Singular Perturbations of Two-Point Boundary Problems for Systems of Ordinary Differential Equations

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Contents

Abstract

Asymptotic solutions of linear systems of ordinary differential equations are employed to discuss the relationship of the solution of a certain "complete" boundary problem.

$$\left\{\begin{aligned} \frac{dx_1}{dt} &= A_{11}(t,\varepsilon)\,x_1(t,\varepsilon)+\cdots+A_{1p}(t,\varepsilon)\,x_p(t,\varepsilon)\\ \varepsilon^{h_2}\frac{dx_2}{dt} &= A_{21}(t,\varepsilon)\,x_1(t,\varepsilon)+\cdots+A_{2p}(t,\varepsilon)\,x_p(t,\varepsilon)\\ &\;\vdots\\ \varepsilon^{h_p}\frac{dx_p}{dt} &= A_{p1}(t,\varepsilon)\,x_1(t,\varepsilon)+\cdots+A_{pp}(t,\varepsilon)\,x_p(t,\varepsilon)\end{aligned}\right\}$$

$$R(\varepsilon)\,x(a,\varepsilon)+S(\varepsilon)\,x(b,\varepsilon)=c(\varepsilon)$$

as $\varepsilon\to 0^+$ and the related "degenerate" problem obtained by setting $\varepsilon=0$. Here the h_i are integers, $0<h_2<\cdots<h_p$, x_i is a vector of dimension n_i, $A_{ij}(t,\varepsilon)$ are matrices of appropriate orders with asymptotic expansions, x is the vector $\begin{pmatrix}x_1\\ \vdots\\ x_p\end{pmatrix}$, R and S are square matrices of order $\sum\limits_{i=1}^{p} n_i$ and $\varepsilon>0$.

It is shown that under certain conditions the solution of the "complete" problem as $\varepsilon\to 0^+$ approaches a solution of the "degenerate" differential system and satisfies n_1 appropriate "degenerate" boundary conditions.

1. Introduction

We are concerned with showing the relationship of the solution of a boundary problem (1.1), (1.2) as $\varepsilon \to 0^+$ to the solutions of a related degenerate problem (1.3), (1.4). The problems are

$$
\begin{aligned}
\frac{d}{dt} x_1(t,\varepsilon) &= A_{11}(t,\varepsilon)\, x_1(t,\varepsilon) + \cdots + A_{1p}(t,\varepsilon)\, x_p(t,\varepsilon)\\
\varepsilon^{h_2} \frac{d}{dt} x_2(t,\varepsilon) &= A_{21}(t,\varepsilon)\, x_1(t,\varepsilon) + \cdots + A_{2p}(t,\varepsilon)\, x_p(t,\varepsilon)\\
&\ \ \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots\\
\varepsilon^{h_p} \frac{d}{dt} x_p(t,\varepsilon) &= A_{p1}(t,\varepsilon)\, x_1(t,\varepsilon) + \cdots + A_{pp}(t,\varepsilon)\, x_p(t,\varepsilon),
\end{aligned} \tag{1.1}
$$

$$
R(\varepsilon)\, x(a,\varepsilon) + S(\varepsilon)\, x(b,\varepsilon) = c(\varepsilon), \tag{1.2}
$$

$$
\begin{aligned}
\frac{d}{dt} x_1 &= A_{11}(t,0)\, x_1 + \cdots + A_{1p}(t,0)\, x_p\\
0 &= A_{21}(t,0)\, x_1 + \cdots + A_{2p}(t,0)\, x_p\\
\vdots\ \ &\qquad \vdots \qquad\qquad\qquad\qquad \vdots\\
0 &= A_{p1}(t,0)\, x_1 + \cdots + A_{pp}(t,0)\, x_p,
\end{aligned} \tag{1.3}
$$

$$
R(0)\, x(a) + S(0)\, x(b) = c(0), \tag{1.4}
$$

where the h_i are integers, $0 < h_2 < h_3 < \cdots < h_p = h$, x_i is a vector of dimension n_i, $m = \sum_{j=2}^{p} n_j$, $A(t,\varepsilon)_{ij}$ are matrices of appropriate orders with asymptotic expansions, x is the vector $\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$, R and S are square matrices of order $(n_1 + m)$ and $\varepsilon > 0$.

Under three hypotheses, H 1, H 2, and H 3, we shall prove Theorem 1 (Section 6), which embodies our results for the problem indicated above. We begin by reducing the problem (1.1), (1.2) to a canonical form (2.15), (6.3). We show that the solution of the canonical boundary problem has a limit as $\varepsilon \to 0^+$ which satisfies the corresponding degenerate differential system and n_1 of the degenerate boundary conditions. A discussion of the hypotheses is given in Section 7.

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2. Preliminary Transformations

If $A_{pp}(t,0)$ is non-singular, $a \leqq t \leqq b$, the last equation of (1.3) may be solved for x_p in terms of $x_1, \ldots, x_{p-1}$ and substituted into the preceeding equations of (1.3) to give a system of the same form as (1.3) in $x_1, \ldots, x_{p-1}$. The last equation of this form is

$$
0 = (A_{p-1,1} - A_{p-1,p} A_{pp}^{-1} A_{p1})\, x_1 + \cdots + (A_{p-1,p-1} - A_{p-1,p} A_{pp}^{-1} A_{p,p-1})\, x_{p-1}.
$$

Thus, if $(A_{p-1,p-1} - A_{p-1,p} A_{pp}^{-1} A_{p,p-1})$ is non-singular, $a \leqq t \leqq b$, we may solve this equation for x_{p-1} in terms of $x_1, \ldots, x_{p-2}$ and repeat the process until we solve the first equation for x_1, which is a differential equation. If this process can be carried out completely, step by step, and $x_p, x_{p-1}, \ldots, x_2$ all eliminated, then E. R. RANG [8] has shown there exists a non-singular transformation, $0 \leqq \varepsilon \leqq \varepsilon_0$, $a \leqq t \leqq b$, of the form

$$(2.1)\quad x(t,\varepsilon) = \big(T_1(t) + \varepsilon\, Q_1(t)\big)\big(T_2(t) + \varepsilon^2 Q_2(t)\big)\dots\big(T_L(t) + \varepsilon^L Q_L(t)\big)\, y(t,\varepsilon)$$

which changes the differential system (1.1) into the form (2.2) with corresponding degenerate differential system (2.3).

