Singular Perturbation of Eigenvalues

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Introduction

This paper deals with the application of Hilbert space methods to self-adjoint eigenvalue problems of the form

$$(1) \qquad (\varepsilon A+\mathscr{B})\,u_\varepsilon=\lambda_\varepsilon C u_\varepsilon,$$

where A, $\mathscr{B}$, C are elliptic differential operators and ε is a small parameter. The order of A is greater than the order of $\mathscr{B}$ which is of greater order than C. Eigenvalues and eigenfunctions of (1) are compared with those of

$$(2) \qquad \mathscr{B}u=\lambda C u.$$

With C the identity operator, such problems have been considered by Moser [22] for ordinary differential equations and by Višik & Lyusternik [25] for partial differential equations. In each of these papers boundary layer techniques are employed to obtain asymptotic expansions for eigenvalues and eigenfunctions of (1).

Kato [16], [17], [18] and Huet [12] have proved theorems on the behavior with respect to ε of the spectrum of a self-adjoint operator A_ε in a Hilbert space. When applied to the problem (1), (2) these theorems yield convergence. In the same studies Kato has obtained asymptotic results under hypotheses which imply that the eigenvalue of A_ε under consideration, say λ_ε, is differentiable (but not necessarily analytic) with respect to ε on an interval $0\leqq\varepsilon\leqq\varepsilon_0$.

In singular perturbations of differential eigenvalue problems it is often not the case that an eigenvalue λ_ε has such differentiability. In this paper fractional powers of positive self-adjoint operators (quadratic interpolation) are used to obtain rate of convergence theorems in the framework of Kato & Huet. These theorems are applied to the problem (1), (2).

In the first section of this paper notation is established. The perturbation theorems are formulated in §2. The main result of the paper is Theorem 2.1 in which the mini-max principle of Poincaré is used to set up the eigenvalue estimates. This same principle was used by Huet to prove a theorem on convergence of eigenvalues under singular perturbation in [12], Theorem 1.12. In Theorems 2.3 and 2.4 interpolation techniques are combined with the variational principle of Kato [15] to obtain supplements to Theorem 8 of [16] and to Theorem 19.3 of [17]. The theorems stated in §2 are proved in §3.

In §4 the theorems of §2 are applied to singular perturbations of differential eigenvalue problems. The applications in §4 were chosen with the intent of being representative of theorems which can be obtained from those of §2 by the techniques used in GREENLEE [10]. The fifth section of the paper consists of remarks.

1. Preliminaries

In this section notation is established for use in the sequel.

Let V_0 be a complex Hilbert space with norm denoted by $|v|_0$ and scalar product by $(v, w)_0$. Let V_1 be a complex Hilbert space which is continuously contained in V_0, written

$$V_1 \underset{c}{\subset} V_0;$$

that is V_1 is a vector subspace of V_0 and the injection of V_1 into V_0 is continuous. Further, assume that V_1 is dense in V_0, and let $|v|_1$, $(v, w)_1$ denote the norm and scalar product in V_1, respectively.

To the Hermitian symmetric bilinear form $(v, w)_1$ there corresponds a linear operator Λ in V_0, herein referred to as *the operator in V_0 associated with $(v, w)_1$*, defined on

$$D(\Lambda) \equiv \{v \in V_1 \mid w \to (v, w)_1 \text{ is continuous on } V_1 \text{ in the topology induced by } V_0\}$$

by

$$(\Lambda v, w)_0 = (v, w)_1, \quad v \in D(\Lambda), \quad \Lambda v \in V_0, \quad v \in V_1.$$

Λ is a positive definite, self-adjoint operator in V_0. For τ real, denote by Λ^τ the positive τ^{th} power of Λ as defined by use of the spectral theorem; Λ^τ is a positive definite, self-adjoint operator in V_0. Furthermore the domain of $\Lambda^{\frac{1}{2}}$, $D(\Lambda^{\frac{1}{2}})$, is V_1 and

$$(v, w)_1 = (\Lambda^{\frac{1}{2}} v, \Lambda^{\frac{1}{2}} w)_0$$

for all $v, w \in V_1$.

Let $S = \Lambda^{\frac{1}{2}}$. For $0 \leqq \tau \leqq 1$, *the τ^{th} interpolation space by quadratic interpolation* between V_1 and V_0, V_τ, is defined as the Hilbert space

$$V_\tau = D(S^\tau)$$

with inner product

$$(v, w)_\tau = (S^\tau v, S^\tau w)_0.$$

Now for any $\tau \in [0, \infty)$ let V_τ be the Hilbert space $D(S^\tau)$ with inner product $(v, w)_\tau = (S^\tau v, S^\tau w)_0$. For $v \in V_0$ and $\tau \in [0, \infty)$, let

$$|v|_{-\tau} = \sup\{|(v, w)_0| \mid w \in V_\tau \text{ and } |w|_\tau \leqq 1\}.$$

Then $|v|_{-\tau} = |S^{-\tau} v|_0$. Let $V_{-\tau}$ be the completion of V_0 in the norm $|v|_{-\tau}$. This defines the Hilbert space V_τ for all real τ. Moreover for $\sigma < \tau$, $V_\tau \underset{c}{\subset} V_\sigma$ with V_τ dense in V_σ. By use of extension by continuity, S^γ is an isometric isomorphism of V_τ onto $V_{\tau-\gamma}$ for all real γ, τ. Moreover, $V_{-\tau}$ is canonically isometrically isomorphic to V_τ^*, the anti-dual of V_τ.

2. Singular Perturbation of Eigenvalues and Eigenvectors

Let H be a complex Hilbert space with inner product (v, w) and norm $|v|$. The eigenvalue problem is to be considered for an operator *formally* given by $A_\varepsilon = \varepsilon A + B$ where ε is a small parameter tending to zero.

Let $b(v, w)$ be a Hermitian symmetric bilinear form defined on a linear manifold $D(b)$ which is dense in H. Further, let the corresponding quadratic form $b(v) \equiv b(v, v)$ be closed and have a positive lower bound. For simplicity, this lower bound may be assumed to be not less than 1,

$$b(v) \geqq (v, v), \qquad v \in D(b)$$

(cf. §5). Since $b(v)$ is closed, $D(b)$, with the inner product

$$(v, w)_0 \equiv b(v, w),$$

is a Hilbert space, V_0. The corresponding norm will be denoted by $|v|_0$.

Let $a(v, w)$ be a Hermitian symmetric bilinear form defined on a linear manifold $D(a)$ which is dense in V_0 (and therefore in H). Further, let the corresponding quadratic form $a(v) \equiv a(v, v)$ be non-negative,

$$a(v) \geqq 0, \qquad v \in D(a),$$

and closed in V_0. Then $D(a)$, with the inner product

$$(v, w)_V \equiv a(v, w) + b(v, w),$$

is a Hilbert space V. The corresponding norm will be denoted by $|v|_V$.

Let B be the operator in H associated with $b(v, w) = (v, w)_0$, that is

$$(Bv, w) = b(v, w), \qquad v \in D(B), \quad Bv \in H, \quad w \in V_0.$$

With the inner product (Bv, Bw), $D(B)$ is a Hilbert space. Further, for $\varepsilon > 0$ let A_ε be the operator in H associated with $\varepsilon a(v, w) + b(v, w)$, that is

$$(A_\varepsilon v, w) = \varepsilon a(v; w) + b(v, w), \qquad v \in D(A_\varepsilon), \quad A_\varepsilon v \in H, \quad w \in V.$$

With the inner product $(A_\varepsilon v, A_\varepsilon w)$, $D(A_\varepsilon)$ is a Hilbert space. Observe that $D(A_\varepsilon) \underset{c}{\subset} V \underset{c}{\subset} V_0 \underset{c}{\subset} H$; $D(B) \underset{c}{\subset} V_0$; $D(A_\varepsilon)$ is dense in each of V, V_0, and H; and that $D(B)$ is dense in V_0 and H.

Assumption 1. *At least the lower part of the spectrum of B consists of isolated eigenvalues each of finite multiplicity. These eigenvalues will be considered as arranged in an increasing sequence, $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_N$ in which each eigenvalue is counted according to multiplicity. Corresponding eigenfunctions $u_1, \ldots, u_N$ will be assumed to be orthonormal in H, that is*

$$(u_i, u_j) = \delta_{ij}, \qquad 1 \leqq i, j \leqq N, \qquad \delta_{ij} = \textit{Kronecker symbol}.$$

Thus for example if $\lambda_i < \lambda_{i+1} = \cdots = \lambda_{i+m} < \lambda_{i+m+1}$ then $u_{i+1}, \ldots, u_{i+m}$ forms an H-orthonormal basis for the eigenspace corresponding to the multiple eigenvalue λ_{i+1} which has multiplicity m.

Let $\mathcal{S}_n = \mathrm{sp}\langle u_1, \dots, u_n\rangle$ be the linear manifold generated by $u_1, \dots, u_n$, and let $\mathfrak{S}_n$ denote an arbitrary n-dimensional subspace of H. Consider the quotient $b(v)/|v|^2$ and adopt the convention that the value of the quotient for $v=0$ is 0. Then by using the spectral theorem, it follows as in the finite dimensional case that for $1 \leqq n \leqq N$,

$$\lambda_n = \max_{v \in \mathcal{S}_n} \frac{b(v)}{|v|^2} = \min_{\mathfrak{S}_n \subset V_0} \max_{v \in \mathfrak{S}_n} \frac{b(v)}{|v|^2}.$$

Recent comments on this mini-max principle can be found in STENGER [24], §5.

Under Assumption 1 the following proposition is included in Theorem 7, p. 440, of KATO [16].

Proposition 2.1. *There exists an ε_0 such that for $0 < \varepsilon \leqq \varepsilon_0$ the lower part of the spectrum of A_ε consists of isolated eigenvalues each of finite multiplicity. If these eigenvalues are arranged in an increasing sequence and counted according to multiplicity, there are at least N terms in the sequence, $\lambda_{1,\varepsilon} \leqq \lambda_{2,\varepsilon} \leqq \cdots \leqq \lambda_{N,\varepsilon}$. Furthermore,*

$$\lambda_{n,\varepsilon} \to \lambda_n \quad \text{as } \varepsilon \downarrow 0, \quad 1 \leqq n \leqq N.$$

In the following, the corresponding eigenfunctions $u_{1,\varepsilon}, \dots, u_{N,\varepsilon}$ will be assumed to be orthonormal in H.

In [16], [17], [18] KATO has given conditions under which $\lambda_{n,\varepsilon} = \lambda_n + O(\varepsilon)$, $\varepsilon \downarrow 0$. The crucial hypothesis used to obtain such an estimate is that the eigenspace corresponding to λ_n is contained in the domain of the bilinear form associated with the "perturbed" operator A_ε. In the present case this means that the eigenspace corresponding to λ_n is contained in V. This criterion is not generally satisfied in singular perturbation problems for eigenvalues of elliptic differential operators, and in such problems it is not usually the case that an estimate of the form $\lambda_{n,\varepsilon} = \lambda_n + O(\varepsilon)$ holds (cf. MOSER [22], VIŠIK & LYUSTERNIK [25], or [18], Chap. 8).

