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Asymptotic Solution of Eigenvalue Problems*

JOSEPH B. KELLER

Institute of Mathematical Sciences, New York University, New York, New York

AND

S. I. RUBINOW

Department of Physics, Stevens Institute of Technology, Hoboken, New Jersey

A method is presented for the construction of asymptotic formulas for the large eigenvalues and the corresponding eigenfunctions of boundary value problems for partial differential equations. It is an adaptation to bounded domains of the method previously devised to deduce the corrected Bohr–Sommerfeld quantum conditions.

When applied to the reduced wave equation in various domains for which the exact solutions are known, it yields precisely the asymptotic forms of those solutions. In addition it has been applied to an arbitrary convex plane domain for which the exact solutions are not known. Two types of solutions have been found, called the "whispering gallery" and "bouncing ball" modes. Applications have also been made to the Schrödinger equation.

1. INTRODUCTION

Recently a corrected form of the Bohr-Sommerfeld quantum conditions has been derived from the Schrödinger equation of quantum mechanics (*1*). The derivation is based upon an analysis of the "classical limit" of the Schrödinger wave function. The corrected conditions contain the appropriate quantum numbers, which are usually integral or half-integral, but may be of some other form. Furthermore, the corrected conditions are applicable to nonseparable systems, for which no such conditions were previously available. Therefore these conditions can be used to determine the energy levels of nonseparable systems. Of course, they are also applicable to separable systems, for which they yield the usual results.

It is clear from the derivation that these quantum conditions are not limited

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to problems of quantum mechanics but are applicable to a large class of eigenvalue problems. By means of them the eigenvalues can be computed. In addition, asymptotic expressions for the eigenfunctions can be obtained by the method used to derive the quantum conditions. In this paper we illustrate the method by applying it to the relatively simple eigenvalue problem associated with the wave equation in a bounded domain. This problem corresponds to a variety of physical problems: the quantum-mechanical motion of a free particle in a box, the motion of sound waves in a room, the vibration of a membrane, etc.

Our analysis begins with the demonstration that, with slight modification, the previous derivation of the corrected quantum conditions can be applied to bounded domains. The new quantum conditions derived in this way depend upon the boundary conditions at the boundaries of the domain. These conditions are then applied to circular, elliptical, rectangular, equilateral, triangular, and spherical domains. Since the eigenvalue problem can be solved for these domains, the present results can be compared with the exact solutions. We find, in every case, that the method yields precisely the asymptotic expression for the exact eigenvalues and the asymptotic form of the exact eigenfunctions. In this way, we obtain new derivations of the asymptotic forms of the Bessel, Mathieu and associated Legendre functions. Some of the results concerning Mathieu functions appear to be new. Closely related derivations have been given by Landauer (*2*, *3*). His procedure utilizes separation of variables, which is not necessary in our method.

We also consider a two-dimensional domain with an arbitrary smooth convex curve as its boundary. This is an example of a nonseparable problem. For it, we find two sets of eigenvalues and their associated eigenfunctions. The eigenfunctions of one set are asymptotically zero except in a thin layer near the boundary. We call these solutions the "whispering gallery" modes because they explain the whispering gallery phenomenon of acoustics. The eigenfunctions of the other set are asymptotically zero except in a thin strip around the minimum diameter of the domain. These solutions might be called the "bouncing ball" modes because they correspond to the motion of a ball bouncing back and forth between opposite sides of the domain.

In the last section we discuss the application of our method to the Schrödinger equation for a particle in a spherically symmetric potential. Finally, we indicate how to use the method for an asymmetric potential.

2. FORMULATION OF THE METHOD

Let us consider a solution u of the reduced wave equation

$$(\Delta + k^2)u = 0. \tag{1}$$

This solution is assumed to be defined in a bounded three-dimensional domain

D on the boundary B of which it satisfies a homogeneous boundary condition. For definiteness we will first consider the vanishing of the normal derivative

$$\partial u/\partial \nu = 0 \qquad \text{on } B. \tag{2}$$

As is well known, Eqs. (1) and (2) have a nontrivial solution only if k has one of a special set of values called the eigenvalues of the problem. We propose to determine the asymptotic form of the eigenfunction u for large values of the eigenvalue k. We will also determine the large eigenvalues asymptotically. This problem is essentially a special case of that treated in Ref. *1*. However in that reference only unbounded domains were considered. The present extension of the results of that paper to bounded domains is also valid for more general equations, but for simplicity we will only consider the reduced wave equation.