$$(2.2)\quad \begin{aligned} \frac{d}{dt} y_1(t,\varepsilon) &= C_{11}(t,\varepsilon)\, y_1(t,\varepsilon) + \cdots C_{1p}(t,\varepsilon)\, y_p(t,\varepsilon)\\ \varepsilon^{h_2}\frac{d}{dt} y_2(t,\varepsilon) &= C_{21}(t,\varepsilon)\, y_1(t,\varepsilon) + \cdots C_{2p}(t,\varepsilon)\, y_p(t,\varepsilon)\\ &\ \vdots\\ \varepsilon^{h_p}\frac{d}{dt} y_p(t,\varepsilon) &= C_{p1}(t,\varepsilon)\, y_1(t,\varepsilon) + \cdots C_{pp}(t,\varepsilon)\, y_p(t,\varepsilon), \end{aligned}$$

$$(2.3)\quad \begin{aligned} \frac{d}{dt} y_1 &= C_{11}(t,0)\, y_1 + \cdots + C_{1p}(t,0)\, y_p\\ 0 &= C_{21}(t,0)\, y_1 + \cdots + C_{2p}(t,0)\, y_p\\ &\ \vdots\\ 0 &= C_{p1}(t,0)\, y_1 + \cdots + C_{pp}(t,0)\, y_p \end{aligned}$$

where $C_{jj}(t,0)$ is non-singular $a \leqq t \leqq b$, $j = 2, \dots, p$, and in particular

$$C_{pp}(t,0) = A_{pp}(t,0) \quad\text{and}\quad C_{p-1,p-1} = A_{p-1,p-1} - A_{p-1,p} A_{pp}^{-1} A_{p,p-1}, \quad \textit{etc.}$$

Further, L may be chosen so large that the elements of $C_{ij}(t,\varepsilon)$ are $O(\varepsilon^\alpha)$ for any particular given large integer $\alpha > 0$, $i \neq j$. We shall have occasion to assume that this has been done.

For an N^{th} order differential system of the form

$$(2.4)\quad \varepsilon^\beta \frac{dz}{dt} = \Big(\sum_{j=0}^{\infty} a_j(t)\, \varepsilon^j\Big) z$$

the most general asymptotic expansions of solutions have been given by H. L. TURRITTIN [9]. In particular, he has given sufficient conditions for the existence of a transformation $z = H(t,\varepsilon)\, w(t,\varepsilon)$, where

$$(2.5)\quad H(t,\varepsilon) = \sum_{k=0}^{K} \varepsilon^{k/r} H_k(t), \qquad r \text{ and } K \text{ suitable positive integers,}$$

which will transform the equation (2.4) into a new equation

$$(2.6)\quad \varepsilon^\beta \frac{dw}{dt} = \{(\delta_{ij}\, \lambda_j(t,\varepsilon)) + \varepsilon^\beta B(t,\varepsilon)\}\, w(t,\varepsilon)$$

where $B(t,\varepsilon) = O(1)$ and the characteristic polynomials $\lambda_j(t,\varepsilon)$ have the form

$$(2.7)\quad \lambda_j(t,\varepsilon) = \sum_{k=0}^{r\beta-1} \varepsilon^{k/r} \lambda_{jk}(t), \qquad j = 1, 2, \dots, N,$$

and $\lambda_j(t,\varepsilon) \equiv \lambda_k(t,\varepsilon)$, or $\lambda_j(t,\varepsilon) \neq \lambda_k(t,\varepsilon)$, $a \leqq t \leqq b$, $0 < \varepsilon \leqq \varepsilon_0$.

If $r > 1$, the problem could have been adjusted in the beginning by introduction of a new parameter $\varepsilon^{1/r} = \varepsilon_1$. We assume that this has been done, so that without loss of generality we may assume $r = 1$.

Further, if the characteristic polynomials $\lambda_j(t, \varepsilon)$ are such that

$$\operatorname{Re}\{\varepsilon^{-\beta}\lambda_1(t,\varepsilon)\} \leqq \cdots \leqq \operatorname{Re}\{\varepsilon^{-\beta}\lambda_N(t,\varepsilon)\}, \quad \text{for} \quad a \leqq t \leqq b \quad \text{and} \quad 0 < \varepsilon \leqq \varepsilon_0,$$

there exists a fundamental matrix solution of (2.6) of the form $W(t,\varepsilon) = F(t,\varepsilon)\,E(t,\varepsilon)$, where when $a \leqq t \leqq b$ and $0 < \varepsilon \leqq \varepsilon_0$

$$F(t,\varepsilon) = \begin{pmatrix} F_{11} \dots F_{1M} \\ \vdots \\ F_{M1} \dots F_{MM} \end{pmatrix}; \qquad E(t,\varepsilon) = \begin{pmatrix} E_1 & 0 & \dots & 0 \\ 0 & E_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & E_M \end{pmatrix},$$

and $F_{ij} \sim \varepsilon^{\beta_{ij}} \sum_{k=0}^{\infty} F_{ijk}(t)\,\varepsilon^k$. Here $\beta_{ij} > 0$ if $i \neq j$; $\beta_{ii} = 0$; and $F_{ii0}(t)$ is non-singular when $a \leqq t \leqq b$. Also $E_i = I_i \exp\left\{\varepsilon^{-\beta} \int_a^t \lambda_{\tau_i}(\sigma,\varepsilon)\,d\sigma\right\}$, where I_i is an identity matrix, and λ_{τ_i} are the distinct characteristic polynomials.

Thus a transformation of the type

$$\text{(2.8)} \qquad y(t,\varepsilon) = \begin{pmatrix} I_1 & 0 & \dots & 0 \\ 0 & H_2(t,\varepsilon) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & H_p(t,\varepsilon) \end{pmatrix} z(t,\varepsilon)$$

will change (2.2) into the form

$$\text{(2.9)} \qquad \begin{aligned} \frac{dz_1}{dt} &= C_{11}(t,\varepsilon)\,z_1 + \{C_{12}(t,\varepsilon)\,H_2(t,\varepsilon)\}\,z_2 + \cdots + \{C_{1p}(t,\varepsilon)\,H_p(t,\varepsilon)\}\,z_p \\ \varepsilon^{h_2}\frac{dz_2}{dt} &= \{H_2^{-1}C_{21}\}\,z_1 + H_2^{-1}\left\{C_{22}H_2 - \varepsilon^{h_2}\frac{d}{dt}H_2\right\}z_2 + \cdots + \{H_2^{-1}C_{2p}H_p\}\,z_p \\ &\;\;\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ \varepsilon^{h_p}\frac{dz_p}{dt} &= \{H_p^{-1}C_{p1}\}\,z_1 + \qquad \cdots \qquad + H_p^{-1}\left\{C_{pp}H_p - \varepsilon^{h_p}\frac{d}{dt}H_p\right\}z_p. \end{aligned}$$

