, \mathrm{d}x \to \int_\omega \varphi^{\alpha\beta} \overline{e_{\alpha\|\beta}}\, \mathrm{d}y \end{aligned}$$

as $\varepsilon \to 0$, on the one hand. Relations (3.7)–(3.12), the weak convergences (5.16), and relations (5.20) and (5.34), together imply that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_\Omega A^{ijkl}(\varepsilon)(e_{k\|l} - 2e_{k\|l}(\varepsilon)) e_{i\|j} \sqrt{g(\varepsilon)}\, \mathrm{d}x \\ &\quad = -\frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau} e_{\sigma\|\tau} e_{\alpha\|\beta}\sqrt{a}\, \mathrm{d}y = -\int_\omega a^{\alpha\beta\sigma\tau}\overline{e_{\sigma\|\tau}}\,\overline{e_{\alpha\|\beta}}\sqrt{a}\, \mathrm{d}y, \end{aligned}$$

on the other hand. Hence

$$\Lambda(\varepsilon) \to \Lambda := \left\{ \int_\omega \varphi^{\alpha\beta} \overline{e_{\alpha\|\beta}}\, dy - \int_\omega a^{\alpha\beta\sigma\tau} \overline{e_{\sigma\|\tau}}\, \overline{e_{\alpha\|\beta}} \sqrt{a}\, dy \right\}. \tag{5.42}$$

For each $\varepsilon > 0, \overline{\boldsymbol{u}(\varepsilon)} \in \boldsymbol{V}(\omega)$; whence, by (5.26),

$$\int_\omega a^{\alpha\beta\sigma\tau} \overline{e_{\sigma\|\tau}} \gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)}) \sqrt{a}\, dy = \int_\omega \varphi^{\alpha\beta} \gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)})\, dy. \tag{5.43}$$

Letting $\varepsilon \to 0$ and using the weak convergences (5.25) in (5.43), we obtain

$$\Lambda = 0, \tag{5.44}$$

where Λ is the limit defined in (5.42). The strong convergences (5.37) then follow from (5.41), (5.42), and (5.44).

In Lemma 3.2 of Ciarlet, Lods and Miara (1996), we showed that

$$\begin{aligned} &\left\| \frac{1}{\varepsilon} \partial_3 e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) + \rho_{\alpha\beta}(\boldsymbol{v}) \right\|_{-1,\Omega} \\ &\quad \leqslant c_5 \left\{ \sum_i \|e_{i\|3}(\varepsilon)(\boldsymbol{v})\|_{0,\Omega} + \varepsilon \sum_\alpha \|v_\alpha\|_{0,\Omega} + \varepsilon \|v_3\|_{1,\Omega} \right\} \end{aligned}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, where $\rho_{\alpha\beta}(\boldsymbol{v})$ is defined as in (3.19). Letting $\boldsymbol{v} = \boldsymbol{u}^{-1}(\varepsilon)$ in this inequality ($\boldsymbol{u}^{-1}(\varepsilon)$) is defined in (5.35)) thus yields

$$\begin{aligned} &\|\partial_3 e_{\alpha\|\beta}(\varepsilon) + \rho_{\alpha\beta}(\boldsymbol{u}^{-1}(\varepsilon))\|_{-1,\Omega} \\ &\quad \leqslant c_5 \left\{ \sum_i \varepsilon \|e_{i\|3}(\varepsilon)\|_{0,\Omega} + \varepsilon \sum_\alpha \|\varepsilon u_\alpha(\varepsilon)\|_{0,\Omega} + \varepsilon \|\varepsilon u_3(\varepsilon)\|_{1,\Omega} \right\} \leqslant c_6 \varepsilon, \end{aligned} \tag{5.45}$$

by Step (ii). Since $e_{\alpha\|\beta}(\varepsilon) \to e_{\alpha\|\beta}$ in $L^2(\Omega)$ (cf. (5.37)),

$$\partial_3 e_{\alpha\|\beta}(\varepsilon) \to \partial_3 e_{\alpha\|\beta} = 0 \quad \text{in} \quad H^{-1}(\Omega), \tag{5.46}$$

by (5.34); therefore (5.45), (5.46) together imply that

$$\rho_{\alpha\beta}(\boldsymbol{u}^{-1}(\varepsilon)) \to 0 \quad \text{in} \quad H^{-1}(\Omega).$$

Another application of Lemma 3.3 thus shows that

$$\boldsymbol{u}^{-1}(\varepsilon) \to \mathbf{o} \quad \text{in} \quad \boldsymbol{H}^1(\Omega),$$

which is exactly the convergence (5.38).

By inequality (3.15), the strong convergences (5.37) imply in particular the strong convergences

$$\overline{e_{\alpha\|\beta}(\varepsilon)} \to \overline{e_{\alpha\|\beta}} \quad \text{in} \quad L^2(\omega),$$

which, combined with those of (5.24), yield the strong convergences (5.39). Hence $(\gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)}))_{\varepsilon>0}$ is a Cauchy sequence in $L^2(\omega)$. Since

$$\begin{aligned}
|\dot{\overline{\boldsymbol{u}(\varepsilon)}} - \dot{\overline{\boldsymbol{u}(\varepsilon')}}|_\omega^M &= |\overline{\boldsymbol{u}(\varepsilon)} - \overline{\boldsymbol{u}(\varepsilon')}|_\omega^M \\
&= \left\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)}) - \gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon')})\|_{0,\omega}^2 \right\}^{1/2},
\end{aligned}$$

the strong convergence (5.40) holds.

