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# THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES AND EIGENFUNCTIONS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS*

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**1. Introduction.** Let $T$ be a realization in $L_2(\Omega)$ of a regular elliptic boundary value problem. By "regular" we mean, roughly, that the underlying differential operator is uniformly elliptic and has smooth coefficients on $\Omega$, a bounded open set in $R^n$, and that the boundary conditions determining $T$ are also regular in some sense. Suppose also for simplicity that $T$ is self-adjoint and semibounded from below. Then $T$ has a complete system of eigenfunctions $\{\varphi_j(x)\}$ in $L_2(\Omega)$ with corresponding eigenvalues $\{\lambda_j\}$, which we may assume to be nondecreasing, tending to $+\infty$:

$$\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots, \qquad \lambda_j \to +\infty. \tag{1.1}$$

The proof of the existence of the $\varphi_j$'s and $\lambda_j$'s was itself a major development in mathematical physics and is associated particularly with the names of Poincaré, Hilbert, Weyl, and Gårding. Gårding's definitive paper for the case of elliptic operators of arbitrary order, with Dirichlet boundary conditions, appeared only in 1953. Of course, the result was "known"—on physical grounds and by virtue of simple examples—long before a rigorous proof was devised. From the beginning it seems to have been recognized that the problem of determining the asymptotic behavior, as $j \to \infty$, of the eigenvalues and eigenfunctions was both of great importance and of considerable difficulty.

In this paper, we review progress made up to now in the problem of determining the asymptotic behavior of eigenvalues and eigenfunctions of elliptic operators. The *regular* case (even for nonself-adjoint systems with nonlocal boundary conditions) has been completely solved in papers by Gårding [47], Browder [22], Agmon [5] and others; these papers all utilize variations of one and the same technique of Tauberian theorems due to Carleman [30], [31]. There are two types of problems, however, on which relatively little progress has been made: (a) problems of improving the asymptotic laws by estimating later terms in an asymptotic expansion, and (b) problems associated with singular elliptic operators, such as the Schrödinger operator; even in cases where such operators have discrete spectra, very little is known in general about the eigenvalues.

Assuming now that $T$ has the eigenvalues (1.1), let us denote by $N(\lambda)$ the number of eigenvalues not exceeding $\lambda$:

$$N(\lambda) = \sum_{\lambda_j \leqq \lambda} 1.$$

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The earliest results on the asymptotic form of $N(\lambda)$ were obtained in 1911 by Weyl [111] for the case of the negative Laplacian $-\Delta$ in two dimensions. Using the theory of integral equations, Weyl derived the formula

$$N(\lambda) \sim \frac{\mu_2(\Omega)}{4\pi} \cdot \lambda \quad \text{as} \quad \lambda \to +\infty, \tag{1.2}$$

where $\mu_2(\Omega)$ denotes the area of $\Omega$. In three dimensions, this becomes

$$N(\lambda) \sim \frac{\mu_3(\Omega)}{6\pi^2} \cdot \lambda^{3/2} \quad \text{as} \quad \lambda \to +\infty. \tag{1.3}$$

The problem was taken up by Courant [36]–[39], who derived and extended the formulas of Weyl on the basis of the "minimax" principles for eigenvalues, which are an outgrowth of the classical Rayleigh-Ritz formulas. The methods of Weyl and Courant were not sufficiently powerful, however, to give information about the eigenfunctions.

In 1934, Carleman [30] introduced Tauberian methods reminiscent of analytic number theory into the study of asymptotic distributions. Besides obtaining Weyl's formulas anew, Carleman also derived the asymptotic law for the *spectral function* $S(x, y; \lambda)$ defined by

$$S(x, y; \lambda) = \sum_{\lambda_j \leqq \lambda} \overline{\varphi_j(x)} \varphi_j(y). \tag{1.4}$$

For $-\Delta$ in two dimensions, the formula is

$$S(x, y; \lambda) \sim \frac{\lambda}{4\pi} \cdot \delta_{xy} \quad \text{as} \quad \lambda \to +\infty; \tag{1.5}$$

in the case $x \neq y$ this is to be interpreted as

$$\lambda^{-1} S(x, y; \lambda) \to 0 \quad \text{as} \quad \lambda \to +\infty.$$

Weyl's formula (1.2) follows formally upon integrating (1.5) over $\Omega$.

The details of Carleman's method will be described in §2. Variations of the method have been used in almost all subsequent developments, and these are discussed in the following sections: the method of Carleman in §2, refinements of the asymptotic law in §3, probabilistic methods in §4, and singular problems in §5.

We mention in passing some related questions which we do not intend to discuss here. First, there is the special case of ordinary differential operators—the Sturm-Liouville theory, for example. Here there are special techniques (e.g., comparison theorems), and the literature is large and still growing. Some of the results have obvious applications to partial differential operators having radial symmetry. Second, there is the problem of asymptotics for eigenvalues of integral operators; except for the celebrated papers of Weyl [92]–[97], there seems to be little contact between the problems for integral and for differential operators. Third, there is the problem of determining sufficient conditions for discreteness of the spectrum of various singular operators. Many such conditions are known, but the asymptotic distribution of the eigenvalues is known only in special cases (see §5). Finally, there are problems of convergence and summability of eigenfunction expansions. The asymptotic behavior of the spectral function (see (1.4)) is important in problems of this type. Discussion of some of these topics and their interrelationships can be found in an article of Levitan [71].

The asymptotic formulas have important applications in several branches of physics; for mathematical discussions of these applications we refer the reader to the articles of Kac [61], [62] and Gelfand and Yaglom [50].

**2. The method of Carleman.** We now outline the generalization of Carleman's method given by Browder [22]; a similar treatment is found in Agmon [5].

Let $\Omega$ be a sufficiently regular, bounded region in $R^n$, and let $A$ be an elliptic operator of order $m$ with coefficients defined in $\Omega$:

$$Au = \sum_{|\alpha| \leqq m} a_\alpha(x)\, D^\alpha u; \tag{2.1}$$

by "elliptic" we mean that the characteristic form

$$A_0(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \tag{2.2}$$

does not vanish for $x \in \Omega$, $\xi \in R^n$, $\xi \neq 0$. Here we are using the usual notation: $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index of nonnegative integers;

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n},$$
$$D_i = (\sqrt{-1})^{-1} \partial/\partial x_i,$$
$$\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}.$$

Now let $T$ be a "realization" in $L_2(\Omega)$ of an elliptic boundary value problem associated with $A$; this means that $T$ is a closed linear operator in $L_2(\Omega)$, with

$$C_0^\infty(\Omega) \subset D(T) \subset W^{m,2}(\Omega),$$

where $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in $\Omega$, and $W^{m,2}(\Omega)$ is the Sobolev space of measurable functions $u$ on $\Omega$, such that $D^\alpha u \in L_2(\Omega)$ for all $\alpha$, $0 \leqq |\alpha| \leqq m$.

Let $s$ denote the smallest integer greater than $n/2m$. Assume that the coefficients $a_\alpha \in C^{ms-m}(\Omega)$, and that $T$ possesses the following "coerciveness" or "regularity" property: $Tu \in W^{k,2}(\Omega)$ implies $u \in W^{m+k,2}(\Omega)$ for $0 \leqq k \leqq ms - m$. It is known, for example, that if the boundary conditions determining $T$ are the null Dirichlet conditions, and if the boundary $\partial\Omega$ and the coefficients $a_\alpha$ are sufficiently smooth, then $T$ possesses the required regularity property (cf. Browder [21, Theorem 4]). The case of general boundary conditions has been investigated by Agmon, Douglis and Nirenberg [6].

For simplicity we will assume also that $T$ is self-adjoint. This implies in particular that $A$ is formally self-adjoint, and that $A_0(x, \xi)$ is real. Under these assumptions, it follows from the general theory of elliptic boundary problems that for $t > 0$,

$$(T^{2s} + tI)^{-1}$$

exists and is a compact operator in $L_2(\Omega)$. We are interested in the kernel of this operator.

