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# ASYMPTOTIC ANALYSIS OF SINGULAR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS*

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**Abstract.** Singularly perturbed systems ordinary differential equations for which the reduced system has a manifold of solutions are called singular singularly perturbed. Boundary value problems for a general class of such systems are examined. Conditions are derived which ensure the existence of a locally unique solution, which can be approximated by an asymptotic expansion. The main tool for our analysis is the theory of boundary value problems on long intervals.

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**Key words.** ordinary differential equations, boundary value problems, singular perturbations, asymptotic expansions

**1. Introduction.** We consider boundary value problems of the form

$$\varepsilon y' = f(y,t,\varepsilon) \qquad 0 \leqq t \leqq 1, \tag{1.1}$$

$$b(y(0), y(1)) = 0, \tag{1.2}$$

where $y$ is an $n$-vector, $\varepsilon$ is small and positive, and $f$ and $b$ are nonlinear mappings.

The asymptotic analysis of (1.1), (1.2) starts with an investigation of the reduced system

$$0 = f(y,t,0). \tag{1.3}$$

If there is a locally unique solution $y = Y(t)$ to (1.3) and the matrix $f_y(Y(t),t,0)$ has just strictly stable and strictly unstable eigenvalues, then (1.1), (1.2) is called a regular singular-perturbation problem. Problems of this type are quite well understood. Combining the results of Vasileva and Butuzov (1973) and Esipova (1975) yields a complete asymptotic analysis, even if equations for "slow" components are added to (1.1).

We want to treat the case of the existence of a solution manifold $y = \phi(\alpha, t)$ of (1.3), with $\alpha$ being an $n_0$-dimensional parameter. We wish to consider problems satisfying an additional assumption:

The matrix $f_y(\phi(\alpha,t),t,0)$ has an $n_0$-dimensional null space with the real parts of the remaining $n - n_0$ eigenvalues being bounded away from zero.

This assumption rules out the existence of turning points, highly oscillatory solutions and boundary layer variables different from $\tau = t/\varepsilon$ and $\sigma = (1-t)/\varepsilon$. Note that regular singularly perturbed systems of the form

$$\varepsilon y' = g(y,z,t,\varepsilon), \qquad z' = h(y,z,t,\varepsilon)$$

obviously fit into our theory if the equations for $z$ are multiplied by $\varepsilon$.

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Singular singularly perturbed problems have received a significant amount of attention recently. General nonlinear initial value problems satisfying the above assumptions have been treated by O'Malley and Flaherty (1980) and Vasileva and Butuzov (1978). In the work by Vasileva and Butuzov also boundary value problems of a special structure are analysed. To the knowledge of the authors no complete classification of problems violating the above assumptions on the eigenstructure of $f_y(\phi(\alpha,t),t,0)$ is available. One of the possible effects in this case is the occurrence of multiple layers. This has been demonstrated on a linear problem by O'Malley (1979).

The asymptotic analysis of (1.1), (1.2) proceeds in two steps: The construction of a formal approximation of the solution and the proof of validity of this approximation.

The first step is contained in §2, where the method of matched asymptotic expansions is used to construct a sequence of formal approximations. The problem defining the first term in the sequence, i.e. the leading term in the asymptotic expansion, is nonlinear in general if (1.1), (1.2) is nonlinear. A key assumption in this paper is the existence of an isolated solution of this problem. It is then shown that this assumption implies the existence and uniqueness of terms of arbitrary order in the asymptotic expansion, since the linear operator in the equations defining these terms is the linearization of the nonlinear problem defining the leading term.

In the proof of the validity of the asymptotic expansion in §4 the contraction principle is used. Thus it is necessary to obtain stability estimates for the linearization of (1.1), (1.2) at the formal approximation, i.e. we have to obtain estimates for the solutions of linear singular singularly perturbed problems, where the coefficients contain boundary layer terms. For the analysis of these problems the theory of boundary value problems on infinite and "long" intervals is used extensively. It is one of the key points of this paper to stress the applicability of this theory to singularly perturbed problems. Some results of this theory are collected in §3. In the proofs given in the Appendix the methods of de Hoog and Weiss (1980) and Markowich (1982, 1983) are used.

The solution of the problem defining the leading term in the expansion is demonstrated on an example from semiconductor theory in §5. The singular perturbation approach to semiconductor problems was originated by Vasileva, Kardosysoev and Stelmakh (1976). Recently the great importance of semiconductor device simulation caused intensive work on the subject. References to some papers using singular perturbation theory are given in §5.

Another nonlinear example can be found in Schmeiser (1985) where a problem modelling large deflections of a thin beam, suggested by Flaherty and O'Malley (1981), is analyzed in the framework of the theory of this paper.

For a class of singular singularly perturbed boundary value problems considered in Vasileva and Butuzov (1978), contraction mapping techniques are employed to establish the validity of the formal asymptotic expansion. The structure of this class is simple enough to a priorily guarantee the existence and uniqueness of an isolated solution to the problem defining the leading term of the expansion and to allow the reduction of the analysis of the linearized problem to that of a scalar second order equation whose coefficients have boundary layers. The stability results employed for this equation have been previously developed in Vasileva (1972).

**2. Asymptotic expansion.** We consider problems of the form (1.1), (1.2) which satisfy

*Hypothesis* H1. Denoting the $n_0$-dimensional solution manifold of the reduced equation by $\phi(\alpha,t)$, the matrix $\phi_\alpha(\alpha,t)$ has constant rank $n_0$. The Jacobian

$f_y(\phi(\alpha,t),t,0)$ has $n_-$ strictly stable and $n_+$ strictly unstable eigenvalues with $n_-+n_++n_0=n$, for $0\leqq t\leqq 1$.

Differentiation of

$$f(\phi(\alpha,t),t,0)=0$$

with respect to $\alpha$ implies

$$\bar{f}_y\phi_\alpha(\alpha,t)=0, \tag{2.1}$$

where from now on the bar above partial derivatives of $f$ indicates the argument $(\phi(\alpha,t),t,0)$. Thus $\bar{f}_y$ has an $n_0$-dimensional null space spanned by the columns of $\phi_\alpha(\alpha,t)$. By adding $n_++n_-$ columns which span the stable and unstable subspaces of $\bar{f}_y$ we obtain a transformation matrix $E(\alpha,t)=(E_\pm(\alpha,t),\phi_\alpha(\alpha,t))$ which block-diagonalizes $\bar{f}_y$:

$$E^{-1}\bar{f}_yE=\Lambda=\begin{pmatrix}\Lambda_- & & \\ & \Lambda_+ & \\ & & 0\end{pmatrix}.$$

Denoting the last $n_0$ rows of $E^{-1}(\alpha,t)$ by $H(\alpha,t)$ it follows that

$$H(\alpha,t)\phi_\alpha(\alpha,t)=I_{n_0} \tag{2.2}$$

and

$$H(\alpha,t)\bar{f}_y=0, \tag{2.3}$$

where $I_r$ is the $r\times r$ identity matrix.

For solutions $y(t,\varepsilon)$ of (1.1),(1.2) we use the ansatz

$$y(t,\varepsilon)\sim\sum_{i=0}^{\infty}\left(\bar{y}_i(t)+L_iy(\tau)+R_iy(\sigma)\right)\varepsilon^i, \tag{2.4}$$

where $\tau=t/\varepsilon$, $\sigma=(1-t)/\varepsilon$ and

$$\lim_{\tau\to\infty}L_iy(\tau)=0,\qquad \lim_{\sigma\to\infty}R_iy(\sigma)=0,\qquad i=0,1,\cdots. \tag{2.5}$$

**2.1. The construction of the leading term.** Substituting (2.4) into (1.1) and setting $\varepsilon=0$ yields

$$\bar{y}_0(t)=\phi(\alpha,t). \tag{2.6}$$

Differential equations determing the as yet unknown parameter $\alpha$ as a function of $t$ are obtained from the relations we get by collecting the terms of order $\varepsilon$ in (1.1), i.e.

$$\bar{y}_0'=\bar{f}_y\bar{y}_1+\bar{f}_\varepsilon. \tag{2.7}$$

Using (2.6), (2.7) reads

$$\phi_\alpha\alpha'+\phi_t=\bar{f}_y\bar{y}_1+\bar{f}_\varepsilon. \tag{2.8}$$

Multiplying (2.8) by $H(\alpha,t)$ and using (2.2) and (2.3) yields the relations for $\alpha$,

$$\alpha'=H\left(\bar{f}_\varepsilon-\phi_t\right),\quad 0\leqq t\leqq 1. \tag{2.9}$$

Note that in practice (2.9) is usually obtained from (2.8) without the explicit use of $H$ by eliminating $\bar{y}_1$ from $n_0$ equations in (2.8).