In order to obtain a rate of convergence theorem which is applicable to a fairly large class of elliptic differential eigenvalue problems, it is helpful to introduce another operator which is associated with this problem. Let $\mathscr{A}$ be the operator in V_0 associated with $a(v, w)$, that is

$$b(\mathscr{A} v, w) = a(v, w), \quad v \in D(\mathscr{A}), \quad \mathscr{A} v \in V_0, \quad w \in V.$$

Then $D(\mathscr{A})$, with the inner product

$$(v, w)_1 \equiv b(v, w) + b(\mathscr{A} v, \mathscr{A} w),$$

is a Hilbert space V_1. The corresponding norm will be denoted by $|v|_1$. For $\varepsilon > 0$, $B^{-1} A_\varepsilon$ is the restriction to $D(A_\varepsilon)$ of $\varepsilon \mathscr{A} + I$, written $B^{-1} A_\varepsilon \subset \varepsilon \mathscr{A} + I$, and $D(A_\varepsilon) \underset{c}{\subset} V_1$ with $D(A_\varepsilon)$ dense in V_1 (cf. [10], Lemma 4.1).

Since $\mathscr{A}$ is a non-negative self-adjoint operator in V_0, $\mathscr{A}$ has non-negative self-adjoint fractional powers, $\mathscr{A}^\tau$, defined for $\tau \geqq 0$ by use of the spectral theorem. Moreover, for $\tau \geqq 0$, $D(\mathscr{A}^\tau) = D(S^\tau)$ where $S = (\mathscr{A}^2 + I)^{\frac{1}{2}}$ (I=identity map on V_0). Thus since $(v, w)_1 = (Sv, Sw)_0$ for all $v, w \in V_1$, we see that $D(\mathscr{A}^\tau)$ coincides for $\tau \geqq 0$ with the space V_τ defined in §1. In particular, for $0 \leqq \tau \leqq 1$, $D(\mathscr{A}^\tau)$ coincides with the τ^{th} interpolation space by quadratic interpolation between V_1 and V_0.

Throughout the rest of this section Assumption 1 and the notation used in Assumption 1 and Proposition 2.1 are assumed.

Theorem 2.1. *i) If for fixed* $\tau\in[0,\frac{1}{2})$ *we have* $D(B)\underset{c}{\subset} V_\tau$, *then for each n with* $1\leqq n\leqq N$,

$$\lambda_{n,\varepsilon}=\lambda_n+o(\varepsilon^{2\tau}) \quad as \quad \varepsilon\downarrow 0.$$

ii) If $j\leqq N$ *and* $\{u_1,\dots,u_j\}\subset V$, *then for each n with* $1\leqq n\leqq j$,

$$\lambda_{n,\varepsilon}=\lambda_n+O(\varepsilon) \quad as \quad \varepsilon\downarrow 0.$$

iii) Assume that $D(B)\underset{c}{\subset} V_\tau$ *with* $D(B)$ *dense in* V_τ *for fixed* $\tau\in[0,1]$. *Let* $\lambda_{i+1}=\dots=\lambda_{i+m}$ $(0\leqq i<i+m\leqq N)$ *be an eigenvalue of B of multiplicity m; let P be the orthogonal projection in H on* $\mathrm{sp}\langle u_{i+1},\dots,u_{i+m}\rangle$; *and for* $0<\varepsilon\leqq\varepsilon_0$, *let* P_ε *be the orthogonal projection in H on* $\mathrm{sp}\langle u_{i+1,\varepsilon},\dots,u_{i+m,\varepsilon}\rangle$. *Then for each* $\gamma\in(\tau-1,\tau]$ *and each* $v\in H$,

$$|P_\varepsilon Pv-Pv|_\gamma=o(\varepsilon^{\tau-\gamma}) \quad as \quad \varepsilon\downarrow 0.$$

Moreover,

$$|P_\varepsilon Pv-Pv|_{\tau-1}=O(\varepsilon) \quad as \quad \varepsilon\downarrow 0.$$

Also, if $\tau\in[0,\frac{1}{2})$,

$$|P_\varepsilon Pv-Pv|=o(\varepsilon^{2\tau}) \quad as \quad \varepsilon\downarrow 0,$$

while if $\tau=\frac{1}{2}$,

$$|P_\varepsilon Pv-Pv|=O(\varepsilon) \quad as \quad \varepsilon\downarrow 0.$$

Note that the estimate given in part *ii)* of Theorem 2.1 applies whenever k linearly independent eigenvectors corresponding to the eigenvalue λ_n and all eigenvectors corresponding to eigenvalues less than λ_n are in V, even if λ_n is of multiplicity greater than k. The conclusions of the following theorem are of a similar nature.

Theorem 2.2. *i) If* $j\leqq N$ *and for fixed* $\tau\in[0,\frac{1}{2})$ *we have* $\mathrm{sp}\langle u_1,\dots,u_j\rangle\subset V_\tau$, *then for each n with* $1\leqq n\leqq j$,

$$\lambda_{n,\varepsilon}=\lambda_n+o(\varepsilon^\tau) \quad as \quad \varepsilon\downarrow 0.$$

ii) Let $j\leqq N$ *and assume that* $\mathrm{sp}\langle u_1,\dots,u_j\rangle\subset V_\tau$ *for fixed* $\tau\in[0,1]$. *Let* $\lambda_{i+1}=\dots=\lambda_{i+m}$ $(0\leqq i\leqq i+m\leqq N)$ *be an eigenvalue of B of multiplicity m, and for* $0<\varepsilon\leqq\varepsilon_0$ *let* P_ε *be the orthogonal projection in H on* $\mathrm{sp}\langle u_{i+1,\varepsilon},\dots,u_{i+m,\varepsilon}\rangle$. *Then for each n such that* $i+1\leqq n\leqq\min(j,i+m)$ *we have*

$$|P_\varepsilon u_n-u_n|_0=\begin{cases} o(\varepsilon^\tau) & as \quad \varepsilon\downarrow 0, \quad \tau\in[0,1) \\ O(\varepsilon) & as \quad \varepsilon\downarrow 0, \quad \tau=1. \end{cases}$$

Observe that by Proposition 2.1, if $\lambda_n(1\leqq n\leqq N)$ is an eigenvalue of B of multiplicity one and J is any open interval containing λ_n but no other point of the spectrum of B, then for ε sufficiently small there is exactly one point $\lambda_{n,\varepsilon}$ of the spectrum of A_ε in J. Moreover, in this case $\lambda_{n,\varepsilon}$ is an eigenvalue of A_ε of multiplicity one.

Theorem 2.3. *Let* $\lambda=\lambda_n$ $(1\leqq n\leqq N)$ *be an eigenvalue of B of multiplicity one with corresponding eigenvector* $u=u_n$, $|u|=1$. *Let the eigenvector* $u_\varepsilon=u_{n,\varepsilon}$ *corresponding* *to* $\lambda_\varepsilon = \lambda_{n,\varepsilon}$ *be chosen such that* $|u_\varepsilon| = 1$ *and* $(u_\varepsilon, u) \geqq 0$. *Then if* $u \in V_\tau$ *we have*

i)
$$\lambda_\varepsilon = \begin{cases} \lambda + o(\varepsilon^{2\tau}) & as \quad \varepsilon \downarrow 0, \quad \tau \in [0, \frac{1}{2}), \\ \lambda + \varepsilon\lambda' + o(\varepsilon^{2\tau}) & as \quad \varepsilon \downarrow 0, \quad \tau \in [\frac{1}{2}, 1), \\ \lambda + \varepsilon\lambda' + O(\varepsilon^2) & as \quad \varepsilon \downarrow 0, \quad \tau = 1, \end{cases}$$

where $\lambda' = a(u)$; *and*

ii)
$$|u_\varepsilon - u|_0 = \begin{cases} o(\varepsilon^\tau) & as \quad \varepsilon \downarrow 0, \quad \tau \in [0, 1), \\ O(\varepsilon) & as \quad \varepsilon \downarrow 0, \quad \tau = 1. \end{cases}$$

Moreover if $\tau \in [0, 1]$ *and* $D(B) \underset{c}{\subset} V_\tau$ *with* $D(B)$ *dense in* V_τ *then*

iii)
$$|u_\varepsilon - u|_\gamma = \begin{cases} o(\varepsilon^{\tau-\gamma}) & as \quad \varepsilon \downarrow 0, \quad \gamma \in (\tau - 1, \tau], \\ O(\varepsilon) & as \quad \varepsilon \downarrow 0, \quad \gamma = \tau - 1, \end{cases}$$

and

iv)
$$|u_\varepsilon - u| = \begin{cases} o(\varepsilon^{2\tau}) & as \quad \varepsilon \downarrow 0, \quad \tau < \frac{1}{2}, \\ O(\varepsilon) & as \quad \varepsilon \downarrow 0, \quad \tau = \frac{1}{2}. \end{cases}$$

Note further that according to Proposition 2.1, if λ_n $(1 \leqq n \leqq N)$ is an eigenvalue of B with multiplicity m and J is an open interval containing λ_n but no other point of the spectrum of B, then for ε sufficiently small the intersection of J and the spectrum of A_ε contains only eigenvalues with total multiplicity exactly m. The following theorem provides additional information under hypotheses which guarantee that the eigenvalue λ_n "completely splits" under this perturbation, i.e., that for ε sufficiently small all eigenvalues $\lambda_{j,\varepsilon}$ converging to λ_n are distinct.