THEOREM 2.1. *There exists a kernel* $G_t(x, y) \in C_0(\Omega \times \Omega)$ *satisfying*:

(i) $(T^{2s} + tI)^{-1}f(x) = (f, G_t(x, \cdot))_{L_2(\Omega)}, \quad f \in L_2(\Omega)$;
(ii) $G_t(x, \cdot) \in W^{ms,2}(\Omega)$;
(iii) $|G_t(x, y)| \leqq \text{const. } t^{(n/2ms)-1}, \quad x, y \in \Omega, \quad t \geqq 1$.

The proof is immediate if we consider the graph space $H_t = D(T^s)$, which is a Hilbert space relative to the inner product

$$((u, v))_t = (T^s u, T^s v) + t(u, v).$$

For it is easy to see that the map $u \to u(x)$, given $x \in \Omega$, is a continuous linear functional on $H_t$, so that by the Riesz representation theorem there exists $G_{t,x} \in H_t$ such that

$$u(x) = ((u, G_{t,x}))_t, \qquad u \in H_t.$$

This is equivalent to (i), with $G_t(x, y) = G_{t,x}(y)$. The remaining properties of $G_t$ follow readily from its construction—joint continuity in $x$ and $y$ follows from the fact that the imbedding $W^{ms,2}(\Omega) \to C_0(\Omega)$ is compact.

Now if $\{\lambda_j\}$ and $\{\varphi_j\}$ are, respectively, the eigenvalues and orthonormalized eigenfunctions of $T$, then $\{(\lambda_j^{2s} + t)^{-1/2}\varphi_j\}$ is clearly an orthonormal sequence in $H_t$. Expanding $G_{t,x}$ in a Fourier series, we find that

$$G_t(x, y) = \sum_{j=1}^{\infty} \frac{\overline{\varphi_j(x)}\varphi_j(y)}{\lambda_j^{2s} + t}, \tag{2.3}$$

with uniform convergence with respect to $y \in \Omega$ for each $x \in \Omega$.

The next, and most involved, step is to obtain the asymptotic behavior of the "Green's" kernel $G_t(x, y)$ as $t \to +\infty$. In [22] this is accomplished by lengthy but elementary calculations with Fourier transforms; the result is the following.

THEOREM 2.2. *Let* $x, y \in \Omega$. *Then, as* $t \to +\infty$,

$$G_t(x, y) \sim \delta_{xy}(2\pi)^{-n}t^{(n/2ms)-1} \int_{R^n} (A_0(x, \eta)^{2s} + 1)^{-1}\, d\eta. \tag{2.4}$$

If $x \neq y$, this is understood to mean that $t^{1-n/2ms}G_t(x, y) \to 0$.

This gives the asymptotic behavior of the sum in (2.3). We can then derive the asymptotic formula for the spectral function from the following Tauberian theorem of Hardy and Littlewood [56].

LEMMA 2.3. *Let* $\sigma(\lambda)$ *be a real-valued nondecreasing function on* $(0, \infty)$, *and let*

$$s(t) = \int_0^\infty \frac{d\sigma(\lambda)}{\lambda + t}, \qquad t > 0.$$

*Suppose that as* $t \to +\infty$,

$$s(t) \sim ct^{\alpha-1} \quad \text{for some} \quad \alpha, \quad 0 < \alpha < 1.$$

*Then, as* $\lambda \to +\infty$,

$$\sigma(\lambda) \sim \frac{c \sin \pi\alpha}{\pi\alpha} \cdot \lambda^{\alpha}.$$

To apply this lemma we define

$$\sigma(\lambda) = \sum_{\lambda_j^{2s} \leqq \lambda} | \varphi_j(x) |^2;$$

then by (2.3),

$$G_t(x, x) = \sum_1^{\infty} \frac{1}{\lambda_j^{2s} + t} | \varphi_j(x)|^2 = \int_0^{\infty} \frac{d\sigma(\lambda)}{\lambda + t}.$$

It follows from (2.4) and the lemma that

$$\sigma(\lambda) \sim (2\pi)^{-n} \frac{2ms}{n\pi} \sin \frac{n\pi}{2ms} \int_{R^n} (A_0(x, \eta)^{2s} + 1)^{-1} \, d\eta \cdot \lambda^{n/2ms}.$$

Since

$$S(x, x; \lambda) = \sum_{|\lambda_j| \leqq \lambda} | \varphi_j(x) |^2 = \sigma(\lambda^{2s}),$$

we obtain the asymptotic formula ((2.6) below) for $S(x, x; \lambda)$. For $x \neq y$ we consider the function

$$\sigma_1(\lambda) = \sum_{\lambda_j^{2s} \leqq \lambda} | \varphi_j(x) + \theta\varphi_j(y) |^2,$$

where $\theta$ is an arbitrary complex number with $| \theta | = 1$. From the Tauberian theorem and the result already obtained we see that for $x \neq y$,

$$\lambda^{-n/2ms} S(x, y; \lambda) \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Finally, to obtain the asymptotic law for $N(\lambda)$ we first integrate (2.3) over $\Omega$:

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{2s} + t} = \int_{\Omega} G_t(x, x) \, dx. \tag{2.5}$$

But by Theorem 2.1 (iii) we can integrate (2.4) (for $x = y$) over $\Omega$ also, so that we obtain the asymptotic behavior of the sum in (2.5). An application of the Tauberian theorem to the function $\sigma_2(\lambda) = N(\lambda^{1/2s})$ then gives the asymptotic formula for $N(\lambda)$. These formulas are summarized in the following theorem.

THEOREM 2.4. *Let $T$ be a self-adjoint realization in $L_2(\Omega)$ of a regular elliptic boundary value problem, as described above. Let $\{\lambda_j\}$ and $\{\varphi_j\}$ be the eigenvalues and eigenfunctions of $T$. Let $s$ be the smallest integer greater than $n/2m$. Then, as* $\lambda \to \infty$,

$$\begin{aligned} S(x, y; \lambda) &= \sum_{|\lambda_j| \leqq \lambda} \overline{\varphi_j(x)}\varphi_j(y) \\ &\sim \delta_{xy}\, h_{mn}\, \lambda^{n/m} \int_{R^n} (A_0(x, \eta)^{2s} + 1)^{-1} \, d\eta \end{aligned} \tag{2.6}$$

*and*

$$N(\lambda) = \sum_{|\lambda_j| \leq \lambda} 1 \tag{2.7}$$
$$\sim h_{mn}\, \lambda^{n/m} \int_\Omega \int_{R^n} (A_0(x,\eta)^{2s} + 1)^{-1}\, d\eta\, dx,$$

*where*

$$h_{mn} = (2\pi)^{-n} \frac{2ms}{n\pi} \sin \frac{n\pi}{2ms}.$$

The integrals occurring in the above expressions can be written in a different form (cf. [4, p. 258]). Define, for $0 < t < \infty$ and $x \in \Omega$,

$$\nu_x(t) = \mu_n \{\xi \colon | A_0(x, \xi) |^s < t\}, \tag{2.8}$$

where $\mu_n$ denotes Lebesgue measure in $R^n$. Since $A_0(x, \xi)$ is homogeneous in $\xi$ of order $m$, we have $\nu_x(t) = t^{n/2ms}\nu_x(1)$. Therefore,

$$\begin{aligned}\int_{R^n} (A_0(x, \eta)^{2s} + 1)^{-1}\, d\eta &= \int_0^\infty (t^2 + 1)^{-1}\, d\nu_x(t) \\ &= \frac{n}{2ms} \nu_x(1) \int_0^\infty (t^2 + 1)^{-1} t^{(n/2ms)-1}\, dt \\ &= \frac{n\pi}{2ms} \nu_x(1) \frac{1}{\sin (n\pi/2ms)}.\end{aligned}$$

Therefore, we have

$$S(x, y; \lambda) \sim \delta_{xy}\, \lambda^{n/m} \nu_x(1), \tag{2.9}$$

$$N(\lambda) \sim \lambda^{n/m} \int_\Omega \nu_x(1)\, dx. \tag{2.10}$$

For the case of a second order operator, with

$$A_0(x, \xi) = \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j,$$

where $a_{ij}(x)$ is assumed positive definite, these formulas take the form

$$S(x, y; \lambda) \sim \delta_{xy}\, \omega_n\, \lambda^{n/2} (\det(a_{ij}(x)))^{1/2}, \tag{2.11}$$

$$N(\lambda) \sim \omega_n\, \lambda^{n/2} \int_\Omega (\det(a_{ij}(x)))^{1/2}\, dx, \tag{2.12}$$

where

$$\omega_n = [(2\sqrt{\pi})^n \Gamma(1 + n/2)]^{-1}.$$

In particular, if $A = A_0 = -\Delta$, we obtain the $n$-dimensional analogues of the formulas (1.2), (1.3) and (1.5).

