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Eigen-oscillations of contrasting non-homogeneous elastic bodies: asymptotic and uniform estimates for eigenvalues

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Estimates of convergence rates for the eigenvalues of spectral stiff elasticity problems are obtained. The bounds in the estimates are expressed in terms of the stiffness ratio h and characteristic properties of the limit spectrum for low and middle frequency ranges. These estimates allow us to distinguish between individual and collective asymptotics of the eigenvalues and eigenvectors and to determine precisely the intervals for the small parameter h where the mathematical model considered provides a suitable approach and accuracy. The results in this paper hold for different boundary conditions, two- and three-dimensional models and scalar problems.

Keywords: eigenfrequencies; linear elasticity; spectral analysis; stiff problems.

1. Preliminary description of the results and auxiliary assertions

In engineering practice many objects and tools are of materials with highly contrasting physical properties, posing serious problems for numerical simulations on different aspects of their behaviour. These objects are as diverse as buffers, shock-absorbers, loudspeakers, dams, just to mention a few. Research into these objects abounds, focusing on eigen-oscillations and induced vibrations. To overcome difficulties arising from discontinuous data, engineers usually use simplified models which can, at first, inherit specific spectral properties of the original complicated problems only over a certain range of its spectrum. To justify such a substitution, namely, to make asymptotic and error analysis and, in particular, to indicate the correct spectrum ranges for modelling, is a task currently facing mathematicians.

It is within this framework that, from the mathematical viewpoint, stiff problems appear. In this paper, we consider the behaviour of a stiff problem for the elasticity system and we describe the model under consideration in Section 1.1. As a sample, of practical applications, the two-dimensional model in this

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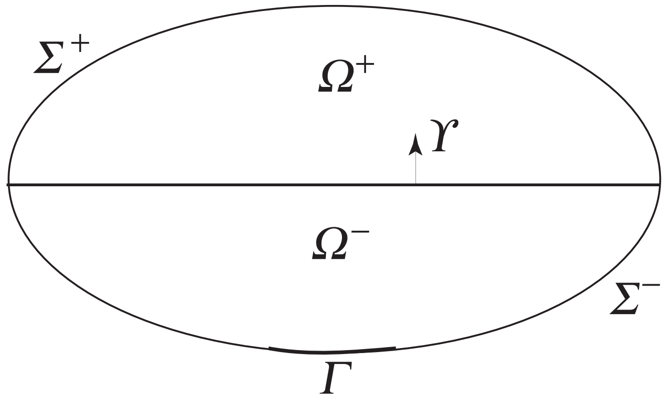

FIG. 1. The case where $\Sigma^+ \neq \emptyset$.

paper can be used in civil engineering when studying dynamic behaviours of constructions containing stiff kernels, such as buildings with bulk machinery inside, originated by horizontal seismic shocks. In the same way, the three-dimensional model is used in problems arising from SSI, i.e. seismic structure interactions.

1.1 *Formulation of the stiff spectral problem*

Let $\Omega^\pm \subset \mathbb{R}^3$ be domains with Lipschitz boundaries and compact closures, $\Omega^+ \cap \Omega^- = \emptyset$. We introduce the two-dimensional sets

$$\Upsilon = \partial\Omega^+ \cap \partial\Omega^-, \ \Sigma^+ = \partial\Omega^+ \setminus \overline{\Upsilon}, \ \Sigma^- = \partial\Omega^- \setminus (\overline{\Upsilon} \cup \overline{\Gamma}) \tag{1.1}$$

where Γ is an open subset of the surface $\partial\Omega^- \setminus \partial\Omega^+$. Considering the junction Ω of the anisotropic heterogeneous elastic bodies $\Omega^\pm$ we assume that the contact zone Υ has a positive area $\mathrm{meas}_2\,\Upsilon$. The junction is clamped over Γ, although this set can be empty (see Fig. 1). The surfaces $\Sigma^\pm$ are free of traction and Σ^+ can also be empty (see Fig. 2, where Ω^+ implies an inclusion in Ω^-). Also, the sets (1.1) are surrounded by simple Lipschitz boundaries.

To shorten notation, we use the matrix form of the linearized elasticity equations (see Nazarov, 2002a, Chapter 2). To this end, we define the differential operator 6×3 matrix

$$D(\nabla_x) = \begin{bmatrix} \partial_1 & 0 & 0 & 0 & \alpha\partial_3 & \alpha\partial_2 \\ 0 & \partial_2 & 0 & \alpha\partial_3 & 0 & \alpha\partial_1 \\ 0 & 0 & \partial_3 & \alpha\partial_2 & \alpha\partial_1 & 0 \end{bmatrix}^\top, \tag{1.2}$$

where $\alpha = 2^{-1/2}$, $\partial_j = \partial/\partial x_j$; $x = (x_1, x_2, x_3)^\top$ denotes a fixed system of Cartesian coordinates and $\top$ stands for transposition. If $u^\pm = (u_1^\pm, u_3^\pm, u_3^\pm)^\top$ is a displacement vector field regarded as a column in $\mathbb{R}^3$, the expression

$$\varepsilon(u^\pm; x) = D(\nabla_x)u^\pm(x)$$

becomes the 6×1 strain column

$$(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \alpha^{-1}\varepsilon_{23}, \alpha^{-1}\varepsilon_{31}, \alpha^{-1}\varepsilon_{12})^\top. \tag{1.3}$$

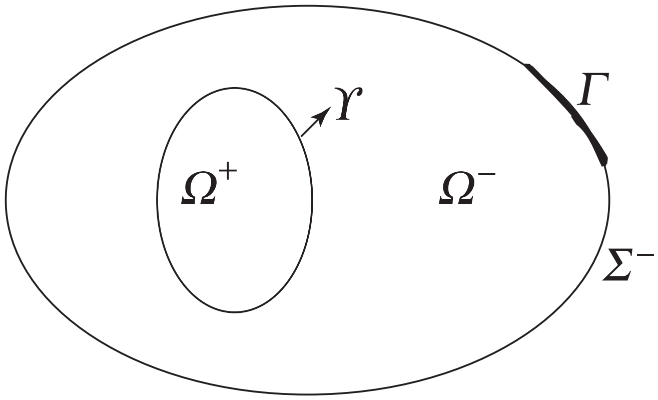

FIG. 2. The case where $\Sigma^+ = \emptyset$.

Here ε_{jk} are the Cartesian components $2^{-1}(\partial_j u_k + \partial_k u_j)$ of the strain tensor induced in $\Omega^\pm$ by the displacement field $u^\pm$. We emphasize that the factors $\alpha^{-1} = 2^{1/2}$ inserted into (1.3) equalize the natural norms of the strain tensor and the strain column. In the same way, we introduce the stress columns

$$\sigma^\pm(u^\pm) = \big(\sigma^\pm_{11}(u^\pm), \sigma^\pm_{22}(u^\pm), \sigma^\pm_{33}(u^\pm), \alpha\sigma^\pm_{23}(u^\pm), \alpha\sigma^\pm_{31}(u^\pm), \alpha\sigma^\pm_{12}(u^\pm)\big)^\top$$

and, hence, Hooke's law for the elastic bodies $\Omega^\pm$ takes the form

$$\sigma^\pm(u^\pm; x) = A^\pm(x)\varepsilon(u^\pm; x) = A^\pm(x)D(\nabla_x)u^\pm(x) \tag{1.4}$$

where $A^\pm$ are symmetric and positive definite 6×6 matrices. In Section 2.1 of Nazarov (2002a) one can find a relation between the matrix and the tensor of elastic moduli as well as a formula for recalculating the matrices $A^\pm$ after an orthogonal transformation of the x-coordinates. Entries of the matrices $A^\pm$ and the densities $\gamma^\pm \geqslant c^\pm > 0$ are measurable bounded functions.

We regard the elastic bodies Ω^+ and Ω^- as hard and soft bodies, respectively. This is expressed by the formula

$$A^-(x) = hA^0(x) \tag{1.5}$$

where $h \in (0, 1]$ is a small parameter, which denotes the stiffness ratio of the two parts $\Omega^\pm$. At the same time, we suppose that A^+, A^0 and $\gamma^\pm$ do not depend on h, i.e. the materials in $\Omega^\pm$ have similar inertia properties. In other words, we assume that

$$C_A||a; \mathbb{R}^6||^2 \geqslant a^\top A^\tau(x)a \geqslant c_A||a; \mathbb{R}^6||^2 \quad \forall a \in \mathbb{R}^6,\ \tau = 0, +;$$
$$C^\pm \geqslant \gamma^\pm(x) \geqslant c^\pm$$

with the positive constants C_A, c_A and $C^\pm, c^\pm$.

In the introduced matrix and column notation (1.2)–(1.4), the spectral boundary value problem describing eigen-oscillations of the elastic junction Ω takes the form

$$\begin{aligned} L^\pm(x,\nabla)u^\pm(x) &:= D(-\nabla)^\top A^\pm(x)D(\nabla)u^\pm(x) = \Lambda\gamma^\pm(x)u^\pm(x), x \in \Omega^\pm, \\ N^\pm(x,\nabla)u^\pm(x) &:= D(n^\pm(x))^\top A^\pm(x)D(\nabla)u^\pm(x) = 0, x \in \Sigma^\pm, \\ u^-(x) &= 0, x \in \Gamma, \\ u^-(x) &= u^+(x), N^-(x,\nabla)u^-(x) = -N^+(x,\nabla)u^+(x), x \in \Upsilon. \end{aligned} \tag{1.6}$$

Here $\nabla = \text{grad}$, $n^{\pm}$ is the outer normal vector to $\partial\Omega^{\pm}$ and $(1.6)_4$ implies ideal adhesion conditions (transmission conditions).

In order to underline the dependence of solutions on the parameter h, we further designate the eigenfrequency and the eigenmode as $\Lambda(h)^{1/2}$ and $u(h,\cdot) = \{u^-(h,\cdot), u^+(h,\cdot)\}$, respectively. In this paper, we study the asymptotic behaviour of the eigen-elements $(\Lambda(h),\ u(h,\cdot))$ of (1.6) as $h \to +0$. In Sections 1.2 and 1.3 we provide certain spectral properties of (1.6) and state our aims and results in a simplified way; in particular we present these results for a one-dimensional scalar problem since explicit computations can be performed and consequently our results can also be simplified (see Section 1.3.1). Finally, Sections 1.4 and 1.5 are devoted to outlining the technique used throughout the paper.

1.2 *The spectrum of the problem*

We proceed by formulating two inequalities of the Friedrichs–Korn and Poincaré–Korn types (see, e.g. Section 3.3 of Duvaut & Lions, 1976; Kondratiev & Oleinik, 1988, and Section 3.1 of Nazarov, 2002a).

LEMMA 1.1 (1) If $\text{meas}_2\Gamma > 0$ and $u^- \in H^1(\Omega^-)^3$ satisfies the Dirichlet conditions $(1.6)_3$, then there holds the inequality

$$||u^-; H^1(\Omega^-)|| \leqslant c_0||D(\nabla)u^-; L_2(\Omega^-)||. \tag{1.7}$$

(2) If $u^{\pm} \in H^1(\Omega^{\pm})^3$ satisfies the orthogonality condition

$$\int_{\Omega^{\pm}} d(x)^{\top} u^{\pm}(x)\,\mathrm{d}x = 0 \in \mathbb{R}^6 \tag{1.8}$$

where columns of the 3×6 matrix $d(x)$ imply rigid motions,

$$d(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & \alpha x_3 & -\alpha x_2 \\ 0 & 1 & 0 & -\alpha x_3 & 0 & \alpha x_1 \\ 0 & 0 & 1 & \alpha x_2 & -\alpha x_1 & 0 \end{bmatrix}, \tag{1.9}$$

there holds the inequality

$$||u^{\pm}; H^1(\Omega^{\pm})|| \leqslant c_{\pm}||D(\nabla)u^{\pm}; L_2(\Omega^{\pm})||. \tag{1.10}$$

The constants $c_{\dots}$ in (1.7) and (1.10) are independent of $u^{\pm}$.

Korn's inequalities provide a base for treating the weak formulation of the spectral problem (1.6): to find $\Lambda(h) \in \mathbb{R}$ and a non-trivial field $u(h,\cdot) \in \mathring{H}^1(\Omega;\Gamma)^3$ such that

$$\begin{gathered}\sum_{\pm}(A^{\pm}D(\nabla)u^{\pm}, D(\nabla)v^{\pm})_{\Omega^{\pm}} = \Lambda(h)\sum_{\pm}(\gamma^{\pm}u^{\pm}, v^{\pm})_{\Omega^{\pm}} \\ \forall\, v = \{v^-, v^+\} \in \mathring{H}^1(\Omega;\Gamma)^3.\end{gathered} \tag{1.11}$$

Here $H^1(\Omega)$ denotes the classic Sobolev space and $\mathring{H}^1(\Omega;\Gamma)$ the subspace composed of functions vanishing on Γ. In the sequel, we indicate the left- and right-hand sides of (1.11) as $(AD(\nabla)u, D(\nabla)v)_{\Omega}$ and $(\gamma u, v)_{\Omega}$, if no confusion arises.

Owing to the Korn inequality and the basic properties of the stiffness matrices $A^{\pm}$, we take

$$\langle u, v\rangle_h = (AD(\nabla)u, D(\nabla)v)_{\Omega} + h(\gamma u, v)_{\Omega} \tag{1.12}$$

as a scalar product in the Hilbert space $\mathcal{H}_h = \mathring{H}^1(\Omega; \Gamma)^3$. We introduce the compact positive symmetric operator $\mathcal{K}_h$ in $\mathcal{H}_h$,

$$\langle \mathcal{K}_h u, v \rangle_h = (\gamma u, v)_\Omega \tag{1.13}$$

and reduce the integral identity (1.11) to the abstract equation

$$\mathcal{K}_h u(h) = M(h) u(h) \in \mathcal{H}_h \tag{1.14}$$

with the new (inverted) spectral parameter

$$M(h) = (h + \Lambda(h))^{-1}. \tag{1.15}$$

Now general results of the functional analysis (see, for instance, Sections 1.4 and 1.5 in Sanchez-Hubert & Sanchez-Palencia, 1989) ensure the following assertion.

THEOREM 1.2 For fixed h, the eigenvalues of the spectral problem (1.11), i.e. problem (1.6), form the sequence

$$0 \leqslant \Lambda_1(h) \leqslant \Lambda_2(h) \leqslant \cdots \leqslant \Lambda_j(h) \leqslant \cdots \to +\infty \tag{1.16}$$

where we adopt the convention of repeated eigenvalues, while the corresponding eigenmodes $u^j(h, \cdot) \in \mathring{H}^1(\Omega; \Gamma)^3$ satisfy the normalization and orthogonality conditions

$$(\gamma u^j, u^k)_\Omega := \sum_{\pm} (\gamma^\pm u^{j\pm}, u^{k\pm})_{\Omega^\pm} = \delta_{j,k}. \tag{1.17}$$

REMARK 1.3 Clearly, $\{M_j(h) = (h + \Lambda_j(h))^{-1} : j = 1, 2, \dots\}$ implies the spectrum of the operator $\mathcal{K}_h$ and, according to (1.11) and (1.17), the corresponding eigenvectors $\mathcal{U}^{jh} = (h + \Lambda_j(h))^{-1/2} u^j \in \mathcal{H}_h$ fulfil the normalization and orthogonality conditions

$$\langle \mathcal{U}^{jh}, \mathcal{U}^{kh} \rangle_h = \delta_{j,k}. \tag{1.18}$$

□

Let us outline simple properties of the eigenvalue sequence in (1.16). Since A^+ and A^0 are symmetric and positive definite matrices, we have

$$\begin{aligned} (A^+ D(\nabla) u^+, D(\nabla) u^+)_{\Omega^+} &\geqslant C_+ ||D(\nabla) u^+; L_2(\Omega^+)||^2, \\ (A^0 D(\nabla) u^-, D(\nabla) u^-)_{\Omega^-} &\geqslant C_0 ||D(\nabla) u^-; L_2(\Omega^-)||^2. \end{aligned} \tag{1.19}$$

Hence, under the condition $\mathrm{meas}_2\, \Gamma > 0$, Lemma 1.1(1) and relation (1.5) lead to the inequality

$$\Lambda_1(h) \geqslant c_0 h \quad \text{with} \quad c_0 > 0. \tag*{(1.20)$_1$}$$

If $\Gamma = \emptyset$, any rigid motion $d(x)^\top a$ with $a \in \mathbb{R}^6$ satisfies problem (1.6) when $\Lambda = 0$ and, thus, by Lemma 1.1(2) where $\Omega^\pm$ is replaced by Ω, we derive that in (1.16)

$$\begin{aligned} &\Lambda_1 = \cdots = \Lambda_6 = 0, \\ &\Lambda_7(h) \geqslant c_0 h \quad \text{with} \quad c_0 > 0. \end{aligned} \tag*{(1.20)$_2$}$$

It should be mentioned that throughout the entire paper, the constants $\mathbf{c}$, c, C, c_i, C_i, $i = 0, 1, 2, \dots$ denote different constants independent of both h and the eigenvalue number k.

On the other hand, considering the minimax principle and the set of eigen-elements $\{\Lambda_i, U_i^-\}_{i=1}^{\infty}$ of the problem

$$(A^0 D(\nabla)U^-, D(\nabla)V^-)_{\Omega^-} = \Lambda(\gamma^- U^-, V^-)_{\Omega^-},$$

$U^-, V^- \in \overset{\circ}{H}{}^1(\Omega; \Gamma)^3$, U^-, V^- vanishing on Υ and extended by zero to Ω^+, i.e. $U^+ = V^+ = 0$ in Ω^+, it can be verified that

$$\Lambda_k(h) \leqslant \Lambda_k h, \quad k = 1, 2, 3, \dots. \tag{1.21}$$

Therefore, from estimates (1.20) and (1.21) we deduce that, for any fixed k, the eigenvalues $\Lambda_k(h)$ are of $\mathrm{O}(h)$.

Let us observe that on account of (1.20) and (1.21), a study of the asymptotic behaviour of the eigen-elements of (1.11) as $h \to +0$ can be performed as has been done in other rigorous mathematical studies for stiff problems in elasticity and acoustics. However, our aim is not only to show the convergence, but to obtain estimates of convergence rates for these eigen-elements; that is, the explicit form of bounds in the estimates and ranges for validity of asymptotic representations which carry important information in order to apply pure mathematical results for a specific small h. In this paper, based on the idea of direct and inverse reductions (see Section 1.4), we obtain first the explicit dependence of the error estimates bounds on the small parameter h and on the eigenvalue number and the ranges of the eigenfrequencies $\Lambda_n(h)^{1/2}$ and the parameter h where the asymptotic formulae hold true. Similar results, though not of the same accuracy as those presented in this paper, have been obtained in the theory of thin plates and rods (see Nazarov, 2002a,b). To confirm the asymptotic precision of our estimates for spatially massive elastic bodies, in Section 1.3 we adapt the results for the scalar problem; in particular for the one-dimensional case where an explicit calculation is available (see Section 1.3.1). We emphasize that, even though asymptotics when $h \to +0$ for the eigen-elements of scalar problems have been studied in previous papers (see Sanchez-Palencia, 1992; Lobo & Sanchez-Palencia, 1989; Lobo & Pérez, 1997; Panasenko, 1980, 1987), the bounds that we obtain in this paper were not previously known even in the case of the Laplacian!

