SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

SINGULAR PERTURBATIONS OF A GENERAL BOUNDARY VALUE PROBLEM*

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Abstract. This paper treats the boundary problem

$$y' = A(t)y + B(t)z,$$

$$\varepsilon z' = C(t)y + D(t)z,$$

$$M(\varepsilon)\begin{pmatrix} y(0, \varepsilon) \\ z(0, \varepsilon) \end{pmatrix} + N(\varepsilon)\begin{pmatrix} y(1, \varepsilon) \\ z(1, \varepsilon) \end{pmatrix} = \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}.$$

The main difference of our approach and that of earlier writers is that we are able to reduce the system to a purely diagonalized form under even less stringent assumptions.

1. Introduction. Consider the boundary value problem consisting of $m + n$ equations

$$(1) \qquad \begin{aligned} y' &= A(t)y + B(t)z, \\ \varepsilon z' &= C(t)y + D(t)z, \end{aligned}$$

and $m + n$ boundary conditions

$$(2) \qquad M(\varepsilon)\begin{pmatrix} y(0, \varepsilon) \\ z(0, \varepsilon) \end{pmatrix} + N(\varepsilon)\begin{pmatrix} y(1, \varepsilon) \\ z(1, \varepsilon) \end{pmatrix} = \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix},$$

on the interval $0 \leqq t \leqq 1$. Here y, c_1 and z, c_2 are respectively real m-dimensional and n-dimensional vectors and A, B, C, D, M, N are square matrices of appropriate orders. We assume that A, B, C, D are continuous functions for $0 \leqq t \leqq 1$ and $M(\varepsilon) = M(0) + 0(\varepsilon)$, $N(\varepsilon) = N(0) + 0(\varepsilon)$, $c_i(\varepsilon) = c_i(0) + 0(\varepsilon)$, $i = 1, 2$, where $0(\varepsilon)$ is a standard order symbol referring to $\varepsilon \to 0+$.

Harris [4], [5] and, more recently, O'Malley [6] have analyzed similar boundary value problems involving powers of ε. Their approach is to reduce (1) to a simpler form:

$$v' = (A - BD^{-1}C + 0(\varepsilon))v + 0(\varepsilon)w,$$

$$\varepsilon w' = 0(\varepsilon)v + (Q^{-1}DQ + 0(\varepsilon))w,$$

by means of the transformation

$$\begin{pmatrix} y \\ z \end{pmatrix} = U(t, \varepsilon)\begin{pmatrix} v \\ w \end{pmatrix}$$

with

$$U(t, \varepsilon) = \begin{pmatrix} I_m & \varepsilon BD^{-1} \\ -QD^{-1}C & Q \end{pmatrix}$$

* Received by the editors September 8, 1971, and in revised form November 17, 1971.

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and Q such that

$$Q^{-1}DQ = \operatorname{diag}[D_-, D_+],$$

where the eigenvalues of the matrices D_- and D_+ have, respectively, negative and positive real parts for $0 \leqq t \leqq 1$. To carry out this transformation, Harris and O'Malley assume that $U(t, \varepsilon)$ and hence BD^{-1}, $D^{-1}C$ and Q are continuously differentiable. Such a Q definitely exists if D is assumed continuously differentiable and its eigenvalues have nonzero real parts for $0 \leqq t \leqq 1$ (cf. [2]). However, as shown by the counterexample in [2], such a Q may not exist if D is continuous but not continuously differentiable.

The main purpose of this paper is to weaken the assumptions of Harris and O'Malley to:

(I) A, B, C, D are continuous and all eigenvalues of D have nonzero real part for $0 \leqq t \leqq 1$.

We shall show in the next section that under assumption (I) we can reduce (1) to a purely diagonalized form

$$v' = (A - BT)v,$$
$$\varepsilon w' = (D + \varepsilon TB)w,$$

by using the transformation

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} I_m & -\varepsilon S \\ -T & I_n + \varepsilon TS \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

in place of the transformation indicated above, where T, S are bounded solutions of

$$T' = \varepsilon^{-1}DT - TA + TBT - \varepsilon^{-1}C,$$
$$S' = [A - BT]S - \varepsilon^{-1}S[D + \varepsilon TB] - \varepsilon^{-1}B,$$

respectively.

