Singular Perturbations of Two Point Boundary Problems

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1. Introduction. We are concerned with showing the relationship of the solution of a "complete" boundary problem

$$\Omega(\epsilon)\frac{d}{dt}x(t,\epsilon) = A(t,\epsilon)x(t,\epsilon), \tag{1.1}$$

$$R(\epsilon)x(a,\epsilon) + S(\epsilon)x(b,\epsilon) = c(\epsilon), \tag{1.2}$$

as $\epsilon \to 0+$ to the solution of a related "degenerate" problem

$$\Omega(0)\frac{d}{dt}x(t) = A(t,0)x(t), \tag{1.3}$$

$$R(0)x(a) + S(0)x(b) = c(0). \tag{1.4}$$

Here Ω, A, R, and S are square matrices of order $n_1 + m$,

$$\Omega(\epsilon) = \operatorname{diag}(I_1 : \epsilon^{h_2}I_2 : \cdots : \epsilon^{h_p}I_p),$$

I_i the unit matrix of order n_i, $m = \sum_{i=2}^{p} n_i$, h_i integers, $0 < h_2 < h_3 < \cdots < h_p = h$; $x(t,\epsilon)$ is a vector of dimension $n_1 + m$ with asymptotic expansion with respect to ϵ; and $\epsilon > 0$.

We define a "regular" problem and show that for a regular problem the solution of the complete boundary problem (1.1), (1.2) has a limit as $\epsilon \to 0+$ which satisfies the degenerate differential system (1.3) and n_1 of the degenerate boundary conditions.

The author has given a different and independent treatment of the same problem in [9]. A discussion of the hypotheses and the relationship of these two treatments is given in section 4.

2. Preliminary transformations. A fundamental hypothesis will be that there exists a transformation $T(t,\epsilon)$ which is non-singular for $a \leqq t \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$, analytic in ϵ and class C^∞ in t, and such that the change of variables $x(t,\epsilon) = T(t,\epsilon)y(t,\epsilon)$ will change the differential system (1.1) into the following type:

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$$\Omega(\epsilon) \frac{d}{dt} y(t, \epsilon) = B(t, \epsilon) y(t, \epsilon), \tag{2.1}$$

where, if $B(t, \epsilon)$ is partitioned $B(t, \epsilon) = (B_{ij})$, $i, j = 1, \cdots, p$, so that B_{ii} is a square matrix of order n_i, the elements of $B_{ij}(t, \epsilon)$ are $O(\epsilon^{h_i+1})$ as $\epsilon \to 0+$, $i \neq j$, and $B_{ii}(t, 0)$ has n_{i1} characteristic roots with negative real part, and n_{i2} characteristic roots with positive real part, $n_{i1} + n_{i2} = n_i$, $a \leqq t \leqq b$. In [9] it was shown under the assumption that

$$A_{pp}(t, 0), \qquad A_{p-1,p-1}(t, 0) - A_{p-1,p}(t, 0) A_{pp}^{-1}(t, 0) A_{p,p-1}(t, 0),$$

and similar matrices were non-singular, $a \leqq t \leqq b$, that such a transformation $T(t, \epsilon)$ could be constructed. These matrices,

$$A_{pp}, \qquad A_{p-1,p-1} - A_{p-1,p} A_{pp}^{-1} A_{p,p-1},$$

etc., can be characterized as those which must be non-singular in order to solve the degenerate differential system (1.3)

$$\begin{aligned}
\frac{d}{dt} x_1 &= A_{11}(t, 0)x_1 + \cdots + A_{1p}(t, 0)x_p, \\
0 &= A_{21}(t, 0)x_1 + \cdots + A_{2p}(t, 0)x_p, \\
&\vdots \\
0 &= A_{p1}(t, 0)x_1 + \cdots + A_{pp}(t, 0)x_p,
\end{aligned}$$

by first solving the last equation for x_p and substituting in the preceding equations and repeating the process until the first equation is solved as a differential system for x_1.

The relationship between the differential system (1.1) and (2.1) is described in the following theorem.