Theorem 2.4. *Let* $\lambda = \lambda_{i+1} = \cdots = \lambda_{i+m}$ $(0 \leqq i < i+m \leqq N)$ *be an eigenvalue of* B *of multiplicity* m *with corresponding eigenvectors* $u_{i+1}, \ldots, u_{i+m}$, $(u_{i+j}, u_{i+k}) = \delta_{jk}$ *for* $1 \leqq j, k \leqq m$. *Assume that for fixed* $\tau \in [\frac{1}{2}, 1]$ *we have* $\mathrm{sp}\langle u_{i+1}, \ldots, u_{i+m}\rangle \subset V_\tau$ *and that* $a(u_{i+j}, u_{i+k}) = \lambda'_j \delta_{jk}$ *with* $\lambda'_1 < \lambda'_2 < \cdots < \lambda'_m$. *Choose the eigenvectors* $u_{i+j,\varepsilon}, \ldots, u_{i+m,\varepsilon}$ *of* A_ε *so that* $(u_{i+j,\varepsilon}, u_{i+k,\varepsilon}) = \delta_{jk}$ *and* $(u_{i+j,\varepsilon}, u_{i+j}) \geqq 0$ *for* $1 \leqq j, k \leqq m$. *Then the following conclusions hold:*

i) We have

$$\lambda_{i+j,\varepsilon} = \begin{cases} \lambda + \varepsilon\lambda'_j + o(\varepsilon^{2\tau}) & as \quad \varepsilon \downarrow 0, \quad \tau \in [\frac{1}{2}, 1) \\ \lambda + \varepsilon\lambda'_j + O(\varepsilon^2) & as \quad \varepsilon \downarrow 0, \quad \tau = 1, \end{cases}$$

and

$$|u_{i+j,\varepsilon} - u_{i+j}|_0 = \begin{cases} o(\varepsilon^{\tau-\frac{1}{2}}) & as \quad \varepsilon \downarrow 0, \quad \tau \in [\frac{1}{2}, 1), \\ O(\varepsilon^{\frac{1}{2}}) & as \quad \varepsilon \downarrow 0, \quad \tau = 1, \end{cases}$$

for $j = 1, \ldots, m$.

ii) If in addition $D(B) \underset{c}{\subset} V$ *with* $D(B)$ *dense in* V, *then*

$$|u_{i+j,\varepsilon} - u_{i+j}|_0 = \begin{cases} o(\varepsilon^{2\tau-1}) & as \quad \varepsilon \downarrow 0, \quad \tau \in [\frac{1}{2}, \frac{3}{4}), \\ O(\varepsilon^{\frac{1}{2}}) & as \quad \varepsilon \downarrow 0, \quad \tau \geqq \frac{3}{4}, \end{cases}$$

and

$$|u_{i+j,\varepsilon} - u_{i+j}|_V = o(1) \quad \text{as} \quad \varepsilon \downarrow 0,$$

for $j = 1, \ldots, m$.

iii) Moreover, if $\tau \in (\frac{1}{2}, 1]$ *and* $D(B) \underset{c}{\subset} V_\tau$ *with* $D(B)$ *dense in* V_τ, *then for each* $\gamma \in (\frac{1}{2}, \tau]$,

$$|u_{i+j,\varepsilon} - u_{i+j}|_\gamma = o(\varepsilon^{\tau-\gamma}) \quad as \quad \varepsilon \downarrow 0$$

for $j = 1, \ldots, m$.

Remark. As noted in [17], p. 108, the assumptions $a(u_{i+j}, u_{i+k}) = \lambda'_j \delta_{jk}$ and $(u_{i+j}, u_{i+k}) = \delta_{jk}$ amount to simultaneous diagonalization of the two matrices $[a(u_{i+j}, u_{i+k})]$ and $[(u_{i+j}, u_{i+k})]$. This is not a restriction.

3. Lemmas. Proofs of Theorems

Recall that $V_{-\tau}$ is a completion of the inner product space obtained by providing V_0 with the norm $|v|_{-\tau}$. It will be important in the sequel to be able to realize this completion as a Hilbert space which is continuously contained in H (cf. ARONSZAJN & GAGLIARDO [5], §3).

Lemma 3.1. *Let* $\tau \in (0, 1]$. *i) If* $D(B) \underset{c}{\subset} V_\tau$ *with* $D(B)$ *dense in* V_τ, *then* $V_{-\tau}$ *can be realized as a Hilbert space which is continuously contained in* H. *ii) If* $V_{-\tau} \underset{c}{\subset} H$, *then* $D(B) \underset{c}{\subset} V_\tau$ *and* $D(B)$ *is dense in* V_τ.

Proof. Let X be the pre-Hilbert space obtained by providing V_0 with the inner product of $V_{-\tau}$. It must be shown that the couple $[X, H]$ is completely compatible in the sense of [5], Definition [3.V].

For $f \in H$, let u be the unique element of $D(B)$ such that $Bu = f$. By hypothesis there exists $K > 0$ such that $|u|_\tau \leqq K|u|_{D(B)} = K|Bu|$. Thus for any $v \in V_0$,

$$\begin{aligned}
|v| &= \sup\{|(v,f)| \,\big|\, f \in H \text{ and } |f| \leqq 1\} \\
&= \sup\{|(v, Bu)| \,\big|\, u \in D(B) \text{ and } |Bu| \leqq 1\} \\
&= \sup\{|(v, u)_0| \,\big|\, u \in D(B) \text{ and } |Bu| \leqq 1\} \\
&\leqq K \sup\{|(v, u)_0| \,\big|\, u \in D(B) \text{ and } |u|_\tau \leqq 1\} = K|v|_{-\tau}
\end{aligned}$$

since $D(B)$ is dense in V_τ.

Now

$$(S^\tau u, S^{-\tau} v)_0 = b(u, v) = (f, v) \qquad \text{for all } v \in V_0,$$

and so

$$|(f, v)| \leqq |S^\tau u|_0 \, |S^{-\tau} v|_0 = |u|_\tau \, |v|_{-\tau} \qquad \text{for all } v \in V_0.$$

Thus since V_0 is dense in $V_{-\tau}$ and B is an isometric isomorphism of $D(B)$ onto H, each $f \in H$ induces a continuous linear functional F on $V_{-\tau}$ given by

$$F(v) = (v, S^{2\tau} u)_{-\tau}, \qquad v \in V_{-\tau}, \qquad u = B^{-1} f.$$

So, if $\{v_n\}$ is a sequence of elements of V_0 such that $v_n \to 0$ in H as $n \to \infty$ and $v_n \to v$ in $V_{-\tau}$ as $n \to \infty$, then for each $f \in H$, $(v_n, f) \to 0$ as $n \to \infty$ and $(v_n, f) = (v_n, S^{2\tau} u)_{-\tau} \to (S^{-\tau} v, S^\tau u)_0$ as $n \to \infty$. Since $D(B)$ is dense in V_τ, therefore $S^\tau D(B)$ is dense in V_0; thus $S^{-\tau} v = 0$ which implies that $v = 0$.

It has been verified that $[X, H]$ is completely compatible, and therefore *i)* follows from Theorem [3.IX] of [5].

ii) Let $u \in D(B)$ and $Bu = f \in H$. Then $b(u, v) = (u, v)_0 = (f, v)$ for all $v \in V_0$ and hence

$$|(u, v)_0| \leqq M |f| |v|_{-\tau} \quad \text{for all } v \in V_0,$$

where M is such that $|g| \leqq M |g|_{-\tau}$ for all $g \in V_{-\tau}$. Thus by the procedure used by LAX [19], p. 623, $u \in V_\tau$ and

$$\begin{aligned} |u|_\tau &= \sup\{|(u, v)_0| \,\big|\, v \in V_0 \quad \text{and} \quad |v|_{-\tau} \leqq 1\} \\ &\leqq M |f| = M |u|_{D(B)}, \end{aligned}$$

that is $D(B) \underset{c}{\subset} V_\tau$.

Now $(S^\tau u, S^{-\tau} v)_0 = (f, v)$ for all $u \in D(B)$ and all $v \in V_0$. Thus since $V_{-\tau} \underset{c}{\subset} H$, $(S^\tau u, w)_0 = (f, S^\tau w)$ for all $u \in D(B)$ and all $w \in V_0$. But $V_{-\tau}$ is dense in H so if $(S^\tau u, w)_0 = 0$ for all $u \in D(B)$, then $w = 0$. Hence $S^\tau D(B)$ is dense in V_0 which implies that $D(B)$ is dense in V_τ.

Corollary 3.1. *If* $\tau \in (0, 1]$ *and* $D(B) \underset{c}{\subset} V_\tau$ *then there exists a constant* $M > 0$ *such that*

$$|v| \leqq M |v|_{-\tau} \quad \text{for all } v \in V_0.$$

Proof. This was shown in the first part of the proof of Lemma 3.1, i).

Now let f, f_ε be given in H, $0 < \varepsilon \leqq \varepsilon_0$. The method of orthogonal projection yields for each $\varepsilon \in (0, \varepsilon_0]$ a unique solution $u_\varepsilon \in D(A_\varepsilon)$ of

$$A_\varepsilon u_\varepsilon = f_\varepsilon \tag{3.1}$$

which is also the unique solution in $V = D(a)$ of

$$\varepsilon a(u_\varepsilon, v) + b(u_\varepsilon, v) = (f_\varepsilon, v) \quad \text{for all } v \in V. \tag{3.2}$$

Similarly let u be the unique solution in $D(B)$ of

$$Bu = f \tag{3.3}$$

and observe that u is also the unique solution in $V_0 = D(b)$ of

$$b(u, v) = (f, v) \quad \text{for all } v \in V_0. \tag{3.4}$$

Proposition 3.1. *Let* u_ε, u *be the solutions of* (3.1), (3.3) *respectively, let* $\tau \in [0, 1]$, *and assume that* $D(B) \underset{c}{\subset} V_\tau$ *with* $D(B)$ *dense in* V_τ. *Then we have:*

i) If for fixed $\gamma \in (\tau - 1, \tau]$, $|f_\varepsilon - f| = o(\varepsilon^{\tau - \gamma})$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - u|_\gamma = o(\varepsilon^{\tau - \gamma}) \quad \text{as} \quad \varepsilon \downarrow 0.$$

ii) If $|f_\varepsilon - f| = O(\varepsilon)$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - u|_{\tau - 1} = O(\varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0.$$

iii) In particular, if $\tau < \frac{1}{2}$ *and* $|f_\varepsilon - f| = o(\varepsilon^{2\tau})$ *as* $\varepsilon \downarrow 0$, *then*

$$|u_\varepsilon - u| = o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0,$$

while if $\tau=\frac{1}{2}$ *and* $|f_\varepsilon-f|=O(\varepsilon)$ *as* $\varepsilon\downarrow 0$, *then*

$$|u_\varepsilon-u|=O(\varepsilon) \quad as \quad \varepsilon\downarrow 0.$$

Proof. In view of Lemma 3.1, *iii*) follows from *i*) and *ii*). For $\gamma\in(0,\tau]$, *i*) was proved in [10], Theorem 4.4: the remaining results are proved by those methods.