The main result is given in the end as a theorem.

2. Reduction into block diagonalization. From our assumption on $D(t)$ it follows that $D(t)$ is invertible and has the constant number p, $0 \leqq p \leqq n$, of eigenvalues with negative real part for $0 \leqq t \leqq 1$. Moreover, since the interval $[0, 1]$ is compact, there exists $\mu > 0$ such that the real part of every eigenvalue of $D(t)$ has absolute value $\geqq 2\mu$. Therefore, by Lemma 1 in [2], the linear equation

$$\varepsilon z' = D(t)z \tag{3}$$

has a fundamental matrix $Z(t) = Z(t, \varepsilon)$ satisfying the inequalities

$$\begin{aligned} |Z(t)PZ^{-1}(s)| &\leqq L \exp(-\mu(t - s)/\varepsilon) \quad \text{for} \quad 1 \geqq t \geqq s \geqq 0, \\ |Z(t)(I_n - P)Z^{-1}(s)| &\leqq L \exp(-\mu(s - t)/\varepsilon) \quad \text{for} \quad 1 \geqq s \geqq t \geqq 0, \end{aligned} \tag{4}$$

where L is a positive constant independent of ε and P is the projection

$$P = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix},$$

where I_p is the unit $p \times p$ matrix.

Since $A(t)$ is continuous and therefore bounded on $[0, 1]$, there exists $\sigma > 0$ such that the norm $\|A(t)\| \leqq \sigma$ and the equation

$$y' = A(t)y \tag{5}$$

has a fundamental matrix $Y(t)$ such that

$$|Y(t)Y^{-1}(s)| \leqq \exp(\sigma|t - s|) \quad \text{for} \quad 0 \leqq t, s \leqq 1. \tag{6}$$

Then we have the following result.

LEMMA. *There exists* $\varepsilon_0 > 0$ *such that the equations*

$$T' = \varepsilon^{-1}D(t)T - TA(t) + TB(t)T - \varepsilon^{-1}C(t), \tag{7}$$

$$S' = [A(t) - B(t)T(t, \varepsilon)]S - \varepsilon^{-1}S[D(t) + \varepsilon T(t, \varepsilon)B(t)] - \varepsilon^{-1}B(t), \tag{8}$$

have respectively solutions $T = T(t, \varepsilon)$, $S = S(t, \varepsilon)$ *which are uniformly bounded for* $0 \leqq t \leqq 1$ *and* $0 < \varepsilon \leqq \varepsilon_0$.

Moreover, for $0 < t < 1$, $T(t, 0) = \lim_{\varepsilon \to 0} T(t, \varepsilon) = D^{-1}(t)C(t)$ *and* $S(t, 0) = \lim_{\varepsilon \to 0} S(t, \varepsilon) = -B(t)D^{-1}(t)$.

Furthermore, the change of variables

$$w = z + T(t, \varepsilon)y, \quad v = y + \varepsilon S(t, \varepsilon)w \tag{9}$$

transforms (1) *into the block diagonal form*:

$$\begin{aligned} v' &= [A(t) - B(t)T(t, \varepsilon)]v, \\ \varepsilon w' &= [D(t) + \varepsilon T(t, \varepsilon)B(t)]w. \end{aligned} \tag{10}$$

Proof. The existence of a bounded solution $T(t, \varepsilon)$ of (7) follows from the theorem in [1]. Clearly, $\lim_{\varepsilon \to 0} T(t, \varepsilon) = D^{-1}(t)C$ for $0 < t < 1$.

To obtain a bounded solution of (8), let $V(t, \varepsilon)$ be a fundamental matrix of the first equation of (10). Since $[0, 1]$ is compact, there exists $\tilde{\sigma} > 0$ such that $\|A(t) - B(t)T(t, \varepsilon)\| \leqq \tilde{\sigma}$ which implies

$$|V(t, \varepsilon)V^{-1}(s, \varepsilon)| \leqq \exp(\tilde{\sigma}|t - s|) \quad \text{for} \quad 0 \leqq t, s \leqq 1.$$