Thus, if TURRITTIN's results apply to the individual equations,

$$\text{(2.10)} \qquad \varepsilon^{h_j}\frac{d}{dt}y_j(t,\varepsilon) = C_{jj}(t,\varepsilon)\,y_j(t,\varepsilon), \qquad j = 2, \dots, p,$$

as we shall assume, the transformations $H_j(t,\varepsilon)$ may be chosen so that

$$H_j^{-1}(t,\varepsilon)\left\{C_{jj}(t,\varepsilon)\,H_j(t,\varepsilon) - \varepsilon^{h_j}\frac{d}{dt}H_j(t,\varepsilon)\right\}$$

has the canonical form shown in (2.6). Also in transformation (2.1) L may be chosen such that the elements of $H_j^{-1}(t,\varepsilon)\,C_{ji}(t,\varepsilon)\,H_i(t,\varepsilon)$ are $O(\varepsilon^{h_i})$ as $\varepsilon \to 0^+$. We may assemble all the successive transformations that have been made into a single transformation $y(t,\varepsilon) = H(t,\varepsilon)\,z(t,\varepsilon)$, where

$$\text{(2.11)} \qquad H(t,\varepsilon) = \sum_{j=0}^{J} H_j(t)\,\varepsilon^j, \qquad J \text{ a suitable large integer,}$$

in such a manner that this transformation will change (1.1) into

$$\begin{aligned} \frac{dz_1}{dt} &= D_{11}(t,\varepsilon)\, z_1 + D_{12}(t,\varepsilon)\, z_2 \\ \varepsilon^h \frac{dz_2}{dt} &= D_{21}(t,\varepsilon)\, z_1 + D_{22}(t,\varepsilon)\, z_2, \end{aligned} \tag{2.12}$$

where

$$D_{12}(t,\varepsilon) = O(\varepsilon), \quad D_{21}(t,\varepsilon) = O(\varepsilon^h), \quad D_{22}(t,\varepsilon) = \big(\delta_{ij}\,\lambda_j(t,\varepsilon)\big) + \varepsilon^h \bar{D}_{22},$$

$\bar{D}_{22}(t,\varepsilon) = O(1)$, and $\lambda_j(t,\varepsilon)$, $j=1,\ldots,m$, are the non-vanishing characteristic polynomials associated with the differential systems (2.10); and where moreover by hypothesis

$$\lambda_1 \equiv \lambda_2 \equiv \cdots \lambda_{\tau_1} \not\equiv \lambda_{\tau_1+1} \equiv \cdots \equiv \lambda_{\tau_2} \not\equiv \cdots \lambda_{\tau_{\gamma-1}} \not\equiv \lambda_{\tau_{\gamma-1}+1} \equiv \cdots \equiv \lambda_{\tau_\gamma}.$$

It is advantageous to make one more transformation on the system (2.12), namely

$$z_1 = P_1(t)\, u_1(t,\varepsilon), \qquad z_2 = \begin{pmatrix} P_{21} & 0 & \ldots & 0 \\ 0 & P_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & P_{2\gamma} \end{pmatrix} u_2(t,\varepsilon) = P_2(t)\, u_2, \tag{2.13}$$

where $P_{2j}(t)$ is a square matrix whose order is the multiplicity of the characteristic polynomial $\lambda_{\tau_j}(t,\varepsilon)$. Thus (2.12) becomes

$$\begin{aligned} \frac{du_1}{dt} &= P_1^{-1}(t)\left\{D_{11}(t,\varepsilon)\, P_1(t) - \frac{d}{dt} P_1(t)\right\} u_1 + \left\{P_1^{-1}(t)\, D_{12}(t,\varepsilon)\, P_2(t)\right\} u_2 \\ \varepsilon^h \frac{du_2}{dt} &= \left\{P_2^{-1}(t)\, D_{21}(t,\varepsilon)\, P_1(t)\right\} u_1 + P_2^{-1}(t)\left\{D_{22}(t,\varepsilon)\, P_2(t) - \varepsilon^h \frac{d}{dt} P_2(t)\right\} u_2. \end{aligned}$$

Choose $P_1(t)$ such that $\frac{d}{dt} P_1(t) = D_{11}(t,0)\, P_1(t)$, $P(a) = I_1$, and $P_{2j}(t)$ such that $\frac{d}{dt} P_{2j}(t) = \big(\bar{D}_{22}(t,0)\big)_{jj} P_{2j}(t)$, $P_{2j}(a) = I_j$, where $\big(\bar{D}_{22}(t,0)\big)_{ij}$ is the ij^{th} block of the matrix $\bar{D}_{22}$ partitioned in the same manner as P_2.

Thus the transformation

$$y(t,\varepsilon) = H(t,\varepsilon)\begin{pmatrix} P_1(t) & 0 \\ 0 & P_2(t) \end{pmatrix} u(t,\varepsilon) \tag{2.14}$$

will change (1.1) into the *canonical differential system*

$$\begin{aligned} \frac{d}{dt} u_1(t,\varepsilon) &= B_{11}(t,\varepsilon)\, u_1(t,\varepsilon) + B_{12}(t,\varepsilon)\, u_2(t,\varepsilon) \\ \varepsilon^h \frac{d}{dt} u_2(t,\varepsilon) &= B_{21}(t,\varepsilon)\, u_1(t,\varepsilon) + B_{22}(t,\varepsilon)\, u_2(t,\varepsilon) \end{aligned} \tag{2.15}$$

with a related *canonical degenerate differential system*

$$\begin{aligned} \frac{du_1}{dt} &= 0 \\ 0 &= \big(\delta_{ij}\,\alpha_j(t)\big)\, u_2 \end{aligned} \tag{2.16}$$

where $\alpha_i(t)$ are the characteristic roots of the matrices $C_{ii}(t,0)$ of the equation (2.2), all of which were assumed to be non-zero on $a \leqq t \leqq b$, which implies in turn that the solution of the canonical degenerate differential system is $u_1 = c_1$, $u_2 = 0$, where c_1 is an arbitrary constant vector of appropriate order.

Further, this conversion of the differential system (1.1) into (2.15) is such that a fundamental matrix solution $W(t, \varepsilon)$ for the system (2.15), when TURRITTIN'S results apply, has the form

$$W(t, \varepsilon) = ([I])\, E(t, \varepsilon)\,\star. \tag{2.17}$$

Since we are treating the boundary problem as a whole, the transformation (2.14) will affect the boundary form (1.2) as well. This effect will be considered in more detail in Section 4.

3. The Canonical Problem

We make the following hypothesis.

H 1: (i) The matrices $A_{ij}(t, \varepsilon)$ indicated in (1.1) *have asymptotic expansions of appropriate high finite orders.*

(ii) The matrices $A_{pp}(t, 0)$, $A_{p-1,p-1}(t, 0) - A_{p-1,p}(t, 0)\, A_{pp}^{-1}(t, 0)\, A_{p,p-1}(t, 0)$ and similar matrices referred to in Section 2 are non-singular on the interval $a \leqq t \leqq b$.

