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Asymptotic Analysis of Linearly Elastic Shells. II. Justification of Flexural Shell Equations

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Abstract

We consider as in Part I a family of linearly elastic shells of thickness 2ε, all having the same middle surface $S = \boldsymbol{\varphi}(\overline{\omega}) \subset \mathbf{R}^3$, where $\omega \subset \mathbf{R}^2$ is a bounded and connected open set with a Lipschitz-continuous boundary, and $\boldsymbol{\varphi} \in \mathscr{C}^3(\overline{\omega}; \mathbf{R}^3)$. The shells are clamped on a portion of their lateral face, whose middle line is $\boldsymbol{\varphi}(\gamma_0)$, where γ_0 is any portion of $\partial\omega$ with *length* $\gamma_0 > 0$. We make an essential geometrical assumption on the middle surface S and on the set γ_0, which states that the space of inextensional displacements

$$\begin{aligned} \boldsymbol{V}_F(\omega) = \{&\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \\ &\eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}, \end{aligned}$$

where $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ are the components of the linearized change is metric tensor of S, contains non-zero functions. This assumption is satisfied in particular if S is a portion of cylinder and $\boldsymbol{\varphi}(\gamma_0)$ is contained in a generatrix of S.

We show that, if the applied body force density is $O(\varepsilon^2)$ with respect to ε, the field $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$, where $u_i(\varepsilon)$ denote the three covariant components of the displacement of the points of the shell given by the equations of three-dimensional elasticity, once "scaled" so as to be defined over the fixed domain $\Omega = \omega \times]-1, 1[$, converges as $\varepsilon \to 0$ in $\boldsymbol{H}^1(\Omega)$ to a limit $\boldsymbol{u}$, which is independent of the transverse variable. Furthermore, the average $\boldsymbol{\zeta} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\, dx_3$, which belongs to the space $\boldsymbol{V}_F(\omega)$, satisfies the (scaled) two-dimensional equations of a "flexural shell", viz.,

$$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\, dy = \int_\omega \left\{ \int_{-1}^{1} f^i dx_3 \right\} \eta_i \sqrt{a}\, dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_F(\omega)$, where $a^{\alpha\beta\sigma\tau}$ are the components of the two-dimensional elasticity tensor of the surface S,

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) = \partial_{\alpha\beta}\eta_3 - \Gamma^{\sigma}_{\alpha\beta}\partial_\sigma\eta_3 + b^{\sigma}_{\beta}(\partial_\alpha\eta_\sigma - \Gamma^{\tau}_{\alpha\sigma}\eta_\tau) + b^{\sigma}_{\alpha}(\partial_\beta\eta_\sigma - \Gamma^{\tau}_{\beta\sigma}\eta_\tau) + b^{\sigma}_{\alpha}|_\beta\eta_\sigma - c_{\alpha\beta}\eta_3$$

are the components of the linearized change of curvature tensor of $S, \Gamma^{\sigma}_{\alpha\beta}$ are the Christoffel symbols of S, b^{β}_{α} are the mixed components of the curvature tensor of S, and f^i are the scaled components of the applied body force. Under the above assumptions, the two-dimensional equations of a "flexural shell" are therefore justified.

Contents

1. The three-dimensional shell problem 164
2. The "scaled" three-dimensional shell problem over a domain independent of ε 165
3. Technical preliminaries 166
4. A generalized Korn inequality for an arbitrary surface 176
5. Asymptotic analysis as $\varepsilon \to 0$ 178
6. Consideration of surface forces 184
7. Conclusions and comments. 186

1. The three-dimensional shell problem

This is the second part of a three-part work, the first (Ciarlet & Lods [1996a]) and third (Ciarlet & Lods [1996b]), henceforth simply referred to as "Part I" and "Part III", being devoted to membrane and Koiter's shell equations. We refer to the "Introduction" of Part I for a general introduction to the asymptotic analysis of "thin" elastic structures and for an extensive list of relevant references.

The assumptions on the set ω and on the mapping $\boldsymbol{\varphi}$, the notation, and the geometrical and mechanical description of the shell are the same as in Sec. 1 of Part I; for this reason, they are not repeated here. The sole difference is that in the present case, the displacement need not vanish along the *whole* lateral face of the shell as in Part I, but only on a *portion* thereof. More specifically, let γ_0 denote a measurable subset of the boundary γ of the set ω with *length* $\gamma_0 > 0$; we now let

$$\Gamma^{\varepsilon}_0 = \gamma_0 \times [-\varepsilon, \varepsilon]\,. \tag{1.1}$$

Note that in Part I, the notation Γ^{ε}_0 always stood for $\gamma \times [-\varepsilon, \varepsilon]$.

The *problem of linearized elasticity* under consideration thus reads as follows (as in Part I, surface forces are considered in a separate section): Let $u^{\varepsilon}_i : \overline{\Omega}^{\varepsilon} \to \mathbf{R}$ denote the three covariant components of the displacement field $u^{\varepsilon}_i \boldsymbol{g}^{i,\varepsilon}$ of the points of the shell $\boldsymbol{\Phi}(\overline{\Omega}^{\varepsilon})$. Then the *unknown* $\boldsymbol{u}^{\varepsilon} = (u^{\varepsilon}_i)$ satisfies

$$(1.2) \qquad \boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) := \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{H}^1(\Omega^\varepsilon); \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma_0^\varepsilon\},$$

$$(1.3) \qquad \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}dx^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}dx^\varepsilon \quad \text{for all } \boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon),$$

where

$$(1.4) \qquad A^{ijkl,\varepsilon} = \lambda^\varepsilon g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu^\varepsilon (g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon})$$

designate the contravariant components of the *three-dimensional elasticity tensor*, $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$ are the *Lamé constants* of the elastic material constituting the shell,

$$(1.5) \qquad e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon) = \tfrac{1}{2}(\partial_i^\varepsilon v_j^\varepsilon + \partial_j^\varepsilon v_i^\varepsilon) - \Gamma_{ij}^{p,\varepsilon} v_p^\varepsilon$$

designate the covariant components of the *linearized strain tensor*, and $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ are the contravariant components of the *applied body force* acting in $\boldsymbol{\Phi}(\Omega^\varepsilon)$.

2. The "scaled" three-dimensional shell problem over a domain independent of ε

Additional comments relevant to the content of this section may be found in Sec. 2 of Part I. Let

$$\Omega = \omega \times]-1, 1[, \quad \Gamma_+ = \omega \times \{1\}, \quad \Gamma_- = \omega \times \{-1\}, \quad \Gamma_0 = \gamma_0 \times [-1, 1],$$

let $x = (x_i)$ denote a generic point in the set $\overline{\Omega}$, and let $\partial_i = \partial/\partial x_i$, $\partial_{ij} = \partial^2/\partial x_i \partial x_j$. With $x^\varepsilon = (x_i^\varepsilon) \in \overline{\Omega}^\varepsilon$, we associate as in (2.1), (2.3)–(2.6) the point $x = (x_i) \in \overline{\Omega}$ defined by $x_\alpha = x_\alpha^\varepsilon$ and $x_3 = (1/\varepsilon)x_3^\varepsilon$. With the unknown $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbf{R}^3$ and the vector field $\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbf{R}^3$ appearing in the three-dimensional problem (1.2), (1.3), we associate the *scaled unknown* $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \bar{\Omega} \to \mathbf{R}^3$ and the *scaled vector fields* $\boldsymbol{v} = (v_i) : \bar{\Omega} \to \mathbf{R}^3$ defined by

$$(2.1) \qquad u_i(\varepsilon)(x) = u_i^\varepsilon(x^\varepsilon), \quad v_i(x) = v_i^\varepsilon(x^\varepsilon) \quad \text{for all } x^\varepsilon \in \overline{\Omega}^\varepsilon.$$

We next make the following *assumptions on the data*, i.e., on the *Lamé constants* and on the *applied forces*: There exist constants $\lambda > 0$ and $\mu > 0$ *independent of* ε, and there exist functions $f^i \in L^2(\Omega)$ *independent of* ε such that

$$(2.2) \qquad \lambda^\varepsilon = \lambda, \quad \mu^\varepsilon = \mu,$$

$$(2.3) \qquad f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^2 f^i(x) \quad \text{for all } x \in \Omega.$$

The assumptions (2.3) on the components of the applied body force are thus *different* from those made in equations (2.3) of Part I.

The scaled unknown $\boldsymbol{u}(\varepsilon)$ then satisfies a *scaled three-dimensional shell problem*, now posed over the set Ω (cf. (2.10), (2.11):

Theorem 2.1. *With the functions* $\Gamma_{ij}^{p,\varepsilon}, g^{\varepsilon}, A^{ijkl,\varepsilon} : \overline{\Omega}^{\varepsilon} \to \mathbf{R}$ *appearing in* (1.3)–(1.5), *let there be associated the functions* $\Gamma_{ij}^{p}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon) : \overline{\Omega} \to \mathbf{R}$ *defined by*

$$\Gamma_{ij}^{p}(\varepsilon)(x) := \Gamma_{ij}^{p,\varepsilon}(x^{\varepsilon}) \quad \text{for all } x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}, \tag{2.4}$$

$$g(\varepsilon)(x) := g^{\varepsilon}(x^{\varepsilon}) \quad \text{for all } x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}, \tag{2.5}$$

$$A^{ijkl}(\varepsilon)(x) := A^{ijkl,\varepsilon}(x^{\varepsilon}) \quad \text{for all } x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}. \tag{2.6}$$

With any vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$, *let there be associated the symmetric tensor* $(e_{i\|j}(\varepsilon)(\boldsymbol{v})) \in \boldsymbol{L}^2(\Omega)$ *defined by*

$$e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) := \tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma_{\alpha\beta}^{p}(\varepsilon) v_p, \tag{2.7}$$

$$e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{2}\left(\partial_\alpha v_3 + \frac{1}{\varepsilon}\partial_3 v_\alpha\right) - \Gamma_{\alpha 3}^{\sigma}(\varepsilon) v_\sigma, \tag{2.8}$$

$$e_{3\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3. \tag{2.9}$$

Then the scaled unknown $\boldsymbol{u}(\varepsilon)$ *defined in* (2.1) *satisfies*

$$\boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega); \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}, \tag{2.10}$$

$$\begin{aligned}&\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{i\|j}(\varepsilon)(\boldsymbol{v}) \sqrt{g(\varepsilon)}\, dx \\ &= \varepsilon^2 \int_\Omega f^i v_i \sqrt{g(\varepsilon)}\, dx \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega).\end{aligned} \tag{2.11}$$

3. Technical preliminaries

As in Part I, we assume that the number $\varepsilon_0 > 0$ (which is such that the "original" three-dimensional problem is well-defined for $0 < \varepsilon \leqq \varepsilon_0$) also satisfies $\varepsilon_0 \leqq 1$, and we use the same rules (described at the beginning of Sec. 3 of Part I) concerning the usage of symbols such as C_1, C_2, etc., or c_1, c_2, etc.. Our first result is analogous to Lemma 3.1 of Part I; the asymptotic behavior of the functions $\Gamma_{\alpha\beta}^{p}(\varepsilon)$ as $\varepsilon \to 0$ is however analyzed to within a higher order.