1.3 *On the scalar spectral problem*

In order to illustrate our results for the elasticity problem (1.6) we outline the analogous problem with the Laplace operator,

$$\begin{aligned} -h\triangle u^-(h,x) &= \Lambda(h)u^-(h,x),\ x \in \Omega^-, \\ -\triangle u^+(h,x) &= \Lambda(h)u^+(h,x),\ x \in \Omega^+, \\ \partial_n u^\pm(h,x) &:= n^\pm(x)^\top \nabla u^\pm(h,x) = 0,\ x \in \Sigma^\pm, \\ u^-(h,x) &= u^+(h,x), h\partial_n u^-(h,x) = \partial_n u^+(h,x),\ x \in \Upsilon. \end{aligned} \tag{1.22}$$

We assume here that $\Gamma = \emptyset$, i.e. the Dirichlet conditions $(1.6)_3$ are skipped; hence, $0 = \Lambda_1(h) < \Lambda_2(h)$ in the eigenvalue sequence (1.16). A slight modification of the approach in Sanchez-Palencia (1992) (see also Lobo & Sanchez-Palencia, 1989; Lobo & Pérez, 1997) establishes the convergence

$$h^{-1}\Lambda_k(h) \to \lambda_k \quad \text{as} \quad h \to +0 \tag{1.23}$$

with conservation of the multiplicity, where

$$0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to +\infty \tag{1.24}$$

denotes the spectrum of the resulting problem

$$\begin{gathered} -\triangle U(x) = \lambda U(x),\ x \in \Omega^-;\ \partial_n U(x) = 0,\ x \in \Sigma^-, \\ \lambda U(x) = (\text{meas}_3\, \Omega^+)^{-1} \int_\Upsilon \partial_n U(x)\, \mathrm{d}s,\ x \in \Upsilon. \end{gathered} \tag{1.25}$$

The boundary condition $(1.25)_2$, which is non-local and of Steklov type, can be derived by repeating (with the suitable simplification) calculations in Section 2.1.

Furthermore, the approximation constructions, developed in Lobo & Pérez (1997) (see also Lobo & Sanchez-Palencia, 1989), uphold the following fact: for any eigenvalue $\beta_k > 0$ of the Neumann spectral problem

$$-\triangle W(x) = \beta W(x),\ x \in \Omega^+;\ \partial_n W(x) = 0,\ x \in \partial\Omega^+, \tag{1.26}$$

there exists an eigenvalue $\Lambda_{\mathcal{N}(h)}(h)$ of problem (1.22) such that

$$|\Lambda_{\mathcal{N}(h)}(h) - \beta_k| \leqslant c_k h^{1/4}. \tag{1.27}$$

We emphasize that, in view of convergence (1.23), the eigenvalue number $\mathcal{N}(h)$ grows indefinitely as $h \to +0$ and, the result (1.27) is of value only in the case where $k > 1$.

The main aim of our paper is to detect the convergence rate in (1.23) and to clarify the dependence on k of bounds in asymptotic accuracy estimates of type (1.27). Again, we emphasize that here and in the sequel, all the constants in the estimates do not depend on either the parameter $h \in (0, 1]$ or the eigenvalue number $k = 1, 2, \ldots$.

The results for the scalar problem (1.22) are literally the same as for the vectorial problem (1.6). We avoid here reproducing the corresponding assertions and refer to Propositions 2.3, 2.4, 3.10 and Theorems 3.8, 3.9 which express connections of diversified levels between the spectra in (1.16) and (1.24) of the original problem (1.22) and the resulting problem (1.25). In Theorems 4.3 and 4.4, we relate the spectrum (1.16) to that of the Neumann problem (1.26). We also refer to Lobo *et al.* (2003a) where precise statements on the eigenvalues of (1.22) have been given without any proof.

Although our results are valid for three- and two-dimensional domains,† the simplest problem is one-dimensional, where

$$\begin{aligned} &\Omega^- = (-T, 0),\, \Omega^+ = (0, T),\, \Sigma^\pm = \{\pm T\},\ \Upsilon = \{0\}; \\ &\triangle U = U'',\, U' := \partial_x U,\, \partial_{n^\pm} U = \pm U', \end{aligned}$$

and we can find explicit solutions of the resulting spectral problems (1.25) and (1.26). These resulting problems are

$$-U''(x) = \lambda U(x),\ x \in (-T, 0);\ U'(-T) = 0,\ \lambda U(0) = T^{-1} U'(0) \tag*{$(1.28)_1$}$$

and

$$-W''(x) = \beta W(x),\ x \in (0, T);\ W'(-T) = 0,\ W'(0) = 0. \tag*{$(1.28)_2$}$$

†To specify the two-dimensional elasticity problem, the derivative $\partial_3 = \partial/\partial x_3$ in the differential operator (1.2) must be annulled. Note that, since the bodies $\Omega^\pm$ are anisotropic, the problem (1.6), in general, does not split into the plane-strain problem and a scalar problem for the deplanation u_3.

In particular, the spectra of problems $(1.28)_1$ and $(1.28)_2$ are composed of the simple eigenvalues

$$\lambda_k = T^{-2} z_k^2 \ \text{ and } \ \beta_k = T^{-2}\pi^2 k^2 \tag{1.29}$$

respectively, where $k = 1, 2, \dots$ and z_k are non-negative roots of the transcendental equation

$$z = -\operatorname{tg} z.$$

It is not difficult to establish that $z_1 = 0$ and, for $k = 2, 3, \dots$, as $k \to +\infty$,

$$z_k = \pi\left(k - \frac{3}{2}\right) + \left[\pi\left(k - \frac{3}{2}\right)\right]^{-1} - \frac{4}{3}\left[\pi\left(k - \frac{3}{2}\right)\right]^{-3} + \mathrm{O}\left(\left[\pi\left(k - \frac{3}{2}\right)\right]^{-5}\right). \tag{1.30}$$

These computations, i.e. (1.29) and (1.30), allow us to be more precise with assertions and formulae obtained in the following sections for the spatial elasticity problem, since they are adapted to the ordinary differential equations (1.22) and expressed in terms of the spectral characteristics in (2.19), (3.18) and (4.8). While modifying constants in the bounds from the restrictions (2.26), (3.19), (4.20) and the estimates (2.34), (3.30), (4.21), by virtue of (1.29) and (1.30), we can replace $1 + \lambda_k$, $1 + d_k^{-1}$ and $1 + \beta_k$ by k^2, k and k^2, respectively (note that the first eigenvalues $\Lambda_1(h)$ and λ_1, β_1 are equal to 0 and, therefore, we need not discuss them). Now Proposition 2.3 and Theorems 3.9 and 4.3, rewritten for the case of dimension one, lead to the following assertions.

There exist constants $\mathbf{h}_q > 0$ *and* $\mathbf{c}_q > 0$ *such that:*

1^0 *under the condition* $h \leqslant \mathbf{h}_1 k^{-2}$, *problem* (1.22) *has at least one eigenvalue* $\Lambda_j(h)$ *satisfying the estimate*

$$|\Lambda_j(h) - h\lambda_k| \leqslant \mathbf{c}_1 h^2 k^4; \tag{1.31}$$

2^0 *under the condition* $h \leqslant \mathbf{h}_2 k^{-3}$, *problem* (1.22) *has exactly one eigenvalue* $\Lambda_k(h)$ *in the segment*

$$\left[\frac{h}{2}(\lambda_{k-1} + \lambda_k), \frac{h}{2}(\lambda_k + \lambda_{k+1})\right] \tag{1.32}$$

and this eigenvalue fulfils the estimate

$$|\Lambda_k(h) - h\lambda_k| \leqslant \mathbf{c}_2 h^2 k^4; \tag{1.33}$$

3^0 *under the condition* $h \leqslant \mathbf{h}_3 k^{-4}$, *problem* (1.22) *has at least one eigenvalue* $\Lambda_p(h)$ *satisfying the estimate*

$$|\Lambda_p(h) - \beta_k| \leqslant \mathbf{c}_3 h^{1/4} k^3. \tag{1.34}$$

1.3.1 *Further discussion for the one-dimensional scalar problem.* The difference between statements 1^0 and 2^0 above is evident and it originates in the fact that the restriction on h in the second statement is much harder than in the first one. Indeed, in the case $h \leqslant \mathbf{h}_1 k^{-2}$ the intervals indicated in (1.31),

$$(h\lambda_k - \mathbf{c}_1 h^2 k^4, h\lambda_k + \mathbf{c}_1 h^2 k^4),\ k = 2, 3, \dots, \tag{1.35}$$

can overlap since $|\lambda_k - \lambda_{k\pm1}| \leqslant c_\pm k$, and thus we cannot guarantee that $j = k$ in (1.31). At the same time, the condition $h \leqslant \mathbf{h}_2 k^{-3}$ with a small $\mathbf{h}_2 > 0$ makes intervals (1.35) disjunct, leaves other eigenvalues out of segment (1.32), and, for fixed k, verifies convergence (1.23) as well as estimating its rate.

Due to (1.29) and (1.30), the restrictions imposed on the parameter h in the first and second assertions above can be turned into the restrictions $\lambda_k \leqslant C_1 h^{-1}$ and $\lambda_k \leqslant c_3 h^{-2/3}$, respectively, which regard h as a fixed small parameter but establish an upper bound for eigenvalues in (1.31) and (1.33). In other words, result 1^0 covers the eigenvalues $\Lambda_j(h)$ in the segment $[0, l_1]$ with a small $l_1 > 0$, i.e. *the whole low-frequency range* of the spectrum (1.16). Since there can exist several eigenvalues verifying estimate (1.31), we relate it to *the collective asymptotics of eigenvalues* in contrast to result 2^0 which, providing *the individual asymptotics of the eigenvalue* $\Lambda_k(h)$ in (1.33), remains valid only on the narrow part $[0, l_3 h^{1/3}]$ of the low-frequency range.

Since formula (1.29) transforms the inequality $h \leqslant \mathbf{h}_3 k^{-4}$ into $\beta_k \leqslant C_4 h^{-1/2}$ with $C_4 = \mathbf{h}_3^{1/2}\pi^2 T^{-2}$, result 3^0 holds true on the part $[l_1, l_4 h^{-1/2}]$ of *the middle-frequency range* $[l_1, l_2 h^{-1}]$ of spectrum (1.16) and always provides collective asymptotics of eigenvalues because, as outlined in Section 4.3, each point $L > 0$ becomes infinitely many times an eigenvalue of the parameter-dependent spectral problem (1.22) in the process when h decreases from 1 to 0.

Asymptotic expressions for eigenvectors can also be naturally divided into *collective* (also known as *quasimodes*) and *individual*. For the sake of brevity, we do not show here asymptotic representations for eigenfunctions of the scalar problem (1.22) but, in analogy, we refer to formulae (2.37) and (4.22) as *collective* asymptotics of the elastic eigenmodes u^k and to formulae (3.36) and (3.37) as *individual*. It is self-evident that individual asymptotics of eigenvectors are available provided the corresponding eigenvalues admit the individual asymptotics as well, i.e. on a narrow part of the low-frequency range. We refer to Lobo & Pérez (1997) for individual and collective asymptotics of eigen elements in a scalar stiff problem, with a Dirichlet condition on the whole boundary, and for more precise results on the structure of the eigenfunctions associated with the low and middle frequencies of the one-dimensional problem.

Specifying lower-order asymptotic terms in the eigenvalue representation,

$$\Lambda_k(h) = h\lambda_k + h^2\lambda_k^{(1)} + h^3\lambda_k^{(2)} + \cdots, \tag{1.36}$$

is usually based on the individual asymptotic forms of eigenvectors (see general procedures in Vainberg & Trenogin, 1974; Sanchez-Hubert & Sanchez-Palencia, 1989; Maz'ya *et al.*, 2000 and others). That is why an amendment of the asymptotics

$$\Lambda_k(h) = h\lambda_k + \mathrm{O}(h^2(1+\lambda_k)^2),$$

which appears in (1.33), can be achieved only within the narrow part of the low-frequency range.

We mention papers Panasenko (1980, 1987) and Lobo & Sanchez-Palencia (1989), where, for the dimension $n > 1$ and for a Dirichlet condition on $\Sigma^\pm$, the whole asymptotic series (1.36) is constructed for problem (1.22), but, only for simple eigenvalues and without estimating the asymptotic accuracy precisely. We note that in Panasenko (1980, 1987) the resulting problem (1.25) is not formulated directly, while the characteristic equation introduced in these papers cannot detect multiplicities of the eigenvalues in (1.24). Indeed, if $\Omega^- \cup \overline{\Omega}^+$ and $\overline{\Omega}^+$ are concentric balls in $\mathbb{R}^3$, then, owing to symmetry properties of spherical functions, the multiplicity $\varkappa_k$ of the eigenvalue λ_k grows indefinitely as $k \to +\infty$. When justifying these asymptotics, in the case where a Dirichlet condition is imposed on the whole boundary of the junction body Ω and $\Sigma^+ \neq \emptyset$, the convergence of the whole spectrum of (1.22) with conservation of the multiplicity can be found in Section 7.2 of Sanchez-Hubert & Sanchez-Palencia (1989).

On the other hand, the Weyl asymptotics for eigenvalues predicts that $\lambda_k = \mathrm{O}(k)$ for $\Omega \subset \mathbb{R}^2$ and $\lambda_k = \mathrm{O}(k^{2/3})$ for $\Omega \subset \mathbb{R}^3$. Such asymptotic behaviour of eigenvalues reduces the growth in k of the bounds in the assertions stated above. However, to our knowledge, these results have not been verified for non-local boundary conditions of Steklov type. Moreover, assertion 2^0 above is crucially based on the lower bound $\mathbf{c}_0 k$ for the differences $|\lambda_k - \lambda_{k\pm1}|$ (see definition (3.18) of d_k appearing in restriction (3.19)) but, in general, it is impossible to obtain information of this kind in many-dimensional domains. Let us observe that $d_k(2(1+\lambda_k))^{-1}$ measures the distance from $(1+\lambda_k)^{-1}$ to the nearest eigenvalue $(1+\lambda_{k\pm p})^{-1}$ with $p \geqslant 1$. Therefore, we emphasize that if two different eigenvalues happen to be close to each other, the value $(1 + d_k^{-1})^{-1}$ is small so that condition (3.19) becomes excessively restrictive and, hence, it is worth using the collective asymptotic forms, for which the condition (2.26) does not contain d_k at all.

1.4 *The direct and inverse reduction*

In order to evaluate the convergence rates of solutions to the elasticity problem (1.6) we use *the procedures of direct and inverse reduction* in singularly perturbed spectral problems. Here we comment upon the procedures by referring to properties of the scalar spectral problem (1.22), but their vectorial elastic analogues can be easily found in the next three sections.

As usual, inverse reduction implies constructing approximations to solutions of the original problem (1.22) from solutions of the resulting problems (1.25) and (1.26). The structures of the approximations are based on the asymptotic ansätze used to derive the resulting problems (see Section 2.1) and, since, in some sense, we reverse the asymptotic analysis, this procedure is called *the inverse reduction.* As a result, an accurate estimation of discrepancies and an application of the classical Lemma 1.4 on ‘almost eigenvalues and eigenvectors’ together with simple algebraic arguments provide inequalities (1.31) and (1.34) in assertions 1^0 and 3^0 of Section 1.3.

The direct reduction adapts a solution of the original problem (1.22) in order to approximate a solution of the resulting problem (1.25). To complete the approximation structures, we can only imitate the asymptotic analysis performed so that the second part of the whole procedure is more complicated than the first part. Again a scrupulous estimation of discrepancies allows us to apply the lemma on ‘almost eigenvalue’ and to obtain an eigenvalue λ_k in a small neighbourhood of the point $h^{-1}\Lambda_j(h)$. Since, in contrast, the inverse reduction has detected an eigenvalue $\Lambda_p(h)$ in the vicinity of the point $h\lambda_k$, we now combine these results and, recalling general properties of parameter-dependent spectra, we are able to conclude with inequality (1.33) and assertion 2^0 in Section 1.3.

The procedure of inverse reduction is a common tool for justifying asymptotic expansions of eigenvalues and eigenfunctions while the direct reduction here is intended to replace proving convergence theorems which, for example, obscure the dependence of convergence rates on the eigenvalue number. The justification scheme including both reductions has been used for miscellaneous problems with singular perturbations, namely, in Maz’ya *et al.* (1985), Nazarov (1993), Kamotskii & Nazarov (1998) and Maz’ya *et al.* (2000) for domains with irregularly perturbed boundaries, in Nazarov (1987a) for partial differential equations with a small parameter in the highest derivatives, and in Nazarov (1987b, 2002a,b, 2003) and Kamotskii & Nazarov (1999) for thin domains. Nevertheless, Nazarov (2002a) is the book where, for the first time, it was shown that direct reduction makes it possible to express explicitly the dependence of the estimate bounds on the eigenvalue number and other properties of the limiting spectrum. Here we adapt the approach in Nazarov (2002a) to the stiff elasticity problem (1.6). As regards the middle frequency vibrations, the kind of results in Section 4 have been proved for the first time in Lobo & Pérez (1997) for a scalar problem, using different techniques. Finally, it should be noted that the results in this paper apply to the Laplace operator and, consequently, because of the explicit bounds depending both on h and k, k being the number of the eigenvalue that we approach, they complement those of Lobo & Sanchez-Palencia (1989) and Panasenko (1980, 1987) for the low frequencies.

The paper is organized as follows. In Section 1.5, we present several lemmas used throughout the paper. We start in Section 2 with a formal asymptotic analysis of the elasticity problem (1.6) and derive the resulting problem describing the low-frequency range of the spectrum in (1.16). The inverse and direct reductions are performed for this range in Sections 2 and 3, respectively, and at the end of Section 3 we prove theorems about the asymptotic behaviour of low-frequencies and the corresponding eigenmodes. The middle-frequency range is considered in Section 4 where we proceed with the inverse reduction. Moreover, we reproduce the direct reduction so as to clarify an interconnection between spectrum (1.16) and the spectrum of the Neumann elasticity problem for the hard body Ω^+.

Finally, it should be mentioned that some of the results in this paper have been announced without proofs in Lobo *et al.* (2003b,a): see Lobo *et al.* (2003b) for the elasticity system in Section 1.1 and Lobo *et al.* (2003a) for the scalar problems in Section 1.3. As a matter of fact, throughout Sections 2–4, we present statements and proofs for the elasticity problem (1.6) since they extend the proofs of results for the scalar problem (1.22).

1.5 *Auxiliary lemmas*

Let us state a couple of known assertions, effective in the framework of both the inverse and direct reductions. The first of them is referred to as the lemma on 'almost eigenvalues and eigenfunctions' (see Vishik & Lyusternik, 1957). We refer to Section 7.1 in Nazarov (2002a) for the proof of the second lemma (see also Lazutkin, 1999 in connection with both lemmas).

LEMMA 1.4 Let $\mathcal{K}$ be a symmetric positive compact operator in the Hilbert space $\mathcal{H}$. Let also m, $\mathcal{Y}$ and δ be such that $m \in \mathbb{R}$, $\mathcal{Y} \in \mathcal{H}$, $||\mathcal{Y}; \mathcal{H}|| = 1$ and

$$\delta = ||\mathcal{K}\mathcal{Y} - m\mathcal{Y}; \mathcal{H}||. \tag{1.37}$$

Then the operator $\mathcal{K}$ has at least one eigenvalue μ satisfying the estimate

$$|\mu - m| \leqslant \delta. \tag{1.38}$$

Moreover, for any $\varepsilon \geqslant \delta$, there exist coefficients a_j such that

$$\left\| \mathcal{Y} - \sum_{\cdots} a_j y^j; \mathcal{H} \right\| < \varepsilon^{-1}\delta \tag{1.39}$$

where $\sum_{\cdots}$ denotes summation over all eigenvalues $\mu_j \in [m - \varepsilon, m + \varepsilon]$ of operator $\mathcal{K}$ and y^j are the corresponding eigenvectors subject to the normalization and orthogonality conditions $\langle y^j, y^k\rangle_{\mathcal{H}} = \delta_{j,k}$.