(iii) There exists a non-singular transformation $x(t, \varepsilon) = \overline{H}(t, \varepsilon)\, u(t, \varepsilon)$, where $\overline{H}(t, \varepsilon) = H_0(t) + \varepsilon H_1(t) + \cdots \varepsilon^J H_J(t)$, $a \leqq t \leqq b$, $0 \leqq \varepsilon \leqq \varepsilon_0$, which will convert (1.1) *into the canonical form* (2.15).

(iv) The m non-vanishing characteristic polynomials $\lambda_j(t, \varepsilon)$ satisfy

$$\mathrm{Re}\{\varepsilon^{-h} \lambda_1(t, \varepsilon)\} \leqq Re\{\varepsilon^{-h} \lambda_2(t, \varepsilon)\} \leqq \cdots \leqq \mathrm{Re}\{\varepsilon^{-h} \lambda_m(t, \varepsilon)\},$$
$$a \leqq t \leqq b, \quad 0 < \varepsilon \leqq \varepsilon_0.$$

If $X(t, \varepsilon)$ is any fundamental matrix for a system of differential equations of the form $\frac{dx}{dt} = A(t, \varepsilon)\, x(t, \varepsilon)$, $\varepsilon > 0$, $a \leqq t \leqq b$, then any particular vector solution $l(t, \varepsilon)$ must be of the form $l(t, \varepsilon) = X(t, \varepsilon)\, l(\varepsilon)$. Thus, if $l(t, \varepsilon)$ is to satisfy the boundary conditions $R(\varepsilon)\, l(a, \varepsilon) + S(\varepsilon)\, l(b, \varepsilon) = c(\varepsilon)$, we must have $\{R(\varepsilon)\, X(a, \varepsilon) + S(\varepsilon)\, X(b, \varepsilon)\}\, l(\varepsilon) = c(\varepsilon)$. Thus, if $\Delta(\varepsilon) = \{R(\varepsilon)\, X(a, \varepsilon) + S(\varepsilon)\, X(b, \varepsilon)\}$, $l(t, \varepsilon)$ will be unique if $\Delta(\varepsilon)$ is non-singular, for in this event

$$l(t, \varepsilon) = X(t, \varepsilon)\, \Delta^{-1}(\varepsilon)\, c(\varepsilon). \tag{3.1}$$

The limit problem is then the computation of the

$$\lim_{\varepsilon \to 0^+} l(t, \varepsilon) = \lim_{\varepsilon \to 0^+} X(t, \varepsilon)\, \Delta^{-1}(\varepsilon)\, c(\varepsilon).$$

To evaluate this limit we need more detailed information about the structure of $\Delta^{-1}(\varepsilon)$.

4. $\Delta^{-1}(\varepsilon)$ for a Canonical Problem

We assume that we are dealing with the canonical differential system (2.15) which has been obtained from (1.1) by the transformation $x(t, \varepsilon) = \overline{H}(t, \varepsilon)\, u(t, \varepsilon)$ of H 1—(iii). This transformation will change the boundary conditions from (1.2) into

$$M(\varepsilon)\, u(a, \varepsilon) + N(\varepsilon)\, u(b, \varepsilon) = c(\varepsilon) \tag{4.1}$$

$\star$ $[\varphi(t)]$ represents a function $\varphi(t, \varepsilon) = \varphi(t) + \varphi_1(t, \varepsilon)\, \varepsilon^{\gamma}$, $\gamma > 0$, $|\varphi_1(t, \varepsilon)| < B$.

where

$$M(\varepsilon) = R(\varepsilon)\,\overline{H}(a,\varepsilon) \quad \text{and} \quad N(\varepsilon) = S(\varepsilon)\,\overline{H}(b,\varepsilon).$$

We make the following hypothesis.

H 2: (i) $R(\varepsilon) = R_0 + \varepsilon R_1(\varepsilon)$, $S(\varepsilon) = S_0 + \varepsilon S_1(\varepsilon)$ *where the elements of* $R_1(\varepsilon)$ *and* $S_1(\varepsilon)$ *are bounded for* $0 \leqq \varepsilon \leqq \varepsilon_0$ *and the rank of* $(R(\varepsilon) : S(\varepsilon)) = n_1 + m$ *for* $0 \leqq \varepsilon \leqq \varepsilon_0$.

(ii) The non-vanishing characteristic polynomials have non-zero real parts, i.e.,

$$\operatorname{Re}\{\varepsilon^{-h}\lambda_1(t,\varepsilon)\} \leqq \cdots \leqq \operatorname{Re}\{\varepsilon^{-h}\lambda_k(t,\varepsilon)\} < 0 < \operatorname{Re}\{\varepsilon^{-h}\lambda_{k+1}(t,\varepsilon)\} \leqq \cdots \leqq \operatorname{Re}\{\varepsilon^{-h}\lambda_m\}$$
$$\text{on} \quad a \leqq t \leqq b, \qquad 0 < \varepsilon \leqq \varepsilon_0.$$

If we choose for the fundamental matrix the one indicated in (2.17), we have

$$\Delta(\varepsilon) = \left\{ M(\varepsilon)\,([I]) + N(\varepsilon)\,([I]) \begin{pmatrix} I_{n_1} & 0 \\ 0 & E_m(b,\varepsilon) \end{pmatrix} \right\}.$$

If $D(\varepsilon) = \det \Delta(\varepsilon) \neq 0$ $0 < \varepsilon \leqq \varepsilon_0$, $\Delta(\varepsilon)$ will be non-singular. We have

$$D(\varepsilon) \sim \Sigma_\alpha\, a_\alpha(\varepsilon)\, e^{\omega_\alpha(\varepsilon)},$$

where

(i) α covers some finite range,

(ii) $\omega_\alpha(\varepsilon)$ are distinct quantities, each of which is of the form $\sum_{j=I}^{J} \varrho_{k_j}(b,\varepsilon)$, where $I, J = 0, 1, \ldots, m$; $J \geqq I$ and $(k_0, k_1, \ldots, k_m)$ is any permutation of $(0, 1, \ldots, m)$; $\varrho_0(b,\varepsilon) \equiv 0$, $\varrho_j(b,\varepsilon) = \varepsilon^{-h} \int_a^b \lambda_j(\sigma,\varepsilon)\, d\sigma$, $j \geqq 1$,

(iii) the coefficient functions $a_\alpha(\varepsilon) \not\equiv 0$, $a_\alpha(\varepsilon) = O(1)$ as $\varepsilon \to 0^+$.