Also, by Theorem 2 in [3], the second equation of (10) has, for all sufficiently small $\varepsilon > 0$, a fundamental matrix $W(t, \varepsilon)$ such that

$$\begin{aligned} |W(t, \varepsilon)PW^{-1}(s, \varepsilon)| &\leqq \tilde{L} \exp(-\mu(t - s)/2\varepsilon) \quad \text{for} \quad 1 \geqq t \geqq s \geqq 0, \\ |W(t, \varepsilon)(I_n - P)W^{-1}(s, \varepsilon)| &\leqq \tilde{L} \exp(-\mu(s - t)/2\varepsilon) \quad \text{for} \quad 1 \geqq s \geqq t \geqq 0, \end{aligned} \tag{4'}$$

where $\tilde{L}$ is a positive constant independent of ε.

It can easily be verified by differentiation that

$$\begin{aligned} S(t, \varepsilon) = &\int_0^t V(t, \varepsilon)V^{-1}(s, \varepsilon)[-\varepsilon^{-1}B(s)]W(s, \varepsilon)(I_n - P)W^{-1}(t, \varepsilon)\, ds \\ &- \int_t^1 V(t, \varepsilon)V^{-1}(s, \varepsilon)[-\varepsilon^{-1}B(s)]W(s, \varepsilon)PW^{-1}(t, \varepsilon)\, ds \end{aligned}$$

is a solution of (8), and for $0 < \varepsilon < \mu/2\tilde{\sigma}$,

$$\|S(t, \varepsilon)\| \leqq \tilde{L}\varepsilon^{-1}\|B\|\left\{\int_0^t \exp[(\tilde{\sigma} - \mu/2\varepsilon)(t - s)]\, ds + \int_t^1 \exp[(\tilde{\sigma} - \mu/2\varepsilon)(s - t)]\, ds\right\}$$
$$\leqq 2\tilde{L}\|B\|(\mu - 2\varepsilon\tilde{\sigma})^{-1}.$$

Thus $S(t, \varepsilon)$ is bounded, and moreover, $\lim_{\varepsilon\to 0} S(t, \varepsilon) = -B(t)D^{-1}(t)$ for $0 < t < 1$. Consequently, the change of variables (9) transforms the system (1) into (10).

3. Theorem and proof. Applying (9) now to the boundary conditions (2), we obtain

$$\tilde{M}(\varepsilon)\begin{pmatrix} v(0, \varepsilon) \\ w(0, \varepsilon) \end{pmatrix} + \tilde{N}(\varepsilon)\begin{pmatrix} v(1, \varepsilon) \\ w(1, \varepsilon) \end{pmatrix} = \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}, \tag{11}$$

where

$$\tilde{M}(\varepsilon) = M(\varepsilon)H(0, \varepsilon), \quad \tilde{N}(\varepsilon) = N(\varepsilon)H(1, \varepsilon)$$

and

$$H(t, \varepsilon) = \begin{pmatrix} I_m & -\varepsilon S(t, \varepsilon) \\ -T(t, \varepsilon) & I_n + \varepsilon T(t, \varepsilon)S(t, \varepsilon) \end{pmatrix}.$$

Clearly, $H(t, \varepsilon)$ is nonsingular for all small ε for which $S(t, \varepsilon)$ and $T(t, \varepsilon)$ exist.

We have now transformed the original problem (1), (2) into a more tractable problem (10), (11), which we treat in the same way as O'Malley, except for a modification due to $D(t)$ not having block diagonal form. One can readily verify by differentiation that the functions

$$\begin{pmatrix} v(t, \varepsilon) \\ w(t, \varepsilon) \end{pmatrix} = \begin{pmatrix} V(t, \varepsilon) & 0 \\ 0 & W(t, \varepsilon)PW^{-1}(0, \varepsilon) + W(t, \varepsilon)(I_n - P)W^{-1}(1, \varepsilon) \end{pmatrix}\begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon) \end{pmatrix} \tag{12}$$

are a solution of (10), where α_1, α_2 are arbitrary constant vectors. It only remains to choose α_1, α_2 to satisfy the boundary conditions (11). Substitution into (11) yields

$$\Delta(\varepsilon)\begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon) \end{pmatrix} = \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix},$$

where

$$\begin{aligned} \Delta(\varepsilon) = {} & \tilde{M}(\varepsilon)\operatorname{diag}[V(0, \varepsilon), W(0, \varepsilon)PW^{-1}(0, \varepsilon) + W(0, \varepsilon)(I_n - P)W^{-1}(1, \varepsilon)] \\ & + \tilde{N}(\varepsilon)\operatorname{diag}[V(1, \varepsilon), W(1, \varepsilon)PW^{-1}(0, \varepsilon) + W(1, \varepsilon)(I_n - P)W^{-1}(1, \varepsilon)]. \end{aligned} \tag{13}$$

