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Asymptotic analysis of a mixed boundary value problem in a multi-structure

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Abstract

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We consider a mixed boundary value problem for the Laplace operator in a union of a three-dimensional domain and thin cylinders. A rigorous procedure is developed to derive the total asymptotic expansion. The asymptotic formulae for the first eigenvalue and corresponding eigenfunction have been obtained.

1. Introduction

The main results, obtained up to the present time, in the mathematical modelling of multi-structures for elliptic boundary-value problems are either exposed or referred to in the book [1] by Ciarlet.

In the present work we deal with the asymptotic analysis of a solution u_ε of a mixed boundary value problem for the Laplace operator in a domain Ω_ε (see Fig. 1), which depends on the small parameter ε. As ε tends to zero we have a union of the elements of different dimensions: a three-dimensional domain and one-dimensional segments. We prescribe Dirichlet data on the bases of thin cylinders; on the remaining part of the surface $\partial\Omega_\varepsilon$ we have Neumann boundary condition.

We would like to point out that the field u_ε outside a neighbourhood of the junction points in Ω_0 is represented by means of some solution of the Neumann boundary value problem with the finite value of the energy integral and of a linear combination of functions with the singular behaviour $O(1/r)$ at the junction points; obviously, the energy integral for the latter functions is infinite. This fact creates an obstacle on the way to use a variational approach developed for the asymptotic analysis of multi-structures by Ciarlet in the book [1].

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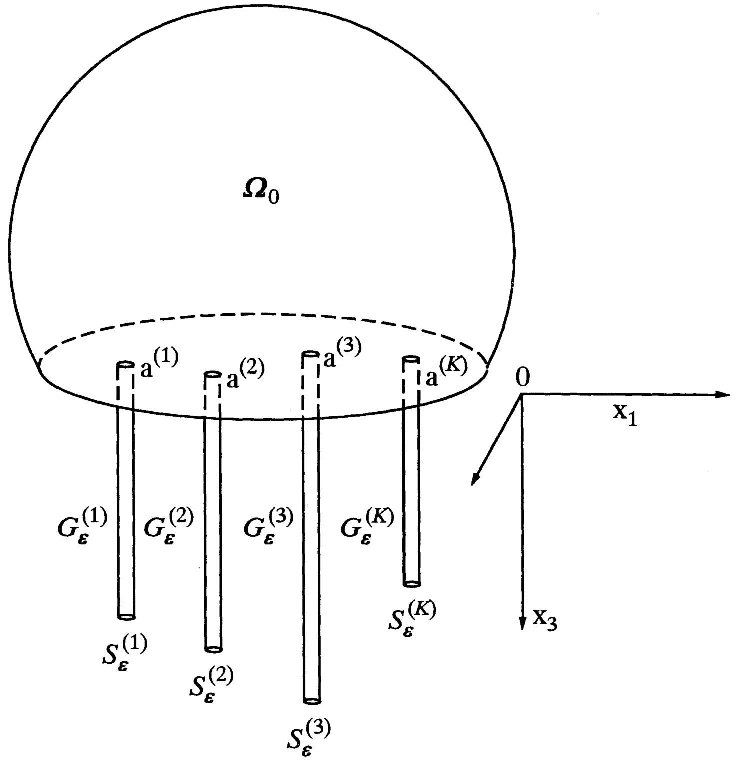

Fig. 1. The multi-structure.

Our main objective is to represent the solution u_ε as an asymptotic series

$$u_\varepsilon(x) \sim \sum_{j=0}^{\infty} \varepsilon^j u_j(x,\varepsilon). \tag{1.1}$$

To perform that we develop a compound asymptotic expansions technique. We mention that the rigorous systematic study of compound asymptotic expansions for solutions of elliptic boundary value problems in domains with singularly perturbed boundaries is presented in the book [2] by Maz'ya, Nazarov and Plamenevskii.

The functions u_j in (1.1) are linear combinations of solutions to so-called model problems which are formulated in scaled variables and related to canonical domains independent of ε.

The behaviour of the solution in Ω_0 outside neighbourhoods of the points $a^{(1)},\dots,a^{(K)}$ is described by means of terms satisfying the model Neumann problem for the three-dimensional element Ω_0 of the given multi-structure. The "junction layer" near $a^{(1)},\dots,a^{(K)}$ is defined by means of solutions of the Neumann boundary value problem in the union of a half-space and a semi-infinite cylinder. The terms of the asymptotic expansion, describing the solution in thin cylinders $G_\varepsilon^{(1)},\dots,G_\varepsilon^{(K)}$ at some distance from the junction points and from the bottom of a cylinder, are obtained by means of the following limiting problems: the Cauchy problem for an ordinary differential equation on a segment and the Neumann boundary value problem on the scaled cross-section of $G_\varepsilon^{(i)}$. Finally, the "bottom layer" near the base $S_\varepsilon^{(j)}$ of a cylinder is defined as a solution of a mixed boundary value problem in a semi-infinite cylinder with the Dirichlet data on the cylinder base and the Neumann boundary condition on the lateral part of the surface.

The information that we need for the use of the solutions of model problems, is presented in Sections 3 and 4.

In Section 5 we construct the principal term $\mathcal{U}$ in the asymptotics of the solution of the mixed boundary value problem in Ω_ε for the Poisson equation with the constant right-hand side. Within an additive constant the function $\mathcal{U}$ can be approximated outside the vicinity of the points a_j as a sum

$$\sum_{j=1}^{K} \mathcal{T}_j \mathbf{N}(x, a^{(j)})$$

where $\mathbf{N}$ is the Neumann function and $\mathcal{T}_j$, $j = 1, \ldots, K$, are explicitly determined constants. Here, one can see the above mentioned fact, that the asymptotic representation of the solution includes the functions $\mathbf{N}(x, a^{(j)})$ with the infinite Dirichlet integral. It is important that this singularity, related to the vicinity of the junction points, is compensated by means of the "junction layer".

In Section 6 we obtain the formal asymptotic expansion (1.1) for the solution u_ε of the original problem with the right-hand side represented in the form of multi-scaled asymptotic expansions. We determine u_j in the following form

$$u_j(x, \varepsilon) = I_j \mathcal{U}(x, \varepsilon) + \Theta_j(x, \varepsilon),$$

where I_j is a constant coefficient.

The function Θ_j includes terms, concentrated near the junction points and in thin cylinders, as well as solutions of the model problem in Ω_0 with the finite energy integral.

The formal asymptotics (1.1) is justified in Section 7, where the estimates for the remainder term are obtained.

In Sections 8 and 9 we analyze the principal term of the asymptotics for u_ε in different zones; we also propose a simplified scheme of calculations and consider some examples.

The last Section 10 is devoted to the asymptotic study of the first eigenvalue λ_1 and the corresponding eigenfunction, based upon an abstract scheme which is given in the Appendix. In particular, for λ_1 we obtain

$$\lambda_1 = \varepsilon^2 (\operatorname{mes} \Omega_0)^{-1} \sum_{j=1}^{K} l_j^{-1} \operatorname{mes}_2 g_j + O(\varepsilon^4),$$

where g_j is the scaled cross-section of a cylinder $G_\varepsilon^{(j)}$ and l_j is the cylinder length.

The asymptotic analysis of elliptic boundary value problems in multi-structures of the type, considered in the present paper, has a lot of applications in solid mechanics (see [1] and bibliography there), and the authors are planning to include the extension of the present results in a different paper related to the analysis of the junction of a three-dimensional domain and one-dimensional elements of a multi-structure for the elasticity equations.

2. Formulation of the problem

Let us consider a set $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon^{(1)} \cup \ldots \cup G_\varepsilon^{(K)}$, where Ω_0 is a finite three-dimensional domain with a smooth boundary (we suppose, that some part of $\partial\Omega_0$ is located in the plane $x_3 = 0$), and

$$G_\varepsilon^{(i)} = \left\{x\colon\ 0 \leqslant x_3 < l_i,\ \left(\left(x_1 - a_1^{(i)}\right)/\varepsilon, \left(x_2 - a_2^{(i)}\right)/\varepsilon\right) \in g_i \subset R^2\right\}, \quad i = 1, \ldots, K,$$

are "thin" cylinders compiled with the flat part of the surface $\partial\Omega_0$ at the points $(a_1^{(i)}, a_2^{(i)}, 0)$, $i = 1, \ldots, K$; here g_i is a finite two-dimensional domain which has smooth boundary and includes the origin; ε is a small positive parameter. Let us mention that the same method of asymptotic analysis can be used for the geometry, where the boundary $\partial\Omega_0$ is flat in the vicinity of the junction points $x = a^{(i)}$, $i = 1, \ldots, K$ (i.e. points $a^{(i)}$ can have different coordinates along the Ox_3 axis).

We consider the following boundary value problem

$$-\Delta_x u_\varepsilon(x) = F(x, \varepsilon), \quad x \in \Omega_\varepsilon, \tag{2.1}$$

$$\frac{\partial u_\varepsilon}{\partial n_x}(x) = P(x, \varepsilon), \quad x \in \partial\Omega_\varepsilon \setminus \bigcup_i \overline{S_\varepsilon^{(i)}}, \tag{2.2}$$

$$u_\varepsilon(x) = \phi^{(i)}(x, \varepsilon), \quad x \in S_\varepsilon^{(i)},\ i = 1, \ldots, K, \tag{2.3}$$

where n_x is the unit outward normal vector with respect to $\partial\Omega_\varepsilon$, $S_\varepsilon^{(i)}$ is a base of a thin cylinder $G_\varepsilon^{(i)}$,

$$S_\varepsilon^{(i)} = \left\{x\colon\ x_3 = l_i,\ \left(\left(x_1 - a_1^{(i)}\right)/\varepsilon, \left(x_2 - a_2^{(i)}\right)/\varepsilon\right) \in g_i\right\}.$$

We suppose that F, P and $\phi^{(i)}$ are such functions that the problem (2.1)–(2.3) is uniquely solvable in a class of functions with the finite energy integral. Our aim is to construct a uniform in Ω_ε asymptotic expansion with respect to the powers of ε for the solution u_ε.

3. The model problems

In the present section we consider an auxiliary problem in Ω_0 and introduce a so-called "junction layer" (the solution of the problem in a union of a half-space and a semi-infinite cylinder) and a "bottom layer" (the solution of the problem in the vicinity of the base of a cylinder).

1^0. The model problem in Ω_0

Let us consider the following formulation

$$-\Delta_x u(x) = f(x), \quad x \in \Omega_0, \tag{3.1}$$

$$\frac{\partial u}{\partial n_x}(x) = p(x) + \sum_{i=1}^{K} T^{(i)} \delta\left(x - a^{(i)}\right), \quad x \in \partial\Omega_0, \tag{3.2}$$

where $T^{(i)}$, $i = 1, \dots, K$, are constants, and f, p are smooth functions which satisfy the relationship

$$\int_{\Omega_0} f(x)\,\mathrm{d}x + \int_{\partial\Omega_0} p(x)\,\mathrm{d}s + \sum_{i=1}^{K} T^{(i)} = 0. \tag{3.3}$$

We seek the solution $u(x)$ in a class of functions satisfying the orthogonality condition

$$\int_{\Omega_0} u(x)\,\mathrm{d}x = 0.$$

The solution of the problem (3.1), (3.2) can be represented by means of the linear combinations

$$u(x) = \mu(x) + \sum_{i=1}^{K} T^{(i)}\mathbf{N}(x, a^{(i)}), \tag{3.4}$$

where $\mu \in \mathbf{H}^1(\Omega_0)$, and $\mathbf{N}$ is the Neumann function.

The function μ is defined as a solution of the boundary value problem

$$-\Delta_x \mu(x) = f(x) + \mathbf{c}, \quad x \in \Omega_0, \tag{3.5}$$

$$\frac{\partial \mu}{\partial n_x}(x) = p(x), \quad x \in \partial\Omega_0, \tag{3.6}$$

where the constant $\mathbf{c}$ is defined by

$$\mathbf{c} = -(\operatorname{mes}\Omega_0)^{-1}\left\{\int_{\Omega_0} f(x)\,\mathrm{d}x + \int_{\partial\Omega_0} p(x)\,\mathrm{d}s\right\} \tag{3.7}$$

and the following orthogonality condition holds

$$\int_{\Omega_0} \mu(x)\,\mathrm{d}x = 0. \tag{3.8}$$

The Neumann function $\mathbf{N}$ is defined as a solution of the boundary value problem

$$-\Delta_x \mathbf{N}(x, y) = \delta(x - y) - (\operatorname{mes}\Omega_0)^{-1}, \quad x \in \Omega_0, \tag{3.9}$$

$$\frac{\partial \mathbf{N}}{\partial n_x}(x, y) = 0, \quad x \in \partial\Omega_0, \tag{3.10}$$

with the orthogonality condition

$$\int_{\Omega_0} \mathbf{N}(x, y)\,\mathrm{d}x = 0. \tag{3.11}$$

Thus, μ can be represented in the form

$$\mu(x) = \int_{\Omega_0} \mathbf{N}(x, y) f(y)\,\mathrm{d}y + \int_{\partial\Omega_0} \mathbf{N}(x, y) p(y)\,\mathrm{d}s_y. \tag{3.12}$$

The function $\mathbf{N}(x, a^{(i)})$ satisfies the boundary value problem

$$\Delta_x \mathbf{N}(x, a^{(i)}) = (\operatorname{mes} \Omega_0)^{-1}, \quad x \in \Omega_0, \tag{3.13}$$

$$\frac{\partial \mathbf{N}}{\partial n_x}(x, a^{(i)}) = \delta(x - a^{(i)}), \quad x \in \partial\Omega_0, \tag{3.14}$$

with the orthogonality condition (3.11), and has the following asymptotic behaviour in a neighbourhood of the points $x = a^{(i)}$

$$\mathbf{N}(x, a^{(i)}) = \frac{1}{2\pi\|x - a^{(i)}\|} + m_i(x), \quad x \to a^{(i)}, \tag{3.15}$$

where m_i is a smooth function.

2^0. The model problem, related to the junction layer

We consider an infinite domain $\omega = R^3_- \cup \Pi$, where $R^3_- = \{X\colon X_3 < 0\}$ is a half-space, and $\Pi = \{X\colon X_3 \geqslant 0,\ (X_1, X_2) \in g\}$ is a semi-infinite cylinder (see Fig. 2); g is a bounded (with compact closure) simply-connected domain in R^2 with a smooth boundary.