For a discussion of the zeros of such exponential sums, see TURRITTIN [*10*].

All terms indicated in (ii) need not be present; however, one is of particular interest, namely the term $a(\varepsilon)\, e^{\omega(\varepsilon)}$ where $\omega(\varepsilon) = \sum_{j=k+1}^{m} \varrho_j(b,\varepsilon)$. (We note that, if $k = m$, then $\omega(\varepsilon) \equiv 0$.) An explicit expression for the leading term of the coefficient function $a(\varepsilon)$ can be given as follows. Let the columns of $R(0)\,\overline{H}(a,\varepsilon)$ and $S(0)\,\overline{H}(b,\varepsilon)$ be α_{i1}, α_{i2} respectively, *i.e.*, $R(0)\,\overline{H}(a,\varepsilon) = (\alpha_{11} : \cdots : \alpha_{n_1+m,1})$ $S(0)\,\overline{H}(b,\varepsilon) = (\alpha_{12} : \cdots \alpha_{n_1+m,2})$ and let $\Omega(\varepsilon)$ be the $n_1 + m^{\text{th}}$ order square matrix

$$\begin{aligned} \Omega = (\alpha_{11} + \alpha_{12} : \cdots : \alpha_{n_1,1} + \alpha_{n_1,2} : \alpha_{n_1+1,1} : \alpha_{n_1+2,1} : \cdots \\ \cdots : \alpha_{n_1+k,1} : \alpha_{n_1+k+1,2} : \cdots : \alpha_{n_1+m,2}). \end{aligned} \tag{4.2}$$

The leading term of $a(\varepsilon)$ is the determinant of $\Omega(\varepsilon)$. In general for ε sufficiently small it can be shown that if $\Omega(\varepsilon)$ is non-singular (see for instance HARRIS [*5*], page 88),

$$\Delta(\varepsilon) = ([\Omega]) \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & E_{m-k}(b,\varepsilon) \end{pmatrix},$$

where

$$E_{m-k}(b,\varepsilon)=\left(\delta_{ij}\exp\left\{\varepsilon^{-h}\int\limits_a^b\lambda_{k+j}(t,\varepsilon)\,dt\right\}\right)$$
$$=\left(\delta_{ij}\exp\{\varrho_{k+j}(b,\varepsilon)\}\right),\qquad j=1,2,\ldots,m-k,$$

and hence

$$\Delta^{-1}(\varepsilon)=\begin{pmatrix} I_{n_1} & 0 & 0\\ 0 & I_k & 0\\ 0 & 0 & E_{m-k}^{-1}(b,\varepsilon)\end{pmatrix}\left([\Omega^{-1}(\varepsilon)]\right). \tag{4.3}$$

It will now be shown that the computation of the limit of (3.1) as $\varepsilon\to0^+$, *i.e.*, the evaluation of $\lim\limits_{\varepsilon\to0^+}X(t,\varepsilon)\,\Delta^{-1}(\varepsilon)\,c(\varepsilon)$, is essentially the question of the limit as $\varepsilon\to0^+$ of the first n_1 rows of the matrix $\Omega^{-1}(\varepsilon)$. To establish the nature of the elements in the first n_1 rows of $\Omega^{-1}(\varepsilon)$ we must examine the transformation $\overline{H}(t,\varepsilon)$ of Hypothesis H 1—(iii). $\overline{H}(t,\varepsilon)$ is a product of four transformations, (2.1), (2.8), (2.13) and $\begin{pmatrix} I & 0\\ 0 & P_0\end{pmatrix}$, where P_0 is a constant matrix which renumbered the last m unknowns so that the characteristic polynomials are ordered by their real parts as shown in H 1—(iv).

$$\overline{H}(t,\varepsilon)=\left\{\prod_{j=1}^{L}\left(T_j(t)+\varepsilon^j Q_j(t)\right)\right\}\begin{pmatrix} I & 0 & \ldots & 0\\ 0 & H_2 & & \vdots\\ \vdots & & \ddots & \\ 0 & \ldots & & H_p\end{pmatrix}\begin{pmatrix} P_1 & 0\\ 0 & P_2\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & P_0\end{pmatrix},$$

where the multiplication indicated by $\prod$ is a polynomial in ε with a non-singular matrix for coefficient of ε^0. This transformation does not change the essential character of the boundary form $R(\varepsilon)\,x(a,\varepsilon)+S(\varepsilon)\,x(b,\varepsilon)=c(\varepsilon)$, since

$$\overline{R}(\varepsilon)=R(\varepsilon)\Big(\prod_{j=1}^{L}T_j(a)+\varepsilon^j Q_j(a)\Big)=\left([R_0\,T(a)]\right)=\left([\overline{R}_0]\right),$$
$$\overline{S}(\varepsilon)=S(\varepsilon)\Big(\prod_{j=1}^{L}T_j(b)+\varepsilon^j Q_j(b)\Big)=\left([S_0\,T(b)]\right)=\left([\overline{S}_0]\right),$$

where

$$T(t)=T_1\,T_2\ldots T_L.$$

Let us consider the effect of the next transformation on $\overline{R}(\varepsilon)$. Partition $\overline{R}(\varepsilon)$ into blocks of columns containing $n_1,n_2,\ldots,n_p$, columns respectively; *i.e.*, $\overline{R}(\varepsilon)=(\overline{R}_1\vdots\overline{R}_2\vdots\cdots\vdots\overline{R}_p)$. Thus

$$\overline{R}(\varepsilon)\begin{pmatrix} I & 0 & \ldots & 0\\ 0 & H_2 & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \ldots & 0 & H_p\end{pmatrix}=\left(\overline{R}_1\vdots\overline{R}_2\,H_2(a,\varepsilon)\vdots\cdots\vdots\overline{R}_p\,H_p(a,\varepsilon)\right),$$

and the matrix $H_i(a,\varepsilon)$ affects only the matrix $\overline{R}_i(\varepsilon)$. Let us consider one such product $\overline{R}(\varepsilon)\,H(a,\varepsilon)$, momentarily dropping the subscript i. $H(a,\varepsilon)$ is a finite product of two types of transformations $\mathscr{L}_j(a,\varepsilon)$ and $\mathscr{M}_j(\varepsilon)$; more precisely (see

