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Asymptotic partial decomposition of domain for spectral problems in rod structures

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Abstract

We consider the vibrations of a bundle composed of two co-linear rods. The thickness of the rods is of order ε, where ε is a small parameter. Let $\varepsilon/2$ be the width of one of the rods, and let ε be the width of the other one. Considering the associated spectral problem, we study the asymptotic behavior, as $\varepsilon \to 0$, of the eigenelements. In particular, we obtain asymptotics for these eigenelements providing correcting terms and precise bounds for convergence rates of the eigenelements. These eigenelements are approached up to the first order by the eigenelements of a one-dimensional Dirichlet problem set in $(-1, 1)$. We also provide alternative approaches via the eigenelements of the spectral problem obtained by asymptotic partial decomposition of domain, which is known to be useful in numerical computations. Finally, we show that the technique developed in the paper can be applied to other spectral problems for thin structures.

Résumé

On considère les vibrations d'un trousseau de deux barres colinéaires. La largeur des barres est d'ordre ε, où ε est un petit paramétre. Soit $\varepsilon/2$ la largeur de l'une des deux barres et soit ε la largeur de l'autre. On considére le problème spectral associé à cette structure et on s'interesse au comportement asymptotique des éléments propres lorsque $\varepsilon \to 0$. En particulier, nous obtenons les approximations asymptotiques de ces éléments propres contenant des termes correcteurs et nous précisons les bornes du taux de convergence. Ces éléments propres sont approchés jusqu'aux termes d'ordre ε par les éléments propres de problème de Dirichlet posé dans l'intervalle $(-1, 1)$. De plus, nous développons une approche alternative via un problème spectral obtenu par la décomposition asymptotique de domaine ; cette méthode est un outil de résolution numérique de plusieurs problèmes posés dans des structures minces.

Keywords: Thin domains; Decomposition of domains; Spectral analysis; Asymptotic analysis

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1. Introduction

In this paper, we study the asymptotic behavior of the eigenelements of a spectral problem associated with the vibrations of a two-rod junction model, depending on the thickness of order of ε, as $\varepsilon \to 0$. Let $\varepsilon/2$ be the thickness of one of the rods and ε be that of the other one. We use the method of asymptotic partial decomposition of domain (cf. [18]) for partial differential equations in thin domains and techniques of spectral perturbation theory (cf. [13], [15] and [19] for an extensive bibliography) in order to approach the eigenelements of rod structures.

The method of asymptotic partial decomposition of domain for the associated stationary problems has been widely studied in the literature over the last few years: we refer to [17] where the case of one structure was considered, to [5] for further application of the method to a model arising in flow–structure interaction and to [18] for more references on the subject. Also, the asymptotic behavior for the eigenelements of spectral problems posed in thin structures have been considered by several authors: let us mention [3,4,7,10,14,18] for further references; see also [6] and [8] for very different studies connected with the vibrations of structures containing very heavy or stiff components concentrated on thin bands. However, we emphasize that this is the first paper where the method of asymptotic partial decomposition of domain is used to approach spectral problems and to obtain correctors for the eigenelements and precise bounds for convergence rates up to a desired order. Also the technique used differs from that in previous papers. We consider here the problem of junction of two co-linear rods for the sake of transparency of the technical details, but we point out that the asymptotic partial decomposition of the model problem can be extended to the case of junction of multiple rods.

It should be noted that thin structures are a very important type of industrial installation. The knowledge of their eigenfrequencies is a crucial issue in civil engineering and in aircraft design, providing a clear motive for proceeding with an asymptotic analysis of such structures. However, often, the analyses existing in the literature give only the leading terms of the eigenelements, or they do not allow the contribution of boundary layers in the neighborhood of junctions to be appreciated, or the complete asymptotic expansions obtained seem to be complex for practical applications. In this paper we focus our analysis not only on the asymptotic expansions of the eigenelements but also on its suitable presentation. This presentation is given here by the above mentioned *method of asymptotic partial domain decomposition* (*MAPDD*, see [18]) which reduces the dimension in the main part of the domain (far from the junction area) and keeps the initial dimension in the remaining part. At the interface, MAPDD prescribes the appropriate interface conditions, e.g., the continuity of the solution and the flux conservation in average. The main theoretical result of the paper is that this eigenvalue problem of hybrid dimension has a solution close to the solution of the initial eigenvalue problem. On the other hand, this *partially decomposed problem* contains all the information about the boundary layers and is suitable for applications in engineering.

To proceed with the proof of the closeness of the solutions of both problems, we first prove the convergence of the spectrum of both problems, the initial problem in the ε-dependent domain and the partially decomposed problem, as $\varepsilon \to 0$, towards the spectrum of a lower-dimensional problem. Then, we construct asymptotic expansions of the eigenelements of the initial problem, and we use the asymptotic expansions of the eigenfunctions, suitably modified, as test functions to construct the so-called *quasimodes* for the operators associated with both problems. Finally, using convergence results on quasimodes (almost eigenvalues, respectively) and on true eigenmodes (eigenvalues, respectively) we obtain the required closeness of the solutions (see the end of Section 2 and Remarks 5.2–5.4 and 6.1 for more details).

Let us describe the outline of the paper. In Section 2 we state the main problem; that is, a spectral problem for the conductivity of a simple rod structure constituted of two thin rods. We consider the Neumann boundary condition on the lateral boundary of the rods and the Dirichlet condition on the bottoms of the rods, namely problem (2.1)–(2.3). We define the *limit spectral problem* as the thickness of the rods tends to zero, that is, the one-dimensional Dirichlet problem (2.9)–(2.12), and study the convergence of the eigenvalues and eigenfunctions of the initial problem towards the eigenvalues and eigenfunctions of the limit problem (cf. Theorem 2.1).

In Section 3, we provide the same asymptotic analysis for the eigenelements of the partially decomposed problem of hybrid dimension, namely, problem (3.5)–(3.11) and we prove the convergence these eigenelements to the eigenelements of the same limit problem (2.9)–(2.12) as in the first section (cf. Theorems 3.1 and 3.2).

In Section 4, an asymptotic expansion of the eigenvalues and the eigenfunctions of the initial problem are constructed. The special attention is focussed at the construction of boundary layers in the neighbourhoods of the junction area (cf. (4.1), (4.2) and (4.53)).

In Section 5, we justify the asymptotic expansion proving the estimates for the difference of the exact solution and the truncated asymptotic expansion (cf. Section 5.1). We also prove that this asymptotic expansion is close to the solution of the spectral partially decomposed problem (cf. Section 5.2). Finally, in this section, we obtain the estimates for the difference of the eigenelements of the initial problem and the partially decomposed problem (cf. Theorem 5.5).

Section 6 is devoted to the brief description of an asymptotic analysis of spectral problems in one rod. Asymptotic expansions and justifications are in Sections 6.1 and 6.2 respectively.

In Appendix A, we prove the auxiliary theorems on the existence, uniqueness and exponential decaying for the solutions of the boundary layer problems used in the construction of the asymptotic expansions. As a matter of fact, in this section we deal with problems posed in unbounded domains, like bands or coupled half-bands, and the results, included for the sake of completeness, can be read independently from the rest of the paper.

2. Statement of the spectral problem

Let ε be a positive parameter, $\varepsilon \in (0,1)$. Let G_ε^+ denote the rectangle $(0,1)\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})$ in the x variable, similarly, G_ε^- be the rectangle $(-1,0)\times(-\frac{\varepsilon}{4},\frac{\varepsilon}{4})$ and G_ε the open domain $G_\varepsilon = G_\varepsilon^+ \cup G_\varepsilon^- \cup (\{0\}\times(-\frac{\varepsilon}{4},\frac{\varepsilon}{4}))$ (cf. Fig. 1). Let $\varGamma_\varepsilon$ be the segment $\varGamma_\varepsilon = \{0\}\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})$, where the junction of the two domains $G_\varepsilon^\pm$ holds. Let us denote by γ_ε^+, S_ε^+ (γ_ε^-, S_ε^-, respectively) the parts of the boundary of G_ε^+ (G_ε^-, respectively) $\gamma_\varepsilon^+ = \{1\}\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})$, $S_\varepsilon^+ = \{0\}\times(-\frac{\varepsilon}{4},\frac{\varepsilon}{4})$ ($\gamma_\varepsilon^- = \{-1\}\times(-\frac{\varepsilon}{4},\frac{\varepsilon}{4})$, $S_\varepsilon^- = \{0\}\times(-\frac{\varepsilon}{4},\frac{\varepsilon}{4})$, respectively). Let n be the unit outward normal to the boundary of G_ε.

Let us denote by (x_1,y_2) the variables $(x_1,x_2\varepsilon^{-1})$ which transform the ε depending domains to the fixed domains. Namely, G_1, $G_1^\pm$, $\gamma_1^\pm$ and $\varGamma_1$ respectively denote the transformed domains of G_ε, $G_\varepsilon^\pm$, $\gamma_\varepsilon^\pm$ and $\varGamma_\varepsilon$, with the change of variable from (x_1,x_2) to (x_1,y_2). That is, they coincide with G_ε, $G_\varepsilon^\pm$, $\gamma_\varepsilon^\pm$ and $\varGamma_\varepsilon$, respectively, in the case where $\varepsilon=1$.

Also, for convenience we introduce an auxiliary variable in $\mathbb{R}^2$, the so-called local variable, $\xi = x\varepsilon^{-1}$, and denote by G^+ (G^- respectively) the half-band $\xi \in (0,+\infty)\times(-\frac{1}{2},\frac{1}{2})$ ($\xi\in(-\infty,0)\times(-\frac{1}{4},\frac{1}{4})$, respectively). We denote by G the domain $G = G^+\cup G^-\cup(\{0\}\times(-\frac{1}{4},\frac{1}{4}))$. Let $S_1^\pm$ be the part of the boundary of $G^\pm$ such that $S_1^\pm = \partial G^+\cap\partial G^-$. That is, $\varGamma_1$, $S_1^\pm$ are the transformed domains of $\varGamma_\varepsilon$, $S_\varepsilon^\pm$ with the change of variable from x to ξ.

We consider the two-rod junction spectral problem:

$$\Delta u_\varepsilon + \lambda_\varepsilon u_\varepsilon = 0, \quad \text{in } G_\varepsilon, \tag{2.1}$$

$$u_\varepsilon = 0 \quad \text{if } x_1 = \pm 1 \ \bigl(\text{i.e. } x\in\gamma_\varepsilon^- \text{ or } \gamma_\varepsilon^+\bigr), \tag{2.2}$$

$$\frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \partial G_\varepsilon \backslash \bigl(\gamma_\varepsilon^+\cup\gamma_\varepsilon^-\bigr), \tag{2.3}$$

where n in (2.3) denotes the outer normal vector to ∂G_ε.

Problem (2.1)–(2.3) has a standard variational formulation in $\{u\in H^1(G_\varepsilon) \mid u=0 \text{ on } \gamma_\varepsilon^\pm\}$ and it has a strictly positive discrete spectrum. For fixed ε, let us denote by $\{\lambda_{\varepsilon,i}\}_{i=1}^\infty$ the sequence of eigenvalues of (2.1)–(2.3), converging towards infinity as $i\to\infty$, with the classical convention of repeated eigenvalues. Let $\{u_{\varepsilon,i}\}_{i=1}^\infty$ be the associated

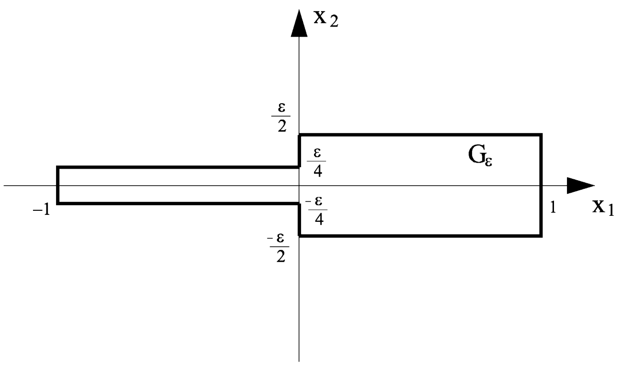

Fig. 1. The two co-linear rods.

eigenfunctions, which are a basis of $\{u \in H^1(G_\varepsilon),\ u = 0 \text{ on } \gamma_\varepsilon^\pm\}$, and we assume that they are normalized in such a way that

$$\int_{G_1} u_{\varepsilon,i}(x_1, y_2)\, u_{\varepsilon,j}(x_1, y_2)\,\mathrm{d}x_1\,\mathrm{d}y_2 = \delta_{i,j}, \tag{2.4}$$

where $\delta_{i,j}$ is the Kronecker symbol and, if no confusion arises, we write indifferently $u_{\varepsilon,i}(x_1, y_2)$ or $u_{\varepsilon,i}(x_1, x_2)$.

By introducing the change of variables from (x_1, x_2) to (x_1, y_2), the variational formulation of (2.1)–(2.3) reads: Find λ_ε, $u_\varepsilon \in \{v \in H^1(G_1) \mid v = 0 \text{ on } \gamma_1^\pm\}$, $u_\varepsilon \neq 0$, such that

$$\int_{G_1} \left(\partial_1 u_\varepsilon \partial_1 v + \frac{1}{\varepsilon^2}\partial_2 u_\varepsilon \partial_2 v\right)\mathrm{d}x_1\,\mathrm{d}y_2 = \lambda_\varepsilon \int_{G_1} u_\varepsilon v\,\mathrm{d}x_1\,\mathrm{d}y_2, \quad \forall v \in H^1(G_1),\ v = 0 \text{ on } \gamma_1^\pm, \tag{2.5}$$

where ∂_1 and ∂_2 are the partial derivatives with respect to x_1 and y_2 respectively.

It is easy to derive the bound for the eigenvalues of (2.5) (cf. [13] for other spectral problems):

$$C \leqslant \lambda_{\varepsilon,i} \leqslant \lambda_{0,i}, \tag{2.6}$$

where C is a constant independent of ε and i and $\lambda_{0,i}$ is the ith eigenvalue of problem (2.8) (or equivalently of (2.9)–(2.12)).

Indeed, the minimax principle gives:

$$\lambda_{\varepsilon,i} = \min_{\substack{E_i \subset H^1(G_1)\\ \dim E_i = i}} \max_{\substack{v \in E_i\\ v \neq 0}} \frac{\int_{G_1} (\partial_1 v)^2 + \frac{1}{\varepsilon^2}(\partial_2 v)^2\,\mathrm{d}x_1\,\mathrm{d}y_2}{\int_{G_1} v^2\,\mathrm{d}x_1\,\mathrm{d}y_2}, \tag{2.7}$$

where the minimum is taken over all the subspaces $E_i \subset \{v \in H^1(G_1) \mid v = 0 \text{ on } \gamma_1^\pm\}$ with $\dim E_i = i$. Then, the right-hand side of inequality (2.6) holds by taking the particular subspace $E_i^* = [v_{0,1}, v_{0,2}, \ldots, v_{0,i}]$, where we consider $v_{0,j}(x_1, y_2) = v_{0,j}(x_1)$ and $\{v_{0,j}\}_{j=1}^i$ the set of eigenfunctions of (2.8) associated with the ith first eigenvalues. On the other hand, the left-hand side of (2.6) holds by using Poincaré inequality on $G_1^\pm$ for the elements of $\{v \in H^1(G_1^\pm) \mid v = 0 \text{ on } \gamma_1^\pm\}$, and the fact that $0 < \varepsilon < 1$.

Therefore, for each fixed $i = 1, 2, \ldots,$ the normalization of the eigenfunctions (2.4), bound (2.6) and Eq. (2.5) allow us to extract converging subsequences, still denoted by ε such that

$$\lambda_{\varepsilon,i} \to \lambda_i^* \quad \text{as } \varepsilon \to 0,$$

and

$$u_{\varepsilon,i} \to u_i^* \quad \text{in } H^1(G_1)\text{-weak as } \varepsilon \to 0,$$

where λ_i^* is a certain positive number and u_i^* is a certain function, $u_i^* \in \{v \in H^1(G_1),\ v = 0 \text{ on } \gamma_1^\pm\}$, u_i^* independent of y_2. Note that this fact holds since the bound $\|\partial_2 u_{\varepsilon,i}\|_{L^2(G_1)} \leqslant C\varepsilon^2$ obtained from (2.5) leads to the conclusion that $\partial_2 u_i^* = 0$. Besides, $u_i^* \neq 0$ since the convergence of $u_{\varepsilon,i}$ towards u_i^* holds in $L^2(G_1)$.

Now, we can identify λ_i^* and u_i^* with an eigenvalue and the corresponding eigenfunction of (2.9)–(2.12) by taking limits, as $\varepsilon \to 0$, in (2.5) for suitable test functions v. Namely, considering $\phi \in H_0^1(-1, 1)$, $v(x_1, y_2) = \phi(x_1)$ in (2.5), and the limit as $\varepsilon \to 0$, we obtain:

$$\int_{G_1} \partial_1 u_i^* \partial_1 v\,\mathrm{d}x_1\,\mathrm{d}y_2 = \lambda_i^* \int_{G_1} u_i^* v\,\mathrm{d}x_1\,\mathrm{d}y_2,$$

that is,

$$\int_{-1}^{0} \partial_1 u_i^* \partial_1 \phi\,\mathrm{d}x_1 + 2\int_0^1 \partial_1 u_i^* \partial_1 \phi\,\mathrm{d}x_1 = \lambda_i^* \left(\int_{-1}^{0} u_i^* \phi\,\mathrm{d}x_1 + 2\int_0^1 u_i^* \phi\,\mathrm{d}x_1\right), \quad \forall \phi \in H_0^1(-1, 1), \tag{2.8}$$

which is the weak formulation of the problem:

$$(Kv_0')' + \lambda_0 K v_0 = 0, \quad x_1 \in (-1,0) \cup (0,1), \tag{2.9}$$

$$v_0(\pm 1) = 0, \tag{2.10}$$

$$[v_0] = 0, \tag{2.11}$$

$$[Kv_0'] = 0, \tag{2.12}$$

where the constant K takes the value $K = 1$ ($K = 2$, respectively) in the interval $(-1, 0)$ ($(0, 1)$, respectively).

Let us denote by $\{\lambda_{0,i}\}_{i=1}^{\infty}$ the set of eigenvalues of (2.8) which can be explicitly computed and are simple, and $\{v_{0,i}(x_1)\}_{i=1}^{\infty}$ the associated eigenfunctions in $H_0^1(-1,1)$. The set of eigenvalues of (2.8) are

$$\left\{(k\pi)^2, \left((2k-1)\pi/2\right)^2\right\}_{k=1}^{\infty},$$

each $(k\pi)^2$ has the associated eigenfunction (up a constant factor) $u_k(x_1)$ defined as $u_k(x_1) = \sin(k\pi x_1)$ if $x_1 \in (-1,0)$ and $u_k(x_1) = 2\sin(k\pi x_1)$ if $x_1 \in (0,1)$, and each $((2k-1)\pi/2)^2$ has the associated eigenfunction (up to a constant factor) $\tilde{u}_k(x_1) = \cos((2k-1)\pi x_1/2)$ in $(-1,1)$.

In the next theorem we prove the convergence of the eigenvalues of (2.5) towards those of (2.8) with conservation of the multiplicity. Also a certain convergence for the corresponding eigenfunctions holds. In this respect, if no confusion arises, for $v \in H_0^1(-1,1)$ we consider $v \in H^1(G_1)$, v being $v(x_1, y_2) = v(x_1)$. In order to derive these results we use a theorem on spectral convergence for abstract operators on a sequence of Hilbert spaces dependent on ε. For the sake of completeness, we introduce this result in Lemma 2.1 and we refer to Section III.1 in [15] for the proof (cf. Section III.9.1 in [2] for an alternative technique which could be applied, and, [9] and [19] for other general methods).

Lemma 2.1. *Let H_ε and H_0 be two separable Hilbert spaces with the scalar products $(\cdot,\cdot)_\varepsilon$ and $(\cdot,\cdot)_0$ respectively. Let $A^\varepsilon \in \mathcal{L}(H_\varepsilon)$ and $A^0 \in \mathcal{L}(H_0)$. Let $\mathcal{W}$ be a subspace of H_0 such that* $\mathrm{Im}(A^0) = \{v \mid v = A^0 u\colon u \in H_0\} \subset \mathcal{W}$. *We assume that the following properties are satisfied*:

(a) *There exists an operator $R^\varepsilon \in \mathcal{L}(H_0, H_\varepsilon)$ and a constant $a > 0$ such that $\|R^\varepsilon f\|_\varepsilon \overset{\varepsilon\to 0}{\to} a\|f\|_0$ for any $f \in \mathcal{W}$.*
(b) *The operators A^ε and A^0 are positive, compact and self-adjoint operators on H^ε and H^0 respectively. Besides, their norms $\|A^\varepsilon\|_{\mathcal{L}(H_\varepsilon)}$ are bounded by a constant independent of ε.*
(c) *For any $f \in \mathcal{W}$, $\|A^\varepsilon R^\varepsilon f - R^\varepsilon A^0 f\|_\varepsilon \overset{\varepsilon\to 0}{\to} 0$.*
(d) *The family of operators A^ε is uniformly compact, i.e., for any sequence f^ε in H_ε such that* $\sup_\varepsilon \|f^\varepsilon\|_\varepsilon$ *is bounded by a constant independent of ε, we can extract a subsequence $f^{\varepsilon'}$ verifying $\|A^{\varepsilon'} f^{\varepsilon'} - R^{\varepsilon'} w^0\|_{\varepsilon'} \to 0$, as $\varepsilon' \to 0$, for certain $w^0 \in \mathcal{W}$.*

Let $\{\mu_i^\varepsilon\}_{i=1}^{\infty}$ and $\{\mu_i^0\}_{i=1}^{\infty}$ be the sequences of the eigenvalues of A^ε and A^0, respectively, with the usual convention of repeated eigenvalues. Let $\{w_i^\varepsilon\}_{i=1}^{\infty}$ and ($\{w_i^0\}_{i=1}^{\infty}$, respectively) be the corresponding eigenfunctions in H_ε which are assumed to be orthonormal (H_0, *respectively*).

Then, for each fixed k there exists a constant C_k independent of ε and $\varepsilon_k > 0$ such that for $\varepsilon \leqslant \varepsilon_k$,

$$\left|\mu_k^\varepsilon - \mu_k^0\right| \leqslant C_k \sup \left\|A^\varepsilon R^\varepsilon u - R^\varepsilon A^0 u\right\|_\varepsilon,$$

where the sup *is taken over all u such that $\|u\|_0 = 1$, u in the eigenspace associated with μ_k^0. In addition, for any eigenvalue μ_k^0 of A^0 with multiplicity s ($\mu_k^0 = \mu_{k+1}^0 = \cdots = \mu_{k+s-1}^0$), and for any w eigenfunction associated with μ_k^0, with $\|w\|_0 = 1$, there exists w^ε, w^ε being a linear combination of eigenfunctions of A^ε $\{w_j^\varepsilon\}_{j=k}^{j=k+s-1}$ associated with $\{\mu_j^\varepsilon\}_{j=k}^{j=k+s-1}$, such that for $\varepsilon \leqslant \varepsilon_k$ the inequality,*

$$\left\|w^\varepsilon - R^\varepsilon w\right\|_\varepsilon \leqslant M_k \left\|A^\varepsilon R^\varepsilon w - R^\varepsilon A^0 w\right\|_\varepsilon, \tag{2.13}$$

holds, where the constant M_k is independent of ε.

The following theorem provides the main convergence results of this section.

Theorem 2.1. *For each fixed i, $i = 1, 2, \ldots$, the eigenvalues $\{\lambda_{\varepsilon,i}\}_\varepsilon$ of* (2.5) *converge towards the ith eigenvalue $\lambda_{0,i}$ of* (2.8) *as $\varepsilon \to 0$. Moreover, for each eigenfunction v_0 of* (2.8) *associated with $\lambda_{0,i}$, there is an eigenfunction*

$\hat{u}_{\varepsilon,i}(x_1, y_2)$ associated with $\lambda_{\varepsilon,i}$ converging towards v_0 in $L^2(G_1)$ as $\varepsilon \to 0$ (note that we identify $v_0(x_1, y_2) = v_0(x_1)$ and the convergence also holds in $H^1(G_1)$-weak). In addition, for each sequence we can extract a subsequence, still denoted by ε, such that $u_{\varepsilon,i} \to v_{0,i}$ in $H^1(G_1)$-weak, as $\varepsilon \to 0$, where $u_{\varepsilon,i}$ are the eigenfunctions of (2.5) *satisfying* (2.4) *and the set $\{v_{0,i}\}_{i=1}^{\infty}$ forms an orthonormal basis of $L^2(-1, 1)$ for the scalar product* (2.15).

Proof. Throughout all the proof, we consider that we have performed the change of variable from x_1, x_2 to x_1, y_2 in problem (2.1)–(2.3), where $y_2 = x_2/\varepsilon$, and then, the weak formulation of (2.1)–(2.3) is given by (2.5).

