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Asymptotic Analysis of Linearly Elastic Shells. III. Justification of Koiter's Shell Equations

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Abstract

We consider as in Parts I and II a family of linearly elastic shells of thickness 2ε, all having the same middle surface $S = \boldsymbol{\varphi}(\bar{\omega}) \subset \mathbf{R}^3$, where $\omega \subset \mathbf{R}^2$ is a bounded and connected open set with a Lipschitz-continuous boundary, and $\boldsymbol{\varphi} \in \mathscr{C}^3(\bar{\omega}; \mathbf{R}^3)$. The shells are clamped on a portion of their lateral face, whose middle line is $\boldsymbol{\varphi}(\gamma_0)$, where γ_0 is a portion of $\partial\omega$ with *length* $\gamma_0 > 0$. For all $\varepsilon > 0$, let u_i^ε denote the covariant components of the displacement $u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the points of the shell, obtained by solving the three-dimensional problem; let ζ_i^ε denote the covariant components of the displacement $\zeta_i^\varepsilon \boldsymbol{a}^i$ of the points of the middle surface S, obtained by solving the two-dimensional model of W.T. Koiter, which consists in finding

$$\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon) \in \boldsymbol{V}_K(\omega) = \left\{\boldsymbol{\eta} = (\boldsymbol{\eta}_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\right\}$$

such that

$$\varepsilon \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, dy + \frac{\varepsilon^3}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, dy$$

$$= \int_\omega p^{i,\varepsilon} \eta_i \sqrt{a}\, dy \qquad \text{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_K(\omega),$$

where $a^{\alpha\beta\sigma\tau}$ are the components of the two-dimensional elasticity tensor of $S, \gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are the components of the linearized change of metric and change of curvature tensors of S, and $p^{i,\varepsilon}$ are the components of the resultant of the applied forces.

Under the same assumptions as in Part I, we show that the fields $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_i^\varepsilon\, \boldsymbol{g}^{i,\varepsilon} dx_3^\varepsilon$ and $\zeta_i^\varepsilon \boldsymbol{a}^i$, both defined on the surface S, have the same principal part as $\varepsilon \to 0$, in $\boldsymbol{H}^1(\omega)$ for the tangential components, and in $\boldsymbol{L}^2(\omega)$ for the normal component; under the same assumptions as in Part II, we show that the same fields again have the same principal part as $\varepsilon \to 0$, in $\boldsymbol{H}^1(\omega)$ for

all their components. For "membrane" and "flexural" shells, the two-dimensional model of W.T. KOITER is therefore justified.

Contents

1. The two-dimensional Koiter shell equations

This is the third part of a three-part work, the first (CIARLET & LODS [1996b]) and second (CIARLET, LODS & MIARA [1996]), henceforth simply referred to as "Part I" and "Part II", devoted to membrane and flexural shell equations. We refer to the "Introduction" of Part I for a general introduction to the asymptotic analysis of "thin" elastic structures and for an extensive list of relevant references.

The assumptions on the set ω and on the mapping $\boldsymbol{\varphi}$, the notation, and the geometrical and mechanical description of the shell are the same as in Sec. 1 of Part I. As in Part II, γ_0 denotes a subset of the boundary γ of ω that satisfies length $\gamma_0 > 0$.

The *two-dimensional linear shell equations of Koiter* (cf. KOITER [1970]) read as follows: Let $\zeta_i^\varepsilon : \bar{\omega} \to \mathbf{R}$ denote the three covariant components of the displacement field of the points of the middle surface $S = \boldsymbol{\varphi}(\bar{\omega})$ of the shell; this means that $\zeta_i^\varepsilon(y)\boldsymbol{a}^i(y)$ is the displacement of the point $\boldsymbol{\varphi}(y)$. Then the *unknown* $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon)$ satisfies