Lemma 3.1. *Let the functions* $\Gamma_{ij}^{p}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$ be defined for $\varepsilon > 0$ *as in* (2.4)–(2.6) *and the functions* $a^{\alpha\beta}, b_{\alpha\beta}, b_\alpha^\sigma, \Gamma_{\alpha\beta}^{\sigma}, a$ *be defined as in* (1.1)–(1.4) *of Part I. In addition, let the covariant derivatives* $b_\beta^\sigma|_\alpha$, *and the covariant components* $c_{\alpha\beta}$ *of the third fundamental form, of the surface S be defined by*

(3.1) $$b^{\sigma}_{\beta}|_{\alpha} = \partial_{\alpha} b^{\sigma}_{\beta} + \Gamma^{\sigma}_{\alpha\tau} b^{\tau}_{\beta} - \Gamma^{\tau}_{\beta\alpha} b^{\sigma}_{\tau}\,,$$

(3.2) $$c_{\alpha\beta} = b^{\sigma}_{\alpha} b_{\sigma\beta}\,.$$

Let all the functions $a^{\alpha\beta}, \ldots, c_{\alpha\beta} \in \mathscr{C}^0(\overline{\omega})$ *be identified with functions in* $\mathscr{C}^0(\overline{\Omega})$. *Then*

(3.3) $$||\Gamma^{\sigma}_{\alpha\beta}(\varepsilon) - (\Gamma^{\sigma}_{\alpha\beta} - \varepsilon x_3 b^{\sigma}_{\beta}|_{\alpha})||_{0,\infty,\overline{\Omega}} \leqq C_1 \varepsilon^2\,,$$

(3.4) $$\Gamma^{3}_{\alpha\beta}(\varepsilon) = b_{\alpha\beta} - \varepsilon x_3 c_{\alpha\beta}\,,$$

(3.5) $$||\Gamma^{\sigma}_{\alpha 3}(\varepsilon) + b^{\sigma}_{\alpha} + \varepsilon x_3 b^{\tau}_{\alpha} b^{\sigma}_{\tau}||_{0,\infty,\overline{\Omega}} \leqq C_1 \varepsilon^2\,,$$

(3.6) $$\Gamma^{3}_{\alpha 3}(\varepsilon) = \Gamma^{p}_{33}(\varepsilon) = 0\,,$$

(3.7) $$||g(\varepsilon) - a||_{0,\infty,\overline{\Omega}} \leqq C_1 \varepsilon\,,$$

(3.8) $$||A^{ijkl}(\varepsilon) - A^{ijkl}(0)||_{0,\infty,\overline{\Omega}} \leqq C_1 \varepsilon\,,$$

(3.9) $$A^{\alpha\beta\sigma 3}(\varepsilon) = A^{\alpha 333}(\varepsilon) = 0$$

for all $0 < \varepsilon \leqq \varepsilon_0$, *where*

(3.10) $$A^{\alpha\beta\sigma\tau}(0) := \lambda a^{\alpha\beta} a^{\sigma\tau} + \mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})\,,$$

(3.11) $$A^{\alpha\beta 33}(0) := \lambda a^{\alpha\beta}, \quad A^{\alpha 3\sigma 3}(0) = \mu a^{\alpha\sigma}, \quad A^{3333}(0) = \lambda + 2\mu\,,$$

(3.12) $$A^{\alpha\beta\sigma 3}(0) = A^{\alpha 333}(0) = 0\,,$$

and finally,

(3.13) $$t_{ij} t_{ij} \leqq C_2 A^{ijkl}(\varepsilon)(x) t_{kl} t_{ij}$$

for all $0 < \varepsilon \leqq \varepsilon_0$, all $x \in \overline{\Omega}$, *and all symmetric tensors* (t_{ij}).

Proof. The vectors $\boldsymbol{a}_i$, $\boldsymbol{a}^i$, $\boldsymbol{g}^{\varepsilon}_i$, $\boldsymbol{g}^{i,\varepsilon}$ and the scalars $g^{\varepsilon}_{ij}, g^{ij,\varepsilon}, a_{\alpha\beta}$ are defined in Sec. 1 of Part I. Let

$$\boldsymbol{g}_i(\varepsilon)(x) = \boldsymbol{g}^{\varepsilon}_i(x^{\varepsilon}), \quad \boldsymbol{g}^i(\varepsilon)(x) = \boldsymbol{g}^{i,\varepsilon}(x^{\varepsilon}) \quad \text{for all } x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}\,,$$

$$g_{ij}(\varepsilon)(x) = g^{\varepsilon}_{ij}(x^{\varepsilon}), \quad g^{ij}(\varepsilon)(x) = g^{ij,\varepsilon}(x^{\varepsilon}) \quad \text{for all } x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}\,,$$

where the points $x^{\varepsilon} \in \overline{\Omega}^{\varepsilon}$ and $x \in \overline{\Omega}$ are in the usual correspondence. By definition (cf. (2.4)),

$$\Gamma^{\sigma}_{\alpha\beta}(\varepsilon) = \boldsymbol{g}^{\sigma}(\varepsilon) \cdot \partial_{\alpha} \boldsymbol{g}_{\beta}(\varepsilon)$$

where

$$\boldsymbol{g}_{\alpha}(\varepsilon) = \boldsymbol{a}_{\alpha} + \varepsilon x_3 \partial_{\alpha} \boldsymbol{a}_3 \quad \text{and} \quad \boldsymbol{g}_3(\varepsilon) = \boldsymbol{a}_3\,.$$

In what follows, remainders such as $O(\varepsilon)$ and $O(\varepsilon^2)$ are meant with respect to the norm $\|\cdot\|_{0,\infty,\overline{\Omega}}$ of the space $\mathscr{C}^0(\overline{\Omega})$.The relations $g_{\alpha\beta}(\varepsilon) = a_{\alpha\beta} + O(\varepsilon)$ and $g_{i3}(\varepsilon) = \delta_{i3}$ imply that $g^{\alpha\beta}(\varepsilon) = a^{\alpha\beta} + O(\varepsilon)$ and $g^{i3}(\varepsilon) = \delta^{i3}$. Consequently,

$$\boldsymbol{g}^\sigma(\varepsilon) = g^{\sigma j}(\varepsilon)\boldsymbol{g}_j(\varepsilon) = a^{\sigma\beta}\boldsymbol{a}_\beta + O(\varepsilon) = \boldsymbol{a}^\sigma + O(\varepsilon)\,.$$

Since the mapping $\boldsymbol{\varphi} : \overline{\omega} \to \mathbf{R}^3$ is by assumption of class $\mathscr{C}^3$ (in fact, it suffices here that $\boldsymbol{\varphi}$ be of class $\mathscr{C}^2$), there exist vectors $\boldsymbol{b}^\sigma$ such that

$$\boldsymbol{g}^\sigma(\varepsilon) = \boldsymbol{a}^\sigma + \varepsilon x_3 \boldsymbol{b}^\sigma + O(\varepsilon^2)\,.$$

To compute the vectors $\boldsymbol{b}^\sigma$, we first observe that

$$0 = \boldsymbol{g}^\sigma(\varepsilon)\cdot\boldsymbol{g}_3(\varepsilon) = \varepsilon x_3 \boldsymbol{b}^\sigma\cdot\boldsymbol{a}_3 + O(\varepsilon^2)\,.$$

Hence the vectors $\boldsymbol{b}^\sigma$ and $\boldsymbol{a}_3$ are orthogonal. We next have (recall that $b^\sigma_\beta = a^{\sigma\tau}b_{\tau\beta}$)

$$\delta^\sigma_\beta = \boldsymbol{g}^\sigma(\varepsilon)\cdot\boldsymbol{g}_\beta(\varepsilon) = \delta^\sigma_\beta + \varepsilon x_3(-b^\sigma_\beta + \boldsymbol{b}^\sigma\cdot\boldsymbol{a}_\beta) + O(\varepsilon^2)\,;$$

therefore $\boldsymbol{b}^\sigma = b^\sigma_\beta\boldsymbol{a}^\beta$, and consequently,

$$\Gamma^\sigma_{\alpha\beta}(\varepsilon) = (\boldsymbol{a}^\sigma + \varepsilon x_3 b^\sigma_\tau\boldsymbol{a}^\tau + O(\varepsilon^2))\cdot(\partial_\alpha\boldsymbol{a}_\beta + \varepsilon x_3\partial_{\alpha\beta}\boldsymbol{a}_3)\,,$$

$$\Gamma^\sigma_{\alpha 3}(\varepsilon) = (\boldsymbol{a}^\sigma + \varepsilon x_3 b^\sigma_\tau\boldsymbol{a}^\tau + O(\varepsilon^2))\cdot\partial_\alpha\boldsymbol{a}_3\,.$$

Combining this expression with the formulas of Gauss and Weingarten, viz.,

$$\partial_\alpha\boldsymbol{a}_\beta = \Gamma^\sigma_{\alpha\beta}\boldsymbol{a}_\sigma + b_{\alpha\beta}\boldsymbol{a}_3, \qquad \partial_\alpha\boldsymbol{a}_3 = -b^\sigma_\alpha\boldsymbol{a}_\sigma\,,$$

and with the definition (3.1) of the covariant derivatives $b^\sigma_\beta|_\alpha$, we find that

$$\Gamma^\sigma_{\alpha\beta}(\varepsilon) = \Gamma^\sigma_{\alpha\beta} - \varepsilon x_3 b^\sigma_\beta|_\alpha + O(\varepsilon^2)\,, \qquad \Gamma^\sigma_{\alpha3}(\varepsilon) = -b^\sigma_\alpha - \varepsilon x_3 b^\tau_\alpha b^\sigma_\tau + O(\varepsilon^2)\,,$$

and thus relations (3.3) and (3.5) are proved. Since $\boldsymbol{g}^3(\varepsilon) = g^{3i}(\varepsilon)\boldsymbol{g}_i(\varepsilon) = \boldsymbol{a}^3$, definition (2.4) implies that

$$\Gamma^3_{\alpha\beta}(\varepsilon) = \boldsymbol{g}^3(\varepsilon)\cdot\partial_\alpha\boldsymbol{g}_\beta(\varepsilon) = \boldsymbol{a}^3\cdot(\partial_\alpha\boldsymbol{a}_\beta + \varepsilon x_3\partial_{\alpha\beta}\boldsymbol{a}_3)\,.$$

Combining this expression with the formulas of Gauss and Weingarten, together with the definition (3.2) of the third fundamental form then yields relation (3.4). Relations (3.6)–(3.13) have been established in Lemma 3.1 of Part I. $\square$

In the next lemma, we analyze the asymptotic behavior as $\varepsilon \to 0$ of the functions $e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v})$ appearing in the definition of the scaled three-dimensional shell problem (2.10), (2.11). To this end, we are naturally led to introduce the three-dimensional analogs (cf. (3.14), (3.15)) of the two-dimensional *change of metric tensor* and *change of curvature tensor*, which play a fundamental role in the definition of the limit problem found by the asymptotic analysis (cf. (5.1), (5.2)).

Lemma 3.2. *Let the functions* $b^\sigma_\beta|_\alpha$ *and* $c_{\alpha\beta}$ *be defined as in* (3.1), (3.2), *let the functions* $\Gamma^\sigma_{\alpha\beta}, \dots, c_{\alpha\beta} \in \mathscr{C}^0(\overline{\omega})$ *be identified with functions in* $\mathscr{C}^0(\overline{\Omega})$, *and for any* $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$ *let the functions* $\gamma_{\alpha\beta}(\boldsymbol{v}) \in L^2(\Omega)$ *and* $\rho_{\alpha\beta}(\boldsymbol{v}) \in H^{-1}(\Omega)$ *be defined by*

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = \tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta} v_\sigma - b_{\alpha\beta} v_3 \,, \tag{3.14}$$

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{v}) = \partial_{\alpha\beta} v_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma v_3 + b^\sigma_\beta(\partial_\alpha v_\sigma - \Gamma^\tau_{\alpha\sigma} v_\tau) \\ + b^\sigma_\alpha(\partial_\beta v_\sigma - \Gamma^\tau_{\beta\sigma} v_\tau) + b^\sigma_\alpha|_\beta v_\sigma - c_{\alpha\beta} v_3 \,.\end{aligned} \tag{3.15}$$

Then, for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, *the functions* $e_{\alpha||\beta}(\varepsilon)(\boldsymbol{v})$ *defined in* (2.7) *satisfy*

$$\left\| \frac{1}{\varepsilon} e_{\alpha||\beta}(\varepsilon)(\boldsymbol{v}) - e^1_{\alpha||\beta}(\varepsilon)(\boldsymbol{v}) \right\|_{0,\Omega} \leqq C_3 \varepsilon \sum_\alpha ||v_\alpha||_{0,\Omega} \,, \tag{3.16}$$

where

$$e^1_{\alpha||\beta}(\varepsilon)(\boldsymbol{v}) := \frac{1}{\varepsilon}\gamma_{\alpha\beta}(\boldsymbol{v}) + x_3 b^\sigma_\beta|_\alpha v_\sigma + x_3 c_{\alpha\beta} v_3 \,, \tag{3.17}$$

$$\begin{aligned}&\left\| \frac{1}{\varepsilon}\partial_3 e_{\alpha||\beta}(\varepsilon)(\boldsymbol{v}) + \rho_{\alpha\beta}(\boldsymbol{v}) \right\|_{-1,\Omega} \\ &\leqq C_3 \left\{ \sum_i ||e_{i||3}(\varepsilon)(\boldsymbol{v})||_{0,\Omega} + \varepsilon \sum_\alpha ||v_\alpha||_{0,\Omega} + \varepsilon ||v_3||_{1,\Omega} \right\} .\end{aligned} \tag{3.18}$$

Proof. Relation (3.16) is a consequence of definition (2.7) and relations (3.3), (3.4). Next, let the functions $e^0_{\alpha||\beta}(\varepsilon)(\boldsymbol{v})$ and $\tilde{\rho}_{\alpha\beta}(\boldsymbol{v}) \in H^{-1}(\Omega)$ be defined by

$$e^0_{\alpha||3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{2}\left(\partial_\alpha v_3 + \frac{1}{\varepsilon}\,\partial_3 v_\alpha \right) + b^\sigma_\alpha v_\sigma \,, \tag{3.19}$$

