COMPUTER METHODS IN APPLIED MECHANICS AND ENGINEERING 26 (1981) 145–172
NORTH-HOLLAND PUBLISHING COMPANY

# TWO-DIMENSIONAL APPROXIMATIONS OF THREE-DIMENSIONAL EIGENVALUE PROBLEMS IN PLATE THEORY

P.G. CIARLET

*Analyse Numérique, Tour 55–65, Université Pierre et Marie Curie, 4, Place Jussieu, 75005 Paris, France*

S. KESAVAN

*TIFR Centre, Indian Institute of Science, Bangalore 560 012, India*

Received 19 November 1979

The eigenvalues and eigenfunctions corresponding to the three-dimensional equations for the linear elastic equilibrium of a clamped plate of thickness $2\epsilon$, are shown to converge (in a specific sense) to the eigenvalues and eigenfunctions of the well-known two-dimensional biharmonic operator of plate theory, as $\epsilon$ approaches zero. In the process, it is found in particular that the displacements and stresses are indeed of the specific forms usually assumed *a priori* in the literature. It is also shown that the limit eigenvalues and eigenfunctions can be equivalently characterized as the leading terms in an asymptotic expansion of the three-dimensional solutions, in terms of powers of $\epsilon$. The method presented here applies equally well to the stationary problem of linear plate theory, as shown elsewhere by P. Destuynder.

## 1. Introduction

It is the purpose of this work (whose results were announced in [1]) to show how the standard biharmonic, two-dimensional eigenvalue problem for a clamped plate (cf. eq. (1.5) below) can be derived *mathematically*, i.e., through a rigorous convergence analysis as the thickness of the plate converges to zero, and without any *a priori* assumption (whether of a mechanical or geometrical nature) from the standard three-dimensional eigenvalue problem of linear elasticity (1.1).

More specifically, we consider (cf. section 2) a *plate* $\Omega^\epsilon = \bar{\omega} \times [-\epsilon, \epsilon]$, $\omega \subset \mathbf{R}^2$, $\epsilon > 0$, *clamped* (the displacement vanishes) on the lateral surface $\Gamma_0^\epsilon = \partial\omega \times [-\epsilon, \epsilon]$. The associated *eigenvalue problem* then consists in finding the *displacement* field $u = (u_i)$, the *stress tensor* field $\sigma = (\sigma_{ij})$, and *real numbers* $\zeta$ which satisfy (with standard notation, subsequently detailed)

$$\begin{aligned} &-\zeta u_i = \partial_j \sigma_{ij} && \text{in } \Omega^\epsilon, \\ &\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{pp}\delta_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \overset{\text{def}}{=} \gamma_{ij}(u) && \text{in } \Omega^\epsilon, \\ &u = 0 && \text{on } \Gamma_0^\epsilon, \\ &\sigma_{i3} = 0 && \text{on } \Gamma_\pm^\epsilon = \omega \times \{\pm\epsilon\}. \end{aligned} \tag{1.1}$$

This problem is put in variational form, the solutions $(u, \sigma)$ being sought in the space

$$\{v \in (H^1(\Omega^\epsilon))^3; v = 0 \quad \text{on } \Gamma_0^\epsilon\} \times \{\tau \in (L^2(\Omega^\epsilon))^9; \tau_{ij} = \tau_{ji}\}.$$

We then define in section 3 a problem equivalent to problem (1.1), but there posed over a domain $\bar{\Omega} = \bar{\omega} \times [-1, 1]$ which does *not* depend on $\epsilon$. Such a transformation also involves appropriate transformations on the unknowns $(\sigma, u, \zeta)$. In this fashion, the 'new' unknowns, denoted $(\sigma^\epsilon, u^\epsilon, \zeta^\epsilon)$, are solutions of variational equations of the form (cf. Proposition 1)

$$\begin{aligned} &\forall \tau \in \Sigma, \quad \mathcal{A}_0(\sigma^\epsilon, \tau) + \epsilon^2 \mathcal{A}_2(\sigma^\epsilon, \tau) + \epsilon^4 \mathcal{A}_4(\sigma^\epsilon, \tau) + \mathcal{B}(\tau, u^\epsilon) = 0, \\ &\forall v \in V, \quad \mathcal{B}(\sigma^\epsilon, v) + \epsilon^2 \zeta^\epsilon \int_\Omega u_\alpha^\epsilon v_\alpha + \zeta^\epsilon \int_\Omega u_3^\epsilon v_3 = 0, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} &\Sigma = \{\tau = (\tau_{ij}) \in (L^2(\Omega))^9;\ \tau_{ij} = \tau_{ji}\}, \\ &V = \{v = (v_i) \in (H^1(\Omega))^3;\ v = 0 \quad \text{on } \Gamma_0 = \partial\omega \times [-1, 1]\}, \end{aligned}$$

and where the bilinear forms $\mathcal{A}_0$, $\mathcal{A}_2$, $\mathcal{A}_4$, $\mathcal{B}$ are *independent of* $\epsilon$ (cf. eqs. (3.9)–(3.12) for their explicit forms). The specific form of eqs. (1.2) is particularly amenable to a 'limit' analysis when $\epsilon$ approaches zero, as our main result shows (cf. Theorem 1 and section 4): let $(\sigma_l^\epsilon, u_l^\epsilon, \zeta_l^\epsilon)_{l \geq 1}$, with $\sigma_l^\epsilon = (\sigma_{lij}^\epsilon)$ and $u_l^\epsilon = (u_{li}^\epsilon)$, be the set of all eigenfunctions and eigenvalues, solutions of problem (1.2). Then the following *convergence results* hold:

$$\lim_{\epsilon \to 0} \zeta_l^\epsilon = \zeta_l^0, \tag{1.3}$$

$$\lim_{\epsilon \to 0} u_{l3}^\epsilon = u_{l3}^0 \quad \textit{in the space } V, \tag{1.4}$$

*where* $(u_{l3}^0, \zeta_l^0)$ *is precisely the lth eigensolution of the two-dimensional eigenvalue problem: find pairs* $(u_3, \zeta) \in H_0^2(\omega) \times \mathbf{R}$ *such that*

$$\forall v \in H_0^2(\omega), \ \frac{E}{3(1-\nu^2)} \int_\omega \Delta u \Delta v = \zeta \int_\omega uv. \tag{1.5}$$

*REMARK.* As expected, the convergence (1.4) holds only for a *subsequence* if $\zeta_l^0$ happens to be a multiple eigenvalue of problem (1.5).

We recognize in (1.5) the *standard biharmonic model in linear plate theory* (up to a multiplicative constant; cf. (3.24)). In particular then, the 'limit' function $u_{l3}^0$ is independent of the variable $x_3$ and, besides, it can be identified with a function in the space $H_0^2(\omega)$ (whereas it is *a priori* 'only' in the space $H^1(\Omega)$).

We also have:

$$\lim_{\epsilon \to 0} u_{l\alpha}^\epsilon = u_{l\alpha}^0 \quad \textit{in the space } V, \quad \alpha = 1, 2, \tag{1.6}$$

*where*

$$u^0_{l\alpha} = -x_3\partial_\alpha u^0_{l3} \tag{1.7}$$

*and*

$$\lim_{\epsilon\to 0} \sigma^\epsilon_{l\alpha\beta} = \sigma^0_{l\alpha\beta} \quad \text{in } L^2(\Omega), \quad \alpha, \beta = 1, 2, \tag{1.8}$$

*where*

$$\sigma^0_{l\alpha\beta} = \frac{E}{(1-\nu^2)}\{(1-\nu)\gamma_{\alpha\beta}(u^0_l) + \nu\gamma_{\mu\mu}(u^0_l)\delta_{\alpha\beta}\}, \tag{1.9}$$

while, as regards the remaining stresses $\sigma^0_{li3}$, we only prove

$$\lim_{\epsilon\to 0} \epsilon\sigma^\epsilon_{l\alpha 3} = 0 \quad \textit{in } L^2(\Omega), \quad \alpha = 1, 2, \tag{1.10}$$

$$\lim \epsilon^2\sigma^\epsilon_{l33} = 0 \quad \textit{in } L^2(\Omega). \tag{1.11}$$

Observe that the limit displacement field $(u^0_{li})$ is precisely a *Kirchhoff–Love* field and that the expressions found for the limit stresses $\sigma^0_{l\alpha\beta}$ are also the standard ones.

The particular form of eqs. (1.2) also suggests that we expand, at least *formally*, any solution $(\sigma^\epsilon_l, u^\epsilon_l, \zeta^\epsilon_l)$ thereof as

$$(\sigma^\epsilon_l, u^\epsilon_l, \zeta^\epsilon_l) = (\sigma^0_l, u^0_l, \zeta^0_l) + \epsilon(\sigma^1_l, u^1_l, \zeta^1_l) + \cdots, \tag{1.12}$$

so that by equating to zero the factors of $\epsilon^p$, $p \geq 0$, in the resulting equations, we find relations to be satisfied by the successive terms. We briefly study this alternate viewpoint in section 5 where the main conclusion (cf. Theorem 2) is that *the first term* $(\sigma^0_l, u^0_l, \zeta^0_l)$ *found in this fashion coincides with the one obtained by the limit analysis* (with a special argument for the stresses $\sigma^0_{li3}$).

The *asymptotic expansion* method, exemplified by (1.12) for the eigenvalue problem, has been shown elsewhere to justify successfully two-dimensional, linear and nonlinear, plate models [2, 3, 4] and also shell models [5]. It is, however, a purely *formal* approach.

The value of the present convergence analysis is that *it justifies the asymptotic expansion method*, in the case of *linear* problems at least. For, although we have presented it in the case of the eigenvalue problem, which has an interest *per se*, it should be emphasized that *the present method applies equally well to the stationary problem* (considered in [2]), as shown by P. Destuynder in his thesis [5], where more general spaces are also introduced for handling the stresses $\sigma^\epsilon_{li3}$.

A further step in the convergence analysis would involve error estimates, but this would require the computation of further terms in the asymptotic expansion, and also the computation of the so-called *corrector functions*, a familiar process in asymptotic analysis [6]. This is done at length in [5] for the stationary clamped plate problem. We also mention the general theory developed in [7, 8] for the homogeneization of eigenvalue problems.

Further references relevant to this paper are [9], in which there is a limit analysis, which bears some resemblance with the one presented here for handling thin inclusions in an elastic body, and [10], where the asymptotic expansion method is applied to the study of flexural vibrations of straight elastic beams.

To conclude this presentation, we also mention one important *practical* motivation which lies behind the present work. There now exist well-established and successful finite element methods (conforming, mixed, hybrid, etc.) for approximating plate problems posed as two-dimensional, fourth-order problems (as in (1.5)), and the corresponding error analysis, for both the stationary and the eigenvalue problem, is well-known (see notably [11, 12, 13] and the references therein). Therefore it makes as much sense to study the *other* approximation, which results in passing from the three-dimensional to the two-dimensional model: the present work, as well as the other works [2–5] previously cited, constitutes a step in this direction.

Let us now briefly review the notation employed: the usual partial derivatives will be denoted $\partial_i v = \partial v/\partial x_i$, $\partial_{\alpha\beta} v = \partial^2 v/\partial x_\alpha \partial x_\beta$. If $\mathcal{O}$ is an open subset of $\mathbf{R}^n$, we denote by $|\cdot|_{0,\mathcal{O}}$ and $\|\cdot\|_{1,\mathcal{O}}$ the associated norms in the space $L^2(\mathcal{O})$ and $H^1(\mathcal{O})$ respectively, with the same notation for vector-valued functions. For brevity, we systematically omit the integration symbols, i.e., we shall simply write $\int v$ instead of $\int v \, dx$.

As a rule, Greek indices $\alpha$, $\beta$, $\mu$, ..., take their values in the set {1, 2}, while Latin indices $i$, $j$, $p$, ..., take their values in the set {1, 2, 3}. We shall also use the repeated index convention for summation in conjunction with the above rule. It should be noted however that the summation rule does *not* apply to the particular indices $k$, $l$ which shall be reserved for numbering eigenvalues and eigenfunctions.