TURRITTIN [9], page 96),

$$H(t,\varepsilon) = \mathscr{L}_1(t,\varepsilon)\,\mathscr{M}_1(\varepsilon)\,\mathscr{L}_2(t,\varepsilon)\,\mathscr{M}_2(\varepsilon)\ldots\mathscr{L}_\beta(t,\varepsilon)\,\mathscr{M}_\beta(\varepsilon),$$

$$\mathscr{L}_j(t,\varepsilon) = L_{j0}(t) + \varepsilon L_{j1}(t) + \cdots \varepsilon^{\beta_j} L_{j\beta_j}(t), \quad L_{j0}(t) \text{ non-singular on } a \leqq t \leqq b,$$

and

$$\mathscr{M}_j(\varepsilon) = (\delta_{kl}\,\varepsilon^{(k_{j_2}-k)\mu_{jk}}),$$

where n is the order of $H(t,\varepsilon)$, k_{j_1} and k_{j_2} positive integers, and

$$\mu_{jk} = \begin{cases} 1, & \text{if } 0 < k_{j_1} \leqq k \leqq k_{j_2} \leqq n \\ 0, & \text{otherwise.} \end{cases}$$

For example, a typical $\mathscr{M}(\varepsilon)$ would be

$$\begin{pmatrix} 1 & 0 & & & \cdots & & & & 0 \\ 0 & \ddots & & & & & & & \\ & & 1 & 0 & & & & & \\ \cdot & & 0 & \varepsilon^l & & & & & \\ & & & & \varepsilon^{l-1} & & & & \\ \cdot & & & & & \ddots & & & \\ & & & & & & \varepsilon & 0 & \\ \cdot & & & & & & 0 & 1 & \\ & & & & & & & \ddots & 0 \\ 0 & & & & \cdots & & & 0 & 1 \end{pmatrix}.$$

Consider $\bar{R}(\varepsilon)\,\mathscr{L}_1(a,\varepsilon)$. Since $\mathscr{L}_1(a,\varepsilon)$ and $\bar{R}(\varepsilon)$ have the same essential features, so does $\bar{R}(\varepsilon)\,\mathscr{L}_1(a,\varepsilon)$, *i.e.* $\bar{R}(\varepsilon)\,\mathscr{L}_1(a,\varepsilon) = ([\bar{R}(0)\,L_{10}(a)])$. Multiplication of $\bar{R}(\varepsilon)\,\mathscr{L}_1(a,\varepsilon)$ on the right by $\mathscr{M}_1(\varepsilon)$ multiplies some columns at the beginning and end by unity and every other column by a different power of ε. For definiteness, let the highest power of ε be l. Multiplication of $\bar{R}(\varepsilon)\,\mathscr{L}_1(a,\varepsilon)\,\mathscr{M}_1(\varepsilon)$ on the right by $\mathscr{L}_a(a,\varepsilon)$ replaces each column by a linear combination of the columns of $\bar{R}\,\mathscr{L}_1\,\mathscr{M}_1$, and each column of $\bar{R}\,\mathscr{L}_1\,\mathscr{M}_1$ has ε^0 for the lowest possible power of ε in the columns. Thus, if n_1+m-n columns of constants were adjoined to the matrix $\bar{R}(\varepsilon)\,\mathscr{L}_1(a,\varepsilon)\,\mathscr{M}_1(\varepsilon)\,\mathscr{L}_2(a,\varepsilon)$ and the determinant of this augmented matrix were evaluated, the lowest power of ε that could occur in the expansion of this determinant would be $\varepsilon^{\frac{l(l+1)}{2}}$ and *not* ε^0 due to certain column dependence of the matrix $\bar{R}(\varepsilon)\,\mathscr{L}_1(a,\varepsilon)\,\mathscr{M}_1(\varepsilon)\,\mathscr{L}_2(a,\varepsilon)$. Multiplication by the remaining $\mathscr{L}$ and $\mathscr{M}$ transformations and a similar argument gives rise to the following statement. Let

$$\det H_i(t,\varepsilon) = \varepsilon^{q_i}[h_i(t)], \qquad h_i(t) \neq 0, \quad a \leqq t \leqq b. \tag{4.4}$$

Any matrix with n_1+m-n_i columns of constants adjoined to the n_i columns of $\bar{R}_i(\varepsilon)\,H_i(a,\varepsilon)$, $i=2,\ldots,p$, will have a determinant which, when expanded in powers of ε, will begin with the term in ε^{q_i} or possibly some higher power of ε. Similar results apply to the matrix $\bar{S}_i(\varepsilon)\,H_i(b,\varepsilon)$, $i=2,\ldots,p$. Further, if we select any k_i columns from $\bar{R}_i(\varepsilon)\,H_i(a,\varepsilon)$, say $a_1, a_2, \ldots, a_{k_i}$, and n_i-k_i columns from $\overline{S}_i(\varepsilon)\,H_i(b,\varepsilon)$, say $a_{k_i+1},\ldots,a_{n_i}$, such that $(a_i,\ldots,a_{n_i})$ is any permutation of $(1,2,\ldots,n_i)$ and adjoin n_1+m-n_i columns of constants, the resulting matrix often has a determinant for which the lowest order of ε that occurs is ε^{q_i}. The last two transformations are of the type already considered and do not change the previous statement.

Let us partition M, N, and Ω in the following manner:

$$M(\varepsilon)=\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix},\qquad N(\varepsilon)=\begin{pmatrix}N_{11} & N_{12}\\ N_{21} & N_{22}\end{pmatrix},\qquad \Omega=\begin{pmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{pmatrix},$$

where M_{11}, N_{11} and Ω_{11} have order n_1 and M_{22}, N_{22}, Ω_{22} have order m. We see that

$$\begin{aligned}M_{11}(0)+N_{11}(0)&=\Omega_{11}(0)\\ M_{21}(0)+N_{21}(0)&=\Omega_{21}(0),\end{aligned}\tag{4.5}$$

i.e., the elements in the first n_1 columns of Ω are $O(1)$ as $\varepsilon\to 0^+$. Thus, often det $\Omega(\varepsilon)=O(\varepsilon^q)$ where $q=\sum_2^p q_i$, (see 4.4), and the cofactor of any element in the first n_1 columns of Ω is $O(\varepsilon^q)$.

We make the hypothesis,

H 3: The matrix $\Omega(\varepsilon)$ as given in (4.2) *satisfies the conditions:*

(i) $\Omega(\varepsilon)$ is non-singular, $0<\varepsilon\leqq\varepsilon_0$,

(ii) the elements in the first n_1 rows of $\Omega^{-1}(\varepsilon)$ are $O(1)$ as $\varepsilon\to 0^+$.

Hypothesis H 3—(i) assures us that $\Delta^{-1}(\varepsilon)$ has the form shown in (4.3).