(1.1)
$$\boldsymbol{\zeta}^\varepsilon \in \boldsymbol{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

$$(1.2)\qquad \varepsilon \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dy + \frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dy$$
$$= \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}\,dy \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}_K(\omega),$$

where (the functions $a, a^{\alpha\beta}, \Gamma^\sigma_{\alpha\beta}, b^\sigma_\alpha, b^\sigma_\alpha|_\beta, c_{\alpha\beta} \in \mathscr{C}^0(\bar{\omega})$ are defined in Sec. 1 of Part I and in (3.1), (3.2) of Part II; ∂_ν is the outer normal derivative operator along γ):

$$(1.3)\qquad a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{(\lambda + 2\mu)}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),$$

$$(1.4)\qquad \gamma_{\alpha\beta}(\boldsymbol{\eta}) = \tfrac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3,$$

(1.5)
$$\rho_{\alpha\beta}(\boldsymbol{\eta}) = \partial_{\alpha\beta}\eta_3 - \Gamma^{\sigma}_{\alpha\beta}\partial_\sigma\eta_3 + b^{\sigma}_{\beta}(\partial_\alpha\eta_\sigma - \Gamma^{\tau}_{\alpha\sigma}\eta_\tau) + b^{\sigma}_{\alpha}(\partial_\beta\eta_\sigma - \Gamma^{\tau}_{\beta\sigma}\eta_\tau) + b^{\sigma}_{\alpha}|_{\beta}\eta_\sigma - c_{\alpha\beta}\eta_3,$$

(1.6)
$$p^{i,\varepsilon} = \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}dx_3^\varepsilon + (h_+^{i,\varepsilon} + h_-^{i,\varepsilon}).$$

We recall that 2ε is the *thickness* of the shell, $\lambda > 0$ and $\mu > 0$ are the *Lamé constants,* assumed to be *independent* of ε, of its constituting material, the functions $a^{\alpha\beta\sigma\tau}$ are the contravariant components of the *elasticity tensor of the surface* S, the functions $\gamma_{\alpha\beta}(\cdot)$ and $\rho_{\alpha\beta}(\cdot)$ are the covariant components of the *change of metric* and *change of curvature tensors of the surface* S, the functions $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ are the contravariant components of the *applied body force* acting in $\boldsymbol{\Phi}(\Omega^\varepsilon)$, and the functions $h_+^{i,\varepsilon}$, $h_-^{i,\varepsilon} \in L^2(\omega)$ are defined by

$$h_+^{i,\varepsilon}(y) = h^{i,\varepsilon}(y,\varepsilon), \quad h_-^{i,\varepsilon}(y) = h^{i,\varepsilon}(y,-\varepsilon) \quad \text{for } y \in \omega,$$

where $h^{i,\varepsilon} \in L^2(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon)$ are the contravariant components of the *applied surface force* acting on the "upper" and "lower" faces $\boldsymbol{\Phi}(\Gamma_+^\varepsilon)$ and $\boldsymbol{\Phi}(\Gamma_-^\varepsilon)$ of the shell (as in Part II, we assume that there are no surface forces acting on the portion $\boldsymbol{\Phi}(\{\gamma - \gamma_0\} \times [-\varepsilon,\varepsilon])$ of the lateral face of the shell).

The *existence* and *uniqueness* of a solution to the variational problem (1.1), (1.2), which follows from the $\boldsymbol{V}_K(\omega)$-ellipticity of the bilinear form found in (1.2), was first established by Bernadou & Ciarlet [1976]. A different, and more natural, proof was subsequently proposed by Ciarlet & Miara [1992], then combined with the first one in Bernadou, Ciarlet & Miara [1994]. Finally, a more "intrinsic" proof, which applies to the more general situation where the mapping $\boldsymbol{\varphi} : \bar{\omega} \to \mathbf{R}^3$ is only in the space $\mathbf{W}^{2,\infty}(\omega)$, has been more more recently given by Blouza & Le Dret [1994a,b]. (Here the variational problem (1.2) has to be formulated differently, for the functions $b^\sigma_\alpha|_\beta$ are no longer defined in this case.) The *regularity* of the solution when $\gamma_0 = \gamma$ has been established by Alexandrescu [1994]; her proof relies on the theory of elliptic systems due to Agmon, Douglis & Nirenberg [1964].