The equations for the layer corrections are

$$\frac{dL_0 y}{d\tau} = f(\phi(\alpha(0),0)+L_0 y,0,0), \quad 0 \leqq \tau < \infty, \quad L_0 y(\infty)=0, \tag{2.10}$$

and

$$\frac{dR_0 y}{d\sigma} = -f(\phi(\alpha(1),1)+R_0 y,1,0), \quad 0 \leqq \sigma < \infty, \quad R_0 y(\infty)=0. \tag{2.11}$$

Equating coefficients of order zero in (1.2) yields

$$b(\phi(\alpha(0),0)+L_0 y(0), \phi(\alpha(1),1)+R_0 y(0))=0. \tag{2.12}$$

We can now state our second fundamental assumption.

*Hypothesis* H2. The boundary value problem (2.9)–(2.12) has an isolated solution.

Because of the boundary conditions at infinity in (2.10), (2.11) stable manifolds naturally enter our discussion. There is an extensive literature on the subject of invariant manifolds. Some basic results, which can be found in Kelley (1967), imply the existence of an $n_-$-dimensional stable manifold for (2.10) and an $n_+$-dimensional stable manifold for (2.11). The boundary conditions at infinity require $L_0 y$ and $R_0 y$ to be trajectories on these manifolds. Then Lemma 3 in Kelley (1967) implies the estimates

$$\begin{aligned} \|L_0 y(\tau)\| &\leqq \text{const}\, e^{-\kappa\tau}, \\ \|R_0 y(\sigma)\| &\leqq \text{const}\, e^{-\kappa\sigma} \end{aligned} \tag{2.13}$$

with a positive constant $\kappa$. Subsequently $\|\cdot\|$ will denote a vector norm or the induced matrix norm.

In general, trying to solve problem (2.9)–(2.12) is quite unpleasant. The differential equations on finite and infinite intervals have to be solved simultaneously because there is a coupling by boundary conditions. In applications, however, some knowledge about the structure of the stable manifolds of (2.10) and (2.11) often enables us to obtain from (2.12) $n_0$ "reduced" boundary conditions for (2.9) alone which do not contain $L_0 y(0)$ and $R_0 y(0)$. This is the case in all of the above mentioned applications, where the problems on the interval $[0,1]$ and the problems on the infinite interval $[0,\infty)$ can be solved consecutively.

Hypothesis H2 implies that the linearized system

$$w' = \left[H_\alpha\langle\cdot, \bar{f}_\varepsilon - \phi_t\rangle + H\left(\bar{f}_{y\varepsilon}\phi_\alpha - \phi_{\alpha t}\right)\right] w, \tag{2.14a}$$

$$\frac{du}{d\tau} = f_y(\phi(\alpha(0),0)+L_0 y,0,0)(u+\phi_\alpha(\alpha(0),0)w(0)), \tag{2.14b}$$

$$\frac{dv}{d\sigma} = -f_y(\phi(\alpha(1),1)+R_0 y,1,0)(v+\phi_\alpha(\alpha(1),1)w(1)), \tag{2.14c}$$

$$b_0(\phi_\alpha(\alpha(0),0)w(0)+u(0))+b_1(\phi_\alpha(\alpha(1),1)w(1)+v(0))=0, \tag{2.14d}$$

$$u(\infty)=0, \qquad v(\infty)=0 \tag{2.15}$$

has only the trivial solution. In (2.14d) $b_0$ and $b_1$ are the partial derivatives of $b$ with respect to the first and second argument respectively at $(\phi(\alpha(0),0)+L_0 y(0),\ \phi(\alpha(1),1) + R_0 y(0))$. For bilinear forms $B$ we use the notation $B\langle\cdot,\cdot\rangle$.

**2.2. The construction of higher order terms.** (2.8) is a linear equation for the smooth part $\bar{y}_1$ of the first order term. The coefficient matrix $\bar{f}_y$ is singular. By (2.9) we have chosen the inhomogeneity $\phi_\alpha\alpha' + \phi_t - \bar{f}_\varepsilon$ in a way that a solution to (2.8) exists. The general solution of (2.8) can be written in the form

$$\bar{y}_1(t)=\phi_\alpha(\alpha,t)\beta_1(t)+\bar{y}_{1p}(t),$$

where $\bar{y}_{1p}(t)$ is a particular solution and $\beta_1$ is an $n_0$-dimensional parameter. To determine $\beta_1(t)$ we equate coefficients of $\varepsilon^2$ in (1.1) and obtain

$$\phi_\alpha\beta_1'+\phi_\alpha'\beta_1+\bar{y}_{1p}'=\bar{f}_y\bar{y}_2+\frac{1}{2}\bar{f}_{yy}\langle\bar{y}_1,\bar{y}_1\rangle+\bar{f}_{y\varepsilon}\bar{y}_1+\frac{1}{2}\bar{f}_{\varepsilon\varepsilon}. \tag{2.16}$$

Analogously to the determination of $\alpha$, multiplying (2.16) by $H$ yields

$$\beta_1'=H\left[\frac{1}{2}\bar{f}_{yy}\langle\bar{y}_1,\bar{y}_1\rangle+\bar{f}_{y\varepsilon}\bar{y}_1-\phi_\alpha'\beta_1\right]+H\left[\frac{1}{2}\bar{f}_{\varepsilon\varepsilon}-\bar{y}_{1p}'\right]. \tag{2.17}$$

Vasileva and Butuzov (1978, p. 68) show that (2.17) is a linear equation for $\beta_1$. Since we require the coefficient matrix explicitly, we will reproduce their argument. In (2.17) we write

$$\frac{1}{2}H\bar{f}_{yy}\langle\bar{y}_1,\bar{y}_1\rangle=\frac{1}{2}H\bar{f}_{yy}\langle\phi_\alpha\beta_1,\phi_\alpha\beta_1\rangle+H\bar{f}_{yy}\langle\phi_\alpha\beta_1,\bar{y}_{1p}\rangle+\frac{1}{2}\bar{f}_{yy}\langle\bar{y}_{1p},\bar{y}_{1p}\rangle, \tag{2.18}$$

which follows from the fact that $\bar{f}_{yy}$ is a symmetric bilinear form. Differentiating (2.3) with respect to $\alpha$ yields

$$H_\alpha\langle\cdot,\bar{f}_y\cdot\rangle+H\bar{f}_{yy}\langle\phi_\alpha\cdot,\cdot\rangle=0. \tag{2.19}$$

Multiplying (2.19) with $\phi_\alpha$ gives

$$H_\alpha\langle\cdot,\bar{f}_y\phi_\alpha\cdot\rangle+H\bar{f}_{yy}\langle\phi_\alpha\cdot,\phi_\alpha\cdot\rangle=0.$$

Due to (2.1) ($\bar{f}_y\phi_\alpha=0$) the first term on the right-hand side of (2.18) disappears and (2.17) can be written in the form

$$\beta_1'=H\left[\bar{f}_{yy}\langle\phi_\alpha\cdot,\bar{y}_{1p}\rangle+\bar{f}_{y\varepsilon}\phi_\alpha-\phi_\alpha'\right]\beta_1+\overline{G}_1, \tag{2.20}$$

where $\overline{G}_1$ depends on $\alpha$ and $t$ only. Next we will show that the coefficient matrix $S$ in (2.20) is equal to that in (2.14a). Obviously

$$S=H\left[\bar{f}_{yy}\langle\phi_\alpha\cdot,\bar{y}_{1p}\rangle-\phi_{\alpha\alpha}\langle\alpha',\cdot\rangle\right]+H\left[\bar{f}_{y\varepsilon}\phi_\alpha-\phi_{\alpha t}\right]. \tag{2.21}$$

Differentiation of (2.2) with respect to $\alpha$ gives

$$H_\alpha\langle\cdot,\phi_\alpha\cdot\rangle+H\phi_{\alpha\alpha}\langle\cdot,\cdot\rangle=0. \tag{2.22}$$

It follows from (2.19), (2.21) and (2.22) that

$$S=H_\alpha\langle\cdot,\phi_\alpha\alpha'-\bar{f}_y\bar{y}_{1p}\rangle+H\left[\bar{f}_{y\varepsilon}\phi_\alpha-\phi_{\alpha t}\right].$$

Using the fact that $\bar{y}_{1p}$ is a particular solution of (2.8) it is now obvious that the coefficient matrices in (2.14a) and (2.20) are the same.