Let us consider the Hilbert space $H^\varepsilon = L^2(G_1)$ with the scalar product $(.,.)_\varepsilon$ the classical scalar product in $L^2(G_1)$. Let the sequence of positive, symmetric and compact operators A^ε be defined on H^ε as follows: for $f \in L^2(G_1)$, we define $A^\varepsilon f = u_f^\varepsilon$ where $u_f^\varepsilon \in \{u \in H^1(G_1),\ u = 0 \text{ on } \gamma_1^\pm\}$ is the unique solution of the problem:

$$\int_{G_1} \left(\partial_1 u_f^\varepsilon \partial_1 v + \frac{1}{\varepsilon^2} \partial_2 u_f^\varepsilon \partial_2 v \right) \mathrm{d}x_1 \, \mathrm{d}y_2 = \int_{G_1} f v \, \mathrm{d}x_1 \, \mathrm{d}y_2, \quad \forall v \in H^1(G_1),\ v = 0 \text{ on } \gamma_1^\pm. \tag{2.14}$$

Obviously, the eigenelements of A^ε are $\{((\lambda_{\varepsilon,i})^{-1}, u_{\varepsilon,i})\}_{i=1}^{\infty}$ where $(\lambda_{\varepsilon,i}, u_{\varepsilon,i})$ are the eigenelements of (2.5), $u_{\varepsilon,i}$ satisfying (2.4).

In the same way, we consider H^0 the Hilbert space $H^0 = L^2(-1, 1)$ with the scalar product,

$$(f, g)_0 = \frac{1}{2} \int_{-1}^{0} f g \, \mathrm{d}x_1 + \int_{0}^{1} f g \, \mathrm{d}x_1, \tag{2.15}$$

and the compact positive and symmetric operator A^0 on H^0 defined by $A^0 f = v_f^0$, for $f \in L^2(-1, 1)$, where $v_f^0 \in H_0^1(-1, 1)$ is the unique solution of the following problem:

$$\frac{1}{2} \int_{-1}^{0} \partial_1 v_f^0 \partial_1 \phi \, \mathrm{d}x_1 + \int_{0}^{1} \partial_1 v_f^0 \partial_1 \phi \, \mathrm{d}x_1 = \frac{1}{2} \int_{-1}^{0} f \phi \, \mathrm{d}x_1 + \int_{0}^{1} f \phi \, \mathrm{d}x_1, \quad \forall \phi \in H_0^1(-1, 1). \tag{2.16}$$

The eigenelements of A^0 are $\{((\lambda_{0,i})^{-1}, v_{0,i})\}_{i=1}^{\infty}$ where $(\lambda_{0,i}, v_{0,i})$ are the eigenelements of (2.8), $v_{0,i}$ satisfying $\|v_{0,i}\|_0 = 1$.

Let the space $\mathcal{W}$ be $\mathcal{W} = H_0^1(-1, 1)$ which obviously satisfies $\mathrm{Im}(A^0) \subset \mathcal{W} \subset H^0$. Let R^ε be the linear continuous operator $R^\varepsilon : H^0 \to H^\varepsilon$ defined by $R^\varepsilon f(x_1, y_2) = f(x_1)$, $R^\varepsilon f \in L^2(G_1)$ for any $f \in L^2(-1, 1)$.

We check the properties (a)–(d) in Lemma 2.1 and then the result in the theorem is a consequence of this lemma.

Property (a) is satisfied for the value of the constant $a = 1$, since $\|R^\varepsilon f\|_{L^2(G_1)}^2 = (f, f)_0$ for any $f \in H_0^1(-1, 1)$.

In order to prove the uniform bound for $\|A^\varepsilon\|_{\mathcal{L}(\mathcal{H}^\varepsilon)}$ we consider, for each $f \in L^2(G_1)$, u_f^ε the solution of (2.14), where taking $v = u_f^\varepsilon$, applying the Poincaré inequality on $\{u \in H^1(G_1^\pm) \mid u = 0 \text{ on } \gamma^\pm\}$ and Cauchy–Schwarz–Buniakowsky inequality, we prove the following inequalities:

$$\|u_f^\varepsilon\|_{L^2(G_1^\pm)} \leqslant C \|\nabla u_f^\varepsilon\|_{L^2(G_1^\pm)}, \tag{2.17}$$

$$\|\partial_1 u_f^\varepsilon\|_{L^2(G_1)} \leqslant C \|f\|_{L^2(G_1)}, \tag{2.18}$$

$$\frac{1}{\varepsilon} \|\partial_2 u_f^\varepsilon\|_{L^2(G_1)} \leqslant C \|f\|_{L^2(G_1)}, \tag{2.19}$$

$$\|u_f^\varepsilon\|_{L^2(G_1)} \leqslant C \|f\|_{L^2(G_1)}, \tag{2.20}$$

where C denotes a constant independent of ε and f. Then, we have:

$$\|A^\varepsilon\|_{\mathcal{L}(\mathcal{H}^\varepsilon)} = \sup_{f \in H^\varepsilon,\ f \neq 0} \frac{\|A^\varepsilon f\|_\varepsilon}{\|f\|_\varepsilon} = \sup_{f \in L^2(G_1),\ f \neq 0} \frac{\|u_f^\varepsilon\|_{L^2(G_1)}}{\|f\|_{L^2(G_1)}} \leqslant C,$$

and property (b) is proved.

Let us prove (c). We consider $f \in H_0^1(-1,1)$ and identify $R^\varepsilon f$ with f by setting $f(x_1, y_2) = f(x_1)$, for all $(x_1, y_2) \in G_1$. Besides, obviously, this extension belongs to the space $\{u \in H^1(G_1) \mid u = 0 \text{ on } \gamma^\pm\}$. Then inequalities (2.17)–(2.20) hold for the solution of (2.14) and, therefore, we can extract a subsequence ε' such that $u_f^{\varepsilon'}$ converges towards some function u_f^* in $H^1(G_1)$-weak, as $\varepsilon \to 0$, with $u_f^* = 0$ on $\gamma^\pm$ and $\partial_2 u_f^* = 0$. Then, taking limits in (2.14) for $v = \phi \in H_0^1(-1,1)$, we obtain (2.16) for $v_f^0 = u_f^*$ and by the uniqueness of solution of (2.16) all the sequence u_f^ε converges towards $v_f^0 \in H_0^1(-1,1)$ the solution of (2.16), namely, $v_f^0 = A^0 f$. Thus, the convergence,

$$\left\| A^\varepsilon R^\varepsilon f - R^\varepsilon A^0 f \right\|_{L^2(G_1)} \stackrel{\varepsilon \to 0}{\to} 0,$$

holds for any $f \in H_0^1(-1,1)$, and property (c) in Lemma 2.1 is proved.

Let us prove property (d). Considering any bounded sequence $f^\varepsilon \in L^2(G_1)$, we can extract a subsequence $f^{\varepsilon'}$ converging weakly towards some f^* in $L^2(G_1)$. Let $u_{f^{\varepsilon'}}^{\varepsilon'}$ be the sequence of solutions of (2.14) for $f = f^{\varepsilon'}$. These solutions also satisfy inequalities (2.17)–(2.20) for $f = f^{\varepsilon'}$ and the right-hand side of these inequalities is bounded by a constant independent of ε. Therefore, we can proceed as in property (c), extracting a subsequence of $\{u_{f^{\varepsilon'}}^{\varepsilon'}\}_{\varepsilon'}$, still denoted by ε', converging as $\varepsilon' \to 0$ towards some function u^* in $H^1(G_1)$-weak, u^* verifying $u^* = 0$ on $\gamma^\pm$, and $\partial_2 u^* = 0$. Thus, we have found $u^* \in H_0^1(-1,1)$ such that the convergence,

$$\left\| A^{\varepsilon'} f^{\varepsilon'} - R^{\varepsilon'} u^* \right\|_{L^2(G_1)} \to 0 \quad \text{as } \varepsilon' \to 0,$$

holds and this shows property (d) in Lemma 2.1. Consequently, we conclude the proof of the two first statements in the theorem.

As regards the proof of the last statement in the theorem, we use the proofs in (2.5)–(2.8) and a classical argument of diagonalization to derive the convergence of a certain subsequence of $u_{\varepsilon,i}$ towards $v_{0,i}$ in $H^1(G_1)$-weak, as $\varepsilon \to 0$, $v_{0,i}$ being an eigenfunction of (2.8) associated with the eigenvalue $\lambda_{0,i}$, for any $i = 1, 2, \ldots$. The fact that the $v_{0,i}$ are orthonormal in $L^2(-1,1)$ for the scalar product (2.15) follows from the strong convergence of $u_{\varepsilon,i}$ towards $v_{0,i}$ in $L^2(G_1)$ and (2.4). The fact that the $\{v_{0,i}\}_{i=1}^\infty$ are a basis of $L^2(-1,1)$ is obtained by contradiction, since all the eigenvalues of (2.8) are simple. Therefore, the theorem is proved. □

Let us note that Theorem 2.1 shows that the first approach as $\varepsilon \to 0$ for the eigenelements of (2.1)–(2.3) is given by the eigenelements of the Dirichlet problem (2.8), which only depends on the x_1 variable, but the theorem does not provide estimates for convergence rates. In Section 4, by using asymptotic expansions we obtain correcting terms for the eigenelements up to any fixed order $O(\varepsilon^J)$. In fact, a complete asymptotic expansion for the eigenelements can be obtained (cf. formulas (4.1) and (4.2)). We justify these asymptotics in Section 5.1. In addition, in Section 5.2 we also prove that the asymptotics provide a good approach of the eigenelements of a problem posed in *decomposed domains*, namely problem (3.3).

In Section 3, we prove that for fixed $\delta \in (0,1)$ or $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, the above mentioned problem, problem (3.3) posed in the three domains, namely, the two 1-D domains $(-1, \delta)$ and $(\delta, 1)$ and the 2-D domain $G_{\varepsilon,\delta} = ((-1, -\delta) \times (-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})) \cup ((\delta, 1) \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}))$, has the same limit problem (2.8) as $\varepsilon \to 0$. Let us note that an important result, from the numerical viewpoint, is that we prove that for a particular $\delta = \delta(\varepsilon) \to 0$, namely $\delta = \tilde{k}\varepsilon|\ln\varepsilon|$ with $\tilde{k}$ a certain constant independent on ε, the eigenelements of (3.3) approach the eigenelements of (2.1)–(2.3), as stated in Theorem 5.5.

In Section 6, we outline the kind of results obtained throughout the previous sections, for another spectral problem in a rod. We observe that, among other techniques for proofs in Sections 5 and 6.2 we combine results on the existence of *quasimodes* from the spectral perturbation theory (cf. Lemma 5.1) along with the convergence of the spectrum previously obtained (cf. Theorems 2.1, 3.1, 3.2, 6.1, 6.2).

3. The spectral problem on the decomposed domains

With the same notation as in Section 2, we consider any fixed δ, $0 < \delta < 1$, and we denote by $G_{\varepsilon,\delta}$ the sub-domain of G_ε, $G_{\varepsilon,\delta} = G_\varepsilon \cap \{|x_1| < \delta\}$ (cf. Fig. 2), and $G_{1,\delta}$ the domain obtained by means of the change $y_2 = x_2\varepsilon^{-1}$.

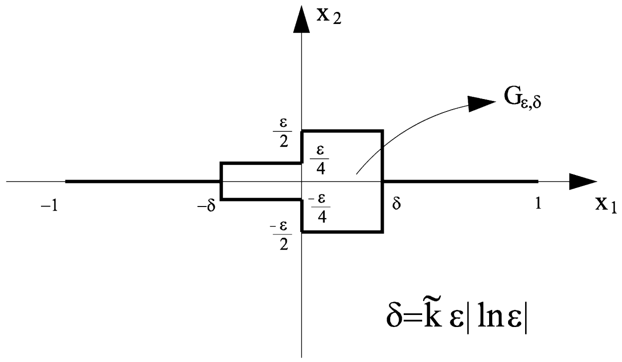

Fig. 2. The decomposed domains.

Let us denote by $H^1_{\mathrm{dec},\delta}(G_\varepsilon)$ the subspace of $\{u \in H^1(G_\varepsilon) \mid u = 0 \text{ on } \gamma_\varepsilon^\pm\}$ defined by:

$$H^1_{\mathrm{dec},\delta}(G_\varepsilon) = \left\{ u \in H^1(G_\varepsilon) \;\middle|\; u = 0 \text{ on } \gamma_\varepsilon^\pm, \text{ and } \frac{\partial u}{\partial x_2}(x_1, x_2) = 0 \text{ for } |x_1| \geqslant \delta \right\},$$

whose elements can be identified with set of three functions $(u^-, u, u^+) \in H^1_{\mathrm{dec},\varepsilon,\delta}$, where we define the space,

$$\begin{aligned} H^1_{\mathrm{dec},\varepsilon,\delta} = \big\{ & u \in H^1(G_{\varepsilon,\delta}),\ u^+ \in H^1(\delta, 1),\ u^- \in H^1(-1, -\delta) \mid \\ & u^+(1) = 0,\ u^-(-1) = 0,\ u^+(\delta) = u(\delta, x_2), u^-(-\delta) = u(-\delta, x_2) \big\}, \end{aligned} \tag{3.1}$$

with the scalar product the usual one in the product of spaces $H^1(-1, -\delta)$, $H^1(G_{\varepsilon,\delta})$ and $H^1(\delta, 1)$. In addition, if no confusion arises we write indifferently u or (u^-, u, u^+) to denote the same element of $H^1_{\mathrm{dec},\delta}(G_\varepsilon)$ or $H^1_{\mathrm{dec},\varepsilon,\delta}$.

Taking into account that the imbedding of $H^1_{\mathrm{dec},\varepsilon,\delta}$ in the Hilbert space,

$$L^2_{\mathrm{dec},\varepsilon,\delta} = L^2(-1, -\delta) \times L^2(G_{\varepsilon,\delta}) \times L^2(\delta, 1), \tag{3.2}$$

is dense and compact, we can consider the weak formulation of the eigenvalue problem (2.1)–(2.3), namely Eq. (2.5) with the change of variables from x_1, y_2 to x_1, x_2, in the couple of spaces $H^1_{\mathrm{dec},\varepsilon,\delta} \subset L^2_{\mathrm{dec},\varepsilon,\delta}$. That is, find $\lambda_{\varepsilon,\mathrm{dec},\delta}$, $u_{\varepsilon,\mathrm{dec},\delta} \in H^1_{\mathrm{dec},\delta}(G_\varepsilon)$, $u_{\varepsilon,\mathrm{dec},\delta} \neq 0$, such that

$$\int_{G_\varepsilon} \nabla u_{\varepsilon,\mathrm{dec},\delta} . \nabla v \,\mathrm{d}x = \lambda_{\varepsilon,\mathrm{dec},\delta} \int_{G_\varepsilon} u_{\varepsilon,\mathrm{dec},\delta} v \,\mathrm{d}x, \quad \forall v \in H^1_{\mathrm{dec},\delta}(G_\varepsilon). \tag{3.3}$$

This problem also has a discrete spectrum which we shall denote by $\{\lambda_{\varepsilon,\mathrm{dec},\delta,i}\}_{i=1}^\infty$, where the convention of repeated eigenvalues has been adopted. Let $\{u_{\varepsilon,\mathrm{dec},\delta,i}\}_{i=1}^\infty$ be the sequence of eigenfunctions in $H^1_{\mathrm{dec},\delta}(G_\varepsilon)$, or equivalently in $H^1_{\mathrm{dec},\varepsilon,\delta}$. In the same way as in Section 2, if no confusion arises, we write indifferently $u_{\varepsilon,\mathrm{dec},\delta,i}(x_1, x_2)$ or $u_{\varepsilon,\mathrm{dec},\delta,i}(x_1, y_2)$, and we assume that these eigenfunctions satisfy the same normalization condition (2.4), which now reads:

$$\int_{G_{1,\delta}} u_{\varepsilon,\mathrm{dec},\delta,i} u_{\varepsilon,\mathrm{dec},\delta,j} \,\mathrm{d}x_1 \,\mathrm{d}y_2 + \frac{1}{2} \int_{-1}^{-\delta} u^-_{\varepsilon,\mathrm{dec},\delta,i} u^-_{\varepsilon,\mathrm{dec},\delta,j} \,\mathrm{d}x_1 + \int_\delta^1 u^+_{\varepsilon,\mathrm{dec},\delta,i} u^+_{\varepsilon,\mathrm{dec},\delta,j} \,\mathrm{d}x_1 = \delta_{i,j}, \tag{3.4}$$

where obviously, $u^+_{\varepsilon,\mathrm{dec},\delta,i}$ ($u^-_{\varepsilon,\mathrm{dec},\delta,i}$, respectively) denote the restriction of $u_{\varepsilon,\mathrm{dec},\delta,i}$ to $y_2 = 0$, $x_1 > \delta$ ($y_2 = 0$, $x_1 < -\delta$, respectively).

It should be noted that the eigenvalue problem (3.3) also has a differential formulation which is:

$$\Delta u_{\varepsilon,\mathrm{dec},\delta} + \lambda_{\varepsilon,\mathrm{dec},\delta} u_{\varepsilon,\mathrm{dec},\delta} = 0, \quad \text{in } G_{\varepsilon,\delta}, \tag{3.5}$$

$$\big(u^-_{\varepsilon,\mathrm{dec},\delta}\big)'' + \lambda_{\varepsilon,\mathrm{dec},\delta} u^-_{\varepsilon,\mathrm{dec},\delta} = 0, \quad \text{in } (-1, -\delta), \tag{3.6}$$

$$\big(u^+_{\varepsilon,\mathrm{dec},\delta}\big)'' + \lambda_{\varepsilon,\mathrm{dec},\delta} u^+_{\varepsilon,\mathrm{dec},\delta} = 0, \quad \text{in } (\delta, 1), \tag{3.7}$$

$$u_{\varepsilon,\mathrm{dec},\delta}(-\delta, x_2) = u^-_{\varepsilon,\mathrm{dec},\delta}(-\delta),$$
$$\frac{2}{\varepsilon}\int_{-\varepsilon/4}^{\varepsilon/4} \frac{\partial u_{\varepsilon,\mathrm{dec},\delta}}{\partial x_1}(-\delta, x_2)\,\mathrm{d}x_2 = \big(u^-_{\varepsilon,\mathrm{dec},\delta}\big)'(-\delta), \tag{3.8}$$

$$u_{\varepsilon,\mathrm{dec},\delta}(\delta, x_2) = u^+_{\varepsilon,\mathrm{dec},\delta}(\delta), \quad \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} \frac{\partial u_{\varepsilon,\mathrm{dec},\delta}}{\partial x_1}(\delta, x_2)\,\mathrm{d}x_2 = \big(u^+_{\varepsilon,\mathrm{dec},\delta}\big)'(\delta), \tag{3.9}$$

$$u^-_{\varepsilon,\mathrm{dec},\delta}(-1) = 0 \quad \text{and} \quad u^+_{\varepsilon,\mathrm{dec},\delta}(1) = 0, \tag{3.10}$$

$$\frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \partial G_{\varepsilon,\delta} \setminus \big\{x \mid |x_1| = \delta\big\}. \tag{3.11}$$

By setting $\varepsilon = 1$ (or performing the change of variable $y_2 = x_2/\varepsilon$, respectively), we denote by $H^1_{\mathrm{dec},\delta}$, $L^2_{\mathrm{dec},\delta}$ the spaces $H^1_{\mathrm{dec},1,\delta}$ $L^2_{\mathrm{dec},1,\delta}$ (transformed spaces of $H^1_{\mathrm{dec},\varepsilon,\delta}$ and $L^2_{\mathrm{dec},\varepsilon,\delta}$, respectively).

With the change of variable from (x_1, x_2) to (x_1, y_2), the formulation (2.5) in the couple of subspaces $H^1_{\mathrm{dec},\delta} \subset L^2_{\mathrm{dec},\delta}$ reads: Find $\lambda_{\varepsilon,\mathrm{dec},\delta}$, $u_{\varepsilon,\mathrm{dec},\delta} \in H^1_{\mathrm{dec},\delta}$, $u_{\varepsilon,\mathrm{dec},\delta} \neq 0$, such that

$$\int_{G_1} \left(\partial_1 u_{\varepsilon,\mathrm{dec},\delta}\partial_1 v + \frac{1}{\varepsilon^2}\partial_2 u_{\varepsilon,\mathrm{dec},\delta}\partial_2 v\right)\mathrm{d}x_1\,\mathrm{d}y_2 = \lambda_{\varepsilon,\mathrm{dec},\delta}\int_{G_1} u_{\varepsilon,\mathrm{dec},\delta} v\,\mathrm{d}x_1\,\mathrm{d}y_2, \quad \forall v \in H^1_{\mathrm{dec},\delta}. \tag{3.12}$$

On account of the definition of (3.1), formulation (3.12) amounts to:

$$\begin{aligned}
&\int_{G_{1,\delta}} \left(\partial_1 u_{\varepsilon,\mathrm{dec},\delta}\partial_1 v + \frac{1}{\varepsilon^2}\partial_2 u_{\varepsilon,\mathrm{dec},\delta}\partial_2 v\right)\mathrm{d}x_1\,\mathrm{d}y_2 + \frac{1}{2}\int_{-1}^{-\delta} \partial_1 u_{\varepsilon,\mathrm{dec},\delta}\partial_1 v\,\mathrm{d}x_1 + \int_{\delta}^{1} \partial_1 u_{\varepsilon,\mathrm{dec},\delta}\partial_1 v\,\mathrm{d}x_1 \\
&\quad = \lambda_{\varepsilon,\mathrm{dec},\delta}\left(\int_{G_{1,\delta}} u_{\varepsilon,\mathrm{dec},\delta} v\,\mathrm{d}x_1\,\mathrm{d}y_2 + \frac{1}{2}\int_{-1}^{-\delta} u_{\varepsilon,\mathrm{dec},\delta} v\,\mathrm{d}x_1 + \int_{\delta}^{1} u_{\varepsilon,\mathrm{dec},\delta} v\,\mathrm{d}x_1\right), \quad \forall v \in H^1_{\mathrm{dec},\delta}.
\end{aligned} \tag{3.13}$$

In the same way as in the proof of (2.6), we use the minimax principle,

$$\lambda_{\varepsilon,i} = \min_{\substack{E_i \subset H^1_{\mathrm{dec},\delta}(G_1) \\ \dim E_i = i}} \max_{\substack{v \in E_i \\ v \neq 0}} \frac{\int_{G_1} (\partial_1 v)^2 + \frac{1}{\varepsilon^2}(\partial_2 v)^2\,\mathrm{d}x_1\,\mathrm{d}y_2}{\int_{G_1} v^2\,\mathrm{d}x_1\,\mathrm{d}y_2}, \tag{3.14}$$

and proceed exactly as in (2.7) to obtain that the eigenvalues of (3.13) satisfy:

$$C \leqslant \lambda_{\varepsilon,\mathrm{dec},\delta,i} \leqslant \lambda_{0,i}, \tag{3.15}$$

where C is a constant independent of ε and i and $\lambda_{0,i}$ is the ith eigenvalue of problem (2.8) (or equivalently of (2.9)–(2.12)).

Therefore, for each fixed $i = 1, 2, \ldots$, the normalization of the eigenfunctions (3.4), bound (3.15), (3.13) allows us to extract converging subsequences, still denoted by ε such that

$$\lambda_{\varepsilon,\mathrm{dec},\delta,i} \to \lambda^*_{\mathrm{dec},i} \quad \text{as } \varepsilon \to 0,$$

and

$$u_{\varepsilon,\mathrm{dec},\delta,i} \to u^*_{\mathrm{dec},i} \quad \text{in } H^1(G_1)\text{-weak as } \varepsilon \to 0,$$

where $\lambda^*_{\mathrm{dec},i}$ is a certain positive number and $u^*_{\mathrm{dec},i}$ is a certain function, $u^*_{\mathrm{dec},i} \in H^1_{\mathrm{dec},\delta}(G_1)$, $u^*_{\mathrm{dec},i} = 0$ on $\gamma^\pm$, and $u^*_{\mathrm{dec},i}$ independent of y_2. Besides, $u^*_{\mathrm{dec},i} \neq 0$ since the convergence of $u_{\varepsilon,i}$ towards $u^*_{\mathrm{dec},i}$ holds in $L^2(G_1)$. Now, considering $\phi \in H^1_0(-1, 1)$, $v(x_1, y_2) = \phi(x_1)$ in (3.13), and the limit as $\varepsilon \to 0$ we obtain:

$$\int_{G_1} \partial_1 u^*_{\mathrm{dec},i}\partial_1 v\,\mathrm{d}x_1\,\mathrm{d}y_2 = \lambda^*_{\mathrm{dec},i}\int_{G_1} u^*_{\mathrm{dec},i} v\,\mathrm{d}x_1\,\mathrm{d}y_2, \tag{3.16}$$

that is, $\{(\lambda^*_{\mathrm{dec},i}, u^*_{\mathrm{dec},i})\}_{i=1}^{\infty}$ are eigenelements of (2.8).