Theorem 1. *Let the matrices $A(t, \epsilon)$ and $B(t, \epsilon)$ be of class C^∞ with respect to t, $a \leqq t \leqq b$, and have asymptotic expansions with respect to ϵ, $0 \leqq \epsilon \leqq \epsilon_0$, with ϵ_0 sufficiently small, and partitionings (A_{ij}), (B_{ij}) so that A_{ii} and B_{ii} are square matrices of order n_i. Further let $B_{ij}(t)$, $= O(\epsilon^{h_i+1})$, $i \neq j$, $B_{ii}(t, 0)$ be non-singular, $i > 1$, $a \leqq t \leqq b$, $0 = h_1 < h_2 < \cdots < h_p = h$.*

A necessary and sufficient condition that a transformation $T(t, \epsilon)$ exist which is non-singular, $a \leqq t \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$, and have an asymptotic expansion with respect to ϵ, such that

$$\Omega(\epsilon) \frac{dx}{dt} = A(t, \epsilon) x(t, \epsilon)$$

and

$$\Omega(\epsilon) \frac{dy}{dt}(t, \epsilon) = B(t, \epsilon) y(t, \epsilon)$$

are equivalent under the transformation $x(t, \epsilon) = T(t, \epsilon)y(t, \epsilon)$ *is that*

$$A_{pp}(t, 0), \qquad A_{p-1,p-1}(t, 0) - A_{p-1,p}(t, 0)A_{pp}^{-1}(t, 0)A_{p,p-1}(t, 0),$$

and similar matrices are non-singular, $a \leqq t \leqq b$.

Proof. The sufficiency has already been proved, see HARRIS [9], but an outline will be given here for completeness.

Let the matrix $A(t, \epsilon)$ be partitioned $(A_{ij}(t, \epsilon))$ so that $A_{ii}(t, \epsilon)$ are square matrices of order n_i and consider the matrix

$$\bar{A}(t, \epsilon) = (\epsilon^{h-h_i}A_{ij}(t, \epsilon)).$$

$\bar{A}(t, \epsilon)$ has an asymptotic expansion $\bar{A}(t, \epsilon) = \bar{A}_0(t) + \bar{A}_1(t)\epsilon + \cdots$. The first matrix $\bar{A}_0(t)$ may be partitioned into the form

$$\bar{A}_0(t) = \begin{bmatrix} 0 & 0 \\ \bar{A}_{21}^0(t) & \bar{A}_{22}^0(t) \end{bmatrix},$$

where in particular $\bar{A}_{22}^0(t) = A_{pp}(t, 0)$ has order n_p.
The matrix

$$P_0(t) = \begin{bmatrix} I & 0 \\ -\bar{A}_{22}^{0^{-1}}\bar{A}_{21}^0 & I \end{bmatrix}$$

is such that

$$P_0^{-1}(t)\bar{A}_0(t)P_0(t) = \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_{22}^0(t) \end{bmatrix},$$

$$P_0^{-1}(t)\bar{A}(t, \epsilon)P_0(t) = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \cdots,$$

and C_i will have the same number of rows of zeros as $\bar{A}_i$. Further a sequence of matrices Q_k, $k = 1, 2, \cdots, h - 1$, may be chosen of the form

$$Q_k = \begin{bmatrix} 0 & Q_{12}^k \\ Q_{21}^k & 0 \end{bmatrix},$$

so that if

$$(2.2) \qquad P(t, \epsilon) = P_0(t)(I + \epsilon Q_1)(I + \epsilon^2 Q_2) \cdots (I + \epsilon^{h-1}Q_{h-1}),$$

$$P^{-1}(t, \epsilon)\bar{A}(t, \epsilon)O(t, \epsilon) = D_0 + \epsilon\, D_1 + \epsilon^2\, D_2 + \cdots,$$

where D_i, $i = 0, \cdots, h - 1$, has block diagonal form

$$\begin{bmatrix} D_{11}^i & 0 \\ 0 & D_{22}^i \end{bmatrix}$$

with D_{22}^i of order n_p and $D_{11}^i = C_{11}^i$, $D_{22}^i = C_{22}^i$.

We note that $D_{11}^i = 0, i = 0, 1, \cdots, h - h_{p-1} - 1$, and if $h - h_p < h$ a short calculation shows that $D_{11}^{h-h_{p-1}}$ has the form

$$D_{11}^{h-h_{p-1}} = \begin{bmatrix} 0 & 0 \\ d_{21} & d_{22} \end{bmatrix},$$

where d_{22} is a square matrix of order n_{p-1} and, in particular,

$$d_{22} = A_{p-1,p-1}(t, 0) - A_{p-1,p}(t, 0)A_{pp}^{-1}(t, 0)A_{p,p-1}(t, 0).$$

The process is now repeated by centering attention on D_{11}^{i}, $i = h - h_p, \cdots, h - 1$, and observing that new transformations do not disturb the diagonalization that has already been accomplished.