Let us list the essential steps in Carleman's method. These same steps are followed in each of the methods described in the sequel:

(i) the Green's kernel $G_t(x, y)$ is constructed for a certain family of operators $\{T_t\}$ associated with $T$;

(ii) the dependence of $G_t(x, y)$ upon $t$ is investigated;

(iii) a Tauberian theorem is invoked to yield asymptotic information about the eigenvalues and eigenfunctions of $T$.

The formulas (2.6) and (2.7) have also been derived (for the case of null Dirichlet boundary conditions) by Gårding [47], who constructed $G_t(x, y)$ by the parametrix method. More general boundary conditions were treated by similar methods by Ehrling [41] and Ton [108]. Nonself-adjoint operators can be treated by using results of Keldysh [64]. The treatment given by Agmon [1]–[5] contains some additional refinements, such as the proof that, in the nonself-adjoint case, the eigenvalues are contained in certain parabolic-shaped regions about the real axis. Agmon also shows that formulas like (2.9) and (2.10) hold separately for the positive and negative eigenvalues in the case of a self-adjoint operator which is not semibounded. Namely, if we set

$$N^+(\lambda) = \sum_{0 \leqq \lambda_j \leqq \lambda} 1$$

and

$$\nu_x^{\;+}(t) = \mu_n\{\xi : 0 < A_0(x, \xi) < t^{1/s}\},$$

we then find that

$$N^+(\lambda) \sim \lambda^{n/m} \int_\Omega \nu_x^{\;+}(1)\, dx.$$

Similar formulas hold for $N^-(\lambda)$, as well as for the spectral functions $S^+(x, y; \lambda)$ and $S^-(x, y; \lambda)$, all of which have obvious definitions. The proof uses a two-sided Tauberian theorem of Pleijel [98].

Beals [13] has shown, by means of a simple perturbation lemma, that the smoothness assumptions made above for the coefficients $a_\alpha(x)$ are unnecessary, at least for the asymptotic formula for eigenvalues. It is sufficient to assume that $a_\alpha(x)$ is bounded and measurable on $\Omega$ for $| \alpha | \leqq m$, and that $a_\alpha(x)$ is continuous on $\Omega$ for $| \alpha | = m$.

We consider next an alternate approach also introduced by Carleman and developed for the case of the Laplacian by Pleijel and Minakshisundaram [76], [78]. General settings are given by Mizohata and Arima [79], [80], and Seeley [101]. The idea is to introduce the following mixed boundary value problem of parabolic type:

$$\begin{aligned} &Tu(x, t) + \frac{\partial}{\partial t} u(x, t) = 0, && x \in \Omega, \quad t > 0, \\ &u \in D(T), && t > 0, \\ &u(\cdot, 0^+) = f \in L_2(\Omega). && \end{aligned} \tag{2.13}$$

This problem has a Green's function $G_t^{\,1}(x, y)$, and the solution is given by

$$u(x, t) = \int_\Omega f(y)\, \overline{G_t^{\,1}(x, y)}\, dy.$$

Let $\{\lambda_j\}$, $\{\varphi_j\}$ again denote the eigenvalues and eigenfunctions of $T$; we now assume that $T$ is semibounded, so that without loss of generality $\lambda_j \geqq 1$ for all $j$. Then $G_t^1$ has the expansion

$$G_t^1(x, y) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\overline{\varphi_j(x)}\varphi_j(y). \tag{2.14}$$

Consider the following function $z_s(x, y)$, which equals $(\Gamma(s))^{-1}$ times the "Mellin" transform of $\overline{G_t^1(x, y)}$:

$$z_s(x, y) = \sum_{j=1}^{\infty} \lambda_j^{-s}\varphi_j(x)\overline{\varphi_j(y)}, \tag{2.15}$$

where $s$ is a complex number. A difficult analysis leads to the result

$$z_s(x, y) = (2\pi)^{-n}\cdot \nu_x(1)\delta_{xy}\frac{n/m}{s - n/m} + g_s(x, y), \tag{2.16}$$

where $g_s(x, y)$ is a holomorphic function of $x$ in the half-plane $\operatorname{Re} s > (n-1)/m$. Setting $x = y$ and integrating over $\Omega$, we obtain a similar result for the generalized "zeta function", $\zeta(s)$, corresponding to $T$:

$$\begin{aligned} \zeta(s) &= \sum_{j=1}^{\infty} \lambda_j^{-s} \\ &= (2\pi)^{-n}\frac{n}{m}\int_\Omega \nu_x(1)\,dx\cdot\frac{1}{s - n/m} + h(s), \end{aligned} \tag{2.17}$$

where $h(s)$ is also holomorphic on $\operatorname{Re} s > (n-1)/m$.

For this approach, the appropriate Tauberian theorem is due to Ikehara [59] (cf. also Wiener [117]).

LEMMA 2.5. *Let $\sigma(t)$ be a nondecreasing function on $t > 1$, and suppose that*

$$s(u) = \int_1^\infty t^{-u}d\sigma(t)$$

*converges for* $\operatorname{Re} u > 1$. *Suppose also that*

$$g(u) = s(u) - \frac{A}{u-1}, \qquad A = \text{const.},$$

*converges to a finite limit as* $\operatorname{Re} u \to 1+$, *uniformly over finite intervals of the line* $\operatorname{Re} u = 1$. *Then,*

$$\lim_{t\to+\infty} t^{-1}\sigma(t) = A.$$

To apply this to (2.16), set

$$\sigma(t) = \sum_{\lambda_j \leqq t^{m/n}} |\varphi_j(x)|^2.$$

Then we obtain

$$s(u) = \sum_{j=1}^{\infty} \lambda_j^{-nu/m} \mid \varphi_j(x)|^2$$

$$= \frac{(2\pi)^{-n}\nu_x(1)}{u-1} + \tilde{g}(u),$$

with $\tilde{g}(u)$ holomorphic on Re $u > 1 - n^{-1}$. Hence the lemma yields (2.9) again:

$$t^{-n/m}S(x, x; t) = (t^{n/m})^{-1}\sigma(t^{n/m}) \rightarrow (2\pi)^{-n}\nu_x(1).$$

The above method clearly has connections with semigroup theory—the Green's function $\overline{G_t^1}$ is the kernel of the semigroup $e^{-tA}$, $t > 0$:

$$e^{-tA}f(x) = \int_\Omega f(y)\, \overline{G_t^1(x, y)}\, dy.$$

The function $\overline{z_s(x, y)}$ on the other hand is the kernel of the operator $A^{-s}$:

$$A^{-s}f(x) = \int_\Omega f(y)\, \overline{z_s(x, y)}\, dy.$$

This connection, and the properties of $A^s$, have been investigated by Seeley [101] (see also Bochner [26], Kotake and Narasimhan [66]).