(viii) *The limit* $\dot{\boldsymbol{\xi}} \in \dot{\boldsymbol{V}}_M^\sharp(\omega)$ *found in Step* (vii) *satisfies the variational equations* (5.11), *which have a unique solution. Consequently, the convergence* (5.40) *holds for the whole family* $(\dot{\overline{\boldsymbol{u}(\varepsilon)}})_{\varepsilon>0}$.

Let $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$. Since $\overline{\boldsymbol{u}(\varepsilon)} \in \boldsymbol{V}(\omega)$,

$$\begin{aligned}
B_M(\overline{\boldsymbol{u}(\varepsilon)}, \boldsymbol{\eta}) &= \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\overline{\boldsymbol{u}(\varepsilon)})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \\
&\to \int_\omega a^{\alpha\beta\sigma\tau}\overline{e_{\alpha\|\beta}}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = L(\boldsymbol{\eta}),
\end{aligned}$$

as $\varepsilon \to 0$, by (5.26) and (5.39). But we also have $\dot{B}_M(\dot{\overline{\boldsymbol{u}(\varepsilon)}}, \dot{\boldsymbol{\eta}}) = B_M(\overline{\boldsymbol{u}(\varepsilon)}, \boldsymbol{\eta})$ and $\dot{L}(\dot{\boldsymbol{\eta}}) = L(\boldsymbol{\eta})$ by definition of the forms $\dot{B}$ and $\dot{L}$; thus

$$\dot{B}_M(\dot{\overline{\boldsymbol{u}(\varepsilon)}}, \dot{\boldsymbol{\eta}}) \to \dot{L}(\dot{\boldsymbol{\eta}}),$$

on the one hand. On the other,

$$\dot{B}_M(\dot{\overline{\boldsymbol{u}(\varepsilon)}}, \dot{\boldsymbol{\eta}}) \to \dot{B}_M^\sharp(\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\eta}}),$$

by (5.40). Hence

$$\dot{B}_M^\sharp(\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\eta}}) = \dot{L}(\dot{\boldsymbol{\eta}}) \quad \text{for all} \quad \dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}(\omega),$$

whence

$$\dot{B}_M^\sharp(\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\eta}}) = \dot{L}_M^\sharp(\dot{\boldsymbol{\eta}}) \quad \text{for all} \quad \dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{V}}_M^\sharp(\omega),$$

by definition of the extensions $\dot{B}_M^\sharp$ and $\dot{L}_M^\sharp$ and their continuity.

These variational equations have one and only one solution, for the space $\dot{V}^{\sharp}_M(\omega)$, the bilinear form $\dot{B}^{\sharp}_M$, and the linear form $\dot{L}^{\sharp}_M$, satisfy all the assumptions of the *Lax-Milgram lemma* (the ellipticity of the bilinear form relies in particular on the uniform positive definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$, established e.g. in Lemma 2.1 of Bernadou, Ciarlet and Miara (1994)).

(ix) *The weak convergences* (5.14) *and* (5.33) *are in fact strong, i.e.,*

$$\dot{u}(\varepsilon) \to \dot{u} \quad \text{in} \quad \dot{V}^{\sharp}_M(\Omega), \tag{5.47}$$

$$\partial_3 u_\alpha(\varepsilon) \to 0 \quad \text{in} \quad L^2(\Omega). \tag{5.48}$$

To establish (5.47), it suffices to show that $(\dot{u}(\varepsilon))_{\varepsilon>0}$ is a Cauchy sequence with respect to the norm $|\cdot|^M_\Omega$ of (4.14). By definition,

$$|\dot{u}(\varepsilon) - \dot{u}(\varepsilon')|^M_\Omega = |u(\varepsilon) - u(\varepsilon')|^M_\Omega$$

$$= \left\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\overline{u(\varepsilon)}) - \gamma_{\alpha\beta}(\overline{u(\varepsilon')})\|^2_{0,\omega} + \sum_i \|\partial_3 u_i(\varepsilon) - \partial_3 u_i(\varepsilon')\|^2_{0,\Omega} \right\}^{1/2};$$

hence, in view of the already established convergences (5.17), (5.33) and (5.39), it suffices in fact to establish the convergences (5.48).

Proving these is equivalent to showing that

$$\partial_3 u'(\varepsilon) \to \mathbf{o} \quad \text{in} \quad L^2(\Omega), \quad \text{where} \quad u'(\varepsilon) := (u_1(\varepsilon), u_2(\varepsilon), 0);$$

by Lemma 3.5, this is in turn equivalent to proving that

$$\partial_3 u'(\varepsilon) \to \mathbf{o} \quad \text{in} \quad H^{-1}(\Omega), \tag{5.49}$$

together with

$$e_{ij}(\partial_3 u'(\varepsilon)) \to 0 \quad \text{in} \quad H^{-1}(\Omega), \tag{5.50}$$

where

$$e_{ij}(v) := \tfrac{1}{2}(\partial_j v_i + \partial_i v_j).$$

By definition of the functions $e_{\alpha\|3}(\varepsilon)$ (cf. (2.6)),

$$\partial_3 u_\alpha(\varepsilon) = 2\varepsilon e_{\alpha\|3}(\varepsilon) - \varepsilon\partial_\alpha u_3(\varepsilon) + 2\varepsilon\Gamma^\sigma_{\alpha 3}(\varepsilon) u_\sigma(\varepsilon); \tag{5.51}$$

hence the boundedness of the sequences $(\Gamma^\sigma_{\alpha 3}(\varepsilon))_{\varepsilon>0}$ in $C^0(\bar{\Omega})$ (cf. (3.5)) and the convergences

$$\varepsilon e_{\alpha\|3}(\varepsilon) \to 0 \quad \text{in} \quad L^2(\Omega) \quad (\text{cf.}(5.37)),$$

$$\varepsilon u_i(\varepsilon) \to 0 \quad \text{in} \quad L^2(\Omega) \quad (\text{cf.}(5.32))$$

imply

$$\partial_3 u_\alpha(\varepsilon) \to 0 \quad \text{in} \quad H^{-1}(\Omega); \tag{5.52}$$

hence the convergence (5.49) is proved. To establish the convergences (5.50), we first notice that

$$e_{33}(\partial_3 \boldsymbol{u}'(\varepsilon)) = \partial_3 e_{33}(\boldsymbol{u}'(\varepsilon)) = 0. \tag{5.53}$$