Finally, given an open subset $\mathcal{O}$ of $\mathbf{R}^n$, $n = 2$ or 3, we shall use the following spaces of symmetric tensors:

$$(L^2(\mathcal{O}))^4_s = \{\tau = (\tau_{\alpha\beta}) \in (L^2(\mathcal{O}))^4;\ \tau_{12} = \tau_{21}\},$$
$$(L^2(\mathcal{O}))^9_s = \{\tau = (\tau_{ij}) \in (L^2(\mathcal{O}))^9;\ \tau_{ij} = \tau_{ji}\}.$$

## 2. The three-dimensional problem

Let $\omega$ be a bounded open subset of the plane, whose boundary $\gamma$ is assumed to be sufficiently smooth for all subsequent purposes. Given a constant $\epsilon > 0$, we let

$$\Omega^\epsilon = \omega \times ]-\epsilon, \epsilon[, \qquad \Gamma_0^\epsilon = \gamma \times [-\epsilon, \epsilon],$$
$$\Gamma_+^\epsilon = \omega \times \{\epsilon\}, \qquad \Gamma_-^\epsilon = \omega \times \{-\epsilon\},$$

so that the boundary of the set $\Omega^\epsilon$ is partitioned into the sets $\Gamma_0^\epsilon$, $\Gamma_+^\epsilon$, $\Gamma_-^\epsilon$.

We are concerned with the problem of finding the *displacement vector* $u = (u_i)$ and the *stress tensor* $\sigma = (\sigma_{ij})$ of a three-dimensional body which occupies the set $\bar{\Omega}^\epsilon$ in its equilibrium position. Because the *thickness* $2\epsilon$ is thought of as being 'small' compared with the dimensions of the set $\omega$, the body under consideration is called a *plate*, and $\omega$ is its *middle surface*.

The plate is assumed to be *clamped*, in the sense that the displacement $u$ is imposed to vanish on the lateral surface $\Gamma_0^\epsilon$.

In linear elasticity theory, the associated evolution problem is governed by the following classical equations ($t$ is time):

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \partial_j \sigma_{ij} + \rho f_i \quad \text{in } \Omega^\epsilon \text{ for all } t, \tag{2.1}$$

$$\sigma_{ij} = a_{ijpq} \gamma_{pq}(u) \quad \text{in } \Omega^\epsilon \text{ for all } t, \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_0^\epsilon \text{ for all } t, \tag{2.3}$$

$$\sigma_{i3} = \pm g_i \quad \text{on } \Gamma_\pm^\epsilon \text{ for all } t, \tag{2.4}$$

where $\rho$ is the density of the constituting material of the plate, $(f_i)$: $\bar{\Omega}^\epsilon \to \mathbf{R}^3$ is the volumic density of the applied body forces, the functions $a_{ijpq}$: $\bar{\Omega}^\epsilon \to \mathbf{R}$ are the elasticity coefficients of the constituting material, the functions

$$\gamma_{ij}(u) = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{2.5}$$

represent the components of the (linearized) strain tensor $\gamma(u) = (\gamma_{ij}(u))$, and $(g_i)$: $\Gamma_+^\epsilon \cup \Gamma_-^\epsilon \to \mathbf{R}^3$ is the density (per unit area) of the applied surface forces on the upper and lower faces, $\Gamma_+^\epsilon$ and $\Gamma_-^\epsilon$.

*REMARKS.* (1) Of course, the functions $f_i$ and $g_i$ may also depend on $t$. (2) In order that the above problem be well-posed, one also needs *initial* conditions, so that an existence theory can be established; cf. Duvaut and Lions [14, p. 124]. We shall be interested in a different problem, however.

We shall confine ourselves to the simplest *constitutive equations* (2.2), where the elasticity coefficients are all expressed in terms of two constants, the *Young modulus* E and the *Poisson coefficient* $\nu$. More specifically, we assume that eqs. (2.2) are of the form

$$\sigma = A^{-1}(\gamma(u)), \tag{2.6}$$

where, for each tensor $X = (X_{ij})$, The tensor $Y = AX = (Y_{ij})$ is defined by

$$Y_{ij} = (AX)_{ij} = \left(\frac{1+\nu}{E}\right) X_{ij} - \frac{\nu}{E} X_{pp} \delta_{ij}. \tag{2.7}$$

We recall that the inequalities

$$E > 0, \qquad 0 < \nu < \tfrac{1}{2} \tag{2.8}$$

hold (for physical reasons).

In the absence of applied forces ($f_i = 0$, $g_i = 0$), the problem of finding *stationary solutions*, i.e., solutions of the form

$$u_i(x, t) = u_i(x)\mathrm{Re}\{e^{i\mu t}\}; \quad \mu \in \mathbf{R}, \tag{2.9}$$

is therefore equivalent to the following *eigenvalue problem* (cf. e.g. Landau and Lifchitz [15, eq. (22, 14)]):

$$-\rho\mu^2 u_i = \partial_j \sigma_{ij} \quad \text{in } \Omega^\epsilon, \tag{2.10}$$

$$\sigma = A^{-1}(\gamma(u)) \quad \text{in } \Omega^\epsilon, \tag{2.11}$$

$$u = 0 \quad \text{on } \Gamma_0^\epsilon, \tag{2.12}$$

$$\sigma_{i3} = 0 \quad \text{on } \Gamma_\pm^\epsilon. \tag{2.13}$$

Notice that the functions $u_i$ and $\sigma_{ij}$ are *independent* of $t$ and that *the number* $\mu$ *is also an unknown.*

In order to write the above problem in variational form, we introduce the spaces

$$V^\epsilon = \{v = (v_i) \in (H^1(\Omega^\epsilon))^3;\ v = 0 \quad \text{on } \Gamma_0^\epsilon\}, \tag{2.14}$$

$$\Sigma^\epsilon = (L^2(\Omega^\epsilon))_s^9. \tag{2.15}$$

Letting

$$\zeta = \rho\mu^2, \tag{2.16}$$

problem (2.10)–(2.13) is, at least *formally*, equivalent to *finding elements* $(\sigma, u, \zeta) \in \Sigma^\epsilon \times V^\epsilon \times \mathbf{R}$ *which satisfy*

$$\forall \tau \in \Sigma^\epsilon, \quad \int_{\Omega^\epsilon} (A\sigma)_{ij}\tau_{ij} - \int_{\Omega^\epsilon} \tau_{ij}\gamma_{ij}(u) = 0, \tag{2.17}$$

$$\forall v \in V^3, \quad \int_{\Omega^\epsilon} \sigma_{ij}\gamma_{ij}(v) - \zeta \int_{\Omega^\epsilon} u_i v_i = 0. \tag{2.18}$$

If we eliminate the unknowns $\sigma_{ij}$ by means of the constitutive equations, the above problem reduces to that of finding elements $(u, \zeta) \in V^\epsilon \times \mathbf{R}$ which satisfy

$$\forall v \in V^\epsilon, \quad \int_{\Omega^\epsilon} (A^{-1}\gamma(u))_{ij}\gamma_{ij}(v) = \zeta \int_{\Omega^\epsilon} u_i v_i. \tag{2.19}$$

*REMARK.* It is crucial for the success of the method subsequently presented, however, that we start out with the so-called *stress-displacement formulation* (2.17)–(2.18), also known as the *Hellinger–Reissner formulation.*

Inequalities (2.8), together with Korn's inequality (cf. Duvaut and Lions [14, chapter 3,

§3.3], or Fichera [16, section 12]), show that the symmetric bilinear form

$$(u, v) \in V^\epsilon \times V^\epsilon \to \int_{\Omega^\epsilon} (A^{-1}\gamma(u))_{ij}\gamma_{ij}(v)$$

appearing in (2.19) is $V^\epsilon$-elliptic, in the sense that

$$\exists \alpha^\epsilon > 0, \quad \forall v \in V^\epsilon, \quad \int_{\Omega^\epsilon} (A^{-1}\gamma(v))_{ij}\gamma_{ij}(v) \geq \alpha^\epsilon \|v\|_{1,\Omega^\epsilon}.$$

This property and the compactness of the injection from $V^\epsilon$ into $(L^2(\Omega^\epsilon))^3$ imply that the symmetric mapping

$$G: \quad u \in V^\epsilon \to Gu \in V^\epsilon$$

defined by

$$\forall v \in V^\epsilon, \quad \int_{\Omega^\epsilon} (A^{-1}\gamma(Gu))_{ij}\gamma_{ij}(v) = \int_{\Omega^\epsilon} u_i v_i,$$

is compact and positive definite. By the spectral theory of such operators (see e.g. Taylor [17]), there exists an increasing sequence of positive eigenvalues (which are the *inverses* of the eigenvalues of the mapping G):

$$0 < \zeta_1 \leq \zeta_2 \leq \cdots \leq \zeta_l \leq \zeta_{l+1} \leq \cdots, \quad \text{with} \lim_{l\to\infty} \zeta_l = \infty, \tag{2.20}$$

associated with eigenfunctions $u_l \in V^\epsilon$, $l \geq 1$, which can be orthonormalized so as to satisfy

$$\int_\Omega u_{ki}u_{li} = \epsilon^{-1}\delta_{kl} \qquad a(u_k, u_l) = \epsilon^{-1}\zeta_k\delta_{kl}, \tag{2.21}$$

for all integers $k,l \geq 1$, and which form a complete set in both Hilbert spaces $V^\epsilon$ and $(L^2(\Omega^\epsilon))^3$.

*REMARK.* The justification of the scaling factor $\epsilon^{-1}$ in the above equations is purely technical; it will appear in Lemma 2.

*REMARK.* In this fashion, we obtain *all* solutions of problem (2.17)–(2.18), which are of the form

$$(\sigma_l, u_l, \zeta_l) \quad \text{with } \sigma_l = A^{-1}\gamma(u_l), \quad l \geq 1. \tag{2.22}$$

If we introduce the *Rayleigh quotient*

$$R(v) = \frac{\int_{\Omega^\epsilon} (A^{-1}\gamma(v))_{ij}\gamma_{ij}(v)}{\int_{\Omega^\epsilon} v_i v_i}, \tag{2.23}$$

which is defined for all $v \in V^\epsilon - \{0\}$, the eigenvalues $\zeta_l$ satisfy the following *minimum principle* and *min–max principle*, respectively (cf. Courant and Hilbert [18]):

$$\zeta_l = \min\left\{R(v);\ \int_{\Omega^\epsilon} v_i u_{ki} = 0,\quad 1 \leq k \leq l-1\right\}, \tag{2.24}$$

$$\zeta^l = \min_{W \in \mathcal{V}_l^\epsilon} \max_{v \in W} R(v), \tag{2.25}$$

where, for each integer $l \geq 1$, $\mathcal{V}_l^\epsilon$ denotes the family of all subspaces of $V^\epsilon$ which are of dimension $l$.