(3.20)

$$\tilde{\rho}_{\alpha\beta}(\boldsymbol{v}) := \partial_{\alpha\beta} v_3 + \partial_\alpha(b^\tau_\beta \boldsymbol{v}_\tau) + \partial_\beta(b^\sigma_\alpha v_\sigma) - \Gamma^\sigma_{\alpha\beta}(\partial_\sigma v_3 + 2 b^\tau_\sigma v_\tau) - b^\sigma_\beta|_\alpha v_\sigma - c_{\alpha\beta} v_3 \,.$$

Then a simple computation shows that

$$\begin{aligned}\partial_3 e^1_{\alpha||\beta}(\varepsilon)(\boldsymbol{v}) + \tilde{\rho}_{\alpha\beta}(\boldsymbol{v}) &= \partial_\alpha e^0_{\beta||3}(\varepsilon)(\boldsymbol{v}) + \partial_\beta e^0_{\alpha||3}(\varepsilon)(\boldsymbol{v}) \\ &\quad - 2\Gamma^\sigma_{\alpha\beta} e^0_{\sigma||3}(\varepsilon)(\boldsymbol{v}) - b_{\alpha\beta} e_{3||3}(\varepsilon)(\boldsymbol{v}) + \varepsilon x_3 c_{\alpha\beta} e_{3||3}(\varepsilon)(\boldsymbol{v}) \\ &\quad + \varepsilon x_3 b^\sigma_\beta|_\alpha (2 e^0_{\sigma||3}(\varepsilon)(\boldsymbol{v}) - \partial_\sigma v_3 - 2 b^\tau_\sigma v_\tau) \,,\end{aligned}$$

and consequently,

$$\begin{aligned}(3.21)\qquad &\| \partial_3 e^1_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) + \tilde{\rho}_{\alpha\beta}(\boldsymbol{v}) \|_{-1,\Omega} \\ &\leqq c_1 \Big\{ \sum_\alpha \| e^0_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) \|_{0,\Omega} + \| e_{3\|3}(\varepsilon)(\boldsymbol{v}) \|_{0,\Omega} \\ &\quad + \varepsilon \sum_\alpha \| v_\alpha \|_{0,\Omega} + \varepsilon \| v_3 \|_{1,\Omega} \Big\}\end{aligned}$$

on the one hand. Since, on the other hand, relation (3.16) implies that

$$(3.22)\qquad \left\| \frac{1}{\varepsilon} \partial_3 e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) - \partial_3 e^1_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) \right\|_{-1,\Omega} \leqq c_2 \varepsilon \sum_\alpha \| v_\alpha \|_{0,\Omega},$$

since definitions (2.8) and (3.19) together with relation (3.5) imply that

$$(3.23)\qquad \| e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) - e^0_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) \|_{0,\Omega} \leqq c_3 \varepsilon \sum_\alpha \| v_\alpha \|_{0,\Omega},$$

and since, finally, an easy computation shows that, for $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$,

$$(3.24)\qquad \rho_{\alpha\beta}(\boldsymbol{v}) = \tilde{\rho}_{\alpha\beta}(\boldsymbol{v}),$$

where $\rho_{\alpha\beta}(\boldsymbol{v})$ and $\tilde{\rho}_{\alpha\beta}(\boldsymbol{v})$ are defined as in (3.15) and (3.20) respectively, the desired inequality (3.18) follows from relations (3.21)–(3.24). □

The next lemma is crucial; it plays an essential rôle in both the proofs of a generalized Korn inequality (Theorem 4.1) and of the convergence of the scaled unknown as $\varepsilon \to 0$ (Theorem 5.1): Consider a sequence $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ of functions in the space $\boldsymbol{V}(\Omega)$ that converges weakly in $\boldsymbol{H}^1(\Omega)$ and strongly in $\boldsymbol{L}^2(\Omega)$, and let $\boldsymbol{u}$ be its limit. We first show that, if it so happens that the corresponding sequences $(e_{i\|j}(\varepsilon)(\boldsymbol{u}(\varepsilon)))_{\varepsilon>0}$ weakly converge in $L^2(\Omega)$, considerable information can then be gathered about the limit $\boldsymbol{u}$ and the functions $\gamma_{\alpha\beta}(\boldsymbol{u})$ and $\rho_{\alpha\beta}(\boldsymbol{u})$ (cf. (3.27)–(3.30)). We also show that, if in addition the corresponding sequences $(\rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)))_{\varepsilon>0}$ converge in $H^{-1}(\Omega)$, then in fact the sequence $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ strongly converges in $\boldsymbol{H}^1(\Omega)$ (cf. (3.32)).

In the following statement, the symbols $\to$ and $\rightharpoonup$ denote strong and weak convergences, repectively, and ∂_ν denotes the outer normal derivative operator along the boundary of ω.

Lemma 3.3. *Let* $\boldsymbol{V}(\Omega)$ *be the space defined in* (2.10) *and let the functions* $e_{i\|j}(\varepsilon)(\boldsymbol{v}) \in L^2(\Omega), \gamma_{\alpha\beta}(\boldsymbol{v}) \in L^2(\Omega), \rho_{\alpha\beta}(\boldsymbol{v}) \in H^{-1}(\Omega)$ *be defined for any function* $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ *as in* (2.7)–(2.9) *and* (3.14), (3.15). *Let* $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ be a sequence of functions $\boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega)$ such that

$$(3.25)\qquad \boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u}\ \ \boldsymbol{H}^1(\Omega), \quad \boldsymbol{u}(\varepsilon) \to \boldsymbol{u}\ \ \textit{in } \boldsymbol{L}^2\Omega),$$

$$(3.26)\qquad \frac{1}{\varepsilon} e_{i\|j}(\varepsilon)\big(\boldsymbol{u}(\varepsilon)\big) \rightharpoonup e^1_{i\|j}\ \textit{in } L^2(\Omega)$$

as $\varepsilon \rightharpoonup 0$. *Then*

$$(3.27)\qquad \boldsymbol{u} = (u_i)\ \ \textit{is independent of the ``transverse'' variable } x_3,$$

$$\overline{\boldsymbol{u}} = (\overline{u}_i) := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\, dx_3 \in H^1(\omega) \times H^1(\omega) \times H^2(\omega), \tag{3.28}$$

$$\overline{u}_i = \partial_\nu \overline{u}_3 = 0 \quad \text{on } \gamma_0 ,$$

$$\gamma_{\alpha\beta}(\boldsymbol{u}) = 0 , \tag{3.29}$$

$$\rho_{\alpha\beta}(\boldsymbol{u}) \in L^2(\Omega), \quad \rho_{\alpha\beta}(\boldsymbol{u}) = -\partial_3 e^1_{\alpha\|\beta} . \tag{3.30}$$

If, in addition to (3.25), (3.26), *there exist functions* $\chi_{\alpha\beta} \in H^{-1}(\Omega)$ *such that*

$$\rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \to \chi_{\alpha\beta} \quad \text{in } H^{-1}(\Omega) \text{ as } \varepsilon \to 0 , \tag{3.31}$$

then

$$\boldsymbol{u}(\varepsilon) \to \boldsymbol{u} \quad \text{in } \boldsymbol{H}^1(\Omega) \text{ as } \varepsilon \to 0 , \tag{3.32}$$

$$\rho_{\alpha\beta}(\boldsymbol{u}) = \chi_{\alpha\beta} \quad \text{and thus} \quad \chi_{\alpha\beta} \in L^2(\Omega) . \tag{3.33}$$

Proof. For the sake of clarity, the proof is divided into six steps, numbered (i) to (vi). We first recall that, in Lemma 3.2 of Part I, we have shown that if $v \in L^2(\Omega), \overline{v}(y) := \frac{1}{2}\int_{-1}^{1} v(y, x_3)dx_3$ is finite for almost all $y \in \omega$, and that the function

$$\overline{v} := \frac{1}{2}\int_{-1}^{1} v\, dx_3 , \tag{3.34}$$

defined in this fashion is in $L^2(\omega)$, or in $H^1(\omega)$ if $v \in H^1(\Omega)$; in particular then, the functions $\overline{u}_i$ of (3.28) are in $H^1(\omega)$ and, by the same lemma, they vanish on γ_0. For notational brevity, we also let

$$e_{i\|j}(\varepsilon) := e_{i\|j}(\varepsilon)\,(\boldsymbol{u}(\varepsilon))$$

throughout the proof.

(i) *We first show that* $\boldsymbol{u} = (u_i)$ *is independent of* x_3 *and that* $\overline{\boldsymbol{u}} = \boldsymbol{0}$ *on* γ_0. Since the sequence $(\boldsymbol{u}(\varepsilon))_{\varepsilon > 0}$ is bounded in $\boldsymbol{H}^1(\Omega)$ by (3.25) and $e_{i\|j}(\varepsilon) \to 0$ *in* $L^2(\Omega)$ by (3.26),

$$\partial_3 u_\alpha(\varepsilon) = \varepsilon\{2e_{\alpha\|3}(\varepsilon) - \partial_\alpha u_3(\varepsilon) + 2\Gamma^\sigma_{\alpha 3}(\varepsilon)u_\sigma(\varepsilon)\} \to 0 \quad \text{in } L^2(\Omega),$$

$$\partial_3 u_3(\varepsilon) = \varepsilon e_{3\|3}(\varepsilon) \to 0 \text{ in } L^2(\Omega).$$

Hence $\partial_3 u_i = 0$, and consequently, by Lemma 3.2 of Part I,

$$\boldsymbol{u}(y, x_3) = \overline{\boldsymbol{u}}(y) \quad \text{for almost all } (y, x_3) \in \Omega,$$

where the function $\overline{\boldsymbol{u}} = (\overline{u}_i)$ of (3.28) is in the space $\boldsymbol{H}^1(\omega)$ and satisfies $\overline{\boldsymbol{u}} = \boldsymbol{0}$ on γ_0 (this last property, which was established in Lemma 3.2 (ii) of

Part I in the case where $\gamma_0 = \gamma$, can be easily extended to the case where $\gamma_0 \subset \gamma$). Thus property (3.27) is established.

(ii) *We next show that* $\overline{u}_3 \in H^2(\omega)$ *and* $\partial_\nu \overline{u}_3 = 0$ *on* γ_0. If $v \in \mathscr{C}^1(\overline{\Omega})$, an integration by parts shows that

$$\int_{-1}^{1} x_3 v(y, x_3) dx_3 = \frac{1}{2} \int_{-1}^{1} (1 - x_3^2) \partial_3 v(y, x_3)\, dx_3 \quad \text{for all} \quad y \in \overline{\omega}.$$

By Lemma 3.2 (i), (ii) of Part I, the mappings $v \in H^1(\Omega) \to \overline{v} \in H^1(\omega)$ and $v \in L^2(\Omega) \to \overline{v} \in L^2(\omega)$ are both continuous, so that the above relation remains valid (for almost all $y \in \omega$) if $v \in H^1(\Omega)$.