Our analysis is based upon the assumption that asymptotically for large values of k, u is of the form

$$u = \sum_{j=1}^{N} e^{ikS_j}\left[A_j + O\left(\frac{1}{k}\right)\right]. \tag{3}$$

The S_j and A_j are functions of position, and N is an integer. Each term in the sum is called a wave. Its phase is S_j and its amplitude is A_j.

We further assume that each term in (3) satisfies (1) asymptotically. Upon inserting (3) into (1) and equating to zero the coefficients of k^2 and k we obtain for each j, omitting the subscripts,

$$(\nabla S)^2 = 1 \tag{4}$$

$$2\nabla S\cdot\nabla A + A\Delta S = 0. \tag{5}$$

Equation (4) is the eiconal equation of geometrical optics, the solution of which can be expressed by means of certain straight lines. These straight lines, which are the characteristics of (4), are the rays of geometrical optics. They are the orthogonal trajectories of the surfaces S = constant, which are called wavefronts or surfaces of constant phase. If t denotes arc length along a ray then (4) implies that along the ray S is given by

$$S(t) = S_0 \pm t. \tag{6}$$

Here S_0 is the value of S at the point from which t is measured. The ambiguity of sign in (6) can be resolved by measuring t positively in the direction in which S increases.

In (5) the only derivative of A which occurs is the directional derivative along a ray. If we denote this by dA/dt, then (5) can be rewritten in the form

$$2\frac{dA}{dt} + A\Delta S = 0. \tag{7}$$

The solution of (7) is

$$A(t) = A_0 \exp\left(-\frac{1}{2}\int_0^t \Delta S\, dt\right) = A_0\left[\frac{G(t)}{G(0)}\right]^{1/2}. \tag{8}$$

The second form of the solution is given in (5). Here A_0 is the value of A at the point $t = 0$ on the ray, $G(t)$ is the Gaussian curvature of the wavefront $S =$ constant at the point t and $G(0)$ is the corresponding curvature at $t = 0$. The expression (8) for A can be rewritten in the form

$$A(t) = A_0\left[\frac{\rho_1\rho_2}{(\rho_1 + t)(\rho_2 + t)}\right]^{1/2}. \tag{9}$$

In (9) ρ_1 and ρ_2 denote the principal radii of curvature of the wavefront at $t =$ 0. Equation (9) can be interpreted as expressing conservation (say, of energy or of probability) within a narrow tube of rays.

Let us now apply the boundary condition (2) to the solution (3). Upon inserting (3) into (2) and equating to zero the coefficient of k we obtain

$$\sum_{j=1}^{N} \frac{\partial S_j}{\partial \nu} e^{ikS_j} A_j = 0 \qquad \text{on } B. \tag{10}$$

We now assume that at every point on the boundary the terms in (10), for which $\partial S_j/\partial\nu \neq 0$, vanish in pairs. By this we mean that for each nonvanishing term, say the jth, there is another term, say the j'th, with $j' \neq j$, such that

$$\frac{\partial S_j}{\partial \nu} e^{ikS_j} A_j + \frac{\partial S_{j'}}{\partial \nu} e^{ikS_{j'}} A_{j'} = 0 \qquad \text{on } B. \tag{11}$$

Physically this assumption corresponds to the hypothesis that each wave or ray which hits the boundary gives rise to a reflected wave or ray. The waves for which $\partial S_j/\partial\nu = 0$ are propagating parallel to the boundary and do not give rise to reflected waves. Since (11) holds for a range of values of k, it follows that

$$S_j = S_{j'} \qquad \text{on } B. \tag{12}$$

From (12) and (4) it follows that $(\partial S_j/\partial\nu)^2 = (\partial S_{j'}/\partial\nu)^2$ on B. Therefore

$$\frac{\partial S_j}{\partial \nu} = \pm\frac{\partial S_{j'}}{\partial \nu} \qquad \text{on } B. \tag{13}$$

If the positive sign applies in (13) then it follows that $S_j = S_{j'}$, not only on the boundary but throughout the entire domain D. It also follows from (11) that $A_j = -A_{j'}$ on B. This fact and (9) then show that $A_j = -A_{j'}$ throughout D. Then the sum of the two terms $A_je^{ikS_j}$ and $A_{j'}e^{ikS_{j'}}$ is identically zero throughout D. Such trivial pairs of terms will be omitted from (1). Therefore, we may conclude that the minus sign applies in (13). Then from (11) it follows that $A_j = A_{j'}$ on B. Thus we have

$$\frac{\partial S_j}{\partial \nu} = -\frac{\partial S_{j'}}{\partial \nu} \qquad \text{on } B, \quad (14)$$

$$A_j = A_{j'} \qquad \text{on } B. \quad (15)$$

Physically, (14) together with (12) implies the law of reflection for the reflected ray. The second conclusion asserts that at the boundary, the amplitude of the reflected wave equals that of the incident wave.