LEMMA 1.5 Let $y^1, \ldots, y^n \in \mathcal{H}$ and $\mathcal{Y}^1, \ldots, \mathcal{Y}^N \in \mathcal{H}$ fulfil the relations

$$\begin{gathered} \langle y^j, y^k\rangle_{\mathcal{H}} = \delta_{j,k}, \; ||\mathcal{Y}^q; \mathcal{H}|| = 1, \\ |\langle \mathcal{Y}^q, \mathcal{Y}^p\rangle_{\mathcal{H}} - \delta_{q,p}| \leqslant \tau, \; \left\| \mathcal{Y}^q - \sum_{j=1}^{n} a_j^q y^j; \mathcal{H} \right\| \leqslant \sigma. \end{gathered} \tag{1.40}$$

(1) Under the condition

$$(1 + \min\{n, N\})(\tau + (2 + \sigma)\sigma) < 1, \tag{1.41}$$

there holds the inequality $n \geqslant N$.

(2) If $n = N$, then the condition

$$n(\tau + (2 + \sigma)\sigma) < 1 \tag{1.42}$$

ensures the existence of the unitary $n \times n$-matrix $\theta = (\theta_q^j)$ such that

$$\left\| y^j - \sum_{q=1}^{n} \theta_q^j \mathcal{Y}^q; \mathcal{H} \right\| \leqslant n(\tau + (3 + \sigma)\sigma). \qquad \square$$

Throughout the paper we apply several versions of Korn's inequalities. Two of them have been presented in Lemma 1.1 and we add the third version, which is formulated here for the whole junction Ω, but we also use it for the units $\Omega^{\pm}$.

LEMMA 1.6 For $u \in H^1(\Omega)^3$, and for γ equal either to $\{\gamma^-, \gamma^+\}$ or to 1, the inequality

$$||u; H^1(\Omega)|| \leqslant c \left\{ ||D(\nabla)u; L_2(\Omega)|| + \left\| \int_{\Omega} d(x)^{\top} u(x)\gamma(x)\, \mathrm{d}x \right\| \right\} \tag{1.43}$$

is valid. $\square$

We point out that, first, inequalities $(1.10)_{\pm}$ follow from $(1.43)_{\pm}$ with $\gamma = 1$ due to conditions $(1.8)_{\pm}$ and, second, the last term in (1.43) can be readily replaced by the norm $||u; L_2(\Omega)||$.

The next two assertions provide us with an extension operator in the Sobolev spaces (see Section 8.1 in Lions & Magenes, 1972) and an inequality of Hardy type for a function with the narrow support. While constructing approximations for eigenmodes in the next sections, we multiply certain extensions with proper cut-off functions, this inequality becoming a crucial tool for deriving precise estimates for Sobolev norms of the products.

LEMMA 1.7 There exists the linear continuous extension operator

$$H^1(\Omega^+) \ni v^+ \mapsto v^- = \widehat{v}^+ \in \mathring{H}^1(\Omega^-; \Gamma) \tag{1.44}$$

and the function $v = \{v^-, v^+\}$ belongs to the Sobolev space $\mathring{H}^1(\Omega; \Gamma)$.

LEMMA 1.8 For any $t > 0$ there holds the estimate

$$t^{1/2}||\chi_t v^-; L_2(\Omega^-)|| + t^{-1/2}||\nabla(\chi_t v^-); L_2(\Omega^-)|| \leqslant c(t)||v^-; H^1(\Omega^-)|| \tag{1.45}$$

where $\chi_t(x) = \chi(t \operatorname{dist}(x, \partial\Omega^-))$ and $\chi \in C_0^{\infty}[0, 1)$ is a cut-off function, $\chi(z) = 1$ as $z \in [0, 1/2]$. The factor $c(t)$ in (1.45) does not depend on $v^- \in H^1(\Omega^-)$ and it can be chosen to be uniformly bounded as $t \geqslant 1$.

Proof. Since the function $\operatorname{dist}(x, \partial\Omega^-)$ belongs to $W^{1,\infty}(\Omega^-)$, inequality (1.45) holds true for a certain factor $c(t)$ independent of v^-. Since $c(t)$ can be chosen to depend continuously on t, it is sufficient to prove the relation $c(t) \leqslant c$ for $t \geqslant t_0^*$, where t_0^* and c are certain fixed positive constants.

According to the obvious estimate

$$t^{-1/2}||\nabla(\chi_t v^-); L_2(\Omega^-)|| \leqslant C\left(||\nabla v^-; L_2(\Omega^-)|| + t^{1/2}||v^-; L_2(\Omega_t^-)||\right)$$

where $\Omega_t^- = \{x \in \Omega^- : \operatorname{dist}(x, \partial\Omega^-) < t^{-1}\}$, we only need to confirm the inequality

$$||v^-; L_2(\Omega_t^-)|| \leqslant ct^{-1/2}||v^-; H^1(\Omega^-)||, \tag{1.46}$$

for sufficiently large t. Inequality (1.46) is proved in Lemma 1.2.4 of Nazarov (2002a) for domains with smooth boundaries, and it is based on the following one-dimensional variant of Hardy's inequality:

$$\delta^{-1}\int_0^\delta |V(t)|^2 \mathrm{d}t \leqslant c(d_0)\int_0^d \left(\left|\frac{\mathrm{d}V}{\mathrm{d}t}(t)\right|^2 + |V(t)|^2\right)\mathrm{d}t \tag{1.47}$$

with the constant $c(d_0)$ independent of $V \in H^1(0, d)$ and $\delta \in (0, d]$, $d \geqslant d_0 > 0$. Here below, we prove (1.46) for the case of a domain with a Lipschitz boundary.

Firstly, we use a localization argument and reduce the problem to a finite number of local maps, which leads us to consider one single local map. That is, we only consider the special sub-domain ω given by the relations

$$y_i \in (\alpha_i^-, \alpha_i^+),\ i = 1, 2; \quad y_3 \in (0, Y(y_1, y_2))$$

where y are Cartesian coordinates, obtained by an orthogonal transformation from the domains ϑ^i in the x-coordinates, $i = 1, 2, \dots, M$, and Y is a function with the global Lipschitz constant L.

Second, we take into account the inequalities

$$\operatorname{dist}(y, \Psi) \leqslant Y(y_1, y_2) - y_3 \leqslant C_L \operatorname{dist}(y, \Psi) \tag{1.48}$$

where $\Psi = \{y : \alpha_i^- < y_i < \alpha_i^+, y_3 = Y(y_1, y_2)\}$ and, for instance, $C_L = (L + 1)$. For sufficiently large t, we define the domains

$$\Omega_t^i = \left\{y \in \Omega^- : \operatorname{dist}(y, \partial\Omega) < \frac{1}{t}\right\} \subset \vartheta_i, \quad \omega_t^i = \left\{y \in \omega : \operatorname{dist}(y, \Psi) < \frac{1}{t}\right\}$$

and

$$\Theta_t^i = \left\{y \in \omega : 0 < Y(y_1, y_2) - y_3 < \frac{C_L}{t}\right\},$$

where $\Omega_t^- = \cup_{i=1}^M \Omega_t^i$, ω' is the rectangle $\omega' = (\omega_1^-, \omega_1^+) \times (\omega_2^-, \omega_2^+)$, and on account of (1.48), $\omega_t^i \subset \Theta_t^i$ for sufficiently large t, $t \geqslant t_0 > 0$. Obviously, for $v \equiv v^-$, we can write

$$\int_{\Omega_t^i} |v(x)|^2\,\mathrm{d}x = \int_{\omega_t^i} |v(y)|^2\,\mathrm{d}y \leqslant \int_{\Theta_t^i} |v(y)|^2\,\mathrm{d}y \leqslant \int_{\omega'} \mathrm{d}y_1\,\mathrm{d}y_2 \int_0^{C_L t^{-1}} |v(y_1, y_2, s)|^2\,\mathrm{d}s\,,$$

where the change of variable $s = Y(y_1, y_2) - y_3$ has been performed in the last integral.

Finally, we apply inequality (1.47) with $\delta = C_L t^{-1}$ and $t \geqslant t_0$, and we consider again $s = Y(y_1, y_2) - y_3$; we obtain the chain of inequalities

$$\int_{\omega_t^i} |v(y)|^2 \,\mathrm{d}y \leqslant \frac{1}{t} c(t_0) \int_{\omega'} \mathrm{d}y_1 \,\mathrm{d}y_2 \int_0^{Y(y_1,y_2)} (\,|v|^2 + |\nabla_y v|^2\,)\,\mathrm{d}y_3 \leqslant \frac{1}{t} c(t_0) ||v^-; H^1(\Omega^-)||^2.$$

Then, since the number of local maps is finite, namely M, we deduce

$$\int_{\Omega_t^-} |v^-|^2 \,\mathrm{d}x \leqslant \frac{1}{t} C(t_0) ||v^-; H^1(\Omega^-)||^2$$

and (1.46) is proved and, therefore, the statement in the lemma holds. □

2. The inverse reduction for the low-frequency range

2.1 *The formal asymptotic analysis*

For a solution of the spectral problem (1.6), we choose the following asymptotic ansätze:

$$\begin{aligned} \Lambda(h) &= h\lambda + \cdots, \\ u^-(h,x) &= U^0(x) + \cdots, \\ u^+(h,x) &= d(x)a^+ + hU^+(x) + \cdots. \end{aligned} \tag{2.1}$$

Let us comment upon our choice. Ansatz $(2.1)_1$ reflects the basic formulae (1.20), (1.21) and, by virtue of (1.11), the corresponding eigenmode u verifies the relation $\langle u, u\rangle_h = \mathrm{O}(h)$. In view of (1.5) and (1.19), this means that the main asymptotic term of u^+ must make zero the energy functional $(A^+D(\nabla)u^+, D(\nabla)u^+)_{\Omega^+}$ and, therefore, u^+ coincides with the rigid motion $d(x)a^+$. However, the main term of u^- can be arbitrary because of assumption (1.5).

We insert (2.1) into problem (1.6) and, recalling (1.5) again, we collect coefficients at the same powers of h. From $(1.6)_{1-}$, $(1.6)_{2-}$ and $(1.6)_3$, it follows that

$$\begin{aligned} L^0(x,\nabla)U^0(x) &:= D(-\nabla)^\top A^0(x)D(\nabla)U^0(x) = \lambda\gamma^-(x)U^0(x),\ x \in \Omega^-, \\ N^0(x,\nabla)U^0(x) &:= D(n^-(x))^\top A^0(x)D(\nabla)U^0(x) = 0,\ x \in \Sigma^-, \\ U^0(x) &= 0,\ x \in \Gamma. \end{aligned} \tag{2.2}$$

In order to obtain a well posed problem (see $(2.2)_4$ below), we consider the first transmission (adhesion) condition in $(1.6)_4$ which, owing to $(2.1)_{2,3}$, yields

$$U^0(x) = d(x)a^+,\ x \in \Upsilon. \tag{2.3}$$

Furthermore, from $(1.6)_{1+}$, $(1.6)_{2+}$ and the second transmission condition, we derive the following Neumann elasticity problem to connect the displacement fields U^0, U^+ with the column $a^+ \in \mathbb{R}^6$ which is still unknown,

$$\begin{aligned} L^+(x,\nabla)U^+(x) &= \lambda\gamma^+(x)d(x)a^+,\ x \in \Omega^+, \\ N^+(x,\nabla)U^+(x) &= 0,\ x \in \Sigma^+, \\ N^+(x,\nabla)U^+(x) &= -N^0(x,\nabla)U^0(x),\ x \in \Upsilon. \end{aligned} \tag{2.4}$$

We point out that the minus in $(2.4)_3$ is due to using the different normals n^+ and $n^- = -n^+$ in the definitions in $(1.6)_2$ and $(2.2)_2$. As is known, the compatibility conditions for problem (2.4) take the form

$$\int_{\Omega^+} d(x)^\top [\lambda^+ \gamma(x) d(x) a^+] \, dx + \int_{\Upsilon} d(x)^\top [-N^0(x, \nabla) U^0(x)] \, ds = 0 \in \mathbb{R}^6 \tag{2.5}$$

and imply the fact that the external forces in (2.4) are self-balanced. If (2.5) holds true, problem (2.4) admits a solution $U^+ \in H^1(\Omega^+)^3$ which becomes unique, e.g. under the orthogonality conditions $(1.8)_+$. The 6×6-matrix

$$\mathbf{d}(\Omega^+) = \int_{\Omega^+} d(x)^\top d(x) \gamma^+(x) \, dx \tag{2.6}$$

is merely the Gram matrix constructed from columns of matrix (1.9) with the help of the scalar product $(\gamma^+ \bullet, \bullet)_{\Omega^+}$ in $L_2(\Omega^+)^6$. Since these columns are linearly independent, $\mathbf{d}(\Omega^+)$ is positive definite and, of course, symmetric. Thus, by (2.5), we have

$$\lambda a^+ = \mathbf{d}(\Omega^+)^{-1} \int_{\Upsilon} d(x)^\top N^0(x, \nabla) U^0(x) \, ds. \tag{2.7}$$

Now, introducing (2.7) into (2.3) furnishes the non-local boundary conditions of Steklov type

$$\lambda U^0(x) = d(x) \mathbf{d}(\Omega^+)^{-1} \int_{\Upsilon} d(x)^\top N^0(x, \nabla) U^0(x) \, ds \tag*{$(2.2)_4$}$$

which completes the resulting problem (2.2).

2.2 *The spectrum of the resulting problem*

To find the weak formulation of the problem (2.2), we introduce the Hilbert space

$$\mathcal{H}_0 = \left\{ V \in \mathring{H}^1(\Omega^-; \Gamma)^3 : V(x) = d(x) a, \ x \in \Upsilon, \ \text{for some cosntant } a \in \mathbb{R}^6 \right\} \tag{2.8}$$

and readily obtain that

$$\begin{gathered} a = \mathbf{d}(\Upsilon)^{-1} \int_{\Upsilon} d(x)^\top V(x) \, ds, \\ \mathbf{d}(\Upsilon) = \int_{\Upsilon} d(x)^\top d(x) \, dx, \end{gathered} \tag{2.9}$$

where, similarly to (2.6), $\mathbf{d}(\Upsilon)$ is a symmetric and positive definite 6×6 Gram matrix. In particular, for $\lambda > 0$ and $U^0 \in \mathring{H}^1(\Omega^-; \Gamma)^3$, condition $(2.2)_4$ provides the inclusion $U^0 \in \mathcal{H}_0$ and the relation

$$\lambda \int_{\Upsilon} d(x)^\top U^0(x) \, ds = \mathbf{d}(\Upsilon) \mathbf{d}(\Omega^+)^{-1} \int_{\Upsilon} d(x)^\top N^0(x, \nabla) U^0(x) \, ds. \tag{2.10}$$

With $V \in \mathcal{H}_0$, we write the Green formula

$$(A^0 D(\nabla) U^0, D(\nabla) V)_{\Omega^-} - \lambda (\gamma^- U^0, V)_{\Omega^-} = (N^0 U^0, V)_{\Upsilon} \tag{2.11}$$

(see $(2.2)_{2,3}$). We transform the right-hand side of (2.11) in accordance with (2.8)–(2.11),

$$(N^0U^0, V)_{\Upsilon} = a^\top \int_{\Upsilon} d(x)^\top N^0(x,\nabla)U^0(x)\,\mathrm{d}s = \lambda a^\top \mathbf{d}(\Omega^+)\mathbf{d}(\Upsilon)^{-1}\int_{\Upsilon} d(x)^\top U^0(x)\,\mathrm{d}s$$
$$= \lambda \left\langle \mathbf{d}(\Omega^+)\mathbf{d}(\Upsilon)^{-1}\int_{\Upsilon} d(x)^\top U^0(x)\,\mathrm{d}s, \mathbf{d}(\Upsilon)^{-1}\int_{\Upsilon} d(x)^\top V(x)\,\mathrm{d}s\right\rangle_{\mathbb{R}^6} =: \lambda q_{\Upsilon}(U^0, V). \tag{2.12}$$

Here q_Υ is a symmetric and non-negative quadratic form on $\mathcal{H}_0$. Thus, we reformulate the resulting spectral problem (2.2): to find a number $\lambda \in \mathbb{R}$ and a non-trivial field $U^0 \in \mathcal{H}_0$ such that

$$(A^0D(\nabla)U^0, D(\nabla)V)_{\Omega^-} = \lambda\{(\gamma^- U^0, V)_{\Omega^-} + q_\Upsilon(U^0, V)\} \quad \forall V \in \mathcal{H}_0. \tag{2.13}$$

Lemma 1.1 and inequality $(1.19)_2$ ensure that

$$\langle U^0, V\rangle_0 = (A^0D(\nabla)U^0, D(\nabla)V)_{\Omega^-} + (\gamma^- U^0, V)_{\Omega^-} + q_\Upsilon(U^0, V) \tag{2.14}$$

is a scalar product in the Hilbert space $\mathcal{H}_0$. Hence, we reduce the integral identity (2.13) to the abstract equation

$$\mathcal{K}_0 U^0 = \mu U^0 \in \mathcal{H}_0 \tag{2.15}$$

with the spectral parameter $\mu = (1+\lambda)^{-1}$ and the positive symmetric compact operator $\mathcal{K}_0 : \mathcal{H}_0 \to \mathcal{H}_0$,

$$\langle \mathcal{K}_0 U^0, V\rangle_0 = (\gamma^- U^0, V)_{\Omega_-} + q_\Upsilon(U^0, V). \tag{2.16}$$

PROPOSITION 2.1 The eigenvalues of the spectral problem (2.13), i.e. problem (2.2), form the sequence

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to +\infty \tag{2.17}$$

while the corresponding eigenmodes $U^{j0} \in \mathring{H}^1(\Omega^-;\Gamma)^3$ satisfy the normalization and orthogonality conditions

$$(\gamma^- U^{j0}, U^{k0})_{\Omega^-} + q_\Upsilon(U^{j0}, U^{k0}) = \delta_{j,k}. \tag{2.18}$$

REMARK 2.2

(1) Lemma 1.1 leads to the following inferences. If $\Gamma = \emptyset$, then $\lambda_1 = \cdots = \lambda_6 = 0$, $\lambda_7 > 0$ (see (1.21)) and $U^{10}, \ldots, U^{60}$ imply rigid motion. In the case $\mathrm{meas}_2\,\Gamma > 0$ we have $\lambda_1 > 0$ in (2.17).
(2) The eigenvectors $\mathcal{U}^{j0} = (1+\lambda_j)^{-1/2}U^{j0}$ corresponding to the eigenvalues $\mu_j = (1+\lambda_j)^{-1}$ of the operator $\mathcal{K}_0$ satisfy the normalization and orthogonality conditions

$$\langle \mathcal{U}^{j0}, \mathcal{U}^{k0}\rangle_0 = \delta_{j,k}.$$

2.3 *The inverse reduction*

Let us take an eigenvalue λ_k of multiplicity $\varkappa_k$,

$$\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+\varkappa_k-1} < \lambda_{k+\varkappa_k}. \tag{2.19}$$

We define the approximate solution to the spectral equation (1.14), obtained from the original problem (1.6), by the formulae

$$\begin{aligned} m_k(h) &= (h + h\lambda_k)^{-1}, \\ v^p &= \{v^{p-}, v^{p+}\} = \{U^{p0} + h\chi^k \widehat{U}^p, da^p + hU^{p+}\},\ p = k, \ldots, k + \varkappa_k - 1. \end{aligned} \tag{2.20}$$

Here the eigenmode U^{p0} satisfies condition (2.18), a^p is the corresponding column in $\mathbb{R}^6$ determined as in (2.9) and $\widehat{U}^p \in \mathring{H}^1(\Omega^-; \Gamma)^3$ the extension (1.44) of the solution $U^{p+} \in H^1(\Omega^+)^3$ to the problem (2.4) with $\lambda = \lambda_k$, $a^+ = a^p$ and $U^0 = U^{p0}$ while $\chi^k = \chi_{1+\lambda_k}$ the cut-off function introduced in Lemma 1.8.