## 3. Statement of the main result: existence of a 'limit' two-dimensional problem for $\epsilon = 0$

Our first task is to define a problem equivalent to problem (2.17)–(2.18), but posed over a domain which does *not* depend on $\epsilon$. Accordingly, we let

$$\Omega = \omega \times ]-1, 1[, \qquad \Gamma_0 = \gamma \times [-1, 1],$$
$$\Gamma_+ = \omega \times \{1\}, \qquad \Gamma_- = \omega \times \{-1\},$$

and, with each point $X \in \bar{\Omega}$, we associate the point $X^\epsilon \in \bar{\Omega}^\epsilon$ through the correspondence

$$X = (x_1, x_2, x_3) \in \bar{\Omega} \to X^\epsilon = (x_1, x_2, \epsilon x_3) \in \bar{\Omega}^\epsilon.$$

With the spaces $V^\epsilon$, $\Sigma^\epsilon$ of (2.14)–(2.15) and their functions $v_i$, $\tau_{ij}$, we associate the spaces $V$, $\Sigma$ and their functions $v_i^\epsilon$, $\tau_{ij}^\epsilon$, defined as follows:

$$V = \{v = (v_i) \in (H^1(\Omega))^3;\ v = 0 \quad \text{on } \Gamma_0\}, \tag{3.1}$$

$$\Sigma = (L^2(\Omega))^9_s, \tag{3.2}$$

$$v_\alpha(X^\epsilon) = v_\alpha^\epsilon(X),\ v_3(X^\epsilon) = \epsilon^{-1} v_3^\epsilon(X), \tag{3.3}$$

$$\tau_{\alpha\beta}(X^\epsilon) = \tau_{\alpha\beta}^\epsilon(X), \quad \tau_{\alpha 3}(X^\epsilon) = \epsilon \tau_{\alpha 3}^\epsilon(X), \quad \tau_{33}(X^\epsilon) = \epsilon^2 \tau_{33}^\epsilon(X), \tag{3.4}$$

and, with the real number $\zeta$ appearing in (2.18), we associate the real number $\zeta^\epsilon$ defined by

$$\zeta = \epsilon^2 \zeta^\epsilon. \tag{3.5}$$

These changes of functions are justified by the invariance, up to an appropriate multiplicative power of $\epsilon$, of one of the integrals found in eqs. (2.17), since one has

$$\int_{\Omega^\epsilon} \sigma_{ij}\gamma_{ij}(v) = \epsilon \int_\Omega \sigma_{ij}^\epsilon \gamma_{ij}(v^\epsilon).$$

This invariance, which in turn causes the breaking of the other integral found in (2.17) into three parts:

$$\int_{\Omega^\epsilon} (A\sigma)_{ij}\tau_{ij} = \epsilon\{\mathscr{A}_0(\sigma^\epsilon, \tau^\epsilon) + \epsilon^2\mathscr{A}_2(\sigma^\epsilon, \tau^\epsilon) + \epsilon^4\mathscr{A}_4(\sigma^\epsilon, \tau^\epsilon)\},$$

plays a crucial role in the limit analysis which follows, as well as in the alternate approach via the asymptotic expansion method; cf. [2, 3, 4].

We define the mapping $A^0$, which transforms any $2\times 2$ tensor $X = (X_{\alpha\beta})$ into a tensor $Y = A^0X = (Y_{\alpha\beta})$, by:

$$Y_{\alpha\beta} = (A^0X)_{\alpha\beta} = \left(\frac{1+\nu}{E}\right)X_{\alpha\beta} - \frac{\nu}{E}X_{\gamma\gamma}\delta_{\alpha\beta}. \tag{3.6}$$

It is then a purely computational matter to obtain the following result:

*PROPOSITION* 1. *Let* $(\sigma^\epsilon, u^\epsilon, \zeta^\epsilon) \in \Sigma \times V \times \mathbf{R}$ *be constructed from a solution* $(\sigma, u, \zeta) \in \Sigma^\epsilon \times V^\epsilon \times \mathbf{R}$ *of problem* (2.17)–(2.18) *through formulas* (3.3)–(3.5). *Then* $(\sigma^\epsilon, u^\epsilon, \zeta^\epsilon)$ *is a solution of the variational problem*:

$$\forall \tau \in \Sigma, \quad \mathscr{A}_0(\sigma^\epsilon, \tau) + \epsilon^2\mathscr{A}_2(\sigma^\epsilon, \tau) + \epsilon^4\mathscr{A}_4(\sigma^\epsilon, \tau) + \mathscr{B}(\tau, u^\epsilon) = 0, \tag{3.7}$$

$$\forall v \in V, \quad \mathscr{B}(\sigma^\epsilon, v) + \epsilon^2\zeta^\epsilon \int_\Omega u^\epsilon_\alpha v_\alpha + \zeta^\epsilon \int_\Omega u^\epsilon_3 v_3 = 0, \tag{3.8}$$

*where, for arbitrary elements* $\sigma, \tau \in \Sigma$ *and* $v \in V$,

$$\mathscr{A}_0(\sigma, \tau) = \int_\Omega (A^0\sigma)_{\alpha\beta}\tau_{\alpha\beta}, \tag{3.9}$$

$$\mathscr{A}_2(\sigma, \tau) = \int_\Omega \left\{2\left(\frac{1+\nu}{E}\right)\sigma_{\alpha 3}\tau_{\alpha 3} - \frac{\nu}{E}(\sigma_{33}\tau_{\gamma\gamma} + \sigma_{\gamma\gamma}\tau_{33})\right\}, \tag{3.10}$$

$$\mathscr{A}_4(\sigma, \tau) = \frac{1}{E}\int_\Omega \sigma_{33}\tau_{33}, \tag{3.11}$$

$$\mathscr{B}(\tau, v) = -\int_\Omega \tau_{ij}\gamma_{ij}(v). \tag{3.12}$$

For each integer $l \geq 1$, we denote by $(\sigma^\epsilon_l, u^\epsilon_l, \zeta^\epsilon_l)$ the solution of problem (3.7)–(3.8) constructed from the solution $(\sigma_l, u_l, \zeta_l)$ of problem (2.17)–(2.18) through formulas (3.3)–(3.5), and consequently orthonormalized by the conditions (cf. (2.21)):

$$\epsilon^2 \int_\Omega u^\epsilon_{k\alpha}u^\epsilon_{l\alpha} + \int_\Omega u^\epsilon_{k3}u^\epsilon_{l3} = \delta_{kl}. \tag{3.13}$$

In view of the terminology used for the eigenvalue problem (2.17)–(2.18), we shall continue to call $\zeta_l^\epsilon$ an *eigenvalue* of, and $(\sigma_l^\epsilon, u_l^\epsilon)$ and *eigenfunction* of, problem (3.7)–(3.8).

We now proceed to analyze, for each integer $l \geq 1$, the 'limit' behavior of the family $(\sigma_l^\epsilon, u_l^\epsilon, \zeta_l^\epsilon)$, $\epsilon > 0$, as $\epsilon$ approaches zero, and we show that this limit behavior is intimately related to the well-known two-dimensional biharmonic model of plate theory.

*THEOREM* 1. (i) *For each integer* $l \geq 1$,

$$\zeta_l^\epsilon \to \zeta_l^0, \tag{3.14}$$

*as $\epsilon$ approaches zero, where $\zeta_l^0$ is the l-th eigenvalue of the two-dimensional eigenvalue problem: find pairs $(u, \zeta) \in H_0^2(\omega) \times \mathbf{R}$ such that*

$$\forall v \in H_0^2(\omega), \quad \frac{E}{3(1-\nu^2)} \int_\omega \Delta u \Delta v = \zeta \int_\omega uv. \tag{3.15}$$

(ii) *If, for a given integer $l \geq 1$, $\zeta_l^0$ is a simple eigenvalue of problem* (3.15), *$\zeta_l^\epsilon$ is also a simple eigenvalue of problem* (3.7)–(3.8) *for $0 < \epsilon \leq \epsilon_0(l)$, and moreover, an eigenfunction $(\sigma_l^\epsilon, u_l^\epsilon)$ (normalized as in* (3.13)) *can be found for all $\epsilon \leq \epsilon_0(l)$, in such a way that*

$$u_l^\epsilon \to u_l^0 = (u_{li}^0) \quad \text{in } V, \tag{3.16}$$

$$\sigma_{l\alpha\beta}^\epsilon \to \sigma_{l\alpha\beta}^0 \quad \text{in } L^2(\Omega), \tag{3.17}$$

$$\epsilon\sigma_{l\alpha 3}^\epsilon \to 0 \quad \text{in } L^2(\Omega), \tag{3.18}$$

$$\epsilon^2\sigma_{l33}^\epsilon \to 0 \quad \text{in } L^2(\Omega), \tag{3.19}$$

*as $\epsilon$ approaches zero. The function $u_{l3}^0$, which can be identified with a function in the space $H_0^2(\omega)$, is an eigenfunction of problem* (3.15) *corresponding to the eigenvalue $\zeta_l^0$, and the functions $u_{l\alpha}^0$ and $\sigma_{l\alpha\beta}^0$ are given by*

$$u_{l\alpha}^0 = -x_3 \partial_\alpha u_{l3}^0, \tag{3.20}$$

$$\sigma_{l\alpha\beta}^0 = \frac{E}{(1-\nu^2)}\{(1-\nu)\gamma_{\alpha\beta}(u_l^0) + \nu\gamma_{\mu\mu}(u_l^0)\delta_{\alpha\beta}\}. \tag{3.21}$$

(iii) *If $\zeta_l^0$ is not simple, properties* (3.16)–(3.21) *still hold, but generally only for a subsequence.*

(iv) *The sequence $(\zeta_l^0)_{l\geq 1}$ comprises all eigenvalues, counting multiplicities, of problem* (3.15), *and the associated sequence $(u_{l3}^0)_{l\geq 1}$ forms a complete set of eigenfunctions, satisfying the normalization condition*

$$\forall k, l \geq 1, \quad \int_\omega u_{k3}^0 u_{l3}^0 = \tfrac{1}{2}\delta_{kl}. \tag{3.22}$$

Because the proof of this theorem is fairly long, it will be given for the sake of clarity in a separate section (the next one), where it is broken down into a series of lemmas and propositions. To conclude this section, we shall simply make a few comments about this result.

A first observation is that the pairs $(u_{l3}^0, \zeta_l^0) = \lim_{\epsilon\to 0}(u_{l3}^\epsilon, \zeta_l^\epsilon)$, $l \geq 1$, are indeed solutions of an eigenvalue problem for the biharmonic operator $\Delta^2$. In order to get exactly the same form as in the literature, it sufficies to go back to the set $\Omega^\epsilon$ actually occupied by the plate, using the second formulas (3.3) and formula (3.5). This shows that the 'new' functions $u_{l3} = \epsilon^{-1} u_{l3}^0$ and the 'new' numbers $\zeta_l = \epsilon^2 \zeta_l^0$ are solutions of

$$\frac{E\epsilon^2}{3(1-\nu^2)}\Delta^2 u = \zeta u. \tag{3.23}$$

If we recall that $2\epsilon = e$ is the thickness of the plate and that $\zeta = \rho\mu^2$ (cf. (2.16)), where $\rho$ is the volumic density of the constituting material of the plate, eq. (3.23) can be equivalently written as

$$\frac{Ee^3}{12(1-\nu^2)}\Delta^2 u = \sigma\mu^2 u, \tag{3.24}$$

where $\sigma = e\rho$ is now the density of the plate, considered as a two-dimensional body. This is the usual equation, as found for instance in [15, eq. (25, 6)].

Secondly, using formulas (3.3), (3.4), (3.20), (3.21), we obtain 'new' functions $u_{l\alpha}$ and $\sigma_{l\alpha\beta}$ (i.e., after returning to the set $\Omega^\epsilon$) of the form:

$$u_{l\alpha} = -x_3\partial_\alpha u_{l3}, \tag{3.25}$$

$$\sigma_{l\alpha\beta} = -\frac{Ex_3}{(1-\nu^2)}\{(1-\nu)\partial_{\alpha\beta}u_{l3} + \nu\Delta u_{l3}\delta_{\alpha\beta}\}. \tag{3.26}$$

Usually such displacement fields ($u_{l3}$ independent of $x_3$, and $u_{l\alpha}$ of the form (3.25)) are themselves derived from the *Kirchhoff–Love hypothesis*, an *a priori* assumption of a *geometrical* nature. Likewise, the *assumption* that the stresses $\sigma_{l\alpha\beta}$ vary linearly as functions of $x_3$ and vanish for $x_3 = 0$ is made *a priori* by [19].