We consider the Neumann boundary value problem in ω

$$-\Delta_X \mathcal{W}(X) = \mathcal{F}(X), \quad X \in \omega, \tag{3.16}$$

$$\frac{\partial \mathcal{W}}{\partial n_X}(X) = \mathcal{P}(X), \quad X \in \partial\omega. \tag{3.17}$$

We suppose that $\mathcal{F}$ is smooth, $\mathcal{P}$ is the normal derivative of a smooth function on R^3; $\mathcal{F}$ and $\mathcal{P}$ have compact supports. We seek the solution which has a finite energy integral over any compact subset of $\overline{\Omega}$, decays like $O(\|X\|^{-2})$ in a half-space as $\|X\| \to +\infty$ and has a linear principal term of the asymptotics at infinity in Π. The following asymptotic relation holds

$$\mathcal{W}(X) \sim \|X\|^{-2} \sum_{k=0}^{\infty} \frac{\mathbf{Y}_{k+1}(\theta)}{\|X\|^k}, \quad \|X\| \to \infty,\ X \in R^3_-; \tag{3.18}$$

here $\theta = (\theta_1, \theta_2)$, such that

$$\pi/2 < \theta_1 \leqslant \pi, \quad 0 < \theta_2 \leqslant 2\pi,$$

and

$$\mathbf{Y}_n(\theta_1, \theta_2) = \sum_{m=0}^{n} \left(A_{nm} \cos(m\theta_2) + B_{nm} \sin(m\theta_2)\right) P_n^{(m)}(\cos\theta_1),$$

$$P_n^{(m)}(t) = \left(1 - t^2\right)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}t^m} P_n(t),$$

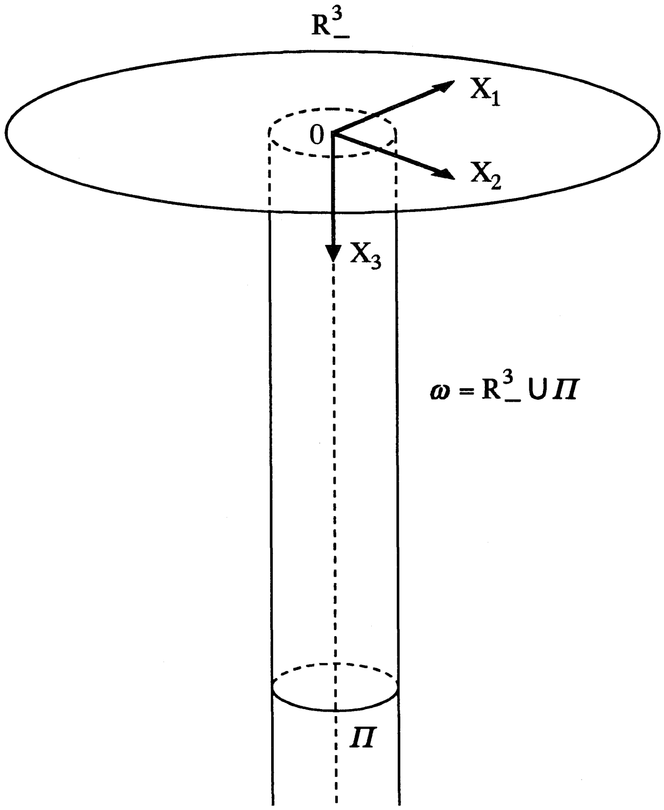

Fig. 2. The model domain for the junction layer.

where P_n are the Legendre polynomials; A_{nm}, B_{nm} are constants; A_{nm} and B_{nm} are equal to zero for even values of $n+m$.

The asymptotics in Π is defined by

$$\mathcal{W}(X) = (\mathrm{mes}_2\, g)^{-1} D X_3 + q + \mathrm{O}(e^{-\alpha X_3}), \quad X_3 \to +\infty, \tag{3.19}$$

where α is a positive constant, and

$$D = -\int_\omega \mathcal{F}(X)\,\mathrm{d}X - \int_{\partial\omega} \mathcal{P}(X)\,\mathrm{d}s. \tag{3.20}$$

The constant q in (3.19) can be determined by means of the special solution Ξ of the homogeneous problem (3.16), (3.17) in ω with the following asymptotic behaviour at infinity

$$\Xi(X) = (\mathrm{mes}_2\, g)^{-1} X_3 + \mathrm{const} + \mathrm{O}(e^{-\alpha X_3}), \quad X_3 \to +\infty,\ X \in \Pi, \tag{3.21}$$

$$\Xi(X) = \frac{1}{2\pi\|X\|} + \mathrm{O}(\|X\|^{-2}), \quad \|X\| \to +\infty,\ X \in R^3_-. \tag{3.22}$$

The use of the Green formula for the functions $\mathcal{W}$ and Ξ in $\omega \cap B_r(O)$ gives as $r \to \infty$

$$q = \int_\omega \Xi(X)\mathcal{F}(X)\,\mathrm{d}X + \int_{\partial\omega} \Xi(X)\mathcal{P}(X)\,\mathrm{d}s. \tag{3.23}$$

3^0. The model problem for the bottom layer in a neighbourhood of the base of a thin cylinder

Let us consider the mixed boundary value problem in a semi-infinite cylinder Π (see Fig. 3)

$$-\Delta_Y v(Y) = G(Y), \quad Y \in \Pi, \tag{3.24}$$

$$\frac{\partial v}{\partial n_Y}(Y) = H(Y), \quad Y \in \partial\Pi \setminus \overline{S}, \tag{3.25}$$

$$v(Y) = \psi(Y), \quad Y \in S, \tag{3.26}$$

where $S = \{Y\colon Y_3 = 0,\ (Y_1, Y_2) \in g \subset R^2\}$ is the base of a cylinder, $\psi \in C^\infty(\overline{S})$; G, H are smooth functions with compact supports in $\overline{\Pi}$ and $\partial\Pi \setminus S$.

The problem (3.24)–(3.26) is uniquely solvable in the class of functions with finite energy integral. The solution admits the asymptotic form

$$v(Y) = C_v + O(e^{-\alpha Y_3}), \quad Y_3 \to +\infty; \tag{3.27}$$

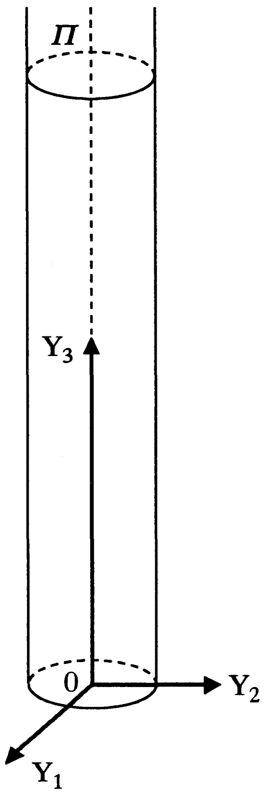

Fig. 3. The model domain for the bottom layer.

here α is a positive constant.

We use the special solution $\xi(Y) = Y_3$ of the homogeneous problem (3.24)–(3.26). Using the Green formula for functions v and ξ in $\{Y \in \Pi\colon\ Y_3 < r\}$ we obtain (as r tends to infinity)

$$C_v = (\operatorname{mes}_2 g)^{-1}\left\{ \int_S \psi(Y)\,\mathrm{d}s_Y + \int_\Pi Y_3 G(Y)\,\mathrm{d}Y + \int_{\partial\Pi\setminus S} Y_3 H(Y)\,\mathrm{d}s \right\}. \tag{3.28}$$

Thus, the function v decays exponentially at infinity if and only if the right-hand sides of the problem (3.24)–(3.26) provide the equality

$$C_v = 0. \tag{3.29}$$

4. A particular solution of a problem in a thin cylinder of a finite length

Let us consider a thin cylinder $G_\varepsilon = \{x\colon\ 0 \leqslant x_3 < l,\ \varepsilon^{-1}(x_1, x_2) \in g \subset R^2\}$. Suppose that a function V_ε satisfies the equation

$$-\Delta_x V_\varepsilon(\zeta, z) = \varepsilon^{-2}\Phi(\zeta, z), \quad \zeta \in g,\ 0 < z < l, \tag{4.1}$$

and the boundary condition on the lateral surface

$$\frac{\partial V_\varepsilon}{\partial n_x}(\zeta, z) = \varepsilon^{-1}\Psi(\zeta, z), \quad \zeta \in \partial g,\ 0 < z < l. \tag{4.2}$$

Here we use a set of new variables $\zeta = \varepsilon^{-1}(x_1, x_2)$, $z = x_3$; n_x is the unit outward normal vector with respect to ∂G_ε. We suppose that the function Φ is smooth in the closure of the cylinder, Ψ is smooth in the closure of the lateral surface.

Due to the change of variables we can write down the following equalities

$$\Delta_x = \varepsilon^{-2}\Delta_\zeta + \frac{\partial^2}{\partial z^2}, \tag{4.3}$$

and on the lateral surface of the cylinder

$$\frac{\partial}{\partial n_x} = \varepsilon^{-1}\frac{\partial}{\partial n_\zeta}. \tag{4.4}$$

Thus, the principal part of the differential operator corresponds to the problem on the scaled cross-section of a cylinder

$$-\Delta_\zeta U(\zeta, z) = \Phi(\zeta, z), \quad \zeta \in g,\ 0 < z < l, \tag{4.5}$$

$$\frac{\partial U}{\partial n_\zeta}(\zeta, z) = \Psi(\zeta, z), \quad \zeta \in \partial g,\ 0 < z < l, \tag{4.6}$$

with a parameter z.

Obviously, the functions Φ and Ψ might violate the solvability condition of the problem (4.5), (4.6). Thus, the asymptotic form of the function V_ε is

$$V_\varepsilon(\zeta, z) \sim \varepsilon^{-2}W_0(z) + U_0(\zeta, z). \tag{4.7}$$

Both terms give the same order in the right-hand side in the equation (4.1). The solvability of the problem can be provided by means of the function W_0

$$\mathrm{mes}_2\, g \frac{\mathrm{d}^2}{\mathrm{d}z^2} W_0(z) + \int_g \Phi(\zeta, z)\,\mathrm{d}z + \int_{\partial g} \Psi(\zeta, z)\,\mathrm{d}s = 0, \quad z \in (0, l). \tag{4.8}$$

The term U_0 from (4.7) satisfies the boundary value problem

$$-\Delta_\zeta U_0(\zeta, z) = \Phi(\zeta, z) - (\mathrm{mes}_2\, g)^{-1}\Big(\int_g \Phi(\zeta, z)\,\mathrm{d}\zeta + \int_{\partial g} \Psi(\zeta, z)\,\mathrm{d}s_\zeta\Big), \tag{4.9}$$

$$\frac{\partial U_0}{\partial n_\zeta}(\zeta, z) = \Psi(\zeta, z), \quad \zeta \in \partial g. \tag{4.10}$$

The problem (4.9), (4.10) is uniquely solvable in a class of functions satisfying the orthogonality condition

$$\int_g U_0(\zeta, z)\,\mathrm{d}\zeta = 0. \tag{4.11}$$

We can continue the formal asymptotic procedure and obtain

$$V_\varepsilon \sim \sum_{k=0}^{\infty} \varepsilon^{2k}\big(\varepsilon^{-2} W_k(z) + U_k(\zeta, z)\big), \tag{4.12}$$

where W_k, $k = 1, 2, \ldots$, are linear functions, and the terms U_k satisfy the problems

$$-\Delta_\zeta U_k(\zeta, z) = \frac{\partial^2}{\partial z^2} U_{k-1}(\zeta, z), \quad \zeta \in g, \tag{4.13}$$

$$\frac{\partial U_k}{\partial n_\zeta}(\zeta, z) = 0, \quad \zeta \in \partial g, \tag{4.14}$$

and have zero mean value over g (see (4.11)).

Further we shall use the asymptotic representation (4.12), where the functions W_k satisfy the equalities

$$W_k(0) = 0, \qquad \frac{\mathrm{d}W_k}{\mathrm{d}z}(0) = 0, \tag{4.15}$$

and, therefore,

$$\begin{aligned} W_0(z) &= -(\mathrm{mes}_2\, g)^{-1} \int_0^z (z - t)\Big\{\int_g \Phi(\zeta, t)\,\mathrm{d}\zeta + \int_{\partial g} \Psi(\zeta, t)\,\mathrm{d}s_\zeta\Big\}\,\mathrm{d}t, \\ W_k &= 0, \; k = 1, 2, \ldots. \end{aligned} \tag{4.16}$$

We should emphasize that, consequently, two kinds of model problems are involved. The first one is the Cauchy problem (4.8), (4.15) for the ordinary differential equation, and the second formulation is the Neumann boundary value problem (see (4.9)–(4.10) and (4.13)–(4.14) on the scaled cross-section of a cylinder).

5. An auxiliary boundary value problem in Ω_ε with a constant right-hand side

Let us consider the following boundary value problem

$$-\Delta_x A(x,\varepsilon) = (\operatorname{mes}\Omega_0)^{-1}, \quad x \in \Omega_\varepsilon, \tag{5.1}$$

$$\frac{\partial A}{\partial n_x}(x,\varepsilon) = 0, \quad x \in \partial\Omega_\varepsilon \setminus \bigcup_i \overline{S_\varepsilon^{(i)}}, \tag{5.2}$$

$$A(x,\varepsilon) = 0, \quad x \in \bigcup_i S_\varepsilon^{(i)}. \tag{5.3}$$

Further we also will be using the solution $\mathcal{N}$ of the problem in Ω_0

$$-\Delta_x \mathcal{N}(x) = (\operatorname{mes}\Omega_0)^{-1}, \quad x \in \Omega_0, \tag{5.4}$$

$$\frac{\partial \mathcal{N}}{\partial n_x}(x) = \sum_{i=1}^{K} T^{(i)} \delta\bigl(x - a^{(i)}\bigr), \quad x \in \partial\Omega_0, \tag{5.5}$$

where the factors $T^{(i)}$ satisfy the equality

$$\sum_{i=1}^{K} T^{(i)} + 1 = 0. \tag{5.6}$$

We seek the solution of the problem (5.4)–(5.5) in a class of functions satisfying the orthogonality condition

$$\int_{\Omega_0} \mathcal{N}(x)\,\mathrm{d}x = 0. \tag{5.7}$$

The following relation holds

$$\mathcal{N}(x) = \sum_{i=1}^{K} T^{(i)} \mathbf{N}\bigl(x, a^{(i)}\bigr), \tag{5.8}$$

where $\mathbf{N}$ is the Neumann function (see (3.9)–(3.11)).