The formulae (2.18), $(2.12)_2$ and (2.13) maintain the estimates

$$\begin{aligned} &||U^{p0}; L_2(\Omega^-)|| + |a^p| \leqslant c, \\ &||D(\nabla)U^{p0}; L_2(\Omega^-)|| \leqslant c\lambda_k^{1/2}. \end{aligned} \tag{2.21}$$

We now proceed to compute the H^1-norm of the other terms in $(2.20)_2$. Since a solution of the Neumann problem (2.4) is defined up to an additive rigid motion and the Gram matrix (2.6) is non-degenerate, we impose the orthogonality condition

$$\int_{\Omega^+} d(x)^\top U^{p+}(x)\gamma^+(x)\, \mathrm{d}x = 0 \in \mathbb{R}^6 \tag{2.22}$$

which determines uniquely the term U^{p+} in (2.20). In the sequel, we find the condition (2.22) to be far more convenient than the alternative condition $(1.8)_+$. Now Lemma 1.6 and inequality $(1.19)_+$ ensure that

$$||U^{p+}; H^1(\Omega^+)|| \leqslant c \sup_{\cdots} |(A^+ D(\nabla)U^{p+}, D(\nabla)W^+)_{\Omega^+}|$$

where the dots stand for '$W^+ \in H^1(\Omega^+)^3 : ||D(\nabla)W^+; L_2(\Omega^+)|| = 1$ and W^+ satisfies $(1.8)_+$'. Using the extension $\widehat{W} \in \mathring{H}^1(\Omega^-; \Gamma)^3$ of the trial function W^+, we apply Green's formulae in $\Omega^\pm$ (see (2.11)) and derive from (2.4) and (2.2) that

$$\begin{aligned} (A^+ D(\nabla)U^{p+}, D(\nabla)W^+)_{\Omega^+} &= \lambda_k(\gamma^+ da^p, W^+)_{\Omega^+} - (N^0 U^{p0}, W^+)_\Gamma \\ = \lambda_k(\gamma^+ da^p, W^+)_{\Omega^+} &+ \lambda_k(\gamma^- U^{p0}, \widehat{W})_{\Omega^-} - (A^0 D(\nabla)U^{p0}, D(\nabla)\widehat{W})_{\Omega^-}. \end{aligned}$$

Hence, by Lemmas 1.7, 1.1(2) and estimate (2.21),

$$||\widehat{U}^p; H^1(\Omega^-)|| \leqslant c||U^{p+}; H^1(\Omega^+)|| \leqslant c(\lambda_k + \lambda_k + \lambda_k^{1/2})(1 + ||D(\nabla)\widehat{W}; L_2(\Omega^-)||) \leqslant c\lambda_k. \tag{2.23}$$

The second step of our treatment concerns evaluating the scalar product (1.12) of the approximate solutions $(2.20)_2$. We readily have

$$\begin{aligned} \langle v^j, v^p\rangle_h = h &\left(A^0 D(\nabla)[U^{j0} + h\chi^k \widehat{U}^j], D(\nabla)[U^{p0} + h\chi^k \widehat{U}^p]\right)_{\Omega^-} \\ &+ h^2 \left(A^+ D(\nabla)U^{j+}, D(\nabla)U^{p+}\right)_{\Omega^+} + h(\gamma^-[U^{j0} + h\chi^k \widehat{U}^j], [U^{p0} + h\chi^k \widehat{U}^p])_{\Omega^-} \\ &+ h\left(\gamma^+[da^j + hU^{j+}], [da^p + hU^{p+}]\right)_{\Omega^+}. \end{aligned} \tag{2.24}$$

The normalization and orthogonality conditions (2.18) together with the integral identity (2.13) and the definition of the quadratic form q_Υ in (2.12) yield

$$\begin{aligned}&h(A^0D(\nabla)U^{j0}, D(\nabla)U^{p0})_{\Omega^-} + h(\gamma^- U^{j0}, U^{p0})_{\Omega^-} + h(\gamma^+ da^j, da^p)_{\Omega^+}\\&= h\lambda_k\{(\gamma^- U^{j0}, U^{p0})_{\Omega^-} + q_\Upsilon(U^{j0}, U^{p0})\} + h(\gamma^- U^{j0}, U^{p0})_{\Omega^-}\\&\quad + h(\gamma^+ da^j, da^p)_{\Omega^+} = h(\lambda_k + 1)\delta_{j,p}.\end{aligned}$$

The remaining terms in (2.24) are estimated by means of inequalities (2.21), (2.23) and (1.45) with $t = 1 + \lambda_k$,

$$\begin{aligned}&h^2|(A^0D(\nabla)\chi^k\widehat{U}^j, D(\nabla)U^{p0})_{\Omega^-}| + h^3|(A^0D(\nabla)\chi^k\widehat{U}^j, D(\nabla)\chi^k\widehat{U}^p)_{\Omega^-}|\\&\quad \leqslant ch^2(1+\lambda_k)^{1/2}\{\lambda_k^{1/2} + h(1+\lambda_k)^{1/2}\lambda_k\}\lambda_k,\\&h^2|(A^+D(\nabla)U^{j+}, D(\nabla)U^{p+})_{\Omega^+}| \leqslant ch^2\lambda_k^2,\\&h^2|(\gamma^- U^{j0}, U^{p0})_{\Omega^-}| + h^3|(\gamma^- U^{j0}, \chi^k\widehat{U}^p)_{\Omega^-}| \leqslant ch^2\{1 + (1+\lambda_k)^{-1/2}\lambda_k\},\\&h^2|(\gamma^+ U^{j+}, da^p)_{\Omega^+}| + h^3|(\gamma^+ U^{j+}, U^{p+})_{\Omega^+}| \leqslant ch^2\lambda_k(1 + h\lambda_k).\end{aligned}$$

It should be noted that, for estimating certain terms in (2.24), it is necessary to interchange the position of the indices j and p in the last inequalities. As a result, we obtain that

$$|\langle v^j, v^p\rangle_h - h(1+\lambda_k)\delta_{j,p}| \leqslant ch^2(1+\lambda_k)^2(1+h\lambda_k). \tag{2.25}$$

Hence, the restriction

$$h \leqslant h_1(1+\lambda_k)^{-1} \tag{2.26}$$

with a small $h_1 > 0$ leads to the formula

$$||v^j; \mathcal{H}_h|| \geqslant \frac{1}{2}h^{1/2}(1+\lambda_k)^{1/2} \tag{2.27}$$

and, moreover, the vectors

$$\mathcal{V}^j = ||v^j; \mathcal{H}_h||^{-1}v^j,\ j = k, \dots, k + \varkappa_k - 1, \tag{2.28}$$

are normalized in $\mathcal{H}_h$ and fulfil the relation

$$\begin{aligned}|\langle \mathcal{V}^j, \mathcal{V}^p\rangle_h - \delta_{j,p}| &\leqslant ||v^j; \mathcal{H}_h||^{-1}||v^p; \mathcal{H}_h||^{-1}\big\{|\langle v^j, v^p\rangle_h\\&\quad - h(1+\lambda_k)\delta_{j,p}| + \delta_{j,p}|\langle v^j, v^j\rangle_h - h(1+\lambda_k)|\big\}\\&\leqslant ch^{-1}(1+\lambda_k)^{-1}h^2(1+\lambda_k)^2 = ch(1+\lambda_k).\end{aligned} \tag{2.29}$$

The next step prepares an application of Lemma 1.4 on ‘almost eigenvalues’. To this end, we compute the norm

$$\begin{aligned}||\mathcal{K}_h\mathcal{V}^j - m_k(h)\mathcal{V}^j; \mathcal{H}_h|| &= (h+h\lambda_k)^{-1}||v^j; \mathcal{H}_h||^{-1}||(h+h\lambda_k)\mathcal{K}_h v^j - v^j; \mathcal{H}_h||\\ &\leqslant ch^{-3/2}(1+\lambda_k)^{-3/2}\sup_{\cdots}|\langle v^j - (h+h\lambda_k)\mathcal{K}_h v^j, W\rangle_h|\\ &= ch^{-3/2}(1+\lambda_k)^{-3/2}\sup_{\cdots}\Big|\sum_{\pm}(A^{\pm}D(\nabla)v^{j\pm}, D(\nabla)W^{\pm})_{\Omega^{\pm}}\\ &\quad -h\lambda_k\sum_{\pm}(\gamma^{\pm}v^{j\pm}, W^{\pm})_{\Omega^{\pm}}\Big|\\ &= ch^{-1/2}(1+\lambda_k)^{-3/2}\sup_{\cdots}\big|(A^0D(\nabla)[U^{j0}+h\chi^k\widehat{U}^j], D(\nabla)W^-)_{\Omega^-}\\ &\quad +(A^+D(\nabla)U^{j+}, D(\nabla)W^+)_{\Omega^+} - \lambda_k(\gamma^-[U^{j0}+h\chi^k\widehat{U}^j], W^-)_{\Omega^-}\\ &\quad -\lambda_k(\gamma^+[da^j+hU^{j+}], W^+)_{\Omega^+}\big|. \end{aligned}\tag{2.30}$$

Here the dots stand for '$W \in \mathring{H}^1(\Omega;\Gamma)^3 : ||W;\mathcal{H}_h|| = 1$'. We detach the rigid motion $d(x)b$ while providing the orthogonality property $(1.8)_+$ of the remainder

$$W^{\perp}(x) = W^+(x) - d(x)a^W. \tag{2.31}$$

Lemmas 1.1(2), 1.6 and definitions (1.12), (1.5) and (1.19) lead to the estimates

$$\begin{aligned}||W^{\perp}; H^1(\Omega^+)|| &\leqslant c_+||D(\nabla)W^{\perp}; L_2(\Omega^+)|| = c_+||D(\nabla)W^+; L_2(\Omega^+)||\\ &\leqslant C||W;\mathcal{H}_h||,\\ ||W^-; H^1(\Omega^-)|| &\leqslant c\left(||D(\nabla)W^-; L_2(\Omega^-)|| + ||W^-; L_2(\Omega^-)||\right)\\ &\leqslant Ch^{-1/2}||W;\mathcal{H}_h||.\end{aligned}\tag{2.32}$$

We multiply $(2.4)_1$ by W^+, integrate by parts in Ω^+, and obtain

$$(A^+D(\nabla)U^{p+}, D(\nabla)W^+)_{\Omega^+} = \lambda_k(\gamma^+da^p, W^+)_{\Omega^+} - (N^0U^{p0}, W^+)_{\Upsilon}.$$

This equality and the Green formula (2.11), where $U^0 = U^{0p}, \lambda = \lambda_k$ and $V = W^-$, cancel certain terms in (2.30) and we continue the calculation,

$$\begin{aligned}||\mathcal{K}_h\mathcal{V}^j - m_k(h)\mathcal{V}^j; \mathcal{H}_h|| &\leqslant ch^{-1/2}(1+\lambda_k)^{-3/2}\sup_{\cdots}\big|h(A^0D(\nabla)\chi^k\widehat{U}^j, D(\nabla)W^-)_{\Omega^-}\\ &\quad -h\lambda_k(\gamma^-\chi^k\widehat{U}^j, W^-)_{\Omega^-} - h\lambda_k(\gamma^+U^{j+}, W^+)_{\Omega^+}\big|.\end{aligned}$$

Note that, by (2.31) and (2.22), $(\gamma^+U^{j+}, W^+)_{\Omega^+} = (\gamma^+U^{j+}, W^{\perp})_{\Omega^+}$. Now, appealing to inequalities (2.21), (2.23), (2.32) and (1.45), we make use of restriction (2.26) and arrive at the final estimate of the norm

$$\begin{aligned}||\mathcal{K}_h\mathcal{V}^j - m_k(h)\mathcal{V}^j; \mathcal{H}_h|| &\leqslant ch^{-1/2}(1+\lambda_k)^{-3/2}\big\{h(1+\lambda_k)^{1/2}\lambda_k h^{-1/2}\\ &\quad +h\lambda_k(1+\lambda_k)^{-1/2}\lambda_k h^{-1/2} + h\lambda_k\lambda_k\big\} \leqslant C.\end{aligned}\tag{2.33}$$

PROPOSITION 2.3 There exist constants $h_1 > 0$ and $c_1 > 0$ such that, for any eigenvalue λ_k of problem (2.2) (see also (2.13)), condition (2.26) provides at least one eigenvalue $\Lambda_j(h)$ of problem (1.6) admitting the estimate

$$|\Lambda_j(h) - h\lambda_k| \leqslant c_1h^2(1+\lambda_k)^2. \tag{2.34}$$

Proof. Since formulae (2.28) and (2.33) verify the hypotheses of Lemma 1.4 for the couple $m_k(h) \in \mathbb{R}$, $v^j \in \mathcal{H}_h$ in (2.2), there appears the eigenvalue $[h + \Lambda_j(h)]^{-1}$ of operator $\mathcal{K}_h$ in (1.13) such that

$$|[h + \Lambda_j(h)]^{-1} - [h + h\lambda_k]^{-1}| \leqslant C.$$

Hence,

$$|\Lambda_j(h) - h\lambda_k| \leqslant Ch^2(1 + \lambda_k) + Ch(1 + \lambda_k)\Lambda_j(h) \tag{2.35}$$

and in the case where $h_1 \leqslant (2C)^{-1}$ we see that $Ch(1 + \lambda_k) \leqslant 1/2$ according to (2.26). We now observe that

$$\begin{gathered} \Lambda_j(h) \leqslant h\lambda_k + Ch^2(1 + \lambda_k) + \frac{1}{2}\Lambda_j(h) \Rightarrow \\ h + \Lambda_j(h) \leqslant h + 2\left(h\lambda_k + \frac{h}{2}\right) = 2(h + h\lambda_k), \\ \Lambda_j(h) \geqslant h\lambda_k - Ch^2(1 + \lambda_k) - \frac{1}{2}\Lambda_j(h) \Rightarrow \\ h + \Lambda_j(h) \geqslant h + \frac{2}{3}\left(h\lambda_k - \frac{h}{2}\right) = \frac{2}{3}(h + h\lambda_k). \end{gathered} \tag{2.36}$$

Using $(2.36)_1$ in (2.35) leads to inequality (2.34) with $c_1 = 2C$. □

2.4 *A multiple eigenvalue*

Proposition 2.3 provides only one eigenvalue $\Lambda_j(h)$ subject to estimate (2.34). To consider carefully the case in (2.19) with $\varkappa_k > 1$, it is worth applying Lemma 1.5(1) after the second part of Lemma 1.4. Indeed, for any $\varepsilon > C$ and $p = k, \dots, k + \varkappa_k - 1$, the coefficients $\alpha_p^j(h)$ are such that, in view of (2.33) and (1.37), inequality (1.39) turns into

$$||\mathcal{V}^j - \sum_{p=K(h)}^{K(h)+X(h)-1} \alpha_p^j(h)\mathcal{U}^{ph}; \mathcal{H}_h|| \leqslant C\varepsilon^{-1} \tag{2.37}$$

where $M_{K(h)}(h), \dots, M_{K(h)+X(h)-1}(h)$ are all eigenvalues of the operator $\mathcal{K}_h$ that satisfy the inclusion

$$M_p(h) \in ([h + h\lambda_k]^{-1} - \varepsilon, [h + h\lambda_k]^{-1} + \varepsilon), \tag{2.38}$$

and $\mathcal{U}^{ph}$ are the corresponding eigenvectors. Note that $\mathcal{V}^j$ and $\mathcal{U}^{ph}$ are normalized in $\mathcal{H}_h$ and $\mathcal{U}^{ph}$ also meets the orthogonality conditions (1.18). Fixing $\varepsilon = \varepsilon_2 l$ with an integer $l \in (0, \varkappa_k]$ and a large $\varepsilon_2 > 0$, we verify hypotheses (1.40) where

$$\begin{aligned} &y^j = \mathcal{U}^{K(h)+j,h}, n = X(h); \mathcal{Y}^p = \mathcal{V}^{k+p}, N = l \leqslant \varkappa_k; \\ &\tau = ch(1 + \lambda_k), \sigma = C\varepsilon^{-1} = C\varepsilon_2^{-1}l^{-1} \end{aligned}$$

(see (2.29) and (2.37)). Assuming

$$h \leqslant h_2 l^{-1}(1 + \lambda_k)^{-1}, \tag{2.39}$$

we have

$$(1+\min\{n,N\})(\tau+(2+\sigma)\sigma) \leqslant 2l(ch(1+\lambda_k)+(2+C\varepsilon_2^{-1}l^{-1})C\varepsilon_2^{-1}l^{-1}) \\ \leqslant 2ch_2+(2+C\varepsilon_2^{-1})2C\varepsilon_2^{-1}.$$

Thus, inequality (1.41) becomes valid if $h_2 \leqslant (8c)^{-1}$ and $\varepsilon_2 \geqslant 8C$.

PROPOSITION 2.4 There exist constants $h_2 > 0$ and $c_2 > 0$ such that, for any eigenvalue λ_k of multiplicity $\varkappa_k$ (see (2.19)) and any integer $l \in (0, \varkappa_k]$, condition (2.39) provides at least l eigenvalues $\Lambda_j(h), \dots, \Lambda_{j+l-1}(h)$ of problem (1.6) admitting the estimate

$$|\Lambda_p(h) - h\lambda_k| \leqslant c_2 l h^2 (1+\lambda_k)^2. \tag{2.40}$$

Proof. Owing to (1.15) and (2.37), inclusion (2.38) implies the estimate

$$|[h+\Lambda_p(h)]^{-1} - [h+h\lambda_k]^{-1}| \leqslant C_2 l \tag{2.41}$$

which, under condition (2.39), can be transformed into (2.40) in the same way as was performed in (2.36). We point out that $c_2 = 2C_2 = 2\varepsilon_2$. □

3. The direct reduction for the low-frequency range

3.1 *Approximate solution to the resulting spectral problem*

Let $\Lambda_j(h)$ be an eigenvalue of problem (1.6) and u^j the corresponding eigenvector subject to the normalization and orthogonality conditions (1.17). Similarly to (2.31), we decompose the restriction u^{j+} of u^j on Ω^+:

$$u^{j+}(x) = u^{j\perp}(x) + d(x)b^j, \tag{3.1}$$

where $b^j \in \mathbb{R}^6$ and $u^{j\perp}$ satisfies (2.22). Lemma 1.7 provides the extension $\widehat{u}^j \in \mathring{H}^1(\Omega^-;\Gamma)^3$ of $u^{j\perp} \in H^1(\Omega^+)^3$. We indicate an approximation solution to the resulting problem (2.15) by the formulae

$$\begin{aligned} \mathbf{m}_j(h) &= (1+h^{-1}\Lambda_j(h))^{-1}, \\ \mathbf{u}^j(x) &= u^{j-}(x) - \chi^j(h,x)\widehat{u}^j(x), x \in \Omega^-, \end{aligned} \tag{3.2}$$

where χ^j is the cut-off function $\chi_{1+h^{-1}\Lambda_j(h)}$ introduced in Lemma 1.8 for $t = 1 + h^{-1}\Lambda_j(h)$. We emphasize that, owing to (3.1), $\mathbf{u}^j(x) = d(x)b^j$ on Υ and, therefore, $\mathbf{u}^j \in \mathcal{H}_0$ (see (2.8)).