If the boundary condition (2) is replaced by

$$u = 0 \qquad \text{on } B, \quad (16)$$

an exactly similar analysis shows that (12) and (14) still hold, but (15) is replaced by

$$A_j = -A_{j'} \qquad \text{on } B. \quad (17)$$

The preceding result (9) for the amplitude A_j of any wave fails by becoming infinite at the two points $t = -\rho_1$ and $t = -\rho_2$ on each ray. These points are centers of curvature of the wavefront corresponding to $t = 0$. The locus of these points for a particular wavefront is called the caustic surface corresponding to that wavefront. The caustic surface is also the envelope of the rays which are normal to the wavefront. It generally consists of two sheets, corresponding to the two centers of curvature on each ray. Points at which the two sheets touch are called focal points. Thus we see that our results are not valid at a caustic.

We now assume that each wave which converges to a caustic gives rise to another wave which diverges from the caustic. The rays of the diverging wave are assumed to be the continuations of those of the converging wave and the phase along these rays is assumed to be the continuation of the phase on the converging rays. (See Fig. 1.) These two assumptions can also be described by stating that the phase $S_{j'}$ of the diverging wave equals the phase S_j of the converging wave at the caustic. It is known that at a regular point (i.e., not a focal point) of a caustic the amplitude $A_{j'}$ of the diverging wave is equal to the amplitude A_j of the converging wave multiplied by the factor $e^{-i\pi/2}$. At a focal point the factor is $e^{-i\pi}$ since a focal point corresponds to a double point of the caustic. These facts are all indicated by (9).

On the basis of the preceding assumptions we see that by following a ray of any wave in the direction of increasing t, we come to a caustic or a boundary. In either case, the ray continues as a ray of another wave. A sequence of waves is encountered in this manner. Since there are, by assumption, only a finite number N of waves in the solution, one of the waves in this sequence must recur.

Therefore a ray orthogonal to a given wavefront is ultimately orthogonal to this same wavefront again. (See Fig. 2.) But the value of S continually increases as a ray is traversed in the positive direction. Therefore, at the second point of intersection of the wavefront and the ray, the value of S_j is greater than its initial value by the length of the ray between intersections. Since S_j is constant on a wavefront, S_j must therefore be multiple valued. The corresponding amplitude A_j may also be multiple valued.

Since the solution u must be single valued we must require each wave to be single valued. If δS_j denotes the difference between two of the values of S_j then the single-valuedness condition may be written as

$$k\delta S_j = 2\pi n_j + i\delta \log A_j \qquad j = 1, \cdots, N. \quad (18)$$

Here n_j is an integer.

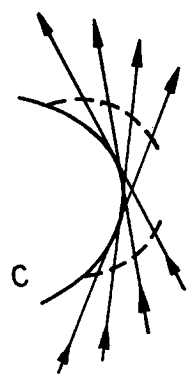

FIG. 1. Rays of a wave converging on a caustic C and the resulting diverging rays. The dashed lines are the converging and diverging wavefronts.

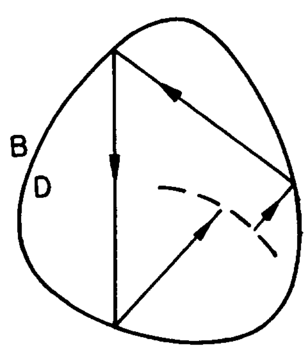

FIG. 2. A ray orthogonal to a particular wavefront, indicated by the dashed line, is shown orthogonal to it again after three reflections from the boundary.

In order to examine the consequences of (18) it is convenient to think of the various multiple-valued functions S_j as branches of a single function S. To represent this function S it is helpful to introduce a certain covering space, i.e., a certain multi-sheeted space analogous to the Riemann surfaces of function theory. However, it is not necessary to have one sheet for each branch of S. It suffices to have one sheet for each of the N distinct branches of ∇S. The various sheets are replicas of the domain D which may be bounded internally by caustics. The sheets corresponding to ∇S_{j_1} and ∇S_{j_2} are joined together along that part of the caustic or boundary where $S_{j_1} = S_{j_2}$. These are just the places where the wave j_1 gives rise to the wave j_2 by reflection or by passing through a caustic. Now the function S may be thought of as being defined on this covering space. It is not single-valued on this space, but its different branches on any sheet differ from each other only by additive constants. We also consider the A_j to be branches of a function A which is also defined on this space. We assume that the number of different branches of ∇A is the same as the number of different branches of ∇S, which is consistent with (5).