2. Compared asymptotic behaviors of the solutions of the three-dimensional and Koiter's equations

We recall that the *three-dimensional shell problem* consists in finding $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ such that (cf. Sec. 1 of Part I, notably for the definitions of the functions g^ε, $g^{ij,\varepsilon}, \Gamma^{p,\varepsilon}_{ij}$) :

(2.1)
$$\boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{H}^1(\Omega^\varepsilon); \boldsymbol{v}^\varepsilon = \mathbf{0} \text{ on } \Gamma_0^\varepsilon\},$$

(2.2)
$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}\, dx^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}\, dx^\varepsilon + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} h^{i,\varepsilon} v_i^\varepsilon d\hat{\Gamma}^\varepsilon \quad \text{for all } \boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon),$$

where

$$\Gamma_0^\varepsilon = \gamma_0 \times [-\varepsilon, \varepsilon], x \tag{2.3}$$

$$A^{ijkl,\varepsilon} = \lambda\, g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}), \tag{2.4}$$

$$e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon) = \tfrac{1}{2}(\partial_i^\varepsilon v_j^\varepsilon + \partial_j^\varepsilon v_i^\varepsilon) - \Gamma_{ij}^{p,\varepsilon} v_p^\varepsilon. \tag{2.5}$$

In (2.1)–(2.5), the unknown functions u_i^ε represent the covariant components of the *displacement field* $u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the point of the shell $\boldsymbol{\Phi}(\overline{\Omega}^\varepsilon)$, the functions $A^{ijkl,\varepsilon}$ represent the contravariant components of the *three-dimensional elasticity tensor* (we recall that the Lamé constants λ and μ are assumed to be independent of ε), the functions $e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon)$ are the contravariant components of the *linearized strain tensor,* and $d\hat{\Gamma}^\varepsilon$ is the area element along the boundary of the set $\boldsymbol{\Phi}(\Omega^\varepsilon)$ (cf. Sec. 6 of Part I).

Consider now either a family of "*membrane*" *shells,* or a family of "*flexural*" shells, in the senses understood in Parts I and II. We show that, *in each case, the asymptotic behavior as* $\varepsilon \to 0$ *of the average across the thickness of the displacement found by solving the three-dimensional problem* (2.1), (2,2) *and the displacement found by solving the two-dimensional equations* (1.1), (1.2) *of Koiter are identical.* To this end, we combine the convergence theorems established in Parts I and II for the "three-dimensional displacement" with results of Destuynder [1985] and Sanchez-Palencia [1989a, 1989b] for the "Koiter's two-dimensional displacement". To begin with, we consider *membrane shells*; we recall in this respect that assumption (2.6) has been discussed in Sec. 7 of Part I.

Theorem 2.1. *Assume that* $\gamma_0 = \gamma$, *that there exists a constant c such that*

$$\left\{\sum_\alpha \| \eta_\alpha \|_{1,\omega}^2 + \| \eta_3 \|_{0,\omega}^2\right\}^{1/2} \leqq c\left\{\sum_{\alpha,\beta} \| \gamma_{\alpha\beta}(\boldsymbol{\eta}) \|_{0,\omega}^2\right\}^{1/2} \tag{2.6}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega)$,

where the space $\boldsymbol{V}_M(\omega)$ *is defined by*

$$\begin{aligned} \boldsymbol{V}_M(\omega) &:= \{\boldsymbol{\eta} = (\eta_i); \eta_\alpha \in H_0^1(\omega), \eta_3 \in L^2(\omega)\} \\ &= H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega), \end{aligned} \tag{2.7}$$

and assume that there exist functions $f^i \in L^2(\Omega), h_+^\alpha, h_-^\alpha \in L^2(\omega)$, *and* $h_+^3, h_-^3 \in H^1(\omega)$ *independent of* ε *such that*

$$f^{i,\varepsilon}(x^\varepsilon) = f^i(x) \text{ for all } x \in \Omega \quad \text{and} \quad h_\pm^{i,\varepsilon}(y) = \varepsilon h_\pm^i(y) \text{ for all } y \in \omega. \tag{2.8}$$