Our immediate objective is to obtain estimates for the vector function $(3.2)_2$. By analogy with (2.21) and (2.32), from (1.17), (1.11) and (1.43) we derive that

$$\begin{aligned} &||u^j; L_2(\Omega)|| \leqslant c, \\ &h^{1/2}||D(\nabla)u^{j-}; L_2(\Omega^-)|| + ||D(\nabla)u^{j+}; L_2(\Omega^+)|| \leqslant c||u^j;\mathcal{H}_h|| = c(h+\Lambda_j(h))^{1/2}, \\ &||\widehat{u}^j; H^1(\Omega^-)|| \leqslant c||u^{j\perp}; H^1(\Omega^+)|| \leqslant c||D(\nabla)u^{j\perp}; L_2(\Omega^+)|| \\ &\quad = c||D(\nabla)u^j; L_2(\Omega^+)|| \leqslant c||u^j;\mathcal{H}_h|| = c(h+\Lambda_j(h))^{1/2}. \end{aligned} \tag{3.3}$$

The upper bound in $(3.3)_3$ can be reduced for the low frequencies.

LEMMA 3.1 The inequalities

$$\begin{aligned} &||b^j; \mathbb{R}^6|| \leqslant c(1 + \Lambda_j(h)^{1/2}), \\ &||\widehat{u}^j; H^1(\Omega^-)|| \leqslant c_0||u^{j\perp}; H^1(\Omega^+)|| \leqslant C(\Lambda_j(h) + h^{1/2}\Lambda_j(h)^{1/2}) \leqslant 2C(h + \Lambda_j(h)) \end{aligned} \tag{3.4}$$

are valid where the constants do not depend on $h \in (0, 1]$ and $j = 1, 2 \ldots$.

Proof. The representation in (3.1) and the orthogonality condition (2.22) for $u^{j\perp}$ yield

$$b^j = \left\{ \int_{\Omega^+} d(x)^\top d(x)\,\mathrm{d}x \right\}^{-1} \int_{\Omega^+} d(x)^\top (u^{j+}(x) - u^{j\perp}(x))\,\mathrm{d}x.$$

Since the first integral above is merely a positive definite and symmetric 6×6 Gram matrix, we derive from $(3.3)_{1,3}$ inequality $(3.4)_1$,

$$||b^j; \mathbb{R}^6|| \leqslant c\left(||u^{j+}; L_2(\Omega^+)|| + ||u^{j\perp}; L_2(\Omega^+)||\right) \leqslant c(1 + \Lambda_j(h)^{1/2}).$$

Now we insert $u^\pm = u^{j\pm}$, $\Lambda(h) = \Lambda_j(h)$ and $v = \{u^{j\perp}, \widehat{u}^j\}$ into the integral identity (1.11). By virtue of (3.3), (1.10), $(1.19)_1$ and (1.20) we have

$$\begin{aligned} (A^+D(\nabla)u^{j\perp}, D(\nabla)u^{j\perp})_{\Omega^+} &= -h(A^0D(\nabla)u^{j-}, D(\nabla)\widehat{u}^j)_{\Omega^-} \\ &\quad + \Lambda_j(h)\{(\gamma^+u^{j+}, u^{j\perp})_{\Omega^+} + (\gamma^-u^{j-}, \widehat{u}^j)_{\Omega^-}\} \\ &\leqslant \delta||D(\nabla)\widehat{u}^j; L_2(\Omega^-)||^2 + c_\delta h^2||D(\nabla)u^{j-}; L_2(\Omega^-)||^2 \\ &\quad + \delta||u^{j\perp}; L_2(\Omega^+)||^2 + c_\delta\Lambda_j(h)^2||u^{j+}; L_2(\Omega^+)||^2 \\ &\quad + \delta||\widehat{u}^j; L_2(\Omega^-)||^2 + c_\delta\Lambda_j(h)^2||u^{j-}; L_2(\Omega^-)||^2 \\ &\leqslant c\delta||D(\nabla)u^{j\perp}; L_2(\Omega^+)||^2 + C_\delta[h\Lambda_j(h) + \Lambda_j(h)^2]. \end{aligned}$$

In the previous algebraic inequality we have used

$$xy \leqslant \delta x^2 + (4\delta)^{-1}y^2,$$

where the factor $\delta > 0$ is arbitrary, thus we can fix $c\delta = 1/2$ and conclude with $(3.4)_2$. □

3.2 *Estimating discrepancies*

According to $(3.2)_2$, we construct $\mathbf{u}^j$ and $\mathbf{u}^p$ from the eigenmodes u^j and u^p, which correspond to the eigenvalues $\Lambda_j(h)$ and $\Lambda_p(h)$. We calculate the scalar product (2.14) in the Hilbert space $\mathcal{H}_0$ associated with the resulting problem (2.13). Recalling condition (1.17) and identity (1.11), we have

$$\begin{aligned} h\langle \mathbf{u}^j, \mathbf{u}^p\rangle_0 &= (hA^0D(\nabla)\mathbf{u}^j, D(\nabla)\mathbf{u}^p)_{\Omega^-} + h(\gamma^-\mathbf{u}^j, \mathbf{u}^p)_{\Omega^-} + hq_\Upsilon(\mathbf{u}^j, \mathbf{u}^p) \\ &= (A^-D(\nabla)u^{j-}, D(\nabla)u^{p-})_{\Omega^-} + R_- \\ &\quad + (A^+D(\nabla)u^{j+}, D(\nabla)u^{p+})_{\Omega^+} + R_+ \\ &\quad + h(\gamma^-u^{j-}, u^{p-})_{\Omega^-} + r_- + h(\gamma^+u^{p+}, u^{j-})_{\Omega^+} + r_+ \\ &= (h + \Lambda_j(h))\delta_{j,p} + R_- + R_+ + r_- + r_+. \end{aligned} \tag{3.5}$$

By virtue of (3.3) and (3.4), the discrepancies R_-, R_+ and r_- satisfy the estimates

$$\begin{aligned}
|R_-| &= h|(A^0D(\nabla)\chi^j\widehat{u}^j, D(\nabla)[u^{p-}-\chi^p\widehat{u}^p])_{\Omega^-} + (A^0D(\nabla)u^{j-}, D(\nabla)\chi^p\widehat{u}^p)_{\Omega^-}| \\
&\leqslant ch\{(1+h^{-1}\Lambda_j(h))^{1/2}(h+\Lambda_j(h))[h^{-1/2}\Lambda_p(h)^{1/2} + (1+h^{-1}\Lambda_p(h))^{1/2}(h+\Lambda_p(h))] \\
&\quad + h^{-1/2}\Lambda_j(h)^{1/2}(1+h^{-1}\Lambda_p(h))^{1/2}(h+\Lambda_p(h))\} \\
&\leqslant c(h+\Lambda_j(h))^{1/2}(h+\Lambda_p(h))^{1/2}\{h+\Lambda_j(h)+\Lambda_p(h)+(h+\Lambda_j(h))(h+\Lambda_p(h))\}, \\
|R_+| &= |(A^+D(\nabla)u^{j\perp}, D(\nabla)u^{p\perp})_{\Omega^+}| \leqslant c(h+\Lambda_j(h))(h+\Lambda_p(h)), \\
|r_-| &= h|(\gamma^-\chi^j\widehat{u}^j, u^{p-}-\chi^p\widehat{u}^p)_{\Omega^-} + (\gamma^-u^{j-}, \chi^p\widehat{u}^p)_{\Omega^-}| \\
&\leqslant ch\{(h+\Lambda_j(h))(1+h+\Lambda_p(h)) + h+\Lambda_p(h)\} \leqslant c(h+\Lambda_j(h))(h+\Lambda_p(h)).
\end{aligned} \tag{3.6}$$

Using formulae (3.1), (3.4) and the definition of the form q_Υ given in (2.12), we observe that

$$\begin{aligned}
r_+ &= hq_\Upsilon(u^{j-}-\widehat{u}^j, u^{p-}-\widehat{u}^p) - h(\gamma^+(db^j+u^{j\perp}), db^p+u^{p\perp})_{\Omega^+} \\
&= hq_\Upsilon(db^j, db^p) - h\langle \mathbf{d}(\Omega^+)b^j, b^p\rangle_{\mathbb{R}^6} \\
&\quad - h(\gamma^+u^{j\perp}, db^p+u^{p\perp})_{\Omega^+} - h(\gamma^+db^j, u^{p\perp})_{\Omega^+}, \\
q_\Upsilon(db^j, db^p) &= \langle \mathbf{d}(\Omega^+)b^j, b^p\rangle_{\mathbb{R}^6}, \\
|r_+| &\leqslant ch\{(h+\Lambda_j(h))(1+\Lambda_p(h)^{1/2}+h+\Lambda_p(h)) \\
&\quad + (1+\Lambda_j(h)^{1/2})(h+\Lambda_p(h))\} \leqslant c(h+\Lambda_j(h))(h+\Lambda_p(h)).
\end{aligned} \tag{3.7}$$

Note that in $(3.6)_1$ we have applied inequality (1.45) with $t = 1+h^{-1}\Lambda_r(h)$ and $r = j, p$. However, to get the same bound in $(3.6)_3$ as in $(3.6)_2$, it has been sufficient to use the evident relation $|\chi^j(h,x)| \leqslant 1$. Moreover, we equalize orders of the bounds in (3.6) and (3.7) by imposing the restriction

$$h+\Lambda_j(h) \leqslant \mathbf{c}. \tag{3.8}$$

Finally, we arrive at the estimate

$$\begin{aligned}
&|\langle \mathbf{u}^j, \mathbf{u}^p\rangle_0 - \delta_{j,p}(1+h^{-1}\Lambda_j(h))| \\
&\leqslant ch^{-1}(h+\Lambda_j(h))^{1/2}(h+\Lambda_p(h))^{1/2}(h+\Lambda_j(h)+\Lambda_p(h)).
\end{aligned} \tag{3.9}$$

We denote

$$\mathbf{U}^j = ||\mathbf{u}^j; \mathcal{H}_0||^{-1}\mathbf{u}^j \tag{3.10}$$

and choose the small constant $\mathbf{c} > 0$ such that estimate (3.8) warrants the inequality

$$||\mathbf{u}^j; \mathcal{H}_0||^2 \geqslant 1+h^{-1}\Lambda_j(h) - Ch^{-1}(h+\Lambda_j(h))^2 \geqslant \frac{1}{2}(1+h^{-1}\Lambda_j(h)). \tag{3.11}$$

LEMMA 3.2 If $\Lambda_j(h)$ and $\Lambda_p(h)$ satisfy (3.8), then

$$|\langle \mathbf{U}^j, \mathbf{U}^p\rangle_0 - \delta_{j,p}| \leqslant c(h+\Lambda_j(h)+\Lambda_p(h)). \tag{3.12}$$

Proof. Taking (3.9) and (3.11) into account, we repeat the calculation in (2.29),

$$\begin{aligned}
&|\langle \mathbf{U}^j, \mathbf{U}^p\rangle_0 - \delta_{j,p}| \leqslant ||\mathbf{u}^j; \mathcal{H}_0||^{-1}||\mathbf{u}^p; \mathcal{H}_0||^{-1}\{|\langle \mathbf{u}^j, \mathbf{u}^p\rangle_0 \\
&-\delta_{j,p}(1+h^{-1}\Lambda_j(h))| + \delta_{j,p}|\langle \mathbf{u}^j, \mathbf{u}^j\rangle_0 - (1+h^{-1}\Lambda_j(h))|\} \leqslant c(h+\Lambda_j(h)+\Lambda_p(h)).
\end{aligned}$$

Let us treat the discrepancy in (1.37) for the value m in $(3.2)_1$, the vector function in (3.10), and the operator $\mathcal{K}_0$ appearing in (2.15) and (2.16). Similarly to (2.30), we have

$$\begin{aligned} &||\mathcal{K}_0\mathbf{U}^j - \mathbf{m}_j(h)\mathbf{U}^j; \mathcal{H}_0|| \\ &\quad = \mathbf{m}_j(h)||\mathbf{u}_j; \mathcal{H}_0||^{-1}||\mathbf{u}^j - (1 + h^{-1}\Lambda_j(h))\mathcal{K}_0\mathbf{u}^j; \mathcal{H}_0|| \\ &\quad = \mathbf{m}_j(h)||\mathbf{u}_j; \mathcal{H}_0||^{-1} \sup_{\dots} \big|(A^0 D(\nabla)\mathbf{u}^j, D(\nabla)V)_{\Omega^-} \\ &\qquad +[1 - (1 + h^{-1}\Lambda_j(h))][(\gamma^-\mathbf{u}^j, V)_{\Omega^-} + q_\Upsilon(\mathbf{u}^j, V)]\big| \end{aligned} \tag{3.13}$$

where the dots stand for '$V \in \mathcal{H}_0 : ||V; \mathcal{H}_0|| = 1$'. We extend V on Ω^+ by $d(x)a$ with the column (2.9) (see (2.8)) and we choose $\{V, da\}$ as a test function in the integral identity (1.11),

$$h(A^0 D(\nabla)u^{j-}, D(\nabla)V)_{\Omega^-} = \Lambda_j(h)\{(\gamma^- u^{j-}, V)_{\Omega^-} + (\gamma^+ u^{j+}, da)_{\Omega^+}\}.$$

We point out that

$$||V; H^1(\Omega^-)|| + ||a; \mathbb{R}^6|| \leqslant c||V; \mathcal{H}_0|| = c \tag{3.14}$$

and then we transform the expression $|\dots|$, which is inside the supreme in (3.13), in the same way as for the scalar product (3.5),

$$\begin{aligned} &(A^0 D(\nabla)\mathbf{u}^j, D(\nabla)V)_{\Omega^-} - h^{-1}\Lambda_j(h)[(\gamma^-\mathbf{u}^j, V)_{\Omega^-} + q_\Upsilon(\mathbf{u}^j, V)] \\ &\quad = (A^0 D(\nabla)u^{j-}, D(\nabla)V)_{\Omega^-} + R^- - h^{-1}\Lambda_j(h)(\gamma^- u^{j-}, V)_{\Omega^-} + r^- \\ &\qquad - h^{-1}\Lambda_j(h)(\gamma^+ u^{j+}, da)_{\Omega^+} + r^+ = R^- + r^- + r^+. \end{aligned}$$

On the basis of (3.4), (3.14) and (1.45), we derive the formulae

$$\begin{aligned} |R^-| &= |(A^0 D(\nabla)\chi^j \widehat{u}^j, D(\nabla)V)_{\Omega^-}| \leqslant c(1 + h^{-1}\Lambda_j(h))^{1/2}(h + \Lambda_j(h)), \\ |r^-| &= h^{-1}\Lambda_j(h)|(\gamma^-\chi^j \widehat{u}^j, V)_{\Omega^-}| \leqslant ch^{-1}\Lambda_j(h)(1 + h^{-1}\Lambda_j(h))^{-1/2}(h + \Lambda_j(h)), \\ r^+ &= h^{-1}\Lambda_j(h)\{(\gamma^+ db^j, da)_{\Omega^+} - q_\Upsilon(db^j, da) + (\gamma^+ u^{j\perp}, da)_{\Omega^+}\} = 0, \end{aligned}$$

where we recall the definition of the form q_Υ and the orthogonality property (2.22) of $u^{j\perp}$. These formulae ensure that the expression $\sup |\dots|$ does not exceed $ch(1 + h^{-1}\Lambda_j(h))^{3/2}$.

Finally, the discrepancy norm (3.13), owing to $(3.2)_1$, (3.11) and (3.9), admits the estimate

$$\begin{aligned} &||\mathcal{K}_0\mathbf{U}^j - \mathbf{m}_j(h)\mathbf{U}^j; \mathcal{H}_0|| \\ &\quad \leqslant c(1 + h^{-1}\Lambda_j(h))^{-1}(1 + h^{-1}\Lambda_j(h))^{-1/2}h(1 + h^{-1}\Lambda_j(h))^{3/2} = ch. \end{aligned} \tag{3.15}$$

We have verified all hypotheses of Lemma 1.4. □

PROPOSITION 3.3 There exist constants $\mathbf{c}$ and $C_3 > 0$ such that if $h \in (0, 1]$ and the eigenvalues $\Lambda_j(h)$ of the original problem (1.6) (see (1.16)) fulfil condition (3.8), then, the problem (2.2) (see (2.13)) has an eigenvalue λ_k in the sequence (2.17) which satisfies the estimate

$$|[1 + \lambda_k]^{-1} - [1 + h^{-1}\Lambda_j(h)]^{-1}| \leqslant C_3 h. \tag{3.16}$$

REMARK 3.4 Interchanging the roles of λ_k and $\Lambda_j(h)$ in the calculations in (2.35) and (2.36), we examine estimate (3.16) and find out that, if the bound $\mathbf{c}$ in (3.8) is reduced to meet the restriction $\mathbf{c} \leqslant (2C_3)^{-1}$, the eigenvalues in Proposition 3.3 satisfy the relationship

$$\frac{2}{3}(1 + h^{-1}\Lambda_j(h)) \leqslant 1 + \lambda_k \leqslant 2(1 + h^{-1}\Lambda_j(h)). \tag{3.17}$$

This transforms (3.16) into the estimate of the same type as (2.34),

$$|\lambda_k - h^{-1}\Lambda_j(h)| \leqslant C_3 h(1 + \lambda_k)(1 + h^{-1}\Lambda_j(h)) \leqslant \frac{3}{2}C_3 h(1 + \lambda_k)^2$$

(see the proof of Proposition 2.3).