Now assume $T(t, \epsilon)$ exists, then we must have

$$T^{-1}(t, \epsilon)\left\{(\epsilon^{h-h_i}A_{ij})T(t, \epsilon) - \epsilon^{h}\frac{d}{dt}T(t, \epsilon)\right\} = (\epsilon^{h-h_i}B_{ij}) \tag{2.3}$$

or

$$\bar{A}(t, \epsilon)T(t, \epsilon) - \epsilon^{h}\frac{d}{dt}T(t, \epsilon) = T(t, \epsilon)\bar{B}(t, \epsilon),$$

where

$$\bar{A} = (\epsilon^{h-h_i}A_{ij}) = \bar{A}_0(t) + \epsilon\bar{A}_1(t) + \epsilon^2\bar{A}_2(t) + \cdots,$$

$$\bar{B} = (\epsilon^{h-h_i}B_{ij}) = \bar{B}_0(t) + \epsilon\bar{B}_1(t) + \epsilon^2\bar{B}_2(t) + \cdots,$$

$$T \simeq T_0(t) + \epsilon T_1(t) + \cdots,$$

and $T_0(t)$ is non-singular, $a \leqq t \leqq b$.

We have

$$\bar{A}_0(t) = \begin{bmatrix} 0 & 0 \\ \bar{A}_{21}^0(t) & \bar{A}_{22}^0(t) \end{bmatrix}, \quad \bar{B}_0(t) = \begin{bmatrix} 0 & 0 \\ 0 & \bar{B}_{22}^0(t) \end{bmatrix}, \quad T_0(t) = \begin{bmatrix} T_{11}^0 & T_{12}^0 \\ T_{21}^0 & T_{22}^0 \end{bmatrix}$$

and from (2.3) we have $\bar{A}_0(t)T_0(t) = T_0(t)\bar{B}_0(t)$, or

$$\begin{bmatrix} 0 & 0 \\ \bar{A}_{21}^0T_{11}^0 + \bar{A}_{22}^0T_{21}^0 & \bar{A}_{21}^0T_{12}^0 + \bar{A}_{22}^0T_{22}^0 \end{bmatrix} = \begin{bmatrix} 0 & T_{12}^0 & \bar{B}_{22}^0 \\ 0 & T_{22}^0 & \bar{B}_{22}^0 \end{bmatrix}.$$

Thus $T_{12}^0\bar{B}_{22}^0 = 0$, and since $\bar{B}_{22}^0(t) = B_{pp}(t, 0)$ is non-singular, $T_{12}^0 \equiv 0$, and T_{22}^0 is non-singular because $T_0(t)$ is non-singular. Thus

$$A_{pp}(t, 0) = \bar{A}_{22}^0(t) = T_{22}^0\bar{B}_{22}^0(T_{22}^0)^{-1}$$

is non-singular, $a \leqq t \leqq b$.

We note that if $B(t, \epsilon)$ is equivalent to $A(t, \epsilon)$ with the transformation $T(t, \epsilon)$, *i.e.* $B = T^{-1}AT - \epsilon^{h}T^{-1}T'$, then A is equivalent to B with the transformation T^{-1}. Further, if B is equivalent to A with the transformation T and C is equivalent to B with the transformation U, then C is equivalent to A with the transformation TU.

Since we have shown that $A_{pp}(t, 0)$ is non-singular, $a \leqq t \leqq b$, we know that $\bar{A}(t, \epsilon)$ is equivalent to $D(t, \epsilon) = D_0(t) + \epsilon D_1(t) + \cdots$, where $D_i(t)$ has block diagonal form, $i = 0, 1, \cdots, h-1$,

$$D_i(t) = \begin{bmatrix} D_{11}^i(t) & 0 \\ 0 & D_{22}^i(t) \end{bmatrix}$$

where in particular

$$D_{22}^0(t) = A_{pp}(t, 0), \qquad D_{11}^i(t) = 0, \qquad i = 0, 1, \cdots, h - h_{p-1} - 1,$$

$$D_{11}^{h-h_{p-1}} = \begin{bmatrix} 0 & 0 \\ d_{21} & d_{22} \end{bmatrix},$$

$$d_{22} = A_{p-1,p-1}(t, 0) - A_{p-1,p}(t, 0) A_{pp}^{-1}(t, 0) A_{p,p-1}(t, 0).$$

Thus $\bar{B}(t, \epsilon)$ is equivalent to $D(t, \epsilon)$ with some transformation

$$U(t, \epsilon) \simeq U_0(t) + \epsilon U_1(t) + \cdots,$$

and

$$D(t, \epsilon) U(t, \epsilon) - \epsilon^h U'(t, \epsilon) = U(t, \epsilon) \bar{B}(t, \epsilon).$$