Following the same idea as in Theorem 2.1, we use Lemma 2.1 to prove the convergence of the eigenvalues of (3.13) towards those of (2.8) with conservation of the multiplicity. Also the convergence for the corresponding eigenfunctions holds in the suitable Hilbert spaces as stated in the following theorem.

Theorem 3.1. *For each fixed i, $i = 1, 2, \ldots,$ the eigenvalues $\{\lambda_{\varepsilon,\mathrm{dec},\delta,i}\}_\varepsilon$ of* (3.13) *converge towards the ith eigenvalue $\lambda_{0,i}$ of* (2.8) *as $\varepsilon \to 0$. Moreover, for each eigenfunction v_0 of* (2.8) *associated with $\lambda_{0,i}$, there is an eigenfunction $\hat{u}_{\varepsilon,\mathrm{dec},\delta,i}(x_1, y_2)$ associated with $\lambda_{\varepsilon,\mathrm{dec},\delta,i}$ converging towards v_0 in $L^2(G_1)$ as $\varepsilon \to 0$ (note that we identify $v_0(x_1, y_2) = v_0(x_1)$ and the convergence in $H^1(G_1)$-weak also holds). In addition, for each sequence we can extract a subsequence still denoted by ε such that $u_{\varepsilon,\mathrm{dec},\delta,i} \to v_{0,i}$ in $H^1(G_1)$-weak, as $\varepsilon \to 0$, where $u_{\varepsilon,\mathrm{dec},\delta,i}$ are the eigenfunctions of* (3.13) *satisfying* (3.4) *and the set $\{v_{0,i}\}_{i=1}^{\infty}$ forms an orthonormal basis of $L^2(-1, 1)$ for the scalar product* (2.15).

Proof. We follow the same steps as in the proof of Theorem 2.1. Let us consider the Hilbert space $H^\varepsilon = L^2_{\mathrm{dec},\delta} = L^2_{\mathrm{dec},1,\delta} = L^2(-1, -\delta) \times L^2(G_{1,\delta}) \times L^2(\delta, 1)$, that is, H^ε is the space defined by (3.2) when $\varepsilon = 1$, with the scalar product $(.,.)_\varepsilon$ the classical one in the space product of Hilbert spaces, namely,

$$(u, v)_\varepsilon = \int_{G_{1,\delta}} uv \, \mathrm{d}x_1 \, \mathrm{d}y_2 + \frac{1}{2} \int_{-1}^{-\delta} uv \, \mathrm{d}x_1 + \int_{\delta}^{1} uv \, \mathrm{d}x_1.$$

For each ε, let A^ε be the positive, symmetric and compact operator defined on H^ε as follows: for $f \in L^2_{\mathrm{dec},\delta}$, we define $A^\varepsilon f = u^\varepsilon_f$ where $u^\varepsilon_f \in H^1_{\mathrm{dec},\delta}(G_1)$ is the unique solution of:

$$\begin{aligned} &\int_{G_1} \left(\partial_1 u^\varepsilon_f \partial_1 v + \frac{1}{\varepsilon^2} \partial_2 u^\varepsilon_f \partial_2 v \right) \mathrm{d}x_1 \, \mathrm{d}y_2 \\ &\quad = \int_{G_{1,\delta}} f v \, \mathrm{d}x_1 \, \mathrm{d}y_2 + \frac{1}{2} \int_{-1}^{-\delta} f v \, \mathrm{d}x_1 + \int_{\delta}^{1} f v \, \mathrm{d}x_1, \quad \forall v \in H^1_{\mathrm{dec},\delta}(G_1). \end{aligned} \tag{3.17}$$

Here, $H^1_{\mathrm{dec},\delta}(G_1)$ has also been identified with the space defined by (3.1) in the case where $\varepsilon = 1$. Clearly, the eigenelements of A^ε are $\{((\lambda_{\varepsilon,\mathrm{dec},\delta,i})^{-1}, u_{\varepsilon,\mathrm{dec},\delta,i})\}_{i=1}^{\infty}$, where $(\lambda_{\varepsilon,\mathrm{dec},\delta,i}, u_{\varepsilon,\mathrm{dec},\delta,i})$ are the eigenelements of (3.13).

On the other hand, we consider the Hilbert space H^0 and operator A^0 those in Theorem 2.1. Namely, H^0 is the Hilbert space $H^0 = L^2(-1, 1)$ with the scalar product (2.15), and the positive, symmetric and compact operator A^0 on H^0 defined as: $A^0 f = v^0_f$, for $f \in L^2(-1, 1)$, where $v^0_f \in H^1_0(-1, 1)$ is the unique solution of problem (2.16) whose eigenelements are $\{((\lambda_{0,i})^{-1}, v_{0,i})\}_{i=1}^{\infty}$ where $(\lambda_{0,i}, v_{0,i})$ are the eigenelements of (2.8). Also, $\mathcal{W} = H^1_0(-1, 1)$ is the same space as in Theorem 2.1. Let R^ε be the linear continuous operator $R^\varepsilon : H^0 \to H^\varepsilon$ defined for any $f \in L^2(-1, 1)$ by: $R^\varepsilon f(x_1, y_2) = f(x_1)$ in the case where $(x_1, y_2) \in G_{1,\delta}$, and $R^\varepsilon f(x_1) = f(x_1)$ if $|x_1| \in (\delta, 1)$.

We check the properties (a)–(d) in Lemma 2.1 and then the result in the theorem is a consequence of this lemma.

Property (a) is satisfied for the value of the constant $a = 1$, since $\|R^\varepsilon f\|_\varepsilon = \int_{G_{1,\delta}} f^2 \, \mathrm{d}x_1 \, \mathrm{d}y_2 + \frac{1}{2} \int_{-1}^{-\delta} f^2 \, \mathrm{d}x_1 + \int_\delta^1 f^2 \, \mathrm{d}x_1 = \frac{1}{2} \int_{-1}^0 f^2 \, \mathrm{d}x_1 + \int_0^1 f^2 \, \mathrm{d}x_1$, for any $f \in H^1_0(-1, 1)$.

In order to prove the uniform bound for $\|A^\varepsilon\|_{\mathcal{L}(\mathcal{H}^\varepsilon)}$ we consider for each $f \in L^2(-1, -\delta) \times L^2(G_{1,\delta}) \times L^2(\delta, 1)$, u^ε_f the solution of (3.17), where taking $v = u^\varepsilon_f$, applying the Poincaré inequalities on $\{u \in H^1(G_1^\pm) \mid u = 0 \text{ on } \gamma^\pm\}$, $\{u \in H^1(-1, -\delta) \mid u(-1) = 0\}$ and $\{u \in H^1(\delta, 1) \mid u(1) = 0\}$, and Cauchy–Schwartz–Buniakowsky inequality, we prove the following inequalities:

$$\|u^\varepsilon_f\|_{L^2(G_1^\pm)} \leqslant C \|\nabla u^\varepsilon_f\|_{L^2(G_1^\pm)}, \tag{3.18}$$

$$\|\partial_1 u^\varepsilon_f\|_{L^2(G_1)} \leqslant C\|f\|_{H^\varepsilon} \leqslant C\big(\|f\|_{L^2(G_{\delta,1})} + \|f\|_{L^2(-1,-\delta)} + \|f\|_{L^2(-1,-\delta)}\big), \tag{3.19}$$

$$\frac{1}{\varepsilon}\|\partial_2 u^\varepsilon_f\|_{L^2(G_1)} \leqslant C\|f\|_{H^\varepsilon} \leqslant C\big(\|f\|_{L^2(G_{\delta,1})} + \|f\|_{L^2(-1,-\delta)} + \|f\|_{L^2(-1,-\delta)}\big), \tag{3.20}$$

$$\|u^\varepsilon_f\|_{L^2(G_1)} \leqslant C\|f\|_{H^\varepsilon} \leqslant C\big(\|f\|_{L^2(G_{\delta,1})} + \|f\|_{L^2(-1,-\delta)} + \|f\|_{L^2(-1,-\delta)}\big), \tag{3.21}$$

where C denotes a constant independent of ε, δ and f. Then, we have:

$$\left\|A^\varepsilon\right\|_{\mathcal{L}(\mathcal{H}^\varepsilon)} = \sup_{f\in H^\varepsilon,\ f\neq 0}\frac{\|A^\varepsilon f\|_\varepsilon}{\|f\|_\varepsilon} = \sup_{f\in H^\varepsilon,\ f\neq 0}\frac{\|u_f^\varepsilon\|_\varepsilon}{\|f\|_\varepsilon}\leqslant C,$$

and property (b) is proved.

Let us prove (c). We consider $f\in H_0^1(-1,1)$ and identify $R^\varepsilon f$ with f by setting $f(x_1,y_2)=f(x_1)$, for all $(x_1,y_2)\in G_{1,\delta}$. Obviously, this extension belongs to the space $H^1_{\mathrm{dec},\delta}(G_1)=\{u\in H^1(G_1)\mid u=0 \text{ on } \gamma^\pm$, and, $\frac{\partial u}{\partial x_2}(x_1,y_2)=0$ for $|x_1|\geqslant\delta\}$. Then inequalities (3.18)–(3.21) hold for solution of (3.17) and, therefore, we can extract a subsequence ε' such that $u_f^{\varepsilon'}$ converges towards some function u_f^* in $H^1(G_1)$-weak, as $\varepsilon\to 0$, with $u_f^*=0$ on $\gamma^\pm$ and $\partial_2 u_f^*=0$. Then, taking limits in (3.17) for $v=\phi\in H_0^1(-1,1)$, we obtain (2.16) for $v_f^0=u_f^*$ and, by the uniqueness of solution of (2.16), all the sequence u_f^ε converges towards $v_f^0\in H_0^1(-1,1)$ the solution of (2.16). Namely, since $u_f^\varepsilon=A^\varepsilon R^\varepsilon f$ and $v_f^0=A^0 f$, we have proved that the convergence,

$$\left\|A^\varepsilon R^\varepsilon f - R^\varepsilon A^0 f\right\|_{L^2_{\mathrm{dec},\delta}}\overset{\varepsilon\to 0}{\to} 0,$$

holds for any $f\in H_0^1(-1,1)$, and property (c) in Lemma 2.1 is verified.

Let us prove property (d). Considering any sequence $f^\varepsilon\in L^2_{\mathrm{dec},\delta}$, such that $\|f^\varepsilon\|_{L^2_{\mathrm{dec},\delta}}\leqslant C$ with C a constant independent of ε, we can extract a subsequence $f^{\varepsilon'}$ converging weakly towards some f^* in $L^2_{\mathrm{dec},\delta}$. Let $u^{\varepsilon'}_{f^{\varepsilon'}}$ be the sequence of solutions of (3.17) for $f=f^{\varepsilon'}$. These solutions also satisfy inequalities (3.18)–(3.21) for $f=f^{\varepsilon'}$ and the right-hand side of these inequalities is bounded by a constant independent of ε. Therefore, we can proceed as in property (c), extracting a subsequence of $\{u^{\varepsilon'}_{f^{\varepsilon'}}\}_{\varepsilon'}$, still denoted by ε', converging as $\varepsilon'\to 0$ towards some function u^* in $H^1(G_1)$-weak, u^* verifying $u^*=0$ on $\gamma^\pm$, and $\partial_2 u^*=0$. Thus, we have found $u^*\in H_0^1(-1,1)$ such that the convergence

$$\left\|A^{\varepsilon'}f^{\varepsilon'} - R^{\varepsilon'}u^*\right\|_{L^2_{\mathrm{dec},\delta}}\to 0\quad\text{as } \varepsilon'\to 0$$

holds true and this shows property (d) in Lemma 2.1 and conclude the proof of the two first assertions in the theorem.

The last assertion in the statement of the theorem, i.e., that the limits $\{v_{0,i}\}_{i=1}^\infty$ of the eigenfunctions form an orthonormal basis of $L^2(-1,1)$ for the scalar product (2.15), is proved by applying the reasoning in (3.15)–(3.16) and the fact that the eigenvalues $\lambda_{0,i}$ are simple. □

Remark 3.1. Note that there is no point in the bounds and convergence results for $\delta=\delta(\varepsilon)\to 0$ throughout the proof of Theorem 3.1. Actually, the fact that δ is a fixed constant for the proofs of the present section above Theorem 3.1 and for properties (a)–(c) in Theorem 3.1 is not important. Note that this fact is due to the identification of the functions of $H^1_{\mathrm{dec},\delta}(G_1)$ with $H^1_{\mathrm{dec},1,\delta}$ defined in (3.1) for $\varepsilon=1$. In contrast, for the proof of property (d), in the case where $\delta(\varepsilon)\to 0$, we cannot derive the result in the same way, since the sequence f^ε that we have taken in Theorem 3.1 can depend on δ. In this case, the result also holds as we state in Theorem 3.2 below.

Theorem 3.2. *Let δ be a parameter depending on ε, $\delta=\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$. For each fixed i, $i=1,2,\ldots,$ the eigenvalues $\{\lambda_{\varepsilon,\mathrm{dec},i}\}_\varepsilon$ of* (3.13) *converge towards the ith eigenvalue $\lambda_{0,i}$ of* (2.8) *as $\varepsilon\to 0$. Moreover, for each eigenfunction v_0 of* (2.8) *associated with $\lambda_{0,i}$, there is an eigenfunction $\hat{u}_{\varepsilon,\mathrm{dec},i}(x_1,y_2)$ associated with $\lambda_{\varepsilon,\mathrm{dec},i}$ converging towards v_0 in $L^2(G_1)$ as $\varepsilon\to 0$ (note that we identify $v_0(x_1,y_2)=v_0(x_1)$ and the convergence in $H^1(G_1)$-weak also holds). In addition, for each sequence we can extract a subsequence still denoted by ε such that $u_{\varepsilon,\mathrm{dec},i}\to v_{0,i}$ in $H^1(G_1)$-weak, as $\varepsilon\to 0$, where $u_{\varepsilon,\mathrm{dec},i}$ are the eigenfunctions of* (3.13) *satisfying* (3.4) *and the set $\{v_{0,i}\}_{i=1}^\infty$ forms an orthonormal basis of $L^2(-1,1)$ for the scalar product* (2.15).

Proof. The proof of Theorem 3.1 holds in the case of the present theorem, where $\delta=\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$ with the only exception of property (d) (see Remark 3.1) which is proved as follows: Considering any sequence $f^\varepsilon\in L^2_{\mathrm{dec},\delta(\varepsilon)}$, such that $\|f^\varepsilon\|_{L^2_{\mathrm{dec},\delta(\varepsilon)}}\leqslant C$, we take $g^\varepsilon=f^\varepsilon$ in $G_{\varepsilon,\delta(\varepsilon)}$ and $g^\varepsilon(x_1,y_2)=f^\varepsilon(x_1)$ for $|x_1|>\delta(\varepsilon)$, and we prove that $\|g^\varepsilon\|_{L^2(G_1)} \leqslant C$ with C a constant independent of ε. Then, we can extract a subsequence $g^{\varepsilon'}$ converging weakly towards some g^* in $L^2(G_1)$. Let $u^{\varepsilon'}_{g^{\varepsilon'}}$ be the sequence of solutions of (3.17) for $f = g^{\varepsilon'}$. These solutions also satisfy inequalities (3.18)–(3.21) for $f = g^{\varepsilon'}$ and the right-hand side of these inequalities is bounded by a constant independent of ε. Therefore, we can proceed as in property (c), extracting a subsequence of $\{u^{\varepsilon'}_{g^{\varepsilon'}}\}_{\varepsilon'}$, still denoted by ε', converging as $\varepsilon' \to 0$ towards some function u^* in $H^1(G_1)$-weak, u^* verifying $u^* = 0$ on $\gamma^\pm$, and $\partial_2 u^* = 0$. Thus, we have found $u^* \in H^1_0(-1,1)$ such that the convergence,

$$\left\| A^{\varepsilon'} f^{\varepsilon'} - R^{\varepsilon'} u^* \right\|_{L^2_{\mathrm{dec},\delta(\varepsilon)}} \to 0 \quad \text{as } \varepsilon' \to 0,$$

holds, and this shows property (d) in Lemma 2.1 concluding the proof of the theorem. □

4. Asymptotics for two coupled thin domains

In this section, we look for complete asymptotic expansions of the eigenvalues and eigenfunctions of (2.1)–(2.3) up to the order J for any positive J. We use the technique developed in [18] for stationary problems. Its validity for the spectral problems in this paper is justified in Section 5. We follow the notation introduced in Section 2.

An asymptotic solution of (2.1)–(2.3) is sought in a form:

$$u_\varepsilon^{(J)}(x) = \sum_{j=0}^{J} \varepsilon^j \left(U_j\left(\frac{x}{\varepsilon}\right) + v_j(x_1) \right), \quad x \in G_\varepsilon, \tag{4.1}$$

$$\lambda_\varepsilon^{(J)} = \sum_{j=0}^{J} \varepsilon^j \lambda_j, \tag{4.2}$$

with $U_0 \equiv 0$. We denote by $\xi = x\varepsilon^{-1}$, and, since in (4.1) only variables ξ and x_1 appear, the derivative with respect to the x_1 variable, $\partial_1 v$, is denoted by v'. Substituting (4.1), (4.2) into (2.1)–(2.3) we obtain the left-hand side of (2.1), that is,

$$\Delta u_\varepsilon^{(J)} + \lambda_\varepsilon^{(J)} u_\varepsilon^{(J)} = \sum_{j=1}^{J} \varepsilon^{j-2} \Delta_\xi U_j(\xi) + \sum_{j=0}^{J} \varepsilon^j v_j'' + \sum_{j=0}^{2J} \varepsilon^j \sum_{p+q=j} \lambda_p (U_q + v_q), \tag{4.3}$$

(under convention that λ_i, U_i or v_i are equal to zero if $i < 0$ or $i > J$). Gathering the terms of order ε^{j-2}, we obtain,

$$\sum_{j=1}^{J} \varepsilon^{j-2} \left\{ \left(\Delta_\xi U_j + \sum_{p+q=j-2} \lambda_p U_q \right) + \left(v_{j-2}'' + \sum_{p+q=j-2} \lambda_p v_q \right) \right\} + r_\varepsilon^{(J)}\left(x_1, \frac{x}{\varepsilon}\right), \tag{4.4}$$

where

$$r_\varepsilon^{(J)}(x,\xi) = \sum_{j=J-1}^{J} \varepsilon^j v_j''(x_1) + \sum_{j=J-1}^{2J} \varepsilon^j \sum_{p+q=j} \lambda_p \big(U_q(\xi) + v_q(x_1) \big), \tag{4.5}$$

Thus, gathering the equations for the ξ and x_1 variables respectively, we choose U_j and v_j satisfying the equations:

$$\Delta_\xi U_j = F_j(\xi), \quad F_j(\xi) = -\sum_{p+q=j-2} \lambda_p U_q(\xi), \tag{4.6}$$

and

$$v_j'' + \lambda_0 v_j = f_j(x_1), \quad f_j = -\sum_{\substack{p+q=j \\ (p,q)\neq(0,j)}} \lambda_p v_q, \tag{4.7}$$

respectively, for $\xi_1 \neq 0$ and $x_1 \neq 0$.

The substitution of (4.1) and (4.2) into condition (2.2) gives the following expression:

$$\sum_{j=0}^{J} \varepsilon^j \left(U_j \left(\pm\frac{1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) + v_j(\pm 1) \right), \tag{4.8}$$

where we require that $v_j(\pm 1) = 0$ and the traces of U_j at the surfaces $\{x_1 = \pm 1\}$ are exponentially small (this will be further explained; cf. also Appendix A).

The substitution into condition (2.3) gives, for the part of boundary $\partial G_\varepsilon \setminus \{\gamma_\varepsilon^- \cup \gamma_\varepsilon^+ \cup \Gamma_\varepsilon\}$,

$$\frac{\partial U_j}{\partial \xi_2} = 0 \quad \text{if } \xi_2 = \pm\frac{1}{2} \text{ and } \xi_1 > 0, \quad \text{or} \quad \text{if } \xi_2 = \pm\frac{1}{4} \text{ and } \xi_1 < 0, \tag{4.9}$$

and on the part of the boundary $\Gamma_\varepsilon \setminus S_\varepsilon$: $\frac{\partial U_j}{\partial \xi_1}|_{\xi_1=0} + v'_{j-1}|_{x_1=0} = 0$, i.e.,

$$\frac{\partial U_j}{\partial \xi_1}(+0, \xi_2) = -v'_{j-1}(+0). \tag{4.10}$$

Let us impose a jump for U_j and $\frac{\partial U_j}{\partial \xi_1}$ on the surface S_1:

$$[U_j] = C_{j1}, \tag{4.11}$$

$$\left[\frac{\partial U_j}{\partial \xi_1} \right] = C_{j2}, \tag{4.12}$$

where C_{j1}, C_{j2} are certain constants to be determined and by $[F]$ we denote $[F] = \lim_{x_1 \to +0} F(x) - \lim_{x_1 \to -0} F(x)$. Consequently, v_j and v'_{j-1} will have the opposite jumps (since the transmission condition should hold for the $u_\varepsilon^{(J)}$):

$$[v_j] = -C_{j1}, \tag{4.13}$$

$$[v'_j] = -C_{j+1,2}. \tag{4.14}$$

Therefore, we have

$$\left[u_\varepsilon^{(J)}\right] = 0 \quad \text{and} \quad \left[\frac{\partial u_\varepsilon^{(J)}}{\partial x_1} \right] = \varepsilon^J \left[v'_J \right]. \tag{4.15}$$

In this way, we get for U_j a sequence of problems set in the unbounded domain G:

$$\Delta_\xi U_j = F_j(\xi), \quad \xi_1 \neq 0, \tag{4.16}$$

$$[U_j] = C_{j1}, \quad \text{if } \xi_1 = 0,\ \xi_2 \in \left[-\frac{1}{4}, \frac{1}{4} \right], \tag{4.17}$$

$$\left[\frac{\partial U_j}{\partial \xi_1} \right] = C_{j2}, \quad \text{if } \xi_1 = 0,\ \xi_2 \in \left[-\frac{1}{4}, \frac{1}{4} \right], \tag{4.18}$$

$$\frac{\partial U_j}{\partial \xi_2} = 0, \quad \text{if } \xi_2 = \pm\frac{1}{2} \text{ and } \xi_1 > 0, \quad \text{or,} \quad \text{if } \xi_2 = \pm\frac{1}{4} \text{ and } \xi_1 < 0, \tag{4.19}$$

$$\frac{\partial U_j}{\partial \xi_1} |_{\xi_1=0} = -v'_{j-1}(+0), \quad \text{if } \xi_1 = 0,\ |\xi_2| > \frac{1}{2}, \tag{4.20}$$

$$\lim_{|\xi_1| \to \infty} U_j(\xi) = 0. \tag{4.21}$$

Integrating (4.16) in $\xi_1 < 0$ ($\xi_1 > 0$, respectively) and using (4.18) (see also Appendix A), we have:

$$\int_{G^- \cup G^+} \Delta_\xi U_j \, \mathrm{d}\xi = \int_{G^- \cup G^+} F_j(\xi) \, \mathrm{d}\xi,$$

i.e.,

$$\int_{S_1^-} \frac{\partial U_j}{\partial \xi_1} \bigg|_{\xi_1=-0} \mathrm{d}\xi_2 - \int_{\Gamma_1} \frac{\partial U_j}{\partial \xi_1} \bigg|_{\xi_1=+0} \mathrm{d}\xi_2 = \int_{G^- \cup G^+} F_j(\xi) \, \mathrm{d}\xi,$$

i.e.,

$$-\int_{S_1}\left[\frac{\partial U_j}{\partial \xi_1}\right]\mathrm{d}\xi_2-\int_{\Gamma_1\backslash S_1^+}\frac{\partial U_j}{\partial \xi_1}\bigg|_{\xi_1=+0}\mathrm{d}\xi_2=\int_{G^-\cup G^+}F_j(\xi)\,\mathrm{d}\xi,$$

$$-C_{j2}\,\mathrm{mes}(S_1)+v'_{j-1}(+0)\,\mathrm{mes}(\Gamma_1\backslash S_1)=\int_{G^-\cup G^+}F_j(\xi)\,\mathrm{d}\xi, \tag{4.22}$$

which gives the value of the constant C_{j2},

$$C_{j2}=v'_{j-1}(+0)-2\int_{G^-\cup G^+}F_j(\xi)\,\mathrm{d}\xi,\quad \text{where } F_j(\xi)=-\sum_{p+q=j-2}\lambda_p U_q(\xi). \tag{4.23}$$

For v_j we have the sequence of problems:

$$(v_j)''+\lambda_0 v_j=f_j(x_1), \tag{4.24}$$

$$v_j(\pm 1)=0, \tag{4.25}$$

$$[v_j]=-C_{j1}, \tag{4.26}$$

$$[v'_j]=-C_{j+1,2}, \tag{4.27}$$

where the last condition can we rewritten in a form (see (4.23)):

$$[v'_j]=-v'_j(+0)-2\int_{G^-\cup G^+}\sum_{p+q=j-1}\lambda_p U_q(\xi)\,\mathrm{d}\xi, \tag{4.28}$$

or equivalently,

$$2v'_j(+0)-v'_j(-0)=-2\int_{G^-\cup G^+}\sum_{p+q=j-1}\lambda_p U_q(\xi)\,\mathrm{d}\xi. \tag{4.29}$$

Multiplying artificially (4.24) by 2 for $x_1>0$, we obtain the problems for the terms $v_j(x_1)$:

$$(Kv'_j)'+\lambda_0 K v_j=Kf_j(x_1), \tag{4.30}$$

$$v_j(\pm 1)=0, \tag{4.31}$$

$$[v_j]=-C_{j1}, \tag{4.32}$$

$$[Kv'_j]=-2\int_{G^-\cup G^+}\sum_{p+q=j-1}\lambda_p U_q(\xi)\,\mathrm{d}\xi, \tag{4.33}$$

where

$$K(x_1)=\begin{cases}2 & \text{for } x_1>0,\\ 1 & \text{for } x_1<0,\end{cases}$$

and

$$f_j=-\sum_{\substack{p+q=j\\(p,q)\neq(0,j)}}\lambda_p v_q.$$

Now, to obtain all the terms in asymptotics (4.1)–(4.2), we start with $j=0$ in the last sequence of problems for v_j. We have (λ_0,v_0) is an eigenelement of the Dirichlet problem (2.9)–(2.12) which has only simple positive eigenvalues; that is, each eigenvalue $\lambda_0>0$ and it is simple (see after (2.8) for explicit formulas for the eigenvalues and eigenfunctions).