3.3 *Advantage of the direct reduction*

Let us fix the eigenvalue λ_k given by Proposition 3.3 and denote its multiplicity by $\varkappa_k$ (see (2.19)). We set

$$\rho_k = \frac{1}{2}\frac{d_k}{1 + \lambda_k},\ d_k = \min\left\{\frac{1 + \lambda_k}{1 + \lambda_{k-1}} - 1, 1 - \frac{1 + \lambda_k}{1 + \lambda_{k+\varkappa_k}}\right\} \tag{3.18}$$

and impose the restriction

$$h \leqslant h_3 \varkappa_k^{-1}(1 + d_k^{-1})^{-1}(1 + \lambda_k)^{-1} \tag{3.19}$$

with a small $h_3 > 0$. Let $\Lambda_p(h)$ admit the inclusion

$$\frac{1}{1 + h^{-1}\Lambda_p(h)} \in \left(\frac{1}{1 + \lambda_k} - \rho_k, \frac{1}{1 + \lambda_k} + \rho_k\right). \tag{3.20}$$

We emphasize that in Proposition 3.3 the eigenvalue $\Lambda_j(h)$ satisfies (3.20) because the bound in (3.16) does not exceed ρ_k in the case where $C_3 h_3 \leqslant 1/2$. Moreover, due to (3.18) we have

$$\begin{aligned} \frac{1}{1 + h^{-1}\Lambda_p(h)} &\geqslant \frac{1}{1 + \lambda_k} - \frac{1}{2}\left(\frac{1}{1 + \lambda_k} - \frac{1}{1 + \lambda_{k+\varkappa_k}}\right) > \frac{1}{2}\frac{1}{1 + \lambda_k} \\ &\Rightarrow h + \Lambda_p(h) < 2h(1 + \lambda_k) \leqslant 2h_3. \end{aligned} \tag{3.21}$$

Hence, condition (3.8) is fulfilled by the assumption $h_3 \leqslant \frac{1}{2}\mathbf{c}$ and Proposition 3.3 provides an eigenvalue λ_q such that

$$\left|\frac{1}{1 + h^{-1}\Lambda_p(h)} - \frac{1}{1 + \lambda_q}\right| \leqslant C_3 h \leqslant C_3 h_3 \frac{d_k}{1 + \lambda_k} \leqslant \rho_k. \tag{3.22}$$

By inclusion (3.20) and definition (3.18), we obtain

$$\frac{1}{1 + \lambda_q} \in \left(\frac{1}{1 + \lambda_k} - 2\rho_k, \frac{1}{1 + \lambda_k} + 2\rho_k\right) \subset \left(\frac{1}{1 + \lambda_{k+\varkappa_k}}, \frac{1}{1 + \lambda_{k-1}}\right)$$

and λ_q thus coincides with λ_k.

REMARK 3.5 If $\Gamma = \emptyset$, formula (1.21) and Remark 2.2(1) show that $\Lambda_j(h) = h\lambda_j = 0$ as $j = 1, \dots, 6$ and, hence, we need not deliberate upon the case $k = 1$ in (3.18). For $\text{meas}_2\,\Gamma > 0$, we can insert the 'phantom' eigenvalue $\lambda_0 = 0$ into definition (3.18) so as to preserve all further considerations. □

Let $\Lambda_{K(h)}, \dots, \Lambda_{K(h)+X(h)-1}(h)$ denote all eigenvalues of problem (1.6) which belong in the interval in (3.20). By virtue of estimate (3.16), the second part of Lemma 1.4 gives the coefficients $\beta_p^j(h)$ in the relations

$$||\mathbf{U}^j - \sum_{p=k}^{k+\varkappa_k-1} \beta_p^j(h)\mathcal{U}^{p0}; \mathcal{H}_0|| \leqslant c\rho_k^{-1}h,\ j = K(h), \dots, K(h) + X(h) - 1, \tag{3.23}$$

where $\mathbf{U}^j$ are the approximate solutions (3.10), normalized in $\mathcal{H}_0$, and $\mathcal{U}^{p0}$ are the eigenvectors of problem (2.2), mentioned in Remark 2.2(2). We are in a position to apply Lemma 1.5(1) where

$$y^j = \mathcal{U}^{k+j,0},\ n = \varkappa_k;\ \mathcal{Y}^p = \mathbf{U}^{K(h)+p},\ N = X(h),$$

and, according to (3.23) and (3.12), (3.21),

$$\sigma = c\rho_k^{-1}h, \tau \leqslant c\max\{(h + \Lambda_j(h) + \Lambda_p(h))\} \leqslant 4ch(1+\lambda_k) = Ch(1+\lambda_k)$$

(the maximum is computed over $j, p = K(h), \dots, K(h) + X(h) - 1$). Using restriction (3.19), we have

$$\begin{aligned}
&(1+\min\{n, N\})(\tau + (2+\sigma)\sigma) \\
&\quad \leqslant (1+\varkappa_k)[Ch(1+\lambda_k) + (2 + 2chd_k^{-1}(1+\lambda_k))2chd_k^{-1}(1+\lambda_k)] \\
&\quad \leqslant \varkappa_k^{-1}(1+\varkappa_k)h_3\big[C(1+d_k^{-1})^{-1} + 4c\big(1 + ch_3\varkappa_k^{-1}d_k^{-1}(1+d_k^{-1})^{-1}\big)(1+d_k^{-1})^{-1}d_k^{-1}\big] \\
&\quad \leqslant 2h_3\{C + 4c(1+ch_3)\}.
\end{aligned}$$

Clearly, diminishing $h_3 > 0$ fulfils the condition and, therefore, Lemma 1.5(1) ensures that $\varkappa_k \geqslant X(h)$. Thus, we have proved the following proposition.

PROPOSITION 3.6 There exist constants $h_3 > 0$ and $C_3 > 0$ such that, for any eigenvalue λ_k of multiplicity $\varkappa_k$, condition (3.19) provides at most $\varkappa_k$ eigenvalues $\Lambda_p(h)$ satisfying inclusion (3.20). All of these eigenvalues are subject to estimate (3.16). □

Clearly, restriction (3.19) is much harder than the restriction (2.39) with $l = \varkappa_k$. Hence, under the condition (3.19), Proposition 2.4 holds true and provides the eigenvalues of the operator $h\mathcal{K}_h$ (see (1.13)),

$$[1 + h^{-1}\Lambda_j(h)]^{-1}, \dots, [1 + h^{-1}\Lambda_{j+\varkappa_k-1}(h)]^{-1}, \tag{3.24}$$

which meet estimate (2.41) with $l = \varkappa_k$. Reducing $h_3 > 0$ yields the inequality $C_2\varkappa_k h < \rho_k$. In other words, we have found $\varkappa_k$ eigenvalues in the interval (3.20) while Proposition 3.6 warrants that this interval does not contain any other eigenvalue of the operator $h\mathcal{K}_h$. Furthermore, our reasoning, as after the formula (3.22), allows us to obtain estimate (3.16) for the eigenvalues in (3.24) which, in comparison with (2.41), skips the factor $l = \varkappa_k$.

3.4 *Final conclusions on the eigenvalues*

We proceed with an auxiliary assertion.

LEMMA 3.7 For the eigenvalue λ_k in (2.19), condition (3.19) guarantees that the segment

$$\Upsilon_k = \left[\frac{1}{2}\left(\frac{1}{1+\lambda_k} + \frac{1}{1+\lambda_{k+\varkappa_k}}\right), \frac{1}{2}\left(\frac{1}{1+\lambda_k} + \frac{1}{1+\lambda_{k-1}}\right)\right] \tag{3.25}$$

contains only the eigenvalues $hM_p(h) = (1 + h^{-1}\Lambda_p(h))^{-1}$ satisfying estimate (3.16).

Proof. Let $hM_p(h)$ belong in segment (3.25). Since $(1 + h^{-1}\Lambda_p(h))^{-1} \geqslant (2 + 2\lambda_k)^{-1}$, the calculation in (3.21) confirms condition (3.8) and the validity of Proposition 3.3 which gives an eigenvalue λ_q satisfying (3.22). If q coincides with k, then the proof is completed. In the case $q \geqslant k + \varkappa_k$ (respectively, $q \leqslant k - 1$) we can take $q = k + \varkappa_k$ (respectively, $q = k - 1$) because the corresponding eigenvalue $(1 + \lambda_q)^{-1}$ becomes closer to $hM_p(h)$. For $q = k - 1$, we have

$$\begin{aligned}
\frac{1}{2}\frac{\lambda_k - \lambda_{k-1}}{(1+\lambda_{k-1})(1+\lambda_k)} &= \frac{1}{2}\left(\frac{1}{1+\lambda_{k-1}} - \frac{1}{1+\lambda_k}\right) \leqslant \left|\frac{1}{1+h^{-1}\Lambda_p(h)} - \frac{1}{1+\lambda_{k-1}}\right| \\
&\leqslant C_3 h \leqslant C_3 h_3 \left(1 + \left\{\frac{1+\lambda_k}{1+\lambda_{k-1}} - 1\right\}^{-1}\right)^{-1} \frac{1}{1+\lambda_k} \\
&= C_3 h_3 \left(1 + \frac{1+\lambda_{k-1}}{\lambda_k - \lambda_{k-1}}\right)^{-1} \frac{1}{1+\lambda_k} < C_3 h_3 \frac{\lambda_k - \lambda_{k-1}}{(1+\lambda_{k-1})(1+\lambda_k)}.
\end{aligned}$$

Since $h_3 \leqslant (2C_3)^{-1}$, we obtain a contradiction. A similar contradiction appears in the case where $q = k + \varkappa_k$,

$$\begin{aligned}
\frac{1}{2}\frac{\lambda_{k+\varkappa_k} - \lambda_k}{(1+\lambda_{k+\varkappa_k})(1+\lambda_k)} &= \frac{1}{2}\left(\frac{1}{1+\lambda_k} - \frac{1}{1+\lambda_{k+\varkappa_q}}\right) \leqslant \left|\frac{1}{1+h^{-1}\Lambda_p(h)} - \frac{1}{1+\lambda_k}\right| \\
&\leqslant C_3 h \leqslant C_3 h_3 \left(1 + \left\{1 - \frac{1+\lambda_k}{1+\lambda_{k+\varkappa_k}}\right\}^{-1}\right)^{-1} \frac{1}{1+\lambda_k} \\
&= C_3 h_3 \left(1 + \frac{1+\lambda_{k+\varkappa_k}}{\lambda_{k+\varkappa_k} - \lambda_k}\right)^{-1} \frac{1}{1+\lambda_k} < C_3 h_3 \frac{\lambda_{k+\varkappa_k} - \lambda_k}{(1+\lambda_{k+\varkappa_k})(1+\lambda_k)}.
\end{aligned}$$

□

THEOREM 3.8 There exist constants $h_3 > 0$ and $C_3 > 0$ such that, for any $\varkappa_k$-multiple eigenvalue λ_k (see (2.19)) of the resulting problem (2.2), condition (3.19) ensures that the inclusion $(1 + h^{-1}\Lambda_p(h))^{-1} \in \Upsilon_k$ occurs only for the eigenvalues $\Lambda_k(h), \dots, \Lambda_{k+\varkappa_k-1}(h)$ of the original problem (1.6). Moreover, these eigenvalues are in the relationship (3.16).

Proof. What remains is to check on the coincidence $j = k$ in (3.24). Note that the functions $(0, 1] \ni h \mapsto \Lambda_q(h)$ are continuous (see, e.g. Kato, 1966) and, as has been verified, the segment

$$\left[\frac{1}{1+\lambda_k} + C_3 h, \frac{1}{2}\left\{\frac{1}{1+\lambda_k} + \frac{1}{1+\lambda_{k-1}}\right\}\right]$$

does not contain the eigenvalues $hM_p(h)$. This means that the total multiplicity of the spectrum of the operator $h\mathcal{K}_h$ on the segment[†]

$$\left[\frac{1}{2}\left\{\frac{1}{1+\lambda_k}+\frac{1}{1+\lambda_{k-1}}\right\}, 1\right] \tag{3.26}$$

is constant for $h > 0$ indicated in (3.19). Let

$$\lambda_1 = \lambda_{i^1} < \lambda_{i^2} < \cdots < \lambda_{i^m} = \lambda_{k-1} \tag{3.27}$$

imply eigenvalues in the subset $\{\lambda_1, \dots, \lambda_{k-1}\}$ of sequence (2.17) listed without repeating multiple eigenvalues. Segment (3.26) can be represented as the union $\Upsilon_{i^1} \cup \Upsilon_{i^2} \cup \cdots \cup \Upsilon_{i^m}$ of segments (3.25). We fix a sufficiently small $h > 0$ such that condition (3.19) with $k = i^q$ is fulfilled for each of λ_{i^q} in (3.27). Proposition 3.6, equipped with the result of the last paragraph in Section 3.3, indicates exactly $\varkappa_q$ eigenvalues $hM_p(h) \in \Upsilon_{i^q}$, $q = 1, \dots, m$. We now finalize the proof by observing that, since the eigenvalues in (2.17) repeat according to their multiplicities, the above-mentioned total multiplicity is equal to $\varkappa_{i^1} + \cdots + \varkappa_{i^m} = k - 1$. Therefore, the eigenvalues $\Lambda_p(h)$ in Υ_k are those in the statement of the theorem. □

We modify restriction (3.19) while reformulating the results for the eigenvalues $\Lambda_p(h)$ themselves.

THEOREM 3.9 With the same h_3 and λ_k as in Theorem 3.8, under the condition

$$h \leqslant \min\{\mathbf{c}, h_3\}\varkappa_k^{-1}(1 + d_k^{-1})^{-1}(1 + \lambda_{k+\varkappa_k})^{-1}, \tag{3.28}$$

the segment

$$\left[\frac{h}{2}(\lambda_k + \lambda_{k-1}), \frac{h}{2}(\lambda_k + \lambda_{k+\varkappa_k})\right] \tag{3.29}$$

contains only the eigenvalues $\Lambda_k(h), \dots, \Lambda_{k+\varkappa_k-1}(h)$ of problem (1.6). There holds the estimate

$$|\Lambda_p(h) - h\lambda_k| \leqslant c_3 h^2 (1 + \lambda_k)^2 \tag{3.30}$$

where $p = k, \dots, k + \varkappa_k - 1$ and $c_3 = \frac{3}{2}C_3$.

Proof. For the eigenvalue $\Lambda_p(h)$ in segment (3.29), we have the relation $h + \Lambda_p(h) \leqslant h + h\lambda_{k+\varkappa_k} \leqslant \min\{\mathbf{c}, h_3\}$, which verifies condition (3.8) in Proposition 3.3. Owing to the left-hand side for the inequality in (3.17), estimate (3.16) transforms into (3.30).

To prove the assertion about segment (3.29), we prove that Λ_p in (3.29) implies that $hM_p(h) \equiv (1 + h^{-1}\Lambda_p(h))^{-1}$ is in the interval (3.25), and, then the assertion holds from Theorem 3.8.

Since the inequality

$$\frac{2}{a+b} \leqslant \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

holds for any constants $a, b > 0$, we have

$$\Lambda_p(h) \geqslant \frac{h}{2}(\lambda_k + \lambda_{k-1}) \Rightarrow 1 + h^{-1}\Lambda_p(h) \geqslant \frac{1}{2}\left[(1+\lambda_k) + (1+\lambda_{k-1})\right]$$

[†] Since eigenvalues of problems (1.6) and (2.2) are non-negative, the spectra of the operators $h\mathcal{K}_h$ and $\mathcal{K}_0$ are situated in [0, 1]. For the case in (1.20), the upper endpoint of the segment Υ_1 has to be shifted up to 1 as well (see Remark 3.5).

$$\Rightarrow \left[1 + h^{-1}\Lambda_p(h)\right]^{-1} \leqslant \frac{1}{2}\left\{\frac{1}{1+\lambda_k} + \frac{1}{1+\lambda_{k-1}}\right\}.$$

Then, we assume that $hM_p(h)$ is not in (3.25). Consequently, $hM_p(h)$ belongs to some Υ_q for $q \geqslant k + \varkappa_k$, and we repeat the reasoning in the proof of Lemma 3.7 searching for a contradiction between the formulae

$$\Lambda_p(h) \leqslant \frac{h}{2}(\lambda_k + \lambda_{k+\varkappa_k}) \quad \text{and} \quad \left|[1 + h^{-1}\Lambda_p(h)]^{-1} - [1 + \lambda_{k+\varkappa_k}]^{-1}\right| \leqslant C_3 h.$$

Indeed, appealing to (3.28), we derive the chain of inequalities

$$\begin{aligned}\frac{1}{2}(\lambda_{k+\varkappa_k} - \lambda_k) &\leqslant |h^{-1}\Lambda_p(h) - \lambda_{k+\varkappa_k}| \leqslant C_3 h(1 + \lambda_{k+\varkappa_k})(1 + h^{-1}\Lambda_p(h)) \\ &\leqslant C_3 h_3 (1 + d_k^{-1})^{-1}\frac{1}{2}(2 + \lambda_k + \lambda_{k+\varkappa_q}) < C_3 h_3 \frac{\lambda_{k+\varkappa_k} - \lambda_k}{1 + \lambda_{k+\varkappa_k}}(1 + \lambda_{k+\varkappa_k}) \\ &= C_3 h_3 (\lambda_{k+\varkappa_k} - \lambda_k)\end{aligned}$$

and recall that $C_3 h_3 \leqslant 1/2$. □

Theorem 3.9 reveals the convergence $h^{-1}\Lambda_k(h) \to \lambda_k$ as $h \to +0$, for a fixed k, and, moreover, formula (3.30) outlines the convergence rate. In particular, any segment $[\mu_-, \mu_+]$ with

$$\lambda_{k-1} < \mu_- < \mu_+ < \lambda_k \tag{3.31}$$

becomes free of the points $h^{-1}\Lambda_p(h)$ for a sufficiently small $h > 0$; the next assertion provides a bound for such h.

PROPOSITION 3.10 The segment $[h\mu_-, h\mu_+]$ (see (3.31)) does not contain an eigenvalue of the problem (1.6) as long as

$$h \leqslant h_4 (1 + \rho^{-1})^{-1}(1 + \lambda_k)^{-2} \tag{3.32}$$

where $\rho = \min\{\lambda_k - \mu_+, \mu_- - \lambda_{k-1}\}$ and h_4 is a certain constant which does not depend on the endpoints $\mu_\pm$, nor on the eigenvalue number k.