For any integer $\beta < h$ we have

$$D_0 U_\beta + D_1 U_{\beta-1} + \cdots + D_\beta U_0 = U_\beta \bar{B}_0 + \cdots + U_0 \bar{B}_\beta . \tag{2.4}$$

Applying (2.4) for $\beta = 0$, we have

$$\begin{bmatrix} 0 & 0 \\ 0 & D_{22}^0 \end{bmatrix} \begin{bmatrix} U_{11}^0 & U_{12}^0 \\ U_{21}^0 & U_{22}^0 \end{bmatrix} = \begin{bmatrix} U_{11}^0 & U_{12}^0 \\ U_{21}^0 & U_{22}^0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \bar{B}_{22}^0 \end{bmatrix},$$

or

$$\begin{bmatrix} 0 & 0 \\ D_{22}^0 U_{21}^0 & D_{22}^0 U_{22}^0 \end{bmatrix} = \begin{bmatrix} 0 & U_{12}^0 \bar{B}_{22}^0 \\ 0 & U_{22}^0 \bar{B}_{22}^0 \end{bmatrix},$$

or $U_{12}^0 \equiv 0$, $U_{21}^0 \equiv 0$, and U_0 has block diagonal form. $U_\beta(t)$ has block diagonal form for $\beta = 0, 1, \cdots, h-1$. To prove this assume $U_j(t)$ has block diagonal form for $j = 0, 1, \cdots, \alpha - 1 \leqq h - 2$. Thus $U_j B_{\alpha-j}$ and $D_j U_{\alpha-j}$ have block diagonal form for $j = 1, 2, \cdots, \alpha - 1$, and, applying (2.4) for $\beta = \alpha$, the off-diagonal blocks of $U_\alpha \bar{B}_0$ and $D_0 U_\alpha$ must be equal, or $U_{12}^\alpha \bar{B}_{22}^0 \equiv 0$, $D_{22}^0 U_{21}^\alpha \equiv 0$, and hence U_α has block diagonal form also.

Therefore for $\beta < h$ we must have

$$D_{11}^0(t) U_{11}^\beta(t) + \cdots + D_{11}^\beta U_{11}^0(t) = U_{11}^\beta \bar{B}_{11}^0 + \cdots U_{11}^0 \bar{B}_{11}^\beta ,$$

and the problem has been reduced to one of the same type and lower order and the same reasoning may be repeated $p - 2$ times and the theorem is proved.

If further there exists a matrix $P_{ii}(t)$, $i \geqq 2$, such that

$$P_{ii}^{-1}(t)B_{ii}(t, 0)P_{ii}(t) = \operatorname{diag}(B_{1,ii}(t): B_{2,ii}(t)),$$

where the characteristic roots of $B_{1,ii}(t)$ have negative real parts and the characteristic roots of $B_{2,ii}(t)$ have positive real parts, Theorem 1 shows the existence of a transformation

$$\tilde{T}_{ii}(t, \epsilon) = P_{ii}(t, \epsilon) + \epsilon P_{1,ii}(t) + \cdots + \epsilon^{q_i} P_{q_i,ii}(t),$$

$q_i = h_i(h_i + 1)/2$, such that the coefficient of ϵ^j in the asymptotic expansion of

$$\tilde{T}_{ii}^{-1}(t, \epsilon)\{\bar{B}_{ii}(t, \epsilon)\tilde{T}_{ii}(t, \epsilon) - \epsilon^{h_i}\tilde{T}'_{ii}(t, \epsilon)\}$$

has the same block diagonal form as $P_{ii}^{-1}(t)B_{ii}(t, 0)P_{ii}(t)$ for $j = 1, 2, \cdots, h_i$. (The existence of such a $P_{ii}(t)$ is known in case $B_{ii}(t, 0)$ is a matrix of analytic functions whose characteristic roots have real parts which are bounded away from zero, see F. Wolf [25].)

The transformation considered in this section will effect the boundary form since we are treating the problem as a whole. This effect will be considered in detail in section 3.

3. The canonical problem. We make the following hypotheses.

H1. (i) *The matrix $A(t, \epsilon)$ indicated in* (1.1) *has an asymptotic expansion with respect to ϵ, $a \leqq t \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$.*

(ii) *The matrices*

$$A_{pp}(t, 0), \qquad A_{p-1,p-1}(t, 0) - A_{p-1,p}(t, 0)A_{pp}^{-1}(t, 0)A_{p,p-1}(t, 0),$$

and similar matrices referred to in section 2 *are nonsingular, $a \leqq t \leqq b$.*

(iii) *There exist matrices $P_{ii}(t)$ such that*

$$P_{ii}^{-1}(t)B_{ii}(t, 0)P_{ii}(t) = \operatorname{diag}(B_{1,ii}(t): B_{2,ii}(t)),$$

where $B_{1,ii}(t)$ is a square matrix of order n_{i1} and $B_{2,ii}(t)$ is a square matrix of order n_{i2}, $n_{i1} + n_{i2} = n_i$, and the real parts of the characteristic roots of $B_{1,ii}(t)$ and $B_{2,ii}(t)$ are less than zero and greater than zero respectively, $i = 2, \cdots, p$.