Instead of the parabolic equation $Tu + \partial u/\partial t = 0$, one can consider the hyperbolic equation $Tu - \partial^2 u/\partial t^2 = 0$, corresponding to a Fourier transform of the spectral function. The second order case has been worked out by Bureau [27] (see also Levitan [68]).

**3. Refinements of the asymptotic law.** In this section we restrict our attention, unless otherwise specified, to the eigenvalue problem for the negative Laplacian $-\Delta$ on a bounded domain $\Omega$ in $R^2$, with either of the boundary conditions

$$u(x) = 0, \qquad x \in \partial\Omega, \tag{3.1}$$

$$\frac{\partial u}{\partial n}(x) = 0, \qquad x \in \partial\Omega. \tag{3.2}$$

In either case (assuming $\partial\Omega$ somewhat smooth), Weyl's formula holds:

$$N(\lambda) = \frac{\mu\lambda}{4\pi} + o(\lambda), \quad \mu = \mu_2(\Omega). \tag{3.3}$$

The following stronger result was conjectured by Weyl:

$$N(\lambda) = \frac{\mu\lambda}{4\pi} \pm \frac{l\sqrt{\lambda}}{4\omega} + o(\sqrt{\lambda}), \tag{3.4}$$

where $l$ = length of $\partial\Omega$, and the $+$ sign is to be taken with the boundary condition (3.2) and the $-$ sign with (3.1).

We can illustrate Weyl's conjecture by invoking a simple connection with the lattice point problem of geometric number theory. Suppose $\Omega$ is a square of side $s$, and the boundary conditions are (3.1). Then the eigenvalues are $\lambda_{nm} = (n^2 + m^2)\pi^2 s^{-2}$, $n, m = 1, 2, 3, \cdots$. Hence, $N(\lambda)$ is the number of integral lattice points inside or on the circle $x^2 + y^2 = s^2\pi^{-2}\lambda$, and lying strictly in the open first quadrant. Now let $M(r)$ denote the number of lattice points inside or on the circle $x^2 + y^2 = r^2$. It is known that

$$M(r) = \pi r^2 + O(r^{2/3})$$

(in fact the exponent 2/3 is too high; the best reported result seems to be 24/37, due to Yin [118]). We conclude that

$$\begin{aligned} N(\lambda) &= \frac{s^2\lambda}{4\pi} - \frac{s\sqrt{\lambda}}{\pi} + O(\lambda^{1/3}) \\ &= \frac{\mu\lambda}{4\pi} - \frac{l\sqrt{\lambda}}{4\pi} + O(\lambda^{1/3}). \end{aligned}$$

The analogous formula for boundary conditions (3.2) also holds. In a recent paper Kuznecov [67] exploits this connection with lattice points to prove Weyl's conjecture (3.4) in all cases of plane domains subject to separation of variables.

For general domains Courant [39] obtained by direct means

$$N(\lambda) = \frac{\mu\lambda}{4\pi} + O(\sqrt{\lambda}\log\lambda). \tag{3.5}$$

Subsequent attempts to improve this result have been based on Tauberian methods. Pleijel [92] has derived various formulas in "pre-Tauberian" form, such as the following (for zero boundary conditions):

$$\sum_{j=1}^{\infty} \exp(-\lambda_j t) = \frac{\mu}{4\pi t} - \frac{l}{8(\sqrt{\pi t})} + o(t^{-1/2}) \tag{3.6}$$

(see also Kac [62]). The investigation of remainder terms in Tauberian theorems is well advanced (cf. Freud [44], Ganelius [45], Marchenko [73]), but (3.6) is not strong enough to yield any new information about $N(\lambda)$. However, Brownell [23], [24] has shown how formulas like (3.6) can lead to asymptotic expansions for $N(\lambda)$ having certain "averaged" error estimates. The terms of such an averaged asymptotic expansion must agree with the terms of a true asymptotic expansion whenever the latter exists.

For *polygonal* domains (with either boundary conditions), Courant's result (3.5) has been improved by Bailey and Brownell [12] and by Fedosov [42] as follows:

$$N(\lambda) = \frac{\mu\lambda}{4\pi} + O(\sqrt{\lambda}) \tag{3.7}$$

(see also [43] for the case of three-dimensional polyhedra). The proof of Bailey and Brownell uses the following stronger form of (3.6):

$$\sum_{1}^{\infty} \exp(-\lambda_j t) = \frac{\mu}{4\pi t} \pm \frac{l}{8\sqrt{\pi t}} + \beta + O\left(\exp\left(-\frac{\delta}{t}\right)\right)$$

for some $\delta > 0$; here $\beta$ is a specific constant depending on $\Omega$. By passing to the limit on the number of sides in a polygon, Kac [62] is led to conjecture that $\beta = 1/6$ for smoothly bounded simply connected domains, and more generally, $\beta = (1 - r)/6$ for smooth domains having $r$ smooth holes. This conjecture has reportedly been proved by McKean and Singer [74].

Recently Grinberg [54] has shown that the Hardy-Littlewood theorem, applied to suitable "pre-Tauberian" formulas, immediately yields an asymptotic formula for the differences $\lambda_k - \mu_k$ of the Dirichlet and Neumann eigenvalues. The formula, in the case of three dimensions, is

$$\sum_{\lambda_k \leqq \lambda} (\lambda_k - \mu_k) \sim \frac{s\lambda^2}{16\pi},$$

where $s$ = area of $\partial\Omega$.

Some negative results related to Weyl's conjecture are also known. Weyl's formula (3.3) is valid for closed Riemann surfaces, as shown by Pleijel and Minakshisundaram [78] (cf. also Seeley [101]). Avakumovic [10] has shown more precisely that

$$N(\lambda) = \frac{\mu\lambda}{4\pi} + O(\sqrt{\lambda}),$$

and also that the remainder $O(\sqrt{\lambda})$ is best possible. Recently, Gromes [55] has shown, by explicit calculation, that the formula (3.4) does not hold for certain regions on a 2-sphere, namely, for the sectors bounded by two meridians meeting at an angle equal to a rational multiple of $\pi$.

Several recent papers have been devoted to obtaining error estimates in the asymptotic formulas for the spectral function $S(x, y; \lambda)$ and its derivatives. The first important developments were published in the theses of Bergendahl [17] and Odhnoff [83]. Further results are due to Nilsson [82] and Peetre [84], [85]; the latter uses the concept of Riesz means rather than Carleman's method. For the case of elliptic operators with $C^\infty$ coefficients, Agmon and Kannai [7] and Hormander [58] have shown that the error term in (2.6) equals

$$O(\lambda^{n/m-(1/2m)+\epsilon})$$

for arbitrary $\epsilon > 0$; the estimate is not shown to be uniform over $\Omega$, however, and therefore does not yield an estimate for $N(\lambda)$. In the case where the leading terms of the given operator have constant coefficients, the exponent $-1/2m$ can be replaced by $-1/m$. Similar results are obtained by Beals [15] for operators with nonsmooth coefficients (see §5).

For the spectral function for $-\Delta$ on $\Omega \subset R^n$, with conditions (3.1), Bureau [27] has obtained

$$S(x, x; \lambda) = \omega_n \lambda^{n/2} + O(\lambda^{(n-1)/2}).$$

For more general second order elliptic operators, Bureau shows that the error in (2.11) is $O(\lambda^{n/2}/\log \lambda)$.