We introduce a cut-off function $\chi \in C^\infty(\overline{R^3_-})$ which is equal to 1 for large $\|X\|$ and vanishes for $\|X\| < \delta$. We assume that the circle $X_1^2 + X_2^2 < \delta^2$ contains $\bar{g}_j$, $j = 1, \ldots, K$, and we extend χ by 0 into the half-cylinder $\{X\colon X_3 > 0,\ X_1^2 + X_2^2 \leqslant \delta^2\}$.

Next, we substitute the function

$$\mathcal{N}(x) \prod_{i=1}^{K} \chi\left(\frac{x - a^{(i)}}{\varepsilon}\right)$$

into the equation (5.1) and boundary conditions (5.2), (5.3).

Further we shall use the following notation

$$X^{(i)} = \varepsilon^{-1}(x - a^{(i)}).$$

Thus, we have

$$-\Delta_x\left(\mathcal{N}(x)\prod_{i=1}^{K}\chi(X^{(i)})\right) = (\operatorname{mes}\Omega_0)^{-1}\prod_{i=1}^{K}\chi(X^{(i)}) - \varepsilon^{-2}\sum_{i=1}^{K}[\Delta_X, \chi(X^{(i)})] \times\left\{\frac{\varepsilon^{-1}T^{(i)}}{2\pi\|X^{(i)}\|} + \mathrm{O}(1)\right\}. \tag{5.9}$$

Outside the vicinity of the junction points the boundary conditions on $\partial\Omega_0$ are satisfied. Within an ε-neighbourhood of the point $x = a^{(i)}$ we have

$$\frac{\partial}{\partial n_x}\left(\mathcal{N}(x)\prod_{i=1}^{K}\chi(X^{(i)})\right) = \varepsilon^{-1}n_X\cdot\nabla_X\left(\chi(X^{(i)})\frac{\varepsilon^{-1}T^{(i)}}{2\pi\|X^{(i)}\|} + \mathrm{O}(1)\right). \tag{5.10}$$

Another smooth cut-off function $\varkappa(x)$, described in original variables (x_1, x_2, x_3), might be useful as well. One can choose a sufficiently small constant d independent of ε. Then, we define $\varkappa(x)$ to be 1 for all x from the set

$$\{x\colon\ x_3 \leqslant 0,\ \|x\| \leqslant d\} \cup \{x\colon\ x_3 \geqslant 0,\ x_1^2 + x_2^2 \leqslant d^2\}$$

and to be 0 outside a d-neighbourhood of this set.

The principal term of the discrepancy in (5.9), (5.10) can be compensated by means of the function

$$\varepsilon^{-1}T^{(i)}\mathcal{W}^{(i)}(X^{(i)})\varkappa(x - a^{(i)}),$$

describing the junction layer in the vicinity of the point $x = a^{(i)}$, where $\mathcal{W}^{(i)}$ satisfies the problem (3.16), (3.17) in $\omega_i = R^3_- \cup \Pi_i$, $\Pi_i = \{X\colon\ X_3 \geqslant 0,\ (X_1, X_2) \in g_i\}$ with the right-hand sides

$$\mathcal{F}^{(i)}(X^{(i)}) = [\Delta_X, \chi(X^{(i)})]\frac{1}{2\pi\|X^{(i)}\|},$$

$$\mathcal{P}^{(i)}(X^{(i)}) = -\frac{\partial}{\partial n_X}\left(\chi(X^{(i)})\frac{1}{2\pi\|X^{(i)}\|}\right). \tag{5.11}$$

The function $A(x,\varepsilon)$ admits in Ω_0 the asymptotic form

$$A(x,\varepsilon) \sim C(\varepsilon) + \mathcal{N}(x)\prod_{i=1}^{K}\chi(X^{(i)}) + \varepsilon^{-1}\sum_{i=1}^{K}T^{(i)}\mathcal{W}^{(i)}(X^{(i)})\varkappa(x - a^{(i)}), \quad \varepsilon \to 0. \tag{5.12}$$

We shall seek the principal term of the value $\mathcal{C}(\varepsilon)$ in the form

$$\mathcal{C}(\varepsilon) \sim \varepsilon^{-2} C_0.$$

The junction layer $\mathcal{W}^{(i)}$ behaves at infinity like a linear function (3.19) in Π_i and decays with the asymptotics (3.18) in a half-space. We obtain on the base of a thin cylinder $G_\varepsilon^{(i)}$

$$\varepsilon^{-1} T^{(i)} \mathcal{W}^{(i)}\big|_{x_3 = l_i} = \varepsilon^{-2} T^{(i)} l_i (\mathrm{mes}_2\, g_i)^{-1} + O(\varepsilon^{-1}). \tag{5.13}$$

Therefore,

$$A(x,\varepsilon)\big|_{x_3 = l_i} = \varepsilon^{-2} C_0 + \varepsilon^{-2} T^{(i)} l_i (\mathrm{mes}_2\, g_i)^{-1} + O(\varepsilon^{-1}). \tag{5.14}$$

By means of the choice of the constant C_0

$$C_0 + T^{(i)} l_i (\mathrm{mes}_2\, g_i)^{-1} = 0, \tag{5.15}$$

we remove the term of the order $O(\varepsilon^{-2})$ in the right-hand side (5.14), and the discrepancy in the Dirichlet boundary condition (5.3) is $O(\varepsilon^{-1})$.

The coefficients $T^{(i)}$ from (5.15) have to satisfy relation (5.6), and, therefore, the value C_0 is defined by

$$C_0 = \left[\sum_{i=1}^{K} l_i^{-1}\, \mathrm{mes}_2\, g_i \right]^{-1}, \tag{5.16}$$

and the constants $T^{(i)}$ satisfy the equalities

$$T^{(k)} = -l_k^{-1}\, \mathrm{mes}_2\, g_k \left[\sum_{i=1}^{K} l_i^{-1}\, \mathrm{mes}_2\, g_i \right]^{-1}, \quad k = 1, 2, \ldots, K. \tag{5.17}$$

We introduce the function $\mathcal{U}$ (see (5.12))

$$\begin{aligned} \mathcal{U}(x,\varepsilon) = {} & \varepsilon^{-2} C_0 \\ & + \sum_{k=1}^{K} T^{(k)} \left\{ \mathbf{N}(x, a^{(k)}) \prod_{i=1}^{K} \chi(X^{(i)}) + \varepsilon^{-1} \varkappa(x - a^{(k)}) \mathcal{W}^{(k)}(X^{(k)}) \right\}. \end{aligned} \tag{5.18}$$

Let us substitute (5.18) into the equation (5.1) and the boundary conditions (5.2), (5.3)

$$
\begin{aligned}
-\Delta_x \mathcal{U}(x,\varepsilon) = {} & \sum_{k=1}^{K} T^{(k)}\Big\{ -[\Delta_x, \chi(X^{(i)})]\mathbf{N}(x,a^{(k)}) + \varepsilon^{-3}[\Delta_X, \chi(X^{(k)})]\frac{1}{2\pi\|X^{(k)}\|} \\
& - \varepsilon^{-1}[\Delta_x, \varkappa(x-a^{(k)})]\mathcal{W}^{(k)}(X^{(k)})\Big\} + (\operatorname{mes}\Omega_0)^{-1}\prod_{i=1}^{K}\chi(X^{(i)}) \\
= {} & \sum_{k=1}^{K} T^{(k)}\Big\{ \varepsilon^{-2}[\Delta_X, \chi(X^{(k)})]\big(m_k(a^{(k)}) + O(\varepsilon)\big) \\
& - \varepsilon^{-1}[\Delta_x, \varkappa(x-a^{(k)})]\Big(\varepsilon^2\frac{\mathbf{Y}_1(\theta)}{\|x-a^{(k)}\|^2} + O(\varepsilon^3)\Big)\Big\} \\
& + (\operatorname{mes}\Omega_0)^{-1}\prod_{i=1}^{K}\chi(X^{(i)}), \quad x \in \Omega_\varepsilon,
\end{aligned} \tag{5.19}
$$

$$
\begin{aligned}
& \frac{\partial \mathcal{U}}{\partial n_x}(x,\varepsilon) = \varepsilon^{-1}\sum_{k=1}^{K} T^{(k)} n_x \cdot \nabla_X \chi(X^{(k)})\big(m_k(a^{(k)}) + O(\varepsilon)\big), \\
& x \in \partial\Omega_\varepsilon \setminus \bigcup_i S_\varepsilon^{(i)},
\end{aligned} \tag{5.20}
$$

$$
\mathcal{U}(x,\varepsilon) = O(\varepsilon^{-1}), \quad x \in \bigcup_i S_\varepsilon^{(i)}. \tag{5.21}
$$

It is important for the further study that due to (5.19)–(5.21) the right-hand side of the problem (5.1)–(5.3) with respect to $A - \mathcal{U}$ corresponds to the structure of asymptotic expansions (6.1)–(6.3) for $k \geqslant 1$.

6. A formal asymptotic expansion of a solution of the problem in Ω_ε

Let us consider problem (2.1)–(2.3) with the right-hand sides defined by means of the multi-scaled asymptotic series

$$
\begin{aligned}
F(x,\varepsilon) \sim {} & \sum_{k=0}^{\infty} \varepsilon^k f_k(x)\prod_{i=1}^{K}\chi(X^{(i)}) \\
& + \varepsilon^{-4}\sum_{k=0}^{\infty}\varepsilon^k\sum_{i=1}^{K}\Big\{\varepsilon\mathcal{F}_k^{(i)}(X^{(i)}) + \varepsilon^2\Upsilon\Big(\frac{x_3}{\varepsilon}\Big)\Phi_k^{(i)}(\zeta^{(i)}, x_3) + G_k^{(i)}(Y^{(i)})\Big\},
\end{aligned} \tag{6.1}
$$

$$\begin{aligned} P(x,\varepsilon) \sim &\sum_{k=0}^{\infty} \varepsilon^k p_k(x) \prod_{i=1}^{K} \chi(X^{(i)}) \\ &+ \varepsilon^{-3} \sum_{k=0}^{\infty} \varepsilon^k \sum_{i=1}^{K} \left\{ \varepsilon \mathcal{P}_k^{(i)}(X^{(i)}) + \varepsilon^2 \Upsilon\Big(\frac{x_3}{\varepsilon}\Big) \Psi_k^{(i)}(\zeta^{(i)}, x_3) + H_k^{(i)}(Y^{(i)}) \right\}, \end{aligned} \tag{6.2}$$

$$\phi^{(i)}(x,\varepsilon) \sim \varepsilon^{-2} \sum_{k=0}^{\infty} \varepsilon^k \phi_k^{(i)}(Y^{(i)}). \tag{6.3}$$

The structure of representations (6.1)–(6.3) is determined by the geometry of the domain Ω_ε. We would like to highlight four types of subsets of Ω_ε:

i) the three-dimensional set Ω_0 excluding neighbourhoods of junction points;

ii) small neighbourhoods (of a radius $O(\varepsilon)$) of the junction points;

iii) thin cylinders $G_\varepsilon^{(i)}$, $i = 1, \dots, K$, excluding neighbourhoods of the junction points $a^{(i)}$, $i = 1, \dots, K$, and of the base domains $S_\varepsilon^{(i)}$, $i = 1, \dots, K$;

iv) small neighbourhoods of $S_\varepsilon^{(i)}$, $i = 1, \dots, K$.

Correlatively, for the cases (ii) and (iv) we have introduced already problems of the junction layer and of the bottom layer types; the auxiliary local scaled variables are in use.

Here f_k are smooth functions defined in Ω_0; p_k are defined and smooth on $\partial\Omega_0$. Functions $\mathcal{F}_k^{(i)}, \mathcal{P}_k^{(i)}$ and $G_k^{(i)}, H_k^{(i)}$, defined in the vicinity junction points and in a neighbourhood of the base of thin cylinders, are smooth in coordinates $X^{(i)}$ and $Y^{(i)}$; we also suppose that $\mathcal{F}_k^{(i)}$, $\mathcal{P}_k^{(i)}$, $G_k^{(i)}$ and $H_k^{(i)}$ have compact supports. Functions $\Phi_k^{(i)}, \Psi_k^{(i)}$, smooth in coordinates $(\zeta^{(i)}, x_3)$, define the right-hand side of the problem in thin cylinders $G_\varepsilon^{(i)}$. Here $\Upsilon(t)$ is a smooth cut-off function: $\Upsilon(t) = 0$ for $t < 0$, and $\Upsilon(t) = 1$ for $t > 1$. We use scaled local variables

$$X^{(i)} = \varepsilon^{-1}(x - a^{(i)}),$$

$$\zeta^{(i)} = \varepsilon^{-1}(x_1 - a_1^{(i)}, x_2 - a_2^{(i)}),$$

$$Y^{(i)} = -(\zeta^{(i)}, \varepsilon^{-1}(x_3 - l^{(i)})).$$

We seek the solution $u(x,\varepsilon)$ (this function is identical to $u_\varepsilon(x)$ from (1.1)) of the problem (2.1)–(2.3) with the right-hand side (6.1)–(6.3) in the form of the asymptotic series

$$u(x,\varepsilon) \sim \sum_{k=1}^{\infty} \varepsilon^k u_k(x,\varepsilon), \tag{6.4}$$

where

$$u_k(x,\varepsilon) = I_k \mathcal{U}(x,\varepsilon) + \Theta_k(x,\varepsilon)$$

and I_k, $k = 1, 2, \dots$, are constant coefficients. The function Θ_k will be described below.

Next, we shall construct the term $u_0(x,\varepsilon)$ of the expansion (6.4). After that one can substitute obtained approximation into the equation and given boundary conditions; the new right-hand side of the problem with respect to the function $u(x,\varepsilon) - u_0(x,\varepsilon)$ will have the same structure (6.1)–(6.3), but the expansions start from $k = 1$. Hence, the formal asymptotic procedure may be repeated again. On this way we obtain the general term $u_k(x,\varepsilon)$ of the asymptotic approximation (6.4).