Suppose we have constructed $\bar{y}_0,\cdots,\bar{y}_{n-1}$, $n\geqq 2$. Then we determine $\bar{y}_n$ from an equation of the form

$$\bar{f}_y\bar{y}_n=F_n(\bar{y}_0,\cdots,\bar{y}_{n-1}), \tag{2.23}$$

where the inhomogeneity satisfies the solvability condition $HF_n=0$. Thus the general solution of (2.23) can be written as

$$\bar{y}_n=\phi_\alpha\beta_n+\bar{y}_{np}.$$

Equating coefficients of $\varepsilon^{n+1}$ in (1.1) gives

$$\phi_\alpha \beta_n' + \phi_\alpha' \beta_n = \bar{f}_y \bar{y}_{n+1} + \bar{f}_{yy}\langle \bar{y}_1, \bar{y}_n\rangle + \bar{f}_{y\varepsilon}\bar{y}_n + \tilde{G}_n,$$

with $\tilde{G}_n$ depending only on $\bar{y}_0, \cdots, \bar{y}_{n-1}$. Multiplying by $H$ and repeating the argument we used for proving the linearity of the equations for $\beta_1$ we get

$$\beta_n' = H\left[\bar{f}_{yy}\langle \bar{y}_{1p}, \phi_\alpha\rangle + \bar{f}_{y\varepsilon}\phi_\alpha - \phi_\alpha'\right]\beta_n + \bar{G}_n, \tag{2.24}$$

which has the same coefficient matrix as (2.20). $\bar{G}_n$, like $\tilde{G}_n$, depends only on $\bar{y}_0, \cdots, \bar{y}_{n-1}$. Suppose we have constructed $\bar{y}_0, \cdots, \bar{y}_{n-1}$; $L_0 y, \cdots, L_{n-1} y$; $R_0 y, \cdots, R_{n-1} y$, where the layer corrections $L_i y$, $R_i y$ satisfy exponential estimates of the type (2.13). Then $L_n y$ and $R_n y$ satisfy

$$\frac{dL_n y}{d\tau} = f_y(\phi(\alpha(0),0) + L_0 y, 0, 0)(L_n y + \phi_\alpha(\alpha(0),0)\beta_n(0)) + L_n G, \tag{2.25a}$$

$$\frac{dR_n y}{d\sigma} = -f_y(\phi(\alpha(1),1) + R_0 y, 1, 0)(R_n y + \phi_\alpha(\alpha(1),1)\beta_n(1)) + R_n G, \tag{2.25b}$$

$$L_n y(\infty) = 0, \qquad R_n y(\infty) = 0, \tag{2.25c}$$

where $L_n G$ and $R_n G$ depend on the terms in the expansion up to order $n-1$. Besides, $L_n G$ and $R_n G$ satisfy exponential estimates of the type (2.13). (2.25a) and (2.25b) differ from (2.14b), (2.14c) only in the terms $L_n G$ and $R_n G$. Equating coefficients of $\varepsilon^n$ in (1.2) yields the boundary conditions

$$b_0(\phi_\alpha(\alpha(0),0)\beta_n(0) + L_n y(0)) + b_1(\phi_\alpha(\alpha(1),1)\beta_n(1) + R_n y(0)) = c_n, \tag{2.26}$$

where $c_n$ depends on the same terms as $L_n G$ and $R_n G$. These boundary conditions differ from (2.14d) only in the right-hand side $c_n$. The unique solvability of the problem (2.24) ((2.20) for $n=1$), (2.25) and (2.26) follows from the fact that (2.14), (2.15) has only the trivial solution. $L_n y$ and $R_n y$ are exponentially decaying functions, which is a consequence of Lemma 3.2 in the next section. Thus, the terms in the formal asymptotic expansion (2.4) can be constructed consecutively up to arbitrary order.

**3. BVP's on infinite and "long" intervals.** In this section we investigate linear problems of the type

$$y' = A(t)y + g(t), \quad t \geqq 0, \quad y \in R^n, \tag{3.1a}$$

$$y(\infty) = 0, \qquad \text{i.e. } \lim_{t\to\infty} y(t) = 0, \tag{3.1b}$$

$$By(0) = \beta. \tag{3.1c}$$

Also we shall derive results on the well-posedness of problems obtained from (3.1) by cutting the infinite interval at a large $T$ and replacing (3.1b) by an appropriate boundary condition at $T$.

The proofs of the results we state will be given in the Appendix and follow along the lines of de Hoog and Weiss (1980) and Markowich (1982, 1983). In these papers instead of (3.1b) the weaker condition $y \in C[0, \infty]$, i.e. $\lim_{t\to\infty} y(t)$ exists and is finite, is used. However, in the singular perturbation context, (3.1b) is relevant (cf. condition (2.5)). Thus the results are slightly different, but the methods of proof are similar.

**3.1. Problems with constant coefficients.** We consider the problem

$$y' = My + g(t), \qquad t \geqq 0, \tag{3.2a}$$

$$y(\infty) = 0, \tag{3.2b}$$

$$By(0) = \beta, \tag{3.2c}$$

where the matrix $M$ has a splitting

$$E^{-1}ME = \Lambda = \begin{pmatrix} \Lambda_- & & \\ & \Lambda_+ & \\ & & 0 \end{pmatrix},$$

with the eigenvalues of the $n_-$-dimensional square matrix $\Lambda_-$ having negative real parts, the eigenvalues of the $n_+$-dimensional matrix $\Lambda_+$ having positive real parts and the zero matrix having dimension $n_0 = n - n_- - n_+$. We denote the column decomposition of $E$ corresponding to the diagonal blocks of $\Lambda$ by $E = (E_-, E_+, E_0)$ and the row decomposition of $E^{-1}$ by $(E^{-1})^T = ((E_-^{-1})^T, (E_+^{-1})^T, (E_0^{-1})^T)$.

To satisfy (3.2b), we pose the following restrictions on $g(t)$:

$$g \in C[0,\infty], \quad g(\infty) = 0 \quad \text{and} \quad \| E_0^{-1} g(t) \| = O(t^{-1-\nu}), \quad \nu > 0. \tag{3.3}$$

Regarding the unique solvability of (3.2), we have

THEOREM 3.1. *The boundary value problem* (3.2) *has a unique solution for all* $\beta \in R^{n_-}$ *and for all* $g(t)$ *satisfying* (3.3), *iff the matrix* $B$ *has* $n_-$ *rows and* $BE_-$ *is regular.*

LEMMA 3.1. *Let the assumptions of Theorem* 3.1 *and*

$$\| g(t) \| = O(e^{-\kappa t}), \qquad \kappa > 0, \tag{3.4}$$

*be valid. Then the solution* $y(t)$ *of* (3.2) *satisfies*

$$\| y(t) \| = O(e^{-\kappa t}), \qquad \kappa > 0. \tag{3.5}$$

(*Henceforth in exponential estimates of the type* (3.4), (3.5) $\kappa$ *will denote a generic constant.*)

Next we consider a problem on the finite interval $[0, T]$:

$$x_T' = Mx_T + g(t), \qquad 0 \leqq t \leqq T, \tag{3.6a}$$

$$\begin{pmatrix} E_+^{-1} \\ E_0^{-1} \end{pmatrix} x_T(T) = \gamma, \tag{3.6b}$$

$$Bx_T(0) = \beta. \tag{3.6c}$$

THEOREM 3.2. *Let the assumptions of Theorem* 3.1 *be valid. Then* (3.6) *has a unique solution* $x_T(t)$ *for all* $T > 0$ *and for all* $g \in C[0, T]$, $\beta \in R^{n_-}$ *and* $\gamma \in R^{n_+ + n_0}$. $x_T$ *satisfies the estimate*

$$\| x_T \|_{[0,T]} \leqq \text{const}\big( T \| g \|_{[0,T]} + \| \beta \| + \| \gamma \| \big), \tag{3.7}$$

*where the norm* $\| \cdot \|_{[t_1, t_2]}$ *on the space* $C[t_1, t_2]$ *is defined by*

$$\| f \|_{[t_1,t_2]} := \sup_{s \in [t_1, t_2]} \| f(s) \|.$$

**3.2. Problems with variable coefficients.** We consider problem (3.1) with

$$A(t)=M+F(t), \quad \text{where } \|F(t)\|=O(e^{-\kappa t}). \tag{3.8}$$

In the Appendix we will show that the general solution of the homogeneous problem (3.1a), (3.1b), with $A(t)$ defined by (3.8), is of the form $y(t)=\Phi_-(t)\eta_-$ with $\eta_-\in R^{n_-}$, where the $n\times n_-$-matrix $\Phi_-(t)$ is defined in the Appendix. We then have

THEOREM 3.3. *The boundary value problem* (3.1) *with* $A(t)$ *given by* (3.8) *has a unique solution for all* $\beta\in R^{n_-}$ *and for all* $g(t)$ *satisfying* (3.3), *iff* $B$ *has* $n_-$ *rows and* $B\Phi_-(0)$ *is regular.*

The analogue to Lemma 3.1 is

LEMMA 3.2. *Let the assumptions of Theorem* 3.3 *and* (3.4) *be valid. Then the solution* $y(t)$ *of* (3.1) *satisfies* (3.5).