Combining relations (5.51), the estimates (3.5), and the convergences (5.37), we next deduce that

$$\{\partial_3 u_\alpha(\varepsilon) + \varepsilon\partial_\alpha u_3(\varepsilon) + 2\varepsilon b_\alpha^\sigma u_\sigma(\varepsilon)\} \to 0 \quad \text{in} \quad L^2(\Omega),$$

whence

$$\partial_{33} u_\alpha(\varepsilon) + \varepsilon\partial_{\alpha 3} u_3(\varepsilon) + 2\varepsilon b_\alpha^\sigma \partial_3 u_\sigma(\varepsilon) \to 0 \quad \text{in} \quad H^{-1}(\Omega).$$

Since $\partial_3 u_3(\varepsilon) \to 0$ in $L^2(\Omega)$ (cf. (5.17)) and $\varepsilon u_\sigma(\varepsilon) \to 0$ in $L^2(\Omega)$ (cf. (5.32)), we have

$$2e_{\alpha 3}(\partial_3 \boldsymbol{u}'(\varepsilon)) = \partial_{33} u_\alpha(\varepsilon) \to 0 \quad \text{in} \quad H^{-1}(\Omega). \tag{5.54}$$

From the relations $e_{\alpha\|\beta}(\varepsilon) \to e_{\alpha\|\beta}(\varepsilon)$ in $L^2(\Omega)$ (cf. (5.37)) and $(\partial_3 e_{\alpha\|\beta} = 0$ (cf. (5.34)), we next infer that $\partial_3 e_{\alpha\|\beta}(\varepsilon) \to 0$ in $H^{-1}(\Omega)$, hence that

$$\{\partial_3 e_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) - \partial_3[\Gamma_{\alpha\beta}^p(\varepsilon) u_p(\varepsilon)]\} \to 0 \quad \text{in} \quad H^{-1}(\Omega), \tag{5.55}$$

by definition of the functions $e_{\alpha\|\beta}(\varepsilon)$ (cf. (2.5)). Because the functions $\Gamma_{\alpha\beta}^p(\varepsilon)$ are in $C^1(\bar{\Omega})$, the functions $\Gamma_{\alpha\beta}^p(\varepsilon) u_p(\varepsilon)$ are in $H^1(\Omega)$,

$$\partial_3[\Gamma_{\alpha\beta}^p(\varepsilon) u_p(\varepsilon)] = [\partial_3 \Gamma_{\alpha\beta}^p(\varepsilon)] u_p(\varepsilon) + \Gamma_{\alpha\beta}^p(\varepsilon) \partial_3 u_p(\varepsilon), \tag{5.56}$$

and

$$\Gamma_{\alpha\beta}^p(\varepsilon) \partial_3 u_p(\varepsilon) \to 0 \quad \text{in} \quad H^{-1}(\Omega), \tag{5.57}$$

since $\partial_3 u_p(\varepsilon) \to 0$ in $H^{-1}(\Omega)$ by (5.17) and (5.52). Finally, the conjunction of (3.4) and (5.32) shows that

$$[\partial_3 \Gamma_{\alpha\beta}^p(\varepsilon)] u_p(\varepsilon) \to 0 \quad \text{in} \quad L^2(\Omega). \tag{5.58}$$

From (5.55)–(5.58), we thus infer that

$$e_{\alpha\beta}(\partial_3 \boldsymbol{u}'(\varepsilon)) = \partial_3 e_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \to 0 \quad \text{in} \quad H^{-1}(\Omega). \tag{5.59}$$

Hence all relations (5.50) are established (cf. (5.53), (5.54), (5.59)).

(x) *The whole family $(\dot{\boldsymbol{u}}(\varepsilon))_{\varepsilon>0}$ converges strongly to $\dot{\boldsymbol{u}}$ in the space $\dot{\boldsymbol{V}}^{\sharp}_M(\Omega)$.*

We have shown that the whole family $(\overline{\boldsymbol{u}(\varepsilon)})_{\varepsilon>0}$ converges in the space $\dot{\boldsymbol{V}}^{\sharp}(\omega)$ (cf. Step (viii)), and that $\partial_3 \boldsymbol{u}(\varepsilon) \to \mathbf{o}$ in $\boldsymbol{L}^2(\Omega)$ for a subsequence (cf. (5.17) and (5.48)). Since the limit of this subsequence is unique, the whole family $(\partial_3 \boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ converges in $\boldsymbol{L}^2(\Omega)$. Hence the family $(\dot{\boldsymbol{u}}(\varepsilon))_{\varepsilon>0}$ is a Cauchy sequence in the Hilbert space $\dot{\boldsymbol{V}}^{\sharp}_M(\Omega)$, and the proof is complete. □