Proof. Let $\Lambda_p(h)$ belong in the segment. Imposing $h_4 \leqslant \mathbf{c}$ and using (3.32), we verify the condition (3.8),

$$h + \Lambda_p(h) \leqslant h(1 + \lambda_k) \leqslant h_4 \leqslant \mathbf{c}.$$

Hence, Proposition 3.3 supplies us with the eigenvalue λ_q satisfying the left inequality in (3.22), i.e.

$$|\Lambda_p(h) - h\lambda_q| \leqslant C_3 h(1 + \lambda_q)(h + \Lambda_p(h)). \tag{3.33}$$

As in the proof of Lemma 3.7, we consider either $\lambda_q = \lambda_{k-1}$, or $\lambda_q = \lambda_k$. According to (3.32), in both cases the bound in (3.33) does not exceed the expression

$$C_3 h^2 (1 + \lambda_k)^2 \leqslant C_3 h_4 h (1 + \rho^{-1})^{-1} < C_3 h_4 \min\{h\lambda_k - h\mu_+, h\mu_- - h\lambda_{k-1}\}.$$

For $h_4 \leqslant C_3^{-1}$, the distance from the point $h^{-1}\Lambda_p(h)$ to the spectrum in (2.17) is smaller than the distance from the endpoints $\mu_\pm$. Thus, we have found a contradiction denying our assumption $h^{-1}\Lambda_p(h) \in [\mu_-, \mu_+]$. □

3.5 *Asymptotic of eigenmodes*

Under condition (3.19), we study the eigenmodes $u^k(h,\cdot),\dots,u^{k+\varkappa_k-1}(h,\cdot)$ corresponding to the eigenvalues mentioned in Theorem 3.8. By Lemma 3.7, inclusion (2.38) is satisfied for $p=k,\dots,k+\varkappa_k-1$ and $\varepsilon=h^{-1}\rho_k$, since $\varepsilon>C$ holds for h_3 sufficiently small, and inequality (2.37) takes the form

$$\|\mathcal{V}^j-\sum_{p=k}^{k+\varkappa_k-1}\alpha_p^j(h)\mathcal{U}^{ph};\mathcal{H}_h\|\leqslant Ch\rho_k^{-1},\ j=k,\dots,k+\varkappa_k-1. \tag{3.34}$$

We now apply Lemma 1.5(2) where, according to (2.29) and (3.34),

$$y^m=\mathcal{U}^{k+m,h},\ \mathcal{Y}^q=\mathcal{V}^{k+q},\ n=N=\varkappa_k;$$
$$\tau=ch(1+\lambda_k),\ \sigma=Ch\rho_k^{-1}.$$

Since, in view of (3.18), restriction (3.19) leads to

$$\begin{aligned}n(\tau+(2+\sigma)\sigma)&=\varkappa_k(ch(1+\lambda_k)+(2+Ch\rho_k^{-1})Ch\rho_k^{-1})\\&\leqslant h_3(1+d_k^{-1})^{-1}(1+\lambda_k)^{-1}(c(1+\lambda_k)+2C(1+Ch_3)\rho_k^{-1})\leqslant h_3(c+4C(1+Ch_3)),\end{aligned}$$

a proper choice of $h_3>0$ verifies hypothesis (1.42). As a result, we get the unitary $\varkappa_k\times\varkappa_k$-matrix $(\theta_q^m(h))$ such that

$$\begin{aligned}\|\mathcal{U}^{k+q,h}-\sum_{m=0}^{\varkappa_k-1}\theta_m^q(h)\mathcal{V}^{k+m};\mathcal{H}_h\|&\leqslant\varkappa_k(ch(\lambda_k+1)+(3+Ch\rho_k^{-1})Ch\rho_k^{-1})\\&\leqslant ch\varkappa_k(\lambda_k+1+\rho_k^{-1})\leqslant ch\varkappa_k(1+d_k^{-1})(1+\lambda_k).\end{aligned}$$

Considering the eigenmodes u^{k+q} in (1.17) and the approximations in $(2.20)_1$, we recall $(2.36)_1$ and we obtain

$$\begin{aligned}&\left\|u^{k+q}-\sum_{m=0}^{\varkappa_k-1}[h+\Lambda_{k+q}(h)]^{1/2}\right\|v^{k+m};\mathcal{H}_h||^{-1}\theta_m^q(h)v^{k+m};\mathcal{H}_h||\\&\leqslant ch\varkappa_k(1+d_k^{-1})(1+\lambda_k)(h+\Lambda_{k+q}(h))^{1/2}\leqslant ch^{3/2}\varkappa_k(1+d_k^{-1})(1+\lambda_k)^{3/2}.\end{aligned}\tag{3.35}$$

In the sum, we replace the factors $[\dots]^{1/2}||\dots||^{-1}$ by 1. We observe that this does not lead to a change in the order of the bound in (3.35) because the sum stores $\varkappa_k$ terms, and the inequalities $|\theta_p^j(h)|\leqslant 1$ (the matrix $\theta(h)$ is unitary) and

$$\begin{aligned}ch^{1/2}(1+\lambda_k)^{1/2}&\leqslant||v^{k+m};\mathcal{H}_h||\leqslant ch^{1/2}(1+\lambda_k)^{1/2},\\&\left|\frac{[h+\Lambda_{k+q}(h)]^{1/2}}{||v^{k+m};\mathcal{H}_h||}-1\right|\leqslant ch^{-1/2}(1+\lambda_k)^{-1/2}\\&\times\left\{\left|[h+\Lambda_{k+q}(h)]^{1/2}-[h+h\lambda_k]^{1/2}\right|+\left|||v^{k+m};\mathcal{H}_h||-h^{1/2}(1+\lambda_k)^{1/2}\right|\right\}\\&\leqslant ch^{-1}(1+\lambda_k)^{-1}\big\{h^{1/2}(1+\lambda_k)^{1/2}\left|[h+\Lambda_{k+q}]^{-1}-[h+h\lambda_k]^{-1}\right|\\&+|\langle v^{k+m},v^{k+m}\rangle_h-h(1+\lambda_k)|\big\}\leqslant ch(1+\lambda_k)\end{aligned}$$

hold (see (2.25), (2.27) and (3.17), (3.16)). We finally arrive at the inequality

$$\left\| u^{k+q} - \sum_{m=0}^{\varkappa_k - 1} \theta_m^q(h) v^{k+m}; \mathcal{H}_h \right\| \leqslant ch^{3/2} \varkappa_k (1 + d_k^{-1})(1 + \lambda_k)^{3/2} \tag{3.36}$$

where $v^k, \ldots, v^{k+\varkappa_k - 1}$ are the approximate eigenmodes $(2.20)_2$.

PROPOSITION 3.11 Under the conditions of Theorem 3.8, estimate (3.36) is valid as $q = 0, \ldots, \varkappa_k - 1$. □

The asymptotic structures in $(2.20)_2$ were sophisticated in order to work accurately on the eigenvalue approximations. We discard excess terms while formulating the final assertion on eigenmodes of problem (1.6).

THEOREM 3.12 There exists $c > 0$ such that, with the same h_3 as in Theorem 3.8, the restriction (3.19) provides the estimates

$$\begin{aligned}
&\left\| u^{k+q,-} - \sum_{\cdots} \theta_m^q(h) U^{k+m,0}; H^1(\Omega^-) \right\| \leqslant ch\varkappa_k (1 + d_k^{-1})(1 + \lambda_k)^{3/2}, \\
&\left\| u^{k+q,+} - \sum_{\cdots} \theta_m^q(h) d a^{k+m}; L_2(\Omega^+) \right\| \leqslant ch\varkappa_k (1 + d_k^{-1})(1 + \lambda_k)^{3/2}, \\
&\left\| D(\nabla)[\, u^{k+q,+} - h \sum_{\cdots} \theta_m^q(h) U^{k+m,+} \,]; L_2(\Omega^+) \right\| \\
&\qquad \leqslant ch^{3/2} \varkappa_k (1 + d_k^{-1})(1 + \lambda_k)^{3/2}
\end{aligned} \tag{3.37}$$

for the eigenmodes $u^k(h, \cdot), \ldots, u^{k+\varkappa_k - 1}(h, \cdot)$ which correspond to the eigenvalues $\Lambda_k(h), \ldots, \Lambda_{k+\varkappa_k - 1}(h)$ and satisfy the normalization and orthogonality conditions (1.17). In (3.37), the dots denote the summation over $m = 0, \ldots, \varkappa_k - 1$; $U^{k0}, \ldots, U^{k+\varkappa_k - 1,0}$ are eigenmodes of problem (2.2) corresponding to the eigenvalue λ_k of multiplicity $\varkappa_k$ and satisfying conditions (2.18); the column a^{k+m} in the rigid motion $d(x)a^{k+m}$ is determined according to (2.9) with $V = U^{k+m,0}$ and $U^{k+m,+}$ is a solution of problem (2.4) with $\lambda = \lambda_k$, $a^+ = a^{k+m}$ and $U^0 = U^{k+m,0}$; $(\theta_m^q(h))$ stands for a unitary $\varkappa_k \times \varkappa_k$-matrix and d_k for the characteristics of the eigenvalues in (3.18).

Proof. To derive (3.37) from (3.36), it is sufficient to recall definition (1.12) of the scalar product in the Hilbert space $\mathcal{H}_h$ and to observe that, by virtue of (2.23) and (1.45),

$$||\chi^k \widehat{U}^{k+m}; H^1(\Omega^-)|| \leqslant c(1 + \lambda_k)^{1/2} \lambda_k, \; ||U^{k+m,+}; H^1(\Omega^+)|| \leqslant c\lambda_k.$$

We also note that $D(\nabla)d(x) = 0$ due to (1.2) and (1.9). □

4. The middle-frequency range of the spectrum

As regards the middle frequencies, in this section we prove that they concentrate asymptotically on the whole positive real axis and only those associated with eigenvalues asymptotically near the eigenvalues of the Neumann problem (4.1) have elastic energy asymptotically non-null which concentrates in the hard body.

4.1 *The Neumann elasticity problem for the hard body*

Let us consider the spectral boundary value problem in Ω^+

$$\begin{aligned} &L^+(x,\nabla)w(x) = \beta\gamma^+(x)w(x),\ x \in \Omega^+,\\ &N^+(x,\nabla)w(x) = 0,\ x \in \partial\Omega^+, \end{aligned} \tag{4.1}$$

where β denotes the spectral parameter and w the corresponding eigenmode. The differential operators L^+ and N^+ are taken from $(1.6)_{1,2}$ while the Neumann boundary condition $(4.1)_2$ results from both, the boundary condition $(1.6)_{2+}$ and the second transmission condition in $(1.6)_4$. Note that relation (1.5), together with the formal passage to the limit as $h \to +0$, provides condition $(4.1)_2$ from $(1.6)_4$.

The weak formulation of problem (4.1) reads: to find $\beta \in \mathbb{R}$ and a non-trivial field $w \in H^1(\Omega^+)^3$ such that

$$(A^+D(\nabla)w, D(\nabla)v)_{\Omega^+} = \beta(\gamma^+w, v)_{\Omega^+} \quad \forall v \in H^1(\Omega^+)^3. \tag{4.2}$$

Introducing the scalar product in the Hilbert space $\mathcal{H}_+ = H^1(\Omega^+)^3$,

$$\langle w, v\rangle_+ = (A^+D(\nabla)w, D(\nabla)v)_{\Omega^+} + (\gamma^+w, v)_{\Omega^+}, \tag{4.3}$$

and the compact positive symmetric operator $\mathcal{K}_+$ in $\mathcal{H}_+$,

$$\langle \mathcal{K}_+w, v\rangle_+ = (\gamma^+w, v)_{\Omega^+}, \tag{4.4}$$

we transform (4.2) into the abstract equation

$$\mathcal{K}_+w = Bw \in \mathcal{H}_+ \tag{4.5}$$

with the spectral parameter $B = (1+\beta)^{-1}$. Then, the Korn inequality $(1.10)_+$ furnishes the following evident assertion.

LEMMA 4.1 The eigenvalues of the spectral problem (4.2), i.e. problem (4.1), form the sequence

$$0 = \beta_1 = \cdots = \beta_6 < \beta_7 \leqslant \cdots \leqslant \beta_j \leqslant \cdots \to +\infty \tag{4.6}$$

while the corresponding eigenmodes $W^j \in H^1(\Omega^+)^3$ satisfy the normalization and orthogonality conditions

$$(\gamma^+W^j, W^p)_{\Omega^+} = \delta_{j,p}. \tag{4.7}$$

□

Clearly, $B_j = (1+\beta_j)^{-1}$ and $W^{j+} = (1+\beta_j)^{-1/2}W^j$ imply an eigenvalue and the associated eigenvector of the operator $\mathcal{K}_+$ in (4.4) and (4.5). In the sequel, we consider the eigenvalue β_k with the number $k \geqslant 7$ and the multiplicity $\varkappa_k$,

$$\beta_{k-1} < \beta_k = \cdots = \beta_{k+\varkappa_k-1} < \beta_{k+\varkappa_k}. \tag{4.8}$$

4.2 *The inverse reduction*

In this section, we proceed with the construction of an approximation of the eigen-elements of (1.22) (see Section 1.4). Lemma 1.7 gives the extension $\widehat{W}^p \in \mathring{H}^1(\Omega^-;\Gamma)^3$ of the eigenvector $W^p \in H^1(\Omega^+)^3$,

$$||\widehat{W}^p; H^1(\Omega^-)|| \leqslant c_0||W^p; H^1(\Omega^+)|| \leqslant c(1+\beta_p)^{1/2}, \tag{4.9}$$

where the last inequality follows from (4.2) and (1.43). We choose the eigenvector approximations in the form

$$w^p = \{w^{p-}, w^{p+}\} = \{\chi^k \widehat{W}^p, W^p\} \tag{4.10}$$

where $p = k, \ldots, k + \varkappa_k - 1$ and $\chi^k(x) = \chi_t(x)$ at $t = h^{-1/2}(1+\beta_k)$, χ_t is the cut-off function introduced in Lemma 1.8 (see Remark 4.2 where the choice of t is explained). At the same time, the eigenvalue β_k is regarded as an approximation of an eigenvalue of the original problem (1.6). In other words, we work with the middle-frequency range of the spectrum in (1.16) and construct asymptotics of eigenvalues $\Lambda_{\mathcal{N}(h)}(h)$ which remain bounded but removed from zero even for a small h. Since, by Theorem 3.9, the total multiplicity of the spectrum on any fixed interval $(0, T)$ grows indefinitely, the eigenvalue number $\mathcal{N}(h)$ must tend to infinity as $h \to +0$.

Dealing with a bounded eigenvalue $\Lambda_{\mathcal{N}(h)}(h)$ inside the middle-frequency range, we need to modify all the properties of the spectral problem (1.11) on the Hilbert spaces exploited in Sections 1.2 and 2.3. To this end, we consider the Hilbert space $\mathcal{H}_1 = H^1(\Omega;\Gamma)^3$ with the scalar product

$$\langle u, v\rangle_1 = (AD(\nabla)u, D(\nabla)v)_\Omega + (\gamma u, v)_\Omega \tag{4.11}$$

(see (1.12)) and define the compact positive symmetric operator $\mathcal{K}_1 : \mathcal{H}_1 \to \mathcal{H}_1$,

$$\langle \mathcal{K}_1 u, v\rangle_1 = (\gamma u, v)_\Omega \tag{4.12}$$

(see (1.13)). Now, we write problem (4.2) as the abstract equation

$$\mathcal{K}_1 u = B(h)u \in \mathcal{H}_1 \tag{4.13}$$

with the new inverse spectral parameter $B(h) = (1 + \Lambda(h))^{-1}$. Clearly, the set $\{(1 + \Lambda_j(h))^{-1} : j = 1, 2, \ldots\}$ implies the spectrum of (4.13) and the eigenvectors $\mathcal{U}^{j1} = (1 + \Lambda_j(h))^{-1/2}u^j$ satisfy the normalization and orthogonality conditions

$$\langle \mathcal{U}^{j1}, \mathcal{U}^{p1}\rangle_1 = \delta_{j,p}. \tag{4.14}$$

Let us introduce useful properties on almost orthogonality of the set of functions $\{w^j\}_{j=1}^\infty$. Arguing in the same way as in Section 2.3, we proceed with the evaluation of the scalar products

$$\begin{aligned}\langle w^j, w^p\rangle_1 &= (A^+D(\nabla)W^j, D(\nabla)W^p)_{\Omega^+} + (\gamma^+W^j, W^p)_{\Omega^+} + \\ &+ h(A^0D(\nabla)\chi^k\widehat{W}^j, D(\nabla)\chi^k\widehat{W}^p)_{\Omega^-} + (\gamma^-\chi^k\widehat{W}^j, \chi^k\widehat{W}^p)_{\Omega^-} = \\ &= (1+\beta_k)\delta_{j,p} + R_- + r_-,\end{aligned}$$

where $j, p = k, \ldots, k + \varkappa_k - 1$. Using inequalities (4.9) and (1.45), we obtain

$$\begin{aligned}|R_-| &\leqslant ch||\nabla(\chi^k\widehat{W}^j); L_2(\Omega^-)|| \times ||\nabla(\chi^k\widehat{W}^p); L_2(\Omega^-)|| \\ &\leqslant ch[h^{-1/4}(1+\beta_k)^{1/2}]^2(1+\beta_k) = ch^{1/2}(1+\beta_k)^2, \\ |r_-| &\leqslant c||\chi^k\widehat{W}^j; L_2(\Omega^-)|| \times ||\chi^k\widehat{W}^p; L_2(\Omega^-)|| \\ &\leqslant c[h^{1/4}(1+\beta_k)^{-1/2}]^2(1+\beta_k) = ch^{1/2}.\end{aligned} \tag{4.15}$$

Thus, under the restriction

$$h \leqslant h_4(1+\beta_k)^{-2} \tag{4.16}$$

with a small $h_4 > 0$, we arrive at the inequalities

$$\begin{aligned} ||w^j;\mathcal{H}_1|| &\geqslant \frac{1}{2}(1+\beta_k)^{1/2}, \\ |\langle \mathcal{W}^j, \mathcal{W}^p\rangle_1 - \delta_{j,p}| &\leqslant ||w^j;\mathcal{H}_1||^{-1}||w^p;\mathcal{H}_1||^{-1} \\ &\quad\times \Big\{|\langle w^j, w^p\rangle_1 - \delta_{j,p}(1+\beta_k)| + \delta_{j,p}|\langle w^j, w^j\rangle_1 - (1+\beta_k)|\Big\} \\ &\leqslant c(1+\beta_k)^{-1}h^{1/2}(1+\beta_k)^2 \leqslant ch^{1/2}(1+\beta_k), \end{aligned} \tag{4.17}$$

where $\mathcal{W}^j = ||w^j;\mathcal{H}_1||^{-1}w^j$ are normalized in $\mathcal{H}_1$.

The next step is to estimate the discrepancies

$$\begin{aligned} ||\mathcal{K}_1\mathcal{W}^j - (1+\beta_k)^{-1}\mathcal{W}^j;\mathcal{H}_1|| &= (1+\beta_k)^{-1}||w^j;\mathcal{H}_1||^{-1}||w^j - (1+\beta_k)\mathcal{K}_1 w^j;\mathcal{H}_1|| \\ &\leqslant c(1+\beta_k)^{-3/2}\sup_{\cdots}\big|(A^+D(\nabla)W^j, D(\nabla)v^+)_{\Omega^+} - \beta_k(\gamma^+W^j, v^+)_{\Omega^+} \\ &\quad + h(A^0D(\nabla)\chi^k\widehat{W}^j, D(\nabla)v^-)_{\Omega^-} - \beta_k(\gamma^-\chi^k\widehat{W}^j, v^-)_{\Omega^-}\big| \end{aligned}$$

where the dots stand for '$v \in \mathcal{H}_h : ||v;\mathcal{H}_1|| = 1$' and, hence, definition (4.11) ensures that

$$||v^+; H^1(\Omega^+)|| \leqslant c,\ ||v^-; L_2(\Omega^-)|| \leqslant c,\ ||\nabla v^-; L_2(\Omega^-)|| \leqslant ch^{-1/2}.$$

Applying the integral identity (4.2) with $\beta = \beta_k$, $w = W^j$ and the inequalities (see (4.15))

$$\begin{aligned} h|(A^0D(\nabla)\chi^k\widehat{W}^j, D(\nabla)v^-)_{\Omega^-}| &\leqslant ch[h^{-1/2}(1+\beta_k)]^{1/2}(1+\beta_k)^{1/2}h^{-1/2} = ch^{1/4}(1+\beta_k), \\ \beta_k|(\gamma^+\chi^k\widehat{W}^j, v^-)_{\Omega^-}| &\leqslant c\beta_k[h^{-1/2}(1+\beta_k)]^{-1/2}(1+\beta_k) \leqslant ch^{1/4}(1+\beta_k), \end{aligned} \tag{4.18}$$

we obtain that

$$||\mathcal{K}_1\mathcal{W}^j - (1+\beta_k)^{-1}\mathcal{W}^j;\mathcal{H}_1|| \leqslant ch^{1/4}(1+\beta_k)^{-1/2}. \tag{4.19}$$

REMARK 4.2 We observe that our choice $t = h^{-1/2}(1+\beta_k)$ in the definition of the cut-off function $\chi^k = \chi_t$ equalizes orders of the discrepancies in (4.18) and reduces them to the lowest possible level.