In terms of this covering space the expression $\delta S_j(P)$, at any point P, can be represented by the line integral

$$\delta S(P) = \oint \nabla S \cdot \mathbf{d\sigma}. \tag{19}$$

In (19) $\delta S(P)$ is represented as an integral along some closed curve on the covering space, starting and ending at P. The vector $\mathbf{d\sigma}$ is the vector element of arc length along this curve. Equation (19) follows from the fact that $\nabla S \cdot \mathbf{d\sigma}$ is just the derivative of S along the curve. The subscript j is omitted in (19) since the same equation holds for all values of j and therefore for the function S. Now the condition (18) can be rewritten as

$$k \oint \nabla S \cdot \mathbf{d\sigma} = 2\pi n + i\delta \log A. \tag{20}$$

Equation (20) must hold for every closed curve on the covering space, with an appropriate integer n in each case, since every such curve corresponds to some δS_j.

Equation (20) will hold for every curve if it holds for each curve in the basis of the fundamental group of the covering space since every closed curve is a linear combination of basis curves with integer coefficients. Therefore, there are a finite number of conditions (20), say q of them, one for each independent closed curve on the covering space, and each condition contains an integer n. The condition (20) can be made more explicit by noting that the phase of A is retarded by $\pi/2$ each time the curve passes through a caustic, so $\log A$ changes by $-i\pi/2$. If m' denotes the number of times a closed curve touches a caustic, and if the only change in log A is that associated with the regular points of the caustics, then (20) becomes

$$k \oint \nabla S \cdot d\boldsymbol{\sigma} = 2\pi\left(n + \frac{m'}{4}\right) \qquad \text{if } \partial u/\partial \nu = 0 \text{ on } B. \quad (21)$$

For the boundary condition (16) we see from (17) that A also changes phase by $-\pi$ each time the curve touches the boundary. If b is the number of times the closed curve touches the boundary, (20) becomes

$$k \oint \nabla S \cdot d\boldsymbol{\sigma} = 2\pi\left(n + \frac{m'}{4} + \frac{b}{2}\right) \qquad \text{if } u = 0 \text{ on } B. \quad (22)$$

3. UTILIZATION OF THE METHOD

To use the foregoing method to solve an eigenvalue problem we must first find in the domain D, for some integer N, a set of N normal congruences[1] of rays which are closed under reflection. This does not mean that each ray is closed, but that each congruence gives rise to another congruence of the set under reflection or passage through a caustic. Next we must consider the covering space associated with this set of N congruences, and determine q, the number of independent closed curves on it. Then we must imbed the set of N congruences of rays in a $q - 1$ parameter family of sets of N congruences, each of which is also closed under reflection. Finally, we must impose the q conditions (21) or (22) from which k and the $q - 1$ parameters can be determined. In this way the eigenvalue k is found.

To determine the phase of the eigenfunction we may arbitrarily assign some value S_0 to some wavefront (i.e., surface orthogonal to a congruence of rays) and then determine S by means of (6). To obtain the amplitude we must find a function A_0 defined on some wavefront such that when the value of A is computed from it by means of (9) and (15) or (17), it returns to the value A_0 (except for a phase factor) after a ray is traversed which returns to the original wavefront.

An alternative procedure, not employing rays, is possible if a $q - 1$ parameter family of phase functions S can be found. These functions must satisfy the eiconal equation (4) and the conditions (12) and (14) on B, and the basis of the fundamental group of the covering space of ∇S must contain q curves. Then the q conditions (21) or (22) determine k and the $q - 1$ parameters. The amplitude is then determined as before.

4. THE CIRCLE

As a first example, let us consider the case in which the domain D is a circle of radius a. To find a set of N normal congruences of rays, let us consider any

[1] A normal congruence of rays is a family of rays orthogonal to any surface.

ray and the successive rays it generates by reflection at the boundary. (See Fig. 3.) It can be seen that all these rays are tangent to a concentric circle, of radius a_0, say. This suggests that we choose as rays all the tangents to the circle of radius a_0, oriented so that they travel in the counter clockwise direction. Then the circle of radius a_0 is a caustic of these rays. Therefore we consider all those rays traveling inward from the outer boundary to the caustic as one normal congruence, and all those traveling outward from the caustic to the boundary as a second congruence. (See Fig. 4.)