(iv) *All functions are class C^∞ with respect to t, $a \leqq t \leqq b$.*

H2.
$$R(\epsilon) = R_0 + \epsilon R_1(\epsilon), \qquad S(\epsilon) = S_0 + \epsilon S_1(\epsilon),$$

where the elements of $R_1(\epsilon)$ and $S_1(\epsilon)$ are bounded for $0 \leqq \epsilon \leqq \epsilon_0$ and the rank of $(R(\epsilon) : S(\epsilon))$ is $n_1 + m$ for $0 \leqq \epsilon \leqq \epsilon_0$.

Hypothesis H1 assures the existence of a non-singular transformation $L(t, \epsilon)$ $a \leqq t \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$, such that the substitution $x(t, \epsilon) = L(t, \epsilon)y(t, \epsilon)$ changes (1.1) into the *canonical* form

$$\epsilon^h \frac{d}{dt} y(t, \epsilon) = \{C(t, \epsilon) + \epsilon^{h+1} D(t, \epsilon)\}y(t, \epsilon), \tag{3.1}$$

where the elements of $D(t, \epsilon)$ are bounded for $0 \leq \epsilon \leq \epsilon_0$, $a \leq t \leq b$, and $C(t, \epsilon)$ is a block diagonal matrix,

$$C(t, \epsilon) = \operatorname{diag}(C_{11} \colon C_{22} \colon \cdots \colon C_{2p-1,2p-1})$$

where $C_{11}(t, \epsilon) = \epsilon^h \bar{C}_{11}(t)$ and $C_{2i-2,2i-2}(t, \epsilon)$ and $C_{2i-1,2i-1}(t, \epsilon)$, $i = 2, \cdots, p$, are matrix polynomials in ϵ of degree at most h for which the lowest order term that is not identically zero is $\epsilon^{h-h_i} B_{1,ii}(t)$ and $\epsilon^{h-h_i} B_{2,ii}(t)$ respectively.

The degenerate canonical differential system corresponding to (3.1) is obtained by multiplying (3.1) by $\epsilon^{-h}\Omega(\epsilon)$ and setting $\epsilon = 0$ to get

$$\begin{aligned} \frac{dy_1}{dt} &= \bar{C}_{11}(t) y_1 \\ 0 &= B_{1,ii}(t) y_{2i-2}(t), \\ 0 &= B_{2,ii}(t) y_{2i-1}(t), \qquad i = 2, 3, \cdots, p. \end{aligned} \tag{3.2}$$

The transformation $L(t, \epsilon)$ will change the boundary form (1.2) into the boundary form

$$M(\epsilon) y(a, \epsilon) + N(\epsilon) y(b, \epsilon) = c(\epsilon), \tag{3.3}$$

where $M(\epsilon) = R(\epsilon) L(a, \epsilon)$ and $N(\epsilon) = S(\epsilon) L(b, \epsilon)$.

If $Y(t, \epsilon)$ is any fundamental matrix for the canonical differential system (3.1), $\epsilon > 0$, $a \leqq t \leqq b$, the boundary problem (3.1), (3.3) will have a unique solution $y(t, \epsilon)$ if $\Delta(\epsilon)$ is non-singular, $\epsilon > 0$, where

$$\Delta(\epsilon) = M(\epsilon) Y(a, \epsilon) + N(\epsilon) Y(b, \epsilon). \tag{3.4}$$

Hence if $\Delta(\epsilon)$ is non-singular the solution of the boundary problem takes the form

$$y(t, \epsilon) = Y(t, \epsilon)\, \Delta^{-1}(\epsilon) c(\epsilon),$$

and the limit problem is

$$\lim_{\epsilon \to 0+} y(t, \epsilon) = \lim_{\epsilon \to 0+} Y(t, \epsilon)\, \Delta^{-1}(\epsilon) c(\epsilon).$$