As regards the remaining stresses $\sigma_{li3}^\epsilon$, it seems that nothing better than (3.18)–(3.19) can be proved, with respect to the norm $|\cdot|_{0,\Omega}$ at least. Using more refined arguments, due to Destuynder [5], one can show that convergence still occurs, but in *larger* spaces. More specifically, one has

$$\sigma_{l\alpha 3}^\epsilon \to \sigma_{l\alpha 3}^0 \quad \text{in } L^2(-1, 1; H^{-1}(\omega)), \tag{3.27}$$

$$\sigma_{l33}^\epsilon \to \sigma_{l33}^0 \quad \text{in } L^2(-1, 1; H^{-2}(\omega)). \tag{3.28}$$

(Consult e.g. [20] for the definition and properties of spaces such as $L^2(\alpha, \beta; X)$), where the functions $\sigma_{li3}^0$ have the following expressions:

$$\sigma^0_{l\alpha 3} = -\frac{E(1-x_3^2)}{2(1-\nu^2)}\,\partial_\alpha \Delta u^0_{l3}, \tag{3.29}$$

$$\sigma^0_{l33} = \frac{x_3 - x_3^3}{2}\,\zeta^0_l u^0_{l3}. \tag{3.30}$$

We shall briefly indicate in section 5 how these last expressions are found.

## 4. Proof of Theorem 1

As is frequently the case in limit problems, the first, and crucial step consists in obtaining *a priori bounds* independent of $\epsilon$ for the various quantities involved. This is the object of Lemmas 1 and 2.

*LEMMA* 1. *For each integer* $l \geq 1$, *there exists a constant* $\delta_l$ *independent of* $\epsilon$ *such that*

$$\zeta^\epsilon_l \leq \delta_l. \tag{4.1}$$

*PROOF.* The inverse mapping of the mapping $A$ of (2.7) takes the form

$$(A^{-1}Y)_{ij} = 2\mu Y_{ij} + \lambda Y_{pp}\delta_{ij}, \tag{4.2}$$

where the constants $\mu$ and $\lambda$, known in elasticity as the *Lamé constants*, are given by

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{4.3}$$

Combining the min–max principle (2.25) expressed in terms of the Rayleigh quotient $R$ of (2.23) and formulas (3.3) and (3.5), we find that

$$\zeta^\epsilon_l = \min_{W \in \mathcal{V}_l} \max_{v \in W} R^\epsilon(v), \tag{4.4}$$

where $\mathcal{V}_l$ denotes the set of all subspaces of $V$ which are of dimension $l$, and

$$R^\epsilon(v) = \frac{N^\epsilon(v,v)}{D^\epsilon(v,v)}, \tag{4.5}$$

where, for arbitrary functions $u$, $v \in V$,

$$N^\epsilon(u,v) = 2\mu \int_\Omega \gamma_{\alpha\beta}(u)\gamma_{\alpha\beta}(v) + \lambda \int_\Omega \gamma_{\alpha\alpha}(u)\gamma_{\beta\beta}(v) + \epsilon^{-2}\lambda \int_\Omega \gamma_{33}(u)\gamma_{\alpha\alpha}(v)$$

$$+ 2\epsilon^{-2}\left\{4\mu \int_\Omega \gamma_{\alpha 3}(u)\gamma_{\alpha 3}(v) + \lambda \int_\Omega \gamma_{\alpha\alpha}(u)\gamma_{33}(v)\right\} + \epsilon^{-4}(\lambda + 2\mu)\int_\Omega \gamma_{33}(u)\gamma_{33}(v), \tag{4.6}$$

$$D^{\epsilon}(u, v) = \epsilon^2 \int_{\Omega} u_\alpha v_\alpha + \int_{\Omega} u_3 v_3. \tag{4.7}$$

With each function $\phi \in H_0^2(\omega)$, we associate the function $v_\phi \in V$ defined by

$$v_\phi = (-x_3\partial_1\phi,\ -x_3\partial_2\phi,\ \phi), \tag{4.8}$$

and we denote by $\mathcal{U}_l$ the set of all subspaces of dimension $l$ of the space $H_0^2(\omega)$. Observing that

$$U \in \mathcal{U}_l \Rightarrow \{v_\phi \in V;\ \phi \in U\} \in \mathcal{V}_l,$$

we deduce from (4.4) that

$$\zeta_l^{\epsilon} \le \min_{U \in \mathcal{U}_l} \max_{\phi \in U} R^{\epsilon}(v_\phi).$$

Using (4.5), we obtain, for all $\phi \in H_0^2(\omega)$,

$$R^{\epsilon}(v_\phi) = \left(2\mu \int_{\Omega} x_3^2 \partial_{\alpha\beta}\phi\partial_{\alpha\beta}\phi + \lambda \int_{\Omega} x_3^2(\Delta\phi)^2\right) \Big/ \left(\epsilon^2 \int_{\Omega} x_3^2\partial_\alpha\phi\partial_\alpha\phi + \int_{\Omega} \phi^2\right) \le r(\phi),$$

with

$$r(\phi) = \frac{1}{3}\left(2\mu \int_{\omega} \partial_{\alpha\beta}\phi\partial_{\alpha\beta}\phi + \lambda \int_{\omega} (\Delta\phi)^2\right) \Big/ \int_{\omega} \phi^2,$$

and consequently,

$$\zeta_l^{\epsilon} \le \min_{U \in \mathcal{U}_l} \max_{\phi \in U} r(\phi).$$

We recognize in $r$ the Rayleigh quotient associated with the positive definite symmetric bilinear form

$$\phi, \psi \in H_0^2(\omega) \to \frac{2\mu}{3} \int_{\omega} \partial_{\alpha\beta}\phi\partial_{\alpha\beta}\psi + \frac{\lambda}{3} \int_{\omega} \Delta\phi\Delta\psi$$

((2.8) and (4.3) imply $\lambda > 0$, $\mu > 0$), and thus,

$$\min_{U \in \mathcal{U}_l} \max_{\phi \in U} r(\phi) = \delta_l,$$

where $\delta_l$ is the $l$th eigenvalue of the operator $((\lambda + 2\mu)/3)\Delta^2 u$ over the space $H_0^2(\omega)$.

*REMARK.* The special form of the 'trial' functions (4.8) is suggested by the study of the stationary problem [2].

Let us next obtain a priori bounds on the eigenfunctions. We recall that the eigenfunctions $u_l$ on $\Omega^\epsilon$ are normalized by the condition $\int_{\Omega^\epsilon} u_{ki}u_{li} = \epsilon^{-1}\delta_{kl}$ (cf. (2.21)), so that the associated eigenfunctions $u_l^\epsilon$ *on* $\Omega$ satisfy

$$D^\epsilon(u_k^\epsilon, u_l^\epsilon) = \epsilon^2 \int_\Omega u_{k\alpha}^\epsilon u_{l\alpha}^\epsilon + \int_\Omega u_{k3}^\epsilon u_{l3}^\epsilon = \delta_{kl}. \tag{4.9}$$

*LEMMA* 2. *For each integer $l \geq 1$, there exists a constant $C_l$ independent of $\epsilon$ such that*

$$|\sigma_{l\alpha\beta}^\epsilon|_{0,\Omega} + \epsilon|\sigma_{l\alpha 3}^\epsilon|_{0,\Omega} + \epsilon^2|\sigma_{l33}^\epsilon|_{0,\Omega} \leq C_l, \tag{4.10}$$

$$\|u_l^\epsilon\|_{1,\Omega} \leq C_l. \tag{4.11}$$

*PROOF.* Let $\tau = \sigma_l$ in (3.7) and $v = u_l$ in (3.8). Taking (4.9) into account, we obtain (no summation on the index $l$)

$$\begin{aligned}\zeta_l^\epsilon &= -\mathcal{B}(\sigma_l^\epsilon, u_l^\epsilon) = \mathcal{A}_0(\sigma_l^\epsilon, \sigma_l^\epsilon) + \epsilon^2\mathcal{A}_2(\sigma_l^\epsilon, \sigma_l^\epsilon) + \epsilon^4\mathcal{A}_4(\sigma_l^\epsilon, \sigma_l^\epsilon) \\ &= \int_\Omega (A\tilde{\sigma}_l^\epsilon)_{ij}\tilde{\sigma}_{lij}^\epsilon,\end{aligned} \tag{4.12}$$

with

$$\tilde{\sigma}_{l\alpha\beta}^\epsilon = \sigma_{l\alpha\beta}^\epsilon, \qquad \tilde{\sigma}_{l\alpha 3}^\epsilon = \epsilon\sigma_{l\alpha 3}^\epsilon, \qquad \tilde{\sigma}_{l33}^\epsilon = \epsilon^2\sigma_{l33}^\epsilon. \tag{4.13}$$

Because the constants $E$ and $\nu$ satisfy inequalities (2.8), it follows that

$$\exists c_0 > 0, \ \forall \tau \in \Sigma, \ \int_\Omega (A\tau)_{ij}\tau_{ij} \geq c_0|\tau|_{0,\Omega}^2, \tag{4.14}$$

and therefore inequality (4.10) is a consequence of relations (4.12)–(4.14) and Lemma 1.

Using Korn's inequality, we next find that Brezzi's condition [21] holds:

$$\exists c_1 > 0, \ \forall v \in V, \ c_1\|v\|_{1,\Omega} \leq \sup_{\tau \in \Sigma} \frac{|\mathcal{B}(\tau, v)|}{|\tau|_{0,\Omega}}. \tag{4.15}$$

An application of this inequality, together with the expressions (3.9)–(3.11) of the bilinear forms $\mathcal{A}_0$, $\mathcal{A}_2$, $\mathcal{A}_4$, leads to

$$c_1\|u_l^\epsilon\|_{1,\Omega} \leq \sup_{\tau \in \Sigma} \frac{|\beta(\tau, u_l^\epsilon)|}{|\tau|_{0,\Omega}},$$

with

$$\begin{aligned}-\mathcal{B}(\tau, u_l^\epsilon) &= \mathcal{A}_0(\sigma_l^\epsilon, \tau) + \epsilon^2\mathcal{A}_2(\sigma_l^\epsilon, \tau) + \epsilon^4\mathcal{A}_4(\sigma_l^\epsilon, \tau) \\ &= \int_\Omega (A^0\tilde{\sigma}_l^\epsilon)\tau_{\alpha\beta} - \frac{\nu}{E}\int_\Omega \tilde{\sigma}_{l33}^\epsilon\tau_{\mu\mu} \\ &\quad + \frac{2(1+\nu)}{E}\epsilon\int_\Omega \tilde{\sigma}_{l\alpha 3}^\epsilon\tau_{\alpha 3} + \frac{\epsilon^2}{E}\int_\Omega \{-\nu\tilde{\sigma}_{l\mu\mu}^\epsilon\tau_{33} + \tilde{\sigma}_{l33}^\epsilon\tau_{33}\}.\end{aligned}$$

Consequently, for all $\tau \in \Sigma$,

$$\frac{|\mathscr{B}(\tau, u_l^\epsilon)|}{|\tau|_{0,\Omega}} \leq (c_2 + c_3\epsilon + c_4\epsilon^2)|\tilde{\sigma}_l^\epsilon|_{0,\Omega} \leq c_5|\tilde{\sigma}_l^\epsilon|_{0,\Omega},$$

for appropriate constants $c_2$, $c_3$, $c_4$, $c_5$, independent of $\epsilon$, and the conclusion follows from the first part of the proof, where it was established that the norms $|\tilde{\sigma}_l^\epsilon|_{0,\Omega}$ are bounded independently of $\epsilon$.

We are now in a position to identify a 'limit' problem (when $\epsilon$ approaches zero) by extracting weakly convergent subsequences. We shall thereafter dispose of these two restrictions (weak convergence; convergence for subsequences); cf. Propositions 3 and 5.

In what follows, weak convergence is denoted by $\rightharpoonup$. Also, in order to avoid introducing further notation, the exponent defining the subsequence considered in the next propositions will still be noted $\epsilon$.