Again we consider the "finite" problem

$$x_T'=A(t)x_T+g(t), \qquad 0\leqq t\leqq T, \tag{3.9a}$$

$$\begin{pmatrix} E_+^{-1} \\ E_0^{-1} \end{pmatrix} x_T(T)=\gamma, \tag{3.9b}$$

$$Bx_T(0)=\beta. \tag{3.9c}$$

THEOREM 3.4. *Let the assumptions of Theorem* 3.3 *be valid. Then* (3.9) *has a unique solution* $x_T(t)$ *for* $T$ *big enough and for all* $g\in C[0,T]$, $\beta\in R^{n_-}$ *and* $\gamma\in R^{n_++n_0}$. $x_T$ *satisfies the estimate* $\|x_T\|_{[0,T]}\leqq \text{const}(T\|g\|_{[0,T]}+\|\beta\|+\|\gamma\|)$.

THEOREM 3.5. *There exists a fundamental solution* $\psi_T(t)$ *of* (3.9a) *which satisfies*

$$\|\psi_T(t)-X_T(t)\|=O(e^{-\kappa t}), \tag{3.10}$$

*where*

$$X_T(t)=E\begin{pmatrix} e^{\Lambda_- t} & & \\ & e^{\Lambda_+(t-T)} & \\ & & I \end{pmatrix}$$

*is a fundamental solution of* (3.6a).

**4. Existence and uniqueness result.** Let $Y_i(t,\varepsilon)$ denote the $i$th partial sum in (2.1), i.e.

$$Y_i(t,\varepsilon)=\sum_{j=0}^{i}\left(\bar{y}_j(t)+L_jy\left(\frac{t}{\varepsilon}\right)+R_jy\left(\frac{1-t}{\varepsilon}\right)\right)\varepsilon^j.$$

Let the space $C^1[0,1]$ be equipped with the norm $\|\cdot\|_*$ defined by

$$\|y\|_*=\|y\|_{[0,1]}+\varepsilon\|y'\|_{[0,1]}.$$

Then $(C^1[0,1], \|\cdot\|_*)$ is a Banach space. Balls in this space will be denoted by

$$B_\delta(y_0)=\left\{ y\in C^1[0,1] \,\middle|\, \|y-y_0\|_*\leqq\delta\right\}.$$

We now prove the main result of this paper.

THEOREM 4.1. *Let* $f$ *and* $b$ *in* (1.1), (1.2) *be infinitely differentiable and let hypotheses* H1, H2 *be valid. Then there are constants* $c$, $\varepsilon_0>0$ *such that for* $0<\varepsilon\leqq\varepsilon_0$ *a solution* $y(t,\varepsilon)$ *to* (1.1), (1.2) *exists which is unique in the ball* $B_{c\varepsilon}(Y_1)$ *and satisfies*

$$\|y-Y_i\|_*=O(\varepsilon^{i+1}), \qquad i\geqq 0.$$

*Proof*. Common methods for proving results like Theorem 4.1 are based on the fixed point theorem for contraction mappings in Banach spaces (cf., e.g. Eckhaus (1979) for general results in perturbation theory and Vasileva and Butuzov (1973), (1978) for applications in singular perturbations). We shall apply a theorem due to Van Harten (1978) stated by Eckhaus (1979, p. 237 f). Using his notation, (1.1), (1.2) is written as

$$L_\varepsilon y=0. \tag{4.1}$$

In our case $L_\varepsilon$ is an operator from the Banach space $(C^1[0,1],\|\cdot\|_*)$ to $(C[0,1]\times R^n, \|\cdot\|_{**})$, where we define the norm $\|\cdot\|_{**}$ as the sum of the supremum norm on $C[0,1]$ and the maximum norm on $R^n$. Our method of constructing the formal approximation $Y_i$ yields

$$L_\varepsilon Y_i=\rho_i, \tag{4.2}$$

where $\|\rho_i\|_{**}=O(\varepsilon^{i+1})$. Subtracting (4.2) from (4.1) yields the following problem for the remainder term $R_i=y-Y_i$:

$$\hat{L}_\varepsilon R_i := L_\varepsilon(R_i+Y_i)-L_\varepsilon Y_i=-\rho_i. \tag{4.3}$$

Denoting the linearization of $L_\varepsilon$ at $Y_i$ by $A_\varepsilon$, we obtain that decomposition

$$\hat{L}_\varepsilon=A_\varepsilon+P_\varepsilon.$$

The above mentioned theorem requires us

a) to obtain an estimate

$$\|A_\varepsilon^{-1}\|_*\leqq\lambda(\varepsilon),$$

b) to prove a Lipschitz condition for $P_\varepsilon$ of the form

$$\|P_\varepsilon v_1-P_\varepsilon v_2\|_{**}\leqq\mu(\varepsilon,\delta)\|v_1-v_2\|_*$$

for $v_1$, $v_2\in B_\delta(0)$, where $\lim_{\delta\to 0}\mu(\varepsilon,\delta)=0$. Then we have to determine $\bar\delta=\bar\delta(\varepsilon)$, such that

$$\lambda(\varepsilon)\mu(\varepsilon,\delta)\leqq 1-\gamma, \qquad \gamma\in(0,1)$$

for all $\delta\leqq\bar\delta$. If

$$\|\rho_i\|_{**}<\frac{\gamma\bar\delta}{\lambda(\varepsilon)}$$

then there is a solution $R_i$ of (4.3) which is unique in the ball $B_{\bar\delta}(0)$ and satisfies

$$\|R_i\|_*\leqq\frac{1}{\gamma}\lambda(\varepsilon)\|\rho_i\|_{**}.$$

We will show that in our case

$$\mu(\varepsilon,\delta)=c_1\delta, \tag{4.4}$$

$$\lambda(\varepsilon)=\frac{c_2}{\varepsilon}. \tag{4.5}$$

Thus, we obtain $\bar\delta=\varepsilon(1-\gamma)/c_1c_2$ and the following condition for $\rho_i$

$$\|\rho_i\|_{**}<\frac{\varepsilon^2\gamma(1-\gamma)}{c_1c_2^2}.$$

This certainly holds for $i \geqq 2$ and $\varepsilon$ small enough. We then have a solution $R_i$ of (4.3), which is unique in $B_{c\varepsilon}(0)$ with $c=(1-\gamma)/c_1c_2$ and satisfies

$$\|R_i\|_* \leqq \frac{c_2}{\gamma\varepsilon}\|\rho_i\|_{**} = O(\varepsilon^i).$$

By the argument

$$\|R_{i-1}\|_* \leqq \|R_i\|_* + \|Y_i - Y_{i-1}\|_* = O(\varepsilon^i)$$

the results of Theorem 4.1 are established. It remains to show (4.4) and (4.5): Using a formula for $P_\varepsilon v_1 - P_\varepsilon v_2$ (Eckhaus (1979, p. 239)) (4.4) follows immediately.

To establish (4.5), we have to analyze the linearization of (1.1), (1.2), which reads

$$\text{(4.6)}\qquad \begin{aligned} &\varepsilon\hat{y}' = f_y(Y_i,t,\varepsilon)\hat{y} + g(t),\\ &\hat{b}_0(Y_i(0),Y_i(1))\hat{y}(0) + \hat{b}_1(Y_i(0),Y_i(1))\hat{y}(1) = \beta \end{aligned}$$

or

$$\text{(4.7)}\qquad \begin{aligned} &\varepsilon\hat{y}' = f_y(\phi(t)+\varepsilon\bar{y}_1 + L_0y + R_0y,t,\varepsilon)\hat{y} + \omega(\varepsilon^2+\varepsilon e^{-\kappa\tau}+\varepsilon e^{-\kappa\sigma})\hat{y} + g(t),\\ &b_0\hat{y}(0) + b_1\hat{y}(1) = \omega(\varepsilon)(\hat{y}(0),\hat{y}(1)) + \beta, \end{aligned}$$

where $\phi(t)$ stands for $\phi(\alpha(t),t)$, $b_0$ and $b_1$ are like in §2 and $\omega$ is used generically to denote linear operators satisfying $\|\omega(r(\varepsilon,t))\| = O(r(\varepsilon,t))$. As in Vasileva and Butuzov (1973) we introduce a partition of the interval $[0,1]$ into three parts $[0,t_0]$, $[t_0,t_1]$ and $[t_1,1]$, where

$$t_0 = -\frac{2}{\kappa}\varepsilon\ln\varepsilon,\qquad t_1 = 1 + \frac{2}{\kappa}\varepsilon\ln\varepsilon,$$

and $\kappa$ is taken from the estimate (2.13). We define

$$\tau_0 = \frac{t_0}{\varepsilon},\qquad \sigma_1 = \frac{1-t_1}{\varepsilon}.$$