We note that $M(\epsilon) = M_0 + \epsilon M_1(\epsilon)$, $N(\epsilon) = N_0 + \epsilon N_1(\epsilon)$, where $M_0 = R_0 L(a, 0)$, $N_0 = S_0 L(b, 0)$. Let M_0 and N_0 be partitioned in the following manner:

$$M_0 = (M_{01} : M_{02} : \cdots : M_{0,2p-1}); \qquad N_0 = (N_{01} : \cdots : N_{0,2p-1}),$$

where M_{0i} and N_{0i} have the same number of columns as $C_{ii}(t, \epsilon)$. If the $n_1 + m$ by m matrix

$$\Lambda = (M_{02} : N_{03} : M_{04} : N_{05} : \cdots : M_{0,2p-2} : N_{0,2p-1}) \tag{3.5}$$

has rank m, there exists a constant square matrix F of order $n_1 + m$ which in non-singular and $F\beta = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$ where I_m is the identity matrix of order m.

We make the following hypothesis:

H3. *The* $n_1 + m$ *by* m *matrix*

$$\Lambda = (M_{02} : N_{03} : M_{04} : N_{05} : \cdots : M_{0,2p-2} : N_{0,2p-1})$$

has rank m.

Due to hypothesis H3, we may replace the boundary form (3.3) by the equivalent boundary form (3.6) obtained by multiplication on the left by F.

$$\bar{M}(\epsilon)y(a, \epsilon) + \bar{N}(\epsilon)y(b, \epsilon) = \bar{c}(\epsilon), \tag{3.6}$$

where $\bar{M}(\epsilon) = FM(\epsilon)$, $\bar{N}(\epsilon) = FN(\epsilon)$, $\bar{c}(\epsilon) = Fc(\epsilon)$, and the corresponding matrix $\bar{\Lambda} = F\Lambda = \begin{pmatrix} 0 \\ I_m \end{pmatrix}$.

The corresponding canonical degenerate boundary form for (3.6) is

$$\bar{M}(0)y(a) + \bar{N}(0)y(b) = \bar{c}(0). \tag{3.7}$$

Let $\bar{M}_{11}(0)$ and $\bar{N}_{11}(0)$ be the matrices which consist of the elements in the first n_1 rows and columns of $\bar{M}(0)$ and $\bar{N}(0)$ respectively, and $\bar{C}_{11}(t)$ the corresponding square matrix of order n_1 indicated in the canonical form (3.2). We make the following hypothesis:

H4. *The boundary problem*

$$\frac{dz}{dt} = \bar{C}_{11}(t)z, \qquad \bar{M}_{11}(0)z(a) + \bar{N}_{11}(0)z(b) = 0,$$

has $z(t) \equiv 0$ *as the only solution.*

Definition. We define the boundary problem (1.1), (1.2) satisfying hypotheses H1, H2, to be a *regular boundary problem* if the corresponding canonical boundary problem (3.1), (3.6) satisfies hypotheses H3, H4, H5.

H5. *The canonical differential system* (3.1) *has a fundamental matrix* $Y(t, \epsilon)$ *of the form*

$$Y(t, \epsilon) = \{I + \epsilon Z(t, \epsilon)\}E(t, \epsilon), \qquad a \leqq t \leqq b, \qquad 0 < \epsilon \leqq \epsilon_0, \tag{3.8}$$

where

$$E(t, \epsilon) = \operatorname{diag}(E_{11}(t) : E_{22}(t, \epsilon) : \cdots : E_{2p-1,2p-1}(t, \epsilon)),$$

$E_{jj}(t, \epsilon)$ *is a fundamental matrix for*

$$\epsilon^h \frac{d}{dt} y_j(t, \epsilon) = C_{jj}(t, \epsilon)y_j(t, \epsilon), \qquad j = 1, 2, \cdots, 2p - 1,$$

and the elements of $Z(t, \epsilon)$ *are bounded,* $a \leqq t \leqq b$, $0 \leqq \epsilon \leqq \epsilon_0$.

Theorem 2. *Under hypotheses H3, H4, and H5 the canonical boundary problem* (3.1), (3.6) *has a unique solution* $y(t, \epsilon)$, $a \leqq t \leqq b$, $0 < \epsilon \leqq \epsilon_0$, *such that*

$$\lim_{\epsilon \to 0+} y(t, \epsilon) = y(t)$$

exists in the open interval $a < t < b$, and uniformly in any closed sub-interval. The function $y(t)$ satisfies the degenerate canonical differential system (3.2). *The limits $y(a + 0)$ and $y(b - 0)$ exist and satisfy the first n_1 boundary conditions of the degenerate boundary form* (3.7).

Proof. In the course of this proof we need the following lemma.