**4. Probabilistic methods.** The equation $\Delta u = \partial u/\partial t$ is subject to treatment also by probabilistic methods based on the Wiener integral. A successful application of these methods to the more general equation

$$\begin{aligned} &\frac{1}{2}\Delta u(x,t) - V(x)u(x,t) = \frac{\partial}{\partial t}u(x,t) \qquad && x \in \Omega, \quad t > 0, \\ (4.1) \qquad &u(x,t) = 0, && x \in \partial\Omega, \quad t > 0, \\ &u(x,0^+) = f(x), && x \in \Omega, \end{aligned}$$

was achieved by Ray [99]. By developing an argument sketched by Kac [61], he derived the asymptotic distribution of eigenvalues and eigenfunctions, including certain singular cases such as the Schrödinger operator, where $\Omega = R^n$ and $V(x) \to +\infty$ as $\|x\| \to \infty$ (cf. §5). The Green's function for this problem can be written as

$$\begin{aligned} K(x, y; t) &= (2\pi t)^{-n/2} \exp\left(-\|x-y\|^2/2t\right) \\ (4.2) \qquad &\cdot E\left\{\exp\left[-\int_0^t V(x + x(\tau))\, d\tau\right] \Big| \; x(t) = y - x\right\}, \end{aligned}$$

where the expression $E\{\cdots\}$ denotes the expected value of

$$\exp\left[-\int_0^t V(x + x(\tau))\, d\tau\right]$$

over all paths $x(\cdot)$ of a Wiener process in $\Omega$, such that $x(0) = 0$ and $x(t) = y - x$.

Following Carleman's recipe, Ray first determines the behavior of $K(x, y; t)$ as $t \to 0^+$:

$$K(x, y; t) = \sum_{j=1}^{\infty} e^{-\lambda_j t/2}\varphi_j(x)\varphi_j(y) \sim \delta_{xy}(2\pi t)^{-n/2}.$$

An application of the following special case of a Tauberian theorem of Karamata [63] then produces the asymptotic formula for $S(x, y; \lambda)$.

LEMMA 4.1. *Let $\sigma(\lambda)$ be nondecreasing for $\lambda > 0$, and suppose that for some $\alpha > 0$,*

$$\int_0^\infty e^{-\lambda t}\, d\sigma(\lambda) \sim ct^{-\alpha} \quad as \quad t \to 0^+.$$

*Then,*

$$\sigma(\lambda) \sim \frac{c\lambda^\alpha}{\Gamma(1+\alpha)} \quad as \quad \lambda \to +\infty.$$

Further applications of this method, as well as relations with physics, are discussed by Kac [61], [62] (see also Gelfand and Yaglom [50]). Relations between the Wiener integral and Schrödinger's equation are given by Babbitt [11] and Nelson [81].

**5. Singular problems.** There are several types of elliptic boundary value problems which, although singular from the point of view of differential equations, are nevertheless regular in an operator-theoretic sense: the corresponding $L_2$ operators have compact resolvents, and hence discrete spectra.

The best known example is the Schrödinger operator in $R^n$:

$$Au = -\Delta u + V(x)u, \qquad x \in R^n. \tag{5.1}$$

If the "potential" function $V(x)$ is real-valued and bounded from below, and if $V(x) \to +\infty$ as $\| x \| \to \infty$, then $A$ determines a unique regular self-adjoint operator $T$ in $L_2(R^n)$. The spectral function for $T$ satisfies the same asymptotic law as in the case of a bounded domain,

$$S(x, y; \lambda) \sim \delta_{xy} \cdot \omega_n \cdot \lambda^{n/2}; \tag{5.2}$$

this can be proved, for example, by Ray's method outlined above. The estimate is no longer uniform in $x$ and $y$, however, and the treatment of $N(\lambda)$ therefore involves extra complications. If $V(x)$ satisfies suitable regularity and growth conditions, it can be shown that

$$N(\lambda) \sim \omega_n \int_{V(x) \leqq \lambda} \{\lambda - V(x)\}^{n/2} \, dx \tag{5.3}$$

(see Ray [99] and Titchmarsh [106]). By setting $V(x) \equiv 0$ on a given domain $\Omega$ and $V(x) \equiv +\infty$ on $R^n - \Omega$, we can consider (5.3) as a formal generalization of Weyl's law.

In the case where $V(x)$ is spherically symmetric, $V(x) = V_1(\| x \|)$, more precise estimates can be obtained by using separation of variables (see Titchmarsh [107], Chaundy and McLeod [32]). Another case of interest, exemplified by the equation for the hydrogen atom, is the case $V(x) \to 0$ as $\| x \| \to \infty$. If $V(x) < 0$ and does not approach zero too rapidly at $\infty$, there will be an infinite sequence of negative eigenvalues approaching zero. (The entire semiaxis $\lambda \geqq 0$ also belongs to the spectrum.) It turns out that the same formula (5.3) holds for the eigenvalues in this case, at least if $V(x)$ is sufficiently close to a spherically symmetric potential (Brownell and Clark [25], McLeod [75]).

The spectral function, even for cases involving essential spectrum, has been investigated by Titchmarsh [106], Levitan [69], Bergendahl [17] and Beals [15] (see also Berezanskii [16]). In the general situation the spectral function is defined by

$$S(x, y; \lambda) = \int_{-\infty}^{\lambda} \sum_{j=1}^{\nu(\lambda)} \overline{\varphi_j(x, \lambda)} \, \varphi_j(y, \lambda) \, d\tau(\lambda)$$

in the usual notation of eigenfunction expansions.

Kostjucenko [65] claims to have obtained the general form which (5.3) takes when $-\Delta$ is replaced by an arbitrary uniformly elliptic operator; no details of the proof are given, however.

Another type of "regular singular" problem was discovered by Rellich [100]. A domain $\Omega$ in $R^n$ is called *quasi-bounded* if

$$\lim_{\|x\| \to \infty, x \in \Omega} \operatorname{dist}(x, \partial\Omega) = 0.$$

Under certain regularity assumptions, the Laplacian operator, with zero boundary conditions, on a quasi-bounded domain $\Omega$, has compact resolvent and discrete spectrum. Let $\rho(x)$ be an arbitrary nonnegative summable function on $\Omega$, and consider the following weighted eigenvalue distribution function (weighted "trace" function):

$$N_\rho(\lambda) = \sum_{\lambda_j \leqq \lambda} \int_\Omega \rho(x)(\varphi_j(x))^2 dx. \tag{5.4}$$

Generalizing the methods of Titchmarsh and Ray, respectively, Hewgill [57] and Clark [33] have shown that

$$N_\rho(\lambda) \sim \omega_n \int_\Omega \rho(x)\, dx \cdot \lambda^{n/2}; \tag{5.5}$$

the spectral function $S(x, y; \lambda)$ still satisfies (5.2). If $\Omega$ has finite measure, (5.5) reduces to Weyl's formula if we put $\rho(x) \equiv 1$ (cf. Glazman and Skacek [52], [102]). In the case where $\mu(\Omega) = \infty$, (5.5) implies that

$$\lambda^{-n/2} N(\lambda) \to +\infty \quad \text{as} \quad \lambda \to +\infty.$$

On the other hand, Hewgill [57] has shown that if $n = 2$ and $\Omega$ satisfies

$$\mu(\Omega \cap [a \leqq \| x \| \leqq a + 1]) = O(a^{-\beta}) \quad \text{as} \quad a \to \infty,$$

then $\lambda^{-2k} N(\lambda)$ is bounded for integers $k > (2\beta)^{-1}$.

Formulas like (5.5) can also be obtained in other cases in which the asymptotic behavior of the spectral function is not uniform. Weighted trace functions of the form (5.4) have been used by Levitan [70] in studying eigenfunction expansions for the Schrödinger operator.

A nonuniformly elliptic operator, acting in a bounded domain $\Omega$, can also lead to a regular operator in $L_2(\Omega)$, provided the breakdown in ellipticity is not too severe. Formulas for $N(\lambda)$ have been derived for second order operators by Solomesc [103]–[105].

The distribution of eigenvalues and eigenfunctions for elliptic operators with nonsmooth coefficients has been thoroughly investigated by Beals [13]–[15]. In [13] and [14] it is shown, by perturbation methods, that the Gårding-Browder formula (2.7) holds under general boundary conditions for elliptic operators (on bounded domains) having uniformly continuous top order coefficients and bounded measurable lower order coefficients. In [15] Beals gives a detailed analysis of the Green's kernel $G_t(x, y)$ under very general conditions encompassing the possibility of unbounded coefficients, unbounded domains, and general nonlocal boundary conditions. Asymptotic formulas are obtained for the spectral function and its derivatives, as well as for the eigenvalues in cases of discrete spectrum. Beals' methods can also be used in certain cases where the coefficients of lower order terms have strong singularities at the boundary (see Clark [35]).