Considering the sequence of problems for U_j (4.16)–(4.21), starting from $j=0$, by the assumption performed on the first term of (4.1), we have that $U_0\equiv 0$ (note that this could also be obtained from definitions, $F_0=0$, $v_{-1}\equiv 0$, and once we have fixed λ_0 and v_0 above, $C_{01}=C_{02}=0$). For $j=1$ we obtain that U_1 satisfies:

$$\Delta_\xi U_1 = 0, \quad \xi_1 \neq 0, \tag{4.34}$$

$$[U_1] = C_{11}, \quad \text{if } \xi_1 = 0,\ \xi_2 \in \left[-\frac{1}{4}, \frac{1}{4}\right], \tag{4.35}$$

$$\left[\frac{\partial U_1}{\partial \xi_1}\right] = C_{12}, \quad \text{if } \xi_1 = 0,\ \xi_2 \in \left[-\frac{1}{4}, \frac{1}{4}\right], \tag{4.36}$$

$$\frac{\partial U_1}{\partial \xi_2} = 0, \quad \text{if } \xi_2 = \pm\frac{1}{2} \text{ and } \xi_1 > 0, \quad \text{or,} \quad \text{if } \xi_2 = \pm\frac{1}{4} \text{ and } \xi_1 < 0, \tag{4.37}$$

$$\left.\frac{\partial U_1}{\partial \xi_1}\right|_{\xi_1 = 0} = -v_0'(+0), \quad \text{for } |\xi_2| > \frac{1}{4}, \tag{4.38}$$

and

$$\lim_{|\xi_1| \to \infty} U_1(\xi) = 0, \tag{4.39}$$

where $C_{12} = v_0'(+0)$, and C_{11} is an unknown but well determined constant (see Remark 4.1).

Then, we determine consecutively $v_1, U_2, v_2, U_3, \ldots$. Every v_j is a solution of problem (4.30)–(4.33).

Let us study this problem (4.30)–(4.33), where we rewrite the transmission condition (4.33) as

$$[K v_j'] = Q_j, \quad Q_j = -2 \int\limits_{G^- \cup G^+} \sum_{p+q=j-1} \lambda_p U_q(\xi)\, \mathrm{d}\xi, \tag{4.40}$$

and we make the change:

$$w_j = \begin{cases} v_j & \text{if } x_1 < 0, \\ v_j + C_{j1} - Q_j x_1(1 - x_1)/2 - C_{j1} x_1^2 & \text{if } x_1 > 0. \end{cases} \tag{4.41}$$

Then, we have the non-homogeneous problem:

$$(K w_j')' + \lambda_0 K w_j = g_j(x_1), \tag{4.42}$$

$$w_j(\pm 1) = 0, \tag{4.43}$$

$$[w_j] = 0, \tag{4.44}$$

$$[K w_j'] = 0, \tag{4.45}$$

where

$$g_j(x_1) = \begin{cases} f_j & \text{for } x_1 < 0, \\ 2 f_j + 2 Q_j - 4 C_{j1} + 2\lambda_0 (C_{j1} - Q_j x_1(1 - x_1)/2 - C_{j1} x_1^2) & \text{for } x_1 > 0. \end{cases} \tag{4.46}$$

This problem (4.42)–(4.46) has a solution if,

$$\int\limits_{-1}^{1} g_j(x_1) v_0(x_1)\, \mathrm{d}x_1 = 0, \tag{4.47}$$

and the set of solutions is $S = \{\overline{w}_j + \text{constant } v_0\}$, where $\overline{w}_j$ is the unique solution of (4.42)–(4.46) such that

$$\int\limits_{-1}^{1} K(x_1) \overline{w_j}(x_1) v_0(x_1)\, \mathrm{d}x_1 = 0. \tag{4.48}$$

So, λ_j is chosen from condition (4.47):

$$-\int\limits_{-1}^{1} K(x_1) \lambda_j v_0(x_1)^2\, \mathrm{d}x_1 = \int\limits_{-1}^{1} K(x_1) \sum_{\substack{p+q=j \\ (p,q) \neq (0,j),(j,0)}} \lambda_p v_q(x_1) v_0(x_1)\, \mathrm{d}x_1$$

$$+\int_0^1 \left(-2Q_j+4C_{j1}-2\lambda_0\left(C_{j1}-Q_jx_1(1-x_1)/2-C_{j1}x_1^2\right)\right)v_0(x_1)\,\mathrm{d}x_1.$$

For instance, for $j=1$, we have:

$$\lambda_1=-\frac{1}{\int_{-1}^1 K(x_1)v_0(x_1)^2\,\mathrm{d}x_1}\int_0^1 \left((4-2\lambda_0)+2\lambda_0x_1^2\right)C_{11}v_0(x_1)\,\mathrm{d}x_1. \tag{4.49}$$

Remark 4.1. For each fixed j, in order to find the constant C_{j1} in problem (4.16)–(4.21), we can solve this problem first with the homogeneous condition in (4.17). Let us denote its solution (such that $\lim_{\xi_1\to-\infty}U_j(\xi)=0$) as $\overline{U}_j$. This solution stabilizes to some constant $\bar{\alpha}_j$ as $\xi_1\to+\infty$. Then, let us define,

$$U_j=\begin{cases}\overline{U_j} & \text{if } \xi_1<0,\\ \overline{U_j}-\overline{\alpha_j} & \text{if } \xi_1>0.\end{cases}$$

Clearly, U_j satisfies (4.16)–(4.21) with $C_{j1}=-\bar{\alpha}_j$. See [12] and [18] in connection with the kind of problems satisfied by functions U_j and for further references. See Appendix A for a complete study on the existence and uniqueness of the specific function U_j used here. In particular, for the following estimates in the present section, namely to derive formulas (4.54), (4.55), (4.57) and (4.58), we only use the fact that $U_j\in H^1(G^\pm)$, $\nabla U_j\in(L^2(G^\pm))^2$ and estimates (4.50).

To justify solution (4.1), (4.2) we have to modify slightly the function $u_\varepsilon^{(J)}$ in (4.1) because its trace on $\gamma_\varepsilon^\pm$ is not exactly equal to zero, and the boundary condition (2.3) is not exactly satisfied on $\Gamma_\varepsilon\backslash S_\varepsilon$.

To fix the first problem, we can multiply every $U_j(\frac{x}{\varepsilon})$ by the function $\eta(\frac{|x_1|}{\delta_\varepsilon})$, where $\delta_\varepsilon=\tilde{k}\varepsilon|\ln\varepsilon|$, $\tilde{k}$ does not depend on ε, and δ_ε and η are such that

$$\left|U_j(\xi)\right|,\left|\nabla U_j(\xi)\right|,\left|\Delta U_j(\xi)\right|<\overline{C}\mathrm{e}^{-\overline{\overline{C}}\delta_\varepsilon\varepsilon^{-1}}\leqslant\overline{C}\varepsilon^J, \tag{4.50}$$

for $|\xi_1|>\frac{\delta_\varepsilon}{3\varepsilon}$, where $\overline{C}$ and $\overline{\overline{C}}$ are certain constants independent of ε. That is,

$$-\overline{\overline{C}}\frac{\delta_\varepsilon}{\varepsilon}\leqslant-J|\ln\varepsilon|,\quad \delta_\varepsilon\geqslant J\left(\overline{\overline{C}}\right)^{-1}\varepsilon|\ln\varepsilon|,$$

which implies $\tilde{k}=J(\overline{\overline{C}})^{-1}$.

It is easy to check that function η can be defined as a function $\eta\in C^1(\mathbb{R})$, $0\leqslant\eta\leqslant1$, $\operatorname{supp}(\eta')\subset[\frac{1}{3},\frac{2}{3}]$, and

$$\eta(t)=\begin{cases}1, & \text{for } t\leqslant\dfrac{1}{3},\\ \dfrac{1}{2}\left(\cos\left(3\pi\left(t-\dfrac{1}{3}\right)\right)+1\right), & \text{for } t\in\left[\dfrac{1}{3},\dfrac{2}{3}\right],\\ 0, & \text{for } t\geqslant\dfrac{2}{3}.\end{cases} \tag{4.51}$$

In addition, in what follows we consider:

$$\delta_\varepsilon=\tilde{k}\varepsilon|\log\varepsilon|,\quad \text{where } \tilde{k}=J\left(\overline{\overline{C}}\right)^{-1} \tag{4.52}$$

with $\overline{\overline{C}}$ a well determined constant independent of ε.

Then, we denote by:

$$\tilde{u}_\varepsilon^{(J)}=\sum_{j=0}^J\varepsilon^j\left(U_j\left(\frac{x}{\varepsilon}\right)\eta\left(\frac{|x_1|}{\delta_\varepsilon}\right)+v_j(x_1)\right), \tag{4.53}$$

and we have:

$$\Delta\tilde{u}_\varepsilon^{(J)} + \lambda_\varepsilon^{(J)}\tilde{u}_\varepsilon^{(J)} = \Delta u_\varepsilon^{(J)} + \lambda_\varepsilon^{(J)} u_\varepsilon^{(J)} + \Delta\left(\sum_{j=0}^{J}\varepsilon^j U_j\left(\frac{x}{\varepsilon}\right)\left(\eta\left(\frac{|x_1|}{\delta_\varepsilon}\right)-1\right)\right)$$
$$+\lambda_\varepsilon^{(J)}\left(\sum_{j=0}^{J}\varepsilon^j U_j\left(\frac{x}{\varepsilon}\right)\left(\eta\left(\frac{|x_1|}{\delta_\varepsilon}\right)-1\right)\right) = r_\varepsilon^{(J)}\left(x_1,\frac{x}{\varepsilon}\right) + \bar{r}_\varepsilon^{(J)}(x),$$

where

$$r_\varepsilon^{(J)} = \Delta u_\varepsilon^{(J)} + \lambda_\varepsilon^{(J)} u_\varepsilon^{(J)}$$

and

$$\bar{r}_\varepsilon^{(J)} = \Delta\left(\sum_{j=0}^{J}\varepsilon^j U_j\left(\frac{x}{\varepsilon}\right)\left(\eta\left(\frac{|x_1|}{\delta_\varepsilon}\right)-1\right)\right) + \lambda_\varepsilon^{(J)}\left(\sum_{j=0}^{J}\varepsilon^j U_j\left(\frac{x}{\varepsilon}\right)\left(\eta\left(\frac{|x_1|}{\delta_\varepsilon}\right)-1\right)\right).$$

On account of Remark 4.1, and bounds (4.50), we can derive the following estimates:

$$\left\| r_\varepsilon^{(J)}\left(x_1,\frac{x}{\varepsilon}\right)\right\|_{L^2(G_\varepsilon)} \leqslant \overline{C}_J\varepsilon^{J-1}\sqrt{\varepsilon} \quad \left\|\nabla_x r_\varepsilon^{(J)}\left(x_1,\frac{x}{\varepsilon}\right)\right\|_{L^2(G_\varepsilon)} \leqslant \overline{C}_J\varepsilon^{J-1}\sqrt{\varepsilon}, \tag{4.54}$$

and

$$\left|\bar{r}_\varepsilon^{(J)}\right| \leqslant \widehat{C}\varepsilon^{J+1}\left(\frac{1}{\delta_\varepsilon}\right)^2 \leqslant \widehat{C}\varepsilon^{J-1}, \tag{4.55}$$

where C_J and $\widehat{C}$ are positive constants independent of ε.

We emphasize that estimates (4.54) and (4.55) come from the exponential rate of the decay (4.50) of functions U_j; it means that $\bar{r}_\varepsilon^{(J)}$ is of order $\mathrm{O}(\varepsilon^{J+1}\delta_\varepsilon^{-2})$ in the part G_ε where $(\eta(|x_1|\varepsilon^{-1})-1)$ differs from zero.

Thus, $\tilde{u}_\varepsilon^{(J)}, \lambda_\varepsilon^{(J)}$ satisfy the following problem (see also (4.15)):

$$-\int_{G_\varepsilon}\nabla\tilde{u}_\varepsilon^{(J)}\cdot\nabla\varphi\,\mathrm{d}x + \lambda_\varepsilon^{(J)}\int_{G_\varepsilon}\tilde{u}_\varepsilon^{(J)}\varphi\,\mathrm{d}x = \int_{G_\varepsilon}\left(r_\varepsilon^{(J)}\left(x_1,\frac{x}{\varepsilon}\right)+\bar{r}_\varepsilon^{(J)}(x)\right)\varphi\,\mathrm{d}x$$
$$+\int_{\Gamma_\varepsilon\setminus S_\varepsilon}\varepsilon^J[v_J']\varphi(0,x_2)\,\mathrm{d}x_2, \quad \forall\varphi\in H^1(G_\varepsilon),\ \varphi=0 \text{ on } \gamma_\varepsilon^{\pm}. \tag{4.56}$$

We observe that the functional of the right-hand side of (4.56) satisfies

$$\left|\Phi(\varphi)\right| \leqslant \widetilde{C}\cdot\left\{\varepsilon^{J-1}\|\varphi\|_{L^2(G_\varepsilon)}\sqrt{\varepsilon} + \varepsilon^J\left|[v_J']\right|\int_{\Gamma_\varepsilon\setminus S_\varepsilon}\left|\varphi(0,x_2)\right|\mathrm{d}x_2\right\}.$$

On the other hand, on account of the equality:

$$\varphi(x) = \varphi(0,x_2) + \int_0^{x_1}\frac{\partial\varphi}{\partial x_1}(t,x_2)\,\mathrm{d}t, \quad \forall(x_1,x_2)\in G_\varepsilon^+,$$

we have

$$\int_{\Gamma_\varepsilon\setminus S_\varepsilon}\left|\varphi(0,x_2)\right|\mathrm{d}x_2 \leqslant \int_{G_\varepsilon^+}\left|\varphi(x)\right|\mathrm{d}x + \int_{G_\varepsilon^+}\left|\int_0^{x_1}\frac{\partial\varphi}{\partial x_1}(t,x_2)\right|\mathrm{d}t\,\mathrm{d}x.$$

Therefore, by Cauchy–Schwartz–Buniakowsky inequality,

$$\int_{\Gamma_\varepsilon\setminus S_\varepsilon}\left|\varphi(0,x_2)\right|\mathrm{d}x_2 \leqslant \sqrt{\varepsilon}\left(\|\varphi\|_{L^2(G_\varepsilon^+)} + \left\|\frac{\partial\varphi}{\partial x_1}\right\|_{L^2(G_\varepsilon^+)}\right) \leqslant \sqrt{\varepsilon}C\|\varphi\|_{H^1(G_\varepsilon)},$$

and we obtain:

$$\left|\Phi(\varphi)\right| \leqslant \tilde{\tilde{C}}\varepsilon^{J-1}\sqrt{\varepsilon}\,\|\varphi\|_{H^1(G_\varepsilon)}, \tag{4.57}$$

where C, $\widetilde{C}$ and $\tilde{\tilde{C}}$ are constants independent of ε.

In addition, taking into account that $\|\nabla_\xi U_j\|_{L^2(G)}$ is bounded by a constant depending on j, the definitions of η and δ_ε, the fact that $U_0 = 0$, and bounds (4.50), we prove that there are constants $C_{1,J}, C_{2,J}$ independent of ε such that for sufficiently small ε, namely $\varepsilon < \varepsilon^*_{J,\lambda_0,v_0}$, we have:

$$C_{1,J}\sqrt{\varepsilon} \leqslant \left\|\nabla_x \tilde{u}^{(J)}\right\|_{L^2(G_\varepsilon)} \leqslant C_{2,J}\sqrt{\varepsilon}. \tag{4.58}$$

Hence, the algorithm of calculation of v_j, U_j, λ_j in the asymptotic expansions (4.1) and (4.2) is as follows: first, we compute v_0, λ_0 from (2.9)–(2.12). Second, from (4.34)–(4.39) we determine U_1, C_{11} and $C_{12} = v_0'(+0)$. Then, we determine λ_1 from (4.49) and therefore v_1 is also determined from (4.30)–(4.33), and consequently we can determine $U_2, \lambda_2, v_2, U_3, \lambda_3, v_3$ and so on. In this way, we can determine all the terms in asymptotics (4.1)–(4.2) for any $J \geqslant 1$ (note that the estimate (4.57) is obtained for $J \geqslant 1$).

5. On the approach of the eigenelements for two coupled thin domains

In this section we justify asymptotic expansions (4.1) and (4.2) for the eigenfunctions and eigenvalues of problem (2.1)–(2.3). We prove that (4.53) and (4.2) provide true approaches of the eigenelements of (2.1)–(2.3) and also of the eigenelements of (3.5)–(3.11) as stated in Theorems 5.1 and 5.3 respectively. In order to perform this proof, we apply a result from the spectral perturbation theory which we introduce below for the sake of completeness, and we refer to [20] for its proof. See Remark 5.4 for further uses of the result.

Lemma 5.1. *Let $A : H \to H$ be a linear, self-adjoint, positive and compact operator on a separable Hilbert space H. Let $u \in H$, with $\|u\|_H = 1$ and λ, $r > 0$ such that $\|Au - \lambda u\|_H \leqslant r$. Then, there exists an eigenvalue λ_i of A satisfying $|\lambda - \lambda_i| \leqslant r$. Moreover, for any $r^* > r$ there is $u^* \in H$, with $\|u^*\|_H = 1$, u^* belonging to the eigenspace associated with all the eigenvalues of the operator A lying on the segment $[\lambda - r^*, \lambda + r^*]$ and such that*

$$\left\|u - u^*\right\|_H \leqslant \frac{2r}{r^*}.$$

As a matter of fact, the results in Lemma 5.1 allow us to assert that the function given in formula (4.53) approaches quasimodes of (2.1)–(2.3) ((3.5)–(3.11) for $\delta = \tilde{k}\varepsilon|\log\varepsilon|$, respectively), as stated in Theorem 5.1 (Theorem 5.3, respectively). Then, in Theorems 5.2 and 5.4, we combine these results of Theorem 5.1 (Theorem 5.3, respectively) with those in Theorem 2.1 (Theorem 3.2, respectively) on the spectral convergence of the eigenelements of (2.1)–(2.3) ((3.5)–(3.11), respectively) towards those of the limit problem (2.8), to prove that we can approach any eigenvalue of (2.8) ((3.3), respectively) with a desired precision, namely $\mathrm{O}(\varepsilon^{J-1})$. Similar results are obtained for the associated eigenfunctions in the x-variable, and, consequently, formulas (4.53) and (4.2) provide correctors. We gather the convergence results for problem (2.1)–(2.3) in Section 5.1 while those for problem (3.5)–(3.11) are in Section 5.2.

Finally, using the previous results, in Theorem 5.5 we prove that the eigenelements of the problem in the decomposed domain (namely, problem (3.3)) also provide a good approach for the eigenelements of the two rods spectral problem and we can use them for numerical computations. Throughout the section, if no confusion arises, we use indifferently either the differential formulation or the variational formulation of the problems under consideration.

5.1. On the two-rod spectral problem

Theorem 5.1. *Let λ_0 be any eigenvalue of* (2.9)–(2.12)*, and v_0 the associated eigenfunction. Let J be any positive integer, $J \geqslant 2$. Let us consider the value $\lambda_\varepsilon^{(J)}$ and the function $\tilde{u}_\varepsilon^{(J)}$ defined by* (4.2) *and* (4.53) *respectively, constructed with the algorithm in Section* 4*. Then, there exists at least one eigenvalue λ^ε of* (2.1)–(2.3) *such that, for sufficiently small ε, namely $\varepsilon < \varepsilon^*_{(J,\lambda_0,v_0)}$,*

$$\left|\lambda^\varepsilon - \lambda_\varepsilon^{(J)}\right| \leqslant C_{(J,\lambda_0,v_0)}\varepsilon^{J-1}, \tag{5.1}$$

where $C_{(J,\lambda_0,v_0)}$ is a certain positive constant independent of ε. In addition, for any sequence $d^\varepsilon \to 0$, such that $\lim_{\varepsilon\to 0}(\varepsilon^{J-1}/d^\varepsilon) = 0$, the interval $[(\lambda_\varepsilon^{(J)})^{-1} - d^\varepsilon, (\lambda_\varepsilon^{(J)})^{-1} + d^\varepsilon]$ contains the values $\{(\lambda_{\varepsilon,l})^{-1}\}_{l=p}^q$, $\{\lambda_{\varepsilon,l}\}_{l=p}^q$ being eigenvalues of (2.1)–(2.3)*, for l ranging between certain natural numbers $p = p(\varepsilon)$ and $q = q(\varepsilon)$, $p(\varepsilon) \leqslant q(\varepsilon)$, and, there is a function $\tilde u_\varepsilon$ in the eigenspace of the associated eigenfunctions $\{u_{\varepsilon,l}\}_{l=p}^q$, with $\|\nabla \tilde u_\varepsilon\|_{L^2(G_\varepsilon)} = 1$, $\tilde u^\varepsilon$ such that*

$$\left\|\nabla \tilde u^\varepsilon - \alpha^\varepsilon \nabla \tilde u_\varepsilon^{(J)}\right\|_{L^2(G_\varepsilon)} \leqslant \frac{2C_{(J,\lambda_0,v_0)}\varepsilon^{J-1}}{d^\varepsilon}, \tag{5.2}$$

for sufficiently small ε, α^ε being the sequence of $\alpha^\varepsilon = (\|\nabla \tilde u_\varepsilon^{(J)}\|_{L^2(G_\varepsilon)})^{-1}$.