Proof of Theorem 2.1. *i*) Let $\mathscr{S}_n=\mathrm{sp}\langle u_1,\dots,u_n\rangle$, the linear manifold spanned by $u_1,\dots,u_n$, and let $\mathscr{S}_{n,\varepsilon}=\mathrm{sp}\langle z_{1,\varepsilon},\dots,z_{n,\varepsilon}\rangle$ where $z_{j,\varepsilon}=(\varepsilon\mathscr{A}+I)^{\tau-\frac{1}{2}}u_j, j=1,\dots,n$. Further, let $\mathfrak{S}_n$ denote an arbitrary n-dimensional subspace of H and, for simplicity of notation, let us adopt the convention that a quotient of quadratic forms assumes the value 0 when both the numerator and denominator are zero. Now

$$(3.5)\qquad \begin{aligned}\lambda_n&=\inf_{\mathfrak{S}_n\subset V}\ \sup_{v\in\mathfrak{S}_n}\frac{b(v)}{|v|^2}\\ &\leqq\min_{\mathfrak{S}_n\subset V}\ \max_{v\in\mathfrak{S}_n}\frac{\varepsilon a(v)+b(v)}{|v|^2}=\lambda_{n,\varepsilon}.\end{aligned}$$

Thus it is sufficient to prove that

$$(3.6)\qquad \lambda_{n,\varepsilon}\leqq\lambda_n+o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon\downarrow 0.$$

For this purpose, first note that

$$(3.7)\qquad \begin{aligned}\lambda_{n,\varepsilon}&\leqq\max_{v\in\mathscr{S}_{n,\varepsilon}}\frac{\varepsilon a(v)+b(v)}{|v|^2}\\ &=\max_{v\in\mathscr{S}_{n,\varepsilon}}\frac{b\big((\varepsilon\mathscr{A}+I)^{\frac{1}{2}}v\big)}{|v|^2}\\ &=\max_{x\in\mathscr{S}_n}\frac{b\big((\varepsilon\mathscr{A}+I)^{\tau}x\big)}{|(\varepsilon\mathscr{A}+I)^{\tau-\frac{1}{2}}x|^2}\\ &\leqq\lambda_n\cdot\max_{x\in\mathscr{S}_n}\frac{b\big((\varepsilon\mathscr{A}+I)^{\tau}x\big)}{b(x)}\cdot\max_{x\in\mathscr{S}_n}\frac{|x|^2}{|(\varepsilon\mathscr{A}+I)^{\tau-\frac{1}{2}}x|^2}.\end{aligned}$$

To estimate the right hand side of the last inequality, first choose $x_\varepsilon\in\mathscr{S}_n$ such that $b(x_\varepsilon)=1$ and

$$(3.8)\qquad b\big((\varepsilon\mathscr{A}+I)^{\tau}x_\varepsilon\big)=\max_{x\in\mathscr{S}_n}\frac{b\big((\varepsilon\mathscr{A}+I)^{\tau}x\big)}{b(x)}.$$

Then $x_\varepsilon=\sum_{i=1}^{n}\alpha_{i,\varepsilon}u_i$ with $\sum_{i=1}^{n}\lambda_i|\alpha_{i,\varepsilon}|^2=1$. Hence, letting E be the resolution of the identity for the self-adjoint operator $\mathscr{A}$, the spectral theorem for functions of a self-adjoint operator gives

$$(3.9)\qquad \begin{aligned}b\big((\varepsilon\mathscr{A}+I)^{\tau}x_\varepsilon\big)&=\int_0^\infty(\varepsilon v+1)^{2\tau}\big(E(dv)\,x_\varepsilon,x_\varepsilon\big)_0\\ &=\varepsilon^{2\tau}\int_0^\infty v^{2\tau}\frac{(\varepsilon v)^{1-2\tau}}{(\varepsilon v+1)^{1-2\tau}}\big(E(dv)\,x_\varepsilon,x_\varepsilon\big)_0+\int_0^\infty(\varepsilon v+1)^{2\tau-1}\big(E(dv)\,x_\varepsilon,x_\varepsilon\big)_0\\ &\leqq b(x_\varepsilon)+n\,\varepsilon^{2\tau}\sum_{i=1}^{n}\lambda_i|\alpha_{i,\varepsilon}|^2\int_0^\infty v^{2\tau}\frac{(\varepsilon v)^{1-2\tau}}{(\varepsilon v+1)^{1-2\tau}}\big(E(dv)\,u_i,u_i\big)_0.\end{aligned}$$

Now $u_i \in D(\mathscr{A}^\tau) = V_\tau$ if and only if $\int_0^\infty v^{2\tau}(E(dv)u_i, u_i)_0 < \infty$. Thus (3.8), (3.9) and Lebesgue's dominated convergence theorem imply that

$$\max_{x \in \mathscr{S}_n} \frac{b((\varepsilon \mathscr{A}+I)^\tau x)}{b(x)} \leqq 1 + o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0. \tag{3.10}$$

Now choose $y_\varepsilon \in \mathscr{S}_n$ such that $|y_\varepsilon| = 1$ and

$$\frac{1}{|(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} y_\varepsilon|^2} = \max_{x \in \mathscr{S}_n} \frac{|x|^2}{|(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} x|^2}. \tag{3.11}$$

Then $y_\varepsilon = \sum_{i=1}^n \beta_{i,\varepsilon} u_i$ with $\sum_{i=1}^n |\beta_{i,\varepsilon}|^2 = 1$. By Corollary 3.1 there is an $M > 0$ such that

$$\big| |w|^2 - |v|^2 \big| \leqq (|w| + |v|)|w - v| \leqq M(|w| + |v|)|w - v|_{-\tau}$$

for all $v, w \in V_0$. Thus it suffices to show that

$$|(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} y_\varepsilon - y_\varepsilon|_{-\tau} = o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0 \tag{3.12}$$

in order to prove that

$$|(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} y_\varepsilon|^2 = |y_\varepsilon|^2 + o(\varepsilon^{2\tau}) = 1 + o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0. \tag{3.13}$$

Now

$$|(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} y_\varepsilon - y_\varepsilon|_{-\tau}^2 \leqq n \sum_{i=1}^n |\beta_{i,\varepsilon}|^2 |(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} u_i - u_i|_{-\tau}^2$$

and, again letting E be the resolution of the identity for the self-adjoint operator $\mathscr{A}$,

$$|(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} u_i - u_i|_{-\tau}^2 = \int_0^\infty [(\varepsilon v+1)^{\tau-\frac{1}{2}} - 1]^2 (v^2+1)^{-\tau} (E(dv) u_i, u_i)_0$$

for each $i = 1, \ldots, n$. Further, since for

$$0 \leqq t \leqq 1 \text{ and } v \geqq 0, \quad (v+1)^t \leqq v^t + 1,$$

$$\begin{aligned}
[(\varepsilon v+1)^{\tau-\frac{1}{2}} - 1]^2 &= [(\varepsilon v+1)^{\frac{1}{2}-\tau} - 1]^2 (\varepsilon v+1)^{2\tau-1} \\
&\leqq [(\varepsilon v+1)^{\frac{1}{2}+\tau} - 1]^2 (\varepsilon v+1)^{2\tau-1} \\
&\leqq (\varepsilon v)^{1+2\tau} (\varepsilon v+1)^{2\tau-1},
\end{aligned}$$

$$|(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} u_i - u_i|_{-\tau}^2 \leqq \varepsilon^{4\tau} \int_0^\infty v^{2\tau} \frac{(\varepsilon v)^{1-2\tau}}{(\varepsilon v+1)^{1-2\tau}} v^{2\tau} (v^2+1)^{-\tau} (E(dv) u_i, u_i)_0.$$

Since $u_i \in V_\tau$, the last expression is $o(\varepsilon^{4\tau})$ as $\varepsilon \downarrow 0$. Thus (3.12) holds and (3.6) follows from (3.7), (3.10), (3.11) and (3.13). Therefore *i*) is proved.

ii) Referring to (3.5), one has

$$\lambda_n \leqq \lambda_{n,\varepsilon} \leqq \max_{v \in \mathscr{S}_n} \frac{\varepsilon a(v) + b(v)}{|v|^2} \leqq \lambda_n + \varepsilon \max_{v \in \mathscr{S}_n} \frac{a(v)}{|v|^2}.$$

iii) Denote by λ the common value of $\lambda_{i+1}=\dots=\lambda_{i+m}$ and let u be any element of $\mathrm{sp}\langle u_{i+1},\dots,u_{i+m}\rangle$ with $|u|=1$. Recall that $A_\varepsilon^{-1}B\subset(\varepsilon\mathscr{A}+I)^{-1}$, $Bu=\lambda u$, and that by Lemma 3.1 there exists $M>0$ such that $|w|\leqq M|w|_{-\tau}$ for all $w\in V_0$. Thus

$$|(A_\varepsilon^{-1}-\lambda^{-1})u|^2=\lambda^{-2}|(A_\varepsilon^{-1}B-I)u|^2\leqq\lambda^{-2}M^2|[(\varepsilon\mathscr{A}+I)^{-1}-I]u|_{-\tau}^2,$$

and if E is the resolution of the identity for $\mathscr{A}$,

$$|[(\varepsilon\mathscr{A}+I)^{-1}-I]u|_{-\tau}^2=\int_0^\infty\left[\frac{1}{\varepsilon\nu+1}-1\right]^2(\nu^2+1)^{-\tau}(E(d\nu)u,u)_0$$

$$=\varepsilon^{4\tau}\int_0^\infty\nu^{2\tau}\frac{(\varepsilon\nu)^{2-4\tau}}{(\varepsilon\nu+1)^{2-4\tau}}\frac{1}{(\varepsilon\nu+1)^{4\tau}}\nu^{2\tau}(\nu^2+1)^{-\tau}(E(d\nu)u,u)_0.$$

Since $u\in V_\tau$ it follows that

$$|(A_\varepsilon^{-1}-\lambda^{-1})u|=\begin{cases}o(\varepsilon^{2\tau}) & \text{as}\quad\varepsilon\downarrow0\quad\text{if}\quad\tau\in[0,\frac12)\\ O(\varepsilon) & \text{as}\quad\varepsilon\downarrow0\quad\text{if}\quad\tau\geqq\frac12.\end{cases}$$

It follows from parts *i*) and *ii*) of the present theorem that $\lambda_{i+1,\varepsilon}^{-1}\geqq\dots\geqq\lambda_{i+m,\varepsilon}^{-1}$ all lie in an interval $(\lambda^{-1}-\delta(\varepsilon),\lambda^{-1}]$ where $\delta(\varepsilon)\geqq0$ and

$$\delta(\varepsilon)=\begin{cases}o(\varepsilon^{2\tau}) & \text{as}\quad\varepsilon\downarrow0\quad\text{if}\quad\tau\in[0,\frac12)\\ O(\varepsilon) & \text{as}\quad\varepsilon\downarrow0\quad\text{if}\quad\tau\geqq\frac12.\end{cases}$$

Choose $\rho>0$ such that the intervals $[\lambda^{-1}-\rho,\lambda^{-1})$, $(\lambda^{-1},\lambda^{-1}+\rho]$ are both contained in the resolvent set of B^{-1}. By Proposition 2.1 there exists an $\varepsilon_1>0$ such that for $0<\varepsilon\leqq\varepsilon_1$ there are no eigenvalues of A_ε^{-1} in either of the intervals $[\lambda^{-1}-\rho,\lambda^{-1}-\delta(\varepsilon)]$, $[\lambda^{-1}+\delta(\varepsilon),\lambda^{-1}+\rho]$. Thus by a lemma of Lyusternik [21], §10,

$$|u-P_\varepsilon u|=\begin{cases}o(\varepsilon^{2\tau}) & \text{as}\quad\varepsilon\downarrow0,\quad\tau\in[0,\frac12)\\ O(\varepsilon) & \text{as}\quad\varepsilon\downarrow0,\quad\tau\geqq\frac12.\end{cases}\tag{3.14}$$