Lemma. *Let $Q(t, \epsilon)$ be a $r \times r$ matrix, continuous in t, $a \leqq t \leqq b$,*

$$Q(t, \epsilon) = Q_0(t) + \epsilon Q_1(t) + \cdots \epsilon^\alpha Q_\alpha(t),$$

$0 < \epsilon$, and let the real parts of the characteristic roots of $Q(t, 0) = Q_0(t)$ be less than -2μ on $a \leqq t \leqq b$, for some $\mu > 0$. Let $\phi(t, s, \epsilon)$ be the solution of

$$\epsilon^\alpha \frac{dX}{dt} = Q(t, \epsilon)X, \qquad X(s) = I$$

on $a \leqq t \leqq b$. Then there exists a constant K such that

$$||\phi(t, s, \epsilon)|| \leqq K \exp \{-\mu(t - s)/\epsilon^\alpha\},$$

for $a \leqq s \leqq t \leqq b$, $0 < \epsilon \leqq \epsilon_0$.

Here $||A||$ is the sum of the magnitudes of all the elements of A. The proof of this lemma is an easy extension of the corresponding lemma in (4).

An application of this lemma yields the following estimates;

$$(3.9) \qquad \begin{aligned} &||E_{2i-2,2i-2}(t, \epsilon)E^{-1}_{2i-2,2i-2}(a, \epsilon)|| \leqq K \exp \{-\mu(t - a)/\epsilon^{h_i}\}, \\ &||E_{2i-1,2i-1}(t, \epsilon)E^{-1}_{2i-1,2i-1}(b, \epsilon)|| \leqq K \exp \{-\mu(b - t)/\epsilon^{h_i}\}, \end{aligned}$$

where $i = 2, 3, \cdots, p$.

Let

$$\bar{E}(a, b, \epsilon) = \operatorname{diag} (I_1 : E^{-1}_{22}(a, \epsilon) : E^{-1}_{33}(b, \epsilon) : \cdots : E^{-1}_{2p-2,2p-2}(a, \epsilon) : E^{-1}_{2p-1,2p-1}(b, \epsilon)),$$

and consider $\Delta(\epsilon)\bar{E}(a, b, \epsilon)$.

$$\begin{aligned} \Delta(\epsilon)\bar{E}(a, b\epsilon) &= \bar{M}(\epsilon)(I + \epsilon Z(a, \epsilon))E(a, \epsilon)\bar{E}(a, b, \epsilon) \\ &\qquad + \bar{N}(\epsilon)(I + \epsilon Z(b, \epsilon))E(b, \epsilon)\bar{E}(a, b, \epsilon) \\ &= (\bar{M}_0 + o(\epsilon)) \operatorname{diag} (E_{11}(a) : I_{22} : E_{33}(a, \epsilon)E^{-1}_{33}(b, \epsilon) : \\ &\qquad \cdots : E_{2p-1,2p-1}(a, \epsilon)E^{-1}_{2p-1,2p-1}(b, \epsilon)) \\ &\quad + (\bar{N}_0 + o(\epsilon)) \operatorname{diag} (E_{11}(b) : E_{22}(b, \epsilon)E^{-1}_{22}(a, \epsilon) : \cdots : I_{2p-1}). \end{aligned}$$

$\Delta(\epsilon)\bar{E}(a, b, \epsilon)$ will have an asymptotic expansion with respect to ϵ, and taking into account the estimates (3.9) we have

$$\begin{aligned} \Delta(\epsilon)\bar{E}(a, b, \epsilon) = (\bar{M}_{01}E_{11}(a) + \bar{N}_{01}E_{11}(b) : \bar{M}_{02} : \bar{N}_{03} : \cdots : \bar{M}_{0,2p-2} : \bar{N}_{0,2p-1}) \\ + o(\epsilon) = \Gamma + o(\epsilon), \end{aligned}$$

where Γ is the constant non-singular matrix

$$\Gamma = (\bar{M}_{01}E_{11}(a) + \bar{N}_{01}E_{11}(b) : \bar{\Lambda}). \tag{3.10}$$