Partial results have been obtained for elliptic boundary value problems in which the eigenvalue parameter occurs only in the boundary conditions. The general theory, as well as estimates for the spectral function, are given by Odhnoff [83]. The asymptotic form of $N(\lambda)$ seems to be known only for disks in $R^2$ (see Eastham [40]). Another type of problem is concerned with equations of the form $Au = \lambda Bu$, where $A$ and $B$ are both elliptic operators. For special cases, the asymptotic distribution problem has been solved by Pleijel [89], [90], [96].

**6. Conclusion.** A glance at the bibliography will convince the reader that our topic is currently the scene of considerable activity. Most of the recent research has been concerned with extending results already known for special operators (especially the Laplacian) to general classes of elliptic operators. These extensions, although far from trivial, could have been predicted in outline on the basis of the work of Gårding, Browder and (later) Agmon. But some of the toughest problems are still quite untouched. Progress on the following problems would be of great interest, at least to the author.

1. A definitive result about the second term in the asymptotic expansion of $N(\lambda)$ for the Laplacian operator in Euclidean regions.

2. A general treatment of Schrödinger-like operators of the form $A + V$, with $A$ uniformly elliptic and $V(x) \to$ const. ($\leqq +\infty$) as $x \to \infty$.

3. A true asymptotic formula for $N(\lambda)$ in the case of quasi-bounded regions.

4. A general study of the effect of "relatively bounded" perturbations upon the asymptotic distribution of eigenvalues.

5. A general analysis of the algebraic structure of the problem.

*Added in proof.* The results of Seeley [101] for elliptic operators defined on $C^\infty$-sections of vector bundles over manifolds have also been derived independently by Greiner [A2], using the parametrix method. Some of the results of McKean and Singer [74] were announced by Berger [A1]. The error estimates of Agmon and Kannai [7] and Hörmander [58] have been extended to operators with Hölder continuous leading coefficients by Beals (unpublished).

## REFERENCES


[1] S. Agmon, *The angular distribution of eigenvalues of nonself-adjoint elliptic boundary value problems of higher order*, Conference on Partial Differential Equations and Continuum Mechanics, University of Wisconsin Press, Madison, Wisconsin, 1961, pp. 9–18.

[2] ———, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math., 15 (1962), pp. 119–147.

[3] ———, *On the asymptotic distribution of eigenvalues of differential problems*, Seminari dell'Instituto Nazionale di Alta Matematica 1962–63, Edizioni Cremonese, Roma, 1964, pp. 556–573.

[4] ———, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, Princeton, New Jersey, 1965.

[5] ———, *On kernels, eigenvalues, and eigenfunctions of operators related to elliptic problems*, Comm. Pure Appl. Math., 18 (1965), pp. 627–663.

[6] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Ibid., 12 (1959), pp. 623–727; *II*, Ibid., 17 (1964), pp. 35–92.

[7] S. Agmon and Y. Kannai, *On the asymptotic behavior of spectral functions and resolvent kernels of elliptic operators*, Israel J. Math., 5 (1967), pp. 1–30.

[8] A. A. Arsenev, *Asymptotic properties of the trace of the spectral function of a self-adjoint elliptic differential operator of second order*, Dokl. Akad. Nauk SSSR, 157 (1964), pp. 761–763.

[9] ———, *On asymptotics of the spectral function of the Schrödinger operator in an unbounded domain*, Uspehi Mat. Nauk, 21 (1966), no. 5 (131), pp. 254–255.

[10] V. G. Avakumovic, *Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten*, Math. Z., 65 (1956), pp. 327–344.

[11] D. Babbitt, *The Wiener integral and the Schrödinger operator*, Trans. Amer. Math. Soc., 116 (1965), pp. 66–78; correction, Ibid., 121 (1966), pp. 549–552.

[12] P. B. Bailey and F. H. Brownell, *Removal of the log factor in the asymptotic estimates of polygonal membrane eigenvalues*, J. Math. Anal. Appl., 4 (1962), pp. 212–239.

[13] R. Beals, *On eigenvalue distributions for elliptic operators without smooth coefficients*, Bull. Amer. Math. Soc., 72 (1966), pp. 701–705.

[14] ———, *Classes of compact operators and eigenvalue distributions for elliptic operators*, Amer. J. Math., to appear.

[15] ———, *Asymptotic behavior of the Green's function and the spectral function of an elliptic operator*, to appear.

[16] Yu. M. Berezanskii, *Eigenfunction Expansions of Self-Adjoint Operators*, Akad. Nauk Ukrainskoi SSR, Naukova Dumka, Kiev, 1965.

[17] G. Bergendahl, *Convergence and summability of eigenfunction expansions connected with elliptic differential operators*, Medd. Lunds Univ. Mat. Sem., 15 (1959), pp. 1–63.

[18] A. Boigelot, *Comportement asymptotique du noyau spectral de $-\Delta$ sous les conditions aux limites de Dirichlet-Neumann dans un ouvert donné*, Bull. Soc. Roy. Sci. Liège, 28 (1959), pp. 175–179.

[19] F. E. Browder, *Le problème des vibrations pour un opérateur aux dérivées partielles self-adjoint et du type elliptique, à coefficients variables*, C. R. Acad. Sci. Paris, 236 (1953), pp. 2140–2142.

[20] ———, *The asymptotic distribution of eigenfunctions and eigenvalues for semi-elliptic differential operators*, Proc. Nat. Acad. Sci. U.S.A., 43 (1957), pp. 270–273.

[21] ———, *On the spectral theory of elliptic differential operators. I*, Math. Ann., 142 (1961), pp. 22–130.

[22] ———, *Asymptotic distribution of eigenvalues and eigenfunctions for non-local elliptic boundary value problems. I*, Amer. J. Math., 87 (1965), pp. 175–195.

[23] F. H. Brownell, *An extension of Weyl's law for eigenvalues*, Pacific J. Math., 5 (1955), pp. 483–489.

[24] ———, *Extended asymptotic eigenvalue distributions for bounded domains in n-space*, J. Math. Mech., 6 (1957), pp. 119–166.

[25] F. H. Brownell and C. W. Clark, *Asymptotic distribution of the eigenvalues of the lower part of the Schrödinger operator spectrum*, Ibid., 10 (1961), pp. 31–70; addendum, Ibid., 10 (1961), pp. 525–528.

[26] S. Bochner, *Zeta functions and Green's functions for linear partial differential operators of elliptic type with constant coefficients*, Ann. of Math., 57 (1953), pp. 32–56.

[27] F. J. Bureau, *Asymptotic representation of the spectral function of self-adjoint elliptic operators of the second order with variable coefficients. I*, J. Math. Anal. Appl., 1 (1960), pp. 423–483; *II*, Ibid., 4 (1962), pp. 181–192.

[28] F. J. Bureau and E. J. Pellicciaro, *Asymptotic behavior of the spectral matrix of the operator of elasticity*, J. Soc. Indust. Appl. Math., 10 (1962), pp. 1–18.

[29] T. Carleman, *Sur les équations intégrales singulières à noyau réel et symétrique*, Uppsala University Årsskrift, 1923.

[30] ———, *Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes*, 8th Scand. Mat. Congress (1934); Collected Works, Malmö, 1960, pp. 471–483.

[31] ———, *Über die asymptotische Verteilung der Eigenwerte partieller Differentialgleichungen*, Ber. der Sächs. Akad. d. Wiss. Leipzig, 88 (1936), pp. 119–132; Collected Works, Malmö, 1960, pp. 483–497.