Remark 5.1. Theorem 5.1 also applies to a 'membrane' family of shells, in which case the space $\boldsymbol{V}^{\sharp}_M(\omega)$ is simply $H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega)$. The convergence result obtained in this fashion is however considerably 'weaker' than that obtained in Ciarlet and Lods (1996a); there, we do *not* assume that the forces are 'admissible', and in addition, we are able to establish the convergence of the family $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ in the space $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$. □

6. Compared Asymptotic Behaviors of the Solutions of the Three-Dimensional and Koiter's Shell Equations

The assumptions on the set ω and the mapping φ are as in Section 1 (we recall in particular that γ_0 denotes a subset of the boundary γ of ω with *length* $\gamma_0 > 0$). We consider the *same* family of shells, defined for all $0 < \varepsilon \leqslant \varepsilon_0$, as in Section 1. The *two-dimensional linear shell equations of Koiter* (cf. Koiter (1970)) then read as follows: Let $\xi^\varepsilon_i : \bar{\omega} \to \boldsymbol{R}$ denote the three covariant components of the displacement field of the points of the middle surface $S = \varphi(\bar{\omega})$ of the shell; this means that $\xi^\varepsilon_i(y)\boldsymbol{a}^i(y)$ is the displacement of the point $\varphi(y)$. Then the *unknown* $\boldsymbol{\xi}^\varepsilon = (\xi^\varepsilon_i)$ satisfies

$$\boldsymbol{\xi}^\varepsilon \in \boldsymbol{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \\ \eta_i = \partial_\nu \eta_3 = 0 \quad \text{on} \quad \gamma_0\}, \tag{6.1}$$

$$\varepsilon \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi}^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\xi}^\varepsilon)\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \\ = L^\varepsilon_K(\boldsymbol{\eta}) \quad \text{for all} \quad \boldsymbol{\eta} \in \boldsymbol{V}_K(\omega), \tag{6.2}$$

where (the functions $a, a^{\alpha\beta}, \Gamma^\sigma_{\alpha\beta}, b^\sigma_\alpha, b^\sigma_\alpha|_\beta, c_{\alpha\beta} \in \mathcal{C}^0(\bar{\omega})$ are defined in Section 1 and in (3.1), (3.2); ∂_ν is the outer normal derivative operator along γ):

$$a^{\alpha\beta\sigma\tau} = \frac{2\lambda\mu}{(\lambda + 2\mu)}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \tag{6.3}$$

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \tfrac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3, \tag{6.4}$$

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{\eta}) &= \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 + b^\sigma_\beta(\partial_\alpha\eta_\sigma - \Gamma^\tau_{\alpha\sigma}\eta_\tau) \\ &\quad + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\sigma_\alpha|_\beta\eta_\sigma - c_{\alpha\beta}\eta_3.\end{aligned} \tag{6.5}$$

We recall that, in (6.1)–(6.5), 2ε is the *thickness* of the shell, $\lambda > 0$ and $\mu > 0$ are the *Lamé constants*, assumed to be *independent* of ε, of its constituting material, the functions $a^{\alpha\beta\sigma\tau}$ are the contravariant components of the *elasticity tensor of the surface* S, the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are the covariant components of the *change of metric* and *change of curvature tensors of the surface* S, associated with a displacement field $\eta_i \boldsymbol{a}^i$ of the surface S. Finally, $L^\varepsilon_K : \boldsymbol{V}_K(\omega^\varepsilon) \to \boldsymbol{R}$ is a continuous linear form that takes into account the *applied forces*. For instance,

$$L^\varepsilon_K(\boldsymbol{\eta}) = \int_\omega \left\{ \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}\, \mathrm{d}x^\varepsilon_3 \right\} \eta_i \sqrt{a}\, \mathrm{d}y \quad \text{for all} \quad \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_K(\omega), \tag{6.6}$$

if the shells are subjected to 'usual' body forces, where $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ denote the contravariant components of their density (in this case, the linear form L^ε found in the corresponding three-dimensional problem is of the form (1.15)).

The *existence* and *uniqueness* of a solution to the variational problem (6.1), (6.2), which follows from the $\boldsymbol{V}_K(\omega)$-ellipticity of the bilinear form found in (6.2), was first established by Bernadou and Ciarlet (1976). A different, and more natural, proof was subsequently proposed by Ciarlet and Miara (1992) and combined with the previous one in Bernadou, Ciarlet and Miara (1994). Finally, a more 'intrinsic' proof, which applies to the more general situation where the mapping $\boldsymbol{\varphi} : \bar{\omega} \to \boldsymbol{R}^3$ is only in the space $\boldsymbol{W}^{2,\infty}(\omega)$, has been more recently given by Blouza and Le Dret (1994a, 1994b) (then the variational problem (6.2) has to be formulated differently, for the functions $b^\sigma_\alpha|_\beta$ are no longer defined in this case).

In Ciarlet and Lods (1996b), we have shown that, for '*flexural*' and '*membrane*' families of shells, the asymptotic behaviors as $\varepsilon \to 0$ of the averages across the thickness of the displacement found by solving the three-dimensional problem (1.11), (1.12) and of the displacement found by solving the two-dimensional equations (6.1), (6.2) are identical. The proofs relied on the convergence theorems established in Ciarlet, Lods and Miara (1996) and Ciarlet and Lods (1996a), compared with prior analyses of the asymptotic behavior of the solution of Koiter's model, due to Destuynder (1985) and Sanchez–Palencia (1989a, 1989b).

If thus remains to study the 'last' case of a '*generalized membrane*' *family of shells*, according to the definition of Section 4. As a first step, we analyze in the next theorem the asymptotic behavior as $\varepsilon \to 0$ of the solution of Koiter's equations in this case.