If the inverse $\Delta^{-1}(\varepsilon)$ exists, then

$$\begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon) \end{pmatrix} = \Delta^{-1}(\varepsilon)\begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}$$

and

$$
(14)\quad \begin{pmatrix} v(t,\varepsilon) \\ w(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} V(t,\varepsilon) & 0 \\ 0 & W(t,\varepsilon)PW^{-1}(0,\varepsilon) + W(t,\varepsilon)(I_n - P)W^{-1}(1,\varepsilon) \end{pmatrix} \cdot \Delta^{-1}(\varepsilon) \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}
$$

is a solution of the boundary value problem (10), (11).

Let us analyze $\Delta(\varepsilon)$. Since $W(1,\varepsilon)PW^{-1}(0,\varepsilon) = 0(\varepsilon)$ and

$$
W(0,\varepsilon)(I_n - P)W^{-1}(1,\varepsilon) = 0(\varepsilon)
$$

as $\varepsilon \to 0$, we can express (13) as

$$
\Delta(\varepsilon) = \tilde{M}(0)\operatorname{diag}[V(0), W(0)PW^{-1}(0)] + \tilde{N}(0)\operatorname{diag}[V(1), W(1)(I_n - P)W^{-1}(1)] + 0(\varepsilon),
$$

where

$$
V(0) = \lim_{\varepsilon\to 0} V(0,\varepsilon), \quad W(0) = \lim_{\varepsilon\to 0} W(0,\varepsilon), \quad \text{etc.}
$$

This is equivalent to

$$
\Delta(\varepsilon) = (\tilde{M}_1(0)V(0) + \tilde{N}_1(0)V(1) : \tilde{M}_2(0)W(0)PW^{-1}(0) + \tilde{N}_2(0)W(1)(I_n - P)W^{-1}(1)) + 0(\varepsilon)
$$

if we partition $\tilde{M}, \tilde{N}$ as

$$
\tilde{M}(\varepsilon) = (\tilde{M}_1(\varepsilon):\tilde{M}_2(\varepsilon)), \quad \tilde{N}(\varepsilon) = (\tilde{N}_1(\varepsilon):\tilde{N}_2(\varepsilon)),
$$

such that $\tilde{M}_1, \tilde{N}_1$ and V have the same number m of columns, and $\tilde{M}_2, \tilde{N}_2$ and W have the same number n of columns. Therefore, for all sufficiently small ε, the inverse $\Delta^{-1}(\varepsilon)$ exists if we make the following assumption:

(II) The matrix

$$
\begin{aligned}
\Delta(0) &= \tilde{M}(0)\operatorname{diag}[V(0), W(0)PW^{-1}(0)] + \tilde{N}(0)\operatorname{diag}[V(1), W(1)(I_n - P)W^{-1}(1)] \\
&= (\tilde{M}_1(0)V(0) + \tilde{N}_1(0)V(1) : \tilde{M}_2(0)W(0)PW^{-1}(0) + \tilde{N}_2(0)W(1)(I_n - P)W^{-1}(1))
\end{aligned}
$$

is nonsingular.

We note that $\Delta(0)$ may be checked immediately since it depends only on the leading coefficients of the problem (1), (2). However, if it were singular, then a higher order analysis of $\Delta(\varepsilon)$ would be necessary to see if it could be nonsingular.