Let $\boldsymbol{\zeta} \in \boldsymbol{V}_M(\omega)$ *denote the unique solution of the two-dimensional membrane shell equations:*

$$\int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, dy = \int_\omega \left\{ \int_{-1}^{1} f^i dx_3 + h_+^i + h_-^i \right\} \eta_i \sqrt{a}\, dy \tag{2.9}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega)$. *Finally, let* $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon) \in \boldsymbol{V}_K(\omega)$ *denote the solution of the two-dimensional shell equations* (1.1), (1.2) *of Koiter, and let* $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) \in \boldsymbol{V}(\Omega^\varepsilon)$ *denote the solution of the three-dimensional problem* (2.1), (2.2). *Then*

$$\zeta_\alpha^\varepsilon \boldsymbol{a}^\alpha \to \zeta_\alpha \boldsymbol{a}^\alpha \text{ in } \boldsymbol{H}^1(\omega), \quad \zeta_3^\varepsilon \boldsymbol{a}^3 \to \zeta_3 \boldsymbol{a}^3 \text{ in } \boldsymbol{L}^2(\omega), \tag{2.10}$$

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon\, \boldsymbol{g}^{\alpha,\varepsilon} dx_3^\varepsilon \to \zeta_\alpha \boldsymbol{a}^\alpha \text{ in } \boldsymbol{H}^1(\omega), \quad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \boldsymbol{g}^{3,\varepsilon} dx_3^\varepsilon \to \zeta_3 \boldsymbol{a}^3 \text{ in } \boldsymbol{L}^2(\omega), \tag{2.11}$$

as $\varepsilon \to 0$.

Proof. There exists a constant c_0 such that (cf., e.g., Lemma 2.1 of Bernadou, Ciarlet & Miara [1994]):

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leqq c_0 a^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta} \tag{2.12}$$

for all $y \in \bar{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$. That equations (2.9) have a unique solution $\boldsymbol{\zeta} \in \boldsymbol{V}_M(\omega)$ is then a consequence of (2.12) combined with assumption (2.6) (which holds for "uniformly elliptic surfaces", such as a portion of a sphere or of an ellipsoid; cf. Ciarlet & Sanchez-Palencia [1996] and Ciarlet & Lods [1996a]).

The convergence $\boldsymbol{\zeta}^\varepsilon \to \boldsymbol{\zeta}$ in $H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ was first established by Destuynder [1985, Th. 7.1]; it was also noted by Sanchez-Palencia [1986b, Th. 4.1] (see also Caillerie & Sanchez-Palencia [1995]), who observed that it is a consequence of general results in perturbation theory (as found for instance in Sanchez-Palencia [1980]). We nevertheless give a simple, "self-contained" proof for the sake of completeness. For brevity, let

$$B_M(\boldsymbol{\zeta}, \boldsymbol{\eta}) := \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, dy, \tag{2.13}$$

$$B_F(\boldsymbol{\zeta}, \boldsymbol{\eta}) := \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\zeta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, dy, \tag{2.14}$$

$$L(\boldsymbol{\eta}) := \int_\omega \left\{ \int_{-1}^{1} f^i dx_3 + h_+^i + h_-^i \right\} \eta_i \sqrt{a}\, dy, \tag{2.15}$$

$$\| L \| := \left\{ \sum_i \left\| \int_{-1}^{1} f^i dx_3 + h_+^i + h_-^i \right\|_{0,\omega}^2 \right\}^{1/2}, \tag{2.16}$$

$$\| \boldsymbol{\eta} \|_{\boldsymbol{V}_M(\omega)} := \left\{ \sum_\alpha \| \eta_\alpha \|_{1,\omega}^2 + \| \eta_3 \|_{0,\omega}^2 \right\}^{1/2}. \tag{2.17}$$

Note that the space $\boldsymbol{V}_K(\omega)$, which here is simply $H_0^1(\omega) \times H_0^1(\omega) \times H^2(\omega)$, is contained in the space $\boldsymbol{V}_M(\omega)$.