THEOREM 4.3 There exist constants $h_4 > 0$ and $c_4 > 0$ such that, for any eigenvalue β_k of multiplicity $\varkappa_k$ (see (4.6) and (4.8)) and any integer $l \in (0, \varkappa_k]$, the condition

$$h \leqslant h_4 l^{-4}(1+\beta_k)^{-2} \tag{4.20}$$

provides at least l eigenvalues $\Lambda_j(h), \ldots, \Lambda_{j+l-1}(h)$ of problem (1.6) admitting the estimate

$$|\Lambda_j(h) - \beta_k| \leqslant c_4 l h^{1/4}(1+\beta_k)^{3/2}. \tag{4.21}$$

Proof. Since condition (4.20) with $l \geqslant 1$ is not weaker than (4.16), formula (4.19) is valid and Lemma 1.4 furnishes, for $\varepsilon > ch^{\frac{1}{4}}(1+\beta_k)^{-\frac{1}{2}}$, the estimates

$$\|\mathcal{W}^j - \sum_{p=K(h)}^{K(h)+X(h)-1}\alpha_p^j(h)\mathcal{U}^{p1};\mathcal{H}_1\| \leqslant c\varepsilon^{-1}h^{1/4}(1+\beta_k)^{-1/2}, \tag{4.22}$$

where $B_{K(h)}(h), \dots, B_{K(h)+X(h)-1}(h)$ imply all eigenvalues of the operator $\mathcal{K}_1$ in (4.13) that satisfy the inclusion

$$B_p(h) \in ([1+\beta_k]^{-1} - \varepsilon, [1+\beta_k]^{-1} + \varepsilon) \tag{4.23}$$

and $\mathcal{U}^{p1}$ are the corresponding eigenvectors subject to (4.14). We take $\varepsilon = \varepsilon_4 l h^{1/4}(1+\beta_k)^{-1/2}$ and apply Lemma 1.5(1) where

$$y^j = \mathcal{U}^{K(h)+j,1},\ n = X(h);\ \mathcal{Y}^p = \mathcal{W}^{k+p},\ N = l \leqslant x_4;$$
$$\tau = ch^{1/2}(1+\beta_k),\ \sigma = c\varepsilon^{-1}h^{1/4}(1+\beta_k)^{-1/2} = c\varepsilon_4^{-1}l^{-1}$$

(see (1.40) to compare with $(4.17)_2$, (4.21)). Now

$$\begin{aligned}(1+\min\{n,N\})(\tau+(2+\sigma)\sigma) &\leqslant (1+l)(ch^{1/2}(1+\beta_k) + (2+c\varepsilon_4^{-1}l^{-1})c\varepsilon_4^{-1}l^{-1}) \\ &\leqslant (1+l)l^{-1}(ch_4^{1/2} + (2+c\varepsilon_4^{-1})c\varepsilon_4^{-1}).\end{aligned}$$

Choosing a small $h_4 > 0$ and a large $\varepsilon_4 > 0$, we verify hypothesis (1.41) of Lemma 1.5(1) which ascertains the relation $X(h) \geqslant l$. In other words, we get l eigenvalues $B_p(h) = (1+\Lambda_p(h))^{-1}$ in (4.23). By our choice of ε, inclusion (4.23) turns into the inequality

$$|\Lambda_p(h) - \beta_k| \leqslant \varepsilon_4 l h^{1/4}(1+\beta_k)^{1/2}(1+\Lambda_p(h)). \tag{4.24}$$

Furthermore, we reduce the bound h_4 in (4.20) so as to obtain $\varepsilon_4 h_4^{1/4} \leqslant 1/2$; we have

$$\Lambda_p(h) \leqslant \beta_k + \varepsilon_4 h_4^{1/4} + \varepsilon_4 h_4^{1/4}\Lambda_p(h) \Rightarrow 1+\Lambda_p(h) \leqslant 1 + 2(\beta_k + 1/2) = 2(1+\beta_k). \tag{4.25}$$

We derive estimate (4.21) with $c_4 = 2\varepsilon_4$, from (4.24), by arguing similarly to (2.36), and, therefore, the theorem is proved. □

4.3 *Discussion*

Theorem 4.3 establishes the inequality $\mathcal{X}_k(h) \geqslant \varkappa_k$ where $\mathcal{X}_k(h)$ is the total multiplicity of spectrum (1.16) in the interval

$$(\beta_k - c_4\varkappa_k h^{1/4}[1+\beta_k]^{3/2}, \beta_k + c_4\varkappa_k h^{1/4}[1+\beta_k]^{3/2}). \tag{4.26}$$

For the eigenvalue $\beta_1 = \cdots = \beta_6 = 0$, this result on the multiplicity has no interest because, as has been proven in Theorems 3.8 and 3.9, each eigenvalue $\Lambda_j(h)$ vanishes asymptotically as $h \to +0$, i.e. the multiplicity $\mathcal{X}_1(h)$ grows indefinitely. Moreover, even in the case $k \geqslant 7$ there exists an infinitesimal sequence $\{h_m^{(k)}\}_{m=1}^\infty$, for which $\mathcal{X}_k(h_m^{(k)}) > \varkappa_k$. Indeed, if $\mathcal{X}_k(h) = \varkappa_k$ as $h \in (0, \mathbf{h}_k]$ with $\mathbf{h}_k > 0$, then, owing to the continuous dependence of $\Lambda_j(h)$ on the parameter h, the number of eigenvalues below the interval (4.26) is fixed—this contradicts the assertion performed after formula (4.10).

For any fixed k, we can also construct the infinitesimal sequences $\{h_m^{k\pm}\}_{m=1}^\infty$ and the continuous functions

$$[h_m^{k-}, h_m^{k+}] \ni h \mapsto \Lambda_{\mathcal{N}_m^k(h)}(h),$$

the ranges of which coincide with the segment $[\beta_{k-1}, \beta_k]$. Thus, for any $\beta_\bullet \in [\beta_{k-1}, \beta_k]$, we find a subsequence $\{h_m^{k\bullet}\}_{m=1}^\infty$ such that $h_m^{k\bullet} \in [h_m^{m-}, h_m^{k+}]$ and $\beta_\bullet$ is an eigenvalue of problem (1.6) with

$h = h_m^{k\bullet}$. In other words, while h decreases to 0, any point on the interval $(0, +\infty)$ becomes an eigenvalue of the problem (1.6) infinitely many times (see Lobo & Pérez, 1997 for an alternative proof using spectral families and Fourier transform).

The above-mentioned facts show that assertions, completed in the same way as Theorems 3.8 and 3.9, cannot hold true on the range of the middle frequencies in (1.16) and, consequently, the direct reduction cannot be performed. Nevertheless, in the next section we specify a certain relation between spectra (4.6) and (1.16) based on the following observation: when the eigenmode approximation (4.10) is 'almost normalized' according to (1.17),

$$\left|\sum_{\pm}(\gamma^{\pm}w^{j\pm}, w^{j\pm})_{\Omega^{\pm}} - 1\right| \leqslant ch^{1/2},$$

the norm

$$||w^{j-}; \mathcal{H}_h^-|| := \left(h||\nabla w^{j-}; L_2(\Omega^-)||^2 + ||w^{j-}; L_2(\Omega^-)||^2\right)^{1/2} \tag{4.27}$$

is bounded by $ch^{1/4}(1 + \beta_j)$ (see (4.15)) and vanishes as $h \to +0$. In other words, the distinguishing feature of approximation (4.10) is nothing more than the concentration of the elastic energy in the hard body Ω^+. In this connection, we emphasize that, for the low-frequency range, the elastic energy of the eigenmode approximation $(2.20)_2$ is distributed on both the units $\Omega^{\pm}$ of the junction Ω (see Lobo & Pérez (1997) for a scalar problem with a Dirichlet condition on $\partial\Omega$).

4.4 *Imitating the direct reduction*

Let us fix the value $\beta_\bullet \in (\beta_{k-1}, \beta_k)$ (see (4.8)) and let $\Lambda_{K(h)}(h), \dots, \Lambda_{K(h)+X(h)-1}(h)$ be eigenvalues of problem (1.6) satisfying the inclusion

$$\frac{1}{1 + \Lambda_j(h)} \in \left(\frac{1}{1+\beta_\bullet} - \rho_\bullet, \frac{1}{1+\beta_\bullet} + \rho_\bullet\right) \tag{4.28}$$

where $\rho_\bullet > 0$ is a certain positive constant to be chosen below. We assume that the linear combination

$$U^\bullet = \sum_{j=K(h)}^{K(h)+X(h)-1} \alpha_j u^j \tag{4.29}$$

fulfils the relations

$$\sum_{j=K(h)}^{K(h)+X(h)-1} \alpha_j^2 = \sum_{\pm}(\gamma^{\pm}U^{\bullet\pm}, U^{\bullet\pm})_{\Omega^{\pm}} = 1, \quad \text{and} \quad ||U^{\bullet -}; \mathcal{H}_h^-|| \leqslant \delta_\bullet(1 + \beta_\bullet) \tag{4.30}$$

with a certain $\delta_\bullet > 0$. Note that the left equality in $(4.30)_1$ follows from the normalization and orthogonality conditions (1.17). We also define

$$d_\bullet = \min\left\{\frac{1+\beta_\bullet}{1+\beta_{k-1}} - 1, 1 - \frac{1+\beta_\bullet}{1+\beta_k}\right\}. \tag{4.31}$$

Our aim in the following theorem is to find out a restriction on $\rho_\bullet$ and $\delta_\bullet$ which ensures the existence of an eigenvalue β_q admitting the inclusion

$$\frac{1}{1+\beta_q} \in \left(\frac{1}{1+\beta_\bullet} - \frac{d_\bullet}{1+\beta_\bullet}, \frac{1}{1+\beta_\bullet} + \frac{d_\bullet}{1+\beta_\bullet}\right). \tag{4.32}$$

Since in view of definition (4.31), the interval in (4.32) is contained in the interval $([1+\beta_k]^{-1}, [1+\beta_{k-1}]^{-1})$, which does not have eigenvalues (4.6). Then, the fact that $(1+\beta_q)^{-1}$ is in the interval (4.32) provides a contradiction.

THEOREM 4.4 There exists $\varepsilon_5 > 0$ such that, for $\beta_\bullet \in (\beta_{k-1}, \beta_k)$ under the condition

$$\delta_\bullet + \rho_\bullet(1+\beta_\bullet)^{1/2} \leqslant \varepsilon_5 \min\{(1+\beta_\bullet)^{-1/2},\ (1+\beta_\bullet)^{-1/2} d_\bullet\}, \tag{4.33}$$

relations $(4.30)_1$ and $(4.30)_2$ cannot be simultaneously valid with the linear combination (4.29) of the eigenmodes which correspond to the eigenvalues $[1+\Lambda_j(h)]^{-1}$ of problem (1.6) in the $\rho_\bullet$-neighbourhood (4.28) of the point $[1+\beta_\bullet]^{-1}$.

Proof. We proceed by contradiction, denying the assertion in the theorem and assuming that (4.28)–(4.30) hold. According to (4.3), (4.11), (1.17), (4.29) and (4.30), we have

$$\begin{aligned}
&\langle U^{\bullet+}, U^{\bullet+}\rangle + = \langle U^\bullet, U^\bullet\rangle_1 + R_-,\\
&|R_-| = |h\langle A^0 D(\nabla)U^{\bullet-}, D(\nabla)U^{\bullet-})_{\Omega^-} + (\gamma^- U^{\bullet-}, U^{\bullet-})_{\Omega^-}| \leqslant c\delta_\bullet^2(1+\beta_\bullet)^2,\\
&|\langle U^\bullet, U^\bullet\rangle_1 - (1+\beta_\bullet)| = |(AD(\nabla)U^\bullet, D(\nabla)U^\bullet)_\Omega - \beta_\bullet(\gamma U^\bullet, U^\bullet)|\\
&\qquad = \left|\sum_{j=K(h)}^{K(h)+X(h)-1} (\Lambda_j(h) - \beta_\bullet)\alpha_j^2\right| \leqslant \max_{K(h)\leqslant q\leqslant K(h)+X(h)-1} |\Lambda_j(h) - \beta_\bullet|.
\end{aligned} \tag{4.34}$$

Then, imposing the restriction

$$\rho_\bullet(1+\beta_\bullet) \leqslant \varepsilon_1 \leqslant \frac{1}{2}, \tag{4.35}$$

we repeat the calculation in (4.25) and from (4.28) we derive that

$$\begin{aligned}
&|\Lambda_q(h) - \beta_\bullet| \leqslant \rho_\bullet(1+\beta_\bullet)(1+\Lambda_q(h)) \Rightarrow 1+\Lambda_q(h) \leqslant 2(1+\beta_\bullet),\\
&|\Lambda_q(h) - \beta_\bullet| \leqslant 2\rho_\bullet(1+\beta_\bullet)^2.
\end{aligned} \tag{4.36}$$

Thus, formulae (4.34) yield

$$|\langle U^{\bullet+}, U^{\bullet+}\rangle_+ - (1+\beta_\bullet)| \leqslant (c\delta_\bullet^2 + 2\rho_\bullet)(1+\beta_\bullet)^2.$$

We observe that, if $\varepsilon_1 > 0$ in (4.35) and $\varepsilon_2 > 0$ in

$$\delta_\bullet^2(1+\beta_\bullet) \leqslant \varepsilon_2 \tag{4.37}$$

are sufficiently small, we obtain the inequality

$$\langle U^{\bullet+}, U^{\bullet+}\rangle_+ \geqslant \frac{1}{2}(1+\beta_\bullet) \tag{4.38}$$

and we can introduce the normalized vector $\mathbf{U}^+ = ||U^{\bullet +}; \mathcal{H}_+||^{-1}U^{\bullet +}$. Note that choosing a suitable ε_5 in (4.33) provides both (4.35) and (4.37).

Now we calculate the discrepancy

$$\begin{aligned}&||\mathcal{K}_+\mathbf{U}^+ - (1+\beta_\bullet)^{-1}\mathbf{U}^+; \mathcal{H}_+|| = (1+\beta_\bullet)^{-1}||U^{\bullet +}; \mathcal{H}_+||^{-1}\\ &\quad \times \sup_{\cdots}\left|(A^+D(\nabla)U^{\bullet +}, D(\nabla)W^+)_{\Omega^+} - \beta_\bullet(\gamma^+U^{\bullet +}, W^+)_{\Omega^+}\right|\end{aligned} \tag{4.39}$$

where the dots stand for '$W^+ \in H^1(\Omega^+)^3 : ||W^+, \mathcal{H}_+|| = 1$'. We decompose the field $W = \{W^-, W^+\} \in \mathring{H}^1(\Omega; \Gamma)^3$ into the series

$$W = \sum_{q=1}^{\infty} a_q u^q \tag{4.40}$$

where $W^- = \chi_\bullet \widehat{W} \in \mathring{H}^1(\Omega^-; \Gamma)^3$, $\widehat{W}$ is the extension of W^+ given by Lemma 1.7 and $\chi_\bullet = \chi_t$ is the cut-off function in Lemma 1.8, $t = 1+\beta_\bullet$. We have the formulae

$$\begin{aligned}&(1+\beta_\bullet)^{1/2}\ ||\ W^-; L_2(\Omega^-)|| + (1+\beta_\bullet)^{-1/2}||\nabla W^-; L_2(\Omega^-)||\\ &\qquad \leqslant c||\widehat{W}; H^1(\Omega^-)|| \leqslant C||W^+; H^1(\Omega^+)||,\\ &\sum_{q=1}^{\infty} a_q^2 = \sum_{\pm}(\gamma^\pm W^\pm, W^\pm)_{\Omega^\pm} \leqslant c_\gamma \sum_{\pm}||W^\pm; L_2(\Omega^\pm)||^2.\end{aligned} \tag{4.41}$$

Owing to (4.29), (4.40), (1.17) and (4.30), (4.41), (4.36), we derive

$$\begin{aligned}&(A^+D(\nabla)U^{\bullet +}, D(\nabla)W^+)_{\Omega^+} - \beta_\bullet(\gamma^+U^{\bullet +}, W^+)_{\Omega^+}\\ &\quad = \sum_{\pm}\left\{(A^\pm D(\nabla)U^{\bullet +}, D(\nabla)W^\pm)_{\Omega^\pm} - \beta_\bullet(\gamma^\pm U^{\bullet \pm}, W^\pm)_{\Omega^\pm}\right\} + r\\ &\quad = \sum_{j=K(h)}^{K(h)+X(h)-1}(\Lambda_j(h) - \beta_\bullet)\alpha_j a_j + r = R + r,\\ |r| &\quad = |h(A^0D(\nabla)U^{\bullet -}, D(\nabla)\chi_\bullet\widehat{W})_{\Omega^-} - \beta_\bullet(\gamma^-U^{\bullet -}, \chi_\bullet\widehat{W})_{\Omega^-}|\\ &\quad \leqslant c\{h^{1/2}\delta_\bullet(1+\beta_\bullet)(1+\beta_\bullet)^{1/2} + \beta_\bullet\delta_\bullet(1+\beta_\bullet)(1+\beta_\bullet)^{-1/2}\}\\ &\quad \times||\widehat{W}; H^1(\Omega^-)|| \leqslant c\delta_\bullet(1+\beta_\bullet)^{3/2},\\ |R| &\quad \leqslant \max_{K(h)\leqslant j\leqslant K(h)+X(h)-1}|\Lambda_j(h) - \beta_\bullet|\left(\sum_{j=K(h)}^{K(h)+X(h)-1}\alpha_j^2\right)^{1/2}\left(\sum_{j=1}^{\infty}a_j^2\right)^{1/2}\\ &\quad \leqslant 2\rho_\bullet(1+\beta_\bullet)^2||W; L_2(\Omega)|| \leqslant c\rho_\bullet(1+\beta_\bullet)^2.\end{aligned}$$

Hence, by applying (4.38), for the discrepancy (4.39) we have the estimate

$$\begin{aligned}&||\mathcal{K}_+\mathbf{U}^+ - (1+\beta_\bullet)^{-1}\mathbf{U}^+; \mathcal{H}_+|| \leqslant c(1+\beta_\bullet)^{-1}(1+\beta_\bullet)^{-1/2}\\ &\qquad \times\{\delta_\bullet(1+\beta_\bullet)^{3/2} + \rho_\bullet(1+\beta_\bullet)^2\} \leqslant c\{\delta_\bullet + \rho_\bullet(1+\beta_\bullet)^{1/2}\}.\end{aligned}$$

Applying Lemma 1.4, we have an eigenvalue β_q such that

$$|(1+\beta_q)^{-1} - (1+\beta_\bullet)^{-1}| \leqslant c\{\delta_\bullet + \rho_\bullet(1+\beta_\bullet)^{1/2}\}.$$

Under restriction (4.33) the last bound is lower than $d_\bullet(1+\beta_\bullet)^{-1}$ and, recalling our comments on the inclusion (4.32), we find the desired contradiction which ends the proof of Theorem 4.4. □

REMARK 4.5 Theorem 4.4 allows us to assert that if $\beta_\bullet$ is different from the eigenvalues of the Neumann spectral problem in Ω^+, (4.1), no combination (4.29) of eigenmodes can have the elastic energy which concentrates asymptotically only in the hard body Ω^+ (see Lobo & Pérez, 1997 for a scalar problem with a Dirichlet condition on $\partial\Omega$). This assertion, together with Theorem 4.3, describes a relation between the spectra in (1.16) and (4.6).

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