Remarkably, as in our analysis of the asymptotic behavior of the scaled three-dimensional unknown, we again need to assume that the applied forces are of a special kind, according to the following definition: Applied forces are '*admissible for Koiter's equations*' if, for each $\varepsilon > 0$, there exist functions $\varphi^{\alpha\beta}(\varepsilon) = \varphi^{\beta\alpha}(\varepsilon) \in L^2(\omega)$ such that the expression $L^\varepsilon_K(\boldsymbol{\eta})$ appearing in the right hand side of Koiter's equations (6.2) is of the form

$$L^\varepsilon_K(\boldsymbol{\eta}) = \varepsilon L_K(\varepsilon)(\boldsymbol{\eta}), \quad \text{with} \quad L_K(\varepsilon)(\boldsymbol{\eta}) := \int_\omega \varphi^{\alpha\beta}(\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta})\,\mathrm{d}y, \tag{6.7}$$

for all $\boldsymbol{\eta} \in V_K(\omega)$, *and* if there exist functions $\varphi^{\alpha\beta} = \varphi^{\beta\alpha} \in L^2(\omega)$ such that

$$\varphi^{\alpha\beta}(\varepsilon) \to \varphi^{\alpha\beta} \quad \text{in} \quad L^2(\omega), \tag{6.8}$$

as $\varepsilon \to 0$. Thus, if the applied forces are 'admissible for Koiter's equations', there exists a constant C_4 such that

$$|L_K(\varepsilon)(\boldsymbol{\eta})| \leqslant C_4|\boldsymbol{\eta}|^M_\omega \quad \text{for all} \quad \boldsymbol{\eta} \in V_K(\omega) \quad \text{and all} \quad 0 < \varepsilon \leqslant \varepsilon_0, \tag{6.9}$$

where $|\cdot|^M_\omega$ is the semi-norm defined in (4.5).

Remark 6.1. Inequality (6.9) is a natural extension to all $0 < \varepsilon \leqslant \varepsilon_0$ of inequality (4.24) of Caillerie and Sanchez–Palencia (1995a). □

THEOREM 6.1 *Let there be given a 'generalized membrane family' subjected to applied forces that are 'admissible for Koiter's equations'. The space* $V_K(\omega)$ *being defined as in* (6.1) *and the semi-norm* $|\cdot|^M_\omega$ *as in* (4.5), *define the space*

$$V^\sharp_{KM}(\omega) := \textit{completion of} \quad V_K(\omega) \quad \textit{for} \quad |\cdot|^M_\omega. \tag{6.10}$$

Let $\boldsymbol{\xi}^\varepsilon \in V_K(\omega)$ *denote for each* $0 < \varepsilon \leqslant \varepsilon_0$ *the solution of Koiter's equations* (6.2). *Then there exists* $\boldsymbol{\xi} \in V^\sharp_{KM}(\omega)$ *such that*

$$\boldsymbol{\xi}^\varepsilon \to \boldsymbol{\xi} \quad \textit{in} \quad V^\sharp_{KM}(\omega) \quad \textit{as} \quad \varepsilon \to 0. \tag{6.11}$$

Let $B^\sharp_{KM}$ *and* $L^\sharp_{KM}$ *denote the unique continuous extensions from* $V_K(\omega)$ *to* $V^\sharp_{KM}(\omega)$ *of the bilinear form* B_M *defined in* (5.4) *and of the linear form L defined in* (5.5), *where the functions* $\varphi^{\alpha\beta} \in L^2(\omega)$ *are those of* (6.8). *Then the limit* $\boldsymbol{\xi}$ *found in* (6.11) *is the unique solution to the equations*

$$B^\sharp_{KM}(\boldsymbol{\xi}, \boldsymbol{\eta}) = L^\sharp_{KM}(\boldsymbol{\eta}) \quad \textit{for all} \quad \boldsymbol{\eta} \in V^\sharp_{KM}(\omega). \tag{6.12}$$

Proof. The following proof is an elaboration over that of Caillerie and Sanchez–Palencia (1995a, Theorem 4.5), who established the weak convergence $\boldsymbol{\xi}^\varepsilon \rightharpoonup \boldsymbol{\xi}$ in $V_{KM}^\sharp(\omega)$; in particular, we extend their proof by establishing the *strong* convergence (6.11).

We first recall that there exists a constant c_0 such that (cf. e.g. Lemma 2.1 of Bernadou, Ciarlet and Miara (1994))

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leqslant c_0 a^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta} \tag{6.13}$$

for all $y \in \bar{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$. By virtue of assumption (6.7), the solution $\boldsymbol{\xi}^\varepsilon$ of Koiter's equations (6.1), (6.2) also satisfies

$$B_M(\boldsymbol{\xi}^\varepsilon, \boldsymbol{\eta}) + \varepsilon^2 B_F(\boldsymbol{\xi}^\varepsilon, \boldsymbol{\eta}) = L_K(\varepsilon)(\boldsymbol{\eta}) \quad \text{for all} \quad \boldsymbol{\eta} \in V_K(\omega), \tag{6.14}$$

where

$$B_F(\boldsymbol{\xi}, \boldsymbol{\eta}) := \frac{1}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\xi}^\varepsilon) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y. \tag{6.15}$$