By virtue of assumptions (2.8), the solution $\boldsymbol{\zeta}^\varepsilon$ of (1.1), (1.2) also satisfies

$$B_M(\boldsymbol{\zeta}^\varepsilon,\boldsymbol{\eta})+\varepsilon^2 B_F(\boldsymbol{\zeta}^\varepsilon,\boldsymbol{\eta})=L(\boldsymbol{\eta})\quad\text{for all }\boldsymbol{\eta}\in\ \boldsymbol{V}_K(\omega).\tag{2.18}$$

Hence letting $\boldsymbol{\eta}=\boldsymbol{\zeta}^\varepsilon$ in (2.18) yields

$$\frac{1}{c^2}\parallel\boldsymbol{\zeta}^\varepsilon\parallel^2_{\boldsymbol{V}_M(\omega)}+\frac{1}{3}\sum_{\alpha,\beta}\parallel\varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon)\parallel^2_{0,\omega}\leqq c_0\parallel L\parallel\ \parallel\boldsymbol{\zeta}^\varepsilon\parallel_{\boldsymbol{V}_M(\omega)},$$

by assumption (2.6) and inequality (2.12). Therefore the family $(\boldsymbol{\zeta}^\varepsilon)_{\varepsilon>0}$ is bounded in $\boldsymbol{V}_M(\omega)$ and the families $(\varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon))_{\varepsilon>0}$ are bounded in $L^2(\omega)$; hence there exists a subsequence, still denoted $(\boldsymbol{\zeta}^\varepsilon)_{\varepsilon>0}$ for convenience, and there exist functions $\tilde{\boldsymbol{\zeta}}\in\boldsymbol{V}_M(\omega)$ and $\rho^{-1}_{\alpha\beta}\in L^2(\omega)$ such that

$$\boldsymbol{\zeta}^\varepsilon\rightharpoonup\tilde{\boldsymbol{\zeta}}\quad\text{in }\boldsymbol{V}_M(\omega),\tag{2.19}$$

$$\varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon)\rightharpoonup\rho^{-1}_{\alpha\beta}\quad\text{in }L^2(\omega).\tag{2.20}$$

(As usual $\rightarrow$ and $\rightharpoonup$ respectively denote strong and weak convergences.)

Fix $\boldsymbol{\eta}\in\boldsymbol{V}_K(\omega)$ in (2.18) and let $\varepsilon\rightarrow 0$; then the weak convergences (2.19), (2.20) yield $B_M(\tilde{\boldsymbol{\zeta}},\boldsymbol{\eta})=L(\boldsymbol{\eta})$. Since the space $\boldsymbol{V}_K(\omega)$ is dense in $\boldsymbol{V}_M(\omega)$, we conclude that $B_M(\tilde{\boldsymbol{\zeta}},\boldsymbol{\eta})=L(\boldsymbol{\eta})$ for all $\boldsymbol{\eta}\in\boldsymbol{V}_M(\omega)$. Hence

$$\tilde{\boldsymbol{\zeta}}=\boldsymbol{\zeta},\tag{2.21}$$

where $\boldsymbol{\zeta}\in\boldsymbol{V}_M(\omega)$ is the unique solution to equations (2.9), and the weak convergence $\boldsymbol{\zeta}^\varepsilon\rightharpoonup\boldsymbol{\zeta}$ in $\boldsymbol{V}_M(\omega)$ holds for the whole family $(\boldsymbol{\zeta}^\varepsilon)_{\varepsilon>0}$.

By virtue of assumption (2.6) combined with inequality (2.12), establishing the strong convergence

$$\boldsymbol{\zeta}^\varepsilon\rightarrow\boldsymbol{\zeta}\quad\text{in }\boldsymbol{V}_M(\omega),\tag{2.22}$$

is equivalent to establishing the convergence

$$B_M(\boldsymbol{\zeta}^\varepsilon-\boldsymbol{\zeta},\boldsymbol{\zeta}^\varepsilon-\boldsymbol{\zeta})\rightarrow 0,$$

which itself follows from the relations

$$0\leqq B_M(\boldsymbol{\zeta}^\varepsilon-\boldsymbol{\zeta},\boldsymbol{\zeta}^\varepsilon-\boldsymbol{\zeta})=B_M(\boldsymbol{\zeta}^\varepsilon,\boldsymbol{\zeta}^\varepsilon)-2B_M(\boldsymbol{\zeta}^\varepsilon,\boldsymbol{\zeta})+B_M(\boldsymbol{\zeta},\boldsymbol{\zeta}),$$