Each of these normal congruences fills out the annular region $a_0 \leqq r \leqq a$. Furthermore an inward traveling ray goes into an outward travelling ray at the caustic and an outward ray reflects into an inward ray at the boundary. Therefore, $N = 2$ and the covering space consists of two replicas of the annular region joined together at their edges. Topologically, this covering space is a torus. Since there are only two linearly independent closed curves on the torus, $q = 2$, and

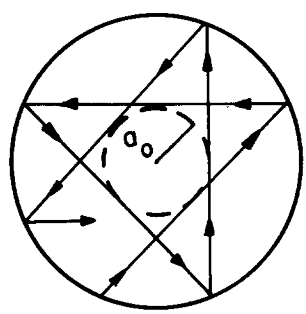

FIG. 3. A ray inside a circular region, and some of the rays which arise from it after several reflections. All of these rays are tangent to a concentric circle of radius a_0.

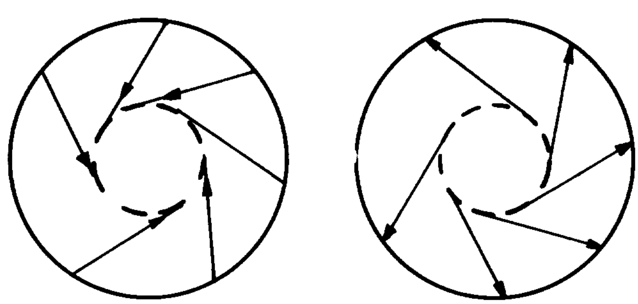

FIG. 4. Two congruences of rays in a circular domain. One consists of the outward directed, counterclockwise traveling tangents to the concentric circle of radius a_0. This circle is the caustic of these rays. The other consists of the corresponding inward directed tangents. Each congruence fills out the annular region between the caustic and the boundary.

we require a one-parameter family of pairs of normal congruences. If we permit a_0 to vary between zero and a we obtain such a family.

Let us now impose the condition (21) on two linearly independent closed curves on the torus. First we choose the circle of radius a_0. Since this is a caustic, the unit vector ∇S is tangential to it and therefore the line integral in (21) is just the length $2\pi a_0$. This path does not cross the caustic, as can be seen by enlarging it slightly, and therefore for it $m' = 0$. Thus (21) becomes, with m in place of n,

$$k \cdot 2\pi a_0 = 2\pi m \qquad m = 0, 1, 2, \cdots. \quad (23)$$

For the second curve we choose that shown in Fig. 5. This consists of two rays, each of length $(a^2 - a_0^2)^{1/2}$, and an arc of the caustic of length $2a_0 \cos^{-1}(a_0/a)$. Since this crosses the caustic once, $m' = 1$. Upon evaluating the integral in (21), taking proper account of directions, (21) becomes

$$2k[(a^2 - a_0^2)^{1/2} - a_0 \cos^{-1}(a_0/a)] = 2\pi(n + \tfrac{1}{4}) \qquad n = 0, 1, 2, \cdots. \quad (24)$$

We now find from (23) that $a_0 = m/k$. When we use this in (24) we obtain the following equation for the eigenvalue k in the case $\partial u/\partial \nu = 0$ on B:

$$[(ka)^2 - m^2]^{1/2} - m \cos^{-1}\left(\frac{m}{ka}\right) = \pi\left(n + \frac{1}{4}\right) \qquad n,m = 0, 1, \cdots. \quad (25)$$

If $u = 0$ on B we must use (22) instead of (21). The first closed curve, the caustic, does not touch the boundary, so for it $b = 0$ and therefore (22) yields (23). The second closed curve touches the boundary once so for it $b = 1$. In this case (22) yields (24) with the term $\tfrac{1}{4}$ on the right side replaced by $\tfrac{3}{4}$.

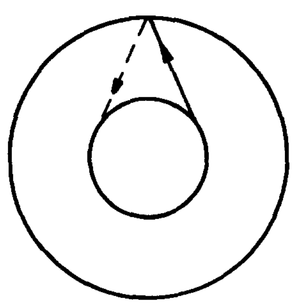

FIG. 5. A closed curve on the toroidal covering space associated with the two ray congruences of Fig. 4. This curve consists of a ray from the caustic to the boundary lying on one sheet, a reflected ray from the boundary to the caustic on the other sheet, and an arc of the caustic between the two points of tangency.