5. Evaluation of $\lim\limits_{\varepsilon\to 0^+} X(t,\varepsilon)\,\Delta^{-1}(\varepsilon)\,c(\varepsilon)$

Combining (2.17) and (4.3), we have the following representation of the unique solution of (2.15) and (4.1):

$$\begin{aligned}l(t,\varepsilon)&=X(t,\varepsilon)\,\Delta^{-1}(\varepsilon)\,c(\varepsilon)\\ &=([I])\begin{pmatrix}I_{n_1} & 0 & 0\\ 0 & E_k(t,\varepsilon) & 0\\ 0 & 0 & E_{m-k}(t,\varepsilon)\end{pmatrix}\begin{pmatrix}I_{n_1} & 0 & 0\\ 0 & I_k & 0\\ 0 & 0 & E_{m-k}^{-1}(b,\varepsilon)\end{pmatrix}([\Omega^{-1}(\varepsilon)])\,c(\varepsilon)\\ &=([I])\begin{pmatrix}I_{n_1} & 0 & 0\\ 0 & E_k(t,\varepsilon) & 0\\ 0 & 0 & E_{m-k}(t,\varepsilon)\,E_{m-k}^{-1}(b,\varepsilon)\end{pmatrix}([\Omega^{-1}(\varepsilon)])\,c(\varepsilon).\end{aligned}$$

We have

$$E_k(t,\varepsilon)=\left(\delta_{ij}\exp\left\{\varepsilon^{-h}\int_a^t\lambda_i(\sigma,\varepsilon)\,d\sigma\right\}\right),\qquad i=1,2,\ldots,k,$$

and

$$E_{m-k}(t,\varepsilon)\,E_{m-k}^{-1}(b,\varepsilon)=\left(\delta_{ij}\exp\left\{-\varepsilon^{-h}\int_t^b\lambda_{i+k}(\sigma,\varepsilon)\,d\sigma\right\}\right),\qquad i=1,2,\ldots,m-k,$$

and hence*

$$E_k(t,\varepsilon) \to O_k, \qquad a < \delta_1 \leqq t \leqq b,$$

$$E_{m-k}(t,\varepsilon)\, E_{m-k}^{-1}(b,\varepsilon) \to O_{m-k}, \qquad a \leqq t \leqq \delta_2 < b,$$

exponentially fast due to H 2—(ii), and uniformly in t for the indicated intervals. Thus

$$(5.1)\quad \lim_{\varepsilon\to 0^+} X(t,\varepsilon)\,\Delta^{-1}(\varepsilon)\,c(\varepsilon) = \begin{pmatrix} \bar{\Omega}_{11}(0) & \bar{\Omega}_{12}(0) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} = \begin{pmatrix} l_1 \\ 0 \end{pmatrix} \quad a < \delta_1 \leqq t \leqq \delta_2 < b,$$

where the vector $l_1 = \bar{\Omega}_{11}(0)\, c_1(0) + \bar{\Omega}_{12}(0)\, c_2(0)$, and

$$\Omega^{-1}(\varepsilon) = \begin{pmatrix} \bar{\Omega}_{11}(\varepsilon) & \bar{\Omega}_{12}(\varepsilon) \\ \bar{\Omega}_{21}(\varepsilon) & \bar{\Omega}_{22}(\varepsilon) \end{pmatrix}, \qquad \Omega_{11}(\varepsilon) \text{ of order } n_1.$$

(We note that, if $k=m$, b may be included, and if $k=0$, a may be included.)

It is clear that the limiting constant vector l

$$(5.2)\qquad l = \lim_{\substack{\varepsilon\to 0^+ \\ a<t<b}} l(t,\varepsilon) = \begin{pmatrix} \bar{\Omega}_{11}(0)\, c_1(0) + \bar{\Omega}_{12}(0)\, c_2(0) \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 \\ 0 \end{pmatrix}$$

is defined for the interval $a \leqq t \leqq b$ and as a function of t is a solution of the degenerate differential system (2.16). We shall now show that this limiting solution l satisfies n_1 degenerate boundary conditions.

6. Boundary Conditions Satisfied by the Limiting Solution

Multiplication of the boundary form (4.1) on the left by any non-singular matrix of constants will give rise to an equivalent boundary form. We have partitioned $\Omega^{-1}(\varepsilon)$ and $\Omega(\varepsilon)$ as follows:

$$\Omega^{-1}(\varepsilon) = \begin{pmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} \\ \bar{\Omega}_{21} & \bar{\Omega}_{21} \end{pmatrix}, \qquad \Omega(\varepsilon) = \begin{pmatrix} \Omega_{11} & \Omega_{21} \\ \Omega_{21} & \Omega_{22} \end{pmatrix};$$

hence

$$\bar{\Omega}_{11}(\varepsilon)\,\Omega_{11}(\varepsilon) + \bar{\Omega}_{12}(\varepsilon)\,\Omega_{21}(\varepsilon) = I_{n_1},$$

and the nature of $\Omega_{11}(0)$, $\Omega_{21}(0)$, and H 3—(ii) imply

$$\bar{\Omega}_{11}(0)\,\Omega_{11}(0) + \bar{\Omega}_{12}(0)\,\Omega_{21}(0) = I_{n_1},$$

and the matrix $(\bar{\Omega}_{11}(0) : \bar{\Omega}_{12}(0))$ has rank n_1. Using (4.5), we have

$$(6.1)\qquad \bar{\Omega}_{11}(0)\,\big(M_{11}(0) + N_{11}(0)\big) + \bar{\Omega}_{12}(0)\,\big(M_{21}(0) + N_{21}(0)\big) = I_{n_1}.$$

Further, there exists constant matrices F_{21} and F_{22} such that the matrix

$$(6.2)\qquad F = \begin{pmatrix} \bar{\Omega}_{11}(0) & \bar{\Omega}_{12}(0) \\ F_{21} & F_{22} \end{pmatrix}$$

is non-singular.

* The symbol O_k here represents the zero square matrix of order k.

Let us replace (4.1) by the equivalent boundary form

$$\overline{M}(\varepsilon)\, u(a, \varepsilon) + \overline{N}(\varepsilon)\, u(b, \varepsilon) = \bar{c}(\varepsilon), \tag{6.3}$$

where $\overline{M}(\varepsilon) = FM(\varepsilon)$, $\overline{N}(\varepsilon) = FN(\varepsilon)$, and $\bar{c}(\varepsilon) = Fc(\varepsilon)$. The degenerate boundary form corresponding to (6.3) is

$$\overline{M}(0)\, u(a) + \overline{N}(0)\, u(b) = \bar{c}(0), \tag{6.4}$$

where

$$\overline{M}(0) = \begin{pmatrix} \overline{M}_{11}(0) & \overline{M}_{12}(0) \\ \overline{M}_{21}(0) & \overline{M}_{22}(0) \end{pmatrix}, \quad \text{and} \quad \overline{N}(0) = \begin{pmatrix} \overline{N}_{11}(0) & \overline{N}_{12}(0) \\ \overline{N}_{21}(0) & \overline{N}_{22}(0) \end{pmatrix}.$$