$$B_M(\boldsymbol{\zeta}^\varepsilon,\boldsymbol{\zeta}^\varepsilon)\leqq L(\boldsymbol{\zeta}^\varepsilon)\quad(\text{cf. }(2.18)),$$

$$B_M(\boldsymbol{\zeta}^\varepsilon,\boldsymbol{\zeta})\rightarrow B_M(\boldsymbol{\zeta},\boldsymbol{\zeta}),\quad L(\boldsymbol{\zeta}^\varepsilon)\rightarrow L(\boldsymbol{\zeta})\quad(\text{cf.}(2.19)\text{ and }(2.21)),$$

$$B_M(\boldsymbol{\zeta},\boldsymbol{\zeta})=L(\boldsymbol{\zeta})\quad(\text{cf. }(2.9)).$$

It is then clear that the convergence (2.22) implies the convergences (2.10). The convergences (2.11) have been established in Theorem 7.1 of Part I. □

Remark. The convergence (2.22) still holds if the functions h^3_+,h^3_- are in $L^2(\omega)$. The stronger assumptions $h^3_+,h^3_-\ \in\ H^1(\omega)$ are needed for establishing the convergences (2.11).

We next consider *flexural shells;* we recall in this respect that the assumption $\boldsymbol{V}_F(\omega)\neq\{\boldsymbol{0}\}$ has been discussed in Sec. 7 of Part II.

Theorem 2.2. *Assume that*

(2.23) $$\text{length } \gamma_0 > 0,$$

and that the space of inextensional displacements

(2.24) $$\boldsymbol{V}_F(\omega) := \{\boldsymbol{\eta} \in \boldsymbol{V}_K(\omega);\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$$

does not reduce to $\{\mathbf{0}\}$; *finally, assume that there exist functions* $f^i \in L^2(\Omega)$ and $h^i_+, h^i_- \in L^2(\omega)$ *such that*

(2.25) $$f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^2 f^i(x) \quad \textit{for all } x \in \Omega\ , \quad h^{i,\varepsilon}_\pm(y) = \varepsilon^3 h^i_\pm(y) \quad \textit{for all } y \in \omega.$$

Let $\boldsymbol{\zeta} \in \boldsymbol{V}_F(\omega)$ *denote the unique solution of the two-dimensional flexural shell equations*

(2.26) $$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,dy = \int_\omega \left\{ \int_{-1}^{1} f^i dx_3 + h^i_+ + h^i_- \right\} \eta_i \sqrt{a}\,dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_F(\omega)$. *Finally, let* $\boldsymbol{\zeta}^\varepsilon \in \boldsymbol{V}_K(\omega)$ *denote the solution of the two-dimensional shell equations* (1.1), (1.2) *of Koiter, and let* $\boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon)$ *denote the solution of the three-dimensional problem* (2.1), (2.2). *Then*

(2.27) $$\zeta^\varepsilon_\alpha \boldsymbol{a}^\alpha \to \zeta_\alpha \boldsymbol{a}^\alpha \ \textit{in } \boldsymbol{H}^1(\omega), \quad \zeta^\varepsilon_3 \boldsymbol{a}^3 \to \zeta_3 \boldsymbol{a}^3 \ \textit{in } \boldsymbol{H}^2(\omega),$$

(2.28) $$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u^\varepsilon_i \boldsymbol{g}^{i,\varepsilon} dx^\varepsilon_3 \to \zeta_i \boldsymbol{a}^i \quad \textit{in } \boldsymbol{H}^1(\omega)$$

as $\varepsilon \to 0$.

Proof. That equations (2.26) have a unique solution $\boldsymbol{\zeta} \in \boldsymbol{V}_F(\omega)$ is a consequence of the $\boldsymbol{V}_K(\omega)$-ellipticity of the bilinear form appearing in Koiter's equations combined with the definition (2.24) of the space $\boldsymbol{V}_F(\omega)$ and the expression of the bilinear form found in (2.26).