Now $P_\varepsilon u=\sum_{j=1}^m c_{j,\varepsilon}u_{i+j,\varepsilon}$ with $\sum_{j=1}^m|c_{j,\varepsilon}|^2\leqq1$ and so

$$A_\varepsilon P_\varepsilon u=\sum_{j=1}^m c_{j,\varepsilon}\lambda_{i+j,\varepsilon}u_{i+j,\varepsilon}=\lambda P_\varepsilon u+\sum_{j=1}^m c_{j,\varepsilon}(\lambda_{i+j,\varepsilon}-\lambda)u_{i+j,\varepsilon}.$$

By parts *i*) and *ii*),

$$\left|\sum_{j=1}^m c_{j,\varepsilon}(\lambda_{i+j,\varepsilon}-\lambda)u_{i+j,\varepsilon}\right|^2=\begin{cases}o(\varepsilon^{4\tau}) & \text{as}\quad\varepsilon\downarrow0,\quad\tau\in[0,\frac12)\\ O(\varepsilon^2) & \text{as}\quad\varepsilon\downarrow0,\quad\tau\geqq\frac12.\end{cases}\tag{3.15}$$

It follows from (3.14) and (3.15) that $A_\varepsilon P_\varepsilon u=z_\varepsilon$ with

$$|z_\varepsilon-\lambda u|=\begin{cases}o(\varepsilon^{2\tau}) & \text{as}\quad\varepsilon\downarrow0,\quad\tau\in[0,\frac12)\\ O(\varepsilon) & \text{as}\quad\varepsilon\downarrow0,\quad\tau\geqq\frac12.\end{cases}$$

Since u satisfies $Bu=\lambda u$, *iii*) now follows from Proposition 3.1 by setting $Pv=u$.

Proof of Theorem 2.2. *i*) The proof proceeds in the same fashion as the proof of part *i*) of Theorem 2.1 through formula (3.11). Then since

$$\big||w|^2-|v|^2\big|\leqq(|w|+|v|)|w-v|_0$$

for all $v, w \in V_0$, it suffices to prove that

$$|(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} y_\varepsilon - y_\varepsilon|_0 = o(\varepsilon^\tau) \quad \text{as} \quad \varepsilon \downarrow 0$$

in order to prove that

$$|(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} y_\varepsilon|^2 = 1 + o(\varepsilon^\tau) \quad \text{as} \quad \varepsilon \downarrow 0.$$

Again,

$$|(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} y_\varepsilon - y_\varepsilon|_0^2 \leqq n \sum_{i=1}^{n} |\beta_{i,\varepsilon}|^2 |(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} u_i - u_i|_0^2,$$

and letting E be the resolution of the identity for $\mathscr{A}$,

$$\begin{aligned} |(\varepsilon \mathscr{A}+I)^{\tau-\frac{1}{2}} u_i - u_i|_0^2 &= \int_0^\infty [(\varepsilon\nu+1)^{\tau-\frac{1}{2}} - 1]^2 (E(d\nu)u_i, u_i)_0 \\ &\leqq \int_0^\infty [(\varepsilon\nu+1)^{\frac{1}{2}} - 1]^2 (\varepsilon\nu+1)^{2\tau-1} (E(d\nu)u_i, u_i)_0 \\ &\leqq \varepsilon^{2\tau} \int_0^\infty \nu^{2\tau} \frac{(\varepsilon\nu)^{1-2\tau}}{(\varepsilon\nu+1)^{1-2\tau}} (E(d\nu)u_i, u_i)_0 \\ &= o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0 \end{aligned}$$

for each $i = 1, \ldots, n$. *i*) follows.

To prove *ii*), one modifies the proof of part *iii*) of Theorem 2.1 in much the same fashion. In the present case, if $\lambda = \lambda_{i+1} = \cdots = \lambda_{i+m}$ and u is one of the vectors u_n, $i+1 \leqq n \leqq \min(j, i+m)$, then

$$\begin{aligned} |(A_\varepsilon^{-1} - \lambda^{-1}) u| &\leqq \lambda^{-1} |[(\varepsilon \mathscr{A}+I)^{-1} - I] u|_0 \\ &= \begin{cases} o(\varepsilon^\tau) & \text{as} \quad \varepsilon \downarrow 0, \quad \tau \in [0,1) \\ O(\varepsilon) & \text{as} \quad \varepsilon \downarrow 0, \quad \tau = 1. \end{cases} \end{aligned}$$

By part *i*) of the present theorem and *ii*) of Theorem 2.1,

$$\delta(\varepsilon) = \begin{cases} o(\varepsilon^\tau) & \text{as} \quad \varepsilon \downarrow 0, \quad \tau \in [0, \frac{1}{2}) \\ O(\varepsilon) & \text{as} \quad \varepsilon \downarrow 0, \quad \tau \geqq \frac{1}{2}. \end{cases}$$

Therefore,

$$|u - P_\varepsilon u| = \begin{cases} o(\varepsilon^\tau) & \text{as} \quad \varepsilon \downarrow 0, \quad \tau \in [0,1) \\ O(\varepsilon) & \text{as} \quad \varepsilon \downarrow 0, \quad \tau = 1. \end{cases}$$

Moreover,

$$\left| \sum_{j=1}^{m} c_{j,\varepsilon} (\lambda_{i+j,\varepsilon} - \lambda) u_{i+j,\varepsilon} \right|^2 = \begin{cases} o(\varepsilon^{2\tau}) & \text{as} \quad \varepsilon \downarrow 0, \quad \tau \in [0, \frac{1}{2}) \\ O(\varepsilon^2) & \text{as} \quad \varepsilon \downarrow 0, \quad \tau \geqq \frac{1}{2}. \end{cases}$$

Hence $A_\varepsilon P_\varepsilon u = z_\varepsilon$ with

$$|z_\varepsilon - \lambda u| = \begin{cases} o(\varepsilon^\tau) & \text{as} \quad \varepsilon \downarrow 0, \quad \tau \in [0,1) \\ O(\varepsilon) & \text{as} \quad \varepsilon \downarrow 0, \quad \tau = 1, \end{cases}$$

and since u satisfies $Bu = \lambda u$, *ii*) now follows from [10], Theorem 4.1.

Proof of Theorem 2.3. It is necessary to calculate $\eta = (A_\varepsilon^{-1} u, u)$ and $\theta = |(A_\varepsilon^{-1} - \eta I) u|$ in order to apply Lemma 4, p. 437, of KATO [16]. Letting E be the resolution of the identity for $\mathscr{A}$,

$$\eta=\lambda^{-2} b\big((\varepsilon\mathscr{A}+I)^{-1} u, u\big)=\lambda^{-2} \int_0^\infty (\varepsilon v+1)^{-1} \big(E(dv) u, u\big)_0$$

since $A_\varepsilon^{-1} B \subset (\varepsilon\mathscr{A}+I)^{-1}$ and $Bu=\lambda u$. Therefore if $u \in V_\tau$ and $\tau \in [0, \frac{1}{2})$,

$$\begin{aligned}
\eta&=\lambda^{-2}\left\{b(u)-\int_0^\infty \varepsilon v(\varepsilon v+1)^{-1}\big(E(dv)u,u\big)_0\right\}\\
&=\lambda^{-2}\left\{\lambda-\varepsilon^{2\tau}\int_0^\infty v^{2\tau}\frac{(\varepsilon v)^{1-2\tau}}{(\varepsilon v+1)^{1-2\tau}}\frac{1}{(\varepsilon v+1)^{2\tau}}\big(E(dv)u,u\big)_0\right\}\\
&=\lambda^{-1}+o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0,
\end{aligned}$$

while if $u \in V_\tau$ and $\tau \geqq \frac{1}{2}$,

$$\begin{aligned}
\eta&=\lambda^{-2}\left\{\lambda-\int_0^\infty \varepsilon v\big(E(dv)u,u\big)_0+\int_0^\infty \varepsilon^2 v^2(\varepsilon v+1)^{-1}\big(E(dv)u,u\big)_0\right\}\\
&=\lambda^{-2}\left\{\lambda-\varepsilon a(u)+\varepsilon^{2\tau}\int_0^\infty v^{2\tau}\frac{(\varepsilon v)^{2-2\tau}}{(\varepsilon v+1)^{2-2\tau}}\frac{1}{(\varepsilon v+1)^{2\tau-1}}\big(E(dv)u,u\big)_0\right\}\\
&=\begin{cases}\lambda^{-1}-\varepsilon\lambda^{-2}\lambda'+o(\varepsilon^{2\tau}) & \text{as} \quad \varepsilon\downarrow 0, \quad \tau\in[\frac{1}{2},1),\\ \lambda^{-1}-\varepsilon\lambda^{-2}\lambda'+O(\varepsilon^{2}) & \text{as} \quad \varepsilon\downarrow 0, \quad \tau=1.\end{cases}
\end{aligned}$$

Thus if $u \in V_\tau$,

$$\theta=\lambda^{-1}\,|[(\varepsilon\mathscr{A}+I)^{-1}-\eta\lambda I]\,u| \leqq \lambda^{-1}\,|[(\varepsilon\mathscr{A}+I)^{-1}-Iu]|+\kappa(\varepsilon)$$

where

$$\kappa(\varepsilon)=\begin{cases}o(\varepsilon^{2\tau}) & \text{as} \quad \varepsilon\downarrow 0, \quad \tau\in[0,\frac{1}{2}),\\ O(\varepsilon) & \text{as} \quad \varepsilon\downarrow 0, \quad \tau\geqq\frac{1}{2}.\end{cases}$$

Moreover, by a now familiar calculation,

$$|[(\varepsilon\mathscr{A}+I)^{-1}-I]\,u| \leqq |[(\varepsilon\mathscr{A}+I)^{-1}-I]\,u|_0=\begin{cases}o(\varepsilon^{\tau}) & \text{as} \quad \varepsilon\downarrow 0, \quad \tau\in[0,1),\\ O(\varepsilon) & \text{as} \quad \varepsilon\downarrow 0, \quad \tau=1.\end{cases}$$

Hence

$$\theta=\begin{cases}o(\varepsilon^{\tau}) & \text{as} \quad \varepsilon\downarrow 0, \quad \tau\in[0,1),\\ O(\varepsilon) & \text{as} \quad \varepsilon\downarrow 0, \quad \tau=1.\end{cases}$$

Lemma 4, p. 437, of [16] now implies that $|\lambda_\varepsilon^{-2}-\eta|=O(\theta^2)$ as $\varepsilon\downarrow 0$ and $|u_\varepsilon-u|=O(\theta)$ as $\varepsilon\downarrow 0$. Thus *i)* follows, and *ii)* is then proved in the same way as part *ii)* of Theorem 2.2. To get *iii)* and *iv)* one proceeds in the same fashion, using the estimate $|[(\varepsilon\mathscr{A}+I)^{-1}-I]\,u| \leqq M\,|[(\varepsilon\mathscr{A}+I)^{-1}-I]\,u|_{-\tau}$ obtained from Lemma 3.1 in the calculation of θ.