Proof. Let us consider $\mathbf{H}^\varepsilon = \{\varphi \in H^1(G_\varepsilon),\ \varphi = 0 \text{ on } \gamma_\varepsilon^\pm\}$, with the norm $\|u\|_{\mathbf{H}^\varepsilon} = \|\nabla u\|_{L^2(G_\varepsilon)}$. Let $\mathcal{A}^\varepsilon$ be the compact, symmetric and positive operator on $\mathbf{H}^\varepsilon$ defined by:

$$\langle \mathcal{A}^\varepsilon u, v\rangle_{\mathbf{H}^\varepsilon} = \int\limits_{G_\varepsilon} uv\,\mathrm{d}x, \quad \forall u, v, \in \mathbf{H}^\varepsilon.$$

We consider the eigenvalue problem for $\mathcal{A}^\varepsilon$ in the space $\mathbf{H}^\varepsilon$: Find $u_\varepsilon \in \mathbf{H}^\varepsilon$, $u_\varepsilon \neq 0$, such that $\mathcal{A}^\varepsilon u_\varepsilon = \mu_\varepsilon u_\varepsilon$, which amounts to,

$$\int\limits_{G_\varepsilon} u_\varepsilon v\,\mathrm{d}x = \mu_\varepsilon \int\limits_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla v\,\mathrm{d}x, \quad \forall v \in \mathbf{H}^\varepsilon. \tag{5.3}$$

Therefore, the eigenelements of $\mathcal{A}^\varepsilon$ are $\{(\lambda_{\varepsilon,i}^{-1}, u_{\varepsilon,i})\}_{i=1}^\infty$, with $(\lambda_{\varepsilon,i}, u_{\varepsilon,i})$ the eigenelements of problem (2.1)–(2.3). Now, for each (λ_0, v_0) eigenelement of problem (2.9)–(2.12) (or equivalently, (2.8)), we consider $\lambda_\varepsilon^{(J)}$ defined by (4.2) and $\tilde u_\varepsilon^{(J)}$ defined by (4.53). From the fact that $\tilde u_\varepsilon^{(J)} \in \mathbf{H}^\varepsilon$ and on account of (4.56), (4.57), the Poincaré inequality on $G_\varepsilon^\pm$ for the elements of $\mathbf{H}^\varepsilon$, and the definition of $\mathcal{A}^\varepsilon$, we have:

$$\left|\left\langle \mathcal{A}^\varepsilon \tilde u_\varepsilon^{(J)} - \left(\lambda_\varepsilon^{(J)}\right)^{-1} \tilde u_\varepsilon^{(J)}, \varphi\right\rangle_{\mathbf{H}^\varepsilon}\right| \leqslant \tilde{\tilde{C}} \varepsilon^{J-1}\sqrt{\varepsilon}\left(\lambda_\varepsilon^{(J)}\right)^{-1}\|\varphi\|_{\mathbf{H}^\varepsilon}, \quad \forall \varphi \in \mathbf{H}^\varepsilon.$$

On the other hand, on account of (4.2), and (4.58) we have that for each fixed (λ_0, v_0) there are constants C_1, C_2, C_3, C_4 independent of ε, such that, for $\varepsilon \leqslant \varepsilon^*_{(J,\lambda_0,v_0)}$,

$$\sqrt{\varepsilon} C_1 \leqslant \left\|\tilde u_\varepsilon^{(J)}\right\|_{\mathbf{H}^\varepsilon} \leqslant \sqrt{\varepsilon} C_2; \quad \text{and} \quad C_3 < \left|\lambda_\varepsilon^{(J)}\right| < C_4. \tag{5.4}$$

Consequently, the estimate:

$$\left\|\mathcal{A}^\varepsilon \tilde u_\varepsilon^{(J)} - \left(\lambda_\varepsilon^{(J)}\right)^{-1} \tilde u_\varepsilon^{(J)}\right\|_{\mathbf{H}^\varepsilon} \leqslant \tilde{\tilde{C}} \varepsilon^{J-1}\sqrt{\varepsilon}, \tag{5.5}$$

holds, and, applying Lemma 5.1 to the operator $\mathcal{A}^\varepsilon$, the Hilbert space $\mathbf{H}^\varepsilon$, the function u, $u = u^{\varepsilon,(J)} = \tilde u_\varepsilon^{(J)}(\|\nabla \tilde u_\varepsilon^{(J)}\|_{L^2(G_\varepsilon)})^{-1}$ and the value λ, $\lambda = (\lambda_\varepsilon^{(J)})^{-1}$, we obtain that there is an eigenvalue $(\lambda^\varepsilon)^{-1}$ of $\mathcal{A}^\varepsilon$ such that

$$\left|\left(\lambda^\varepsilon\right)^{-1} - \left(\lambda_\varepsilon^{(J)}\right)^{-1}\right| \leqslant C_J \varepsilon^{J-1} \tag{5.6}$$

for a certain constant C_J independent of ε and for sufficiently small ε.

In the same way, for any d^ε, $d^\varepsilon > C_J \varepsilon^{J-2}$, there is a function u^ε in the eigenspace of all the eigenvalues of $\mathcal{A}^\varepsilon$ in $[(\lambda_\varepsilon^{(J)})^{-1} - d^\varepsilon, (\lambda_\varepsilon^{(J)})^{-1} + d^\varepsilon]$, with $\|u^\varepsilon\|_{\mathbf{H}^\varepsilon} = 1$, and such that

$$\left\|u^\varepsilon - u^{\varepsilon,(J)}\right\|_{\mathbf{H}^\varepsilon} \leqslant \frac{2C_J \varepsilon^{J-1}}{d^\varepsilon}. \tag{5.7}$$

Therefore, formulas (5.4) and (5.6) ensure a bound for $\lambda_\varepsilon^{(J)}$ and λ^ε independent of ε, for sufficiently small ε, and they give the approach for the eigenvalues of (2.1)–(2.3) by (4.2),

$$\left|\lambda^\varepsilon - \lambda_\varepsilon^{(J)}\right| \leqslant \widetilde{C}_J \varepsilon^{J-1}, \tag{5.8}$$

for another constant $\widetilde{C}_J$ independent of ε. Also, (5.7) gives the approach for the corresponding eigenfunctions for any $d^\varepsilon \to 0$, $d^\varepsilon > C_J \varepsilon^{J-1}$, as stated in the theorem, and the proof is concluded. □

The following theorem shows that for each $(\lambda_{0,i}, v_{0,i})$ eigenelement of (2.9)–(2.12), the eigenvalue and associated eigenfunction of (2.1)–(2.3) satisfying (5.7) and (5.8), in the case where $(\lambda_0, v_0) \equiv (\lambda_{0,i}, v_{0,i})$, is precisely the ith eigenelement of (2.1)–(2.3). That is, (4.53) and (4.2) provide true asymptotic expansions of the eigenfunctions and eigenvalues respectively of (2.1)–(2.3). We use Theorems 2.1 and 5.1 for this proof.

Theorem 5.2. *For each fixed $i = 1, 2, \dots,$ let $\lambda_{0,i}$ be the ith eigenvalue of* (2.9)–(2.12)*, and $v_{0,i}$ the associated eigenfunction. Let J be any positive integer, $J \geqslant 2$. Let us consider the value $\lambda_{\varepsilon,i}^{(J)}$ and the function $\tilde{u}_{\varepsilon,i}^{(J)}$ defined by* (4.2) *and* (4.53) *respectively, constructed with the algorithm in Section* 4 *for $(\lambda_0, v_0) \equiv (\lambda_{0,i}, v_{0,i})$. Then, the ith eigenvalue $\lambda_{\varepsilon,i}$ of* (2.1)–(2.3) *satisfies*:

$$\left|\lambda_{\varepsilon,i} - \lambda_{\varepsilon,i}^{(J)}\right| \leqslant C_{(J,i)} \varepsilon^{J-1}, \tag{5.9}$$

for sufficiently small ε, namely $\varepsilon < \varepsilon^*_{(J,\lambda_{0,i},v_{0,i})}$, where $C_{(J,i)} = C_{(J,\lambda_{0,i},v_{0,i})}$ is a certain constant independent of ε.

In addition, there is an eigenfunction $\tilde{u}_{\varepsilon,i}$, associated with $\lambda_{\varepsilon,i}$, satisfying $\|\nabla \tilde{u}_{\varepsilon,i}\|_{L^2(G_\varepsilon)} = 1$ and such that, for any strictly positive r, with $r < J - 1$, and for sufficiently small ε, the inequality,

$$\left\|\nabla \tilde{u}_{i,\varepsilon} - \alpha^\varepsilon \nabla \tilde{u}_{\varepsilon,i}^{(J)}\right\|_{L^2(G_\varepsilon)} \leqslant \widehat{C}_{(J,\lambda_{0,i},v_{0,i},r)} \varepsilon^{J-1-r}, \tag{5.10}$$

holds, where $\widehat{C}_{(J,\lambda_{0,i},v_{0,i},r)}$ is a constant independent of ε and $\alpha^\varepsilon = (\|\nabla \tilde{u}_{\varepsilon,i}^{(J)}\|_{L^2(G_\varepsilon)})^{-1}$.

Proof. On account of Theorem 2.1, $\lambda_{\varepsilon,i} \to \lambda_{0,i}$ as $\varepsilon \to 0$. Also, from the construction of $\lambda_{\varepsilon,i}^{(J)}$ in Section 4, $\lambda_{\varepsilon,i}^{(J)}$ is a well determined sequence of ε once that we have fixed the eigenfunction $v_{0,i}$, and $\lambda_{\varepsilon,i}^{(J)} \to \lambda_{0,i}$ as $\varepsilon \to 0$. On the other hand, Theorem 5.1 ensures that (5.1) holds for a certain eigenvalue of (2.5) λ_ε. Let the number of these eigenvalue be $k(i,\varepsilon)$, i.e., $\lambda_\varepsilon = \lambda_{\varepsilon,k(i,\varepsilon)}$. Then, inequality (5.9) will be proved once we prove that $k(i,\varepsilon) = i$ for sufficiently small ε, that is, $\lambda_{\varepsilon,k(i,\varepsilon)} = \lambda_{\varepsilon,i}$ for sufficiently small ε.

We prove this assertion in two steps. In a first step we prove that, for sufficiently small ε, we can take $k(i,\varepsilon)$ equal to a constant k independent of ε. In another step we prove that this constant k coincides with i: i.e., $k = i$.

Let us start with the proof of the second step assuming that the first step is already done, and prove that $k(i,\varepsilon) = k = i$.

Let us assume that there is a fixed $k \neq i$ such that the $\lambda_{\varepsilon,k(i,\varepsilon)}$ in Theorem 5.1 satisfies $\lambda_{\varepsilon,k(i,\varepsilon)} = \lambda_{\varepsilon,k}$. Let us denote by α the constant $\alpha = |\lambda_{0,k} - \lambda_{0,i}|$, which is assumed to be a strictly positive constant, i.e., $\alpha > 0$. Let β be a fixed constant, $\beta < \alpha/4$. Because of the above argument, (5.1) ensures the existence of ε_i^* such that

$$\left|\lambda_{\varepsilon,i}^{(J)} - \lambda_{\varepsilon,k}\right| < \beta, \quad \forall \varepsilon < \varepsilon_i^*.$$

Also, because of Theorem 2.1, there is $\varepsilon_{0,k}$ such that

$$|\lambda_{\varepsilon,k} - \lambda_{0,k}| < \beta, \quad \forall \varepsilon < \varepsilon_{0,k}.$$

On the other hand, the convergence of $\lambda_{\varepsilon,i}^{(J)}$ towards $\lambda_{0,i}$ gives the existence of $\varepsilon_{0,i}$ such that

$$\left|\lambda_{\varepsilon,i}^{(J)} - \lambda_{0,i}\right| < \beta, \quad \forall \varepsilon < \varepsilon_{0,i}.$$

Gathering the three last inequalities, for $\varepsilon < \min(\varepsilon_i^*, \varepsilon_{0,i}, \varepsilon_{0,k})$ we have:

$$\alpha = |\lambda_{0,k} - \lambda_{0,i}| \leqslant |\lambda_{\varepsilon,k} - \lambda_{0,k}| + \left|\lambda_{\varepsilon,i}^{(J)} - \lambda_{\varepsilon,k}\right| + \left|\lambda_{\varepsilon,i}^{(J)} - \lambda_{0,i}\right| < 3\beta < 3\alpha/4 < \alpha,$$

which is a contradiction. Therefore, $\alpha = 0$ and $\lambda_{0,i} = \lambda_{0,k}$ which implies $k = i$ and $\lambda_{\varepsilon,k(i,\varepsilon)} = \lambda_{\varepsilon,i}$ as we need to prove (5.9) in the statement of the theorem.

Now let us prove the first step, namely $k(i,\varepsilon)$ is a constant k independent of ε, for sufficiently small ε. Consider the case where $\{k(i,\varepsilon)\}_\varepsilon$ is uniformly bounded with respect to ε. Then, there is $K_1 \in \mathbb{N}$ such that $k(i,\varepsilon) \leqslant K_1$, for any $\varepsilon > 0$, and therefore $k(i,\varepsilon)$ takes a finite number of integer values that belong to the set $\{1, 2, \dots, K_1\}$. Let us fix k_0 one of these values such that $k(i,\varepsilon_j) = k_0$ for a some subsequence $\varepsilon_j \to 0$. Then, rewriting the proof of the previous step above we prove that $k(i,\varepsilon_j) = i$. Thus, in the case where $\{k(i,\varepsilon)\}_\varepsilon$ is uniformly bounded with respect to ε, for sufficiently small ε, $k(i,\varepsilon) = i$. Let us consider then the case where $\{k(i,\varepsilon)\}_\varepsilon$ is not uniformly bounded with respect to ε. Thus, there exists a sequence $\varepsilon_j \to 0$ such that $k(i,\varepsilon_j) \to \infty$. Consequently, for any fixed natural i_0 and for sufficiently small ε_j, namely $\varepsilon_j < \varepsilon^*(i_0)$, we have $\lambda_{\varepsilon_j,k(i,\varepsilon_j)} > \lambda_{\varepsilon_j,i_0}$. We can take $i_0 = i+1$, and, since $\lambda^{(J)}_{\varepsilon_j,i} \to \lambda_{0,i}$, the difference $|\lambda^{(J)}_{\varepsilon_j,i} - \lambda_{\varepsilon_j,k(i,\varepsilon_j)}|$ does not converge towards zero. This contradicts the assertion in Theorem 5.1 (cf. (5.1)) and our assumption.

Therefore we have proved $k(i,\varepsilon) = i$ and (5.9 is satisfied, for sufficiently small ε.

As regards the eigenfunctions, since the eigenvalues of (2.9)–(2.12) are simple, Theorem 2.1 ensures that, for any $d > 0$ such that $[\lambda_{0,i} - d, \lambda_{0,i} + d]$ does not contain more eigenvalues of (2.9)–(2.12) than $\lambda_{0,i}$, and for sufficiently small ε, there is only one simple eigenvalue $\lambda_{\varepsilon,i}$ in $[\lambda_{0,i} - d, \lambda_{0,i} + d]$. On the other hand, in the first part of the theorem (i.e., estimate (5.9)) we have proved that the eigenvalue in Theorem 5.1 satisfying (5.1) is precisely $\lambda_{\varepsilon,i}$. Now we prove that taking $d^\varepsilon = \varepsilon^r$, with $0 < r < J-1$, there are not more inverses of eigenvalues of (2.1)–(2.3) in the interval $[(\lambda^{(J)}_{\varepsilon,i})^{-1} - d^\varepsilon, (\lambda^{(J)}_{\varepsilon,i})^{-1} + d^\varepsilon]$ than $(\lambda_{\varepsilon,i})^{-1}$ and therefore the result on the eigenfunctions in (5.2) in Theorem 5.1 holds now for an eigenfunction $\tilde{u}_{\varepsilon,i}$ associated with $\lambda_{\varepsilon,i}$, such that $\|\nabla \tilde{u}_{\varepsilon,i}\|_{L^2(G_\varepsilon)} = 1$. That is, this eigenfunction associated with $\lambda_{\varepsilon,i}$ satisfies:

$$\left\| \nabla \tilde{u}_{\varepsilon,i} - \alpha^\varepsilon \nabla \tilde{u}^{(J)}_{\varepsilon,i} \right\|_{L^2(G_\varepsilon)} \leqslant C\varepsilon^{J-1-r},$$

for a certain constant C independent on ε, and this proves (5.10).

Let us proceed by contradiction, by assuming that the interval $[(\lambda^{(J)}_{\varepsilon,i})^{-1} - d^\varepsilon, (\lambda^{(J)}_{\varepsilon,i})^{-1} + d^\varepsilon]$ contains more than one eigenvalue $(\lambda_{\varepsilon,i})^{-1}$, for instance two eigenvalues $(\lambda_{\varepsilon,i})^{-1}$ and $(\lambda_{\varepsilon,i+1})^{-1}$. Then, on account of bounds (5.4) and asymptotics (4.2) we prove that for sufficiently small ε, they belong to the interval $[\lambda_{0,i} - d, \lambda_{0,i} + d]$. Indeed, for sufficiently small ε, we have:

$$|\lambda_{\varepsilon,i} - \lambda_{0,i}| \leqslant \left|\lambda_{\varepsilon,i} - \lambda^{(J)}_{\varepsilon,i}\right| + \left|\lambda^{(J)}_{\varepsilon,i} - \lambda_{0,i}\right| \leqslant C\varepsilon^r \leqslant d.$$

In the same way, the last inequalities also hold for $\lambda_{\varepsilon,i+1}$ and this is in contradiction with the d taken which avoids the existence of two eigenvalues of (2.1)–(2.3) in $[\lambda_{0,i} - d, \lambda_{0,i} + d]$. $\square$

Remark 5.1. Note that estimate (5.9) can also be improved. Indeed, for $\lambda^{(J+2)}_\varepsilon$, we obtain:

$$\left|\lambda_{\varepsilon,i} - \lambda^{(J+2)}_{\varepsilon,i}\right| \leqslant C_{J+2,i}\varepsilon^{J+1}.$$

On the other hand, for $\varepsilon \leqslant 1$,

$$\left|\lambda^{(J)}_{\varepsilon,i} - \lambda^{(J+2)}_{\varepsilon,i}\right| = \left| \sum_{j=J+1}^{J+2} \varepsilon^j \lambda_j \right| \leqslant \max\left\{|\lambda_{J+1}|, |\lambda_{J+2}|\right\}\varepsilon^{J+1},$$

and, therefore,

$$\left|\lambda_{\varepsilon,i} - \lambda^{(J)}_{\varepsilon,i}\right| \leqslant \left(C_{J+2,i} + \max\left\{|\lambda_{J+1}|, |\lambda_{J+2}|\right\}\right)\varepsilon^{J+1}.$$

In particular, for $J = 0$,

$$\left|\lambda_{\varepsilon,i} - \lambda_{0,i}\right| = \mathrm{O}(\varepsilon).$$

The same remark is valid for further estimates (5.15) and (6.30).

5.2. On the spectral problem in the decomposed domain

Throughout this section, we consider $u^{(J)}_\varepsilon$ and $\delta = \delta(\varepsilon) = \delta_\varepsilon = \tilde{k}\varepsilon|\ln\varepsilon|$, with the constant $\tilde{k}$ independent on ε (see (4.52)) introduced in Section 4 such that $u^{(J)}_\varepsilon \in H^1_{\mathrm{dec},\delta}(G_\varepsilon)$ and estimates (4.53)–(4.57) are satisfied. We also use the notations introduced in Section 3.

Theorem 5.3. *Let λ_0 be any eigenvalue of* (2.8)*, and v_0 the associated eigenfunction. Let J be any positive integer, $J \geqslant 2$. Let us consider the value $\lambda^{(J)}_\varepsilon$ and the function $\tilde{u}^{(J)}_\varepsilon$ defined by* (4.2) *and* (4.53) *respectively, constructed with the algorithm in Section* 4*. Then, there exists at least one eigenvalue $\lambda^\varepsilon_{\mathrm{dec}}$ of* (3.3) *such that, for sufficiently small ε, namely $\varepsilon < \varepsilon^*_{(J,\lambda_0,v_0)}$,*

$$\left|\lambda^\varepsilon_{\mathrm{dec}} - \lambda^{(J)}_\varepsilon\right| \leqslant C_{(J,\lambda_0,v_0)}\varepsilon^{J-1}, \tag{5.11}$$

where $C_{(J,\lambda_0,v_0)}$ is certain positive constant independent of ε. In addition, for any sequence $d^\varepsilon \to 0$, such that $\lim_{\varepsilon\to 0}(\varepsilon^{J-1}/d^\varepsilon) = 0$, the interval $[(\lambda_\varepsilon^{(J)})^{-1} - d^\varepsilon, (\lambda_\varepsilon^{(J)})^{-1} + d^\varepsilon]$ contains the values $\{(\lambda_{\varepsilon,\mathrm{dec},l})^{-1}\}_{l=p}^{q}$, $\{\lambda_{\varepsilon,\mathrm{dec},l}\}_{l=p}^{q}$ being eigenvalues of (3.3)*, for l ranging between certain natural numbers $p = p(\varepsilon)$ and $q = q(\varepsilon)$, $p(\varepsilon) \leqslant q(\varepsilon)$, and, there is a function $\tilde{u}^\varepsilon_{\mathrm{dec}}$ in the eigenspace associated of the eigenfunctions $\{u_{\varepsilon,\mathrm{dec},l}\}_{l=p}^{q} \subset H^1_{\mathrm{dec},\delta}(G_\varepsilon)$, with $\|\nabla\tilde{u}^\varepsilon_{\mathrm{dec}}\|_{L^2(G_\varepsilon)} = 1$, $\tilde{u}^\varepsilon_{\mathrm{dec}}$ such that*

$$\left\|\nabla\tilde{u}^\varepsilon_{\mathrm{dec}} - \alpha^\varepsilon\nabla\tilde{u}_\varepsilon^{(J)}\right\|_{L^2(G_\varepsilon)} \leqslant \frac{2C_{(J,\lambda_0,v_0)}\varepsilon^{J-1}}{d^\varepsilon}, \tag{5.12}$$

for sufficiently small ε, α^ε being the sequence of constants $\alpha^\varepsilon = (\|\nabla\tilde{u}_\varepsilon^{(J)}\|_{L^2(G_\varepsilon)})^{-1}$.

Proof. Let us consider $\mathbf{H}^\varepsilon = \{\varphi \in H^1(G_\varepsilon), \varphi = 0 \text{ on } \gamma_\varepsilon^\pm \text{ and } \frac{\partial\varphi}{\partial x_2} = 0 \text{ for } |x_2| \geqslant \delta\}$, i.e., $\mathbf{H}^\varepsilon = H^1_{\mathrm{dec},\delta}(G_\varepsilon)$ (cf. Section 3), with the norm $\|u\|_{\mathbf{H}^\varepsilon} = \|\nabla u\|_{L^2(G_\varepsilon)}$. Let $\mathcal{A}^\varepsilon$ be the compact, symmetric and positive operator on $\mathbf{H}^\varepsilon$ defined by:

$$\langle\mathcal{A}^\varepsilon u, v\rangle_{\mathbf{H}^\varepsilon} = \int_{G_\varepsilon} uv\,\mathrm{d}x, \quad \forall u, v, \in \mathbf{H}^\varepsilon.$$

We consider the eigenvalue problem for $\mathcal{A}^\varepsilon$ in the space $\mathbf{H}^\varepsilon$: Find $u_\varepsilon \in \mathbf{H}^\varepsilon$, $u_\varepsilon \neq 0$, such that $\mathcal{A}^\varepsilon u_\varepsilon = \mu_\varepsilon u_\varepsilon$, which amounts to,

$$\int_{G_\varepsilon} u_\varepsilon v\,\mathrm{d}x = \mu_\varepsilon\int_{G_\varepsilon}\nabla u_\varepsilon\cdot\nabla v\,\mathrm{d}x, \quad \forall v \in \mathbf{H}^\varepsilon. \tag{5.13}$$

Therefore, the eigenelements of $\mathcal{A}^\varepsilon$ are $\{((\lambda_{\varepsilon,\mathrm{dec},i})^{-1}, u_{\varepsilon,\mathrm{dec},i})\}_{i=1}^\infty$, with $(\lambda_{\varepsilon,\mathrm{dec},i}, u_{\varepsilon,\mathrm{dec},i})$ the eigenelements of problem (3.3). Now, for each (λ_0, v_0) eigenelement of problem (2.9)–(2.12) (or equivalently, (2.8)), we consider $\lambda_\varepsilon^{(J)}$ defined by (4.2) and $\tilde{u}_\varepsilon^{(J)}$ defined by (4.53). From the fact that $\tilde{u}_\varepsilon^{(J)} \in \mathbf{H}^\varepsilon$ and on account of (4.56), (4.57), the Poincaré inequality on $G_\varepsilon^\pm$ for the elements of $\mathbf{H}^\varepsilon$, and the definition of $\mathcal{A}^\varepsilon$, we have:

$$\left|\left\langle\mathcal{A}^\varepsilon\tilde{u}_\varepsilon^{(J)} - \left(\lambda_\varepsilon^{(J)}\right)^{-1}\tilde{u}_\varepsilon^{(J)}, \varphi\right\rangle_{\mathbf{H}^\varepsilon}\right| \leqslant \tilde{\tilde{C}}\varepsilon^{J-1}\sqrt{\varepsilon}\left(\lambda_\varepsilon^{(J)}\right)^{-1}\|\varphi\|_{\mathbf{H}^\varepsilon}, \quad \forall\varphi \in \mathbf{H}^\varepsilon,$$

and consequently (cf. (5.4)),

$$\left\|\mathcal{A}^\varepsilon\tilde{u}_\varepsilon^{(J)} - \left(\lambda_\varepsilon^{(J)}\right)^{-1}\tilde{u}_\varepsilon^{(J)}\right\|_{\mathbf{H}^\varepsilon} \leqslant \tilde{\tilde{C}}\varepsilon^{J-1}\sqrt{\varepsilon}. \tag{5.14}$$

Then, we rewrite the proof in Theorem 5.1 (see from (5.4) to (5.8)) with minor modifications to obtain the results in the statement of the theorem. □

Theorem 5.4. *For each fixed $i = 1, 2, \ldots$, let $\lambda_{0,i}$ be the ith eigenvalue of* (2.8)*, and $v_{0,i}$ the associated eigenfunction. Let J be any positive integer, $J \geqslant 2$. Let us consider the value $\lambda_{\varepsilon,i}^{(J)}$ and the function $\tilde{u}_{\varepsilon,i}^{(J)}$ defined by* (4.2) *and* (4.53) *respectively, constructed with the algorithm in Section* 4 *for $(\lambda_0, v_0) \equiv (\lambda_{0,i}, v_{0,i})$. Then, the ith eigenvalue $\lambda_{\varepsilon,\mathrm{dec},i}$ of* (3.3) *satisfies,*

$$\left|\lambda_{\varepsilon,\mathrm{dec},i} - \lambda_{\varepsilon,i}^{(J)}\right| \leqslant C_{(J,i)}\varepsilon^{J-1}, \tag{5.15}$$

for sufficiently small ε, namely $\varepsilon < \varepsilon^*_{(J,\lambda_{0,i},v_{0,i})}$, where $C_{(J,i)} = C_{(J,\lambda_{0,i},v_{0,i})}$ is a certain constant independent of ε.