Let

$$\overline{\boldsymbol{u}}^1(\varepsilon) = (\overline{u}_i^1(\varepsilon)) := \frac{1}{2\varepsilon} \int_{-1}^{1} (1 - x_3^2)\ \partial_3 \boldsymbol{u}(\varepsilon)\ dx_3\,. \tag{3.35}$$

Hence we may also write

$$\overline{\boldsymbol{u}}^1(\varepsilon) = \frac{1}{\varepsilon} \int_{-1}^{1} x_3\ \boldsymbol{u}(\varepsilon)\ dx_3\,, \tag{3.36}$$

and Lemma 3.2(ii) of Part I implies that

$$\overline{\boldsymbol{u}}^1(\varepsilon) \in \boldsymbol{H}^1(\omega) \quad \text{and} \quad \overline{\boldsymbol{u}}^1(\varepsilon) = \boldsymbol{0} \quad \text{on } \gamma_0\,.$$

By assumption (3.26), $e_{3\|3}(\varepsilon) = \frac{1}{\varepsilon}\ \partial_3 u_3(\varepsilon) \to 0$ in $L^2(\Omega)$, and thus, by definition (3.35),

$$\overline{u}_3^1(\varepsilon) \to 0 \quad \text{in } L^2(\omega)\,. \tag{3.37}$$

By assumptions (3.25), (3.26) combined with relation (3.5),

$$\begin{aligned} \frac{1}{\varepsilon} \partial_3 u_\alpha(\varepsilon) &= 2e_{\alpha\|3}(\varepsilon) - \partial_\alpha u_3(\varepsilon) + 2\Gamma_{\alpha 3}^\sigma(\varepsilon) u_\sigma(\varepsilon) \\ &\rightharpoonup \left\{ -\partial_\alpha u_3 - 2b_\alpha^\sigma u_\sigma \right\} \quad \text{in } L^2(\Omega)\,, \end{aligned}$$

and thus (a strongly continuous linear mapping is also continuous for the weak topologies; cf., e.g., Brezis [1983, Th. III.9]) again by definition (3.35),

$$\overline{u}_\alpha^1(\varepsilon) \rightharpoonup \frac{1}{2} \int_{-1}^{1} (x_3^2 - 1)(\partial_\alpha u_3 + 2b_\alpha^\sigma u_\sigma) dx_3 \quad \text{in } L^2(\omega)\,. \tag{3.38}$$

Since the functions u_i are independent of x_3, since $\overline{\partial_\alpha u_3} = \partial_\alpha \overline{u}_3$ by Lemma 3.2 (ii) of Part I, and since $\frac{1}{2} \int_{-1}^{1} (x_3^2 - 1)\, dx_3 = -\frac{2}{3}$, it follows that

$$\overline{u}_\alpha^1(\varepsilon) \rightharpoonup \overline{u}_\alpha^1 := -\tfrac{2}{3}(\partial_\alpha \overline{u}_3 + 2b_\alpha^\beta \overline{u}_\beta) \quad \text{in } L^2(\omega)\,. \tag{3.39}$$

Our next objective consists in showing that, *in fact, the sequences* $(\overline{u}_\alpha^1(\varepsilon))_{\varepsilon>0}$ *weakly converge in* $H^1(\omega)$. To this end, it suffices to establish that the sequences $\left(\frac{1}{2}\left(\partial_\alpha \overline{u}_\beta^1(\varepsilon) + \partial_\beta \overline{u}_\alpha^1(\varepsilon)\right)\right)_{\varepsilon>0}$ weakly converge in $L^2(\omega)$ (by virtue of the two-dimensional Korn inequality, the norm $\|\cdot\|_{1,\omega}$ is equivalent to the norm $(\eta_\alpha) \to \{\sum_{\alpha,\beta} \| \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) \|^2_{0,\omega)}\}^{1/2}$ over the space $\{(\eta_\alpha) \in \boldsymbol{H}^1(\omega);$ $\eta_\alpha = 0$ on $\gamma_0\})$. A simple computation, based in particular on relations (3.17) and (3.36), shows that

$$\frac{1}{2}(\partial_\alpha \overline{u}^1_\beta(\varepsilon) + \partial_\beta \overline{u}^1_\alpha(\varepsilon)) = 2\overline{x_3 e^1_{\alpha\|\beta}(\varepsilon)} - 2b^\sigma_\beta|_\alpha \overline{x_3^2 u_\sigma(\varepsilon)} - 2e_{\alpha\beta}\overline{x_3^2 u_3(\varepsilon)} + b_{\alpha\beta}\overline{u}^1_3(\varepsilon) + \Gamma^\sigma_{\alpha\beta}\overline{u}^1_\sigma(\varepsilon),$$

where

$$e^1_{\alpha\|\beta}(\varepsilon) := e^1_{\alpha\|\beta}(\varepsilon)(\boldsymbol{u}(\varepsilon)), \tag{3.40}$$

and the notations of (3.17), (3.34), (3.35) have been used. By (3.16) and assumption (3.25),

$$\left\{\frac{1}{\varepsilon} e_{\alpha\|\beta}(\varepsilon) - e^1_{\alpha\|\beta}(\varepsilon)\right\} \to 0 \quad \text{in } L^2(\Omega);$$

and thus, by assumption (3.26),

$$e^1_{\alpha\|\beta}(\varepsilon) \rightharpoonup e^1_{\alpha\|\beta} \quad \text{in } L^2(\Omega);$$

consequently,

$$\overline{x_3 e^1_{\alpha\|\beta}(\varepsilon)} \rightharpoonup \overline{x_3 e^1_{\alpha\|\beta}} \quad \text{in } L^2(\omega).$$

By assumption (3.25), both sequences $(b^\sigma_\beta|_\alpha \overline{x_3^2 u_\sigma(\varepsilon)})_{\varepsilon>0}$ and $(c_{\alpha\beta}\overline{x_3^2 u_3(\varepsilon)})_{\varepsilon>0}$ strongly converge in $L^2(\omega)$; by (3.37) the sequence $(b_{\alpha\beta}\overline{u}^1_3(\varepsilon))_{\varepsilon>0}$ strongly converges in $L^2(\omega)$ and by (3.38), the sequence $(\Gamma^\sigma_{\alpha\beta}\overline{u}^1_\sigma(\varepsilon))_{\varepsilon>0}$ weakly converges in $L^2(\omega)$.

Hence the sequence $(\overline{u}^1_\alpha(\varepsilon))_{\varepsilon>0}$ weakly converges in $H^1(\omega)$; thus relation (3.39) implies that $\overline{u}^1_\alpha \in H^1(\omega)$, and therefore that $\partial_\alpha \overline{u}_3 \in H^1(\omega)$ since $b^\beta_\alpha \overline{u}_\beta \in H^1(\omega)$; we have shown in this fashion that

$$\overline{u}_3 \in H^2(\omega).$$

By (3.36), $\overline{u}^1_\alpha(\varepsilon) = 0$ on γ_0, and thus $\overline{u}^1_\alpha(\varepsilon) \rightharpoonup \overline{u}^1_\alpha$ in $H^1(\omega)$ implies that $\overline{u}^1_\alpha = 0$ on γ_0. Since $\overline{u}_i = 0$ on γ_0, relation (3.39) shows that $\partial_\alpha \overline{u}_3 = 0$ on γ_0; hence

$$\partial_\nu \overline{u}_3 = 0 \quad \text{on } \gamma_0,$$

and relations (3.28) are established.

(iii) *We next prove that* $\gamma_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \to 0$ *in* $L^2(\Omega)$ *as* $\varepsilon \to 0$ *and that* $\gamma_{\alpha\beta}(\boldsymbol{u}) = 0$. By definition (3.14), the functions $e^1_{\alpha\|\beta}(\varepsilon)$ defined in (3.40) may be also written as (cf. (3.17))

$$e^1_{\alpha\|\beta}(\varepsilon) = \frac{1}{\varepsilon}\gamma_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) + x_3 b^\sigma_\beta|_\alpha u_\sigma(\varepsilon) + x_3 c_{\alpha\beta}\boldsymbol{u}_3(\varepsilon).$$

Hence

$$||\gamma_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) - e_{\alpha||\beta}(\varepsilon)||_{0,\Omega} \leqq \varepsilon \left|\left| e^1_{\alpha||\beta}(\varepsilon) - \frac{1}{\varepsilon}\, e_{\alpha||\beta}(\varepsilon) \right|\right|_{0,\Omega} + \varepsilon ||x_3 b^\sigma_\beta|_\alpha u_\sigma(\varepsilon)||_{0,\Omega} + \varepsilon ||x_3 c_{\alpha\beta} u_3(\varepsilon)||_{0,\Omega}\,,$$

and thus, by inequality (3.16) and assumptions (3.25), (3.26),

$$\gamma_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \to 0 \quad \text{in } L^2(\Omega)\,.$$

By the same assumptions,

$$\gamma_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \rightharpoonup \gamma_{\alpha\beta}(\boldsymbol{u}) \quad \text{in } L^2(\Omega)\,.$$

Hence $\gamma_{\alpha\beta}(\boldsymbol{u}) = 0$, as was to be proved.

(iv) *We next prove that* $\rho_{\alpha\beta}(\boldsymbol{u}) = -\partial_3 e^1_{\alpha||\beta}$ *in* $L^2(\Omega)$. From inequality (3.18) and assumptions (3.25), (3.26), we infer that

$$\left\{ \frac{1}{\varepsilon}\, \partial_3 e_{\alpha||\beta}(\varepsilon) + \rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \right\} \to 0 \quad \text{in } H^{-1}(\Omega)\,.$$

The operator $\partial_3 : L^2(\Omega) \to H^{-1}(\Omega)$ being also continuous for the weak topologies, we deduce from assumption (3.26) that

$$\frac{1}{\varepsilon}\, \partial_3 e_{\alpha||\beta}(\varepsilon) \rightharpoonup \partial_3 e^1_{\alpha||\beta} \quad \text{in } H^{-1}(\Omega)\,,$$

whence

$$\left\{ \rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) + \partial_3 e^1_{\alpha||\beta} \right\} \rightharpoonup 0 \quad \text{in } H^{-1}(\Omega)\,.$$

Since, again by assumptions (3.25), (3.26),

$$\rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \rightharpoonup \rho_{\alpha\beta}(\boldsymbol{u}) \quad \text{in } H^{-1}(\Omega)\,,$$

the desired equality $\rho_{\alpha\beta}(\boldsymbol{u}) = -\partial_3 e^1_{\alpha||\beta}$ holds in $H^{-1}(\Omega)$; that it is in fact an equality in $L^2(\Omega)$ follows from the relation $\overline{u}_3 \in H^2(\omega)$ established in Step (ii), which, together with the independence of u_3 with respect to x_3 established in Step (i), implies that $u_3 \in H^2(\Omega)$.

(v) *We next prove that, under the additional assumption* (3.31), *the sequence* $(\boldsymbol{u}_3(\varepsilon))_{\varepsilon>0}$ *converges strongly in* $H^1(\Omega)$. We first note that

$$\partial_3 u_3(\varepsilon) = \varepsilon e_{3||3}(\varepsilon) \to 0 = \partial_3 u_3 \quad \text{in } L^2(\Omega)\,. \tag{3.41}$$

By a *lemma of J.-L. Lions* (mentioned for the first time in Magenes & Stampacchia [1958] and proved in Duvaut & Lions [1972, p. 110] for domains with smooth boundaries, then extended to arbitrary domains by Borchers & Sohr [1990]) and Amrouche & Girault [1994], the mapping $v \in L^2(\Omega) \to (v, (\partial_i v)) \in H^{-1}(\Omega) \times \boldsymbol{H}^{-1}(\Omega)$ is an isomorphism. In order to prove that

$$\partial_\alpha u_3(\varepsilon) \to \partial_\alpha u_3 \quad \text{in } L^2(\Omega)\,, \tag{3.42}$$

it therefore suffices to prove that

$$(3.43) \qquad \partial_\alpha u_3(\varepsilon) \to \partial_\alpha u_3 \quad \text{in } H^{-1}(\Omega)\,,$$

$$(3.44) \qquad \partial_{\alpha i} u_3(\varepsilon) \to \partial_{\alpha i} u_3 \quad \text{in } H^{-1}(\Omega)\,.$$

Relation (3.43) follows from assumption (3.25). Relation (3.41) likewise implies that

$$\partial_{\alpha 3} u_3(\varepsilon) \to 0 = \partial_{\alpha 3} u_3 \quad \text{in } H^{-1}(\Omega)\,.$$

From definition (3.15) and assumptions (3.25), we infer that

$$\rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \rightharpoonup \rho_{\alpha\beta}(\boldsymbol{u}) \quad \text{in } H^{-1}(\Omega)\,,$$

and thus, by assumption (3.31),

$$\rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \to \chi_{\alpha\beta} = \rho_{\alpha\beta}(\boldsymbol{u}) \quad \text{in } H^{-1}(\Omega)\,,$$

which proves (3.33).

Since $u_i(\varepsilon) \to u_i$ in $H^{-1}(\Omega)$ and $\partial_\alpha u_i(\varepsilon) \to \partial_\alpha u_i$ in $H^{-1}(\Omega)$ by assumption (3.25), and since

$$\begin{aligned}\partial_{\alpha\beta} u_3(\varepsilon) =&\rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) + \Gamma^\sigma_{\alpha\beta}\partial_\sigma u_3(\varepsilon) - b^\sigma_\beta(\partial_\alpha u_\sigma(\varepsilon) - \Gamma^\tau_{\alpha\sigma} u_\tau(\varepsilon))\\ &- b^\sigma_\alpha(\partial_\beta u_\sigma(\varepsilon) - \Gamma^\tau_{\beta\sigma} u_\tau(\varepsilon)) - b^\sigma_\alpha|_\beta u_\sigma(\varepsilon) + c_{\alpha\beta} u_3(\varepsilon)\,,\end{aligned}$$

relation (3.43) shows that

$$\partial_{\alpha\beta} u_3(\varepsilon) \to \partial_{\alpha\beta} u_3 \quad \text{in } H^{-1}(\Omega)\,.$$

Hence relations (3.44), and consequently relations (3.42), are proved.