With (2.13) we obtain

$$\|L_0y(\tau)\| = O(\varepsilon^2),\quad \tau \geqq \tau_0,\qquad \|R_0y(\sigma)\| = O(\varepsilon^2),\qquad \sigma \geqq \sigma_1.$$

Introducing the independent variables $\tau$ and $\sigma$ on the short intervals $[0,t_0]$ and $[t_1,1]$, we obtain a problem equivalent to (4.7),

$$\text{(4.8a)}\qquad \frac{d\hat{y}_1}{d\tau} = f_y(\phi(0)+L_0y,0,0)\hat{y}_1 + \omega(\varepsilon\ln\varepsilon)\hat{y}_1 + g(\varepsilon\tau),\qquad 0 \leqq \tau \leqq \tau_0,$$

$$\text{(4.8b)}\qquad \frac{d\hat{y}_2}{d\sigma} = -f_y(\phi(1)+R_0y,1,0)\hat{y}_2 + \omega(\varepsilon\ln\varepsilon)\hat{y}_2 + g(1-\varepsilon\sigma),\qquad 0 \leqq \sigma \leqq \sigma_1,$$

$$\text{(4.8c)}\qquad \varepsilon\hat{y}_3' = f_y(\phi(t)+\varepsilon\bar{y}_1,t,\varepsilon)\hat{y}_3 + \omega(\varepsilon^2)\hat{y}_3) + g(t),\qquad t_0 \leqq t \leqq t_1,$$

$$\text{(4.8d)}\qquad b_0\hat{y}_1(0) + b_1\hat{y}_2(0) = \omega(\varepsilon)(\hat{y}_1(0),\hat{y}_2(0) + \beta,$$

$$\text{(4.8e)}\qquad \hat{y}_1(\tau_0) = \hat{y}_3(t_0),\qquad \hat{y}_2(\sigma_1) = \hat{y}_3(t_1).$$

The transformation

$$\hat{y}_3 = E\xi,\qquad \text{where } \xi = \begin{pmatrix} \xi_\pm \\ \xi_0 \end{pmatrix},$$

changes (4.8c) to

$$\varepsilon\xi' = \left(E^{-1}f_y(\phi(t)+\varepsilon\bar{y}_1,t,\varepsilon)E - \varepsilon E^{-1}E'\right)\xi + \omega(\varepsilon^2)\xi + E^{-1}g(t). \tag{4.9}$$

From

$$f_y(\phi(t)+\varepsilon\bar{y}_1,t,\varepsilon) = \bar{f}_y + \varepsilon\left(\bar{f}_{yy}\langle\cdot,\bar{y}_1\rangle + \bar{f}_{y\varepsilon}\right) + \omega(\varepsilon^2).$$

It follows on using (2.19) and (2.1) that (4.9) can be written as

$$\varepsilon\xi'_\pm = \Lambda_\pm\xi_\pm + \omega(\varepsilon)\xi + E_\pm^{-1}g(t), \tag{4.10a}$$

$$\begin{aligned}\xi_0' = {} & \left(H\bar{f}_{yy}\langle\phi_\alpha\cdot,\bar{y}_{1p}\rangle + H\bar{f}_{y\varepsilon}\phi_\alpha - H\phi_\alpha'\right)\xi_0 \\ & + H\left[\bar{f}_{yy}\langle E_\pm\cdot,\bar{y}_1\rangle + \bar{f}_{y\varepsilon}E_\pm - E'_\pm\right]\xi_\pm + \omega(\varepsilon)\xi + \frac{1}{\varepsilon}Hg(t).\end{aligned} \tag{4.10b}$$

With the decoupling transformation

$$\xi_0 = \tilde{\xi}_0 + \varepsilon H\left[\bar{f}_{yy}\langle E_\pm\cdot,\bar{y}_1\rangle + \bar{f}_{y\varepsilon}E_\pm - E'_\pm\right]\Lambda_\pm^{-1}\xi_\pm, \qquad \tilde{\xi} = \begin{pmatrix}\xi_\pm \\ \tilde{\xi}_0\end{pmatrix}$$

(4.10b) becomes

$$\tilde{\xi}_0' = \left(H\bar{f}_{yy}\langle\phi_\alpha\cdot,\bar{y}_{1p}\rangle + H\bar{f}_{y\varepsilon}\phi_\alpha - H\phi_\alpha'\right)\tilde{\xi}_0 + \omega(\varepsilon)\tilde{\xi} + \omega\left(\frac{1}{\varepsilon}\right)g(t). \tag{4.11}$$

Introducing the transformation

$$\begin{aligned}\hat{y}_1 &= \mu + \phi_\alpha(0)\tilde{\xi}_0(t_0), \\ \hat{y}_2 &= \nu + \phi_\alpha(1)\tilde{\xi}_0(t_1),\end{aligned} \tag{4.12}$$

up to $O(\varepsilon\ln\varepsilon)$ terms, (4.8) takes the form

$$\tilde{\xi}_0' = \left[H_\alpha\langle\cdot,\bar{f}_\varepsilon - \phi_t\rangle + H\left(\bar{f}_{y\varepsilon} - \phi_{\alpha t}\right)\right]\tilde{\xi}_0 + k_1(t), \qquad t_0 \leqq t \leqq t_1, \tag{4.13a}$$

$$\frac{d\mu}{d\tau} = f_y(\phi(0)+L_0y,0,0)(\mu+\phi_\alpha(0)\tilde{\xi}_0(t_0)) + k_2(\tau), \qquad 0 \leqq \tau \leqq \tau_0, \tag{4.13b}$$

$$\frac{d\nu}{d\sigma} = -f_y(\phi(1)+R_0y,1,0)(\nu+\phi_\alpha(1)\tilde{\xi}_0(t_1)) + k_3(\sigma), \qquad 0 \leqq \sigma \leqq \sigma_1, \tag{4.13c}$$

$$b_0(\mu(0)+\phi_\alpha(0)\tilde{\xi}_0(t_0)) + b_1(\nu(0)+\phi_\alpha(1)\tilde{\xi}_0(t_1)) = \beta_1, \tag{4.13d}$$

$$\varepsilon\xi'_\pm = \Lambda_\pm\xi_\pm + k_4(t), \qquad t_0 \leqq t \leqq t_1 \tag{4.14a}$$

$$\begin{aligned}&\mu(\tau_0)+\phi_\alpha(0)\tilde{\xi}_0(t_0) = E(t_0)\tilde{\xi}(t_0)+\beta_2,\ \nu(\sigma_1)+\phi_\alpha(1)\tilde{\xi}_0(t_1) \\ &= E(t_1)\tilde{\xi}(t_1)+\beta_3.\end{aligned} \tag{4.14b}$$

where we now employ arbitrary inhomogeneities $k_i$ and $\beta_j$. The equality of the coefficient matrices in (4.11) and (4.13a) has been proven in §2. Thus, the homogeneous equations (4.13) correspond to (2.14), and the solvability of (2.14), (2.15) will now be employed to conclude unique solvability of (4.13), (4.14). The general solutions of the differential equations in (4.13), (4.14) have the form

$$\begin{aligned}\mu(\tau)&=M(\tau)\eta+P(\tau)\tilde{\xi}_0(t_0)+\mu_p(\tau),\\ \nu(\sigma)&=N(\sigma)\delta+Q(\sigma)\tilde{\xi}_0(t_1)+\nu_p(\sigma),\\ \xi_\pm(t)&=G_\pm(t)\gamma_\pm+\xi_{\pm p}(t),\\ \tilde{\xi}_0(t)&=G_0(t)\gamma_0+\tilde{\xi}_{0p}(t).\end{aligned}\tag{4.15}$$