The first convergence result similar to (2.27) is due to Sanchez-Palencia [1989a, Th. 2.1], who noticed that the *weak* convergence $\boldsymbol{\zeta}^\varepsilon \rightharpoonup \boldsymbol{\zeta}$ in $\boldsymbol{V}_K(\omega)$, i.e., in $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, holds, as a consequence of general results in perturbation theory. We show that in fact the stronger result (2.35) holds. In addition to the notation (2.13)–(2.16), we let

(2.29) $$\|\, \boldsymbol{\eta} \,\|_{\boldsymbol{V}_K(\omega)} = \left\{ \sum_\alpha \|\, \eta_\alpha \,\|^2_{1,\omega} + \|\, \eta_3 \,\|^2_{2,\omega} \right\}^{1/2}.$$

As shown in Bernadou & Ciarlet [1976], Ciarlet & Miara [1992], Bernadou, Ciarlet & Miara [1994], Blouza & Le Dret [1994a, b], there exists a constant c_1 such that

(2.30) $$\|\, \boldsymbol{\eta} \,\|_{\boldsymbol{V}_K(\omega)} \leqq c_1 \left\{ \sum_{\alpha,\beta} \|\, \gamma_{\alpha\beta}(\boldsymbol{\eta}) \,\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\, \rho_{\alpha\beta}(\boldsymbol{\eta}) \,\|^2_{0,\omega} \right\}^{1/2}$$

for all $\boldsymbol{\eta} \in \boldsymbol{V}_K(\omega)$ (in particular, the assumption (2.23) is needed here).

By virtue of assumptions (2.25), the solution $\boldsymbol{\zeta}^\varepsilon$ of (1.1), (1.2) also satisfies

$$\frac{1}{\varepsilon^2} B_M(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{\eta}) + B_F(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{\eta}) = \boldsymbol{L}(\eta) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}_K(\omega). \tag{2.31}$$

Hence letting $\boldsymbol{\eta} = \boldsymbol{\zeta}^\varepsilon$ in (2.31) and combining inequalities (2.12) and (2.30), we obtain (without loss of generality, we may assume $\varepsilon \leqq 1$):

$$\frac{1}{3c_1^2} \| \boldsymbol{\zeta}^\varepsilon \|^2_{\boldsymbol{V}_K(\omega)} \leqq \sum_{\alpha,\beta} \left| \frac{1}{\varepsilon} \gamma_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon) \right|^2_{0,\omega} + \frac{1}{3} \sum_{\alpha,\beta} \| \rho_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon) \|^2_{0,\omega} \leqq c_0 \, \|L\| \, \| \boldsymbol{\zeta}^\varepsilon \|_{\boldsymbol{V}_K(\omega)} \, .$$

Therefore there exists a subsequence, still denoted $(\boldsymbol{\zeta}^\varepsilon)_{\varepsilon>0}$ for convenience, and there exists a function $\tilde{\boldsymbol{\zeta}} \in \boldsymbol{V}_K(\omega)$, such that

$$\boldsymbol{\zeta}^\varepsilon \rightharpoonup \tilde{\boldsymbol{\zeta}} \quad \text{in } \boldsymbol{V}_K(\omega), \tag{2.32}$$

$$\gamma_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon) \to 0 \quad \text{in } L^2(\omega). \tag{2.33}$$

The weak convergence (2.32) implies that $\gamma_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon) \rightharpoonup \gamma_{\alpha\beta}(\tilde{\boldsymbol{\zeta}})$ in $L^2(\omega)$; hence $\gamma_{\alpha\beta}(\tilde{\boldsymbol{\zeta}}) = 0$ by (2.33), and thus $\tilde{\boldsymbol{\zeta}} \in \boldsymbol{V}_F(\omega)$. Fix $\boldsymbol{\eta} \in \boldsymbol{V}_F(\omega)$ in (2.31) and let $\varepsilon \to 0$; then the weak convergence (2.32) yields $B_F(\tilde{\boldsymbol{\zeta}}, \boldsymbol{\eta}) = L(\boldsymbol{\eta})$. Hence

$$\tilde{\boldsymbol{\zeta}} = \boldsymbol{\zeta}, \tag{2.34}$$

where $\boldsymbol{\zeta} \in \boldsymbol{V}_F(\omega)$ is the unique solution to equations (2.26), and the weak convergence (2.32) holds for the whole family $(\boldsymbol{\zeta}^\varepsilon)_{\varepsilon>0}$.