We introduce the function

$$\mathbf{f}_0(x) = f_0(x) - (\operatorname{mes} \Omega_0)^{-1} I_0, \tag{6.5}$$

where the constant I_0 will be determined below. Thus, the principal term $u_0(x,\varepsilon)$ of the solution of the problem (2.1)–(2.3) with the right-hand side

$$\begin{aligned} &f_0(x) \prod_{i=1}^{K} \chi(X^{(i)}) \\ &\quad + \varepsilon^{-4} \sum_{i=1}^{K} \Big\{ \varepsilon \mathcal{F}_0^{(i)}(X^{(i)}) + \varepsilon^2 \Upsilon\Big(\frac{x_3}{\varepsilon}\Big) \Phi_0^{(i)}(\zeta^{(i)}, x_3) + G_0^{(i)}(Y^{(i)}) \Big\}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} &p_0(x) \prod_{i=1}^{K} \chi(X^{(i)}) \\ &\quad + \varepsilon^{-3} \sum_{i=1}^{K} \Big\{ \varepsilon \mathcal{P}_0^{(i)}(X^{(i)}) + \varepsilon^2 \Upsilon\Big(\frac{x_3}{\varepsilon}\Big) \Psi_0^{(i)}(\zeta^{(i)}, x_3) + H_0^{(i)}(Y^{(i)}) \Big\}, \end{aligned} \tag{6.7}$$

and

$$\varepsilon^{-2} \phi_0(Y^{(i)}), \tag{6.8}$$

allows the following asymptotic form

$$u_0(x,\varepsilon) \sim I_0 \mathcal{U}(x,\varepsilon) + \Theta_0(x,\varepsilon). \tag{6.9}$$

The function Θ_0, related to the behaviour of the solution in thin cylinders, will be defined below.

Let us consider terms of (6.6), (6.7), concentrated in Ω_0. By $\mu_0(x)$ we denote the smooth solution of the problem

$$-\Delta_x \mu_0(x) = \mathbf{f}_0(x), \quad x \in \Omega_0, \tag{6.10}$$

$$\frac{\partial}{\partial n_x} \mu_0(x) = p_0(x), \quad x \in \partial\Omega_0, \tag{6.11}$$

satisfying the orthogonality condition

$$\int_{\Omega_0} \mu_0(x)\, \mathrm{d}x = 0. \tag{6.12}$$

Clearly, for the right-hand sides the following equality holds

$$\int_{\Omega_0} \mathbf{f}_0(x)\,\mathrm{d}x + \int_{\partial\Omega_0} p_0(x)\,\mathrm{d}x = 0. \tag{6.13}$$

One can multiply μ_0 by a smooth cut-off function

$$\prod_{i=1}^{K} \chi(X^{(i)})$$

and extend as zero into neighbourhoods of the points $x = a^{(i)}$, and into thin cylinders $G_\varepsilon^{(i)}$, $i = 1, \ldots, K$. Next, we substitute

$$\mu_0(x) \prod_{i=1}^{K} \chi(X^{(i)})$$

in (6.10), (6.11) and obtain

$$\begin{aligned} -\Delta_x\left(\mu_0(x)\prod_{i=1}^{K}\chi(X^{(i)})\right) &= \mathbf{f}_0(x)\prod_{i=1}^{K}\chi(X^{(i)}) \\ &\quad - \varepsilon^{-2}\sum_{i=1}^{K}\left[\Delta_X, \chi(X^{(i)})\right]\left(\mu_0(a^{(i)}) + O(\varepsilon)\right), \end{aligned} \tag{6.14}$$

$$\begin{aligned} \frac{\partial}{\partial n_x}\left(\mu_0(x)\prod_{i=1}^{K}\chi(X^{(i)})\right) &= p_0(x)\prod_{i=1}^{K}\chi(X^{(i)}) \\ &\quad + \varepsilon^{-1}\sum_{i=1}^{K} n_x \cdot \nabla_X \chi(X^{(i)})\left(\mu_0(a^{(i)}) + O(\varepsilon)\right). \end{aligned} \tag{6.15}$$

By comparison of above relations with (6.6), (6.7) one can see that the function

$$\mu_0(x) \prod_{i=1}^{K} \chi(X^{(i)})$$

is determined by the first terms on the right-hand sides, concentrated in Ω_0 outside ε-neighbourhoods of the junction points.

Next, we are going to study the neighbourhoods of the junction points. We introduce the junction layer $\varepsilon^{-1}\mathcal{W}_0^{(i)}(X^{(i)})$, $i = 1, \ldots, N$, which satisfies the problem (3.16), (3.17) in Ω_i, where the right-hand sides are

$$\mathcal{F}_0^{(i)}(X^{(i)}), \quad \mathcal{P}_0^{(i)}(X^{(i)}).$$

As we mentioned above, the solution $\mathcal{W}_0^{(i)}$ is sought in a class of functions, vanishing in a half-space R_-^3 like $O(\|X\|^{-2})$, with the linear principal term of the asymptotics at infinity in a semi-cylinder Π_i.

Following (3.19)–(3.23) we can write the asymptotic equality

$$\begin{aligned} \mathcal{W}_0^{(i)}(X^{(i)}) &= -\Big\{\int_{\omega_i} \mathcal{F}_0^{(i)}(X)\,\mathrm{d}X + \int_{\partial\omega_i} \mathcal{P}_0^{(i)}(X)\,\mathrm{d}s_X\Big\}(\operatorname{mes} g_i)^{-1}X_3^{(i)} \\ &\quad + \int_{\omega_i} \Xi^{(i)}(X)\mathcal{F}_0^{(i)}(X)\,\mathrm{d}X + \int_{\partial\omega_i} \Xi^{(i)}(X)\mathcal{P}_0^{(i)}(X)\,\mathrm{d}s_X \\ &\quad + O\big(e^{-\alpha X_3^{(i)}}\big), \quad X_3^{(i)} \to +\infty. \end{aligned} \tag{6.16}$$

Here $\Xi^{(i)}$ are special solutions of the problem formulated in Section 3, part 2^0 (see (3.21)–(3.23)).

Therefore, on the base of a thin cylinder $G_\varepsilon^{(i)}$ one can obtain the asymptotic relation

$$\begin{aligned} &\varepsilon^{-1}\mathcal{W}_0^{(i)}(X^{(i)})\big|_{x_3=l_i} \\ &\quad = -\varepsilon^{-2}\Big\{\int_{\omega_i} \mathcal{F}_0^{(i)}(X)\,\mathrm{d}X + \int_{\partial\omega_i} \mathcal{P}_0^{(i)}(X)\,\mathrm{d}s_X\Big\}(\operatorname{mes} g_i)^{-1}l_i + O(\varepsilon^{-1}). \end{aligned} \tag{6.17}$$

Let us note that the principal part in (6.17) has the same order as the first term in the expansion (6.3).

As in Section 5 we multiply the junction layer approximation by the cut-off function $\varkappa(x - a^i)$ and substitute

$$\varepsilon^{-1}\varkappa(x - a^i)\mathcal{W}_0^{(i)}(X^{(i)})$$

into the equation (2.1) and the boundary condition (2.2).

Thus,

$$\begin{aligned} &-\Delta_x\{\varepsilon^{-1}\varkappa(x - a^{(i)})\mathcal{W}_0^{(i)}(X^{(i)})\} \\ &\quad = -\varepsilon^{-1}[\Delta_x, \varkappa(x - a^{(i)})]\Big(\varepsilon^2\frac{\mathbf{Y}_1(\theta)}{|x - a^{(i)}|^2} + O(\varepsilon^3)\Big), \end{aligned} \tag{6.18}$$

$$\begin{aligned} &\frac{\partial}{\partial n_x}\{\varepsilon^{-1}\varkappa(x - a^{(i)})\mathcal{W}_0^{(i)}(X^{(i)})\} \\ &\quad = \varepsilon^{-1}n_x \cdot \nabla\Big\{\varkappa(x - a^{(i)})\Big(\varepsilon^2\frac{\mathbf{Y}_1(\theta)}{|x - a^{(i)}|^2} + O(\varepsilon^3)\Big)\Big\}. \end{aligned} \tag{6.19}$$

Here $\mathbf{Y}_k$, $k = 1, 2, \ldots$, are spherical harmonics (see (3.18)).

Let us now consider the problem in a thin cylinder $G_\varepsilon^{(i)}$ (see Section 4). We introduce a function

$$\varepsilon^{-2}W_0^{(i)}(x_3) + U_0^{(i)}(\zeta^{(i)}, x_3), \tag{6.20}$$

where $W_0^{(i)}$ satisfies the equation

$$\frac{\mathrm{d}^2 W_0^{(i)}}{\mathrm{d}z^2}(z) = -(\mathrm{mes}_2\, g_i)^{-1}\left\{ \int_{g_i} \Phi_0^{(i)}(\zeta, z)\,\mathrm{d}\zeta + \int_{\partial g_i} \Psi_0^{(i)}(\zeta, z)\,\mathrm{d}s_\zeta \right\}, \tag{6.21}$$

on the interval $(0, l_i)$, and the following Cauchy data

$$W_0^{(i)}(0) = \frac{\mathrm{d}W_0^{(i)}}{\mathrm{d}z}(0) = 0, \tag{6.22}$$

at the left end $z = 0$.

The second term $U_0^{(i)}$ in (6.20) is defined as a solution of the Neumann boundary value problem on a scaled cross-section of a cylinder $G_\varepsilon^{(i)}$

$$\begin{aligned} -\Delta_\zeta U_0^{(i)}(\zeta, z) = \Phi_0^{(i)}(\zeta, z) - (\mathrm{mes}_2\, g_i)^{-1}\bigg(\int_{g_i} \Phi_0^{(i)}(\zeta, z)\,\mathrm{d}\zeta \\ + \int_{\partial g_i} \Psi_0^{(i)}(\zeta, z)\,\mathrm{d}s_\zeta \bigg), \quad \zeta \in g_i, \end{aligned} \tag{6.23}$$

$$\frac{\partial U_0^{(i)}}{\partial n_\zeta}(\zeta, z) = \Psi_0^{(i)}(\zeta, z), \quad \zeta \in \partial g_i, \tag{6.24}$$

satisfying the orthogonality condition

$$\int_{g_i} U_0^{(i)}(\zeta, z)\,\mathrm{d}z = 0.$$

Thus, one can see that the function (6.20) is determined by the terms $\varepsilon^{-2}\Phi_0^{(i)}$, $\varepsilon^{-1}\Psi_0^{(i)}$ of the expansion (6.1), (6.2).

Next, we multiply (6.20) by a smooth cut-off function $\Upsilon(x_3/\varepsilon)$ and extend

$$\Upsilon\left(\frac{x_3}{\varepsilon}\right)\left\{\varepsilon^{-2} W_0^{(i)}(x_3) + U_0^{(i)}(\zeta^{(i)}, x_3)\right\}$$

as zero into Ω_0.

The substitution in (2.1)–(2.3) gives

$$\begin{aligned} &-\Delta_x\left(\Upsilon\left(\frac{x_3}{\varepsilon}\right)\left\{\varepsilon^{-2} W_0^{(i)}(x_3) + U_0^{(i)}(\zeta^{(i)}, x_3)\right\}\right) \\ &= \Upsilon\left(\frac{x_3}{\varepsilon}\right)\left(\varepsilon^{-2}\Phi_0^{(i)}(\zeta^{(i)}, x_3) + \mathrm{O}(1)\right) \\ &\quad - \left[\Delta_x, \Upsilon\left(\frac{x_3}{\varepsilon}\right)\right]\left\{\varepsilon^{-2} W_0^{(i)}(x_3) + U_0^{(i)}(\zeta^{(i)}, x_3)\right\}, \quad x \in G_\varepsilon^{(i)}, \end{aligned} \tag{6.25}$$

$$\begin{aligned}&\frac{\partial}{\partial n_x}\big(\varepsilon^{-2}W_0^{(i)}(x_3)+U_0^{(i)}(\zeta^{(i)},x_3)\big)\\&\quad=\Upsilon\Big(\frac{x_3}{\varepsilon}\Big)\varepsilon^{-1}\Psi_0^{(i)}(\zeta^{(i)},x_3),\quad x\in\partial G_\varepsilon^{(i)}\setminus S_\varepsilon^{(i)},\end{aligned}\tag{6.26}$$

$$\begin{aligned}&\Upsilon\Big(\frac{x_3}{\varepsilon}\Big)\{\varepsilon^{-2}W_0^{(i)}(x_3)+U_0^{(i)}(\zeta^{(i)},x_3)\}\big|_{x\in S_\varepsilon^{(i)}}\\&\quad=-\varepsilon^{-2}(\mathrm{mes}_2\,g_i)^{-1}\int_0^{l_i}(l_i-t)\\&\qquad\times\Big\{\int_{g_i}\Phi_0^{(i)}(\zeta,t)\,\mathrm{d}\zeta+\int_{\partial g_i}\Psi_0^{(i)}(\zeta,t)\,\mathrm{d}s_\zeta\Big\}\,\mathrm{d}t+\mathrm{O}(1).\end{aligned}\tag{6.27}$$

Let us note that the term

$$[\Delta_x,\Upsilon]\Big(\frac{x_3}{\varepsilon}\Big)\{\varepsilon^{-2}W_0^{(i)}(x_3)+U_0^{(i)}(\zeta^{(i)},x_3)\}$$

from the right-hand side (6.25) has a support in the ε-neighbourhood of the point $x=a^{(i)}$, where due to (4.15) $W_0^{(i)}(x_3)=\mathrm{O}(\varepsilon^2)$.

Also we would like to mention that the principal part of the right-hand side in (6.27) has the same order as the first term in (6.3).

The boundary condition on the base of a thin cylinder can be satisfied by means of the bottom layer $\varepsilon^{-2}v_0^{(i)}(Y^{(i)})$, which is the solution of a mixed boundary value problem of the type (3.24)–(3.26) in a semi-infinite cylinder $\Pi^{(i)}$. Obviously, the right-hand side of the problem (3.24)–(3.26), corresponding to the expansions (6.1)–(6.3), (6.17), (6.27), does not satisfy, in general, the condition (3.29) of the exponential vanishing of $v_0^{(i)}$ at infinity. Let us introduce the function $\varepsilon^{-1}\Xi^{(i)}(X^{(i)})$, a solution of the homogeneous problem (3.16), (3.17), defined in (3.21), (3.22). We remind the reader that the function $\Xi^{(i)}(X)$ has a linear principal term of the asymptotics (3.21) in a cylinder and decays as $\mathrm{O}(|X|^{-1})$ in a half-space $\mathbb{R}^3_-$.