(vi) *We finally prove that the sequences* $(u_\alpha(\varepsilon))_{\varepsilon>0}$ *converge strongly in* $H^1(\Omega)$. By virtue of Korn's inequality applied to functions in the space $\boldsymbol{V}(\Omega)$, this is equivalent to proving that

$$e_{ij}(\tilde{\boldsymbol{u}}(\varepsilon)) \to e_{ij}(\tilde{\boldsymbol{u}}) \quad \text{in } L^2(\Omega)\,,$$

where

$$e_{ij}(\boldsymbol{v}) := \tfrac{1}{2}(\partial_j v_i + \partial_i v_j)\,,$$

$$\tilde{\boldsymbol{u}}(\varepsilon) := (u_1(\varepsilon), u_2(\varepsilon), 0), \quad \tilde{\boldsymbol{u}} = (u_1, u_2, 0)\,.$$

In Step (iii), we have shown that

$$\gamma_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \to 0 \quad \text{in } L^2(\Omega)\,;$$

hence

$$\begin{aligned}e_{\alpha\beta}(\tilde{\boldsymbol{u}}(\varepsilon)) &= \gamma_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) + \Gamma^\sigma_{\alpha\beta} u_\sigma(\varepsilon) + b_{\alpha\beta} u_3(\varepsilon)\\ &\to \{\Gamma^\sigma_{\alpha\beta} u_\sigma + b_{\alpha\beta} u_3\} = e_{\alpha\beta}(\tilde{\boldsymbol{u}}) \quad \text{in } L^2(\Omega)\,,\end{aligned}$$

by assumption (3.25). Next,

$$\begin{aligned}e_{\alpha 3}(\tilde{\boldsymbol{u}}(\varepsilon)) &= \tfrac{1}{2}\partial_3 u_\alpha(\varepsilon)\\ &= \tfrac{1}{2}\varepsilon\{2e_{\alpha\|3}(\varepsilon) - \partial_\alpha u_3(\varepsilon) + 2\Gamma^\sigma_{\alpha 3}(\varepsilon) u_\sigma(\varepsilon)\} \to 0 = \tfrac{1}{2}\partial_3 u_\alpha = e_{\alpha 3}(\tilde{\boldsymbol{u}})\,,\end{aligned}$$

by Step (i); finally,

$$e_{33}(\tilde{\boldsymbol{u}}(\varepsilon)) = 0 = e_{33}(\tilde{\boldsymbol{u}}),$$

and the proof is complete. □

4. A generalized Korn inequality for an arbitrary surface

The key to the convergence theorem of Sec. 5 is the *generalized Korn inequality* (4.3), which involves the functions $e_{i||j}(\varepsilon)(\boldsymbol{v})$ defined in (2.7)–(2.9), instead of the "traditional" functions $\frac{1}{2}(\partial_i v_j + \partial_j v_i)$. The "constant" C/ε appearing in this inequality, together with assumption (2.3), will yield the fundamental *a priori bounds* that the family of scaled unknowns $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ satisfies (cf. Step (i) of the proof of Theorem 5.1).

Remarks. (1) We have seen in Part I that, if $\gamma_0 = \gamma$ and the surface S is regular and "uniformly elliptic", the "constant" C/ε can be replaced by a "genuine" constant, i.e., one that is *independent of* ε (the norm appearing in the left-hand side must however be slightly modified). This is why the applied body forces were $O(1)$ in Part I, instead of $O(\varepsilon^2)$ here (compare assumptions (2.3) of Part I, and assumptions (2.3) here).

(2) This generalized Korn inequality is valid for an *arbitrary* surface $S = \boldsymbol{\varphi}(\overline{\omega})$ (the only requirements are that the set ω and the mapping $\boldsymbol{\varphi}$ satisfy the assumptions of Sec. 1 of Part I), irrespectively of whether the space $\boldsymbol{V}_F(\omega)$ introduced in the asymptotic analysis of Sect. 5 reduces to $\{\boldsymbol{0}\}$ or not.

(3) Another Korn inequality with a "constant" also of the form C/ε was established in KOHN & VOGELIUS [1985] (see also ACERBI, BUTTAZZO & PERCIVALE [1988]). It holds however over the "variable" domain Ω^ε and besides, it involves the "traditional" functions $\frac{1}{2}(\partial_i^\varepsilon v_j^\varepsilon + \partial_j^\varepsilon v_i^\varepsilon)$.

Theorem 4.1. *Let the space* $\boldsymbol{V}(\Omega)$ *be defined as*

$$\boldsymbol{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega);\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\}, \tag{4.1}$$

where

$$\Gamma_0 = \gamma_0 \times [-1, 1] \quad \text{and} \quad \text{length } \gamma_0 > 0. \tag{4.2}$$

Then there exist $0 < \varepsilon_1 \leqq \varepsilon_0$ *and* $C > 0$ *such that, for all* $0 < \varepsilon \leqq \varepsilon_1$,

$$||\boldsymbol{v}||_{1,\Omega} \leqq \frac{C}{\varepsilon}\left\{\sum_{i,j} ||e_{i||j}(\varepsilon)(\boldsymbol{v})||^2_{0,\Omega}\right\}^{1/2} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega), \tag{4.3}$$

where the symmetric tensor $(e_{i||j}(\varepsilon)(\boldsymbol{v}))$ *is defined by*

$$e_{\alpha||\beta}(\varepsilon)(\boldsymbol{v}) := \tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^p_{\alpha\beta}(\varepsilon) v_p, \tag{4.4}$$

$$e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{2}\left(\partial_\alpha v_3 + \frac{1}{\varepsilon}\partial_3 v_\alpha\right) - \Gamma^\sigma_{\alpha 3}(\varepsilon) v_\sigma\,, \tag{4.5}$$

$$e_{3\|3}(\varepsilon)(\boldsymbol{v}) := \frac{1}{\varepsilon}\partial_3 v_3\,. \tag{4.6}$$

Proof. Assume that inequality (4.3) is false. Then there exist $\varepsilon_m > 0$ and $\boldsymbol{v}^m = (v_i^m) \in \boldsymbol{V}(\Omega), m = 0, 1, \ldots$ (the Latin letters m and n are used here for indexing sequences), such that

$$\varepsilon_m \to 0 \quad \text{as } m \to \infty\,, \tag{4.7}$$

$$\|\boldsymbol{v}^m\|_{1,\Omega} = 1 \quad \text{for all } m\,, \tag{4.8}$$

$$\frac{1}{\varepsilon_m} e_{i\|j}(\varepsilon_m)(\boldsymbol{v}^m) \to 0 \quad \text{in } L^2(\Omega) \text{ as } m \to \infty\,. \tag{4.9}$$

By (4.8), there exist a subsequence $(\boldsymbol{v}^n)_{n=0}^\infty$ of the sequence $(\boldsymbol{v}^m)_{m=0}^\infty$ and a function $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ such that

$$\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \text{ in } \boldsymbol{H}^1(\Omega), \quad \boldsymbol{v}^n \to \boldsymbol{v} \quad \text{in } L^2(\Omega) \text{ as } n \to \infty\,. \tag{4.10}$$

By (4.9),

$$\frac{1}{\varepsilon_n}\partial_3 e_{\alpha\|\beta}(\varepsilon_n)(\boldsymbol{v}^n) \to 0 \quad \text{in } H^{-1}(\Omega)\,,$$

and, by inequality (3.18),

$$\left\|\rho_{\alpha\beta}(\boldsymbol{v}^n) + \frac{1}{\varepsilon_n}\partial_3 e_{\alpha\|\beta}(\varepsilon_n)(\boldsymbol{v}^n)\right\|_{-1,\Omega}$$

$$\leqq C_3\left(\sum_i \|e_{i\|3}(\varepsilon_n)(\boldsymbol{v}^n)\|_{0,\Omega} + \varepsilon_n \sum_\alpha \|v_\alpha^n\|_{0,\Omega} + \varepsilon_n \|v_3^n\|_{1,\Omega}\right);$$

hence, by (4.7)–(4.9),

$$\rho_{\alpha\beta}(\boldsymbol{v}^n) \to 0 \quad \text{in } H^{-1}(\Omega) \text{ as } n \to \infty\,. \tag{4.11}$$

We may thus apply Lemma 3.3:

$$\boldsymbol{v} = (v_i) \text{ is independent of } x_3\,, \tag{4.12}$$

$$\overline{\boldsymbol{v}} = (\overline{v}_i) := \frac{1}{2}\int_{-1}^{1} \boldsymbol{v}\, dx_3 \in H^1(\omega) \times H^1(\omega) \times H^2(\omega), \tag{4.13}$$

$$\overline{v}_i = \partial_\nu \overline{v}_3 = 0 \text{ on } \gamma_0\,,$$

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = \rho_{\alpha\beta}(\boldsymbol{v}) = 0\,, \tag{4.14}$$

$$v^n \to v \quad \text{in } \boldsymbol{H}^1(\Omega) \text{ as } n \to \infty\,. \tag{4.15}$$

Relations (4.12)–(4.14) and the *rigid displacement lemma for a general surface* proved in Bernadou & Ciarlet [1976] (see also Bernadou, Ciarlet & Miara [1994] and Blouza & Le Dret [1994a] for various improvements) together imply that $\overline{v} = \mathbf{0}$, whence that $v = \mathbf{0}$ by (4.12). But this contradicts (4.8) and (4.15), and the proof is complete. □

5. Asymptotic analysis as $\varepsilon \to 0$

We now establish our main results, namely that *the scaled three-dimensional solutions $\boldsymbol{u}(\varepsilon)$ converge in $\boldsymbol{H}^1(\Omega)$ toward a limit $\boldsymbol{u}$, and that this limit, which is independent of the "transverse" variable x_3, can be identified with the solution of a two-dimensional variational problem* (cf. (5.7), (5.8)) *posed over the set ω*. This limit problem will be identified in Sec. 7 as a *scaled two-dimensional "flexural" shell problem*.

Although this is not stated here as a formal assumption, it is clear that the "interesting" case covered by the next theorem is that where the space $\boldsymbol{V}_F(\omega)$ of "inextensional displacements" defined in (5.4) contains *nonzero* functions; this point will be discussed in Sec. 7.

The functions $\gamma_{\alpha\beta}(\cdot)$ and $\rho_{\alpha\beta}(\cdot)$ defined in (5.1) and (5.2) respectively represent the covariant components of the *change of metric* and *change of curvature* tensors of the surface S. Their "three-dimensional extensions" have already been introduced (with identical notations) in (3.14), (3.15). The functions $\Gamma^\sigma_{\alpha\beta}, b_{\alpha\beta}, \ldots, c_{\alpha\beta}$ have been defined in Sec. 1 of Part I and in (3.1), (3.2). The functions $a^{\alpha\beta\sigma\tau}$ defined in (5.9) are the contravariant components of the *elasticity tensor* of the surface S.

Theorem 5.1. *Let the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ and $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ be defined for any $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ by*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \tfrac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3\,, \tag{5.1}$$

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{\eta}) = {} & \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 + b^\sigma_\beta(\partial_\alpha\eta_\sigma - \Gamma^\tau_{\alpha\sigma}\eta_\tau) \\ & + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\sigma_\alpha|_\beta\eta_\sigma - c_{\alpha\beta}\eta_3\,.\end{aligned} \tag{5.2}$$

Assume that

$$\operatorname{length}\gamma_0 > 0\,, \tag{5.3}$$

and let the space $\boldsymbol{V}_F(\omega)$ be defined by

$$\begin{aligned}\boldsymbol{V}_F(\omega) := \{ & \boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \\ & \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \textit{ in } \omega, \eta_i = \partial_\nu\eta_3 = 0 \textit{ on } \gamma_0\}.\end{aligned} \tag{5.4}$$

Let $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) \in \boldsymbol{V}(\Omega)$ *denote for* $0 < \varepsilon \leqq \varepsilon_0$ *the solution of the scaled variational problem* (2.10), (2.11). *Then there exists* $\boldsymbol{u} = (u_i) \in \boldsymbol{V}(\Omega)$ *such that*

$$\boldsymbol{u}(\varepsilon) \to \boldsymbol{u} \quad \text{in } \boldsymbol{H}^1(\Omega) \text{ as } \varepsilon \to 0\,, \tag{5.5}$$

$$\boldsymbol{u} \text{ is independent of the ``transverse'' variable } x_3\,, \tag{5.6}$$

$$\overline{\boldsymbol{u}} = (\overline{u}_i) := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\, dx_3 \in \boldsymbol{V}_F(\omega)\,, \tag{5.7}$$

and the function $\overline{\boldsymbol{u}}$ *satisfies the following two-dimensional variational problem:*

$$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\overline{\boldsymbol{u}})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\, dy = \int_\omega \left\{\int_{-1}^{1} f^i dx_3\right\}\eta_i\sqrt{a}\, dy \tag{5.8}$$

$$\text{for all } \boldsymbol{\eta} = \eta_i \in \boldsymbol{V}_F(\omega),$$

where

$$a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{(\lambda + 2\mu)}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})\,. \tag{5.9}$$

Proof. The proof is divided into six steps, numbered (i) to (vi). We let

$$e_{i\|j}(\varepsilon) := e_{i\|j}(\varepsilon)(\boldsymbol{u}(\varepsilon))$$

throughout the proof.