Substituting the limiting solution l into the left-hand side of (6.4), we get

$$\overline{M}(0) \begin{pmatrix} l_1 \\ 0 \end{pmatrix} + \overline{N}(0) \begin{pmatrix} l_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \{\overline{M}_{11}(0) + \overline{N}_{11}(0)\}\, l_1 \\ \{\overline{M}_{21}(0) + \overline{N}_{21}(0)\}\, l_1 \end{pmatrix}.$$

By direct computation we have

$$\overline{M}_{11}(0) = \overline{\Omega}_{11}(0)\, M_{11}(0) + \overline{\Omega}_{12}(0)\, M_{21}(0),$$

$$\overline{N}_{11}(0) = \overline{\Omega}_{11}(0)\, N_{11}(0) + \overline{\Omega}_{12}(0)\, N_{21}(0),$$

and hence by (6.1)

$$\overline{M}_{11}(0) + \overline{N}_{11}(0) = I_{n_1}.$$

Further,

$$\bar{c}(0) = \begin{pmatrix} \bar{c}_1(0) \\ \bar{c}_2(0) \end{pmatrix} = \begin{pmatrix} \overline{\Omega}_{11}(0) & \overline{\Omega}_{12}(0) \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix},$$

so

$$\bar{c}_1(0) = \overline{\Omega}_{11}(0)\, c_1(0) + \overline{\Omega}_{12}(0)\, c_2(0) = l_1.$$

Thus, the limiting solution l satisfies the first n_1 degenerate boundary conditions of (6.4) corresponding to the boundary form (6.3).

Without loss of generality we may assume that the boundary form (1.2) has been replaced by the equivalent one obtained by multiplication on the left by F as given in (6.2). Further, the solution of the canonical problem will provide the solution of the original problem through the transformation (2.14).

$$y(t, \varepsilon) = H(t, \varepsilon) \begin{pmatrix} P_1(t) & 0 \\ 0 & P_2(t)\, P_0 \end{pmatrix} l(t, \varepsilon). \tag{6.5}$$

The transformation $H(t, \varepsilon)$ was defined for $a \leqq t \leqq b$, $0 \leqq \varepsilon \leqq \varepsilon_0$, and $\lim_{\varepsilon \to 0^+} H(t, \varepsilon) = H(t, 0)$ exists, $a \leqq t \leqq b$. Thus the limiting solution for the problem (1.1), (1.2) will be

$$y(t) = H(t, 0) \begin{pmatrix} P_1(t) & 0 \\ 0 & P_2(t)\, P_0 \end{pmatrix} l, \qquad a < t < b. \tag{6.6}$$

Theorem 1. *Under hypotheses H 1, H 2, H 3, the two-point boundary problem* (1.1), (1.2) *has a unique solution* $y(t, \varepsilon)$ *on* $a \leqq t \leqq b$, $0 < \varepsilon \leqq \varepsilon_0$, *such that the* $\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = y(t)$ *exists on the open interval* $a < t < b$, *and uniformly on any closed* *sub-interval* $a \leqq \delta_1 \leqq t \leqq \delta_2 < b$. *The function* $y(t)$ *satisfies the degenerate differential system* (1.3). *The limits,* $y(a+0)$ *and* $y(b-0)$ *exist and satisfy the first* n_1 *boundary conditions of the degenerate boundary form* (1.4).

7. Remarks

(1) H 1 is a condition of regularity imposed on the differential system to ensure that solutions to the degenerate differential system exist and that asymptotic solutions exist for small ε and t in the interval $a \leqq t \leqq b$. H 2 is a regularity condition on the boundary form and a restriction on the characteristic polynomials $\lambda_j(t, \varepsilon)$ which allows us to determine the essential character of the matrix $\Delta(\varepsilon)$. H 3 defines a "regular" problem and assures the existence of the limiting solution. H 3 could be replaced by the assumption that the coefficient function $a(\varepsilon)$ for which $\det \Omega(\varepsilon)$ is the leading term is $\neq 0$. $a(\varepsilon)$ is the determinant of some matrix $\mathscr{A}(\varepsilon)$. If $\mathscr{A}(\varepsilon)$ is non-singular, $0 < \varepsilon \leqq \varepsilon_0$, $\Delta(\varepsilon)$ has essentially the same form as before, and the limiting solution will exist only if the first n_1 rows of $\mathscr{A}^{-1}(\varepsilon)$ are $O(1)$ as $\varepsilon \to 0^+$.

Elementary considerations show that Ω is singular for the following example, but the limiting solution exists, satisfies the degenerate differential system and the first boundary condition:

$$\left\{ \begin{aligned} \frac{dx_1}{dt} &= x_2(t, \varepsilon) \\ \varepsilon \frac{dx_2}{dt} &= \varepsilon x_1(t, \varepsilon) + x_2(t, \varepsilon) \end{aligned} \right\}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(0, \varepsilon) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(1, 0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Here we have

$$\Omega = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad a(\varepsilon) = \varepsilon[1],$$

$$\lim_{\varepsilon \to 0^+} X(t, \varepsilon)\, \Delta^{-1}(\varepsilon)\, c = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}, \qquad 0 \leqq t \leqq \delta < 1,$$

and

$$(1\ 0) \begin{pmatrix} c_1 \\ 0 \end{pmatrix} + (0\ 0) \begin{pmatrix} c_1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}.$$

However, if we consider the same differential system with the boundary conditions

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x(0, \varepsilon) + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x(1, \varepsilon) = c,$$

$$\Omega = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a(\varepsilon) = \varepsilon[1],$$

and we see that the $\lim_{\varepsilon \to 0^+} X(t, \varepsilon)\, \Delta^{-1}(\varepsilon)\, c$ does not exist.

(2) The results of this paper are most closely related to the work of G. G. CHAPIN Jr. [*2*], W. R. WASOW [*11*], and I. S. GRADSTEIN [*4*], who consider single N^{th} order differential equations. CHAPIN and WASOW consider the case where α

conditions are specified at one point and $N-\alpha$ at another point. CHAPIN's differential equation is more general than that of WASOW, but his assumptions on the characteristic roots (when his problem is restricted to that of WASOW) are more stringent than WASOW's. GRADSTEIN considers an initial value problem which is essentially contained in Theorem 1 of this paper. For our treatment of the initial value problem, we set $R\equiv I$, $S\equiv 0$, and the requirement that Ω be non-singular (or $\mathscr{A}(\varepsilon)$) implies that all the characteristic polynomials have negative real parts. A similar treatment of this type of initial value problem for non-linear systems is given by J. J. LEVIN & N. LEVINSON [7] without recourse to asymptotic expansions.

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