Proof of Theorem 2.4. This theorem is proved by employing some of the techniques used above, together with the method of Theorem 19.3, p. 117, KATO [17]. Accordingly, only some essential estimates corresponding to estimates in the proof of KATO's theorem are given here and the reader is referred to [17], pp. 117–121, for the rest of the details.

Assuming the hypotheses of *i*), the estimate

$$(A_\varepsilon^{-1} u_{i+j}, u_{i+k}) = \begin{cases} (\lambda^{-1} - \varepsilon\lambda^{-2}\lambda'_j)\,\delta_{jk} + o(\varepsilon^{2\tau}) & \text{as } \varepsilon\downarrow 0, \quad \tau\in[\tfrac{1}{2},1), \\ (\lambda^{-1} - \varepsilon\lambda^{-2}\lambda'_j)\,\delta_{jk} + O(\varepsilon^{2}) & \text{as } \varepsilon\downarrow 0, \quad \tau=1, \end{cases}$$

is obtained as in the evaluation of η in the proof of Theorem 2.3. Thus the estimates in part II of the proof of Theorem 19.3 of [17] become

$$\eta_j = \begin{cases} -\lambda^{-1} + \varepsilon\lambda^{-2}\lambda'_j + o(\varepsilon^{2\tau}) & \text{as } \varepsilon\downarrow 0, \quad \tau\in[\tfrac{1}{2},1), \\ -\lambda^{-1} + \varepsilon\lambda^{-2}\lambda'_j + O(\varepsilon^{2}) & \text{as } \varepsilon\downarrow 0, \quad \tau=1, \end{cases}$$

$$\theta_j = \begin{cases} o(\varepsilon^{2\tau}) & \text{as } \varepsilon\downarrow 0, \quad \tau\in[\tfrac{1}{2},1), \\ O(\varepsilon^{2}) & \text{as } \varepsilon\downarrow 0, \quad \tau=1, \end{cases}$$

$$\mu_j = \begin{cases} -\lambda^{-1} + \varepsilon\lambda^{-2}\lambda'_j + o(\varepsilon^{2\tau}) & \text{as } \varepsilon\downarrow 0, \quad \tau\in[\tfrac{1}{2},1), \\ -\lambda^{-1} + \varepsilon\lambda^{-2}\lambda'_j + O(\varepsilon^{2}) & \text{as } \varepsilon\downarrow 0, \quad \tau=1, \end{cases}$$

and

$$|\psi_j - u_{i+j}| = \begin{cases} o(\varepsilon^{2\tau-1}) & \text{as } \varepsilon\downarrow 0, \quad \tau\in[\tfrac{1}{2},1), \\ O(\varepsilon) & \text{as } \varepsilon\downarrow 0, \quad \tau=1. \end{cases} \tag{3.16}$$

In part III of the proof of KATO's theorem one has

$$\sum_{j=1}^{m} \theta_j^2 \leq \sum_{j=1}^{m} |(A_\varepsilon^{-1} - \lambda^{-1})\, u_{i+j}|^2 = \begin{cases} o(\varepsilon^{2\tau}) & \text{as } \varepsilon\downarrow 0, \quad \tau\in[\tfrac{1}{2},1), \\ O(\varepsilon^{2}) & \text{as } \varepsilon\downarrow 0, \quad \tau=1, \end{cases}$$

and

$$\sum_{j=1}^{m} \theta_j^2 = O(\varepsilon^2) \quad \text{as} \quad \varepsilon\downarrow 0 \quad \text{if} \quad D(B) \underset{c}{\subset} V$$

with $D(B)$ dense in V ($V_{\frac{1}{2}} = V$ up to equivalent norms). These estimates are obtained as in the estimate for θ in the proof of Theorem 2.3. Thus

$$-\lambda_{i+j,\varepsilon}^{-1} = \begin{cases} -\lambda^{-1} + \varepsilon\lambda^{-2}\lambda'_j + o(\varepsilon^{2\tau}) & \text{as } \varepsilon\downarrow 0, \quad \tau\in[\tfrac{1}{2},1), \\ -\lambda^{-1} + \varepsilon\lambda^{-2}\lambda'_j + O(\varepsilon^{2}) & \text{as } \varepsilon\downarrow 0, \quad \tau=1, \end{cases} \tag{3.17}$$

and

$$|u_{i+j,\varepsilon} - \psi_j| = \begin{cases} o(\varepsilon^{\tau-\frac{1}{2}}) & \text{as } \varepsilon\downarrow 0, \quad \tau\in[\tfrac{1}{2},1), \\ O(\varepsilon^{\frac{1}{2}}) & \text{as } \varepsilon\downarrow 0, \quad \tau=1, \\ O(\varepsilon^{\frac{1}{2}}) & \text{as } \varepsilon\downarrow 0, \quad D(B) \underset{c}{\subset} V \text{ and } D(B) \text{ dense in } V. \end{cases} \tag{3.18}$$

i) and *ii*) follow from (3.16), (3.17), (3.18), and the methods used in the proofs of the preceding theorems. Since $\tau - \gamma < 2\tau - 1$ if $\gamma > \frac{1}{2}$, *iii*) follows similarly.

4. Singular Perturbation of Elliptic Eigenvalue Problems

The elliptic eigenvalue problems to be considered will be formulated within the theory of Bessel potentials (cf. ARONSZAJN [4], ARONSZAJN & SMITH [8], ADAMS, ARONSZAJN & SMITH [1]). The Bessel kernel of order $\alpha > 0$ on R^n is the function given by

$$G_\alpha(x) = G_\alpha^{(n)}(x) = \frac{1}{2^{(n+\alpha-2)/2}\,\pi^{n/2}\,\Gamma(\alpha/2)}\, K_{(n-\alpha)/2}(|x|)\,|x|^{(\alpha-n)/2}$$

where K_ν is the modified Bessel function of the 3rd kind. For $0<\alpha<1$, let

$$C(n,\alpha)=\frac{2^{-2\alpha+1}\,\pi^{(n+2)/2}}{\Gamma(\alpha+1)\,\Gamma(\alpha+n/2)\sin\pi\alpha}.$$

Further let D be a domain in R^n and let u be a complex valued function in $C^\infty(D)$. The standard α-norm over D, $|u|_{\alpha,D}$, is defined as follows,

$$|u|_{0,D}^2=\int_D |u(x)|^2\,dx,$$

and for $0<\alpha<1$,

$$|u|_{\alpha,D}^2=|u|_{0,D}^2+\frac{1}{C(n,\alpha)\,G_{2n+2\alpha}(0)}\int_D\int_D \frac{G_{2n+2\alpha}(x-y)}{|x-y|^{n+2\alpha}}\,|u(x)-u(y)|^2\,dx\,dy.$$

For arbitrary $\alpha\geqq 0$, let $m=[\alpha]$ be the greatest integer $\leqq\alpha$ and let $\beta=\alpha-m$. Then

$$|u|_{\alpha,D}^2=\sum_{k=0}^{m}\binom{m}{k}\sum_{|i|\leqq k}|D_i u|_{\beta,D}^2.$$

The space $\dot{P}^\alpha(D)$ is the perfect functional completion in the sense of ARONSZAJN & SMITH [7] of the functions in $C^\infty(D)$ for which $|u|_{\alpha,D}$ is finite. For $D=R^n$, $\check{P}^\alpha(D)$ is denoted simply by P^α and $|u|_{\alpha,R^n}$ by $\|u\|_\alpha$. $P^\alpha(D)$ is defined as the space of all restrictions to D of functions in P^α with the norm

$$\|u\|_{\alpha,D}=\inf\|\tilde{u}\|_\alpha$$

with the infimum taken over all $\tilde{u}\in P^\alpha$ such that $\tilde{u}=u$ except on a subset of D of 2α-capacity zero. For all domains D to be considered in the present work, $\check{P}^\alpha(D)=P^\alpha(D)$ with equivalent norms (cf. [1] or [4]). It should be noted that for such domains D, $P^\alpha(D)$ is the class of corrections (cf. [1], §0) of functions in the class $W^{\alpha,2}(D)$ (cf. LIONS & MAGENES [20], §2). Denote the closure of $C_0^\infty(D)$ in $P^\alpha(D)$ by $P_0^\alpha(D)$.

Now let m', m be positive integers with $m'>m$, and assume that D is a bounded domain in R^n of class $C^{2m'}$. For v, $w\in P^{m'}(D)$ (note that $\check{P}^{m'}(D)=P^{m'}(D)$ with equivalent norms) let

$$a(v,w)=\sum_{|i|,|j|\leqq m'}\int_D a_{ij}(x)\,D_j v\,\overline{D_i w}\,dx$$

with $a_{ij}=\bar{a}_{ji}\in C^{|i|}(\bar{D})$, and for v, $w\in P^m(D)$ let

$$b(v,w)=\sum_{|i|,|j|\leqq m}\int_D b_{ij}(x)\,D_j v\,\overline{D_i w}\,dx$$

with $b_{ij}=\bar{b}_{ji}\in C^{|i|}(\bar{D})$. Denote by A, $\mathscr{B}$ the associated formal differential operators over D, that is

$$A=\sum_{|i|,|j|\leqq m'}(-1)^{|i|}D_i(a_{ij}D_j\,\cdot),$$

$$\mathscr{B}=\sum_{|i|,|j|\leqq m}(-1)^{|i|}D_i(b_{ij}D_j\,\cdot).$$

The theorems of Section 2 will be applied to the following types of problems. Let V_0 be a closed subspace of $P^m(D)$ such that $P_0^m(D)\subset V_0\subset P^m(D)$ and V a closed subspace of $P^{m'}(D)$ such that $P_0^{m'}(D)\subset V\subset P^{m'}(D)$ with $V\subset V_0$ and V

dense in V_0. Each of V, V_0 is to be determined by a normal system of boundary operators with smooth coefficients (cf. ARONSZAJN & MILGRAM [6]). Restrict $a(v, w)$ (respectively $b(v, w)$) to V (respectively V_0) and assume that $a(v) \equiv a(v,v) \geqq 0$ for all $v \in V$. Further assume that there exists a constant $\beta > 0$ such that

$$b(v) \geqq \beta |v|^2_{m,D} \quad \text{for all } v \in V_0$$

and that there exist constants α, $\delta > 0$ such that for $0 < \varepsilon \leqq \varepsilon_0$,

$$\varepsilon a(v) + b(v) \geqq \varepsilon \alpha |v|^2_{m',D} + \delta |v|^2_{m,D} \quad \text{for all } v \in V.$$

The hypotheses assumed on $a(v, w)$ and $b(v, w)$ in Section 2 are satisfied as long as H is chosen such that $V_0 \underset{c}{\subset} H$ with V_0 dense in H.