*PROPOSITION* 2. *There exists a subsequence (the same for all integers $l \geq 1$) of the family $(\sigma_l^\epsilon, u_l^\epsilon, \zeta_l^\epsilon) \in \Sigma \times V \times \mathbf{R}$, $\epsilon > 0$, and there exist elements $((\sigma_{l\alpha\beta}^0), u_l^0, \zeta_l^0) \in (L^2(\Omega))_s^4 \times V \times \mathbf{R}$, $l \geq 1$, such that, for each integer $l \geq 1$,*

$$\zeta_l^\epsilon \to \zeta_l^0, \tag{4.16}$$

$$u_l^\epsilon \rightharpoonup u_l^0 \quad \textit{in } V, \tag{4.17}$$

$$\sigma_{l\alpha\beta}^\epsilon \rightharpoonup \sigma_{l\alpha\beta}^0 \quad \textit{in } L^2(\Omega), \tag{4.18}$$

$$\epsilon\sigma_{l\alpha 3}^\epsilon \rightharpoonup 0 \quad \textit{in } L^2(\Omega), \tag{4.19}$$

$$\epsilon^2\sigma_{l33}^\epsilon \rightharpoonup 0 \quad \textit{in } L^2(\Omega), \tag{4.20}$$

*as $\epsilon$ approaches zero. The function $u_{l3}^0$, which can be identified with a function in the space $H_0^2(\omega)$, satisfies*

$$\forall v \in H_0^2(\omega), \quad \frac{E}{3(1-\nu^2)}\int_\omega \Delta u_{l3}^0 \Delta v = \zeta_l^0 \int_\omega u_{l3}^0 v, \tag{4.21}$$

*and the functions $u_{l\alpha}^0$ and $\sigma_{l\alpha\beta}^0$ are given by*

$$u_{l\alpha}^0 = -x_3\partial_\alpha u_{l3}^0, \tag{4.22}$$

$$\sigma_{l\alpha\beta}^0 = \frac{E}{(1-\nu^2)}\{(1-\nu)\gamma_{\alpha\beta}(u_l^0) + \nu\gamma_{\mu\mu}(u_l^0)\delta_{\alpha\beta}\}. \tag{4.23}$$

*PROOF.* For convenience, the proof is divided into several steps.

*Step* 1. *Proof of relations* (4.16)–(4.20). The existence of a subsequence satisfying (4.16)–(4.18), and

$$\epsilon\sigma_{l\alpha 3}^\epsilon \rightharpoonup \chi_{l\alpha 3} \quad \text{in } L^2(\Omega),$$

$$\epsilon^2\sigma_{l33}^\epsilon \rightharpoonup \chi_{l33} \quad \text{in } L^2(\Omega),$$

for each $l \geq 1$, follows from Lemmas 1 and 2. That we may choose the same subsequence for all integers $l \geq 1$ follows by an application of the diagonal procedure. It therefore remains to prove that $\chi_{li3} = 0$.

*For notational brevity, we henceforth suppress the index $l$ and the exponent $^0$ throughout the following proof.* Explicitly, eqs. (3.8) read:

$$\forall v \in V \quad \int_\Omega \{\sigma^\epsilon_{\alpha\beta}\partial_\alpha v_\beta + \sigma^\epsilon_{3\alpha}\partial_3 v_\alpha + \sigma^\epsilon_{\alpha 3}\partial_\alpha v_3 + \sigma^\epsilon_{33}\partial_3 v_3\} = \epsilon^2\zeta^\epsilon \int_\Omega u^\epsilon_\alpha v_\alpha + \zeta^\epsilon \int_\Omega u^\epsilon_3 v_3. \tag{4.24}$$

Letting $v_3 = 0$ and multiplying by $\epsilon$, we find that

$$\epsilon \int_\Omega \sigma^\epsilon_{\alpha\beta}\partial_\alpha v_\beta + \int_\Omega (\epsilon\sigma^\epsilon_{3\alpha})\partial_3 v_\alpha = \epsilon^2\zeta^\epsilon \int_\Omega u^\epsilon_\alpha v_\alpha,$$

so that passing to the limit for $\epsilon \to 0$ (for each $v \in V$) gives us:

$$\forall v_\alpha \in H^1(\Omega), \quad v_\alpha = 0 \text{ on } \Gamma_0, \quad \int_\Omega \chi_{\alpha 3}\partial_3 v_\alpha = 0$$

(by Lemmas 1 and 2, $|\sigma^\epsilon_{\alpha\beta}|_{0,\Omega}$ and $|\zeta^\epsilon|$ are bounded independently of $\epsilon$). The above variational equations are equivalent to the equations

$$\left.\begin{aligned} &\partial_3\chi_{\alpha 3} = 0 \quad \text{in } \Omega, \\ &\chi_{\alpha 3} = 0 \qquad \text{on } \Gamma_+ \cup \Gamma_-, \end{aligned}\right\}$$

for $\alpha = 1, 2$, which have the unique solutions $\chi_{\alpha 3} = 0$. To see this, observe that, because $\partial_3\chi_{\alpha 3} = 0 \in L^2(\Omega)$, we may define traces of each function $\chi_{\alpha 3}$ on the faces $\Gamma_+$ and $\Gamma_-$ as functions in the spaces $L^2(\Gamma_+)$ and $L^2(\Gamma_-)$, respectively, and these traces vanish because of the imposed boundary condition on $\Gamma_+ \cup \Gamma_-$. From $\partial_3\chi_{\alpha 3} = 0$, we also infer that the function $\chi_{\alpha 3}$ can be identified with a function in the space $L^2(\omega)$, which vanishes since it is then equal to the traces on $\Gamma_+$ and $\Gamma_-$.

Next letting $v_\alpha = 0$ in (4.24) and multiplying by $\epsilon^2$, we similarly obtain by passing to the limit,

$$\forall v \in H^1(\Omega), \quad v = 0 \text{ on } \Gamma_0 \quad \int_\Omega \chi_{33}\partial_3 v = 0,$$

from which we deduce that $\chi_{33} = 0$, and thus relations (4.19)–(4.20) are proved.

*Step 2. The functions $u_i$ and $\sigma_{\alpha\beta}$ are of the following form:*

$$u_3 \in H^2_0(\omega),$$

$$\exists u'_\alpha \in H^1_0(\omega) \quad u_\alpha = u'_\alpha - x_3\partial_\alpha u_3,$$

$$\sigma_{\alpha\beta} = \frac{E}{(1-\nu^2)}\{(1-\nu)\gamma_{\alpha\beta}(u) + \nu\gamma_{\mu\mu}(u)\delta_{\alpha\beta}\}.$$

Explicitly, eqs. (3.7) read

$$\forall \tau \in \Sigma,\ \int_\Omega (A^0\sigma^\epsilon)_{\alpha\beta}\tau_{\alpha\beta} + \frac{2(1+\nu)}{E}\epsilon\int_\Omega (\epsilon\sigma^\epsilon_{\alpha 3})\tau_{\alpha 3} - \frac{\nu}{E}\int_\Omega (\epsilon^2\sigma^\epsilon_{33})\tau_{\mu\mu} - \frac{\nu}{E}\epsilon^2\int_\Omega \sigma^\epsilon_{\mu\mu}\tau_{33} + \frac{\epsilon^2}{E}\int_\Omega (\epsilon^2\sigma^\epsilon_{33})\tau_{33} - \int_\Omega \tau_{ij}\gamma_{ij}(u^\epsilon) = 0, \tag{4.25}$$

so that, passing to the limit for each $\tau \in \Sigma$ gives us

$$\forall \tau \in \Sigma,\ \int_\Omega (A^0\sigma)_{\alpha\beta}\tau_{\alpha\beta} - \int_\Omega \tau_{ij}\gamma_{ij}(u) = 0. \tag{4.26}$$

Reproducing the arguments given in Steps 1 and 2 of the proof of Theorem 4 of [2], we first let $\tau_{i3} = 0$ in (4.26), which then reduces to

$$\forall (\tau_{\alpha\beta}) \in (L^2(\Omega))^4_s,\ \int_\Omega (A^0\sigma)_{\alpha\beta}\tau_{\alpha\beta} - \int_\Omega \tau_{\alpha\beta}\partial_\alpha u_\beta = 0. \tag{4.27}$$

This immediately gives

$$\gamma_{\alpha\beta}(u) = (A^0\sigma)_{\alpha\beta},$$

but this last relation is just another form of relation (4.23).

Next we successively let $\tau_{\alpha\beta} = \tau_{33} = 0$, and $\tau_{\alpha i} = 0$, in (4.26), which reduces to

$$\forall \tau_{\alpha\beta} \in L^2(\Omega),\ \int_\Omega \tau_{\alpha 3}(\partial_\alpha u_3 + \partial_3 u_\alpha) = 0,$$

$$\forall \tau_{33} \in L^2(\Omega),\ \int_\Omega \tau_{33}\partial_3 u_3 = 0.$$

This immediately gives

$$\partial_\alpha u_3 + \partial_3 u_\alpha = 0, \tag{4.28}$$

$$\partial_3 u_3 = 0. \tag{4.29}$$

From relation (4.29), combined with the fact that $u_3 \in V$, we deduce that the function $u_3$ can be identified with a function of the space $H^1_0(\omega)$.

By combining (4.28) and (4.29), we obtain $\partial_{33}u_\alpha = 0$. Thus there exist functions $u'_\alpha$, $u''_\alpha \in H^1_0(\omega)$ such that $u_\alpha = u'_\alpha + x_3 u''_\alpha$. Going back to (4.28), we get $\partial_\alpha u_3 = -u''_\alpha \in H^1_0(\omega)$ and thus we have shown that $u_3$ *can be identified with a function in the space* $H^2_0(\omega)$ and that *the functions* $u_\alpha$ *are of the form* $u_\alpha = u'_\alpha - x_3\partial_\alpha u_3$, *with* $u'_\alpha \in H^1_0(\omega)$.

Notice in passing that we have proved a *regularity result* regarding the function $u_3$ (*a priori* 'only' in the space $H^1(\Omega)$), which may be viewed as the first step towards the transformation into a *fourth-order* problem with respect to the function $u_3$.

*Step* 3: *The functions $u'_\alpha \in H^1_0(\omega)$ found in the previous step vanish.* Let the functions $\tau_{\alpha\beta} \in L^2(\Omega)$ in (4.27) be of the form $\tau'_{\alpha\beta}$, with $\tau'_{\alpha\beta} \in L^2(\omega)$. Using the fact that the functions $u_\alpha$ must be of the form $u_\alpha = u'_\alpha - x_3\partial_\alpha u_3$, $u_3 \in H^2_0(\omega)$, we find that

$$\forall(\tau'_{\alpha\beta}) \in (L^2(\omega))^4_s,\quad \int_\omega (A^0 n)_{\alpha\beta}\tau'_{\alpha\beta} - 2\int_\omega \tau'_{\alpha\beta}\partial_\alpha u'_\beta = 0, \tag{4.30}$$

where we let

$$n = (n_{\alpha\beta}) \quad \text{with } n_{\alpha\beta} = \int_{-1}^{1} \sigma_{\alpha\beta}\,\mathrm{d}x_3. \tag{4.31}$$

Notice that the functions $n_{\alpha\beta}$ belong to the space $L^2(\omega)$ if the functions $\sigma_{\alpha\beta}$ belong to the space $L^2(\Omega)$.

Next, let the functions $v \in V$ in (4.24) be of the particular form $(v'_1, v'_2, 0)$ with $v'_\alpha \in H^1_0(\omega)$. This yields

$$\forall v'_\alpha \in H^1_0(\omega),\quad \int_\Omega \sigma^\epsilon_{\alpha\beta}\partial_\alpha v'_\beta = \epsilon^2\zeta^\epsilon \int_\Omega u^\epsilon_\alpha v'_\alpha,$$

so that, by passing to the limit,

$$\forall v'_\alpha \in H^1_0(\omega),\quad \int_\omega n_{\alpha\beta}\partial_\alpha v'_\beta = 0. \tag{4.32}$$

It is then an easy matter to establish that problem (4.30) and (4.32) has a unique solution in the space $(L^2(\omega))^4_s \times (H^1_0(\omega))^2$, which is consequently $n_{\alpha\beta} = 0$, $u'_\alpha = 0$ (let $\tau'_{\alpha\beta} = n_{\alpha\beta}$ in (4.30) and $v'_\alpha = u_\alpha$ in (4.32)).