(i) *A priori bounds and extraction of weakly convergent sequences: The norms* $\|\frac{1}{\varepsilon}e_{i\|j}(\varepsilon)\|_{0,\Omega}$ *and* $\|u_i(\varepsilon)\|_{1,\Omega}$ *are bounded independently of* $0 < \varepsilon \leqq \varepsilon_0$. Consequently, *there exists a subsequence,* still denoted $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ for convenience, *and there exist functions* $e^1_{i\|j} \in L^2(\Omega)$ *and* $u_i \in H^1(\Omega)$ *satisfying* $u_i = 0$ on Γ_0 *such that*

$$u_i(\varepsilon) \rightharpoonup u_i \text{ in } H^1(\Omega), \quad u_i(\varepsilon) \to u_i \text{ in } L^2(\Omega)\,, \tag{5.10}$$

$$\frac{1}{\varepsilon}e_{i\|j}(\varepsilon) \rightharpoonup e^1_{i\|j} \quad \text{in } L^2(\Omega)\,. \tag{5.11}$$

(Recall that $\to$ and $\rightharpoonup$ denote strong and weak convergences, respectively.)

From inequality (3.7) and from inequality (1.5) of Part I ($0 < a_0 \leqq a(y)$ for all $y \in \overline{\omega}$), we infer that there exist constants g_0 and g_1 such that

$$0 < g_0 \leqq g(\varepsilon)(x) \leqq g_1 \quad \text{for all } 0 < \varepsilon \leqq \varepsilon_0 \text{ and all } x \in \overline{\Omega}\,. \tag{5.12}$$

From the variational equations (2.11), inequality (3.13), and the generalized Korn inequality (4.3), we infer that

$$\begin{aligned}\varepsilon^2 C^{-2}\|\boldsymbol{u}(\varepsilon)\|^2_{1,\Omega} &\leqq \sum_{i,j}\|e_{i\|j}(\varepsilon)\|^2_{0,\Omega}\\ &\leqq C_2 g_0^{-1/2}\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)\, e_{i\|j}(\varepsilon)\sqrt{g(\varepsilon)}dx\\ &= \varepsilon^2 C_2 g_0^{-1/2}\int_\Omega f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}dx\\ &\leqq \varepsilon^2 C_2 g_0^{-1/2} g_1^{1/2}\left\{\sum_i \|f^i\|^2_{0,\Omega}\right\}^{1/2}\|\boldsymbol{u}(\varepsilon)\|_{0,\Omega}\,;\end{aligned}$$

hence the assertions follow.

(ii) *The limit function $\boldsymbol{u} = (u_i)$ and the "average" $\overline{\boldsymbol{u}} = (\overline{u}_i)$ defined in* (5.7) *satisfy the following properties:*

(5.13) $$\boldsymbol{u} \text{ is independent of } x_3\,,$$

(5.14) $$\overline{\boldsymbol{u}} = \frac{1}{2}\int_{-1}^{1}\boldsymbol{u}\; dx_3 \in \boldsymbol{V}_F(\omega)\,,$$

where $\boldsymbol{V}_F(\omega)$ is the space defined in (5.4). These properties immediately follow from the convergences (5.10), (5.11) and Lemma 3.3.

(iii) *The limit functions $e^1_{i\|j}$ of* (5.11) *are related to the limit function $\boldsymbol{u}$ of* (5.10) *by*

(5.15) $$-\partial_3 e^1_{\alpha\|\beta} = \rho_{\alpha\beta}(\boldsymbol{u}) \quad in\ L^2(\Omega)\,,$$

(5.16) $$e^1_{\alpha\|3} = 0,$$

(5.17) $$e^1_{3\|3} = -\frac{\lambda}{\lambda+2\mu}a^{\alpha\beta}e^1_{\alpha\|\beta}\,.$$

Relation (5.15) follows from Lemma 3.3. Let $\boldsymbol{v} = (v_i)$ be an arbitrary function in the space $\boldsymbol{V}(\Omega)$. Then

(5.18) $$\varepsilon e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}) \to 0 \quad \text{in } L^2(\Omega)\,,$$

(5.19) $$\varepsilon e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}) \to \frac{1}{2}\partial_3 v_\alpha \quad \text{in } L^2(\Omega)\,,$$

(5.20) $$\varepsilon e_{3\|3}(\varepsilon)(\boldsymbol{v}) = \partial_3 v_3 \quad \text{for all } \varepsilon > 0\,.$$

Using the variational equations (2.11) and relations (3.12), we may write

$$\begin{aligned}&\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)e_{i\|j}(\varepsilon)(\boldsymbol{v})\sqrt{g(\varepsilon)}\; dx\\ &= \int_\Omega\left\{A^{\alpha\beta\sigma\tau}(\varepsilon)\left[\frac{1}{\varepsilon}e_{\sigma\|\tau}(\varepsilon)\right] + A^{\alpha\beta 33}(\varepsilon)\left[\frac{1}{\varepsilon}e_{3\|3}(\varepsilon)\right]\right\}\left\{\varepsilon e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v})\right\}\sqrt{g(\varepsilon)}\; dx\end{aligned}$$

$$+ \int_\Omega \left\{ 4A^{\alpha 3\sigma 3}(\varepsilon)\left[\frac{1}{\varepsilon}e_{\sigma||3}(\varepsilon)\right]\right\}\left\{\varepsilon e_{\alpha||3}(\varepsilon)(\boldsymbol{v})\right\}\sqrt{g(\varepsilon)}\,dx$$

$$+ \int_\Omega \left\{ A^{33\sigma\tau}(\varepsilon)\left[\frac{1}{\varepsilon}e_{\sigma||\tau}(\varepsilon)\right] + A^{3333}(\varepsilon)\left[\frac{1}{\varepsilon}e_{3||3}(\varepsilon)\right]\right\}\left\{\varepsilon e_{3||3}(\varepsilon)(\boldsymbol{v})\right\}\sqrt{g(\varepsilon)}\,dx$$

$$= \varepsilon^2 \int_\Omega f^i v_i \sqrt{g(\varepsilon)}\,dx.$$

Keep $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$ *fixed* and let $\varepsilon \to 0$. Using relations (3.8), (3.10), (3.11), (5.18)–(5.20) and the weak convergences (5.11), we obtain

$$\int_\Omega \left\{ 2\mu a^{\alpha\sigma} e^1_{\sigma||3}\partial_3 v_\alpha + \left[\lambda a^{\sigma\tau} e^1_{\sigma||\tau} + (\lambda + 2\mu)e^1_{3||3}\right]\partial_3 v_3 \right\}\sqrt{a}\,dx = 0.$$

Letting $\boldsymbol{v}$ vary in $\boldsymbol{V}(\Omega)$ then yields relations (5.16), (5.17).

(iv) *The function* $\overline{\boldsymbol{u}} := (\overline{u}_i)$, *which belongs to the space* $\boldsymbol{V}_F(\omega)$ (Step (ii)), *satisfies the variational equations* (5.8). Consequently, since these equations have a *unique* solution (as observed in Ciarlet & Sanchez-Palencia [1995], this is an immediate corollary of Theorem 6.1-2 of Bernadou & Ciarlet [1976], of Theorem 2.1 of Bernadou, Ciarlet & Miara[1994], or of Blouza & Le Dret [1994a, 1994b]), *the convergences* (5.10) *hold for the whole family* $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ (if the function $\overline{\boldsymbol{u}}$ is unique, so is the function $\boldsymbol{u}$, by (5.13)). Note that the same conclusion will also be reached for the families $(e_{i||j}(\varepsilon))_{\varepsilon>0}$, but not before Step (v).

Given an arbitrary element $\boldsymbol{\eta} = (\eta_i)$ in the space $\boldsymbol{V}_F(\omega)$, let the function $\boldsymbol{v}(\varepsilon) = (v_i(\varepsilon))$ be defined a.e. in Ω for all $\varepsilon > 0$ by

(5.21) $$v_\alpha(\varepsilon) := \eta_\alpha - \varepsilon x_3 \theta_\alpha, \quad \text{with} \quad \theta_\alpha := \partial_\alpha \eta_3 + 2b^\sigma_\alpha \eta_\sigma\,,$$

(5.22) $$v_3(\varepsilon) := \eta_3\,.$$

(This idea is due to Miara & Sanchez-Palencia [1996].)

First, we clearly have

(5.23) $$\boldsymbol{v}(\varepsilon) \in \boldsymbol{V}(\Omega) \quad \text{for all } \varepsilon > 0.$$

Next, we show that, for a fixed element $\boldsymbol{\eta} \in \boldsymbol{V}_F(\omega)$,

(5.24) $$\frac{1}{\varepsilon}\, e_{\alpha||\beta}(\varepsilon)(\boldsymbol{v}(\varepsilon)) \to \{-x_3\rho_{\alpha\beta}(\boldsymbol{\eta})\} \quad \textit{in } L^2(\Omega) \textit{ as } \varepsilon \to 0,$$

(5.25) $$\frac{1}{\varepsilon}\, e_{\alpha||3}(\varepsilon)(\boldsymbol{v}(\varepsilon))_{\varepsilon>0} \text{ converges in } L^2(\Omega),$$

(5.26) $$e_{3||3}(\varepsilon)(\boldsymbol{v}(\varepsilon)) = 0 \quad \text{for all } \varepsilon > 0,$$

(5.27) $$\boldsymbol{v}(\varepsilon) \to \boldsymbol{\eta} \quad \text{in } \boldsymbol{H}^1(\Omega),$$

(only $\boldsymbol{v}(\varepsilon) \to \boldsymbol{\eta}$ in $\boldsymbol{L}^2(\Omega)$ is in fact needed here; but the stronger convergence (5.27) is needed for proving Theorem 6.2). Using the relations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$, we find that

$$\begin{aligned} e^1_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}(\varepsilon)) &= -x_3\left\{\tfrac{1}{2}(\partial_\alpha\theta_\beta + \partial_\beta\theta_\alpha) - \Gamma^\sigma_{\alpha\beta}\theta_\sigma - b^\sigma_\beta|_\alpha\eta_\sigma - c_{\alpha\beta}\eta_3\right\} - \varepsilon x_3^2 b^\sigma_\beta|_\alpha\theta_\sigma \\ &= -x_3\tilde{\rho}_{\alpha\beta}(\boldsymbol{\eta}) - \varepsilon x_3^2 b^\sigma_\beta|_\alpha\theta_\sigma, \end{aligned}$$

where the functions $e^1_{\alpha\|\beta}(\varepsilon)(\cdot)$ and $\tilde{\rho}_{\alpha\beta}(\cdot)$ are defined as in (3.17) and (3.20), respectively. Using equality (3.24), we then find that

$$e^1_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v}(\varepsilon)) = -x_3\rho_{\alpha\beta}(\boldsymbol{\eta}) - \varepsilon x_3^2 b^\sigma_\beta|_\alpha\theta_\sigma,$$

which, combined with (3.16). proves (5.24). Another computation shows that

$$\frac{1}{\varepsilon}\, e_{\alpha\|3}(\varepsilon)(\boldsymbol{v}(\varepsilon)) = -\frac{1}{\varepsilon}(\Gamma^\sigma_{\alpha3}(\varepsilon) + b^\sigma_\alpha)\eta_\sigma + x_3\Gamma^\sigma_{\alpha3}(\varepsilon)\theta_\sigma,$$

which, combined with relations (3.5), proves (5.25). Relations (5.26), (5.27) clearly hold.