Hence letting $\boldsymbol{\eta} = \boldsymbol{\xi}^\varepsilon$ in (6.14) yields

$$(|\boldsymbol{\eta}^\varepsilon|_\omega^M)^2 + \frac{1}{3} \sum_{\alpha,\beta} \|\varepsilon \rho_{\alpha\beta}(\boldsymbol{\xi}^\varepsilon)\|_{0,\omega}^2 \leqslant c_0 C_4 |\boldsymbol{\xi}^\varepsilon|_\omega^M, \tag{6.16}$$

by (6.9) and (6.13). Therefore the family $(\boldsymbol{\xi}^\varepsilon)_{\varepsilon>0}$ is bounded in the space $V_{KM}^\sharp(\omega)$ and the families $(\varepsilon \rho_{\alpha\beta}(\boldsymbol{\xi}^\varepsilon))_{\varepsilon>0}$ are bounded in $L^2(\omega)$; in particular then, there exists a subsequence, still denoted $(\boldsymbol{\xi}^\varepsilon)_{\varepsilon>0}$ for convenience, and there exist $\tilde{\boldsymbol{\xi}} \in V_{KM}^\sharp(\omega)$ and $\rho_{\alpha\beta}^{-1} \in L^2(\omega)$ such that (as usual $\rightharpoonup$ denotes weak convergence)

$$\boldsymbol{\xi}^\varepsilon \rightharpoonup \tilde{\boldsymbol{\xi}} \quad \text{in} \quad V_{KM}^\sharp(\omega), \tag{6.17}$$

$$\varepsilon \rho_{\alpha\beta}(\boldsymbol{\xi}^\varepsilon) \rightharpoonup \rho_{\alpha\beta}^{-1} \quad \text{in} \quad L^2(\omega). \tag{6.18}$$

Fix $\boldsymbol{\eta} \in V_K(\omega)$ in (6.14) and let $\varepsilon \to 0$; then assumption (6.8) and the weak convergences (6.17), (6.18) yield $B_{KM}^\sharp(\tilde{\boldsymbol{\xi}}, \boldsymbol{\eta}) = L(\boldsymbol{\eta})$. Since the space $V_K(\omega)$ is dense in $V_{KM}^\sharp(\omega)$, we conclude that $B_{KM}^\sharp(\tilde{\boldsymbol{\xi}}, \boldsymbol{\eta}) = L_{KM}^\sharp(\boldsymbol{\eta})$ for all $\boldsymbol{\eta} \in V_{KM}^\sharp(\omega)$. Hence

$$\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi},$$

where $\boldsymbol{\xi}$ is the unique solution to the equations (6.12) (the Lax-Milgram lemma applies to these), and the weak convergence

$$\boldsymbol{\xi}^\varepsilon \rightharpoonup \boldsymbol{\xi} \quad \text{in} \quad V_{KM}^\sharp(\omega) \quad \text{as} \quad \varepsilon \to 0, \tag{6.19}$$

holds for the whole family.

By virtue of inequality (6.13) and by the definitions of the norm $|\cdot|_\omega^M$, of the bilinear form B_M, and of its extension $B_{KM}^\sharp$, establishing the strong convergence (6.11) is equivalent to establishing the convergence

$$B_{KM}^\sharp(\boldsymbol{\xi}^\varepsilon - \boldsymbol{\xi}, \boldsymbol{\xi}^\varepsilon - \boldsymbol{\xi}) \to 0,$$

which itself follows from the relations

$$0 \leqslant B_{KM}^\sharp(\boldsymbol{\xi}^\varepsilon - \boldsymbol{\xi}, \boldsymbol{\xi}^\varepsilon - \boldsymbol{\xi}) = B_M(\boldsymbol{\xi}^\varepsilon, \boldsymbol{\xi}^\varepsilon) - 2B_{KM}^\sharp(\boldsymbol{\xi}^\varepsilon, \boldsymbol{\xi}) + B_{KM}^\sharp(\boldsymbol{\xi}, \boldsymbol{\xi}),$$

$$B_M(\boldsymbol{\xi}^\varepsilon, \boldsymbol{\xi}^\varepsilon) \leqslant L_K(\varepsilon)(\boldsymbol{\xi}^\varepsilon) \text{ (cf. (6.14))},$$

$$\begin{aligned} L_K(\varepsilon)(\boldsymbol{\xi}^\varepsilon) &= L(\boldsymbol{\xi}^\varepsilon) + \int_\omega (\varphi^{\alpha\beta}(\varepsilon) - \varphi^{\alpha\beta})\gamma_{\alpha\beta}(\boldsymbol{\xi}^\varepsilon)\,\mathrm{d}y \\ &\to L_{KM}^\sharp(\boldsymbol{\xi}) \quad \text{as} \quad \varepsilon \to 0 \text{ (cf. (6.8) and (6.19))}, \end{aligned}$$

$$B_{KM}^\sharp(\boldsymbol{\xi}^\varepsilon, \boldsymbol{\xi}) \to B_{KM}^\sharp(\boldsymbol{\xi}, \boldsymbol{\xi}) \text{ (cf. (6.19))},$$

$$B_{KM}^\sharp(\boldsymbol{\xi}, \boldsymbol{\xi}) = L_{KM}^\sharp(\boldsymbol{\xi}) \text{ (as shown } \textit{supra}\text{)}.$$

Hence the proof is complete. □

Combining this theorem with the convergence theorem established in Section 5 allows us to show that, *for 'generalized membrane families of the first kind' at least, the asymptotic behaviors as $\varepsilon \to 0$ of the average of the three-dimensional solution and of the solution of Koiter's model are the same.*

THEOREM 6.2 *Let there be given a 'generalized membrane family of the first kind' subjected to applied forces that are 'admissible', according to the definitions of Section 4. Let $\boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon)$ denote the solution of the three-dimensional problem* (1.11), (1.12), *and let $\boldsymbol{\xi}^\varepsilon \in \boldsymbol{V}_K(\omega)$ denote the solution of Koiter's equations* (6.1), (6.2) *with a right-side of the form* (6.7), *the functions $\varphi^{\alpha\beta}(\varepsilon)$ being those of* (4.16).