*Step* 4: *The function $u_3 \in H^2_0(\omega)$ is a solution of the variational problem* (4.21). Let the functions $\tau_{\alpha\beta}$ in (4.27) be of the form $x_3\tau''_{\alpha\beta}$, with $\tau''_{\alpha\beta} \in L^2(\omega)$. Using the fact that the functions $u_\alpha$ are of the form $u_\alpha = -x_3\partial_\alpha u_3$, $u_3 \in H^2_0(\omega)$, we find that

$$\forall(\tau''_{\alpha\beta}) \in (L^2(\omega))^4_s,\quad \int_\omega (A^0 m)_{\alpha\beta}\tau''_{\alpha\beta} + \tfrac{2}{3}\int_\omega \tau''_{\alpha\beta}\partial_{\alpha\beta}u_3 = 0, \tag{4.33}$$

where we let

$$m = (m_{\alpha\beta}) \quad \text{with } m_{\alpha\beta} = \int_{-1}^{1} x_3\sigma_{\alpha\beta}\,\mathrm{d}x_3. \tag{4.34}$$

Next, let the functions $v \in V$ in (4.24) be of the form $(-x_3\partial_1 v, -x_3\partial_2 v, v)$, with $v \in H^2_0(\omega)$. This yields

$$\forall v \in H^2_0(\omega),\quad -\int_\Omega x_3\sigma^\epsilon_{\alpha\beta}\partial_{\alpha\beta}v = -\epsilon^2\zeta^\epsilon \int_\Omega x_3 u^\epsilon_\alpha \partial_\alpha v + \zeta^\epsilon \int_\Omega u^\epsilon_3 v_3,$$

so that, by passing to the limit,

$$\forall v \in H_0^2(\omega), \quad -\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}v = 2\zeta \int_\omega u_3 v. \tag{4.35}$$

The variational equations (4.33) are equivalent to

$$m_{\alpha\beta} = -\frac{2E}{3(1-\nu^2)}\{(1-\nu)\partial_{\alpha\beta}u_3 + \nu\Delta u_3\delta_{\alpha\beta}\},$$

so that, upon replacing $m_{\alpha\beta}$ by this expression in (4.35), we obtain

$$\forall v \in H_0^2(\omega), \quad \frac{2E}{3(1-\nu^2)}\int_\omega \{\nu\Delta u_3\Delta v + (1-\nu)\partial_{\alpha\beta}u_3\partial_{\alpha\beta}v\} = 2\zeta \int_\omega u_3 v, \tag{4.36}$$

and eqs. (4.21) follow by observing that

$$\forall u,\ v \in H_0^2(\omega), \quad \int_\omega \partial_{12}u\partial_{12}v = \int_\omega \partial_{11}u\partial_{22}v.$$

As a technical preliminary for establishing the strong convergence, we prove Lemma 3.

*LEMMA* 3. *For each integer* $l \geqslant 1$,

$$\zeta_l^0 = \mathcal{A}_0(\tilde{\sigma}_l^0, \tilde{\sigma}_l^0), \tag{4.37}$$

*where* $\mathcal{A}_0$ *is the bilinear form defined in* (3.9) *and* $\tilde{\sigma}_l$ is the element of the space $\Sigma$ defined by

$$\tilde{\sigma}_{l\alpha\beta}^0 = \sigma_{l\alpha\beta}^0, \qquad \tilde{\sigma}_{li3}^0 = 0. \tag{4.38}$$

*PROOF.* With the definition of the bilinear form $\mathcal{A}_0$ and of the element $\tilde{\sigma}_l^0 \in \Sigma$, and with the specific forms found in (4.22)–(4.23) for the functions $u_{l\alpha}^0$ and $\sigma_{l\alpha\beta}^0$, we easily get

$$\mathcal{A}_0(\tilde{\sigma}_l^0, \tilde{\sigma}_l^0) = \frac{2E}{3(1-\nu^2)}\int_\omega (\Delta u_{l3}^0)^2.$$

Using (4.21) and the fact that the function $u_{l3}^0$ is independent of the variable $x_3$, we next obtain

$$\mathcal{A}_0(\tilde{\sigma}_l, \tilde{\sigma}_l) = 2\zeta_l^0 \int_\omega (u_{l3}^0)^2 = \zeta_l^0 \int_\Omega (u_{l3}^0)^2.$$

Because weak convergence in the space $V$ implies strong convergence in the space $(L^2(\Omega))^3$ (by Rellich' theorem), we infer from Proposition 2 that (no summation for the index $l$)

$$\int_\Omega (u_{l3}^0)^2 = \lim_{\epsilon\to 0}\left\{\epsilon^2 \int_\Omega u_{l\alpha}^\epsilon u_{l\alpha}^\epsilon + \int_\Omega (u_{l3}^\epsilon)^2\right\},$$

and thus the normalization condition (4.9) implies

$$\int_\Omega (u^0_{l3})^2 = 1,$$

which completes the proof.

*PROPOSITION* 3. *The subsequence considered in Proposition* 2 *satisfies*

$$u^\epsilon_l \to u^0_l \quad \text{in } V, \tag{4.39}$$

$$\sigma^\epsilon_{l\alpha\beta} \to \sigma^0_{l\alpha\beta} \quad \text{in } L^2(\Omega), \tag{4.40}$$

$$\epsilon\sigma^3_{l\alpha 3} \to 0 \quad \text{in } L^2(\Omega), \tag{4.41}$$

$$\epsilon^2\sigma^\epsilon_{l33} \to 0 \quad \text{in } L^2(\Omega), \tag{4.42}$$

*i.e., all weak convergences established in Proposition 2 are in fact strong convergences, the functions* $u^0_{li}$ *and* $\sigma^0_{l\alpha\beta}$ *being as in* (4.21)–(4.23).

*PROOF.* Introducing the elements $\tilde{\sigma}^\epsilon_l \in V$ and $\tilde{\sigma}^0_l \in V$ of (4.13) and (4.38) respectively, we note that proving (4.40)–(4.42) is the same as proving

$$(\tilde{\sigma}^\epsilon_l - \tilde{\sigma}^0_l) \to 0 \quad \text{in } \Sigma = (L^2(\Omega))^9_s. \tag{4.43}$$

We recall once and for all that the repetition of the particular index $l$ does not entail summation.

Using the definitions (3.9)–(3.11) of the bilinear forms $\mathscr{A}_0$, $\mathscr{A}_2$, $\mathscr{A}_4$, we have, for any integer $l \geq 1$ and all $\epsilon > 0$,

$$\begin{aligned} X^\epsilon_l &\overset{\text{def}}{=} \int_\Omega (A(\tilde{\sigma}^\epsilon_l - \tilde{\sigma}^0_l))_{ij}(\tilde{\sigma}^\epsilon_l - \tilde{\sigma}^0_l)_{ij} \\ &= \mathscr{A}_0(\sigma^\epsilon_l - \tilde{\sigma}^0_l, \sigma^\epsilon_l - \tilde{\sigma}^0_l) + \epsilon^2\mathscr{A}_2(\sigma^\epsilon_l - \tilde{\sigma}^0_l, \sigma^\epsilon_l - \tilde{\sigma}^0_l) + \epsilon^4\mathscr{A}_4(\sigma^\epsilon_l - \tilde{\sigma}^0_l, \sigma^\epsilon_l - \tilde{\sigma}^0_l), \end{aligned}$$

which reduces to

$$\begin{aligned} X^\epsilon_l = {} & \mathscr{A}_0(\sigma^\epsilon_l, \sigma^\epsilon_l) + \epsilon^2\mathscr{A}_2(\sigma^\epsilon_l, \sigma^\epsilon_l) + \epsilon^4\mathscr{A}_4(\sigma^\epsilon_l, \sigma^\epsilon_l) - 2\mathscr{A}_0(\sigma^\epsilon_l, \tilde{\sigma}^0_l) \\ & - 2\epsilon^2\mathscr{A}_2(\sigma^\epsilon_l, \tilde{\sigma}^0_l) + \mathscr{A}_0(\tilde{\sigma}^0_l, \tilde{\sigma}^0_l), \end{aligned}$$

in view of the specific form of the functions $\tilde{\sigma}_{lij}$. Using next relations (3.7)–(3.8), (4.9), and Lemma 3, we find

$$X^\epsilon_l = \zeta^\epsilon_l - 2\mathscr{A}_0(\sigma^\epsilon_l, \tilde{\sigma}^0_l) + \frac{2\nu}{E}\int_\Omega (\epsilon^2\sigma^\epsilon_{l33})\sigma^0_{l\mu\mu} + \zeta^0_l,$$

and, finally, using (4.16) and the weak convergences (4.18) and (4.20), we conclude that

$$\lim_{\epsilon\to 0} X_l^\epsilon = 2\zeta_l^0 - 2\mathcal{A}_0(\sigma_l^0, \tilde{\sigma}_l^0) = 0,$$

by another application of Lemma 3, since $\mathcal{A}_0(\tilde{\sigma}_l^0, \tilde{\sigma}_l^0) = \mathcal{A}_0(\sigma_l^0, \tilde{\sigma}_l^0)$.

Applying inequality (4.14), we obtain

$$c_0|\tilde{\sigma}_l^\epsilon - \tilde{\sigma}_l^0|_{0,\Omega}^2 \leqslant X_l^\epsilon,$$

and thus relation (4.43) is proved.

Let us next establish the strong convergences of the family $(u_l^\epsilon)$, $\epsilon > 0$. Given an arbitrary element $\tau \in \Sigma$, we can write

$$\begin{aligned}
-\mathcal{B}(\tau, u_l^\epsilon - u_l^0) &= -\mathcal{B}(\tau, u_l^\epsilon) - \int_\Omega \tau_{\alpha\beta}\partial_\alpha u_{l\beta}^0 \\
&= \int_\Omega (A^0(\sigma_l^\epsilon - \sigma_l^0))_{\alpha\beta}\tau_{\alpha\beta} - \frac{\nu}{E}\epsilon^2 \int_\Omega \sigma_{l\mu\mu}^\epsilon \tau_{33} \\
&\quad + \frac{2(1+\nu)}{E}\epsilon^2 \int_\Omega \sigma_{l\alpha 3}^\epsilon \tau_{\alpha 3} - \frac{\nu}{E}\epsilon^2 \int_\Omega \sigma_{l33}^\epsilon \tau_{\mu\mu} + \frac{\epsilon^4}{E}\int_\Omega \sigma_{l33}^\epsilon \tau_{33},
\end{aligned}$$

by using successively relations (4.27)–(4.29) and equations (3.7). The convergences (4.40)–(4.42) then imply that there exists a constant $c_6$ independent of $\epsilon$ such that

$$\forall \tau \in \Sigma \quad |\mathcal{B}(\tau, u_l^\epsilon - u_l^0)| \leqslant c_6|\tau|_{0,\Omega}\eta(\epsilon), \quad \text{with } \lim_{\epsilon\to 0}\eta(\epsilon) = 0.$$

From inequality (4.15), we deduce that

$$c_1\|u_l^\epsilon - u_l^0\|_{1,\Omega} \leqslant \sup_{\tau\in\Sigma} \frac{|\mathcal{B}(\tau, u_l^\epsilon - u_l^0)|}{|\tau|_{0,\Omega}} \leqslant c_6\eta(\epsilon),$$

and the proof is complete.