[32] T. W. Chaundy and J. B. McLeod, *Eigenvalues in problems with spherical symmetry and an associated integral*, Quart. J. Math. Oxford Ser. (2), 14 (1963), pp. 205–212.

[33] C. CLARK, *An asymptotic formula for the eigenvalues of the Laplace operator in an unbounded domain*, Bull. Amer. Math. Soc., 72 (1966), pp. 709–713.

[34] C. CLARK AND D. E. HEWGILL, *One can hear whether a drum has finite area*, Proc. Amer. Math. Soc., 18 (1967), pp. 236–237.

[35] C. CLARK, *The asymptotic distribution of eigenvalues for certain singular elliptic boundary value problems*, to appear.

[36] R. COURANT, *Über die Abhängigkeit der Schwingungszahlen einer Membran von ihrer Begrenzung und über asymptotische Eigenwertverteilung*, Göttinger Nachr., (1919), pp. 255–264.

[37] ———, *Über die Eigenwerte bei den Differentialgleichungen der mathematischen Physik*, Math. Z., 7 (1920), pp. 1–57.

[38] ———, *Über die Schwingungen eingespannter Platten*, Ibid., 15 (1922), pp. 195–200.

[39] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, vol. I, Interscience, New York, 1953.

[40] M. S. P. EASTHAM, *The distribution of the eigenvalues in certain eigenvalue problems*, Quart. J. Math. Oxford Ser. (2), 15 (1964), pp. 251–257.

[41] G. EHRLING, *On a type of eigenvalue problems for certain elliptic differential operators*, Math. Scand., 2 (1954), pp. 267–285.

[42] B. V. FEDOSOV, *Asymptotic formulas for the eigenvalues of the Laplace operator in the case of a polygonal domain*, Dokl. Akad. Nauk SSSR, 151 (1963), pp. 786–789.

[43] ———, *Asymptotic formulas for the eigenvalues of the Laplace operator in the case of a polyhedron*, Ibid., 157 (1964), pp. 536–538.

[44] G. FREUD, *Restglied eines Tauberschen Satzes III*, Acta Math. Acad. Sci. Hungar., 5 (1954), pp. 275–289.

[45] T. GANELIUS, *On the remainder in a Tauberian theorem*, Kungl. Fysiogr. Sällsk. i Lund Förh., 24 (1954), no. 20, 6 pp.

[46] L. GÅRDING, *Dirichlet's problem and the vibration problem for linear elliptic partial differential equations with constant coefficients*, Proc. Symposium on Spectral Theory and Differential Problems, Oklahoma Agricultural and Mechanical College, Stillwater, Oklahoma, 1951, pp. 291–299.

[47] ———, *On the asymptotic distribution of the eigenvalues and eigenfunctions of elliptic differential operators*, Math. Scand., 1 (1953), pp. 237–255.

[48] ———, *On the asymptotic properties of the spectral function belonging to a self-adjoint semi-bounded extension of an elliptic differential operator*, Kungl. Fysiogr. Sällsk. i Lund Förh., 24 (1954), no. 21, 18 pp.

[49] M. G. GASYMOV AND B. M. LEVITAN, *The asymptotic behavior of the spectral function of the Schrödinger operator near a planar part of the boundary*, Izv. Akad. Nauk SSSR Ser. Mat., 28 (1964), pp. 527–552.

[50] I. M. GELFAND AND A. M. YAGLOM, *Integration in function spaces and applications to quantum physics*, J. Mathematical Phys., 1 (1960), pp. 48–69.

[51] I. M. GLAZMAN, *Direct Methods in the Qualitative Spectral Analysis of Singular Differential Operators*, Fizmatgiz, Moscow, 1963.

[52] I. M. GLAZMAN AND B. YA. SKACEK, *On the discrete part of the spectrum of the Laplacian in limit cylindrical domains*, Dokl. Akad. Nauk SSSR, 147 (1962), pp. 760–763.

[53] V. N. GORCAKOV, *Asymptotic properties of the spectral function of hypoelliptic operators*, Ibid., 160 (1965), pp. 746–749.

[54] S. I. GRINBERG, *Some asymptotic formulas for the eigenvalues of the first and third boundary value problems*, Teor. Funkcii Funkcional. Anal. i Prilozen, 3 (1966), pp. 117–134.

[55] D. GROMES, *Über die asymptotische Verteilung der Eigenwerte des Laplace-Operators für Gebeite auf der Kugeloberfläche*, Math. Z., 94 (1966), pp. 110–121.

[56] G. H. HARDY AND J. E. LITTLEWOOD, *Notes on the theory of series XI: On Tauberian theorems*, Proc. London Math. Soc. (2), 30 (1930), pp. 23–37.

[57] D. E. HEWGILL, *On the eigenvalues of the Laplacian operator in an unbounded domain*, Arch. Rational Mech. Anal., to appear.

[58] L. HÖRMANDER, *On the Riesz means of spectral functions and eigenfunction expansions of elliptic operators*, to appear.

[59] S. IKEHARA, *An extension of Landau's theorem in the analytic theory of numbers*, J. Math. and Phys., 10 (1931), pp. 1–12.

[60] D. S. JONES, *The eigenvalues of* $\nabla^2 u + \lambda u = 0$ *when the boundary conditions are given on semi-infinite domains*, Proc. Cambridge Philos. Soc., 49 (1953), pp. 668–684.

[61] M. KAC, *On some connections between probability theory and differential and integral equations*, Proc. 2nd Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, 1951, pp. 189–215.

[62] ———, *Can one hear the shape of a drum?* Amer. Math. Monthly, 73 (1966), pp. 1–23.

[63] J. KARAMATA, *Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stieltjessche Transformation betreffen*, J. Reine Angew. Math., 164 (1931), pp. 27–39.

[64] M. V. KELDYSH, *On the eigenvalues and eigenfunctions of some classes of nonself-adjoint operators*, Dokl. Akad. Nauk SSSR, 77 (1951), pp. 11–14.

[65] A. G. KOSTJUCENKO, *The asymptotic distribution of the eigenvalues of elliptic operators*, Ibid., 158 (1964) pp. 41–44.

[66] T. KOTAKE AND M. S. NARASIMHAN, *Regularity theorems for fractional powers of a linear elliptic operator*, Bull. Soc. Math. France, 90 (1962), pp. 449–471.

[67] N. V. KUZNECOV, *The asymptotic growth of eigenfrequencies of plane membranes subject to separation of variables*, Differencial'nye Uravnenija, 2 (1966), pp. 1385–1402.

[68] B. M. LEVITAN, *On expansion in eigenfunctions of the Laplace operator*, Mat. Sb. (N.S.), 35 (77) (1954), pp. 267–316.

[69] ———, *On the asymptotic behavior of the spectral function and expansion in eigenfunctions of the equation* $\Delta u + (\lambda - q(x))u = 0$, Amer. Math. Soc. Transl. (2), 20 (1962), pp. 1–53.

[70] ——— *On eigenfunction expansions of the Schrödinger equation in the case of an unbounded potential*, Dokl. Akad. Nauk SSSR, 103 (1955), pp. 191–194.

[71] ———, *Certain questions in the spectral theory of self-adjoint differential operators*, Uspehi Mat. Nauk (N.S.), 11 (1956), no. 6 (72), pp. 117–144.

[72] ——— *On the asymptotic behavior of Green's function and its expansion in eigenfunctions of Schrödinger's equation*, Mat. Sb. (N.S.), 41 (83) (1957), pp. 439–458.

[73] V. A. MARCHENKO, *Tauberian theorems and the spectral analysis of differential operators*, Izv. Akad. Nauk SSSR Ser. Mat., 19 (1955), pp. 381–422.

[74] H. P. MCKEAN, JR. AND I. SINGER, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry, to appear.

[75] J. B. MCLEOD, *The distribution of the eigenvalues for the hydrogen atom and similar cases*, Proc. London Math. Soc. (3), 11 (1961), pp. 139–158.