Then the completions $\boldsymbol{V}_M^\sharp(\omega)$ of (5.1) *and $\boldsymbol{V}_{KM}^\sharp(\omega)$ of* (6.10) *coincide; furthermore,*

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{u}^\varepsilon\,\mathrm{d}x_3^\varepsilon \to \boldsymbol{\xi} \quad \textit{in} \quad \boldsymbol{V}_M^\sharp(\omega), \tag{6.20}$$

$$\boldsymbol{\xi}^\varepsilon \to \boldsymbol{\xi} \quad \textit{in} \quad \boldsymbol{V}_M^\sharp(\omega), \tag{6.21}$$

as $\varepsilon \to 0$, where $\boldsymbol{\xi}$ is the unique solution to equations (5.6).

Proof. It is classical that the space $V_K(\omega)$ of (6.1) is dense in the space $V(\omega)$ of (4.10), the latter being equipped with $\|\cdot\|_{1,\omega}$. Since there exists c such that

$$|\eta|_\omega^M \leqslant c\|\eta\|_{1,\omega} \quad \text{for all} \quad \eta \in V(\omega),$$

the two completions $V_M^\sharp(\omega)$ and $V_{KM}^\sharp(\omega)$ are identical (likewise, $B_M^\sharp = B_{KM}^\sharp$ and $L_M^\sharp = L_{KM}^\sharp$). The convergence (6.20) immediately follows from (5.2), since

$$\overline{u(\varepsilon)} = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u^\varepsilon \, \mathrm{d}x_3^\varepsilon,$$

by (2.1). The convergence (6.21) has been established in (6.11). □

7. Conclusions and Comments

7.1 A *major conclusion* is that the asymptotic behavior as $\varepsilon \to 0$ of a family of shells of thickness 2ε, with the same middle surface $S = \varphi(\bar{\omega})$, clamped along a portion of their lateral face whose trace on S is $\varphi(\gamma_0)$ with $\gamma_0 \subset \gamma = \partial\omega$, and subjected to applied forces that are $O(1)$ with respect to ε, critically depends on *whether the space of inextensional displacements*

$$V_F(\omega) = \{\eta = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);$$
$$\eta_i = \partial_\nu \eta_3 = 0 \quad \text{on} \quad \gamma_0, \gamma_{\alpha\beta}(\eta) = 0 \quad \text{in} \quad \omega\}$$

contains non-zero functions or not.

If $V_F(\omega) \neq \{\mathbf{o}\}$, the family is a *'flexural'* one. Then the averages across the thickness of the three-dimensional solution, *times* ε^2, converge in $\boldsymbol{H}^1(\omega)$ as $\varepsilon \to 0$ towards the solution of the *two-dimensional equations of a 'flexural' shell*, which is posed over the space $V_F(\omega)$; cf. Ciarlet, Lods and Miara (1996) (see also the Introduction).

If $V_F(\omega) = \{\mathbf{o}\}$, the family is either a *'membrane'* or a *'generalized membrane'* one. A *'membrane family'* occurs when *the surface S is regular and elliptic and* $\gamma_0 = \gamma$ (cf. Theorems 4.1–4.3). Then the averages across the thickness of the three-dimensional solution converges in $H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ as $\varepsilon \to 0$ towards the solution of the *two-dimensional equations of a 'membrane' shell*, which is posed over the space

$$V_M(\omega) = H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega);$$

cf. Ciarlet and Lods (1996a) (see also the Introduction and Section 4).

If $V_F(\omega) = \{\mathbf{o}\}$ and the family is *not* a 'membrane' one, it is called a *'generalized membrane family'*. Under the *additional* assumption that *the applied forces are 'admissible'* in the sense of relations (4.16), (4.17) (these express in particular that the applied forces are $O(1)$ with respect to ε), *we have shown here that the averages across the thickness of the three-dimensional solutions*, viz.,

$$\overline{\boldsymbol{u}(\varepsilon)} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}(\varepsilon)\,\mathrm{d}x_3 = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{u}^\varepsilon\,\mathrm{d}x_3^\varepsilon,$$

converge as $\varepsilon \to 0$ (cf. (5.2) or (5.8)) *toward the solution of a variational problem* (cf. (5.6) or (5.11)). More specifically, we have shown that these convergences occur in '*abstract Hilbert spaces*', viz.,

$$\boldsymbol{V}_M^\sharp(\omega) = \text{completion of} \quad \boldsymbol{V}(\omega) \quad \text{for} \quad |\cdot|_\omega^M,$$

$$\dot{\boldsymbol{V}}_M^\sharp(\omega) = \text{completion of} \quad \boldsymbol{V}(\omega)/\boldsymbol{V}_0(\omega) \quad \text{for} \quad |\cdot|_\omega^M,$$

depending on whether the family is of the *first, or second*, kind, where

$$\boldsymbol{V}(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{H}^1(\omega); \boldsymbol{\eta} = \boldsymbol{o} \quad \text{on} \quad \gamma_0\},$$

$$\boldsymbol{V}_0(\omega) = \{\boldsymbol{\eta} \in \boldsymbol{V}(\omega); \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \quad \text{in} \quad \omega\},$$

$$|\boldsymbol{\eta}|_\omega^M = \left\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2\right\}^{1/2}.$$

In so doing, *we have justified the formal asymptotic approach of* Caillerie and Sanchez–Palencia (1995b).