From Theorem 3.5 we obtain for $M(\tau)$ and $N(\sigma)$,

$$\left\| M(\tau)-E(0)\begin{pmatrix}\exp(\Lambda_-(0)\tau) & & \\ & \exp(\Lambda_+(0)(\tau-\tau_0)) & \\ & & I\end{pmatrix}\right\| = O(e^{-\kappa\tau}),\tag{4.16a}$$

$$\left\| N(\sigma)-E(1)\begin{pmatrix}\exp(-\Lambda_-(1)(\sigma-\sigma_1)) & & \\ & \exp(-\Lambda_+(1)\sigma) & \\ & & I\end{pmatrix}\right\| = O(e^{-\kappa\sigma}),\tag{4.16b}$$

and it follows from the proof of Theorem 3.4 that

$$\begin{aligned}\|\mu_p\|_{[0,\tau_0]}&\leqq \text{const}\,\tau_0\|k_2\|_{[0,\tau_0]}=\text{const}\ln\frac{1}{\varepsilon}\|k_2\|_{[0,\tau_0]},\\ \|\nu_p\|_{[0,\sigma_1]}&\leqq \text{const}\,\sigma_1\|k_3\|_{[0,\sigma_1]}=\text{const}\ln\frac{1}{\varepsilon}\|k_3\|_{[0,\sigma_i]}.\end{aligned}$$

Due to (2.1), $f_y(\phi(0)+L_0y,0,0)\phi_\alpha(0)$ and $f_y(\phi(1)+R_0y,1,0)\phi_\alpha(1)$ decay exponentially. Lemma 3.2 therefore implies

$$\|P(\tau)\|=O(e^{-\kappa\tau}),\qquad \|Q(\sigma)\|=O(e^{-\kappa\sigma}).\tag{4.17}$$

Standard results yield

$$G_\pm(t)=\begin{pmatrix}\exp\left(\Lambda_-(t_0)\dfrac{t-t_0}{\varepsilon}\right) & \\ & \exp\left(\Lambda_+(t_1)\dfrac{t-t_1}{\varepsilon}\right)\end{pmatrix}+O(\varepsilon),\tag{4.18a}$$

$$\|\xi_{\pm p}\|_{[t_0,t_1]}\leqq \text{const}\,\|k_4\|_{[t_0,t_1]}\tag{4.18b}$$

and

$$\|\tilde{\xi}_{0p}\|_{[t_0,t_1]}\leqq \text{const}\|k_1\|_{[t_0,t_1]}.$$

Substituting the representation (4.15) into the boundary conditions (4.13d) and (4.14b), we obtain

$$\begin{aligned}&b_0[M(0)\eta+P(0)G_0(t_0)\gamma_0+\phi_\alpha(0)G_0(t_0)\gamma_0]\\ &\qquad+b_1[N(0)\delta+Q(0)G_0(t_1)\gamma_0+\phi_\alpha(1)G_0(t_1)\gamma_0]=\bar{\beta}_1,\\ &M(\tau_0)\eta+P(\tau_0)G_0(t_0)\gamma_0+\phi_\alpha(0)G_0(t_0)\gamma_0-E_\pm(t_0)G_\pm(t_0)\gamma_\pm-\phi_\alpha(t_0)G_0(t_0)\gamma_0=\bar{\beta}_2,\\ &N(\sigma_1)\delta+Q(\sigma_1)G_0(t_1)\gamma_0+\phi_\alpha(1)G_0(t_1)\gamma_0-E_\pm(t_1)G_\pm(t_1)\gamma_\pm-\phi_\alpha(t_1)G_0(t_1)\gamma_0=\bar{\beta}_3,\end{aligned}$$

where the $\bar{\beta}_j$ contain the contributions of the $\beta_j$ and the particular solutions. Denoting the coefficient matrix of this linear system by $C(\varepsilon)$, the estimates (4.16)–(4.18) imply that $C(\varepsilon)=C(0)+\omega(\varepsilon\ln\varepsilon)$, where

$$C(0)=\left[\begin{array}{c|c|c|c} b_0M(0) & b_1N(0) & 0 & S \\ 0\ E_+(0)\ E_0(0) & 0 & -E_-(0)\ \ 0 & 0 \\ 0 & E_-(1)\ 0\ E_0(1) & 0\ \ -E_+(1) & 0 \end{array}\right] \tag{4.19}$$

and

$$S=b_0[P(0)+\phi_\alpha(0)]G_0(0)+b_1[Q(0)+\phi_\alpha(1)]G_0(1).$$

Here the ordering of the unknowns is $\eta$, $\delta$, $\gamma_\pm$, $\gamma_0$, with the partitions $\eta^T=(\eta_-^T,\eta_+^T,\eta_0^T)$, $\delta^T=(\delta_-^T,\delta_+^T,\delta_0^T)$, $\gamma_\pm^T=(\gamma_-^T,\gamma_+^T)$. Permutation of the columns in (4.19), corresponding to the ordering $\eta$, $\delta_+$, $\gamma_0$, $-\gamma_-$, $\eta_+$, $\eta_0$, $\delta_-$, $-\gamma_+$, $\delta_0$ yields the matrix

$$\begin{pmatrix} \bar{C} & X_1 & X_2 \\ 0 & E(0) & 0 \\ 0 & 0 & E(1) \end{pmatrix} \tag{4.20}$$

where $\bar{C}$, $X_1$, $X_2$ are square matrices. It follows on employing Theorem 3.3 that the unique solvability of (2.14), (2.15) is equivalent to $C$ being nonsingular, which implies that the matrix (4.20) is nonsingular. Thus, $C(\varepsilon)$ is nonsingular as well and has a bounded inverse for $\varepsilon$ small enough. This and the estimates on the particular solutions imply that (4.13), (4.14) is uniquely solvable for small $\varepsilon$ and that

$$\|\tilde{\xi}\|_{[t_0,t_1]},\ \|\mu\|_{[0,\tau_0]},\ \|\nu\|_{[0,\sigma_1]}$$
$$\leqq \text{const}\left(\sum_{i=1}^{3}\|\beta_i\|+\|k_1\|_{[t_0,t_1]}+\|k_4\|_{[t_0,t_1]}+\ln\frac{1}{\varepsilon}\left(\|k_2\|_{[0,\tau_0]}+\|k_3\|_{[0,\sigma_1]}\right)\right).$$

This allows the application of a contraction mapping argument to obtain unique solvability of (4.8), which is equivalent to (4.6). We thus have unique solvability of (4.6) for $\varepsilon$ small enough, with the estimate

$$\|\hat{y}\|_{[0,1]}\leqq\text{const}\left(\|\beta\|+\varepsilon^{-1}\|g\|_{[0,1]}\right).$$

The differential equation in (4.6) can be used to obtain an estimate on $\varepsilon\|\hat{y}'\|_{[0,1]}$ which finally yields

$$\|\hat{y}\|_*\leqq\text{const}\left(\|\beta\|+\varepsilon^{-1}\|g\|_{[0,1]}\right)\leqq\text{const}\,\varepsilon^{-1}\|(\beta,g)\|_{**}.$$

This establishes (4.5) and completes the proof of the theorem.

**5. Example.** We consider the fundamental semiconductor device equations for the case of a symmetric $p-n$ junction with piecewise constant doping. The singular perturbation approach for this problem was originated by Vasileva, Kardosysoev and Stelmakh (1976). We assume that recombination-generation effects are negligible and that the total current is kept at a prescribed value $J$. For the scaling which leads to the formulation as a singular perturbation problem, see also Markowich et al. (1982). The governing equations are

$$\begin{aligned} \varepsilon\psi' &= E, \\ \varepsilon E' &= n-p-1, \\ \varepsilon n' &= nE+\frac{\varepsilon J}{2}, \\ \varepsilon p' &= -pE-\frac{\varepsilon J}{2}. \end{aligned} \tag{5.1}$$

$$\begin{aligned} &\psi(0)=0, \\ &n(0)=p(0), \\ &p(1)=\frac{1}{2}(-1+\sqrt{1+4\delta^4})=p_1, \\ &n(1)=p_1+1. \end{aligned} \tag{5.2}$$

The variables $\psi$, $E$, $n$ and $p$ are scaled and proportional to the potential, the electric field, the electron density and the hole density in the device. $\varepsilon$ and $\delta$ result from the scaling. $\varepsilon$ is equal to the Debye length, and is small when the doping is large. Thus (5.1) is singularly perturbed in this situation.

In Vasileva and Stelmakh (1977) and Vasileva and Butuzov (1978) (5.1), (5.2) is considered with $\delta=0$. Smith (1980) treats the case of a symmetric $p-n$ junction where the doping and the given electron and hole current densities are not constant. In all these papers results are proven which are more or less equivalent to the one we will obtain below by an application of the general result Theorem 4.1.