Though we have proved that there exists a subsequence $(\zeta_l^\epsilon, u_{l3}^\epsilon)$, $\epsilon > 0$, which, for all integers $l \geqslant 1$, converges to a solution $(\zeta_l^0, u_{l3}^0)$ of the two-dimensional eigenvalue problem: find $(u, \zeta) \in H_0^2(\omega) \times \mathbf{R}$ such that (cf. (4.21))

$$\forall v \in H_0^2(\omega) \quad \frac{E}{3(1-\nu^2)}\int_\omega \Delta u \Delta v = \zeta \int_\omega uv, \tag{4.44}$$

nothing tells us so far whether $\zeta_l^0$ is indeed precisely *the lth* eigenvalue (counting multiplicities) of this problem. This question is answered in the next proposition. There we shall make essential use of the symmetry, positive definiteness, and compactness of the operator

$$g:\ u \in H_0^2(\omega) \to gu \in H_0^2(\omega), \tag{4.45}$$

defined by

$$\forall v \in H_0^2(\omega) \quad \frac{E}{3(1-\nu^2)} \int_\omega \Delta(gu)\Delta v = \int_\omega uv. \tag{4.46}$$

*PROPOSITION* 4. *The sequence* $(\zeta_l^0)_{l\geq 1}$ *comprises all eigenvalues, counting multiplicities, of problem* (4.21), *and the associated sequence* $(u_{l3}^0)_{l\geq 1}$ *forms a complete set of eigenfunctions (in both spaces* $L^2(\omega)$ *and* $H_0^2(\omega)$*), satisfying the orthonormalization condition*

$$\forall k, l \geq 1, \quad \int_\omega u_{k3}^0 u_{l3}^0 = \tfrac{1}{2}\delta_{kl}. \tag{4.47}$$

*PROOF.* In what follows, we argue with the *subsequence* considered in Propositions 2 and 3. We first note that the orthonormalization condition (4.47) is an immediate consequence of the orthonormalization condition (4.9).

*Step* 1. *The eigenvalues* $\zeta_l^0$, $l \geq 1$, *satisfy*

$$0 < \zeta_1^0 \leq \zeta_2^0 \cdots \leq \zeta_l^0 \leq \zeta_{l+1}^0 \leq \cdots, \quad \text{with } \lim_{l\to\infty} \zeta_l^0 = \infty. \tag{4.48}$$

Since $0 < \zeta_1^\epsilon \leq \zeta_2^\epsilon \leq \cdots$ for each $\epsilon > 0$, we deduce that $0 \leq \zeta_1^0 \leq \zeta_2^0 \cdots$, and because the mapping $g$ of (4.45) is positive definite, we have $\zeta_1^0 > 0$.

To prove that $\lim_{l\to\infty}\zeta_l^0 = \infty$, assume the contrary: the mapping $g$ being compact, this would imply that the corresponding eigenvectors $u_{l3}^0$, $l \geq 1$, span a finite dimensional space, but this contradicts the orthonormalization condition (4.47).

*Step* 2. *Let* $\zeta$ *be an eigenvalue of problem* (4.44). *Then there exists an integer* $l \geq 1$ *such that* $\zeta = \zeta_l^0$. To prove this, assume the contrary, and let $u_3 \in H_0^2(\omega)$ be a corresponding eigenfunction, which can be chosen so as to satisfy

$$\forall l \geq 1, \quad \int_\omega u_3 u_{l3}^0 = 0 \quad \text{and} \int_\omega (u_3)^2 = 1/2. \tag{4.49}$$

For each $\epsilon > 0$, let $(\rho^\epsilon, w^\epsilon) \in \Sigma \times V$ be defined as the unique solution of the equations

$$\forall \tau \in \Sigma, \quad \mathcal{A}_0(\rho^\epsilon, \tau) + \epsilon^2 \mathcal{A}_2(\rho^\epsilon, \tau) + \epsilon^4 \mathcal{A}_4(\rho^\epsilon, \tau) + \mathcal{B}(\tau, w^\epsilon) = 0, \tag{4.50}$$

$$\forall v \in V, \quad \mathcal{B}(\rho^\epsilon, v) + \zeta \int_\Omega u_3 v_3 = 0. \tag{4.51}$$

Proceeding exactly as in Propositions 2 and 3 (the argument is even simpler because there is no need for the analog of Lemma 1), one can show that, for the whole subsequence heretofore considered (as shown below, the limit is unique),

$$w^\epsilon \to w^0 \quad \text{in } V, \tag{4.52}$$

with

$$w_3^0 \in H_0^2(\omega) \quad \text{and} \quad \forall v \in H_0^2(\omega), \ \frac{E}{3(1-\nu^2)} \int_\omega \Delta w_3^0 \Delta v = \zeta \int_\omega u_3 v, \tag{4.53}$$

and $w_\alpha^0 = -x_3 \partial_\alpha w_3^0$. Because $w_3^0 = u_3$ is a solution of problem (4.53), and because the solution of this problem is unique, we conclude that

$$w_3^0 = u_3. \tag{4.54}$$

By virtue of Step 1, there exists an integer $m \geq 0$ such that

$$\zeta < \zeta_{m+1}^0. \tag{4.55}$$

For each $\epsilon > 0$, let

$$v^\epsilon = w^\epsilon - \sum_{k=1}^m D^\epsilon(w^\epsilon, u_k^\epsilon) u_k^\epsilon \quad (v^\epsilon = w^\epsilon \text{ if } m = 0), \tag{4.56}$$

where the bilinear form $D^\epsilon(\cdot, \cdot)$ is defined as in (4.7) and $u_k^\epsilon \in V$ is (together with $\zeta_k^\epsilon \in \mathbf{R}$ and $\sigma_k^\epsilon \in \Sigma$) the $k$th eigenfunction solution of problem (3.7)–(3.8). By construction,

$$D^\epsilon(v^\epsilon, u_l^\epsilon) = 0 \quad \text{for } 1 \leq l \leq m,$$

and therefore, with the notation of (4.5)–(4.7), the minimum principle (2.24) gives us

$$\forall \epsilon > 0, \ \zeta_{m+1}^\epsilon \leq R^\epsilon(v^\epsilon) = \frac{N^\epsilon(v^\epsilon, v^\epsilon)}{D^\epsilon(v^\epsilon, v^\epsilon)}. \tag{4.57}$$

Let us study the behavior of $R^\epsilon(v^\epsilon)$ as $\epsilon$ approaches zero. Using (4.56) and the bilinearity and symmetry of the form $N^\epsilon(\cdot, \cdot)$, we obtain

$$\begin{aligned} N^\epsilon(v^\epsilon, v^\epsilon) = {} & N^\epsilon(w^\epsilon, w^\epsilon) - 2 \sum_{k=1}^m D^\epsilon(w^\epsilon, u_k^\epsilon) N^\epsilon(w^\epsilon, u_k^\epsilon) \\ & + \sum_{k,l=1}^m D^\epsilon(w^\epsilon, u_k^\epsilon) D^\epsilon(w^\epsilon, u_l^\epsilon) N^\epsilon(u_k^\epsilon, u_l^\epsilon). \end{aligned} \tag{4.58}$$

First, we have, for each integer $k$,

$$D^\epsilon(w^\epsilon, u_k^\epsilon) = \epsilon^2 \int_\Omega w_\alpha^\epsilon u_{k\alpha}^\epsilon + \int_\Omega w_3^\epsilon u_{k3}^\epsilon,$$

and thus

$$\lim_{\epsilon \to 0} D^\epsilon(w^\epsilon, u_k^\epsilon) = 0, \tag{4.59}$$

since $w^\epsilon \to w^0$ and $u_k^\epsilon \to u_k^0$ in $V$ (cf. (4.39) and (4.52)), and since $\int_\Omega w_3^0 u_{k3}^0 = 2\int_\omega u_3 u_{k3}^0 = 0$ (cf. (4.49) and (4.54)).

Next, it is an easy matter to establish that, for arbitrary elements $u^\epsilon$, $v^\epsilon \in V$, we can write

$$N^\epsilon(u^\epsilon, v^\epsilon) = -\mathscr{B}(\tau^\epsilon, u^\epsilon), \tag{4.60}$$

where $\tau^\epsilon \in \Sigma$ is the unique element in the space $\Sigma$ which satisfies

$$\forall \tau \in \Sigma,\quad \mathscr{A}_0(\tau^\epsilon, \tau) + \epsilon^2 \mathscr{A}_2(\tau^\epsilon, \tau) + \epsilon^4 \mathscr{A}_4(\tau^\epsilon, \tau) + \mathscr{B}(\tau, v^\epsilon) = 0. \tag{4.61}$$

Applying (4.60)–(4.61) in conjunction with (4.50)–(4.51), we obtain

$$N^\epsilon(w^\epsilon, w^\epsilon) = -\mathscr{B}(\rho^\epsilon, w^\epsilon) = \zeta \int_\Omega u_3 w_3^\epsilon,$$

so that, by (4.49) and (4.52)

$$\lim_{\epsilon\to 0} N^\epsilon(w^\epsilon, w^\epsilon) = \zeta. \tag{4.62}$$

Applying similarly (4.60)–(4.61) in conjunction with (3.7)–(3.8), we get

$$N^\epsilon(w^\epsilon, u_k^\epsilon) = -\mathscr{B}(\sigma_k^\epsilon, w^\epsilon) = \epsilon^2 \zeta_k^\epsilon \int_\Omega u_{k\alpha}^\epsilon w_\alpha^\epsilon + \zeta_k^\epsilon \int_\Omega u_{k3}^\epsilon w_3^\epsilon,$$

and thus, by (4.16), (4.39), (4.49), (4.52), and (4.54),

$$\lim_{\epsilon\to 0} N^\epsilon(w^\epsilon, u_k^\epsilon) = \zeta_k^0 \int_\Omega u_{k3}^0 u_3 = 0. \tag{4.63}$$

A third application of (4.60)–(4.61) yields

$$N^\epsilon(u_k^\epsilon, u_l^\epsilon) = -\mathscr{B}(\sigma_l^\epsilon, u_k^\epsilon) = \zeta_l^\epsilon \delta_{kl}, \tag{4.64}$$

and consequently,

$$\lim_{\epsilon\to 0} N^\epsilon(u_k^\epsilon, u_l^0) = \zeta_l^0 \delta_{kl}. \tag{4.65}$$

Combining (4.58), (4.59), (4.62), (4.63) and (4.65), we conclude that

$$\lim_{\epsilon\to 0} N^\epsilon(v^\epsilon, v^\epsilon) = \zeta. \tag{4.66}$$

Since it is clear from (4.56), (4.59) and the boundedness of the functions $u_k^\epsilon$ (cf. (4.11)) in the space $V$ that

$$\lim_{\epsilon\to 0} (v^\epsilon - w^\epsilon) = 0 \quad \text{in } V,$$

we also have

$$\lim_{\epsilon\to 0} D^\epsilon(v^\epsilon, v^\epsilon) = \lim_{\epsilon\to 0}\left\{\epsilon^2\int_\Omega v^\epsilon_\alpha v^\epsilon_\alpha + \int_\Omega (v^\epsilon_3)^2\right\} = \int_\Omega (u_3)^2 = 1, \tag{4.67}$$

so that the conjunction of (4.57), (4.66), (4.67) yields

$$\zeta^0_{m+1} \leqslant \zeta,$$

which contradicts inequality (4.55), and thus Step 2 is proved.

*Step* 3. *The sequence* $(u^0_{l3})_{l\geqslant 1}$ *forms a complete set.* Otherwise there would exist an eigenfunction $u_3 \in H^2_0(\omega)$, i.e., a solution of problem (4.44), together with a corresponding eigenvalue $\zeta$, orthogonal to all functions $(u^0_{l3})_{l\geqslant 1}$ in the sense of (4.49). Therefore the same argument as in Step 2 (which only relied on conditions (4.49) and on Step 1) can be repeated, and the assertion follows.

Notice that we can now assert that the sequence $(\zeta^0_l)_{l\geqslant 1}$ comprises all eigenvalues of problem (4.21), *including multiplicities*, and thus that $\zeta^0_l$ is indeed the $l$th eigenvalue of the same problem.