Keep the function $\boldsymbol{\eta} \in \boldsymbol{V}_F(\omega)$ fixed, let $\boldsymbol{v} = \boldsymbol{v}(\varepsilon)$ in the variational equations (2.11), where $\boldsymbol{v}(\varepsilon) = (v_i(\varepsilon))$ is defined in (5.21), (5.22), and let $\varepsilon \to 0$. Using relations (3.7), (3.8), (5.9), (5.16), (5.17), (5.24)–(5.27), and the weak convergences (5.11), we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_\Omega A^{ijkl}(\varepsilon)e_{k\|l}(\varepsilon)e_{i\|j}(\varepsilon)(\boldsymbol{v})\sqrt{g(\varepsilon)}\,dx \\ &= \lim_{\varepsilon\to 0}\Bigg(\int_\Omega \left\{A^{\alpha\beta\sigma\tau}(\varepsilon)\left[\frac{1}{\varepsilon}e_{\sigma\|\tau}(\varepsilon)\right] + A^{\alpha\beta33}(\varepsilon)\left[\frac{1}{\varepsilon}e_{3\|3}(\varepsilon)\right]\right\}\left\{\frac{1}{\varepsilon}e_{\alpha\|\beta}(\varepsilon)(\boldsymbol{v})\right\}\sqrt{g(\varepsilon)}\,dx \\ &\quad + \int_\Omega \left\{A^{\alpha3\sigma3}(\varepsilon)\left[\frac{1}{\varepsilon}e_{\sigma\|3}(\varepsilon)\right]\right\}\left\{\frac{1}{\varepsilon}e_{\alpha\|3}(\varepsilon)(\boldsymbol{v})\right\}\sqrt{g(\varepsilon)}\,dx \\ &\quad + \int_\Omega \left\{A^{33\sigma\tau}(\varepsilon)\left[\frac{1}{\varepsilon}e_{\sigma\|\tau}(\varepsilon)\right] + A^{3333}(\varepsilon)\left[\frac{1}{\varepsilon}e_{3\|3}(\varepsilon)\right]\right\}\left\{\frac{1}{\varepsilon}e_{3\|3}(\varepsilon)(\boldsymbol{v})\right\}\sqrt{g(\varepsilon)}\,dx\Bigg) \\ &= -\frac{1}{2}\int_\Omega x_3 a^{\alpha\beta\sigma\tau}e^1_{\sigma\|\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dx \\ &= \lim_{\varepsilon\to 0}\int_\Omega f^i v_i(\varepsilon)\sqrt{g(\varepsilon)}\,dx = \int_\Omega f^i\eta_i\sqrt{a}\,dx = \int_\omega\left\{\int_{-1}^{1} f^i dx_3\right\}\eta_i\sqrt{a}\,dy. \end{aligned}$$

We have yet to take into account relation (5.15), viz., $\rho_{\alpha\beta}(\boldsymbol{u}) = -\partial_3 e^1_{\alpha\|\beta}$ in $L^2(\Omega)$. Since the function $\boldsymbol{u}$ is independent of x_3 (Step (ii)), relation (5.15) implies that

$$e^1_{\alpha\|\beta} = \theta_{\alpha\beta} - x_3\rho_{\alpha\beta}(\overline{\boldsymbol{u}}) \quad \text{with } \theta_{\alpha\beta} \in L^2(\omega). \tag{5.28}$$

Therefore

$$-\frac{1}{2}\int_{\Omega} x_3 a^{\alpha\beta\sigma\tau} e^1_{\sigma\|\tau}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dx = \frac{1}{2}\int_{\Omega} x_3^2 a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\overline{\boldsymbol{u}})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dx$$

$$= \frac{1}{3}\int_{\omega} a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\overline{\boldsymbol{u}})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dy,$$

and thus we have established that $\overline{\boldsymbol{u}}$ satisfies the variational equations (5.8).

(v) *The weak convergences* (5.11) *are in fact strong* i.e.,

$$\frac{1}{\varepsilon}e_{i\|j}(\varepsilon) \to e^1_{i\|j} \quad \textit{in } L^2(\Omega), \tag{5.29}$$

and besides, the limits $e^1_{i\|j}$ *are unique;* hence *each convergence* (5.29) *holds for the whole family* $\left(\frac{1}{\varepsilon}e_{i\|j}(\varepsilon)\right)_{\varepsilon>0}$

Using inequalities (3.13) and (5.12), and letting $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in the variational equations (2.11), we obtain

$$C_2^{-1}g_0^{1/2}\sum_{i,j}\left|\left|\frac{1}{\varepsilon}e_{i\|j}(\varepsilon) - e^1_{i\|j}\right|\right|^2_{0,\Omega} \leqq \Lambda(\varepsilon), \tag{5.30}$$

where

$$\Lambda(\varepsilon) := \int_{\Omega} A^{ijkl}(\varepsilon)\left(\frac{1}{\varepsilon}e_{k\|l}(\varepsilon) - e^1_{k\|l}\right)\left(\frac{1}{\varepsilon}e_{i\|j}(\varepsilon) - e^1_{i\|j}\right)\sqrt{g(\varepsilon)}\,dx$$
$$= \int_{\Omega} f^i u_i(\varepsilon)\sqrt{g(\varepsilon)}\,dx + \int_{\Omega} A^{ijkl}(\varepsilon)\left(e^1_{k\|l} - \frac{2}{\varepsilon}e_{k\|l}(\varepsilon)\right)e^1_{i\|j}\sqrt{g(\varepsilon)}\,dx.$$

The convergences (3.7), (3.8), together with the convergences $u_i(\varepsilon) \to u_i$ in $L^2(\Omega)$ (cf. (5.10); in the proof of Theorem 6.2, the "other" convergences $u_i(\varepsilon) \rightharpoonup u_i$ in $H^1(\Omega)$ obtained in (5.10) are needed) and the weak convergences (5.11) imply that

$$\Lambda := \lim_{\varepsilon\to 0}\Lambda(\varepsilon) = \int_{\Omega} f^i u_i\sqrt{a}\,dx - \int_{\Omega} A^{ijkl}(0)e^1_{k\|l}e^1_{i\|j}\sqrt{a}\,dx. \tag{5.31}$$

Using (3.10)–(3.12), (5.13), and (5.16), (5.17), we further infer that

$$\Lambda = \int_{\omega}\left\{\int_{-1}^{1} f^i dx_3\right\}\overline{u}_i\sqrt{a}\,dy - \frac{1}{2}\int_{\Omega} a^{\alpha\beta\sigma\tau}e^1_{\sigma\|\tau}e^1_{\alpha\|\beta}\sqrt{a}\,dx.$$

By (5.28),

$$\int_{\Omega} a^{\alpha\beta\sigma\tau}e^1_{\sigma\|\tau}e^1_{\alpha\|\beta}\sqrt{a}\,dx = 2\int_{\omega} a^{\alpha\beta\sigma\tau}\theta_{\sigma\tau}\theta_{\alpha\beta}\sqrt{a}\,dy + \frac{2}{3}\int_{\omega} a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\overline{\boldsymbol{u}})\rho_{\alpha\beta}(\overline{\boldsymbol{u}})\sqrt{a}\,dy,$$

and thus, since $\overline{\boldsymbol{u}}$ satisfies the variational equations (5.8) (cf. Step (iv)),

$$\Lambda = -\int_\omega a^{\alpha\beta\sigma\tau}\theta_{\sigma\tau}\theta_{\alpha\beta}\sqrt{a}\,dy.$$

Since $\Lambda \geqq 0$ (cf. inequality (5.30) and definition (5.31)) and since the tensor $(a^{\alpha\beta\sigma\tau}(y))$ is uniformly (with respect to $y \in \overline{\omega}$) positive-definite (cf. Lemma 2.1 of Bernadou, Ciarlet & Miara [1994]), we conclude that $\theta_{\alpha\beta} = 0$. These relations have two consequences: First,

$$\Lambda = 0,$$

and thus the strong convergences (5.29) hold; secondly, the functions $e^1_{\alpha\|\beta}$ are uniquely determined, since they are given by

$$e^1_{\alpha\|\beta} = -x_3\rho_{\alpha\beta}(\overline{\boldsymbol{u}}),$$

and the function $\overline{\boldsymbol{u}}$ is unique (cf. Step (iv)). That the functions $e^1_{i\|3}$ are uniquely determined then follows from relations (5.16), (5.17).

(vi) *The weak convergences in* (5.10) *are in fact strong*, i.e.,

$$u_i(\varepsilon) \to u_i \quad \textit{in } H^1(\Omega). \tag{5.32}$$

By Lemma 3.3, it remains to show that each family $\left(\rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon))\right)_{\varepsilon>0}$ strongly converges in $H^{-1}(\Omega)$. Since

$$\frac{1}{\varepsilon}\partial_3 e_{\alpha\|\beta}(\varepsilon) \to \partial_3 e^1_{\alpha\|\beta} \quad \text{in } \ H^{-1}(\Omega)$$

as a consequence of Step (v), and since

$$\rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) + \frac{1}{\varepsilon}\partial_3 e_{\alpha\|\beta}(\varepsilon) \to 0 \quad \text{in } \ H^{-1}(\Omega)$$

as a consequence of inequality (3.18), it follows that

$$\rho_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) \to \{-\partial_3 e^1_{\alpha\|\beta}\} \quad \text{in } \ H^{-1}(\Omega),$$

and thus the strong convergences (5.32) are established. $\square$

6. Consideration of surface forces

We refer to Sec. 6 of Part I for the new notation unexplained here. We now assume that the applied body force vanishes, but that *surface forces* are acting on the "upper" and "lower" faces $\boldsymbol{\Phi}(\Gamma^\varepsilon_+)$ and $\boldsymbol{\Phi}(\Gamma^\varepsilon_-)$ of the shell. Note that there are *no* surface forces acting on the portion $\boldsymbol{\Phi}(\{\gamma - \gamma_0\} \times [-\varepsilon, \varepsilon])$ of the lateral face of the shell.

Let $h^{i,\varepsilon} \in L^2(\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-)$ denote the contravariant components of the applied surface force density and let the space $\boldsymbol{V}(\Omega^\varepsilon)$ be defined as in (1.2). Then the unknown $\boldsymbol{u}^\varepsilon = (u^\varepsilon_i) \in \boldsymbol{V}(\Omega^\varepsilon)$ satisfies

$$(6.1) \qquad \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}\, dx^\varepsilon = \int_{\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-} h^{i,\varepsilon} v^\varepsilon_i d\hat{\Gamma}^\varepsilon$$

for all $\boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon)$, where $d\hat{\Gamma}^\varepsilon = (\det \nabla^\varepsilon \boldsymbol{\Phi})|\nabla^\varepsilon \boldsymbol{\Phi}^{-T}\boldsymbol{n}^\varepsilon| d\Gamma^\varepsilon$.

In addition to (2.2), we assume that there exist functions $h^i \in L^2(\Gamma_+ \cup \Gamma_-)$ *independent of* ε such that

$$(6.2) \qquad h^{i,\varepsilon}(x^\varepsilon) = \varepsilon^3 h^i(x) \quad \text{for all } \Gamma_+ \cup \Gamma_-.$$

Then the scaled unknown satisfies a *scaled three-dimensional shell* problem with a "new" right-hand side (cf. (6.4); note that the space $\boldsymbol{V}(\Omega)$ of (6.3) is the same as in Theorem 2.1):

Theorem 6.1. *The scaled unknown* $\boldsymbol{u}(\varepsilon)$ *defined in* (2.1) *satisfies*

$$(6.3) \qquad \boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega); \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0\},$$

$$(6.4) \qquad \int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{i\|j}(\varepsilon)(\boldsymbol{v})\sqrt{g(\varepsilon)}\, dx = \varepsilon^2 \int_{\Gamma_+\cup\Gamma_-} h^i v_i \sigma(\varepsilon) d\Gamma \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(\Omega),$$

where the notation is the same as in Theorem 2.1, the function $\sigma(\varepsilon) : \Gamma_+ \cup \Gamma_- \to \mathbf{R}$ is defined by

$$(6.5) \qquad \sigma(\varepsilon)(x) := (\det \nabla^\varepsilon \boldsymbol{\Phi}(x^\varepsilon))\, |\, \nabla^\varepsilon \boldsymbol{\Phi}(x^\varepsilon)^{-T} \boldsymbol{n}^\varepsilon(x^\varepsilon)| \quad \text{for all } x^\varepsilon \in \Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-,$$

and $d\Gamma$ *is the area element along the boundary of the set* Ω.

In Sec. 6 of Part I, special care (see in particular Lemma 5.2 there) had to be taken, because it could only be concluded that (for some subsequence) $u_3(\varepsilon) \rightharpoonup u_3$ in $L^2(\Omega)$ in the first step of the proof of the convergence theorem. By contrast, the "other" generalized Korn inequality established in Theorem 4.1, combined with assumptions (6.2), allows us to conclude here that $u_3(\varepsilon) \rightharpoonup u_3$ in $H^1(\Omega)$. The changes in the proof of the convergence theorem then become minor, and for this reason, are omitted (only Lemma 6.1 of Part I needs to be used). The behavior of the solution of problem (6.3), (6.4) as $\varepsilon \to 0$ is then described in the following analog of Theorem 5.1:

Theorem 6.2. *Assume that relation* (5.3) *holds. Define the functions* $h^i_+ \in L^2(\omega)$ *and* $h^i_- \in L^2(\omega)$ *by*

$$(6.6) \qquad h^i_+(y) := h^i(y, 1), \quad h^i_-(y) := h^i(y, -1) \quad \text{for } y \in \omega,$$

where the functions h^i *are those of* (6.2).