The limit problems found in this fashion, which are posed over the same abstract spaces, constitute the **two-dimensional equations of a 'generalized membrane shell'**, *of the first, or second, kind.* These equations are '*two-dimensional*' because the problems are posed over completions of Sobolev spaces of functions defined on the two-dimensional set ω; the shell is a '*generalized membrane shell*' because the bilinear form, $B_M^\sharp$ or $\dot{B}_M^\sharp$, occurring in these equations are extensions of the bilinear form B_M of (5.4), or $\dot{B}_M$ of (5.9), and B_M is classically the bilinear form of a 'membrane shell' (cf. e.g. Niordson (1985, Equation (10.3))).

7.2 The definitions of 'flexural', 'membrane', or 'generalized membrane', families only depend on the 'geometry' of the middle surface S of the shell (*via* the mapping φ) and on the set γ_0. This reflects the observation that *the asymptotic behavior of the three-dimensional solution is entirely governed by* S *and* γ_0.

7.3 An inspection of the proof of Theorem 5.1 shows that it in fact provides further worthy information about the asymptotic behavior of the scaled three-dimensional solutions $\boldsymbol{u}(\varepsilon)$ and their averages $\overline{\boldsymbol{u}(\varepsilon)}$, namely,

$$e_{i\|j}(\varepsilon) \to e_{i\|j} \quad \text{in} \quad L^2(\Omega),$$

$$\gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)}) \to \overline{e_{\alpha\|\beta}} \quad \text{in} \quad L^2(\omega),$$

$$\partial_3 \boldsymbol{u}(\varepsilon) \to \mathbf{o} \quad \text{in} \quad \boldsymbol{L}^2(\Omega),$$

$$\varepsilon \boldsymbol{u}(\varepsilon) \to \mathbf{o} \quad \text{in} \quad \boldsymbol{H}^1(\Omega).$$

A noticeable feature of these convergences is that they occur in spaces that are more 'decent' than the somewhat 'exotic' spaces $\boldsymbol{V}_M^\sharp(\omega)$ or $\dot{\boldsymbol{V}}_M^\sharp(\omega)$.

For a 'flexural' or a 'membrane' family, the three-dimensional limit was found to be independent of x_3. In a sense, the convergence $\partial_3 \boldsymbol{u}(\varepsilon) \to \mathbf{o}$ in $\boldsymbol{L}^2(\Omega)$ is a 'weaker' expression of the same property.

The convergence $\varepsilon \boldsymbol{u}(\varepsilon) \to \mathbf{o}$ in $\boldsymbol{H}^1(\Omega)$ constitutes an improvement over Ciarlet, Lods and Miara (1996), whose convergence theorem (which still applies when $\boldsymbol{V}_F(\omega) = \{\mathbf{o}\}$ as here) only predicts that $\mathbf{u}(\varepsilon) \to \mathbf{o}$ in $\boldsymbol{H}^1(\Omega)$.

7.4 In Ciarlet and Lods (1996b), we have justified Koiter's equations when they are applied to a 'flexural' or 'membrane' shell. Here, *we have justified the two-dimensional Koiter's shell equations in a new situation, namely when they are applied to a 'generalized membrane shell of the first kind' and the forces are 'admissible'*. For relations (6.20), (6.21) together show that in this case *the averages* $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{u}^\varepsilon \, \mathrm{d}x_3^\varepsilon$ *found by solving the three-dimensional problem and the solution* $\boldsymbol{\xi}^\varepsilon$ *of Koiter's two-dimensional equations have again the same principal part* (the solution of the limit variational problem) *in the space* $\boldsymbol{V}_M^\sharp(\omega)$ as $\varepsilon \to 0$.

7.5 The limit variational problem satisfies all the assumptions of the *Lax-Milgram lemma*. It otherwise possesses *two unusual features*: First, *the space* ($\boldsymbol{V}_M^\sharp(\omega)$ *or* $\dot{\boldsymbol{V}}_M^\sharp(\omega)$) *in which its solution is sought may not necessarily be a space of distributions*. Second, *its solution may no longer exist if the data undergo arbitrarily small, yet arbitrarily smooth, perturbations*! Such a variational problem falls in the category of '*sensitive problems*' recently introduced and studied by Lions and Sanchez–Palencia (1994, 1995).

7.6 While forces applied to a 'flexural' or 'membrane' family of shells were of the most general form, body forces (to fix ideas) applied to a 'generalized membrane family' can no longer be accounted for by a linear form of the form (1.15) with *arbitrary* functions $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$. They must be 'admissible' in the sense of Section 4, essentially in order that the associated linear form be (uniformly) continuous with respect to the norm $|\cdot|_\omega^M$. The same kind of restriction was also encountered by Lions and Sanchez–Palencia (1994, 1995) in their analysis of 'sensitive' problems.

7.7 As we shall see in Ciarlet and Lods (1996d), a noteworthy feature of the present asymptotic analysis is that, without much further ado, it also provides valuable information about '*membrane effects in flexural shells*', i.e. those occurring in a 'flexural' family (at least when the applied forces are again assumed to be 'admissible'). Such 'membrane effects', which were first identified in the formal approach of Caillerie and Sanchez–Palencia (1995b), are thereby also justified.

In Ciarlet and Lods (1996d), we shall also discuss the important special case of a *plate* (which is a 'flexural' shell), and conclude our comparison, begun in Ciarlet, Lods and Miara (1996), with the 'traditional' asymptotic analysis of plates (cf. Ciarlet (1990)), where different scalings and assumptions are allowed on the 'horizontal' and 'vertical' components of the displacement and forces.

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