[76] S. MINAKSHISUNDARAM, *A generalization of Epstein zeta functions*, Canad. J. Math., 1 (1949), pp. 320–327.

[77] ———, *Lattice point problems and eigenvalue problems*, Proc. Symposium on Spectral Theory and Differential Problems, Oklahoma Agricultural and Mechanical College, Stillwater, Oklahoma, 1951, pp. 325–332.

[78] S. MINAKSHISUNDARAM AND Å. PLEIJEL, *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds*, Canad. J. Math., 1 (1949), pp. 242–256.

[79] S. MIZOHATA, *Sur les propriétés asymptotiques des valeurs propres pour les opérateurs elliptiques*, J. Math. Kyoto Univ., 4 (1965), pp. 399–428.

[80] S. MIZOHATA AND R. ARIMA, *Propriétés asymptotiques des valeurs propres des opérateurs elliptiques auto-adjoints*, Ibid., 4 (1964), pp. 245–254.

[81] E. NELSON, *Feynman integrals and the Schrödinger equation*, J. Mathematical Phys., 5 (1964), pp. 332–343.

[82] N. NILSSON, *Some estimates for eigenfunction expansions and spectral functions corresponding to elliptic differential operators*, Math. Scand., 9 (1961), pp. 107–121.

[83] J. ODHNOFF, *Operators generated by differential problems with eigenvalue parameter in equation and boundary condition*, Medd. Lunds Univ. Mat. Sem., 14 (1959), pp. 1–79.

[84] J. PEETRE, *Remark on eigenfunction expansions for elliptic operators with constant coefficients*, Math. Scand., 15 (1964), pp. 83–92.

[85] ———, *Some estimates for spectral functions*, Math. Z., 92 (1966), pp. 146–153.

[86] Å. PLEIJEL, *Propriétés asymptotiques des fonctions fondamentales du problème des vibrations dans un corps élastique*, Ark. Mat. Astr. Fys., 26 (1939), no. 19, 9 pp.

[87] ———, *Propriétés asymptotiques des fonctions et valeurs propres de certaines problèmes de vibrations*, Ibid., 27A (1940), no. 13, 101 pp.

[88] ———, *Sur la distribution des valeurs propres de problèmes régis par l'équation* $\Delta u + \lambda k(x, y)u = 0$, Ibid., 29B (1943), no. 7, 8 pp.

[89] ———, *Asymptotic relations for the eigenfunctions of certain boundary problems of polar type*, Amer. J. Math., 70 (1948), pp. 892–907.

[90] ———, *On the eigenvalues and eigenfunctions of elastic plates*, Comm. Pure Appl. Math., 3 (1950), pp. 1–10.

[91] ———, *Green's functions and asymptotic distribution of eigenvalues and eigenfunctions*, Proc. Symposium on Spectral Theory and Differential Problems, Oklahoma Agricultural and Mechanical College, Stillwater, Oklahoma, 1951, pp. 439–454.

[92] ———, *Sur les valeurs et les fonctions propres des membranes vibrantes*, Medd. Lunds Univ. Mat. Sem. Tome Supplémentaire, 1952, pp. 173–180.

[93] ———, *On Green's functions and the eigenvalue distribution of the three-dimensional membrane equation*, Tolfte Skandinaviska Matematikerkongressen, Lund, 1954, pp. 222–240.

[94] ———, *A study of certain Green's functions with applications in the theory of vibrating membranes*, Ark. Mat., 2 (1954), pp. 553–569.

[95] ———, *On the problem of improving Weyl's law for the asymptotic eigenvalue distribution*, Convagno Internazionale sulle Equazioni Lineari alle Derivate Partiali, Trieste, 1954, Edizioni Cremonese, Rome, 1955, pp. 69–75.

[96] ———, *On the eigenfunctions of the membrane equation in a singular case*, Proc. Conference on Differential Equations, University of Maryland, College Park, Maryland, 1956, pp. 197–207.

[97] ———, *Certain indefinite differential eigenvalue problems—the asymptotic distribution of their eigenfunctions*, Conference on Partial Differential Equations and Continuum Mechanics, University of Wisconsin Press, Madison, Wisconsin, 1961, pp. 19–37.

[98] ———, *A bilateral Tauberian theorem*, Ark. Mat., 4 (1963), pp. 561–571; remark, Ibid., pp. 573–574.

[99] D. RAY, *On spectra of second-order differential operators*, Trans. Amer. Math. Soc., 77 (1954), pp. 299–321.

[100] F. RELLICH, *Das Eigenwertproblem von* $\Delta u + \lambda u = 0$ *in Halbröhren*, Essays Presented to R. Courant, Interscience, New York, 1948, pp. 329–344.

[101] R. T. SEELEY, *The powers* $A^s$ *of an elliptic operator* $A$, to appear.

[102] B. YA. SKACEK, *Asymptotic behavior of the negative part of the spectrum of higher order singular operators*, Dopovidi Akad. Nauk Ukrain. RSR, (1964), pp. 14–17.

[103] I. A. SOLOMESC, *Eigenvalues of some degenerate elliptic equations*, Mat. Sb. (N.S.), 54 (96) (1961), pp. 295–310.

[104] ———, *The asymptotic behavior of eigenvalues of bilinear forms associated with certain elliptic equations which degenerate on the boundary*, Dokl. Akad. Nauk SSSR, 144 (1962), pp. 727–729.

[105] ———, *On the asymptotic behavior of eigenvalues of bilinear forms associated with de-*

*generate elliptic operators*, Vestnik Leningrad Univ. Ser. Mat. Meh. Astronom., 19 (1964), no. 1, pp. 163–166.

[106] E. C. TITCHMARSH, *Eigenfunction Expansions*, vols. I and II, Oxford University Press, Oxford, 1953 and 1958.

[107] ———, *On the eigenvalues in problems with spherical symmetry I*, Proc. Roy. Soc. Ser. A, 245 (1958), pp. 147–155; *II*, Ibid., 251 (1959), pp. 46–54; *III*, Ibid., 252 (1959), pp. 436–444.

[108] BUI AN TON, *Asymptotic distribution of eigenvalues and eigenfunctions for general elliptic boundary value problems*, Trans. Amer. Math. Soc., 122 (1966), pp. 516–546.

[109] J. S. DE WET AND F. MANDL, *On the asymptotic distribution of eigenvalues*, Proc. Roy. Soc. Ser. A, 200 (1949–1950), pp. 572–580.

[110] H. WEYL, *Über die asymptotische Verteilung der Eigenwerte*, Göttinger Nachr., (1911), pp. 110–117.

[111] ———, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen*, Math. Ann., 71 (1912), pp. 441–479.

[112] ———, *Über die Abhängigkeit der Eigenschwingungen einer Membran von deren Begrenzung*, J. Reine Angew. Math., 141 (1912), pp. 1–11.

[113] ———, *Über das Spektrum der Hohlraumstrahlung*, Ibid., 141 (1912), pp. 163–181.

[114] ———, *Über die Randwertaufgabe der Strahlungstheorie und asymptotische Spektralgesetze*, Ibid., 143 (1913), pp. 177–202.

[115] ———, *Das asymptotische Verteilungsgesetz der Eigenschwingungen eines beliebig gestalteten elastischen Körpers*, Rend. Circ. Mat. Palermo, 39 (1915), pp. 1–49.

[116] ———, *Ramifications, old and new, of the eigenvalue problem*, Bull. Amer. Math. Soc., 56 (1950), pp. 115–139.

[117] N. WIENER, *Tauberian theorems*, Ann. of Math., 33 (1932), pp. 1–100.

[118] WEN-LIN YIN, *The lattice points in a circle*, Sci. Sinica, 11 (1962), pp. 10–15.

[A1] M. BERGER, *Sur le spectre d'une variété Riemanienne*, C. R. Acad. Sci. Paris, 263 (1966), pp. 13–16.

[A2] P. GREINER, *On zeta functions connected with elliptic differential operators*, to appear.