Let $\boldsymbol{u}(\varepsilon)$ *denote for* $0 < \varepsilon \leqq \varepsilon_0$ *the solution of the scaled variational problem* (6.3), (6.4). *Then there exists a function* $\boldsymbol{u} \in \boldsymbol{V}(\Omega)$ *that satisfies relations* (5.5)–(5.7), *and the function* $\overline{\boldsymbol{u}} \in V_F(\omega)$ *satisfies the two-dimensional variational problem*

(6.7) $$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\overline{\boldsymbol{u}})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dy = \int_\omega (h^i_+ + h^i_-)\eta_i\sqrt{a}\,dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_F(\omega)$.

7. Conclusions and comments

7.1. Assume that both body and surface forces, satisfying assumptions (2.3) and (6.2) respectively, are acting on the shell and let $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) \in \boldsymbol{V}(\Omega)$ denote the scaled unknown (cf. (2.1) and (2.10)) that satisfies the corresponding scaled three dimensional shell problem. Theorems 5.1 and 6.2 together imply that there exists a function $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$ vanishing on $\Gamma_0 = \gamma_0 \times [-1, 1]$ such that

(7.1) $$\boldsymbol{u}(\varepsilon) \to \boldsymbol{u} \quad \text{in } \mathbf{H}^1(\Omega) \text{ as } \varepsilon \to 0,$$

(7.2) $\boldsymbol{u}$ is independent of the transverse variable x_3,

(7.3) $$\boldsymbol{\zeta} := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\, dx_3 \in \boldsymbol{V}_F(\omega),$$

(7.4) $$\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dy = \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}\,dy$$

for all $\boldsymbol{\eta} = (\boldsymbol{\eta}_i) \in \boldsymbol{V}_F(\omega)$,

where

(7.5) $$\boldsymbol{V}_F(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega, \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\},$$

the tensors $(\gamma_{\alpha\beta}(\boldsymbol{\eta})), (\rho_{\alpha\beta}(\boldsymbol{\eta})), (a^{\alpha\beta\sigma\tau})$ are defined in (5.1), (5.2), (5.9), and

(7.6) $$p^{i,\varepsilon} := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon} dx_3^\varepsilon + (h^{i,\varepsilon}_+ + h^{i,\varepsilon}_-),$$

where $h^{i,\varepsilon}_+ := \varepsilon h^i_+, h^{i,\varepsilon}_- := \varepsilon h^i_-$ and the functions h^i_+, h^i_- are defined in (6.6).

Naturally, the interesting case covered by Theorems 5.1 and 6.2 occurs when the space $\boldsymbol{V}_F(\omega)$ *of inextensional displacements* defined in (7.5) satisfies.

(7.7) $$\boldsymbol{V}_F(\omega) \neq \{\mathbf{0}\}.$$

Relation (7.7), whose importance was noted by Sanchez-Palencia [1989a, 1990], *is essentially an assumption on the allowed geometries of the surface S and allowed sets* $\gamma_0 \subset \gamma$ (which in turn define the portion $\boldsymbol{\Phi}(\gamma_0 \times [-\varepsilon, \varepsilon])$ of the lateral surface where the "original" three-dimensional shell is clamped).

As observed by SANCHEZ-PALENCIA [1989a], it is satisfied for instance when S is a portion of a cylinder, and $\boldsymbol{\varphi}(\gamma_0)$ is contained in a generatrix of S.

Under the essential assumption (7.7), *we have therefore justified by a convergence result* (cf. (7.1)) *the two-dimensional variational equations* (7.4) *of a "flexural" shell.* In so doing, *we have also justified the formal asymptotic approach of* SANCHEZ-PALENCIA [1990] (see also MIARA & SANCHEZ PALENCIA [1996] and CAILLERIE & SANCHEZ PALENCIA [1995]) in the *"non-inhibited" case*, according to the terminology of SANCHEZ-PALENCIA [1989a].

7.2. The *existence and uniqueness* of a solution to the two-dimensional flexural shell equations (7.4) is a consequence of the uniform positive definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$ (cf., e.g., BERNADOU, CIARLET & MIARA [1994, Lemma 2.1]), of the definition of the space $\boldsymbol{V}_F(\omega)$, and of the existence of a constant $c > 0$ such that

(7.8)

$$\left\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega}\right\}^{1/2} \geqq c \left\{\sum_{\alpha} \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\right\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_K(\omega)$, where

$$\boldsymbol{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\} \ .$$

Relation (7.8) was first proved by BERNADOU & CIARLET [1976]; see also BERNADOU, CIARLET & MIARA [1994] and BLOUZA & LE DRET [1994a, 1994b].

7.3. As in equations (7.8) of Part I, we can easily deduce a more "intrinsic" convergence from the convergences (7.1), viz.,

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon} dx_3^\varepsilon \to \zeta_i \, \boldsymbol{a}^i \quad \text{in } \mathbf{H}^1(\omega). \tag{7.9}$$

7.4. *If the space* $\boldsymbol{V}_F(\omega)$ *reduces to* $\{0\}$ (the "inhibited" case of SANCHEZ-PALENCIA [1989b]), Theorems 5.1 and 6.2 can still be applied, but the only information they provide is that $\boldsymbol{u}(\varepsilon) \to \mathbf{0}$ in $\mathbf{H}^1(\Omega)$ as $\varepsilon \to 0$.

This convergence can be considerably refined when the *fundamental assumption* (5.1) *of Part I holds* (the "well-inhibited" case of SANCHEZ-PALENCIA [1989b]), i.e., *when* $\gamma_0 = \gamma$ *and the surface S is regular and elliptic* (cf. Theorems 4.2, 4.3 of Part. I). For, under assumptions (2.3), (6.2) as here on the applied forces, Theorems 5.1 and 6.2 of Part I show that

$$\frac{1}{\varepsilon^2} u_\alpha(\varepsilon) \to u_\alpha \text{ in } H^1(\Omega), \quad \frac{1}{\varepsilon^2} u_3(\varepsilon) \to u_3 \text{ in } L^2(\Omega) \quad \text{as } \varepsilon \to 0,$$

where $\boldsymbol{u} = (u_i)$ is independent of x_3 and $\boldsymbol{\zeta} := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u} dx_3$ satisfies the *membrane shell equations* (7.4) of Part I. However, this does not exhaust all the situations where $\boldsymbol{V}_F(\omega) = \{\mathbf{0}\}$, the remaining ones being treated in CIARLET & LODS [1996c].

7.5. *A plate is an example of a flexural shell,* for it is readily verified in this case that (with $\boldsymbol{\varphi}(y_1, y_2) := (y_1, y_2, 0)$ for $(y_1, y_2) \in \overline{\omega}$)

$$V_F(\omega) = \{\boldsymbol{\eta} = (0, 0, \eta_3); \eta_3 \in H^2(\omega), \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\} \neq \{\mathbf{0}\}.$$

Besides, a plane is *not* an elliptic surface in the sense understood in Part I.

If the applied forces satisfy assumptions (2.3) and (6.2) (the functions $f^{i,\varepsilon}, f^i, h^{i,\varepsilon}, h^i$ now represent *Cartesian* components), Theorems 5.1 and 6.2 together imply that there exists a function $u_3 \in H^2(\Omega)$ vanishing on $\Gamma_0 = \gamma_0 \times [-1, 1]$ such that

(7.10) $$\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) \to (0, 0, u_3) \quad \text{in } \mathbf{H}^1(\Omega) \text{ as } \varepsilon \to 0,$$

(7.11) u_3 is independent of the transverse variable x_3,

(7.12) $$\zeta_3 := \frac{1}{2} \int_{-1}^{1} u_3 \, dx_3 \in V_3(\omega),$$

(7.13) $$\frac{\varepsilon^3}{3} \int_\omega b^{\alpha\beta\sigma\tau} \, \partial_{\sigma\tau} \zeta_3 \partial_{\alpha\beta} \eta_3 \, dy = \int_\omega p^{3,\varepsilon} \eta_3 \, dy \quad \text{for all } \eta_3 \in V_3(\omega),$$

where

(7.14) $$V_3(\omega) := \{\eta_3 \in H^2(\omega); \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

(7.15) $$b^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{(\lambda + 2\mu)} \delta^{\alpha\beta} \delta^{\sigma\tau} + 2\mu(\delta^{\alpha\sigma} \delta^{\beta\tau} + \delta^{\alpha\tau} \delta^{\beta\sigma}),$$

(7.16) $$p^{3,\varepsilon} := \int_{-\varepsilon}^{\varepsilon} f^{3,\varepsilon} dx_3^\varepsilon + (h_+^{3,\varepsilon} + h_-^{3,\varepsilon}).$$

However, this result is only a special case of the "classical" convergence theorem for a plate; cf., e.g., Ciarlet [1990, Thm. 3.3-1]. For, when the asymptotic analysis is directly applied to a linearly elastic *plate* as in this reference, more "freedom" is allowed, as regards both the choice of scalings of the displacement field and the choice of the asymptotic assumptions on the applied forces. More specifically, *another* scaled unknown $\tilde{\boldsymbol{u}}(\varepsilon) = (\tilde{u}_i(\varepsilon))$ is defined in this case by (compare with (2.1))

(7.17) $$\tilde{u}_\alpha(\varepsilon)(x) = \varepsilon u_\alpha^\varepsilon(x^\varepsilon) \ , \quad \tilde{u}_3(\varepsilon)(x) = u_3^\varepsilon(x^\varepsilon)$$

for all $x^\varepsilon \in \overline{\Omega}^\varepsilon$, and the applied forces are such that there exist functions $\tilde{f}_i \in L^2(\Omega)$ and $\tilde{h}^i \in L^2(\Gamma_+ \cup \Gamma_-)$ *independent of* ε such that (compare with (2.3) and (6.2); it is assumed that relations (2.2) still hold)

(7.18) $$f^{\alpha,\varepsilon}(x^\varepsilon) = \varepsilon \tilde{f}^\alpha(x), \quad f^{3,\varepsilon}(x^\varepsilon) = \varepsilon^2 \tilde{f}^3(x) \quad \text{for all } x \in \Omega \ ,$$

(7.19) $$h^{\alpha,\varepsilon}(x^\varepsilon) = \varepsilon^2 \tilde{h}^\alpha(x), \quad h^{3,\varepsilon}(x^\varepsilon) = \varepsilon^3 \tilde{h}^3(x) \quad \text{for all } x \in \Gamma_+ \cup \Gamma_- \ .$$

The scalings (2.1), cruder than (7.17), and assumptions (2.3) and (6.2), cruder than (7.18), (7.19), (which are unavoidable for a "genuine" shell; cf. Ciarlet [1992b] and Miara & Sanchez-Palencia [1995]) have two consequences: First, the "horizontal" components $f^{\alpha,\varepsilon}$ and $h^{\alpha,\varepsilon}$ of the applied forces do not contribute to the definition of the limit two-dimensional problem (7.13) satisfied by the unknown ζ_3 (compare with Ciarlet [1990, Thm. 3.5-1(a)]). Secondly, the components of the limit displacement field (0, 0, ζ_3) found here are all of order 0 with respect to ε. Therefore, the "horizontal" components of order 1 with respect to ε of the displacement field (once descaled) are necessarily "ignored" by the present approach; these components, together with the "transverse" component of order 0, form a *Kirchhoff-Love displacement field;* cf. Ciarlet [1990, Thm. 3.5-1(c)].

Another major difference is that the asymptotic analysis of a linearly elastic plate yields a limit problem that includes *simultaneously* the *flexural* and the *membrane* equations (cf. Ciarlet [1990, Thm. 3.5-1(a), (b)]). As hinted at *supra,* this is so because "horizontal" components of order 1 of the limit displacement problem can be also recovered, thanks to the "refined" scalings (7.17) and assumptions (7.18), (7.19). By contrast, the asymptotic analysis of a linearly elastic shell yields *either* the *flexural* equations (as here), or the *membrane* equations (cf. Part I), according to the geometry of the middle surface of the shell and the set where a boundary condition is imposed.

Other subtle "asymptotic differences" appear when "a shell becomes a plate", i.e., when the mapping $\boldsymbol{\varphi}$ is imbedded in a one-parameter family of smooth mappings $\boldsymbol{\varphi}(t) : \overline{\omega} \to \mathbf{R}^3, 0 \leqq t \leqq 1$, in such a way that $\boldsymbol{\varphi}(1) = \boldsymbol{\varphi}$ and $\boldsymbol{\varphi}(0)(y_1, y_2) = (y_1, y_2, 0), (y_1, y_2) \in \overline{\omega}$. Unexpected behaviors then occur when $t \to 0$; see in this respect Ciarlet [1992a, 1992b] and Lods [1996].

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