Dirichlet Eigenvalue Problems. Let V be $P_0^{m'}(D)$ with norm $(a(v)+b(v))^{\frac{1}{2}}$, V_0 be $P_0^m(D)$ with norm $(b(v))^{\frac{1}{2}}$, and let $H = L^2(D) = P^0(D)$ with the usual norm. Then A_ε, the operator in $L^2(D)$ associated with $\varepsilon a(v, w) + b(v, w)$ restricted to V, is given by $\varepsilon A + \mathscr{B}$ on $D(A_\varepsilon)$. Here $D(A_\varepsilon) = P^{2m'}(D) \cap P_0^{m'}(D)$ by the regularity results of NIRENBERG [23] & AGMON [2], Chapter 9. $|A_\varepsilon v|_{0,D}$ is equivalent to $|v|_{2m',D}$ on $P^{2m'}(D) \cap P_0^{m'}(D)$. Also the operator B in $L^2(D)$ associated with $b(v, w)$ (restricted to V_0) is given by $\mathscr{B}$ with domain $P^{2m}(D) \cap P_0^m(D)$, and $|Bv|_{0,D}$ is equivalent to $|v|_{2m,D}$ on $P^{2m}(D) \cap P_0^m(D)$.

It follows from Rellich's Theorem (cf. [2], p. 30) that the spectrum of B is discrete, that is, it consists of a sequence of eigenvalues each of finite multiplicity. As before, these eigenvalues are considered as arranged in an increasing sequence counted according to multiplicity,

$$0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n \leqq \cdots \to \infty .$$

The spectrum of A_ε is also discrete and arranged in an increasing sequence, each eigenvalue counted according to multiplicity,

$$0 < \lambda_{1,\varepsilon} \leqq \lambda_{2,\varepsilon} \leqq \cdots \leqq \lambda_{n,\varepsilon} \leqq \cdots \to \infty .$$

Now let $\mathscr{A}$ be the operator in V_0 associated with $a(v, w)$, that is $a(v, w) = b(\mathscr{A} v, w)$. Then as in [10], Proposition 6.1 and Theorem 6.2, $V_1 = D(\mathscr{A})$ is $P^{2m'-m}(D) \cap P_0^{m'}(D)$ with an equivalent norm. Furthermore, it follows from the interpolation results of GREENLEE [10] or GRISVARD [11] that for $\tau \in [0, 1/4(m'-m))$, V_τ is $P_0^{m+2(m'-m)\tau}(D) = P^{m+2(m'-m)\tau}(D) \cap P_0^m(D)$ with an equivalent norm. So $D(B) \underset{c}{\subset} V_\tau$ with $D(B)$ dense in V_τ for all $\tau \in [0, 1/4(m'-m))$. (It also follows that $D(B)$ is not contained in $V_{1/4(m'-m)}$). Thus according to Theorem 2.1, for each $n = 1, 2, \ldots$ and all $\tau < 1/4(m'-m)$,

$$\lambda_{n,\varepsilon} = \lambda_n + o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0,$$

and, with P_ε, P as in Theorem 2.1, we find for any $v \in L^2(D)$

$$|P_\varepsilon P v - P v|_{m,D} = o(\varepsilon^\tau) \quad \text{as} \quad \varepsilon \downarrow 0$$

$$|P_\varepsilon P v - P v|_{0,D} = o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0,$$

and

$$|P_\varepsilon P v - P v|_{\alpha,D} = o(1) \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{for all } \alpha < m + \tfrac{1}{2}.$$

Moreover, if λ is an eigenvalue of B of multiplicity one and u_ε, u are as in Theorem 2.3, then for all $\tau<1/4(m'-m)$

$$|u_\varepsilon-u|_{m,D}=o(\varepsilon^\tau) \quad \text{as} \quad \varepsilon\downarrow 0,$$

$$|u_\varepsilon-u|_{0,D}=o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon\downarrow 0,$$

and

$$|u_\varepsilon-u|_{\alpha,D}=o(1) \quad \text{as} \quad \varepsilon\downarrow 0 \quad \text{for all } \alpha<m+\tfrac{1}{2}.$$

Uniform estimates follow from the Sobolev theorems (cf. [8], p. 424).

Neumann Eigenvalue Problems. Let V be $P^{m'}(D)$ with norm $(a(v)+b(v))^{\frac{1}{2}}$, V_0 be $P^m(D)$ with norm $(b(v))^{\frac{1}{2}}$, and let $H=L^2(D)$ with the usual norm. Then A_ε is $\varepsilon A+\mathscr{B}$ with $D(A_\varepsilon)$ being a closed subspace of $P^{2m'}(D)$ determined by m' natural boundary conditions of orders $m',\ldots,2m'-1$. B is $\mathscr{B}$ with $D(B)$ being a closed subspace of $P^{2m}(D)$ determined by m natural boundary conditions of orders $m,\ldots,2m-1$. $V_1=D(\mathscr{A})$ is the closed subspace of $P^{2m'-m}(D)$ determined by the boundary conditions of orders $m',\ldots,2m'-m-1$ associated with A_ε, with an equivalent norm. It follows from the interpolation results of [11] that for $\tau\in[0,\frac{1}{2}+1/4(m'-m))$, V_τ is $P^{m+2(m'-m)\tau}(D)$ with an equivalent norm.

Thus if $2m>m'$, $D(B)\underset{c}{\subset} V_\tau$ for all $\tau\in[0,\frac{1}{2}+1/4(m'-m))$. Hence by Theorem 2.1, for each $n=1,2,\ldots,$

$$\lambda_{n,\varepsilon}=\lambda_n+O(\varepsilon) \quad \text{as} \quad \varepsilon\downarrow 0,$$

and with P_ε, u_n as in Theorem 2.2,

$$|P_\varepsilon u_n-u_n|_{m,D}=o(\varepsilon^\tau) \quad \text{as} \quad \varepsilon\downarrow 0 \quad \text{for all } \tau<\tfrac{1}{2}+1/4(m'-m).$$

However, $D(B)$ is dense in V_τ only for $\tau\leqq 1/4(m'-m)$. Thus if P_ε, P are as in Theorem 2.1, for any $v\in L^2(D)$,

$$|P_\varepsilon Pv-Pv|_{\alpha,D}=o(1) \quad \text{as} \quad \varepsilon\downarrow 0 \quad \text{for all } \alpha\leqq m+\tfrac{1}{2}.$$

Moreover, if λ is an eigenvalue of B of multiplicity one and λ', u_ε, u are as in Theorem 2.3, then for all $\tau<\frac{1}{2}+1/4(m'-m)$,

$$\lambda_\varepsilon=\lambda+\varepsilon\lambda'+o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon\downarrow 0,$$

$$|u_\varepsilon-u|_{m,D}=o(\varepsilon^\tau) \quad \text{as} \quad \varepsilon\downarrow 0,$$

and

$$|u_\varepsilon-u|_{\alpha,D}=o(1) \quad \text{as} \quad \varepsilon\downarrow 0 \quad \text{for all } \alpha\leqq m+\tfrac{1}{2}.$$

If $\lambda=\lambda_{i+1}=\cdots=\lambda_{i+l}$ is an eigenvalue of B of multiplicity l which "completely splits" under this perturbation in the sense of Theorem 2.4, and if λ'_j, $u_{i+j,\varepsilon}$, u_{i+j} are as in Theorem 2.4, then for all $\tau<\frac{1}{2}+1/4(m'-m)$,

$$\lambda_{i+j,\varepsilon}=\lambda+\varepsilon\lambda'_j+o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon\downarrow 0,$$

and

$$|u_{i+j,\varepsilon}-u_{i+j}|_{m,D}=o(\varepsilon^{\tau-\frac{1}{2}}) \quad \text{as} \quad \varepsilon\downarrow 0.$$

If $2m\leqq m'$, then $D(B)\underset{c}{\subset} V_{m/2(m'-m)}$. If one assumes that $b_{ij}\in C^{|i|+m'-2m+1}(\bar{D})$, then all eigenfunctions of B are in $P^{m'+1}(D)$ and hence in V_τ for all $\tau<\frac{1}{2}+$

$1/4(m'-m)$. With this additional assumption all the conclusions obtained above for the case $2m>m'$ also hold in the case $2m\leqq m'$.

Relative Dirichlet Eigenvalue Problems. Let k be an integer with $0<k<m$, and for $v, w\in P^k(D)$ let

$$c(v,w)=\sum_{|i|,|j|\leqq k}\int_D c_{ij}(x)\,D_j v\,\overline{D_i w}\,dx$$

where $c_{ij}=\bar{c}_{ji}\in C^{|i|}(\bar{D})$. Let C be the associated formal differential operator over D,

$$C=\sum_{|i|,|j|\leqq k}(-1)^{|i|}D_i(c_{ij}D_j\cdot).$$

Further, assume that there exists $\gamma>0$ such that

$$c(v)\geqq\gamma\,|v|_{k,D}^2\qquad\text{for all } v\in P_0^k(D).$$

Let V be $P_0^{m'}(D)$ with norm $(a(v)+b(v))^{\frac{1}{2}}$, V_0 be $P_0^m(D)$ with norm $(b(v))^{\frac{1}{2}}$, and let H be $P_0^k(D)$ with norm $(c(v))^{\frac{1}{2}}$. Then variational methods applied to the Rayleigh quotients

$$\frac{\varepsilon\,a(v)+b(v)}{c(v)},\qquad\frac{b(v)}{c(v)}$$

give the eigenvalues and eigenfunctions of

$$(\varepsilon A+\mathscr{B})\,u_\varepsilon=\lambda_\varepsilon C u_\varepsilon,\qquad \mathscr{B}u=\lambda C u$$

where $u_\varepsilon\in P^{2m'}(D)\cap P_0^{m'}(D)$ and $u\in P^{2m}(D)\cap P_0^m(D)$ respectively. In this type of problem it is important that the boundary conditions associated with C are included in those associated with $\varepsilon A+\mathscr{B}$ and $\mathscr{B}$ (cf. ARONSZAJN [3]).