The following asymptotic equality holds (see (3.21), (6.17), (6.27))

$$\begin{aligned}&\Big\{\varepsilon^{-1}\varkappa(x-a^{(i)})\big(W_0^{(i)}(X^{(i)})+b_0^{(i)}\Xi^{(i)}(X^{(i)})\big)\\&\qquad+\Upsilon\Big(\frac{x_3}{\varepsilon}\Big)\big(\varepsilon^{-2}W_0^{(i)}(x_3)+U_0^{(i)}(\zeta^{(i)},x_3)\big)\Big\}\Big|_{S_\varepsilon^{(i)}}\\&\quad=\varepsilon^{-2}\Big\{-l_i(\mathrm{mes}_2\,g_i)^{-1}\Big[\int_{\omega_i}\mathcal{F}_0^{(i)}(X)\,\mathrm{d}X+\int_{\partial\omega_i}\mathcal{P}_0^{(i)}(X)\,\mathrm{d}s_X\Big]\\&\qquad+b_0^{(i)}l_i(\mathrm{mes}_2\,g_i)^{-1}-(\mathrm{mes}_2\,g_i)^{-1}\int_0^{l_i}(l_i-t)\Big[\int_{g_i}\Phi_0^{(i)}(\zeta,t)\,\mathrm{d}\zeta\\&\qquad+\int_{\partial g_i}\Psi_0^{(i)}(\zeta,t)\,\mathrm{d}s_\zeta\Big]\,\mathrm{d}t\Big\}+\mathrm{O}(\varepsilon^{-1}),\end{aligned}\tag{6.28}$$

where $b_0^{(i)}$ is a constant, which provides zero for the value (3.28), and therefore the exponential decay at infinity for $v_0^{(i)}$.

Thus, the function $v_0^{(i)}$ satisfies the problem (3.24)–(3.26) with the right-hand sides

$$
\begin{aligned}
&G(Y) = G_0^{(i)}(Y), \quad Y \in \Pi^{(i)}, \\
&H(Y) = H_0^{(i)}(Y), \quad Y \in \Pi^{(i)} \setminus \overline{S}^{(i)}, \\
&\psi(Y) = \phi_0^{(i)}(Y) + (\operatorname{mes}_2 g_i)^{-1} \int_0^{l_i} (l_i - t)\Big\{ \int_{g_i} \Phi_0^{(i)}(\zeta, t)\, \mathrm{d}\zeta \\
&\qquad + \int_{\partial g_i} \Psi_0^{(i)}(\zeta, t)\, \mathrm{d}s_\zeta \Big\}\, \mathrm{d}t \\
&\qquad + (\operatorname{mes}_2 g_i)^{-1} l_i \Big\{ \int_{\omega_i} \mathcal{F}_0^{(i)}(X)\, \mathrm{d}X + \int_{\partial\omega_i} \mathcal{P}_0^{(i)}(X)\, \mathrm{d}s_X \Big\} \\
&\qquad - b_0^{(i)} l_i (\operatorname{mes}_2 g_i)^{-1},
\end{aligned}
\tag{6.29}
$$

where

$$
\begin{aligned}
b_0^{(i)} = l_i^{-1} \Big[&\int_{S^{(i)}} \phi_0^{(i)}(Y)\, \mathrm{d}Y + \int_{\Pi^{(i)}} Y_3 G_0^{(i)}(Y)\, \mathrm{d}Y \\
&+ \int_{\partial\Pi^{(i)} \setminus S^{(i)}} Y_3 H_0^{(i)}(Y)\, \mathrm{d}s_Y \\
&+ \int_0^{l_i} (l_i - t) \Big\{ \int_{g_i} \Phi_0^{(i)}(\zeta, t)\, \mathrm{d}\zeta + \int_{\partial g_i} \Psi_0^{(i)}(\zeta, t)\, \mathrm{d}s_\zeta \Big\}\, \mathrm{d}t \\
&+ l_i \Big\{ \int_{\omega_i} \mathcal{F}_0^{(i)}(X)\, \mathrm{d}X + \int_{\partial\omega_i} \mathcal{P}_0^{(i)}(X)\, \mathrm{d}s_X \Big\} \Big].
\end{aligned}
\tag{6.30}
$$

Now, we substitute the sum

$$
\begin{aligned}
&\sum_{i=1}^{K} \Big\{ \varepsilon^{-1} \varkappa(x - a^{(i)}) \big(\mathcal{W}_0^{(i)}(X^{(i)}) + b_0^{(i)} \Xi^{(i)}(X^{(i)}) \big) \\
&\qquad + \Upsilon\Big(\frac{x_3}{\varepsilon}\Big) \big(\varepsilon^{-2} W_0^{(i)}(x_3) + U_0^{(i)}(\zeta^{(i)}, x_3) \big) + v_0^{(i)}(Y^{(i)}) \Upsilon\Big(\frac{x_3}{\varepsilon}\Big) \Big\} \\
&\equiv \sum_{i=1}^{K} \mathcal{A}^{(i)}(x, \varepsilon)
\end{aligned}
$$

into the equation (2.1) and the boundary condition (2.2) in Ω_0.

Thus, we have

$$-\Delta_x \sum_{i=1}^{K} \mathcal{A}^{(i)}(x,\varepsilon) = -\sum_{i=1}^{K} [\Delta_x, \varkappa(x-a^{(i)})] \frac{b_0^{(i)}}{2\pi\|x-a^{(i)}\|} + \mathrm{O}(\varepsilon), \quad x \in \Omega_0, \tag{6.31}$$

$$\frac{\partial}{\partial n_x} \sum_{i=1}^{K} \mathcal{A}^{(i)}(x,\varepsilon) = \mathrm{O}(\varepsilon), \quad x \in \partial\Omega_0. \tag{6.32}$$

Due to (6.31) we have to turn back to the function $\mathcal{U}$, defined by the equality (5.18); we should emphasize that the integral over Ω_0 of the principal term in (6.31) is not equal to zero (the right-hand side is not self-balanced).

$$\int_{\Omega_0} \sum_{i=1}^{K} [\Delta_x, \varkappa(x-a^{(i)})] \frac{b_0^{(i)}}{2\pi\|x-a^{(i)}\|}\, \mathrm{d}x = \sum_{i=1}^{K} b_0^{(i)}. \tag{6.33}$$

Finally, the function Θ_0 from (6.4) can be written down as

$$\begin{aligned} \Theta_0(x,\varepsilon) = \sum_{i=1}^{K} \Big\{ & \varepsilon^{-1}\varkappa(x-a^{(i)})\big(\mathcal{W}_0^{(i)}(X^{(i)}) + b_0^{(i)}\Xi^{(i)}(X^{(i)})\big) \\ & + \Upsilon\Big(\frac{x_3}{\varepsilon}\Big)\big(\varepsilon^{-2}W_0^{(i)}(x_3) + U_0^{(i)}(\zeta^{(i)}, x_3) + v_0^{(i)}(Y^{(i)})\big)\Big\} \\ & + M_0(x)\prod_{i=1}^{K}\chi(X^{(i)}), \end{aligned} \tag{6.34}$$

where $M_0(x) = \alpha_0(x) + \mu_0(x)$, α_0 is a smooth solution of the problem

$$-\Delta\alpha_0(x) = \sum_{i=1}^{K} [\Delta_x, \varkappa(x-a^{(i)})] \frac{b_0^{(i)}}{2\pi\|x-a^{(i)}\|} - (\operatorname{mes}\Omega_0)^{-1}\sum_{i=1}^{K} b_0^{(i)}, \quad x \in \Omega_0, \tag{6.35}$$

$$\frac{\partial\alpha_0}{\partial n_x}(x) = 0, \quad x \in \partial\Omega_0, \tag{6.36}$$

satisfying the orthogonality condition

$$\int_{\Omega_0} \alpha_0(x)\, \mathrm{d}x = 0. \tag{6.37}$$

Let us define the constant I_0 from (6.5) as follows

$$I_0 = \int_{\Omega_0} f_0(x)\, \mathrm{d}x + \int_{\partial\Omega_0} p_0(x)\, \mathrm{d}s + \sum_{i=1}^{K} b_0^{(i)}. \tag{6.38}$$

Let us substitute the sum

$$I_0\mathcal{U}(x,\varepsilon)+\Theta_0(x,\varepsilon)$$

into the equation (2.1) and boundary conditions (2.2), (2.3). Due to (6.14), (6.15), (6.18), (6.19), (6.25)–(6.27), (6.31), (6.32) we obtain

$$\begin{aligned}
&\Delta_x[I_0\mathcal{U}(x,\varepsilon)+\Theta_0(x,\varepsilon)]\\
&\quad= f_0(x)\prod_{i=1}^{K}\chi(X^{(i)})-\varepsilon^{-2}\sum_{i=1}^{K}[\Delta_X,\chi(X^{(i)})]\big(\mu_0(a^{(i)})+\alpha_0(a^{(i)})+O(\varepsilon)\big)\\
&\quad-\sum_{i=1}^{K}[\Delta_x,\varkappa(x-a^{(i)})]\Big(\varepsilon\frac{Y_1(\theta)}{\|x-a^{(i)}\|^2}+O(\varepsilon^2)\Big)\\
&\quad+\sum_{i=1}^{K}[\varepsilon^{-3}\mathcal{F}_0^{(i)}(X^{(i)})+O(\varepsilon^{-2})]\\
&\quad+\sum_{i=1}^{K}\Big[\varepsilon^{-4}G_0^{(i)}(Y^{(i)})+\Upsilon\Big(\frac{x_3}{\varepsilon}\Big)(\varepsilon^{-2}\Phi_0^{(i)}(\zeta^{(i)},x_3)+O(1))\\
&\quad-\Big[\Delta_x,\Upsilon\Big(\frac{x_3}{\varepsilon}\Big)\Big]\{\varepsilon^{-2}W_0^{(i)}(x_3)+U_0^{(i)}(\zeta^{(i)},x_3)+O(1)\}\Big],\quad x\in\Omega_\varepsilon,
\end{aligned}\tag{6.39}$$

$$\begin{aligned}
&\frac{\partial}{\partial n_x}\{I_0\mathcal{U}(x,\varepsilon)+\Theta_0(x,\varepsilon)\}\\
&\quad= p_0(x)\prod_{i=1}^{K}\chi(X^{(i)})+\varepsilon^{-1}\sum_{i=1}^{K}n_x\cdot\nabla_X\chi(X^{(i)})\big(\mu_0(a^{(i)})+\alpha_0(a^{(i)})+O(\varepsilon)\big)\\
&\quad+\sum_{i=1}^{K}n_x\cdot\nabla\Big\{\varkappa(x-a^{(i)})\Big(\varepsilon\frac{\mathbf{Y}_1(\theta)}{\|x-a^{(i)}\|^2}+O(\varepsilon^2)\Big)\Big\}\\
&\quad+\sum_{i=1}^{K}[\varepsilon^{-2}\mathcal{P}_0^{(i)}(X^{(i)})+O(\varepsilon^{-1})]\\
&\quad+\sum_{i=1}^{K}\Big[\varepsilon^{-3}H_0^{(i)}(Y^{(i)})+\Upsilon\Big(\frac{x_3}{\varepsilon}\Big)(\varepsilon^{-1}\Psi_0^{(i)}(\zeta^{(i)},x_3)+O(\varepsilon))\Big],\\
&\quad x\in\partial\Omega_\varepsilon\setminus\bigcup_i S_\varepsilon^{(i)},
\end{aligned}\tag{6.40}$$

$$\{I_0\mathcal{U}(x,\varepsilon)+\Theta_0(x,\varepsilon)\}\big|_{S_\varepsilon^{(i)}}=\varepsilon^{-2}\phi_0^{(i)}+O(\varepsilon^{-1}).\tag{6.41}$$

A simple check gives that

$$F(x,\varepsilon) + \Delta_x\{I_0\mathcal{U}(x,\varepsilon) + \Theta_0(x,\varepsilon)\},$$
$$P(x,\varepsilon) - \frac{\partial}{\partial n_x}\{I_0\mathcal{U}(x,\varepsilon) + \Theta_0(x,\varepsilon)\},$$
$$\psi^{(i)}(x,\varepsilon) - \{I_0\mathcal{U}(x,\varepsilon) + \Theta_0(x,\varepsilon)\}\big|_{S_\varepsilon^{(i)}}$$

have the structure (6.1)–(6.3) for $k \geqslant 1$.

Taking the next step and repeating the above described algorithm, we obtain the general term u_k (see the asymptotic expansion (6.4)), where the coefficient I_k is defined by

$$I_k = \int_{\Omega_0} f_k(x)\,\mathrm{d}x + \int_{\partial\Omega_0} p_k(x)\,\mathrm{d}s + \sum_{i=1}^{K} b_k^{(i)}, \tag{6.42}$$

and

$$\begin{aligned}\Theta_k(x,\varepsilon) = \sum_{i=1}^{K}\Big\{&\varepsilon^{-1}\varkappa(x-a^{(i)})\big(\mathcal{W}_k^{(i)}(X^{(i)}) + b_k^{(i)}\Xi^{(i)}(X^{(i)})\big)\\ &+\Upsilon\Big(\frac{x_3}{\varepsilon}\Big)\big(\varepsilon^{-2}W_k^{(i)}(x_3) + U_k^{(i)}(\zeta^{(i)},x_3) + \varepsilon^{-2}v_k^{(i)}(Y^{(i)})\big)\Big\}\\ &+ M_k(x)\prod_{i=1}^{K}\chi(X^{(i)}).\end{aligned} \tag{6.43}$$

Here $\mathcal{W}_k^{(j)}$ is a junction layer, satisfying the problem of the type (3.16), (3.17), with the asymptotics (3.18)–(3.20); $\Xi^{(i)}$ is a solution of a homogeneous problem (3.16), (3.17) with the asymptotic behaviour (3.21), (3.22); $v_k^{(i)}$ is a bottom layer, introduced in the Section 3.3; the coefficients $b_k^{(i)}$ are defined from the condition (similar to (6.30)) that provides an exponential vanishing of $v_k^{(i)}$ at infinity in $\Pi^{(i)}$; as shown in Section 4, the sum $\varepsilon^{-2}W_k^{(i)}(x_3) + U_k^{(i)}(\zeta^{(i)},x_3)$ is defined by terms of the right-hand side expansion, concentrated in a thin cylinder $G_\varepsilon^{(i)}$; $M_k(x)$ is a smooth solution of the problem of the type (6.10), (6.11) with the orthogonality conditions (6.12), (6.13).