Thus

$$\begin{aligned} Y(t, \epsilon)\, \Delta^{-1}(\epsilon) &= Y(t, \epsilon)\bar{E}(a, b, \epsilon)(\Delta(\epsilon)\bar{E}(a, b, \epsilon))^{-1} \\ &= (I + \epsilon Z(t, \epsilon))E(t, \epsilon)\bar{E}(a, b, \epsilon)(\Gamma^{-1} + o(\epsilon)), \end{aligned}$$

and, for $a < t < b$,

$$\lim_{\epsilon \to 0+} Y(t, \epsilon)\, \Delta^{-1}(\epsilon) = \begin{bmatrix} E_{11}(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{11}^{-1} & 0 \\ -\Gamma_{21}\Gamma_{11}^{-1} & I \end{bmatrix} = \begin{bmatrix} E_{11}(t)\Gamma_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$\lim_{\epsilon \to 0+, a<t<b} y(t, \epsilon) = y(t) = \begin{bmatrix} y_1(t) \\ 0 \end{bmatrix} = \begin{bmatrix} E_{11}(t)\Gamma_{11}^{-1} \quad \bar{c}_1(0) \\ 0 \end{bmatrix}.$$

Clearly, $y(t)$ satisfies the degenerate canonical differential system (3.2) and since $E_{11}(t)$ was defined and continuous for $a \leqq t \leqq b$, the limits $y(a + 0)$ and $y(b - 0)$ exist.

Noting that

$$\Gamma_{11}^{-1} = \{\bar{M}_{11}(0)E_{11}(a) + \bar{N}_{11}(0)E_{11}(b)\}^{-1},$$

we see that $y(a + 0)$ and $y(b - 0)$ satisfy the first n_1 boundary conditions (3.7) and in fact $y_1(t)$ is the unique solution of the boundary problem

$$\frac{dy_1(t)}{dt} = \bar{C}_{11}(t)y_1(t), \qquad \bar{M}_{11}(0)y_1(a) + \bar{N}_{11}(0)y_1(b) = \bar{c}_1(0).$$

Theorem 3. *Under hypotheses* $H1$, $H2$, $H3$, $H4$, *and* $H5$, *a regular boundary problem* (1.1), (1.2) *has a solution* $x(t, \epsilon)$, $a \leqq t \leqq b$, $0 < \epsilon \leqq \epsilon_0$, *such that* $\lim_{\epsilon \to 0+} x(t, \epsilon) = x(t)$, $a < t < b$, *and uniformly on any closed sub-interval. The function* $x(t)$ *satisfies the degenerate differential system* (1.3). *The limits* $x(a + 0)$ *and* $x(b - 0)$ *exist and satisfy the first* n_1 *boundary conditions for the equivalent boundary form obtained from* (1.2) *by multiplication on the left by* F.

Proof. Let $y(t, \epsilon)$ be the unique solution of the canonical boundary problem (3.1), (3.6) as in Theorem 2. The function $x(t, \epsilon) = L(t, \epsilon)y(t, \epsilon)$ is the unique solution for the regular boundary problem (1.1), (1.2). Hence

$$\lim_{\substack{\epsilon \to 0+ \\ a<t<b}} x(t, \epsilon) = L(t, 0)y(t) = x(t),$$

and $x(a + 0) = L(a, 0)y(a + 0)$, $x(b - 0) = L(b, 0)y(b - 0)$, and $x(a + 0)$ and $x(b - 0)$ satisfy the first n_1 boundary conditions of the equivalent boundary form obtained from (1.2) by multiplication on the left by the matrix F whose existence was a consequence of H3.

4. Remarks. H1 and H2 are conditions of regularity imposed upon the boundary problem together with conditions to insure the reduction of the differential system to canonical form. These conditions are given in the most convenient form but it is clear that the order of differentiability could be lowered and only finite asymptotic expansions are required. H3, H4, and H5 define a "regular problem" and allow the existence of the limiting solution to be proved, the selection of n_1 appropriate boundary conditions, and the characterization of the problem as a "perturbation" problem.

H5 is a restriction on the fundamental matrix for the canonical differential system. When no "turning point" problems arise, and the asymptotic expansions of solutions for differential systems such as H. L. TURRITTIN's (22) are available, H5 is no essential further restriction on the problem. However, when turning point problems are present and the powerful method of "related equations" applies, as developed by R. E. LANGER, [10]–[15], C. C. LIN, [17], [18], Y. SIBUYA, [19]–[21], and others, frequently the existence of a fundamental matrix satisfying the conditions of H5 may be proved.

The problem has been studied recently by the author [9]. This previous treatment assumes a more restrictive canonical form for the differential system so that TURRITTIN's asymptotic expansions may be utilized, hence the possibility of "turning points" was excluded. The present paper allows the inclusion of certain classes of turning point problems. However, the present treatment assumes that the real parts of the characteristic roots of certain matrices are bounded away from zero, whereas in the previous treatment the corresponding assumption is placed on the "characteristic polynomials" and in some cases allow this possibility by considering higher order effects. The two treatments are parallel, but neither includes the other.

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