The reduced equations

$$0=\bar{E}, \quad 0=\bar{n}-\bar{p}-1, \quad 0=\bar{n}\bar{E}, \quad 0=-\bar{p}\bar{E} \tag{5.3}$$

have the solution

$$\bar{\psi}=\alpha_1, \quad \bar{E}=0, \quad \bar{n}=\alpha_2+1, \quad \bar{p}=\alpha_2. \tag{5.4}$$

The Jacobian of the right-hand side of (5.1) at the solution of (5.3) is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & \alpha_2+1 & 0 & 0 \\ 0 & -\alpha_2 & 0 & 0 \end{pmatrix}$$

with the eigenvalues $\lambda_{1,2}=0$ and $\lambda_{3,4}=\pm\sqrt{2\alpha_2+1}$. Obviously, the matrix has rank 2. Thus hypothesis H1 is fulfilled with the assumption $\alpha_2 \geqq 0$ which is natural because $\bar{p}=\alpha_2$ denotes a density. As in chapter 2 we find differential equations for $\alpha_1$ and $\alpha_2$:

$$\begin{aligned} &\text{a)} \qquad \alpha_1' = -\frac{J}{2\alpha_2+1}, \\ &\text{b)} \qquad \alpha_2' = -\frac{J}{2(2\alpha_2+1)}. \end{aligned} \tag{5.5}$$

The equations for the left layer terms are

$$(5.6)\quad \begin{aligned} &\text{a)} && \frac{dL\psi}{d\tau}=LE,\\ &\text{b)} && \frac{dLE}{d\tau}=Ln-Lp,\\ &\text{c)} && \frac{dLn}{d\tau}=(\alpha_2(0)+1+Ln)LE,\\ &\text{d)} && \frac{dLp}{d\tau}=-(\alpha_2(0)+Lp)LE. \end{aligned}$$

The zeroth order right layer terms disappear. This is due to the fact that the last condition in (5.2) is consistent with (5.3), and will be proved by showing that there is a unique solution of (5.5), (5.6) satisfying the boundary conditions

$$(5.7)\quad \begin{aligned} &\text{a)} && \alpha_1(0)+L\psi(0)=0,\\ &\text{b)} && Ln(0)+1=Lp(0),\\ &\text{c)} && \alpha_2(1)=p_1,\\ &\text{d)} && L\psi(\infty)=LE(\infty)=Lp(\infty)=Ln(\infty)=0. \end{aligned}$$

Introducing $L\psi$ as a new independent variable in (5.6c), (5.6d) and solving the resulting linear equations yields with (5.7d)

$$(5.8)\quad \begin{aligned} Ln&=(\alpha_2(0)+1)(e^{L\psi}-1),\\ Lp&=\alpha_2(0)(e^{-L\psi}-1). \end{aligned}$$

Now we substitute (5.8) in (5.6b) and solve this equation again with $L\psi$ as independent variable. We obtain

$$(5.9)\quad LE=-\sqrt{2}\left[(\alpha_2(0)+1)e^{L\psi}+\alpha_2(0)e^{-L\psi}-L\psi-2\alpha_2(0)-1\right]^{1/2}\operatorname{sgn} L\psi,$$

where the sign of $LE$ is determined by the condition that the remaining equation for $L\psi$ has to have decaying solutions. (5.8), (5.9) represent the stable manifold of (5.6). Using (5.8) in (5.7b–c) gives

$$(5.10)\quad \begin{aligned} &(\alpha_2(0)+1)e^{-\alpha_1(0)}=\alpha_2(0)e^{\alpha_1(0)},\\ &\alpha_2(1)=p_1. \end{aligned}$$

The solution of (5.5), (5.10) is

$$(5.11)\quad \begin{aligned} \alpha_1(t)&=\sqrt{1+4\delta^4+2J(1-t)}-\sqrt{1+4\delta^4+2J}+\ln\frac{1+\sqrt{1+4\delta^4+2J}}{\sqrt{4\delta^4+2J}},\\ \alpha_2(t)&=\frac{1}{2}\left(-1+\sqrt{1+4\delta^4+2J(1-t)}\right). \end{aligned}$$

(5.7a) and (5.11) yield an initial value for $L\psi$:

$$L\psi(0)=-\ln\frac{1+\sqrt{1+4\delta^4+2J}}{\sqrt{4\delta^4+2J}}.$$

The proof of the isolatedness of this solution, i.e. the invertibility of the linearization of (5.5)–(5.7), is now outlined. We denote the linearization of the equation (a, b) by $(a, b)_1$ and proceed as follows: The terminal value problem $(5.5b)_1$, $(5.7c)_1$ has a unique solution. The equations $(5.6c, d)_1$ can be integrated similarly to (5.6c, d), and jointly with $(5.7a, b)_1$ lead to an initial condition for $(5.5a)_1$. It remains to determine a decaying solution of $(5.6a, b)_1$ with an initial value defined by $(5.7a)_1$. Writing $(5.6a, b)_1$ as a scalar second order equation, we can use the results of Fife (1974) on such problems to conclude existence and uniqueness of the desired solution.

Thus Hypothesis H2 has been verified and the validity of the formal approximation follows from Theorem 4.1.

## Appendix: Proofs of the Results in §3.

*Proof of Theorem* 3.1. The general solution of (3.2a) is

$$y(t)=Y(t)\eta+y_p(t),$$

where $Y(t)=Ee^{\Lambda t}$ and

$$y_p(t)=H^\delta g(t)=E\begin{pmatrix}\int_\delta^t e^{\Lambda_-(t-s)}E_-^{-1}g(s)\,ds\\ \int_\infty^t e^{\Lambda_+(t-s)}E_+^{-1}g(s)\,ds\\ \int_\infty^t E_0^{-1}g(s)\,ds\end{pmatrix}\quad \text{with } \delta\geqq 0.$$

LEMMA A.1. *$H^\delta g(\infty)=0$ holds for $g(t)$ satisfying* (3.3).

*Proof.*

a) $$\left\|\int_\delta^t e^{\Lambda_-(t-s)}E_-^{-1}g(s)\right\|\leqq \text{const}\int_\delta^t e^{-\kappa(t-s)}\|g(s)\|\,ds$$

$$=\text{const}\left(\int_\delta^{(t+\delta)/2}+\int_{(t+\delta)/2}^t\right)e^{-\kappa(t-s)}\|g(s)\|\,ds$$

$$\leqq \text{const}\left(\left(e^{-\kappa(t-\delta)/2}-e^{-\kappa(t-\delta)}\right)\|g\|_{[\delta,(t+\delta)/2]}+\|g\|_{[(t+\delta)/2,t]}\right)$$

$$\to 0 \text{ for } t\to\infty$$

b) $$\left\|\int_\infty^t e^{\Lambda_+(t-s)}E_+^{-1}g(s)\,ds\right\|\leqq \text{const}\|g\|_{[t,\infty]}\int_t^\infty e^{\kappa(t-s)}\,ds$$

$$\leqq \text{const}\|g\|_{[t,\infty]}\to 0 \text{ for } t\to\infty$$

c) $$\left\|\int_\infty^t E_0^{-1}g(s)\,ds\right\|\leqq \text{const}\int_\infty^t s^{-1-\nu}\,ds=\text{const}\,t^{-\nu}\to 0 \text{ for } t\to\infty.$$

In $Y(t)$ the components corresponding to zero eigenvalues and eigenvalues with positive real parts do not satisfy (3.2b). Therefore, the general solution of (3.2a),(3.2b) is

$$y(t)=E_-e^{\Lambda_-t}\eta_-+H^\delta g(t). \tag{A.1}$$

Substituting (A.1) into (3.2c) shows the validity of Theorem 3.1.

*Proof of Lemma* 3.1. The proof is similar to that of Lemma A.1 and therefore omitted.

*Proof of Theorem* 3.2. The general solution of (3.6a) reads

$$x_T(t)=X_T(t)\eta+H_T^\delta g(t), \tag{A.2}$$

where
$$X_T(t)=E\begin{pmatrix} e^{\Lambda_- t} & & \\ & e^{\Lambda_+(t-T)} & \\ & & I\end{pmatrix}$$

and

$$H_T^\delta g(t)=E\begin{pmatrix} \int_\delta^t e^{\Lambda_-(t-s)}E_-^{-1}g(s)\,ds \\ \int_T^t e^{\Lambda_+(t-s)}E_+^{-1}g(s)\,ds \\ \int_T^t E_0^{-1}g(s)\,ds\end{pmatrix}.$$

Similarly to the proof of Lemma A.1 we derive the estimate

$$\|H_T^\delta g\|_{[0,T]}\leqq \text{const}\|g\|_{[0,T]} \text{ for all } g\in C[0,T]. \tag{A.3}$$

Applying the boundary conditions (3.6b), (3.6c) to (A.2) yields

$$BX_T(0)\eta+BH_T^\delta g(0)=\beta,$$
$$\begin{pmatrix} E_+^{-1} \\ E_0^{-1}\end{pmatrix} X_T(T)\eta+\begin{pmatrix} E_+^{-1} \\ E_0^{-1}\end{pmatrix} H_T^\delta g(T)=\gamma.$$

The coefficient matrix in these equations for $\eta$ is

$$\begin{pmatrix} BE_- & BE_+e^{-\Lambda_+T} & BE_0 \\ 0 & I_{n_+} & 0 \\ 0 & 0 & I_{n_0}\end{pmatrix}.$$

Obviously, this matrix is regular since $BE_-$ is regular, and has a bounded inverse for all $T>0$. This fact together with (A.3) yields unique solvability and the estimate (3.7).