By the methods used previously to determine $D(\mathscr{A})$, we find for this problem that $D(A_\varepsilon)$ is $P^{2m'-k}(D)\cap P_0^{m'}(D)$, $D(B)$ is $P^{2m-k}(D)\cap P_0^m(D)$, and $V_1=D(\mathscr{A})$ is $P^{2m'-m}(D)\cap P_0^{m'}(D)$, each with an equivalent norm. Thus for $\tau\in[0, 1/4(m'-m))$, V_τ is $P^{m+2(m'-m)\tau}(D)\cap P_0^m(D)$ with an equivalent norm, and thus $D(B)\underset{c}{\subset} V_\tau$ with $D(B)$ dense in V_τ for all $\tau\in[0, 1/4(m'-m))$. Therefore by Theorem 2.1, for each $n=1,2,\ldots$ and all $\tau<1/4(m'-m)$,

$$\lambda_{n,\varepsilon}=\lambda_n+o(\varepsilon^{2\tau})\quad\text{as}\quad\varepsilon\downarrow 0,$$

and with P_ε, P as in Theorem 2.1, for any $v\in P_0^k(D)$,

$$|P_\varepsilon Pv-Pv|_{m,D}=o(\varepsilon^\tau)\quad\text{as}\quad\varepsilon\downarrow 0,$$

$$|P_\varepsilon Pv-Pv|_{k,D}=o(\varepsilon^{2\tau})\quad\text{as}\quad\varepsilon\downarrow 0,$$

and

$$|P_\varepsilon Pv-Pv|_{\alpha,D}=o(1)\quad\text{as}\quad\varepsilon\downarrow 0\quad\text{for all } \alpha<m+\tfrac{1}{2}.$$

Also, if λ is an eigenvalue of B (that is, of $\mathscr{B}u=\lambda Cu$) of multiplicity one, and u_ε, u are as in Theorem 2.3, then for all $\tau<1/4(m'-m)$,

$$|u_\varepsilon-u|_{m,D}=o(\varepsilon^\tau)\quad\text{as}\quad\varepsilon\downarrow 0,$$

$$|u_\varepsilon-u|_{k,D}=o(\varepsilon^{2\tau})\quad\text{as}\quad\varepsilon\downarrow 0,$$

and

$$|u_\varepsilon-u|_{\alpha,D}=o(1)\quad\text{as}\quad\varepsilon\downarrow 0\quad\text{for all } \alpha<m+\tfrac{1}{2}.$$

Some "Mixed" Eigenvalue Problems. Let $c(v, w)$, C be as in the previous example, with

$$c(v) \geqq \gamma |v|_{k,D}^2 \quad \text{for all } v \in P_0^k(D), \quad \gamma > 0,$$

where k is now an integer with $0 \leqq k < m$. Let V be $P^{m'}(D) \cap P_0^m(D)$ with norm $(a(v)+b(v))^{\frac{1}{2}}$, V_0 be $P_0^m(D)$ with norm $(b(v))^{\frac{1}{2}}$, and H be $P_0^k(D)$ with norm $(c(v))^{\frac{1}{2}}$. As in the previous example, variational methods applied to the Rayleigh quotient $b(v)/c(v)$ yield the eigenvalues and eigenfunctions of

$$\mathscr{B} u = \lambda C u, \quad u \in P^{2m}(D) \cap P_0^m(D),$$

and $D(B)$ is $P^{2m-k}(D) \cap P_0^m(D)$. Variational methods applied to the Rayleigh quotient $(\varepsilon a(v)+b(v))/c(v)$ give the eigenvalues and eigenfunctions of

$$(\varepsilon A + \mathscr{B}) u_\varepsilon = \lambda_\varepsilon C u_\varepsilon,$$

with u_ε in a closed subspace of $P^{2m'}(D) \cap P_0^m(D)$ which is determined by $m'-m$ natural boundary conditions of orders $m', \ldots, 2m'-m-1$. $V_1 = D(\mathscr{A})$ is the closed subspace of $P^{2m'-m}(D) \cap P_0^m(D)$ which is determined by these same natural boundary conditions of orders $m', \ldots, 2m'-m-1$.

Thus if $2m-k > m'$, $D(B) \underset{c}{\subset} V_\tau$ with $D(B)$ dense in V_τ for all $\tau < \frac{1}{2} + 1/4(m'-m)$. Hence by Theorem 2.1, for each $n = 1, 2, \ldots$

$$\lambda_{n,\varepsilon} = \lambda_n + O(\varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0,$$

and with P_ε, P as in Theorem 2.1, for any $v \in P_0^k(D)$ and all $\tau < \frac{1}{2} + 1/4(m'-m)$,

$$|P_\varepsilon P v - P v|_{m,D} = o(\varepsilon^\tau) \quad \text{as} \quad \varepsilon \downarrow 0,$$

$$|P_\varepsilon P v - P v|_{k,D} = O(\varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0,$$

and

$$|P_\varepsilon P v - P v|_{\alpha,D} = o(1) \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{for all } \alpha < m' + \tfrac{1}{2}.$$

Moreover, if λ is an eigenvalue of B of multiplicity one and λ', u_ε, u are as in Theorem 2.3, then for all $\tau < \frac{1}{2} + 1/4(m'-m)$,

$$\lambda_\varepsilon = \lambda + \varepsilon \lambda' + o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0,$$

$$|u_\varepsilon - u|_{m,D} = o(\varepsilon^\tau) \quad \text{as} \quad \varepsilon \downarrow 0,$$

$$|u_\varepsilon - u|_{k,D} = O(\varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0,$$

and

$$|u_\varepsilon - u|_{\alpha,D} = o(1) \quad \text{for all } \alpha < m' + \tfrac{1}{2}.$$

If $\lambda = \lambda_{i+1} = \cdots = \lambda_{i+l}$ is an eigenvalue of B of multiplicity l which "completely splits" under this perturbation in the sense of Theorem 2.4, then for all $\tau < \frac{1}{2} + 1/4(m'-m)$

$$\lambda_{i+j,\varepsilon} = \lambda + \varepsilon \lambda'_j + o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0,$$

$$|u_{i+j,\varepsilon} - u_{i+j}|_{m,D} = o(\varepsilon^{2\tau-1}) \quad \text{as} \quad \varepsilon \downarrow 0,$$

and

$$|u_{i+j,\varepsilon} - u_{i+j}|_{\alpha,D} = o(1) \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{for all } \alpha < m' + \tfrac{1}{2}.$$

If $2m-k \leqq m'$, then $D(B)=V_{(m-k)/2(m'-m)}$ with an equivalent norm. Assume that $b_{ij} \in C^{|i|+m'-2m+1}(\bar{D})$ and $c_{ij} \in C^{|i|+m'-2k+1}(\bar{D})$, so that all eigenfunctions of B are in $P^{m'+1}(D)$ and hence in V_τ for all $\tau<\frac{1}{2}+1/4(m'-m)$. Then by Theorem 2.1, for each $n=1, 2, \ldots$,

$$\lambda_{n,\varepsilon}=\lambda_n+O(\varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0,$$

and with P_ε, P as in Theorem 2.1, for any $v \in P_0^k(D)$,

$$|P_\varepsilon Pv-Pv|_{\alpha,D}=o(\varepsilon^{(2m-k-\alpha)/2(m'-m)}) \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{for all } \alpha \in [m, 2m-k],$$

and

$$|P_\varepsilon Pv-Pv|_{k,D}=\begin{cases} o(\varepsilon^{(2m-k)/(m'-m)} & \text{as } \varepsilon \downarrow 0 \quad \text{if } 2m-k<m', \\ O(\varepsilon) & \text{as } \varepsilon \downarrow 0 \quad \text{if } 2m-k=m'. \end{cases}$$

In addition, with P_ε, u_n as in Theorem 2.2, it follows from Theorem 2.2 that

$$|P_\varepsilon u_n-u_n|_{m,D}=o(\varepsilon^\tau) \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{for all } \tau<\tfrac{1}{2}+1/4(m'-m).$$

If λ is an eigenvalue of B of multiplicity one and λ', u_ε, u are as in Theorem 2.3, then for all $\tau<\frac{1}{2}+1/4(m'-m)$,

$$\lambda_\varepsilon=\lambda+\varepsilon\lambda'+o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0,$$

$$|u_\varepsilon-u|_{\alpha,D}=o(\varepsilon^{(2m-k-\alpha)/2(m'-m)}) \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{for all } \alpha \in [m, 2m-k],$$

and

$$|u_\varepsilon-u|_{k,D}=\begin{cases} o(\varepsilon^{(2m-k)/(m'-m)}) & \text{as } \varepsilon \downarrow 0 \quad \text{if } 2m-k<m', \\ O(\varepsilon) & \text{as } \varepsilon \downarrow 0 \quad \text{if } 2m-k=m'. \end{cases}$$

Also if $\lambda=\lambda_{i+1}=\cdots=\lambda_{i+l}$ is an eigenvalue of B of multiplicity l which "completely splits" under this perturbation in the sense of Theorem 2.4 and if λ_j', $u_{i+j,\varepsilon}$, u_{i+j} are as in Theorem 2.4, then for all $\tau<\frac{1}{2}+1/4(m'-m)$,

$$\lambda_{i+j,\varepsilon}=\lambda+\varepsilon\lambda_j'+o(\varepsilon^{2\tau}) \quad \text{as} \quad \varepsilon \downarrow 0,$$

$$|u_{i+j,\varepsilon}-u_{i+j}|_{m,D}=o(\varepsilon^{\tau-\frac{1}{2}}) \quad \text{as} \quad \varepsilon \downarrow 0,$$

and, if $2m-k=m'$,

$$|u_{i+j,\varepsilon}-u_{i+j}|_{m',D}=o(1) \quad \text{as} \quad \varepsilon \downarrow 0.$$

5. Concluding Remarks

As noted in [16], §6, it is possible to weaken some of the assumptions on the forms $a(v, w)$ and $b(v, w)$. The condition $b(v) \geqq |v|^2$ may be replaced by $b(v) \geqq -\delta|v|^2$, $\delta \geqq 0$. Then one considers $b'(v)=b(v)+(\delta+1)|v|^2$, which amounts to a change of origin for all spectra. The condition $a(v) \geqq 0$ can be replaced by $a(v) \geqq -\eta\,|v|^2-\theta b(v)$, η, $\theta \geqq 0$, by considering $a'(v)=a(v)+\eta\,|v|^2+\theta b(v)$.

Proposition 3.1 can be used to obtain some rate of convergence theorems which supplement the results of FRIEDMAN [9], §2, in the self adjoint case. Further recent results on singular perturbation of elliptic boundary value problems can be found in HUET [13] and [14]. The techniques of [10] and [14] can be used to reduce the regularity assumptions in §4 of the present paper.

In each of the preceding theorems in which the hypothesis $D(B) \underset{c}{\subset} V_\tau$ with $D(B)$ dense in V_τ was not used, explicit bounds for the rates of convergence can be obtained in terms of the constants of the problem and the interpolation norms by examining the proofs. Some bounds of this type were given in the problems considered in [10], §2. The assumption that $D(B) \underset{c}{\subset} V_\tau$ with $D(B)$ dense in V_τ is used in conjunction with Lemma 3.1 and the closed graph theorem to assert the existence of a bound.

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