As a consequence of the above proposition, we finally show that, except for eigenfunctions corresponding to multiple eigenvalues, convergence occurs in fact for the *whole* family $(\sigma^\epsilon_l, u^\epsilon_l, \zeta^\epsilon_l)$, $\epsilon > 0$ (for a given integer $l \geqslant 1$).

*PROPOSITION* 5. *For each integer* $l \geqslant 1$, *the whole family* $\zeta^\epsilon_l$, $\epsilon > 0$, *converges to* $\zeta^0_l$ *as* $\epsilon$ *approaches zero.*

*If, for a given integer* $l \geqslant 1$, *the eigenvalue* $\zeta^0_l$ *is simple, then* $\zeta^\epsilon_l$ *is also a simple eigenvalue for* $0 < \epsilon \leqslant \epsilon_0(l)$, *and besides, a normalized eigenfunction* $(\sigma^\epsilon_l, u^\epsilon_l)$ *can be found for all* $\epsilon \leqslant \epsilon_0(l)$, *such that the convergence properties* (4.39)–(4.42) *hold for the entire family* $(\sigma^\epsilon_l, u^\epsilon_l)$, $\epsilon > 0$.

*PROOF.* If, instead of the whole family (i.e., indexed by all $\epsilon > 0$), we start out with any subfamily thereof, the arguments given in the previous propositions hold *verbatim*: they yield the existence of a further subsequence (again indexed by $\epsilon$ for notational convenience) such that $\zeta^\epsilon_l$ converges to $\zeta^0_l$. Since the limit is unique, we conclude that, for each $l$, the entire family converges.

Since $\lim_{\epsilon\to 0} \zeta^\epsilon_l = \zeta^0_l$ for each $l$, there exists a number $\epsilon_0(l) > 0$ such that the multiplicity of the eigenvalue $\zeta^\epsilon_l$ is less than or equal to that of $\zeta^0_l$ for $\epsilon \leqslant \epsilon_0(l)$. In particular, $\zeta^\epsilon_l$ is simple for $\epsilon \leqslant \epsilon_0(l)$ if $\zeta^0_l$ is simple.

If $\zeta^0_l$ is simple, let $u_{l3} \in H^2_0(\omega)$ be one of the two corresponding normalized eigenfunctions. For $\epsilon \leqslant \epsilon_0(l)$, $\zeta^\epsilon_l$ is also a simple eigenvalue, with only two corresponding normalized eigenfunctions, $u^\epsilon_l$ and $-u^\epsilon_l$. For each $\epsilon \geqslant \epsilon_0(l)$, let $v^\epsilon_{l3}$ be so chosen that

$$v^\epsilon_{l3} = \pm u^\epsilon_{l3} \quad \text{and} \quad \int_\Omega u_{l3} v^\epsilon_{l3} \geqslant 0. \tag{4.68}$$

Clearly, $v^\epsilon_{l3}$ is still an eigenfunction of problem (4.21), and therefore the previous arguments can be repeated: given any subfamily, there exists a further subsequence (still indexed by $\epsilon$) such that $v_{l3}^{\epsilon}$ converges in $H^1(\Omega)$ to $+u_{l3}$ or $-u_{l3}$. Since the convergence can only be towards $+u_{l3}$ in view of the inequality in (4.68), the limit is unique, and therefore we conclude that the whole family defined by (4.68) converges.

Notice that this last proposition completes the proof of Theorem 1.

## 5. Relation with the asymptotic expansion method

We recall that we have shown in section 3 that, over the set $\Omega$, the three-dimensional eigenvalue problem consists in finding elements $(\sigma^{\epsilon}, u^{\epsilon}, \zeta^{\epsilon}) \in \Sigma \times V \times \mathbf{R}$, solution of

$$\forall \tau \in \Sigma, \quad \mathcal{A}_0(\sigma^{\epsilon}, \tau) + \epsilon^2 \mathcal{A}_2(\sigma^{\epsilon}, \tau) + \epsilon^4 \mathcal{A}_4(\sigma^{\epsilon}, \tau) + \mathcal{B}(\tau, u^{\epsilon}) = 0, \tag{5.1}$$

$$\forall v \in V; \quad \mathcal{B}(\sigma^{\epsilon}, v) + \epsilon^2 \zeta^{\epsilon} \int_{\Omega} u_{\alpha}^{\epsilon} v_{\alpha} + \zeta^{\epsilon} \int_{\Omega} u_3^{\epsilon} v_3 = 0, \tag{5.2}$$

$$\epsilon^2 \int_{\Omega} u_{\alpha}^{\epsilon} u_{\alpha}^{\epsilon} + \int_{\Omega} (u_3^{\epsilon})^2 = 1, \tag{5.3}$$

where the bilinear forms $\mathcal{A}_0$, $\mathcal{A}_2$, $\mathcal{A}_4$, $\mathcal{B}$ (whose expressions are given in (3.9)–(3.12)) are independent of $\epsilon$. This observation, together with the fact that $\epsilon$ may be thought of as a 'small' parameter, suggests that *we construct a formal series of 'approximations' of any solution* $(\sigma^{\epsilon}, u^{\epsilon}, \zeta^{\epsilon})$ *of* (5.1)–(5.3) *in the form*

$$\sigma^{\epsilon} = \sigma^0 + \epsilon \sigma^1 + \epsilon^2 \sigma^2 + \cdots, \tag{5.4}$$

$$u^{\epsilon} = u^0 + \epsilon u^1 + \epsilon^2 u^2 + \cdots, \tag{5.5}$$

$$\zeta^{\epsilon} = \zeta^0 + \epsilon \zeta^1 + \epsilon^2 \zeta^2 + \cdots, \tag{5.6}$$

where the elements $(\sigma^0, u^0, \zeta^0)$, $(\sigma^1, u^1, \zeta^1), \ldots,$ are obtained as solutions of equations obtained by equating to zero the coefficients of $1, \epsilon, \ldots,$ in the resulting series when expressions (5.4)–(5.6) are put in eqs. (5.1)–(5.3).

In this fashion, we find that the first term $(\sigma^0, u^0, \zeta^0)$ should be a solution of

$$\forall \tau \in \Sigma, \quad \mathcal{A}_0(\sigma^0, \tau) + \mathcal{B}(\tau, u^0) = 0, \tag{5.7}$$

$$\forall v \in V, \quad \mathcal{B}(\sigma^0, v) + \zeta^0 \int_{\Omega} u_3^0 v_3 = 0, \tag{5.8}$$

$$\int_{\Omega} (u_3^0)^2 = 1. \tag{5.9}$$

Proceeding exactly as in the proof of Theorem 5 of [2], we can prove the following theorem (as is natural, the normalization (5.9) plays no role in what follows).

*THEOREM* 2. *All solutions* $(\sigma^0, u^0, \zeta^0)$ *of problem* (5.7)–(5.8) *are of the following form: the function* $u_3^0$, *which is independent of the variable* $x_3$, *and the real number* $\zeta^0$ *are solutions of the two-dimensional eigenvalue problem: find* $(u, \zeta) \in H_0^2(\omega) \times \mathbf{R}$ *such that*

$$\forall v \in H_0^2(\omega) \quad \frac{E}{3(1-\nu^2)} \int_\omega \Delta u \Delta v = \zeta \int_\omega uv, \tag{5.10}$$

*and the other unknowns are then given by*:

$$u_\alpha^0 = -x_3 \partial_\alpha u_3^0, \tag{5.11}$$

$$\sigma_{\alpha\beta}^0 = -\frac{Ex_3}{(1-\nu^2)}\{(1-\nu)\partial_{\alpha\beta} u_3^0 + \nu \Delta u_3^0 \delta_{\alpha\beta}\}, \tag{5.12}$$

$$\sigma_{\alpha 3}^0 = -\frac{E(1-x_3^2)}{2(1-\nu^2)} \partial_\alpha \Delta u_3^0, \tag{5.13}$$

$$\sigma_{33}^0 = \frac{x_3 - x_3^3}{2} \zeta^0 u_3^0. \tag{5.14}$$

In other words, *the first term in the asymptotic expansion coincides with that found by the 'direct' limit analysis as* $\epsilon$ *approaches zero* (with some extra analysis for the functions $\sigma_{i3}^0$, as mentioned at the end of Sect. 3).

*REMARK.* This is what justified our use of the same superscript $^0$ in both cases.

For the sake of brevity, we shall not proceed any further in the study of the asymptotic expansions (5.4)–(5.6). We shall simply mention that, though it is fairly easy to prove that all terms of odd order vanish in the expansions (5.4)–(5.6) when the corresponding limit eigenvalue $\zeta^0$ is simple, it is a much more delicate matter to obtain the *corrector functions* which necessarily appear in the computation of the subsequent terms, and which play an essential role for obtaining *error estimates*. For details, we refer to [7] and, especially, to the work of Destuynder [5].

## References


[1] P.G. Ciarlet and S. Kesavan, Approximation bidimensionnelle du problème de valeurs propres pour une plaque, C.R. Acad. Sci. Paris 289 (1979) 579–582.

[2] P.G. Ciarlet and P. Destuynder, A justification of the two-dimensional linear plate model, J. Mécanique 18 (1979) 315–344.

[3] P.G. Ciarlet and P. Destuynder, A justification of a nonlinear model in plate theory, Comput. Meths. Appl. Mech. Engrg. 17/18 (1979) 227–258.

[4] P.G. Ciarlet, A justification of the von Kármán equations, Arch. Rational Mech. Anal. 73 (1980) 349–389.

[5] P. Destuynder, Sur une justification mathématique des théories de plaques et de coques en elasticité linéaire, Dissertation, Université Pierre et Marie Curie, Paris (1979).

[6] J.L. Lions, Perturbations Singulières dans les Problèmes aux Limites et en Contrôle Optimal, Lecture Notes in Mathematics, Vol. 323 (Springer, Berlin, 1973).

[7] S. Kesavan, Sur l'Approximation de problèmes Linéaires et non-linéaires de valeurs propres, Dissertation, Université Pierre et Marie Curie, Paris (1979).
[8] S. Kesavan, Homogeneization of eigenvalue problems: part 1, Appl. Math. Optim. 5 (1979) 153–167.
[9] D. Caillerie, The effect of a thin inclusion of high rigidity in an elastic body, Math. Meths. Appl. Sci., to appear.
[10] A. Rigolot, Approximation asymptotique des vibrations de flexion des poutres droites élastiques, J. Mécanique 16 (1977) 493–529.
[11] I. Babuška and A.K. Aziz, Survey lectures on the mathematical foundations of the finite element method, in: The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, ed., A.K. Aziz (Academic Press, New York, 1972) pp. 3–359.
[12] P.G. Ciarlet, The Finite Element Method for Elliptic Problems (North-Holland, Amsterdam, 1978).
[13] G. Strang and G.J. Fix, An Analysis of the Finite Element Method (Prentice-Hall, Englewood Cliffs, 1973).
[14] G. Duvaut and J.L. Lions, Les Inéquations en Mécanique et en Physique (Dunod, Paris, 1972).
[15] L. Landau and E. Lifchitz, Théorie de l'Elasticité (Mir, Moscow, 1967).
[16] G. Fichera, Existence theorems in elasticity-Boundary value problems of elasticity with unilateral constraints, in: Encyclopedia of Physics, ed., S. Flügge, Vol. VI a/2: Mechanics of Solids II, ed., C. Truesdell (Springer, Berlin, 1972) pp. 347–424.
[17] A.E. Taylor, Introduction to Functional Analysis (Wiley, New York, 1958).
[18] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 1 (Interscience, New York, 1953).
[19] B. Fraeijs de Veubeke, A Course in Elasticity, Applied Mathematical Sciences Series, Vol. 29 (Springer, Berlin, 1979).
[20] J.L. Lions and E. Magenes, Problèmes aux Limites Non Homogènes et Applications, Vol. 1 (Dunod, Paris, 1968).
[21] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, Rev. Française Automat. Informat. Recherche Opérationnelle, Sér. Rouge Anal. Numér. R-2 (1974) 129–151.