7. The remainder estimate

The following proposition holds

Theorem 1. *The solution $u(x,\varepsilon)$ of the problem* (2.1)–(2.3) *with the right-hand sides* (6.1)–(6.3) *has the asymptotic representation* (6.4). *A finite sum of the series* (6.4)

$$\mathrm{s}_k = \sum_{i=0}^{k} \varepsilon^i u_i(x,\varepsilon) \tag{7.1}$$

admits the estimate

$$\int_{\Omega_\varepsilon} |\nabla(u - \mathrm{s}_k)|^2 \, dx + \int_{\Omega_\varepsilon} |u - \mathrm{s}_k|^2 \, dx \leqslant \mathrm{const}\, \varepsilon^{2(k-1)}, \tag{7.2}$$

where the constant does not depend on ε.

Before the proof of the theorem we present an auxiliary statement.

Lemma 1. *For the solution of the problem* (2.1)–(2.3) *with the right sides of the form* (6.1)–(6.3) *the following inequality is valid*

$$\|u\|_{H^1(\Omega_\varepsilon)} \leqslant \mathrm{const}\, \varepsilon^{-\sigma} \left\{ \|F\|_{L_2(\Omega_\varepsilon)} + \|P\|_{L_2(\partial\Omega_\varepsilon \setminus (\cup_i S_\varepsilon^{(i)}))} + \sum_{i=1}^{N} \|\phi^{(i)}\|_{H^1(S_\varepsilon^{(i)})} \right\}, \tag{7.3}$$

where the constant does not depend on ε; σ *is an absolute constant* (*its exact value is not in use*).

Proof. By virtue of a standard technique one can derive the inequality

$$\|u\|_{L_2(\partial\Omega_\varepsilon)} + \|u\|_{L_2(\Omega_\varepsilon)} \leqslant \mathrm{const}\, \varepsilon^{-1} \Big(\|\nabla u\|_{L_2(\Omega_\varepsilon)} + \|u\|_{L_2\left(\bigcup_i S_\varepsilon^{(i)}\right)} \Big), \tag{7.4}$$

which is valid for all functions $u \in H^1(\Omega_\varepsilon)$; here the constant does not depend on ε. Using (7.4), we arrive to the result of the lemma. □

Now, we start the proof of the Theorem.

Proof. Let us choose a large enough number n of terms of the asymptotic series (6.4) ($n > k$). Then we have

$$\begin{aligned} &\int_{\Omega_\varepsilon} |\nabla(u - \mathrm{s}_k)|^2 \, dx + \int_{\Omega_\varepsilon} |u - \mathrm{s}_k|^2 \, dx \\ &\quad \leqslant 2\Big(\int_{\Omega_\varepsilon} |\nabla(u - \mathrm{s}_n)|^2 \, dx + \int_{\Omega_\varepsilon} |u - \mathrm{s}_n|^2 \, dx \\ &\qquad + \int_{\Omega_\varepsilon} |\nabla(\mathrm{s}_n - \mathrm{s}_k)|^2 \, dx + \int_{\Omega_\varepsilon} |\mathrm{s}_n - \mathrm{s}_k|^2 \, dx \Big). \end{aligned} \tag{7.5}$$

We apply the inequality (7.3) to estimate the first two terms. Let us denote by F_n, P_n, $\phi_n^{(i)}$ the right-hand sides of the problem (2.1)–(2.3) for $u - \mathrm{s}_k$. Due to the asymptotic procedure of deriving of the series (6.4) we have the estimate

$$\|F_n\|_{L_2(\Omega_\varepsilon)} + \|P_n\|_{L_2(\partial\Omega_\varepsilon)} + \sum_{i=1}^{N} \|\phi_n^{(i)}\|_{H^1(S_\varepsilon^{(i)})} \leqslant \mathrm{const}\, \varepsilon^{n-1}. \tag{7.6}$$

By virtue of (7.3) we obtain

$$\left(\int_{\Omega_\varepsilon} |\nabla(u - \mathrm{s}_n)|^2 \, dx + \int_{\Omega_\varepsilon} |u - \mathrm{s}_n|^2 \, dx \right)^{1/2} \leqslant \mathrm{const}\, \varepsilon^{n-1-\sigma}, \tag{7.7}$$

where the constant does not depend on ε.

To estimate the last two terms in (7.5) we define the order (with respect to ε) of the $k+1$ term of the series (6.4).

$$\int_{\Omega_\varepsilon} |\nabla(\mathrm{s}_n - \mathrm{s}_k)|^2 \,\mathrm{d}x + \int_{\Omega_\varepsilon} |\mathrm{s}_n - \mathrm{s}_k|^2 \,\mathrm{d}x \leqslant \mathrm{const}\, \varepsilon^{2(k-1)}. \tag{7.8}$$

Choosing $n \geqslant k + \sigma$ and taking into account (7.5), (7.7), (7.8), we arrive at (7.2). □

8. The analysis of the asymptotics in different zones and a simplified scheme of calculations

In the present section we perform the analysis of the principal term of the formal asymptotics (6.4) for a particular case of the formulation (2.1)–(2.3). As an illustration we consider a mixed boundary value problem in Ω_ε with the right side, represented in the form

$$\begin{aligned} -\Delta_x u(x,\varepsilon) = f(x) + \sum_{i=1}^{K} \{ & \varepsilon^{-3} \mathcal{F}^{(i)}(X^{(i)}) \\ & + \varepsilon^{-2} \Phi^{(i)}(\zeta^{(i)}, x_3) + \varepsilon^{-4} G^{(i)}(Y^{(i)}) \}, \quad x \in \Omega_\varepsilon, \end{aligned} \tag{8.1}$$

$$\begin{aligned} \frac{\partial u}{\partial n_x}(x,\varepsilon) = p(x) + \sum_{i=1}^{K} \{ & \varepsilon^{-2} \mathcal{P}^{(i)}(X^{(i)}) + \varepsilon^{-1} \Psi^{(i)}(\zeta^{(i)}, x_3) \\ & + \varepsilon^{-3} H^{(i)}(Y^{(i)}) \}, \quad x \in \partial\Omega_\varepsilon \setminus \bigcup_i \bar{S}_\varepsilon^{(i)}, \end{aligned} \tag{8.2}$$

$$u(x,\varepsilon) = \varepsilon^{-2} \phi^{(i)}(Y^{(i)}), \quad i = 1, \ldots, K, \ x \in S_\varepsilon^{(i)}. \tag{8.3}$$

Here $X^{(i)}$, $Y^{(i)}$, $\zeta^{(i)}$ are local stretched variables, introduced in Section 6; $\mathcal{F}^{(i)}, \mathcal{P}^{(i)}$ and G_i, H_i define the right-hand sides of the problems for the junction layer and for the bottom layer, respectively (as above, we suppose that these functions have compact supports); functions Φ_i and Ψ_i depend on the "fast" variable on the scaled cross-section of a cylinder and on the "slow" variable along the axis of a cylinder; we assume that these functions vanish in the vicinity of the junction points. We also suppose that $\partial\Omega_0$ is a smooth boundary, smooth functions f, p vanish in neighbourhoods of the junction points and can be extended as zero into thin cylinders.

In the sense of the physical interpretation the solution of the problem (8.1)–(8.3) gives the temperature in Ω_ε with prescribed distribution of internal sources, the values of the temperature on the bases $S_\varepsilon^{(i)}$, $i = 1, \ldots, K$, of thin cylinders and prescribed heat flux through the lateral surface.

(i) *A three-dimensional set Ω_0*

As described in Section 3, the principal part of the solution of the problem (8.1)–(8.3) outside neighbourhoods of the junction points in Ω_0 satisfies the following equation in Ω_0 and the Neumann boundary condition on $\partial\Omega_0$.

$$-\Delta_x u_0(x,\varepsilon) = f(x), \quad x \in \Omega_0, \tag{8.4}$$

$$\frac{\partial u_0}{\partial n_x}(x,\varepsilon) = p(x) + \sum_{i=1}^{K} \mathcal{T}^{(i)}\delta\left(x - a^{(i)}\right), \quad x \in \partial\Omega_0. \tag{8.5}$$

It is assumed that the elements of the right-hand side satisfy the relation

$$\sum_{i=1}^{K} \mathcal{T}^{(i)} + \int_{\Omega_0} f(x)\,\mathrm{d}x + \int_{\partial\Omega_0} p(x)\,\mathrm{d}s = 0 \tag{8.6}$$

which is the "heat flux balance" condition for Ω_0.

We are looking for a solution with the zero mean value over Ω_0

$$\int_{\Omega_0} u_0(x,\varepsilon)\,\mathrm{d}x = 0. \tag{8.7}$$

Let us mention that the function u_0 is defined within an arbitrary additive constant involving the small parameter ε.

Thus, outside neighbourhoods $V(a^{(i)})$ of the junction points $a^{(i)}$, $i = 1,\dots,K$, in Ω_0 the asymptotic relations hold

$$u = \varepsilon^{-2}C + \mathrm{O}(\varepsilon^{-1}), \quad x \in \Omega_0 \setminus \left(\bigcup_i \overline{V}(a^{(i)})\right), \tag{8.8}$$

$$\nabla u = \nabla\mu(x) + \sum_{i=1}^{K} \mathcal{T}^{(i)}\nabla\mathrm{N}(x, a^{(i)}) + \mathrm{O}(\varepsilon), \quad x \in \Omega_0 \setminus \left(\bigcup_i \overline{V}(a_{(i)})\right), \tag{8.9}$$

where μ is a smooth solution of problem (3.5)–(3.7), satisfying the orthogonality condition (3.8); N is the Neumann function; C, $\mathcal{T}^{(i)}$, $i = 1,\dots,K$, are unknown constant.

We would like to emphasize that the representation (8.9) involves terms with an infinite value of energy integral. To describe the behaviour of the solution in the vicinity of the points $a^{(i)}$, $i = 1,\dots,K$, we shall consider the problem of the junction layer type.

(ii) *Neighbourhoods of the junction points*

We introduce the junction layer functions $\varepsilon^{-1}\mathcal{R}^{(i)}(X^{(i)})$, $i = 1,2,\dots,K$, which satisfy the following equation and boundary condition in a union of a half-space and a semi-infinite cylinder (see Sections 3, 6).

$$-\Delta_X \mathcal{R}^{(i)}(X^{(i)}) = \mathcal{F}^{(i)}(X^{(i)}), \quad x \in \omega_i, \tag{8.10}$$

$$\frac{\partial \mathcal{R}^{(i)}}{\partial n_X}(X^{(i)}) = \mathcal{P}^{(i)}(X^{(i)}), \quad x \in \partial\Omega_i. \tag{8.11}$$

To match the solution of this problem with singular terms of the right-hand side (8.9), we seek $\mathcal{R}^{(i)}$ in the class of functions which behave at infinity in R^3_- like

$$\frac{\mathcal{T}^{(i)}}{2\pi\|X\|} + \mathrm{O}(\|X\|^{-2}),$$

and increase at most linearly in a semi-infinite cylinder $\Pi^{(i)}$.

$$\begin{aligned}\mathcal{R}^{(i)}(X^{(i)}) = -X_3^{(i)}(\mathrm{mes}_2\, g_i)^{-1}\Big\{ \int_{\omega_i} \mathcal{F}^{(i)}(X^{(i)})\,\mathrm{d}X^{(i)} \\ + \int_{\partial\omega_i} \mathcal{P}^{(i)}(X^{(i)})\,\mathrm{d}s_X - \mathcal{T}^{(i)}\Big\} + \mathrm{O}(1), \quad X_3^{(i)} \to +\infty.\end{aligned} \tag{8.12}$$

Let us note that in terms of notations of Sections 3, 6 the function $\mathcal{R}^{(i)}$ can be represented as the sum

$$\mathcal{R}^{(i)}(X^{(i)}) = \mathcal{W}^{(i)}(X^{(i)}) + \mathcal{T}^{(i)}\Xi^{(i)}(X^{(i)}).$$

(iii) *A thin rod* $G_\varepsilon^{(i)}$

In the middle zone of a cylinder $G_\varepsilon^{(i)}$ the following asymptotic equality holds

$$u(x,\varepsilon) = \varepsilon^{-2}w^{(i)}(x_3) + \mathrm{O}(\varepsilon^{-1}), \quad x \in G_\varepsilon^{(i)} \setminus \big(\overline{V}(a^{(i)}) \cup \overline{V}(b^{(i)})\big). \tag{8.13}$$

Here $V(a^{(i)})$, $V(b^{(i)})$ are neighbourhoods of the points $a^{(i)}, b^{(i)} \in S_\varepsilon^{(i)}$.

One can easily derive the representation for $w^{(i)}$ by virtue of (6.20), (8.8), (8.12)

$$\begin{aligned}w^{(i)}(x_3) = -x_3(\mathrm{mes}_2\, g_i)^{-1}\Big\{ \int_{\omega_i} \mathcal{F}^{(i)}(X)\,\mathrm{d}X \\ + \int_{\partial\omega_i} \mathcal{P}^{(i)}(X)\,\mathrm{d}s_X - \mathcal{T}^{(i)}\Big\} + W^{(i)}(x_3) + C,\end{aligned} \tag{8.14}$$

where the function $W^{(i)}$ satisfies the Cauchy problem (6.21), (6.22) on $(0, l^{(i)})$.

Obviously, the function $w^{(i)}$ satisfies the problem

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} w^{(i)}(z) = \mathcal{B}^{(i)}(z), \quad z \in (0, l^{(i)}), \tag{8.15}$$

$$w^{(i)}(0) = C, \tag{8.16}$$

$$\frac{\mathrm{d}w^{(i)}}{\mathrm{d}z}(0) = \mathcal{Q}^{(i)} \tag{8.17}$$

where

$$\mathcal{B}^{(i)}(z) = -(\mathrm{mes}_2\, g_i)^{-1}\Big\{ \int_{g_i} \Phi^{(i)}(\zeta, z)\,\mathrm{d}\zeta + \int_{\partial g_i} \Psi^{(i)}(\zeta, z)\,\mathrm{d}s_\zeta \Big\}, \quad z \in (0, l^{(i)}), \tag{8.18}$$

$$\mathcal{Q}^{(i)} = (\mathrm{mes}_2\, g_i)^{-1}\Big\{ -\int_{\omega_i} \mathcal{F}^{(i)}(X)\,\mathrm{d}X - \int_{\partial\omega_i} \mathcal{P}^{(i)}(X)\,\mathrm{d}s_X + \mathcal{T}^{(i)}\Big\}. \tag{8.19}$$

The solution of the problem (8.15)–(8.17) can be represented as follows

$$w^{(i)} = \int_0^z (z-t)\mathcal{B}^{(i)}(t)\,\mathrm{d}t + \mathcal{Q}^{(i)}z + C.$$

Unknown constants $\mathcal{T}^{(i)}$ and C will be obtained by means of the heat flux balance conditions in part (v) of the present section.