In addition, there is an eigenfunction $\tilde{u}_{\varepsilon,\mathrm{dec},i}$ associated with $\lambda_{\varepsilon,\mathrm{dec},i}$, $\tilde{u}_{\varepsilon,\mathrm{dec},i} \in H^1_{\mathrm{dec},\delta}(G_\varepsilon)$, verifying $\|\nabla\tilde{u}_{\varepsilon,\mathrm{dec},i}\|_{L^2(G_\varepsilon)} = 1$ and such that, for any strictly positive r, with $r < J - 1$, and for sufficiently small ε, the inequality,

$$\left\|\nabla\tilde{u}_{\varepsilon,\mathrm{dec},i} - \alpha^\varepsilon\nabla\tilde{u}_{\varepsilon,i}^{(J)}\right\|_{L^2(G_\varepsilon)} \leqslant \widehat{C}_{(J,\lambda_{0,i},v_{0,i},r)}\varepsilon^{J-1-r} \tag{5.16}$$

holds, where $\widehat{C}_{(J,\lambda_{0,i},v_{0,i},r)}$ is a constant independent of ε and $\alpha^\varepsilon = (\|\nabla\tilde{u}_{\varepsilon,i}^{(J)}\|_{L^2(G_\varepsilon)})^{-1}$.

Proof. We proceed as in Theorem 5.2 with the suitable modifications as we outline here below.

On account of Theorem 3.2, $\lambda_{\varepsilon,\text{dec},i} \to \lambda_{0,i}$ as $\varepsilon \to 0$. Also, from the construction of $\lambda_{\varepsilon,i}^{(J)}$ in Section 4, $\lambda_{\varepsilon,i}^{(J)}$ is a well determined sequence of ε once that we have fixed the eigenfunction $v_{0,i}$, and $\lambda_{\varepsilon,i}^{(J)} \to \lambda_{0,i}$ as $\varepsilon \to 0$. On the other hand, Theorem 5.3 ensures that (5.11) holds for a certain eigenvalue $\lambda_{\text{dec}}^{\varepsilon}$ of (3.3) which we denote by $\lambda_{\varepsilon,\text{dec},k(i,\varepsilon)}$. Then, inequality (5.15) in the statement of the theorem holds once we prove that $\lambda_{\varepsilon,\text{dec},k(i,\varepsilon)} = \lambda_{\varepsilon,\text{dec},i}$.

In order to prove this equality, we proceed as in Theorem 5.2; thus, there is no restriction if we assume that there is a fixed $k \neq i$ such that the $\lambda_{\varepsilon,\text{dec},k(i,\varepsilon)}$ in Theorem 5.3 satisfies $\lambda_{\varepsilon,\text{dec},k(i,\varepsilon)} = \lambda_{\varepsilon,\text{dec},k}$. Let us denote by α the constant $\alpha = |\lambda_{0,k} - \lambda_{0,i}|$, which is assumed to be $\alpha > 0$. Let β be a fixed constant, $\beta < \alpha/4$. Because of the above argument, (5.11) ensures the existence of ε_i^* such that

$$\left|\lambda_{\varepsilon,i}^{(J)} - \lambda_{\varepsilon,\text{dec},k}\right| < \beta, \quad \forall \varepsilon < \varepsilon_i^*.$$

Also, because of Theorem 3.2, there is $\varepsilon_{0,k}$ such that

$$|\lambda_{\varepsilon,\text{dec},k} - \lambda_{0,k}| < \beta, \quad \forall \varepsilon < \varepsilon_{0,k}.$$

On the other hand, the convergence of $\lambda_{\varepsilon,i}^{(J)}$ towards $\lambda_{0,i}$ gives the existence of $\varepsilon_{0,i}$ such that

$$|\lambda_{\varepsilon,i}^{(J)} - \lambda_{0,i}| < \beta, \quad \forall \varepsilon < \varepsilon_{0,i}.$$

Gathering the three last inequalities, for $\varepsilon < \min(\varepsilon_i^*, \varepsilon_{0,i}, \varepsilon_{0,k})$ we have:

$$\alpha = |\lambda_{0,k} - \lambda_{0,i}| \leqslant |\lambda_{\varepsilon,\text{dec},k} - \lambda_{0,k}| + \left|\lambda_{\varepsilon,i}^{(J)} - \lambda_{\varepsilon,\text{dec},k}\right| + \left|\lambda_{\varepsilon,i}^{(J)} - \lambda_{0,i}\right| < 3\beta < 3\alpha/4 < \alpha,$$

which is a contradiction. Therefore, $\alpha = 0$ and $\lambda_{0,i} = \lambda_{0,k}$ which implies $k = i$ and $\lambda_{\varepsilon,\text{dec},k(i,\varepsilon)} = \lambda_{\varepsilon,\text{dec},i}$, as was necessary to prove (5.15).

As regards the eigenfunctions, since the eigenvalues of (2.8) are simple, Theorem 3.2 ensures that, for any $d > 0$ such that $[\lambda_{0,i} - d, \lambda_{0,i} + d]$ does not contain more eigenvalues of (2.8) than $\lambda_{0,i}$, and for sufficiently small ε, there is only one simple eigenvalue $\lambda_{\varepsilon,\text{dec},i}$ in $[\lambda_{0,i} - d, \lambda_{0,i} + d]$. On the other hand, in the first part of the theorem (i.e., estimate (5.15)) we have proved that the eigenvalue in Theorem 5.3 satisfying (5.11) is precisely $\lambda_{\varepsilon,\text{dec},i}$. Now, by rewriting the last part of the proof in Theorem 5.2 with minor modifications, we prove that taking $d^\varepsilon = \varepsilon^r$, with $0 < r < J - 1$, there are not more inverses of eigenvalues of (3.3) in the interval $[(\lambda_{\varepsilon,i}^{(J)})^{-1} - d^\varepsilon, (\lambda_{\varepsilon,i}^{(J)})^{-1} + d^\varepsilon]$ than $(\lambda_{\varepsilon,\text{dec},i})^{-1}$, and therefore, the result on the eigenfunctions in (5.12) in Theorem 5.3 holds now for an eigenfunction $\tilde{u}_{\varepsilon,\text{dec},i}$ associated with $\lambda_{\varepsilon,\text{dec},i}$, such that $\|\nabla \tilde{u}_{\varepsilon,\text{dec},i}\|_{L^2(G_\varepsilon)} = 1$. That is, this eigenfunction associated with $\lambda_{\varepsilon,\text{dec},i}$ satisfies:

$$\left\|\nabla \tilde{u}_{\varepsilon,\text{dec},i} - \alpha^\varepsilon \nabla \tilde{u}_{\varepsilon,i}^{(J)}\right\|_{L^2(G_\varepsilon)} \leqslant C\varepsilon^{J-1-r},$$

for a certain constant C independent on ε, and this proves (5.16). Therefore, the results stated in the theorem are proved. □

Combining the results in Theorems 5.2 and 5.4 we readily obtain:

Theorem 5.5. *For each fixed $i = 1, 2, \ldots$, let $\lambda_{\varepsilon,i}$ ($\lambda_{\varepsilon,\text{dec},i}$, respectively) be the ith eigenvalue of (2.5) ((3.3), respectively). Let J be any positive integer, $J \geqslant 2$. The inequality*

$$|\lambda_{\varepsilon,\text{dec},i} - \lambda_{\varepsilon,i}| \leqslant C_{(J,i)}\varepsilon^{J-1}, \tag{5.17}$$

holds for sufficiently small ε, namely $\varepsilon < \varepsilon_{(J,i)}^*$, where $C_{(J,i)}$ is a certain constant independent of ε.

In addition, there is an eigenfunction $\tilde{u}_{\varepsilon,i} \in H^1(G_\varepsilon)$, associated with $\lambda_{\varepsilon,i}$ ($\tilde{u}_{\varepsilon,\text{dec},i} \in H^1_{\text{dec},\delta}(G_\varepsilon)$, respectively, associated with $\lambda_{\varepsilon,\text{dec},i}$, respectively), both functions of the x variable, verifying $\|\nabla \tilde{u}_{\varepsilon,i}\|_{L^2(G_\varepsilon)} = 1$ and $\|\nabla \tilde{u}_{\varepsilon,\text{dec},i}\|_{L^2(G_\varepsilon)} = 1$ respectively, and such that, for any strictly positive r, with $r < J - 1$, and for sufficiently small ε, the inequality,

$$\|\nabla \tilde{u}_{\varepsilon,\text{dec},i} - \nabla \tilde{u}_{\varepsilon,i}\|_{L^2(G_\varepsilon)} \leqslant \widehat{C}_{(J,i,r)}\varepsilon^{J-1-r}, \tag{5.18}$$

holds, where $\widehat{C}_{(J,i,r)}$ is a constant independent of ε.

Remark 5.2. Note that for the proofs of theorems in Section 5.2 we have used the δ introduced in Section 4 depending on ε, namely $\delta_\varepsilon = \tilde{k}\varepsilon|\ln\varepsilon|$, where $\tilde{k}$ does not depend on ε (see (4.52) for its dependence on J).

Remark 5.3. Results in Theorem 5.5 allow us to assert that the eigenvalues and the associated eigenfunctions of the problem posed in the decomposed domain (3.5)–(3.11) are good approaches (up to a desired order) of the low frequencies and the associated eigenfunctions of problem (2.1)–(2.3), and they can be used for numerical computations. Note that the fact that the eigenvalues of the limiting problem (2.8) are simple is important for the results on the eigenfunctions in Theorems 5.2, 5.4 and 5.5.

Remark 5.4. It should be pointed out that Lemma 5.1 has been used recently in many papers in connection with high frequency vibrations (see, for instance, [6,7,13]). It has been used in [15] to prove Lemma 2.1, and, for instance, in [8] to derive asymptotics for low frequencies of another ε-depending spectral problem avoiding Lemma 2.1. In this paper, we combine both results of the spectral perturbation theory, namely Lemmas 2.1 and 5.1, to justify asymptotics in Section 4 and obtain precise estimates for the differences between eigenelements and the truncated asymptotic expansions and between the eigenelements of (2.1)–(2.3) and (3.5)–(3.11).

6. Asymptotics and convergence for the spectrum in the case of a simple rod

In this section we show that the technique developed to approach the eigenelements of (2.1)–(2.3) can also be applied to other models where the method of asymptotic partial decomposition of domains works (cf. [18]). The eigenvalue problem, which we deal with here, is a simple model for a rod of thickness of order of ε with fixed ends, but the size of the two lateral segments where the rod is fixed are different. Namely, one of these extreme segments is of width ε and the other one $\varepsilon/2$. Since we follow the steps in Sections 2–5, for brevity, we just outline the main results avoiding proofs.

In the same way, we also continue with the same notations introduced in the previous sections, with the following modifications: The domain where we pose the eigenvalue problem is G_ε^+; the part of the boundary where we consider a Dirichlet condition is now $\gamma_\varepsilon^+ \cup (\{0\} \times (-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}))$ and for simplicity it is called again $\gamma_\varepsilon^+ \cup \gamma_\varepsilon^-$, that is, the segment $\{0\} \times (-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$ is denoted by γ_ε^- throughout this section. We consider the spectral problem:

$$\Delta u_\varepsilon + \lambda_\varepsilon u_\varepsilon = 0, \quad \text{in } G_\varepsilon^+, \tag{6.1}$$

$$u_\varepsilon = 0 \quad \text{if } x \in \gamma_\varepsilon^- \text{ or } \gamma_\varepsilon^+, \tag{6.2}$$

$$\frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \partial G_\varepsilon^+ \backslash (\gamma_\varepsilon^+ \cup \gamma_\varepsilon^-), \tag{6.3}$$

which has a standard variational formulation in $\{u \in H^1(G_\varepsilon^+) \mid u = 0 \text{ on } \gamma_\varepsilon^\pm\}$ and it has a strictly positive discrete spectrum. For fixed ε, let us denote by $\{\lambda_{\varepsilon,i}\}_{i=1}^\infty$ the sequence of eigenvalues of (6.1)–(6.3), converging towards infinity as $i \to \infty$, with the classical convention or repeated eigenvalues. Let $\{u_{\varepsilon,i}\}_{i=1}^\infty$ be the associated eigenfunctions, which are a basis of $\{u \in H^1(G_\varepsilon^+),\ u = 0 \text{ on } \gamma_\varepsilon^\pm\}$, and we assume that they are normalized in such a way that

$$\int_{G_1^+} u_{\varepsilon,i}(x_1, y_2) u_{\varepsilon,j}(x_1, y_2)\, dx_1\, dy_2 = \delta_{i,j}, \tag{6.4}$$

where $\delta_{i,j}$ is the Kronecker symbol and, if no confusion arises, we write indifferently $u_{\varepsilon,i}(x_1, y_2)$ or $u_{\varepsilon,i}(x_1, x_2)$.

By introducing the change of variable from (x_1, x_2) to (x_1, y_2), the variational formulation of (6.1)–(6.3) reads: Find λ_ε, $u_\varepsilon \in \{v \in H^1(G_1^+) \mid v = 0 \text{ on } \gamma_1^\pm\}$, $u_\varepsilon \neq 0$, such that

$$\int_{G_1^+} \left(\partial_1 u_\varepsilon \partial_1 v + \frac{1}{\varepsilon^2} \partial_2 u_\varepsilon \partial_2 v \right) dx_1\, dy_2 = \lambda_\varepsilon \int_{G_1^+} u_\varepsilon v\, dx_1\, dy_2, \quad \forall v \in H^1(G_1^+),\ v = 0 \text{ on } \gamma_1^\pm, \tag{6.5}$$

where ∂_1 and ∂_2 are the partial derivatives with respect to x_1 and y_2 respectively.

As in Section 2 we can prove that, for fixed i, the eigenvalues $\lambda_{\varepsilon,i}$ are of order O(1) and we find that the limit eigenvalue problem is the classical Dirichlet problem in the x_1 variable in the interval $(0,1)$: Find λ, $v \in H_0^1(0,1)$, $v \neq 0$ satisfying

$$\int_0^1 v'(x_1)w'(x_1)\,\mathrm{d}x_1 = \lambda \int_0^1 v(x_1)w(x_1)\,\mathrm{d}x_1, \quad \forall w \in H_0^1(0,1). \tag{6.6}$$

Problem (6.6) has also simple eigenvalues $\{\lambda_{0,i}\}_{i=1}^{\infty}$ which can be explicitly computed and $\{v_{0,i}(x_1)\}_{i=1}^{\infty}$ the associated eigenfunctions in $H_0^1(0,1)$. These explicit formulas for the eigenelements of (6.6) are: The set of eigenvalues are $\{(k\pi)^2\}_{k=1}^{\infty}$, each $(k\pi)^2$ has the associated eigenfunction (up to product by a constant) $u_k(x_1)$ defined as $u_k(x_1) = \sin(k\pi x_1)$.

The result of convergence of the spectrum of (6.5) towards that of (6.6) with conservation of the multiplicity is stated in the following theorem

Theorem 6.1. *For each fixed i, $i = 1,2,\ldots,$ the eigenvalues $\{\lambda_{\varepsilon,i}\}_\varepsilon$ of* (6.5) *converge towards the ith eigenvalue $\lambda_{0,i}$ of* (6.6) *as $\varepsilon \to 0$. Moreover, for each eigenfunction v_0 of* (6.6) *associated with $\lambda_{0,i}$ there is an eigenfunction $\hat{u}_{\varepsilon,i}(x_1,y_2)$ associated with $\lambda_{\varepsilon,i}$ converging towards v_0 in $L^2(G_1^+)$ as $\varepsilon \to 0$* (*note that we identify $v_0(x_1,y_2) = v_0(x_1)$ and the convergence also holds in $H^1(G_1)$-weak*). *In addition, for each sequence we can extract a subsequence, still denoted by ε, such that $u_{\varepsilon,i} \to v_{0,i}$ in $H^1(G_1^+)$-weak, as $\varepsilon \to 0$, where $u_{\varepsilon,i}$ are the eigenfunctions of* (6.5) *satisfying* (6.4) *and the set $\{v_{0,i}\}_{i=1}^{\infty}$ forms an orthonormal basis of $L^2(0,1)$.*

The proof of Theorem 6.1 holds by rewriting the proof of Theorem 2.1 with minor modifications.

Now, following the notations of Section 3, we consider any fixed δ, with $0 < \delta < 1$, or $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and we denote by $G_{\varepsilon,\delta}^+$ the sub-domain of G_ε^+, $G_{\varepsilon,\delta}^+ = G_\varepsilon \cap \{0 < x_1 < \delta\}$, Let us denote by $H^1_{\mathrm{dec},\delta}(G_\varepsilon^+)$ the subspace of $\{u \in H^1(G_\varepsilon^+) \mid u = 0 \text{ on } \gamma_\varepsilon^\pm\}$ defined by,

$$H^1_{\mathrm{dec},\delta}(G_\varepsilon^+) = \left\{u \in H^1\big(G_\varepsilon^+\big) \mid u = 0 \text{ on } \gamma_\varepsilon^\pm, \text{ and } \frac{\partial u}{\partial x_2}(x_1,x_2) = 0 \text{ for } x_1 \geqslant \delta\right\},$$

whose elements can be identified with pairs of functions $(u,u^+) \in H^{1+}_{\mathrm{dec},\varepsilon,\delta}$, where we define the space:

$$H^{1+}_{\mathrm{dec},\varepsilon,\delta} = \big\{u \in H^1\big(G_{\varepsilon,\delta}^+\big),\ u^+ \in H^1(\delta,1) \mid u^+(1) = 0,\ u^+(\delta) = u(\delta,x_2)\big\}, \tag{6.7}$$

with the scalar product the usual one in the product of spaces $H^1(G_{\varepsilon,\delta}^+)$ and $H^1(\delta,1)$. In addition, if no confusion arises we write indifferently u or (u,u^+) to denote the same element of $H^1_{\mathrm{dec},\delta}(G_\varepsilon^+)$ or $H^{1+}_{\mathrm{dec},\varepsilon,\delta}$.

We consider the weak formulation of the eigenvalue problem (6.1)–(6.3) (namely, Eq. (6.5) with the change of variables from x_1, y_2 to x_1, x_2) in the couple of spaces $H^{1+}_{\mathrm{dec},\varepsilon,\delta} \subset L^{2+}_{\mathrm{dec},\varepsilon,\delta}$ where $L^{2+}_{\mathrm{dec},\varepsilon,\delta} = L^2(G_{\varepsilon,\delta}^+) \times L^2(\delta,1)$: Find $\lambda_{\varepsilon,\mathrm{dec}}$, $u_{\varepsilon,\mathrm{dec}} \in H^1_{\mathrm{dec},\delta}(G_\varepsilon^+)$, $u_{\varepsilon,\mathrm{dec}} \neq 0$, such that

$$\int_{G_\varepsilon^+} \nabla u_{\varepsilon,\mathrm{dec}}.\nabla v\,\mathrm{d}x = \lambda_{\varepsilon,\mathrm{dec}} \int_{G_\varepsilon^+} u_{\varepsilon,\mathrm{dec}} v\,\mathrm{d}x, \quad \forall v \in H^1_{\mathrm{dec},\delta}\big(G_\varepsilon^+\big). \tag{6.8}$$

This problem also has a discrete spectrum which we denote by $\{\lambda_{\varepsilon,\mathrm{dec},i}\}_{i=1}^{\infty}$. Let $\{u_{\varepsilon,\mathrm{dec},i}\}_{i=1}^{\infty}$ be the sequence of eigenfunctions in $H^1_{\mathrm{dec},\delta}(G_\varepsilon^+)$, or equivalently in $H^{1+}_{\mathrm{dec},\varepsilon,\delta}$. If no confusion arises, we write indifferently $u_{\varepsilon,\mathrm{dec},i}(x_1,x_2)$ or $u_{\varepsilon,\mathrm{dec},i}(x_1,y_2)$, and we assume that these eigenfunctions satisfy the same normalization condition (6.4), which now reads:

$$\int_{G_{1,\delta}^+} u_{\varepsilon,\mathrm{dec},i} u_{\varepsilon,\mathrm{dec},j}\,\mathrm{d}x_1\,\mathrm{d}y_2 + \int_\delta^1 u^+_{\varepsilon,\mathrm{dec},i} u^+_{\varepsilon,\mathrm{dec},j}\,\mathrm{d}x_1 = \delta_{i,j}, \tag{6.9}$$

where, obviously, $u^+_{\varepsilon,\mathrm{dec},i}$ denote the restriction of $u_{\varepsilon,\mathrm{dec},i}$ to $y_2 = 0$, $x_1 > \delta$.

Note that, for the sake of brevity, in this section we use the same notation, namely $(\lambda_{\varepsilon,\mathrm{dec}}, u_{\varepsilon,\mathrm{dec}})$ to denote the eigenelements of (6.8) either when the parameter δ is fixed or $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ (see Section 3 for comparison).

By setting $\varepsilon = 1$, we denote by $H^{1+}_{\mathrm{dec},\delta}$ and $L^{2+}_{\mathrm{dec},\delta}$ the spaces $H^{1+}_{\mathrm{dec},1,\delta}$ and $L^{2+}_{\mathrm{dec},1,\delta}$ respectively. With the change of variable from (x_1, x_2) to (x_1, y_2), the formulation (6.8) reads: Find $\lambda_{\varepsilon,\mathrm{dec}}$, $u_{\varepsilon,\mathrm{dec}} \in H^1_{\mathrm{dec},\delta}$, $u_{\varepsilon,\mathrm{dec}} \neq 0$, such that

$$\int\limits_{G_1^+} \left(\partial_1 u_{\varepsilon,\mathrm{dec}} \partial_1 v + \frac{1}{\varepsilon^2} \partial_2 u_{\varepsilon,\mathrm{dec}} \partial_2 v\right) \mathrm{d}x_1\, \mathrm{d}y_2 = \lambda_{\varepsilon,\mathrm{dec}} \int\limits_{G_1^+} u_{\varepsilon,\mathrm{dec}} v \,\mathrm{d}x_1\, \mathrm{d}y_2, \quad \forall v \in H^{1+}_{\mathrm{dec},\delta}. \tag{6.10}$$

On account of the definition of (6.7), formulation (6.10) amounts to:

$$\begin{aligned} &\int\limits_{G_{1,\delta}^+} \left(\partial_1 u_{\varepsilon,\mathrm{dec}} \partial_1 v + \frac{1}{\varepsilon^2} \partial_2 u_{\varepsilon,\mathrm{dec}} \partial_2 v\right) \mathrm{d}x_1\, \mathrm{d}y_2 + \int\limits_{\delta}^{1} \partial_1 u_{\varepsilon,\mathrm{dec}} \partial_1 v \,\mathrm{d}x_1 \\ &\quad = \lambda_{\varepsilon,\mathrm{dec}} \left(\int\limits_{G_{1,\delta}^+} u_{\varepsilon,\mathrm{dec}} v \,\mathrm{d}x_1\, \mathrm{d}y_2 + \int\limits_{\delta}^{1} u_{\varepsilon,\mathrm{dec}} v \,\mathrm{d}x_1 \right), \quad \forall v \in H^{1+}_{\mathrm{dec},\delta}. \end{aligned} \tag{6.11}$$

Then, we can prove that the limiting eigenvalue problem of (6.10) is (6.6) and we have the following result:

Theorem 6.2. *For each fixed i, $i = 1, 2, \ldots,$ and either for fixed δ, $0 < \delta < 1$, or $\delta = \delta(\varepsilon) \to 0$, the eigenvalues $\{\lambda_{\varepsilon,\mathrm{dec},i}\}_\varepsilon$ of* (6.11) *converge towards the ith eigenvalue $\lambda_{0,i}$ of* (6.6) *as $\varepsilon \to 0$. Moreover, for each eigenfunction v_0 of* (6.6) *associated with $\lambda_{0,i}$, there is an eigenfunction $\hat{u}_{\varepsilon,\mathrm{dec},i}(x_1, y_2)$ associated with $\lambda_{\varepsilon,i}$ converging towards v_0 in $L^2(G_1^+)$ as $\varepsilon \to 0$ (note that we identify $v_0(x_1, y_2) = v_0(x_1)$ and the convergence in $H^1(G_1^+)$-weak also holds). In addition, for each sequence we can extract a subsequence, still denoted by ε, such that $u_{\varepsilon,\mathrm{dec},i} \to v_{0,i}$ in $H^1(G_1^+)$-weak, as $\varepsilon \to 0$, where $u_{\varepsilon,\mathrm{dec},i}$ are the eigenfunctions of* (6.11) *satisfying* (6.9) *and the set $\{v_{0,i}\}_{i=1}^{\infty}$ forms an orthonormal basis of $L^2(0,1)$.*

The proof of Theorem 6.2 holds by rewriting the proof of Theorems 3.1 and 3.2 with minor modifications.