By virtue of inequality (2.30) combined with the strong convergence (2.33) and the relations $\gamma_{\alpha\beta}(\boldsymbol{\zeta}) = 0$, establishing the strong convergence

$$\boldsymbol{\zeta}^\varepsilon \to \boldsymbol{\zeta} \quad \text{in } \boldsymbol{V}_K(\omega) \tag{2.35}$$

is equivalent to establishing the convergence

$$B_F(\boldsymbol{\zeta}^\varepsilon - \boldsymbol{\zeta}, \boldsymbol{\zeta}^\varepsilon - \boldsymbol{\zeta}) \to 0,$$

which itself follows from the relations

$$0 \leqq B_F(\boldsymbol{\zeta}^\varepsilon - \boldsymbol{\zeta}, \boldsymbol{\zeta}^\varepsilon - \boldsymbol{\zeta}) = B_F(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{\zeta}^\varepsilon) - 2B_F(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{\zeta}) + B_F(\boldsymbol{\zeta}, \boldsymbol{\zeta}),$$

$$B_F(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{\zeta}^\varepsilon) \leqq L(\boldsymbol{\zeta}^\varepsilon) \quad (\text{cf. } (2.31)),$$

$$B_F(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{\zeta}) \to B_F(\boldsymbol{\zeta}, \boldsymbol{\zeta}), \quad \text{and} \quad L(\boldsymbol{\zeta}^\varepsilon) \to L(\boldsymbol{\zeta}) \quad (\text{cf.} (2.32) \text{ and } (2.34)),$$

$$B_F(\boldsymbol{\zeta}, \boldsymbol{\zeta}) = L(\boldsymbol{\zeta}) \quad (\text{cf. } (2.26)).$$

It is then clear that the convergence (2.35) implies the convergences (2.27) (the assumption $\boldsymbol{\varphi} \in \mathscr{C}^3(\bar{\omega})$ is used for the last one). The convergences (2.28) have been established in Theorem 7.1 of Part II. □

3. Conclusions and comments

We have therefore justified the two-dimensional Koiter shell equations when they are applied to a "membrane" shell ($\gamma_0 = \gamma$ and assumption (2.6) holds) or to *a "flexural" shell* ($\boldsymbol{V}_F(\omega) \neq \{\boldsymbol{0}\}$). For in essence, relations (2.10), (2.11) and (2.27), (2.28) show that in each case the averages across the thickness of the tangential components $u^\varepsilon_\alpha \boldsymbol{g}^{\alpha,\varepsilon}$and normal component $u^\varepsilon_3 \boldsymbol{g}^{3,\varepsilon}$ of the "three-dimensional" displacement vector $u^\varepsilon_i \boldsymbol{g}^{i,\varepsilon}$ and the tangential components $\zeta^\varepsilon_\alpha \boldsymbol{a}^\alpha$ and normal component $\zeta^\varepsilon_3 \boldsymbol{a}^3$ of the two-dimensional displacement vector $\zeta^\varepsilon_i \boldsymbol{a}^i$ found by solving Koiter's equations have the *same principal parts,* either in $\boldsymbol{H}^1(\omega) \times L^2(\omega)$ (membrane shells) or in $\boldsymbol{H}^1(\omega)$ (flexural shell).

Our asymptotic analysis also shows that *Koiter's equations are as good as the membrane shell equations* (2.9) *when they are applied to a "membrane" shell, and as good as the flexural shell equations* (2.26) *when they are applied to a "flexural" shell.* These truly remarkable properties of Koiter's two-dimensional shell equations, which also hold for the remaining "generalized membrane" shells (cf. Ciarlet & Lods [1996c]), certainly explain their "improbable success" in the manifold numerical simulations where they are so often blithely applied.

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