(iv) *Neighbourhoods of the bases of thin cylinders*

The trace of the function $w^{(i)}$ on the base $S_\varepsilon^{(i)}$ of a thin cylinder has the form

$$w^{(i)}(l^{(i)}) = \int_0^{l^{(i)}} (l^{(i)}-t)\mathcal{B}^{(i)}(t)\,\mathrm{d}t + \mathcal{Q}^{(i)}l^{(i)} + C. \tag{8.20}$$

The Dirichlet condition (8.3) on the base $S_\varepsilon^{(i)}$ can be satisfied by means of the bottom layer $\varepsilon^{-2}v^{(i)}$ (see (3.24)–(3.26)). Due to (6.29), (8.14), (8.20) the function $v^{(i)}(Y^{(i)})$ (here $Y^{(i)}$ are scaled coordinates, introduced in the vicinity of $S_\varepsilon^{(i)}$; see Section 3) satisfies the boundary value problem

$$-\Delta_Y v^{(i)}(Y) = G^{(i)}(Y), \quad Y \in \Pi^{(i)}, \tag{8.21}$$

$$\frac{\partial v^{(i)}}{\partial n_Y}(Y) = H^{(i)}(Y), \quad Y \in \Pi^{(i)} \setminus \overline{S}^{(i)}, \tag{8.22}$$

$$v^{(i)}(Y) = \phi^{(i)}(Y) - \int_0^{l^{(i)}} (l^{(i)}-t)\mathcal{B}^{(i)}(t)\,\mathrm{d}t - \mathcal{Q}^{(i)}l^{(i)} - C. \tag{8.23}$$

The solution of the problem (8.21)–(8.23) vanishes at infinity if and only if the following equality holds (see (3.28), (3.29))

$$\begin{aligned}&\int_{\Pi^{(i)}} Y_3 G^{(i)}(Y)\,\mathrm{d}Y + \int_{\partial\Pi^{(i)}\setminus\overline{S}^{(i)}} Y_3 H^{(i)}(Y)\,\mathrm{d}s_Y + \int_{S^{(i)}} \phi^{(i)}(Y)\,\mathrm{d}s_Y \\ &\quad - \mathcal{Q}^{(i)}l^{(i)}\,\mathrm{mes}_2\, g_i - C\,\mathrm{mes}_2\, g_i - \mathrm{mes}_2\, g_i \int_0^{l^{(i)}} (l^{(i)}-t)\mathcal{B}^{(i)}(t)\,\mathrm{d}t = 0.\end{aligned} \tag{8.24}$$

Let us note that in the sense of the temperature distribution the last relation gives the balance condition for the principal part of the heat flux through the surface of a thin rod $G_\varepsilon^{(i)}$. In this case the principal part of the temperature is defined by the solution of the problem (8.15), (8.17), (8.20), where we prescribe the value of the heat flux at the end $z = 0$ and the value of the temperature at $z = l^{(i)}$.

(v) *The calculation of the constants* C, $\mathcal{T}^{(i)}$, $i = 1, \dots, K$

To calculate unknown constants from (8.8), (8.9) we shall use the system of equations which consists of the "heat flux balance" conditions for all elements of the multi-structure. By virtue of (8.19), (8.24) one can obtain

$$\begin{aligned}
&\int_{\omega_i} \mathcal{F}^{(i)}(X)\,\mathrm{d}X + \int_{\partial\omega_i} \mathcal{P}^{(i)}(X)\,\mathrm{d}s_X - \mathcal{T}^{(i)} \\
&\quad + \left(l^{(i)}\right)^{-1}\left(\int_{\Pi^{(i)}} Y_3 G^{(i)}(Y)\,\mathrm{d}Y + \int_{\partial\Pi^{(i)}\setminus\overline{S}^{(i)}} Y_3 H^{(i)}(Y)\,\mathrm{d}s_Y\right. \\
&\quad \left. + \int_{S^{(i)}} \phi^{(i)}(Y)\,\mathrm{d}s_Y\right) \\
&\quad - C\left(l^{(i)}\right)^{-1} \mathrm{mes}_2\, g_i - \mathrm{mes}_2\, g_i \left(l^{(i)}\right)^{-1} \int_0^{l^{(i)}} \left(l^{(i)} - t\right) \mathcal{B}^{(i)}(t)\,\mathrm{d}t = 0, \\
&\quad i = 1, 2, \dots, K.
\end{aligned} \tag{8.25}$$

The system of equations (8.6), (8.25) with respect to unknown constants C, $\mathcal{T}^{(i)}$, $i = 1, 2, \dots, K$, can be easily solved. As a result we derive

$$C = I_0 \left[\sum_{i=1}^{K} l_i^{-1}\, \mathrm{mes}_2\, g_i\right]^{-1}, \tag{8.26}$$

where the constant I_0 was defined by (6.38); the distribution of sources with intensities $\mathcal{T}^{(i)}$ at the points $x = a^{(i)}$, $i = 1, \dots, K$, is determined by the relation

$$\mathcal{T}^{(i)} = I_0 T^{(i)} + b_0^{(i)}, \tag{8.27}$$

which agrees with the representation (6.9); again we use notations introduced in Sections 5 and 6 (see (5.17), (6.30)).

Thus, simple formulae (8.26), (8.27) enable us to obtain the constants C, $\mathcal{T}^{(i)}$, that we need to determine the solution of the problem in Ω_0 (see (8.8), (8.9)). The heat flux values $\mathcal{Q}^{(i)}$ at the junction points $a^{(i)}$, $i = 1, \dots, K$, are determined by (8.19).

9. Some examples of the energy calculation

Section 6 (see (6.4)) contains the total asymptotic expansion of the solution of the problem (2.1)–(2.3) for the general form of the right-hand sides F, P, $\phi^{(i)}$ which have multi-scaled representation (6.1)–(6.3).

Here we consider some particular cases, where we derive simple estimates of the energy integral for the solution $u(x, \varepsilon)$.

1^0. The influence of the right-hand side concentrated in Ω_0 outside neighbourhoods of the junction points

Let us consider the following boundary value problem

$$-\Delta_x u(x,\varepsilon) = f(x), \quad x \in \Omega_\varepsilon, \tag{9.1}$$

$$\frac{\partial u}{\partial n_x}(x,\varepsilon) = p(x), \quad x \in \partial\Omega_\varepsilon \setminus \bigcup_i \overline{S_\varepsilon^{(i)}}, \tag{9.2}$$

$$u(x,\varepsilon) = 0, \quad x \in \bigcup_i S_\varepsilon^{(i)}. \tag{9.3}$$

Here f is a smooth function, defined in a domain independent of ε and containing Ω_ε; we suppose that f vanishes outside Ω_0; p is a trace of a normal derivative of a smooth function which vanishes in the vicinity of the points $x = a^{(k)}$, $k = 1,\dots,K$, and which is equal to zero outside Ω_0.

Let us suppose at the beginning that the right-hand side in (9.1)–(9.2) is nonbalanced, i.e.,

$$\int_{\Omega_0} f(x)\,\mathrm{d}x + \int_{\partial\Omega_0} p(x)\,\mathrm{d}s \neq 0. \tag{9.4}$$

The following asymptotic equality holds

$$\begin{aligned} u(x,\varepsilon) \sim &\left\{\int_{\Omega_0} f(x)\,\mathrm{d}x + \int_{\partial\Omega_0} p(x)\,\mathrm{d}s\right\} \\ &\times\left[\varepsilon^{-2}C + \sum_{k=1}^{K} \mathcal{T}^{(k)}\{\chi(X^{(k)})\mathbf{N}(x,a^{(k)}) \right. \\ &\left. + \varepsilon^{-1}\varkappa(x-a^{(k)})\mathcal{W}^{(k)}(X^{(k)})\}\right], \end{aligned} \tag{9.5}$$

where

$$C = \left[\sum_{i=1}^{K} l_i^{-1}\,\mathrm{mes}_2\, g_i\right]^{-1}, \qquad \mathcal{T}^{(k)} = -l_k^{-1} C\,\mathrm{mes}_2\, g_k \quad k = 1,\dots,K. \tag{9.6}$$

We remind the reader that $\mathbf{N}$ is the Neumann function, $\mathcal{W}^{(k)}$ are the junction layer functions satisfying the problem (3.16), (3.17) with the right sides of the form (5.11); χ and $\varkappa$ are the cut-off functions.

From (9.5) and from the Green formula

$$\int_{\Omega_\varepsilon} |\nabla u|^2\,\mathrm{d}x = \int_{\Omega_\varepsilon} u(x,\varepsilon)f(x)\,\mathrm{d}x + \int_{\partial\Omega_\varepsilon} u(x,\varepsilon)p(x)\,\mathrm{d}s$$

we obtain that the energy integral for the solution of the problem (9.1)–(9.4) satisfies the asymptotic equality

$$\|\nabla u\|^2_{L_2(\Omega_\varepsilon)} \sim \varepsilon^{-2}\left\{ \int_{\Omega_0} f(x)\,\mathrm{d}x + \int_{\partial\Omega_0} p(x)\,\mathrm{d}s \right\} \left[\sum_{i=1}^{K} l_i^{-1} \operatorname{mes}_2 g_i \right]^{-1}. \tag{9.7}$$

Now, let us consider the case where the following equality holds

$$\int_{\Omega_0} f(x)\,\mathrm{d}x + \int_{\partial\Omega_0} p(x)\,\mathrm{d}s = 0. \tag{9.8}$$

We call the right-hand sides, satisfying (9.8), self-balanced.

Then the solution $u(x,\varepsilon)$ admits the asymptotic relation

$$\begin{aligned} u(x,\varepsilon) \sim{}& \mu(x) \prod_{i=1}^{K} \chi(X^{(i)}) + \sum_{i=1}^{K} \varkappa(x-a^{(i)}) \mathcal{W}^{(i)}(X^{(i)}) \\ &+ \varepsilon \sum_{i=1}^{K} \varkappa(x-a^{(i)}) b^{(i)} \Xi^{(i)}(X^{(i)}) \\ &+ D\left[C + \sum_{i=1}^{K} \mathcal{T}^{(i)} \{ \varepsilon^2 \chi(X^{(i)}) \mathrm{N}(x,a^{(i)}) + \varepsilon \varkappa(x-a^{(i)}) \mathcal{W}^{(i)}(X^{(i)}) \} \right], \end{aligned} \tag{9.9}$$

where

$$b^{(i)} = l_i^{-1} \mu(a^{(i)}) \operatorname{mes}_2 g_i; \qquad D = \sum_{i=1}^{K} b^{(i)}.$$

Here a smooth function $\mu(x)$ is a solution of the problem (6.10)–(6.12) with the right-hand side f, p; junction layer $\mathcal{W}^{(i)}$ describes a neighbourhood of the junction point $x = a^{(i)}$ and satisfies the problem (3.16), (3.17) with the right-hand side

$$\mathcal{F}(X^{(i)}) = \mu(a^{(i)}) \Delta_X \chi(X^{(i)}), \qquad \mathcal{P}(X^{(i)}) = -\mu(a^{(i)}) \frac{\partial}{\partial n_X} \chi(X^{(i)}); \tag{9.10}$$

$\Xi^{(i)}$ are special solutions of the homogeneous problem (3.16), (3.17), introduced in Section 3 (see (3.21), (3.22)).

The energy integral is defined by

$$\|\nabla u\|^2_{L_2(\Omega_\varepsilon)} = \int_{\Omega_0} |\nabla \mu(x)|^2\,\mathrm{d}x + \mathrm{O}(\varepsilon). \tag{9.11}$$

Thus, we have obtained the asymptotic estimates for the energy integral for nonbalanced and self-balanced right-hand sides. One can see that in the first case this value is $\mathrm{O}(\varepsilon^{-2})$, and it is determined by

$$\int_{\Omega_0} f(x)\,\mathrm{d}x + \int_{\partial\Omega_0} p(x)\,\mathrm{d}s$$

and by the magnitude of the constant C from (9.6).

2^0. The influence from the Dirichlet data at the base of thin cylinders

Let us consider the problem, where the Dirichlet data at the base of thin cylinders is the only non-zero part in the right-hand side.

$$\Delta_x u(x,\varepsilon) = 0, \quad x \in \Omega_\varepsilon, \tag{9.12}$$

$$\frac{\partial u}{\partial n_x}(x,\varepsilon) = 0, \quad x \in \partial\Omega_\varepsilon \setminus \Big(\bigcup_i \overline{S_\varepsilon^{(i)}}\Big), \tag{9.13}$$

$$u(x,\varepsilon) = \phi^{(i)}(Y^{(i)}), \quad x \in S_\varepsilon^{(i)},\ i = 1,\dots,K. \tag{9.14}$$

Then the solution $u(x,\varepsilon)$ has the asymptotic form

$$\begin{aligned} u(x,\varepsilon) \sim{}& \sum_{i=1}^{K} \Upsilon\Big(\frac{x_3}{\varepsilon}\Big) v^{(i)}(Y^{(i)}) \\ &+ D\Big[C + \sum_{i=1}^{K} \mathcal{T}^{(i)}\{\varepsilon^2 \chi(X^{(i)}) \mathrm{N}(x, a^{(i)}) + \varepsilon \varkappa(x - a^{(i)}) \mathcal{W}^{(i)}(X^{(i)})\}\Big] \\ &+ \varepsilon \sum_{i=1}^{K} \varkappa(x - a^{(i)}) b^{(i)} \Xi^{(i)}(X^{(i)}), \end{aligned} \tag{9.15}$$

where

$$b^{(i)} = l_i^{-1} \int_{S_i} \phi^{(i)}(Y)\,\mathrm{d}s_Y, \quad D = \sum_{i=1}^{K} b^{(i)};$$

the values C, $\mathcal{T}^{(i)}$ are defined by the equalities (9.6); Υ is a smooth cut-off function introduced at the beginning of Section 6. The bottom layer $v^{(i)}$ is the solution, which vanishes exponentially at infinity, of the boundary value problem

$$\Delta_Y v^{(i)}(Y) = 0, \quad Y \in \Pi_i, \tag{9.16}$$

$$\frac{\partial v^{(i)}}{\partial n_Y}(Y) = 0, \quad Y \in \partial\Pi_i \setminus \overline{S}_i, \tag{9.17}$$

$$v^{(i)}(Y) = \phi^{(i)}(Y) - (\mathrm{mes}_2\, S_i)^{-1} \int_{S_i} \phi^{(i)}(Y)\,\mathrm{d}s_Y, \quad Y \in S_i,\ i = 1,\dots,K. \tag{9.18}$$

In contrast with the previous case, the principal term of the energy integral is defined by the bottom layer, concentrated in the vicinity of the base of thin cylinders, and has the asymptotic form

$$\|\nabla u\|^2_{L_2(\Omega_\varepsilon)} \sim \varepsilon \sum_{i=1}^{K} \int_{\Pi_i} |\nabla v^{(i)}|^2\,\mathrm{d}Y + \mathrm{O}(\varepsilon^2). \tag{9.19}$$

10. The asymptotics of the first eigenvalue and the corresponding eigenfunction

The results of the previous sections enable us to formulate the following statement for the first eigenvalue of the problem in Ω_ε.