6.1. Asymptotics for the eigenelements of (6.1)–(6.3)

Following the technique in Section 4 and Section VI of [18], we look for an asymptotic expansion of the solution of (6.1)–(6.3) in a form:

$$u_\varepsilon^{(J)}(x) = \sum_{j=0}^{J} \varepsilon^j \left(U_j\left(\frac{x}{\varepsilon}\right) + v_j(x_1) \right), \quad x \in G_\varepsilon^+, \tag{6.12}$$

for the eigenfunctions, and

$$\lambda_\varepsilon^{(J)} = \sum_{j=0}^{J} \varepsilon^j \lambda_j, \tag{6.13}$$

for the eigenvalues. In (6.12) we consider the variable $\xi = x\varepsilon^{-1}$, $U_0(\xi) = 0$ and $U_j(\xi)$ are exponentially decaying boundary layer functions such that, for any j, there exist constants C_{j1} and C_{j2} satisfying the condition,

$$\left|U_j(\xi)\right| \leqslant C_{j1} \mathrm{e}^{-C_{j2}\xi_1}, \tag{6.14}$$

as $\xi \to \infty$.

Substituting (6.12), (6.13) into (6.1)–(6.3), gathering separately terms in the ξ variable, in the x_1 variable, and of the different powers of ε, we determine equations satisfied by λ_j, $v_j(x_1)$ and $U_{j+1}(\xi)$ recursively:

First, we obtain (λ_0, v_0) as an eigenelement of the Dirichlet problem (6.6), i.e., it satisfies:

$$v_0'' + \lambda_0 v_0 = 0 \quad \text{for } x_1 \in (0,1); \quad v_0(0) = v_0(1) = 0. \tag{6.15}$$

Second, for $j=1$ we obtain U_1 and constant c_1 from the general equations satisfied by U_j: U_j satisfies the following problem posed in G^+:

$$\Delta_\xi U_j = F_j(\xi), \quad F_j(\xi) = -\sum_{p+q=j-2} \lambda_p U_q(\xi) \quad \text{if } \xi \in G^+, \tag{6.16}$$

$$\frac{\partial U_j}{\partial \xi_2} = 0 \quad \text{if } \xi_2 = \pm\frac{1}{2}, \tag{6.17}$$

$$\frac{\partial U_j}{\partial \xi_1} = -v'_{j-1}(0) \quad \text{if } \xi_1 = 0, \frac{1}{4} < |\xi_2| < \frac{1}{2}, \tag{6.18}$$

$$U_j = c_j \quad \text{if } \xi_1 = 0, \frac{1}{4} < |\xi_2| < \frac{1}{2}, \tag{6.19}$$

where, c_j in (6.19) is the well determined constant such that (6.14) is satisfied. See Remark 4.1 and Appendix A in connection with the existence and properties of these solutions. Here and in what follows we adopt the convention that λ_i, U_i or v_i are equal to zero if $i<0$ or $i>J$. Third, we obtain λ_1, v_1 from the equations satisfied by λ_j, v_j which are:

$$v''_j + \lambda_0 v_j = f_j(x_1), \quad x_1 \in (0,1), \text{ where } f_j = -\sum_{\substack{p+q=j\\(p,q)\neq(0,j)}} \lambda_p v_q, \tag{6.20}$$

$$v_j(0) = -c_j, \quad v_j(1) = 0. \tag{6.21}$$

Since, problem (6.20)–(6.21) is a non homogeneous problem associated with (6.15), we homogenize the boundary conditions adding function $-c_j(1-x_1)$ to the function v_j: $\tilde v_j = c_j(1-x_1) + v_j$, and then, we apply the Fredholm alternative to the problem satisfied by $\tilde v_j$ and we obtain λ_j from the solvability condition. Since v_j is determined up to an eigenfunction of (6.15), we fix $\tilde v_j$ by the orthogonality condition $\int_0^1 \tilde v_j v_0 \,\mathrm{d}x_1 = 0$. In this way, λ_j is given by,

$$\lambda_j \int_0^1 (v_0)^2 \,\mathrm{d}x_1 = \lambda_0 c_j \int_0^1 (1-x_1) v_0 \,\mathrm{d}x_1 - \sum_{\substack{p+q=j\\(p,q)\neq(0,j),(j,0)}} \lambda_p \int_0^1 v_q v_0 \,\mathrm{d}x_1. \tag{6.22}$$

Then, we follow the process to obtain U_2, λ_2, v_2 from (6.16)–(6.22) as well as all the U_j, λ_j, v_j up to a desired order.

To justify solution (6.12), (6.13) we have to modify slightly the function $u_\varepsilon^{(J)}$ in (6.12) because its trace on γ_ε^+ is not exactly equal to zero, and the boundary condition (6.3) is not exactly satisfied on $\Gamma_\varepsilon \setminus \gamma_\varepsilon^-$.

To fix the first problem, we can multiply every $U_j(\frac{x}{\varepsilon})$ by function $\eta(\frac{|x_1|}{\delta_\varepsilon})$, where η is given by (4.51), $\delta_\varepsilon = \tilde k \varepsilon |\ln \varepsilon|$, $\tilde k$ does not depend on ε, and δ_ε is such that

$$\left|U_j(\xi)\right|, \left|\nabla U_j(\xi)\right|, \left|\Delta U_j(\xi)\right| < \overline{C} \mathrm{e}^{-\overline{\overline{C}} \delta_\varepsilon \varepsilon^{-1}} \leqslant \overline{C} \varepsilon^J,$$

for $|\xi_1| > \frac{\delta_\varepsilon}{3\varepsilon}$, where $\overline{C}$ and $\overline{\overline{C}}$ are certain constants independent of ε. That is,

$$-\overline{\overline{C}} \frac{\delta_\varepsilon}{\varepsilon} \leqslant -J|\ln \varepsilon|, \qquad \delta_\varepsilon \geqslant J\big(\overline{\overline{C}}\big)^{-1} \varepsilon |\ln \varepsilon|,$$

which implies

$$\tilde k = J\big(\overline{\overline{C}}\big)^{-1}. \tag{6.23}$$

Then, we denote by:

$$\tilde u_\varepsilon^{(J)} = \sum_{j=0}^{J} \varepsilon^j \left(U_j\left(\frac{x}{\varepsilon}\right) \eta\left(\frac{|x_1|}{\delta_\varepsilon}\right) + v_j(x_1) \right), \tag{6.24}$$

and we have

$$\Delta \tilde u_\varepsilon^{(J)} + \lambda_\varepsilon^{(J)} \tilde u_\varepsilon^{(J)} = r_\varepsilon^{(J)}\left(x_1, \frac{x}{\varepsilon}\right) + \bar r_\varepsilon^{(J)}(x)$$

(see formulas (4.5), (4.54) and (4.55) for the definition and estimates for functions $\tilde{u}_\varepsilon^{(J)}$, $\bar{r}_\varepsilon^{(J)}$ and $\bar{r}_\varepsilon^{(J)}$). Then, proceeding as in (4.53)–(4.57) we have that $\tilde{u}_\varepsilon^{(J)}$, $\lambda_\varepsilon^{(J)}$ satisfy the following problem:

$$-\int\limits_{G_\varepsilon^+} \nabla \tilde{u}_\varepsilon^{(J)} \cdot \nabla \varphi \,\mathrm{d}x + \lambda_\varepsilon^{(J)} \int\limits_{G_\varepsilon^+} \tilde{u}_\varepsilon^{(J)} \varphi \,\mathrm{d}x = \int\limits_{G_\varepsilon^+} \left(r_\varepsilon^{(J)}\left(x_1, \frac{x}{\varepsilon}\right) + \bar{r}_\varepsilon^{(J)}(x) \right) \varphi \,\mathrm{d}x$$

$$+ \int\limits_{\Gamma_\varepsilon \setminus \gamma_\varepsilon^-} \varepsilon^J v_J'(0) \varphi(0, x_2) \,\mathrm{d}x_2, \quad \forall \varphi \in H^1(G_\varepsilon^+),\ \varphi = 0 \text{ on } \gamma_\varepsilon^\pm, \tag{6.25}$$

and the functional of the right-hand side of (6.25) satisfies:

$$\left|\Phi(\varphi)\right| \leqslant \widehat{C} \varepsilon^{J-1} \sqrt{\varepsilon} \|\varphi\|_{H^1(G_\varepsilon)}, \quad \forall \varphi \in H^1(G_\varepsilon^+),\ \varphi = 0 \text{ on } \gamma_\varepsilon^\pm, \tag{6.26}$$

where $\widehat{C}$ is a constant independent of ε.

In addition, as for (4.58), it can be proved that there are constants C_1^*, C_2^* independent of ε, such that for ε sufficiently small, namely $\varepsilon < \varepsilon^*_{J,\lambda_0,v_0}$, we have:

$$C_1^* \sqrt{\varepsilon} \leqslant \left\| \nabla_x \tilde{u}_\varepsilon^{(J)} \right\|_{L^2(G_\varepsilon)} \leqslant C_2^* \sqrt{\varepsilon}. \tag{6.27}$$

6.2. Justification of asymptotics

We follow the technique in Section 5 for problem (6.1)–(6.3) posed in G_ε^+. Throughout this section, the parameter δ also depends on ε. More specifically, $\delta = \tilde{k}\varepsilon|\log \varepsilon|$, with the constant $\tilde{k}$ depending on the order of the approach that we obtain, that is, $\tilde{k}$ depends on J and they are related by (6.23). We only state the main results.

Theorem 6.3. *Let λ_0 be any eigenvalue of* (6.15), *and v_0 the associated eigenfunction. Let J be any positive integer, $J \geqslant 2$. Let us consider the value $\lambda_\varepsilon^{(J)}$ and the function $\tilde{u}_\varepsilon^{(J)}$ defined by* (6.13) *and* (6.24) *respectively, constructed with the algorithm in Section* 6.1. *Then, there exists at least one eigenvalue λ^ε of* (6.1)–(6.3) *such that, for sufficiently small ε, namely $\varepsilon < \varepsilon^*_{(J,\lambda_0,v_0)}$,*

$$\left|\lambda^\varepsilon - \lambda_\varepsilon^{(J)}\right| \leqslant C_{(J,\lambda_0,v_0)} \varepsilon^{J-1}, \tag{6.28}$$

where $C_{(J,\lambda_0,v_0)}$ is a certain positive constant independent of ε. In addition, for any sequence $d^\varepsilon \to 0$, such that $\lim_{\varepsilon\to 0}(\varepsilon^{J-1}/d^\varepsilon) = 0$, the interval $[(\lambda_\varepsilon^{(J)})^{-1} - d^\varepsilon, (\lambda_\varepsilon^{(J)})^{-1} + d^\varepsilon]$ contains the values $\{(\lambda_{\varepsilon,l})^{-1}\}_{l=p}^q$, where $\{\lambda_{\varepsilon,l}\}_{l=p}^q$ are eigenvalues of (6.1)–(6.3), *for l ranging between certain natural numbers $p = p(\varepsilon)$ and $q = q(\varepsilon)$, $p(\varepsilon) \leqslant q(\varepsilon)$, and, there is a function $\tilde{u}_\varepsilon$ in the eigenspace of the associated eigenfunctions $\{\tilde{u}_{\varepsilon,l}\}_{l=p}^q$, with $\|\nabla \tilde{u}_\varepsilon\|_{L^2(G_\varepsilon^+)} = 1$, $\tilde{u}^\varepsilon$ such that*

$$\left\| \nabla \tilde{u}^\varepsilon - \alpha^\varepsilon \nabla \tilde{u}_\varepsilon^{(J)} \right\|_{L^2(G_\varepsilon^+)} \leqslant \frac{2C_{(J,\lambda_0,v_0)}\varepsilon^{J-1}}{d^\varepsilon}, \tag{6.29}$$

for sufficiently small ε, α^ε being the sequence of constants $\alpha^\varepsilon = (\|\nabla \tilde{u}_\varepsilon^{(J)}\|_{L^2(G_\varepsilon^+)})^{-1}$.

The proof of Theorem 6.3 holds rewriting the proof of Theorem 5.1 with minor modifications.

The following theorem proves that for each $(\lambda_{0,i}, v_{0,i})$ eigenelement of (6.15), the eigenvalue and associated eigenfunction of (6.1)–(6.3) satisfying (6.28) and (6.29), in the case where $(\lambda_0, v_0) \equiv (\lambda_{0,i}, v_{0,i})$, is precisely the ith eigenelement of (6.1)–(6.3). That is, (6.24) and (6.13) provide true asymptotic expansions of the eigenfunctions and eigenvalues respectively of (6.1)–(6.3). We use Theorems 6.1 and 6.3 for this proof.

Theorem 6.4. *For each fixed $i = 1, 2, \ldots$, let $\lambda_{0,i}$ be the ith eigenvalue of* (6.15), *and $v_{0,i}$ the associated eigenfunction. Let J be any positive integer, $J \geqslant 2$. Let us consider the value $\lambda_{\varepsilon,i}^{(J)}$ and the function $\tilde{u}_{\varepsilon,i}^{(J)}$ defined by* (6.13) *and* (6.24) *respectively, constructed with the algorithm in Section* 6.1 *for $(\lambda_0, v_0) \equiv (\lambda_{0,i}, v_{0,i})$. Then, the ith eigenvalue $\lambda_{\varepsilon,i}$ of* (6.1)–(6.3) *satisfies*:

$$\left|\lambda_{\varepsilon,i} - \lambda_{\varepsilon,i}^{(J)}\right| \leqslant C_{(J,i)} \varepsilon^{J-1}, \tag{6.30}$$

for sufficiently small ε*, namely* $\varepsilon < \varepsilon^*_{(J,\lambda_{0,i},v_{0,i})}$*, where* $C_{(J,i)} = C_{(J,\lambda_{0,i},v_{0,i})}$ *is a certain constant independent of* ε*.*

In addition, there is an eigenfunction $\tilde{u}_{\varepsilon,i}$*, associated with* $\lambda_{\varepsilon,i}$*, verifying* $\|\nabla\tilde{u}_{\varepsilon,i}\|_{L^2(G_\varepsilon^+)} = 1$ *and such that, for any strictly positive* r*, with* $r < J-1$*, and for sufficiently small* ε*, the inequality,*

$$\big\|\nabla\tilde{u}_{i,\varepsilon} - \alpha^\varepsilon\nabla\tilde{u}^{(J)}_{\varepsilon,i}\big\|_{L^2(G_\varepsilon^+)} \leqslant \widehat{C}_{(J,\lambda_{0,i},v_{0,i},r)}\varepsilon^{J-1-r} \tag{6.31}$$

holds, where $\widehat{C}_{(J,\lambda_{0,i},v_{0,i},r)}$ *is a constant independent of* ε *and* $\alpha^\varepsilon = (\|\nabla\tilde{u}^{(J)}_{\varepsilon,i}\|_{L^2(G_\varepsilon^+)})^{-1}$*.*

The proof of Theorem 6.4 holds rewriting the proof of Theorem 5.2 with minor modifications.

In what follows we consider $u_\varepsilon^{(J)}$ and $\delta = \delta(\varepsilon) = \delta_\varepsilon = \tilde{k}\varepsilon|\ln\varepsilon|$ in Section 6.1 such that $u_\varepsilon^{(J)} \in H^1_{\mathrm{dec},\delta}(G_\varepsilon^+)$ in (6.24) and estimates (6.26) are satisfied $\forall\varphi \in H^1_{\mathrm{dec},\delta}(G_\varepsilon^+)$.

Theorem 6.5. *Let* λ_0 *be any eigenvalue of* (6.6)*, and* v_0 *the associated eigenfunction. Let* J *be any positive integer,* $J \geqslant 2$*. Let us consider the value* $\lambda_\varepsilon^{(J)}$ *and the function* $\tilde{u}_\varepsilon^{(J)}$ *defined by* (6.13) *and* (6.24) *respectively, constructed with the algorithm in Section* 6.1*. Then, there exists at least one eigenvalue* $\lambda^\varepsilon_{\mathrm{dec}}$ *of* (6.8) *such that, for sufficiently small* ε*, namely* $\varepsilon < \varepsilon^*_{(J,\lambda_0,v_0)}$*,*

$$\big|\lambda^\varepsilon_{\mathrm{dec}} - \lambda_\varepsilon^{(J)}\big| \leqslant C_{(J,\lambda_0,v_0)}\varepsilon^{J-1}, \tag{6.32}$$

where $C_{(J,\lambda_0,v_0)}$ *is a certain positive constant independent of* ε*. In addition, for any sequence* $d^\varepsilon \to 0$*, such that* $\lim_{\varepsilon\to 0}(\varepsilon^{J-1}/d^\varepsilon) = 0$*, the interval* $[(\lambda_\varepsilon^{(J)})^{-1} - d^\varepsilon, (\lambda_\varepsilon^{(J)})^{-1} + d^\varepsilon]$ *contains the values* $\{(\lambda_{\varepsilon,\mathrm{dec},l})^{-1}\}_{l=p}^q$*,* $\{\lambda_{\varepsilon,\mathrm{dec},l}\}_{l=p}^q$ *being eigenvalues of* (6.8)*, for certain natural numbers* $p = p(\varepsilon)$ *and* $q = q(\varepsilon)$*,* $p(\varepsilon) \leqslant q(\varepsilon)$*, and, there is a function* $\tilde{u}^\varepsilon_{\mathrm{dec}}$ *in the eigenspace associated to the eigenfunctions* $\{u_{\varepsilon,\mathrm{dec},l}\}_{l=p}^q \subset H^1_{\mathrm{dec},\delta}(G_\varepsilon^+)$ *, with* $\|\nabla\tilde{u}^\varepsilon_{\mathrm{dec}}\|_{L^2(G_\varepsilon^+)} = 1$*,* $\tilde{u}^\varepsilon_{\mathrm{dec}}$ *such that*

$$\big\|\nabla\tilde{u}^\varepsilon_{\mathrm{dec}} - \alpha^\varepsilon\nabla\tilde{u}_\varepsilon^{(J)}\big\|_{L^2(G_\varepsilon^+)} \leqslant \frac{2C_{(J,\lambda_0,v_0)}\varepsilon^{J-1}}{d^\varepsilon}, \tag{6.33}$$

for sufficiently small ε*,* α^ε *being the sequence of constants* $\alpha^\varepsilon = (\|\nabla\tilde{u}_\varepsilon^{(J)}\|_{L^2(G_\varepsilon^+)})^{-1}$*.*

The proof of Theorem 6.5 holds rewriting the proof of Theorem 5.3 with minor modifications.

Theorem 6.6. *For each fixed* $i = 1, 2, \ldots,$ *let* $\lambda_{0,i}$ *be the* i*th eigenvalue of* (6.6)*, and* $v_{0,i}$ *the associated eigenfunction. Let* J *be any positive integer,* $J \geqslant 2$*. Let us consider the value* $\lambda^{(J)}_{\varepsilon,i}$ *and the function* $\tilde{u}^{(J)}_{\varepsilon,i}$ *defined by* (6.13) *and* (6.24) *respectively, constructed with the algorithm in Section* 6.1 *for* $(\lambda_0, v_0) \equiv (\lambda_{0,i}, v_{0,i})$*. Then, the* i*th eigenvalue* $\lambda_{\varepsilon,\mathrm{dec},i}$ *of* (6.8) *satisfies*:

$$\big|\lambda_{\varepsilon,\mathrm{dec},i} - \lambda^{(J)}_{\varepsilon,i}\big| \leqslant C_{(J,i)}\varepsilon^{J-1}, \tag{6.34}$$

for sufficiently small ε*, namely* $\varepsilon < \varepsilon^*_{(J,\lambda_{0,i},v_{0,i})}$*, where* $C_{(J,i)} = C_{(J,\lambda_{0,i},v_{0,i})}$ *is a certain constant independent of* ε*.*

In addition, there is an eigenfunction $\tilde{u}_{\varepsilon,\mathrm{dec},i}$*, associated with* $\lambda_{\varepsilon,\mathrm{dec},i}$*,* $\tilde{u}_{\varepsilon,\mathrm{dec},i} \in H^1_{\mathrm{dec},\delta}(G_\varepsilon^+)$*, satisfying* $\|\nabla\tilde{u}_{\varepsilon,\mathrm{dec},i}\|_{L^2(G_\varepsilon^+)} = 1$ *and such that, for any strictly positive* r*, with* $r < J-1$*, and for sufficiently small* ε*, the inequality*:

$$\big\|\nabla\tilde{u}_{\varepsilon,\mathrm{dec},i} - \alpha^\varepsilon\nabla\tilde{u}^{(J)}_{\varepsilon,i}\big\|_{L^2(G_\varepsilon^+)} \leqslant \widehat{C}_{(J,\lambda_{0,i},v_{0,i},r)}\varepsilon^{J-1-r} \tag{6.35}$$

holds, where $\widehat{C}_{(J,\lambda_{0,i},v_{0,i},r)}$ *is a constant independent of* ε *and* $\alpha^\varepsilon = (\|\nabla\tilde{u}^{(J)}_{\varepsilon,i}\|_{L^2(G_\varepsilon^+)})^{-1}$*.*

The proof of Theorem 6.6 holds rewriting the proof of Theorem 5.4 with minor modifications.

Theorem 6.7. *For each fixed* $i = 1, 2, \ldots,$ *let* $\lambda_{\varepsilon,i}$ *(*$\lambda_{\varepsilon,\mathrm{dec},i}$*, respectively) be the* i*th eigenvalue of* (6.6) ((6.8)*, respectively). Let* J *be any positive integer,* $J \geqslant 2$*. The inequality,*

$$|\lambda_{\varepsilon,\mathrm{dec},i} - \lambda_{\varepsilon,i}| \leqslant C_{(J,i)}\varepsilon^{J-1}, \tag{6.36}$$

holds for sufficiently small ε*, namely* $\varepsilon < \varepsilon^*_{(J,i)}$*, where* $C_{(J,i)}$ *is a certain constant independent of* ε*.*

In addition, there is an eigenfunction $\tilde{u}_{\varepsilon,i} \in H^1(G_\varepsilon^+)$, associated with $\lambda_{\varepsilon,i}$ ($\tilde{u}_{\varepsilon,\mathrm{dec},i} \in H^1_{\mathrm{dec},\delta}(G_\varepsilon^+)$, associated with $\lambda_{\varepsilon,\mathrm{dec},i}$, respectively), both functions of the x variable, satisfying $\|\nabla \tilde{u}_{\varepsilon,i}\|_{L^2(G_\varepsilon^+)} = 1$ and $\|\nabla \tilde{u}_{\varepsilon,\mathrm{dec},i}\|_{L^2(G_\varepsilon^+)} = 1$ respectively, and such that, for any strictly positive r, with $r < J-1$, and for sufficiently small ε, the inequality,

$$\left\| \nabla \tilde{u}_{\varepsilon,\mathrm{dec},i} - \nabla \tilde{u}_{\varepsilon,i} \right\|_{L^2(G_\varepsilon^+)} \leqslant \widehat{C}_{(J,i,r)} \varepsilon^{J-1-r} \tag{6.37}$$

holds, where $\widehat{C}_{(J,i,r)}$ is a constant independent of ε.

Remark 6.1. Let us observe that all the remarks on asymptotic expansions and proofs throughout Sections 2–5 apply to the problem in this section. Also, it should be noted that restrictions for J and bounds for convergence rates in Section 5 (Section 6.2, respectively) rely on the estimates (4.57) ((6.26), respectively) and (4.58) ((6.27), respectively), and consequently improvements of these estimates imply improvements for convergence rates.

Appendix A. Elliptic equation in an unbounded domain with outlets at infinity

In this section, for the sake of completeness, we provide proofs for general results on existence, uniqueness and behavior at infinity for solutions of certain problems for the Laplace operator posed on infinite half-bands, with mixed boundary conditions on the boundary, which appear throughout the paper. Appendices A.1 and A.2 contain the case of one half-band and two coupled half-bands respectively. We note that this section can be read independently from the rest of the paper and that references [1,3,11,16–18] are used for proofs in this section.

A.1. The case of one half-band

Consider the problem,

$$\begin{aligned}
&\Delta U = F(\xi) && \text{for } \xi \in (0,+\infty) \times \left(-\frac{1}{2}, \frac{1}{2}\right),\\
&U = c && \text{for } \xi_1 = 0,\ \xi_2 \in \left(-\frac{1}{4}, \frac{1}{4}\right),\\
&\frac{\partial U}{\partial \xi_2} = 0 && \text{for } \xi_2 = \pm\frac{1}{2},\\
&\frac{\partial U_l}{\partial \xi_1} = \beta && \text{for } \xi_1 = 0,\ |\xi_2| \geqslant \frac{1}{4},
\end{aligned} \tag{A.1}$$

where F is a given measurable function satisfying the following condition:

$$\exists c_1, c_2 > 0\colon \forall \xi \in (0,+\infty) \times \left(-\frac{1}{2}, \frac{1}{2}\right), \quad \left|F(\xi)\right| \leqslant c_1 \mathrm{e}^{-c_2 \xi_1}, \tag{A.2}$$

β is a given constant, c is a constant such that $U \to 0$ if $\xi_1 \to +\infty$.