We next analyze the form of the solution within $[0, 1]$ as $\varepsilon \to 0$. In view of (4′) it follows from (12) that for $0 < t < 1$,

$$
x(t) \equiv \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \equiv \lim_{\varepsilon\to 0} \begin{pmatrix} v(t,\varepsilon) \\ w(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} \lim_{\varepsilon\to 0} V(t,\varepsilon) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1(0) \\ \alpha_2(0) \end{pmatrix} = \begin{pmatrix} \lim_{\varepsilon\to 0} V(t,\varepsilon)\,\alpha_1(0) \\ 0 \end{pmatrix},
$$

that is, $x(t)$ satisfies the degenerate system of (10):

$$x_1' = [A(t) - B(t)T(t,0)]x_1 = [A(t) - B(t)D^{-1}(t)C(t)]x_1,$$
$$0 = D(t)x_2.$$

Also, $x(t)$ satisfies the first m boundary conditions of $\Delta^{-1}(0)\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}$. In fact, on partitioning $\tilde{M}(0)$, $\tilde{N}(0)$, $\Delta(0)$, $\Delta^{-1}(0)$ after the first m rows and columns as

$$\tilde{M}(0) = \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{pmatrix}, \qquad \tilde{N}(0) = \begin{pmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{pmatrix},$$

$$\Delta(0) = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}, \qquad \Delta^{-1}(0) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

we find that the first m rows of $\Delta^{-1}(0)\tilde{M}(0)x(0) + \Delta^{-1}(0)\tilde{N}(0)x(1)$ are

$$[d_{11}(\tilde{M}_{11}V(0) + \tilde{N}_{11}V(1)) + d_{12}(\tilde{M}_{21}V(0) + \tilde{N}_{21}V(1))]\alpha_1(0)$$
$$= (d_{11}\delta_{11} + d_{12}\delta_{12})\alpha_1(0) = \alpha_1(0),$$

that is, they are the first m rows of $\Delta^{-1}(0)\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}$.

To sum up, we have proved the following theorem.

THEOREM. *Let assumptions* (I), (II) *hold. Then for all sufficiently small* ε *the boundary value problem* (10), (11) *has the solution*

$$\begin{pmatrix} v(t,\varepsilon) \\ w(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} V(t,\varepsilon) & 0 \\ 0 & W(t,\varepsilon)PW^{-1}(0,\varepsilon) + W(t,\varepsilon)(I_n - P)W^{-1}(1,\varepsilon) \end{pmatrix} \Delta^{-1}(\varepsilon)\begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}$$

for $0 \leqq t \leqq 1$, *where* $\Delta(\varepsilon)$ *is given by* (13).

Moreover, as $\varepsilon \to 0$ *this solution* $(v(t,\varepsilon), w(t,\varepsilon)) \to (x_1(t), x_2(t))$ *for* $0 < t < 1$, *where* $(x_1(t), x_2(t))$ *is the solution of the degenerate system*

$$x_1' = [A(t) - B(t)D^{-1}(t)C(t)]x_1,$$
$$0 = D(t)x_2,$$

and the first m equations of

$$\Delta^{-1}(0)\tilde{M}(0)\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \Delta^{-1}(0)\tilde{N}(0)\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \Delta^{-1}(0)\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}.$$

Returning to the original variables, for all sufficiently small ε *the boundary value problem* (1), (2) *has the solution*

$$\begin{pmatrix} y(t,\varepsilon) \\ z(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} I_m & -\varepsilon S(t,\varepsilon) \\ -T(t,\varepsilon) & I_n + \varepsilon T(t,\varepsilon)S(t,\varepsilon) \end{pmatrix}\begin{pmatrix} v(t,\varepsilon) \\ w(t,\varepsilon) \end{pmatrix}$$

for $0 \leqq t \leqq 1$. *Moreover, for* $0 < t < 1$, *this solution tends, as* $\varepsilon \to 0$, *to the solution* $(\bar{y}(t), \bar{z}(t))$ *of the degenerate boundary value problem consisting of*

$$\bar{y}' = A(t)\bar{y} + B(t)\bar{z},$$
$$0 = C(t)\bar{y} + D(t)\bar{z}$$

and the first m equations of

$$\Delta^{-1}(0)M(0)\begin{pmatrix}\bar{y}(0)\\ \bar{z}(0)\end{pmatrix} + \Delta^{-1}(0)N(0)\begin{pmatrix}\bar{y}(1)\\ \bar{z}(1)\end{pmatrix} = \Delta^{-1}(0)\begin{pmatrix}c_1(0)\\ c_2(0)\end{pmatrix}.$$

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