*Proof of Theorem* 3.3. For solutions of (3.1a), (3.1b)

$$y(t)=E_-e^{\Lambda_- t}\eta_-+H^\delta Fy(t)+H^\delta g(t)$$

must hold, which can be written as

$$(I-H^\delta F)y(t)=E_-e^{\Lambda_- t}\eta_-+H^\delta g(t), \tag{A.4}$$

where $I-H^\delta F$ is considered as an operator from the space $A_\delta=\{f\in C[\delta,\infty]\,|\,f(\infty)=0\}$ to itself.

*Lemma* A.2. $\|H^\delta Ff(t)\|\leqq \text{const}\, e^{-\kappa t}\|f\|_{[\delta,\infty]}$, $t\geqq\delta$, *holds for all* $f\in A_\delta$ (*with the constants independent of* $\delta$).

*Proof*. The proof is similar to that of Lemma A.1 and is therefore omitted.

It follows from Lemma A.2 that

$$\|H^\delta F\|_{[\delta,\infty]}\leqq \text{const}\, e^{-\kappa\delta}.$$

Hence

$$\| H^{\delta_1} F \|_{[\delta_1, \infty]} \leqq \frac{1}{2} \tag{A.5}$$

for some $\delta_1$ big enough. Thus, the operator $I - H^{\delta_1} F$ on $A_{\delta_1}$ is invertible and (A.4) yields

$$y(t) = \Phi_{-}(t)\eta + \tilde{H}g(t), \ t \geqq \delta_1, \tag{A.6}$$

where $\Phi_{-}(t) = (I - H^{\delta_1}F)^{-1} E_{-} e^{\Lambda_{-} t}$ and $\tilde{H}g(t) = (I - H^{\delta_1}F)^{-1} H^{\delta_1} g(t)$. As solutions of (3.1a) $\Phi_{-}$ and $\tilde{H}g$ can be extended to $[0, \infty)$. Applying the boundary conditions (3.1c) to (A.6) confirms the results of Theorem 3.3.

*Proof of Lemma* 3.2. The result immediately follows from Lemma 3.1, if we can prove

LEMMA A.3. $\|((I - H^{\delta_1}F)^{-1} - I) f(t)\| \leqq \text{const}\, e^{-\kappa t}$ *holds for all* $f \in A_{\delta_1}$.

*Proof*. We use the identity $(I - H^{\delta_1}F)^{-1} = \sum_{i=0}^{\infty} (H^{\delta_1}F)^i$. It follows from Lemma A.2 and from (A.5) that

$$\left\| (H^{\delta_1}F)^i f(t) \right\| \leqq \text{const}\, e^{-\kappa t} 2^{1-i} \| f \|_{[\delta_1, \infty]} \text{ for } i \geqq 1.$$

Thus

$$\left\| \left( (I - H^{\delta_1}F)^{-1} - I \right) f(t) \right\| \leqq \sum_{i=1}^{\infty} \text{const}\, e^{-\kappa t} 2^{1-i} \| f \|_{[\delta_1, \infty]} = \text{const}\, e^{-\kappa t}.$$

*Proof of Theorem* 3.4. Solutions of (3.9a) satsify

$$x_T(t) = X_T(t)\eta + H_T^{\delta} F x_T(t) + H_T^{\delta} g(t),$$

which implies

$$\left( I - H_T^{\delta} F \right) x_T(t) = X_T(t)\eta + H_T^{\delta} g(t),$$

where we consider $I - H_T^{\delta}F$ as an operator from $C[\delta, T]$ to itself. An analogous result to Lemma A.2 is

LEMMA A.4. *For* $f \in C[\delta, T]$ *the estimate*

$$\left\| H_T^{\delta} F f(t) \right\| \leqq \text{const}\, e^{-\kappa t} \| f \|_{[\delta, T]}, \qquad t \geqq \delta$$

*holds* (*with the constants independent of* $\delta$ *and* $T$).

*Proof*. Again the proof is similar to that of Lemma A.1 and is therefore omitted. Lemma A.4 yields

$$\| H_T^{\delta_2} F \|_{[\delta_2, T]} \leqq \frac{1}{2}$$

for $\delta_2$ big enough and all $T \geqq \delta_2$. Thus, $(I - H_T^{\delta_2}F)$ is invertible and we have

$$x_T(t) = \psi_T(t)\eta + \tilde{H}_T g(t), \qquad \delta_2 \leqq t \leqq T, \tag{A.7}$$

where $\psi_T(t) = (I - H_T^{\delta_2}F)^{-1} X_T(t)$ and $\tilde{H}_T g(t) = (I - H_T^{\delta_2}F)^{-1} H_T^{\delta_2} g(t)$. $\psi_T$ and $\tilde{H}_T g$ are defined on $[0, T]$ by continuation. An argument as in the proof of Lemma A.3 yields the estimate

$$\| \psi_T(t) - X_T(t) \| \leqq \text{const}\, e^{-\kappa t}. \tag{A.8}$$

Let $\psi_T^-$ be defined by $\psi_T^-(t)=(I-H_T^{\delta_2}F)^{-1}E_-e^{\Lambda\pm t}$ for $t\in[\delta_2,T]$, and by continuation for $t\in[0,\delta_2]$. We will now prove that

$$\text{(A.9)} \qquad \lim_{T\to\infty}\|\psi_T^- - \Phi_-\|_{[0,T]}=0.$$

We define $Z=\psi_T^- - \Phi_-$ and substitute $\bar\delta=\max\{\delta_1,\delta_2\}$ for $\delta$ in the definitions of $H^\delta$ and $H_T^\delta$. The definitions of $\psi_T^-$ and $\Phi_-$ imply that $Z$ is defined on $[0,\bar\delta]$ by continuation as a solution of $Z'=A(t)Z$. Thus, it suffices to show that

$$\text{(A.10)} \qquad \lim_{T\to\infty}\|Z\|_{[\bar\delta,T]}=0.$$

The definitions of $\psi_T^-$ and $\Phi_-$ on $[\bar\delta,T]$ yield

$$Z=H_T^{\bar\delta}F\psi_T^- - H^{\bar\delta}F\Phi_- = H_T^{\bar\delta}FZ+H_T^{\bar\delta}F\phi_- - H^{\bar\delta}F\phi_-.$$

Thus

$$Z=\left(I-H_T^{\bar\delta}F\right)^{-1}\left(H_T^{\bar\delta}F\Phi_- - H^{\bar\delta}F\Phi_-\right).$$

This implies (A.10), since obviously

$$\lim_{T\to\infty}\left\|H_T^{\bar\delta}F\Phi_- - H^{\bar\delta}F\Phi_-\right\|_{[\bar\delta,T]}=0.$$

Now we apply the boundary conditions (3.9b), (3.9c) to (A.7) and obtain equations for $\eta$ with the coefficient matrix

$$C(T)=\begin{pmatrix} B\psi_T(0) \\ E_+^{-1}\psi_T(T) \\ E_0^{-1}\psi_T(T) \end{pmatrix}.$$

The relations (A.8) and (A.9) imply that

$$\text{(A.11)} \qquad \lim_{T\to\infty}C(T)=\begin{pmatrix} B\Phi_-(0) & D \\ 0 & I_{n_+ + n_0}\end{pmatrix},$$

where $D$ is some rectangular matrix. The matrix $B\Phi_-(0)$ is regular which implies regularity of $C(T)$ and boundedness of $\|C^{-1}(T)\|$ for $T$ big enough. An argument similar to the proof of Lemma A.3 and (A.3) yields the estimate

$$\|\tilde H_T g\|_{[0,T]}\leqq \text{const } T\|g\|_{[0,T]}.$$

This completes the proof of Theorem 3.4.

*Proof of Theorem* 3.5. It follows from the proof of Theorem 3.4 that $\psi_T(t)$ in (A.7) is a fundamental solution of (3.9a). The inequality (3.10) has been shown during the proof of Theorem 3.4.

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