We consider first the auxiliary problem:

$$\begin{aligned}
&\Delta \widetilde{U} = F(\xi) && \text{for } \xi \in (0,+\infty) \times \left(-\frac{1}{2}, \frac{1}{2}\right),\\
&\widetilde{U} = 0 && \text{for } \xi_1 = 0, \xi_2 \in \left(-\frac{1}{4}, \frac{1}{4}\right),\\
&\frac{\partial \widetilde{U}}{\partial \xi_2} = 0 && \text{for } \xi_2 = \pm\frac{1}{2},\\
&\frac{\partial \widetilde{U}}{\partial \xi_1} = \beta && \text{for } \xi_1 = 0, |\xi_2| \geqslant \frac{1}{4}.
\end{aligned} \tag{A.3}$$

Applying change (see (4.51)),

$$\widetilde{\widetilde{U}} = -\beta \xi_1 \eta(\xi_1) + \widetilde{U}(\xi),$$

we obtain a problem of the same type but with the homogeneous Neumann condition:

$$\begin{aligned}
&\Delta\widetilde{\widetilde{U}} = \widetilde{F}(\xi) && \text{for } \xi \in (0,+\infty)\times\left(-\frac{1}{2},\frac{1}{2}\right),\\
&\widetilde{\widetilde{U}} = 0 && \text{for } \xi_1 = 0,\ \xi_2 \in \left(-\frac{1}{4},\frac{1}{4}\right),\\
&\frac{\partial\widetilde{\widetilde{U}}}{\partial\xi_2} = 0 && \text{for } \xi_2 = \pm\frac{1}{2},\\
&\frac{\partial\widetilde{\widetilde{U}}}{\partial\xi_1} = 0 && \text{for } \xi_1 = 0,\ |\xi_2| \geqslant \frac{1}{4},
\end{aligned}$$

where $\widetilde{F}(\xi) = F(\xi) - \beta(\xi_1\eta(\xi_1))''$ and it satisfies the same condition of the exponential decaying (A.2).

Let us study the existence and uniqueness of the solution of problem (A.3) in the space:

$$H = \left\{ u \in H^1\left((0,R)\times\left(-\frac{1}{2},\frac{1}{2}\right)\right) \quad \forall R \in \mathbb{R}^+,\ u = 0 \text{ for } \xi_1 = 0,\ |\xi_2| < \frac{1}{4},\right.$$
$$\left.\text{and } \nabla u \in L^2\left((0,+\infty)\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)\right\},$$

supplied with the norm $\|\nabla u\|_{L^2((0,+\infty)\times(-\frac{1}{2},\frac{1}{2}))}$ and with the inner product $\int_{(0,+\infty)\times(-\frac{1}{2},\frac{1}{2})}\nabla u \nabla\varphi\,\mathrm{d}\xi$. Note that for $u \in H$, then for all $\xi_1 \in (0,+\infty)$, the inequality

$$\int_{-1/2}^{1/2} \left|u(\xi_1,\xi_2)\right| \mathrm{d}\xi_2 = \int_{-1/2}^{1/2} \left| \int_0^{\xi_1} \frac{\partial u}{\partial \xi_1}(s,\xi_2)\,\mathrm{d}s \right| \mathrm{d}\xi_2 \leqslant \sqrt{\xi_1}\,\|\nabla u\|_{L^2((0,+\infty)\times(-\frac{1}{2},\frac{1}{2}))},$$

leads us to the fact that the trace contribution on $\xi_1 = R$ in the Green formula tends to zero as $R \to \infty$ due to the inequality (A.2).

Define such solution as a function of H satisfying for any $\varphi \in H$ the equation:

$$-\int_{(0,+\infty)\times(-\frac{1}{2},\frac{1}{2})} \nabla\widetilde{\widetilde{U}}\cdot\nabla\varphi\,\mathrm{d}\xi = \int_{(0,+\infty)\times(-\frac{1}{2},\frac{1}{2})} \widetilde{F}(\xi)\varphi(\xi)\,\mathrm{d}\xi. \tag{A.4}$$

Let us represent $\widetilde{F}(\xi)$ in the following form,

$$\widetilde{F}(\xi) = F_1(\xi_1) + F_2(\xi),$$

where

$$F_1(\xi_1) = \int_{-1/2}^{1/2} \widetilde{F}(\xi_1,\xi_2)\,\mathrm{d}\xi_2, \qquad F_2(\xi) = \frac{\partial}{\partial\xi_2}\int_{-1/2}^{\xi_2} F_2(\xi_1,t)\,\mathrm{d}t = \widetilde{F}(\xi) - F_1(\xi_1),$$

and

$$\int_{-1/2}^{\xi_2} F_2(\xi_1,t)\,\mathrm{d}t = 0 \quad \text{if } \xi_2 = \pm\frac{1}{2}.$$

Denote $\widehat{F}_2(\xi) = \int_{-1/2}^{\xi_2} F_2(\xi_1,t)\,\mathrm{d}t$, $\hat{f} = \int_0^{+\infty} F_1(\xi_1)\,\mathrm{d}\xi_1$ and $\widehat{F}_1(\xi_1) = -\int_{\xi_1}^{+\infty} F_1(t)\,\mathrm{d}t$. Then (A.4) can be rewritten as

$$-\int_{(0,+\infty)\times(-\frac{1}{2},\frac{1}{2})} \nabla\widetilde{\widetilde{U}}\cdot\nabla\varphi\,\mathrm{d}\xi$$
$$= -\int_{(0,+\infty)\times(-\frac{1}{2},\frac{1}{2})} \widehat{F}_2(\xi)\frac{\partial\varphi}{\partial\xi_2}(\xi)\,\mathrm{d}\xi - \int_{(0,+\infty)\times(-\frac{1}{2},\frac{1}{2})} \widehat{F}_1(\xi_1)\frac{\partial\varphi}{\partial\xi_1}(\xi)\,\mathrm{d}\xi - \int_{-1/2}^{1/2} \varphi(0,\xi_2)\widehat{F}_1(0)\,\mathrm{d}\xi_2,$$

i.e.,

$$-\int\limits_{(0,+\infty)\times(-\frac{1}{2},\frac{1}{2})} \nabla\widetilde{\widetilde{U}}\cdot\nabla\varphi\,\mathrm{d}\xi = -\sum_{i=1}^{2}\int\limits_{(0,+\infty)\times(-\frac{1}{2},\frac{1}{2})} \widehat{F}_i(\xi)\frac{\partial\varphi}{\partial\xi_i}(\xi)\,\mathrm{d}\xi + \hat{f}\int\limits_{-1/2}^{1/2}\varphi(0,\xi_2)\,\mathrm{d}\xi_2. \tag{A.5}$$

Theorem A.1. *There exists a unique solution $\widetilde{\widetilde{U}}$ to problem* (A.5).

Proof. Consider the right-hand side of (A.5) as a linear functional $-\Phi(\varphi)$ defined on H. Let us prove that this functional is bounded (continuous) in H.

$$\left|\Phi(\varphi)\right| \leqslant \sum_{i=1}^{2}\left\|\widehat{F}_i\right\|_{L^2((0,+\infty)\times(-\frac{1}{2},\frac{1}{2}))}\left\|\frac{\partial\varphi}{\partial\xi_i}\right\|_{L^2((0,+\infty)\times(-\frac{1}{2},\frac{1}{2}))} + \left|\hat{f}\right|\left|\int\limits_{-1/2}^{1/2}\varphi(0,\xi_2)\,\mathrm{d}\xi_2\right|.$$

The trace theorem gives the estimate:

$$\left|\int\limits_{-1/2}^{1/2}\varphi(0,\xi_2)\,\mathrm{d}\xi_2\right| \leqslant c_3\|\varphi\|_{H^1((0,1)\times(-\frac{1}{2},\frac{1}{2}))},$$

and from Poincaré–Friedrichs inequality for the square $(0,1)\times(-\frac{1}{2},\frac{1}{2})$ we obtain that

$$\left|\int\limits_{-1/2}^{1/2}\varphi(0,\xi_2)\,\mathrm{d}\xi_2\right| \leqslant c_4\|\nabla\varphi\|_{L^2((0,1)\times(-\frac{1}{2},\frac{1}{2}))} \leqslant c_4\|\nabla\varphi\|_{L^2((0,+\infty)\times(-\frac{1}{2},\frac{1}{2}))},$$

where c_3 and c_4 are some positive constants. So the functional Φ is continuous. Applying the Riesz representation theorem we obtain the existence and uniqueness of solution of (A.5). □

Theorem A.2. *Let $\widetilde{\widetilde{U}}$ be a solution of* (A.5)*. Then there exist constants c_5, c_6, c_7 such that $c_6, c_7 > 0$ and*

$$\left|\widetilde{\widetilde{U}}(\xi) - c_5\right| \leqslant c_6\mathrm{e}^{-c_7\xi_1}.$$

Proof. Let us reduce problem (A.5) to a problem with periodicity condition at the lateral boundary. To this end, let us extend the domain $(0,+\infty)\times(-\frac{1}{2},\frac{1}{2})$ by reflection to $(0,+\infty)\times(-\frac{1}{2},\frac{3}{2})$ and extend the right-hand side $\widetilde{F}$ as an even function with respect to the line $\xi_2 = \frac{1}{2}$. Then we obtain the equivalent problem:

$$\begin{aligned}
&\Delta\widetilde{\widetilde{U}} = \begin{cases} \widetilde{F}(\xi) & \text{for } \xi_2 \in \left(-\dfrac{1}{2},\dfrac{1}{2}\right), \\[2ex] \widetilde{F}(\xi_1, 1-\xi_2) & \text{for } \xi_2 \in \left(\dfrac{1}{2},\dfrac{3}{2}\right), \end{cases} \\
&\widetilde{\widetilde{U}} = 0 \quad \text{for } \xi_1 = 0,\ \xi_2 \in \left(-\frac{1}{4},\frac{1}{4}\right)\cup\left(\frac{3}{4},\frac{5}{4}\right), \\
&\widetilde{\widetilde{U}} \text{ is 2-periodic in } \xi_2.
\end{aligned} \tag{A.6}$$

We can now apply the result of [11] that every 2-periodic in ξ_2 solution of Eq. (A.6) set in half-space $(0,+\infty)\times\mathbb{R}$ can either have a linear or an exponential growth as $\xi_1 \to +\infty$, or it stabilizes to some constant. Theorem 2 in [11] leaves only this last possibility. So Theorem A.2 is proved. □

So finally we have proved the existence and uniqueness of a solution of problem (A.3) from the space H and it stabilizes to some constant c_5, i.e.,

$$\lim_{\xi_1\to+\infty}\widetilde{U}(\xi) = c_5.$$

Now we can find a solution (U, c) to problem (A.1) as $U = \widetilde{U} - c_5$, $c = -c_5$. The uniqueness of this solution follows from Theorem A.1 because if U is a solution to (A.1) then $U - c$ is a solution to (A.3).

Remark A.1. If F is smooth, then applying results [1] we can prove the exponential decaying of ∇U as well as of higher derivatives.

A.2. The case of two coupled half-bands

A similar analysis to that in Appendix A.1 can be developed with respect to the problem in a domain with two infinite outlets at infinity. Consider $\Pi_- = (-\infty, 0) \times (-\frac{1}{4}, \frac{1}{4})$, $\Pi_+ = (0, +\infty) \times (-\frac{1}{2}, \frac{1}{2})$, $\Pi = \Pi_- \cup \Pi_+$ and the problem:

$$\begin{aligned}
&\Delta U = F(\xi) && \text{for } \xi \in \Pi,\\
&[U] = \beta_1, \quad \left[\frac{\partial U}{\partial \xi_1}\right] = \beta_2 && \text{for } \xi_1 = 0, \xi_2 \in \left(-\frac{1}{4}, \frac{1}{4}\right),\\
&\frac{\partial U}{\partial \xi_2} = 0 && \text{for } \xi_1 < 0,\ \xi_2 = \pm\frac{1}{4} \text{ and } \xi_1 > 0,\ \xi_2 = \pm\frac{1}{2},\\
&\frac{\partial U}{\partial \xi_1} = \beta_3 && \text{for } \xi_1 = 0,\ |\xi_2| \geqslant \frac{1}{4},
\end{aligned} \tag{A.7}$$

where F is a given measurable function satisfying the condition,

$$\exists c_1, c_2 > 0\colon\ \forall \xi \in \Pi, \quad \left|F(\xi)\right| \leqslant c_1 \mathrm{e}^{-c_2|\xi_1|},$$

β_3 is a given constant, β_1, β_2 are constants such that $U \to 0$ if $|\xi_1| \to +\infty$.

Let us find a necessary condition for the existence of such a solution. Integrating the equation in Π_- and Π_+ we obtain:

$$\int_\Pi F(\xi)\,\mathrm{d}\xi = -\int_{-1/4}^{1/4} \left[\frac{\partial u}{\partial \xi_1}\right] \mathrm{d}\xi_2 - \int_{(-\frac{1}{2},\frac{1}{2})\setminus(-\frac{1}{4},\frac{1}{4})}^{1/4} \frac{\partial u}{\partial \xi_1}\,\mathrm{d}\xi_2,$$

i.e.,

$$\int_\Pi F(\xi)\,\mathrm{d}\xi = -\frac{1}{2}\beta_2 - \frac{1}{2}\beta_3. \tag{A.8}$$

Let us prove that (A.8) is sufficient for the existence of the solution. Consider the following auxiliary problem:

$$\begin{aligned}
&\Delta \widetilde{U} = F(\xi) && \text{for } \xi \in \Pi,\\
&[\widetilde{U}] = 0, \quad \left[\frac{\partial \widetilde{U}}{\partial \xi_1}\right] = \beta_2 && \text{for } \xi_1 = 0, \xi_2 \in \left(-\frac{1}{4}, \frac{1}{4}\right),\\
&\frac{\partial \widetilde{U}}{\partial \xi_2} = 0 && \text{for } \xi_1 < 0,\ \xi_2 = \pm\frac{1}{4} \text{ and } \xi_1 > 0,\ \xi_2 = \pm\frac{1}{2},\\
&\frac{\partial \widetilde{U}}{\partial \xi_1} = \beta_3 && \text{for } \xi_1 = 0,\ |\xi_2| \geqslant \frac{1}{4},
\end{aligned}$$

where $\widetilde{U}$ is sought in the space $\widehat{H} = \{u \in H^1(\Pi \cap \{|\xi_1| < R\})\ \forall R \in \mathbb{R}^+, \nabla u \in L^2(\Pi)\}$ with a semi-norm $\|\nabla u\|_{L^2(\Pi)}$. Every function here is defined up to an additive arbitrary constant (see [16]) and so we consider the classes of equivalence of functions differing by constant.

This problem can be reduced to:

$$
\begin{aligned}
&\Delta\widetilde{\widetilde{U}} = \widetilde{F}(\xi) && \text{for } \xi \in \Pi,\\
&\big[\widetilde{\widetilde{U}}\big] = 0, \quad \left[\frac{\partial\widetilde{\widetilde{U}}}{\partial\xi_1}\right] = 0 && \text{for } \xi_1 = 0, \xi_2 \in \left(-\frac{1}{4}, \frac{1}{4}\right),\\
&\frac{\partial\widetilde{\widetilde{U}}}{\partial\xi_2} = 0 && \text{for } \xi_1 < 0,\ \xi_2 = \pm\frac{1}{4} \text{ and } \xi_1 > 0,\ \xi_2 = \pm\frac{1}{2},\\
&\frac{\partial\widetilde{\widetilde{U}}}{\partial\xi_1} = 0 && \text{for } \xi_1 = 0,\ |\xi_2| \geqslant \frac{1}{4},
\end{aligned}
\tag{A.9}
$$

where

$$
\begin{aligned}
&\widetilde{\widetilde{U}}(\xi) = \widetilde{U}(\xi) + \chi^+(\xi_1)\big(-\beta_3\xi_1\eta(\xi_1)\big) + \chi^-(\xi_1)\big((\beta_2-\beta_3)\xi_1\eta(-\xi_1)\big),\\
&\chi^+(\xi_1) = \begin{cases} 1, & \text{if } \xi_1 > 0,\\ 0, & \text{if } \xi_1 \leqslant 0, \end{cases} \qquad \chi^-(\xi_1) = \begin{cases} 1, & \text{if } \xi_1 < 0,\\ 0, & \text{if } \xi_1 \geqslant 0, \end{cases}\\
&\int_{\Pi} \widetilde{F}(\xi)\,\mathrm{d}\xi = \int_{\Pi} F(\xi)\,\mathrm{d}\xi + \frac{1}{2}\beta_2 + \frac{1}{2}\beta_3.
\end{aligned}
$$

Let us prove that the condition,

$$
\int_{\Pi} \widetilde{F}(\xi)\,\mathrm{d}\xi = 0 \tag{A.10}
$$

is necessary and sufficient for existence and uniqueness (up to an arbitrary additive constant) of a solution $\widetilde{\widetilde{U}}$ of (A.9) in $\widehat{H}$.

Theorem A.3. *Let $\widetilde{F}$ be a measurable function defined in Π such that*

$$
\exists c_1, c_2 > 0\colon\ \forall\xi \in \Pi, \quad \big|F(\xi)\big| \leqslant c_1 \mathrm{e}^{-c_2|\xi_1|}.
$$

Then condition (A.10) *is necessary and sufficient for the existence of solution of* (A.9) *in $\widehat{H}$. This solution is unique up to an additive constant.*

Proof. Consider the subspace $\widehat{H}_0 = \{u \in \widehat{H}\colon \int_{\Pi_1} u(\xi)\,\mathrm{d}\xi = 0\}$ of $\widehat{H}$, where $\Pi_1 = (0,1) \times (-\frac{1}{2}, \frac{1}{2})$. Let us make the same decomposition of $\widetilde{F}$ as in the first section:

$$
\widetilde{F}(\xi) = F_1(\xi_1) + F_2(\xi),
$$

where

$$
F_1(\xi_1) = \begin{cases} \displaystyle\int_{-1/2}^{1/2} \widetilde{F}(\xi_1,\xi_2)\,\mathrm{d}\xi_2, & \text{if } \xi_1 > 0,\\ \displaystyle 2\int_{-1/4}^{1/4} \widetilde{F}(\xi_1,\xi_2)\,\mathrm{d}\xi_2, & \text{if } \xi_1 < 0, \end{cases} \qquad F_2(\xi) = \widetilde{F}(\xi) - F_1(\xi_1).
$$

Denote:

$$
\widehat{F}_1(\xi_1) = \begin{cases} \displaystyle\int_{-\infty}^{\xi_1} F_1(t)\,\mathrm{d}t, & \text{if } \xi_1 < 0,\\ \displaystyle -\int_{\xi_1}^{+\infty} F_1(t)\,\mathrm{d}t, & \text{if } \xi_1 > 0, \end{cases} \qquad \widehat{F}_2(\xi) = \begin{cases} \displaystyle\int_{-1/2}^{\xi_2} F_2(\xi_1,t)\,\mathrm{d}t, & \text{if } \xi_1 > 0,\\ \displaystyle\int_{-1/4}^{\xi_2} F_2(\xi_1,t)\,\mathrm{d}t, & \text{if } \xi_1 < 0. \end{cases}
$$

$$\hat{f}_- = \int_{-\infty}^{0} F_1(t)\,\mathrm{d}t, \quad \widehat{f_+} = \int_{0}^{+\infty} F_1(t)\,\mathrm{d}t.$$

Then the right-hand side in the variational formulation can be represented as

$$-\Phi(\varphi) = \int_{\Pi} \widetilde{F}(\xi)\varphi(\xi)\,\mathrm{d}\xi = -\int_{\Pi} \sum_{i=1}^{2} \widehat{F}_i \frac{\partial \varphi}{\partial \xi_i}\,\mathrm{d}\xi + \int_{-1/4}^{1/4} \hat{f}_-\varphi(0,\xi_2)\,\mathrm{d}\xi_2 + \int_{-1/2}^{1/2} \hat{f}_+\varphi(0,\xi_2)\,\mathrm{d}\xi_2.$$

As in the first section, applying the trace theorem, we obtain that

$$\left|\int_{-1/2}^{1/2} \varphi(0,\xi_2)\,\mathrm{d}\xi_2\right|, \left|\int_{-1/4}^{1/4} \varphi(0,\xi_2)\,\mathrm{d}\xi_2\right| \leqslant c_3 \|\varphi\|_{H^1(\Pi_1)}.$$

Consider now the identity:

$$-\int_{\Pi} \nabla\widetilde{\widetilde{U}} \cdot \nabla\varphi\,\mathrm{d}\xi = \int_{\Pi} \widetilde{F}\varphi\,\mathrm{d}\xi, \quad \forall \varphi \in \widehat{H}_0. \tag{A.11}$$

Then applying the Poincaré inequality, we obtain that for $\varphi \in \widehat{H}_0$:

$$\|\varphi\|_{L^2(\Pi_1)} \leqslant \|\nabla\varphi\|_{L^2(\Pi_1)},$$

and therefore

$$\left|\int_{-1/2}^{1/2} \varphi(0,\xi_2)\,\mathrm{d}\xi_2\right|, \left|\int_{-1/4}^{1/4} \varphi(0,\xi_2)\,\mathrm{d}\xi_2\right| \leqslant \sqrt{2}c_3 \|\nabla\varphi\|_{L^2(\Pi_1)} \leqslant \sqrt{2}c_3 \|\nabla\varphi\|_{L^2(\Pi)}.$$

So $|\Phi(\varphi)| \leqslant c_4 \|\nabla\varphi\|_{L^2(\Pi)}$ for $\varphi \in \widehat{H}_0$. Applying the Riesz theorem we obtain the existence and uniqueness of $\widetilde{\widetilde{U}} \in \widehat{H}_0$ such that (A.11) holds true.

Now let us prove that such $\widetilde{\widetilde{U}}$ is a solution of the identity:

$$-\int_{\Pi} \nabla\widetilde{\widetilde{U}} \cdot \nabla\varphi\,\mathrm{d}\xi = \int_{\Pi} \widetilde{F}\varphi\,\mathrm{d}\xi, \quad \forall \varphi \in \widehat{H}, \tag{A.12}$$

if (A.10) holds.

Let φ be a function of $\widehat{H}$. Present it in a form $\varphi = \int_{\Pi_1} \varphi(\xi)\,\mathrm{d}\xi + \varphi_1$. Then $\varphi_1 \in \widehat{H}_0$ and so

$$-\int_{\Pi} \nabla\widetilde{\widetilde{U}} \cdot \nabla\varphi_1\,\mathrm{d}\xi = \int_{\Pi} \widetilde{F}\varphi_1\,\mathrm{d}\xi.$$

Moreover

$$\int_{\Pi} \widetilde{F}\varphi\,\mathrm{d}\xi = \int_{\Pi} \widetilde{F}\varphi_1\,\mathrm{d}\xi + \int_{\Pi} \widetilde{F}\left(\int_{\Pi_1} \varphi(t)\,\mathrm{d}t\right)\mathrm{d}\xi = \int_{\Pi} \widetilde{F}\varphi_1\,\mathrm{d}\xi,$$

because $\int_{\Pi} \widetilde{F}\,\mathrm{d}\xi = 0$ and $\nabla\varphi = \nabla\varphi_1$, then (A.12) holds true. So we have proved that condition $\int_{\Pi} \widetilde{F}\,\mathrm{d}\xi = 0$ is sufficient for the existence of solution of (A.12).

This solution is unique up to an arbitrary additive constant. Indeed if U_1, U_2 are two solutions, then

$$\int_{\Pi} \nabla(U_1 - U_2) \cdot \nabla\varphi\,\mathrm{d}\xi = 0, \quad \forall \varphi \in \widehat{H},$$

and taking $\varphi = U_1 - U_2$, we have $\nabla(U_1 - U_2) = 0$ a.e. on Π. So $U_1 - U_2 = const$.

The necessity of (A.10) for the existence of solution is trivial: taking $\varphi = 1$ in (A.12), we obtain $\int_{\Pi} \widetilde{F}\,\mathrm{d}\xi = 0$. □

Final remarks. In the present paper we have considered a model example of an eigenvalue problem set in thin rod structures. Applying the dimensional reduction and the boundary layer theory we have constructed an asymptotic expansion for the eigenelements. High order estimates are obtained for the difference of exact and approximate solutions.

The closeness of the partially decomposed problem to the initial problem is also proved. This partially decomposed problem reduces the dimension at a distance $\tilde{k}\varepsilon|\ln\varepsilon|$ from the junction surface, where the boundary layer takes place, $\tilde{k}$ being a certain constant, and keeps the initial formulation for the remaining part of the domain. The interface conditions are the continuity of the solution and the flow conservation in average. This hybrid formulation seems to be comprehensible and useful for engineers and its solution is very close to the solution of the initial problem.

As pointed out in the introduction, we emphasize that the technique in this paper can be extended to the case of a junction of multiple rods.

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