Theorem 2. *The first eigenvalue* λ_1 *of the problem*

$$-\Delta_x u(x,\varepsilon) = \lambda u(x,\varepsilon), \quad x \in \Omega_\varepsilon, \tag{10.1}$$

$$\frac{\partial u}{\partial n_x}(x,\varepsilon) = 0, \quad x \in \partial\Omega_\varepsilon \setminus \bigcup_i \overline{S_\varepsilon^{(i)}}, \tag{10.2}$$

$$u = 0, \quad x \in \bigcup_i S_\varepsilon^{(i)}, \tag{10.3}$$

has the asymptotic representation

$$\lambda_1 = \varepsilon^2 B + \mathrm{O}(\varepsilon^4), \tag{10.4}$$

where the constant B *is defined by the equality*

$$B = (\operatorname{mes} \Omega_0)^{-1} \sum_{i=1}^{K} l_i^{-1} \operatorname{mes}_2 g_i. \tag{10.5}$$

Proof. We shall use the statement A1, proved in the Appendix, about an estimate for eigenvalues of spectral problems. In our particular case of the problem (10.1)–(10.3) the notations, used in the Appendix, have the form: $X = L_2(\Omega_\varepsilon)$, $\mathcal{H} = W_2^1(\Omega_\varepsilon)$, H is a subspace of functions from $W_2^1(\Omega_\varepsilon)$ vanishing at $S_\varepsilon^{(i)}, i = 1,\ldots,K$; $a(u,v) = (\nabla u, \nabla v)_{L_2(\Omega_\varepsilon)}$; H_0 coincides with the space of functions cA, where A is the solution of the auxiliary problem from Section 5.

Due to Theorem A1 the following inequality holds

$$\frac{\nu}{1+q\nu} \leqslant \lambda_1 \leqslant \nu, \tag{10.6}$$

where

$$\nu = \frac{\|\nabla u\|^2_{L_2(\Omega_\varepsilon)}}{\|u\|^2_{L_2(\Omega_\varepsilon)}}, \quad u \in H_0. \tag{10.7}$$

Here q is a positive constant, independent of ε.

The relation (10.7) can be reduced to the form

$$\nu = \frac{(\operatorname{mes} \Omega_0)^{-1} \int_{\Omega_\varepsilon} A(x,\varepsilon)\,\mathrm{d}x}{\int_{\Omega_\varepsilon} A(x,\varepsilon)^2\,\mathrm{d}x}. \tag{10.8}$$

Using (5.18), (6.4), (10.8), we find that

$$\nu = \varepsilon^2 (\operatorname{mes} \Omega_0)^{-1} \sum_{i=1}^{K} l_i^{-1} \operatorname{mes}_2 g_i + \mathrm{O}(\varepsilon^4), \tag{10.9}$$

which gives (10.5). □

Let us denote by $\mathbf{w}$ the normalized eigenfunction of the problem (10.1)–(10.3). Then we formulate the statement.

Lemma 2. *The eigenfunction* $\mathbf{w}$ *can be represented in the form*

$$\mathbf{w} = \pm\|A\|^{-1}_{L_2(\Omega_\varepsilon)}\mathcal{U} + w. \tag{10.10}$$

The remainder w *satisfies the inequality*

$$\|w\|_{L_2(\Omega_\varepsilon)} + \|\nabla w\|_{L_2(\Omega_\varepsilon)} \leqslant \mathrm{const}\,\varepsilon^2, \tag{10.11}$$

with a constant, independent of ε.

The proof of the lemma is presented in the Appendix.

Lemma 2 and equality (5.18) enable us to formulate the theorem.

Theorem 3. *For the normalized eigenfunction* $\mathbf{w}$ *of the problem* (10.1)–(10.3) *the following asymptotic representation is valid*

$$\begin{aligned}\mathbf{w} \sim \pm(\mathrm{mes}\,\Omega_0)^{-1/2}\Bigg[1 - \varepsilon^2\sum_{k=1}^{K} l_k^{-1}\,\mathrm{mes}_2\, g_k\{\chi(X^{(k)})\mathrm{N}(x, a^{(k)})\\ + \varepsilon^{-1}\varkappa(x - a^{(k)})\mathcal{W}^{(k)}(X^{(k)})\}\Bigg],\end{aligned} \tag{10.12}$$

where N *is the Neumann function,* $\mathcal{W}^{(k)}$ *are solutions of the problem* (3.16), (3.17) *with the right-hand side*

$$\mathcal{F} = [\Delta_X, \chi(X^{(k)})]\,\frac{1}{2\pi\|X^{(k)}\|}, \qquad \mathcal{P} = -\frac{\partial}{\partial n_X}\left(\chi(X^{(k)})\,\frac{1}{2\pi\|X^{(k)}\|}\right),$$

the functions $\mathcal{W}^{(k)}$ *describe the junction layer in a neighbourhood of the points* $x = a^{(k)}$, $k = 1, \dots, K$, *and vanish as* $\mathrm{O}(\|X\|^{-2})$ *in* R^3_-. *The remainder, corresponding to* (10.12), *is* $\mathrm{O}(\varepsilon^2)$ *with respect to the norm in* $H^1(\Omega_\varepsilon)$.

Appendix

1⁰. An abstract scheme for the asymptotics of eigenvalues of problems with a small parameter

Let X and $\mathcal{H}$ be Hilbert spaces with the scalar products $(\cdot,\cdot)_X$, $(\cdot,\cdot)_{\mathcal{H}}$ and with the norms

$$\|x\|_X = (x, x)_X^{1/2}, \qquad \|h\|_{\mathcal{H}} = (h, h)_{\mathcal{H}}^{1/2}.$$

We suppose that the space H is densely and compactly embedded in the space X.

Let us consider on $\mathcal{H}$ a semi-linear form $a(\cdot,\cdot)$ such that

1) $a(h, h) \geqslant 0$, $\forall h \in \mathcal{H}$,

2) $a(h,h) \geqslant c_0\|h\|^2_{\mathcal{H}} - c_1\|h\|^2_X$, $\forall h \in \mathcal{H}$, where c_0, c_1 are positive constants.

Due to the properties 1), 2) the form a can degenerate only on a finite-dimensional subspace $X_0 \subset \mathcal{H}$. We denote by n the dimension of X_0.

If P: $X \to X_0$ is an orthogonal projector, then the following inequality is valid for some positive constant q

$$\|h - Ph\|^2_X \leqslant qa(h,h), \quad \forall h \in \mathcal{H}. \tag{A.1}$$

Further we shall suppose that the constant q is the best one in (A.1).

Let H be a closed subspace in $\mathcal{H}$, dense in X.

We are interested in eigenvalues of the spectral problem induced by the relation of quadratic forms

$$\frac{a(h,h)}{\|h\|^2_X}, \quad h \in H. \tag{A.2}$$

We assign the numbers (taking into account multiplicities) in a nondecreasing order

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \dots.$$

Our objective is to obtain the estimates of λ_i, $i = 1, 2, \dots$, in terms of eigenvalues of some finite-dimensional spectral problem.

Let us formulate this problem.

Suppose that

$$H_1 = \{h \in H\colon\ Ph = 0\}.$$

Thus, H_1 is a closed subspace in H with the codimension n. Let

$$H_0 = \{h \in H\colon\ a(h,g) = 0,\ \forall g \in H_1\}.$$

One can see that $\dim H_0 = n$ and H is the direct sum of H_0 and H_1.

Since the value (A.2) can be bounded below by $1/q$ for all $0 \neq h \in H_1$, then $\lambda_{n+1} \geqslant 1/q$.

The auxiliary finite-dimensional spectral problem is induced by the ratio

$$\frac{a(h,h)}{\|h\|^2_X}, \quad h \in H_0. \tag{A.3}$$

We denote by $0 \leqslant \nu_1 \leqslant \dots \leqslant \nu_n$ the eigenvalues of the problem.

Thus, we can formulate the theorem.

Theorem A.1. *The following inequalities hold*

$$\frac{\nu_j}{1 + q\nu_j} \leqslant \lambda_j \leqslant \nu_j, \quad \textit{for } j = 1, \dots, n. \tag{A.4}$$

Proof. The second inequality in (A.4) is a consequence of the variational principle of definition of eigenvalues and of the inclusion $H_0 \subset H$.

Now, let us prove the left inequality in (A.4). We introduce the spectrum distribution functions for the problems induced by the relations (A.2), (A.3):

$$n(\lambda) = \sum_{\lambda_j < \lambda} 1, \qquad n_0(\lambda) = \sum_{\nu_j < \lambda} 1.$$

Suppose that $\lambda > 0$ and $n_0(\lambda) = k$. Thus, if $k < n$ then there exists a subspace M of the dimension $n - k$ such that

$$\|h\|_X^2 \leqslant \frac{1}{\lambda} a(h, h), \quad \forall h \in M.$$

In the case $k \geqslant n$ we put $M = \{0\}$. If $h = h_0 + h_1$, $h_0 \in M, h_1 \in H_1$ then

$$\begin{aligned} \|h\|_X^2 &= \|h - Ph\|_X^2 + \|Ph_0\|_X^2 \leqslant \|h - Ph\|_X^2 + \|h_0\|_X^2 \\ &\leqslant q a(h, h) + \frac{1}{\lambda} a(h_0, h_0) \leqslant \left(q + \frac{1}{\lambda}\right) a(h, h). \end{aligned} \tag{A.5}$$

Since the codimension of the space $M \oplus H_1$ in H is equal to k, the inequality (A.5) yields

$$n\left(\frac{\lambda}{1 + q\lambda}\right) \leqslant n_0(\lambda) \quad \text{for all } \lambda > 0.$$

Therefore, we obtain that if $s > n_0(\lambda)$ then $\lambda_s \geqslant \lambda/1 + q\lambda$, and we arrive at the left inequality in (A.4). □

2^0. The estimate for the eigenfunction

Let us denote by A the solution of the problem (5.1)–(5.3). Also let u_1 be a normalized eigenfunction corresponding to the first eigenvalue of the problem

$$-\Delta u(x, \varepsilon) = \lambda u(x, \varepsilon), \quad x \in \Omega_\varepsilon, \tag{A.6}$$

$$\frac{\partial}{\partial n} u(x, \varepsilon) = 0, \quad x \in \partial\Omega_\varepsilon \setminus \bigcup_i \overline{S_\varepsilon^{(i)}}, \tag{A.7}$$

$$u(x, \varepsilon) = 0, \quad x \in S_\varepsilon^{(i)}, \ i = 1, \ldots, K. \tag{A.8}$$

Let us formulate the relevant statement

Theorem A2. *The following representation holds*

$$u_1(x, \varepsilon) = \frac{1}{b} e^{i\phi} A(x, \varepsilon) + A'(x, \varepsilon)$$

with $b = \|A\|_{L_2(\Omega_\varepsilon)}$, $\phi \in [0, 2\pi)$. The remainder A' satisfies the inequality

$$\|A'\|_{L_2(\Omega_\varepsilon)} + \|\nabla A'\|_{L_2(\Omega_\varepsilon)} \leqslant C\varepsilon^2.$$

Proof. We denote by $\{\lambda_j\}_{j\geqslant 1}$ a nondecreasing sequence of eigenvalues of (A.6)–(A.8) and by $\{u_j\}$ the corresponding sequence of the orthogonal eigenfunctions. Due to Theorem A.1 we obtain that

$$\lambda_1 = \varepsilon^2 B + \mathrm{O}(\varepsilon^4) \quad \text{and} \quad \lambda_2 \geqslant c_0,$$

where c_0 is a positive constant, independent on ε; the constant B is defined by (10.5).

We represent the function A in the form of the Fourier series

$$A = b\sum_{j=1}^{\infty} a_j u_j.$$

Thus,

$$\sum_{j=1}^{\infty} |a_j|^2 = 1. \tag{A.9}$$

Using Theorem A1 we derive that

$$\left(\varepsilon^2 B + \mathrm{O}(\varepsilon^4)\right)|a_1|^2 + \sum_{j=2}^{\infty} \lambda_j |a_j|^2 = \varepsilon^2 B + \mathrm{O}(\varepsilon^4). \tag{A.10}$$

Relations (A.9)–(A.10) imply that

$$\sum_{j=2}^{\infty} |a_j|^2 = \mathrm{O}(\varepsilon^2), \qquad |a_1|^2 = 1 + \mathrm{O}(\varepsilon^2),$$

and we obtain the asymptotic equality

$$\sum_{j=2}^{\infty} \lambda_j |a_j|^2 = \mathrm{O}(\varepsilon^4). \tag{A.11}$$

Using (A.9) and the inequality $\lambda_2 \geqslant c_0$ we obtain

$$\sum_{j=2}^{\infty} |a_j|^2 = \mathrm{O}(\varepsilon^4), \qquad |a_1|^2 = 1 + \mathrm{O}(\varepsilon^4).$$

Hence, we get

$$\frac{1}{b} A = \mathrm{e}^{\mathrm{i}\theta} u_1 + u', \quad \text{where } \theta \in [0, 2\pi),$$

and

$$\|u'\|_{L_2(\Omega_\varepsilon)} + \|\nabla u'\|_{L_2(\Omega_\varepsilon)} \leqslant C\varepsilon^2.$$

At this point the proof of the theorem is completed. □

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