TABLE I

COMPARISON OF ASYMPTOTIC AND EXACT EIGENVALUES FOR A CIRCULAR REGION OF RADIUS a[a]

m	n	j_{mn} Approx.	j_{mn} Exact	Fractional error	j'_{mn} Approx.	j'_{mn} Exact	Fractional error
0	1	2.356	2.405	0.0204	3.927	3.832	−0.0248
	2	5.498	5.520	0.0040	7.069	7.016	−0.0076
	3	8.639	8.654	0.0017	10.210	10.173	−0.0036
	4	11.781	11.792	0.0009	13.352	13.324	−0.0021
1	1	3.795	3.832	0.0097	2.115	1.841	−0.1488
	2	6.997	7.016	0.0027	5.405	5.331	−0.0139
	3	10.161	10.173	0.0012	8.581	8.536	−0.0053
	4	13.311	13.324	0.0010	11.739	11.706	−0.0028
2	1	5.101	5.136	0.0068	3.300	3.054	−0.0806
	2	8.401	8.417	0.0019	6.771	6.706	−0.0097
	3	11.609	11.620	0.0010	10.010	9.969	−0.0041
	4	14.788	14.796	0.0005	13.200	13.170	−0.0023
3	1	6.346	6.380	0.0053	4.439	4.201	−0.0567
	2	9.745	9.761	0.0016	8.076	8.015	−0.0076
	3	13.005	13.015	0.0008	11.384	11.346	−0.0033
	4	16.216	16.223	0.0004	14.614	—	—

[a] In the third and sixth columns are shown the values of ka computed from (26) and (25), respectively, for the values of m and n listed in the first two columns. These results are labeled "approx." In the fourth and seventh columns are shown the values of j_{mn} and j'_{mn}, the corresponding exact eigenvalues. These are the nth zeroes of the mth Bessel function or of its derivative, respectively. The differences between the exact and approximate eigenvalues, divided by the exact eigenvalues, are listed in the fifth and eighth columns.

When a_0 is eliminated from (24) by means of (23), the following equation for the eigenvalue k results in this case of $u = 0$ on B.

$$[(ka)^2 - m^2]^{1/2} - m \cos^{-1}\left(\frac{m}{ka}\right) = \pi\left(n + \frac{3}{4}\right) \qquad n,m = 0, 1, \cdots. \tag{26}$$

In Table I the values of ka determined from (25) and (26) are shown for various values of m and n. The exact values, obtained by solving the problems exactly, are also shown for comparison. The agreement between the two sets of values is surprisingly good, considering that only small values of m and n, and therefore of ka, are tabulated—although the theory is based on ka being large. For large values of n and m the exact equations for the eigenvalues coincide precisely with (25) and (26), as we shall show.

Equations (25) and (26) can be solved explicitly in the limiting case in which $m \ll ka$ and the opposite case in which $m \approx ka$. In the former case (25) and

(26) become, respectively,

$$ka = \pi\left(n + \frac{m}{2} + \frac{1}{4}\right) + \cdots \qquad \partial u/\partial \nu \text{ on } B, \quad (27)$$

$$ka = \pi\left(n + \frac{m}{2} + \frac{3}{4}\right) + \cdots \qquad u = 0 \text{ on } B. \quad (28)$$

In this case the radius of the caustic $a_0 = (m/ka)a$ is nearly zero.

In the opposite case we introduce the small quantity ϵ defined by

$$m/ka = 1 - \epsilon. \quad (29)$$

When (29) is used in (25) and (26) simple equations for ϵ result. Once ϵ is found (29) yields for ka the results

$$ka = m + \frac{m^{1/3}}{2}\left[3\pi\left(n + \frac{1}{4}\right)\right]^{2/3} + \cdots \qquad \partial u/\partial \nu = 0 \text{ on } B, \quad (30)$$

$$ka = m + \frac{m^{1/3}}{2}\left[3\pi\left(n + \frac{3}{4}\right)\right]^{2/3} + \cdots \qquad u = 0 \text{ on } B. \quad (31)$$

In this case the caustic nearly coincides with the boundary since from (23) and (29), $a_0 = a(1 - \epsilon)$. As we shall see, the solution u is practically zero except in the region between the caustic and the boundary. The existence of this type of eigenfunction of a circular domain was first discovered by Rayleigh (*4*) in order to explain the "whispering gallery" phenomenon of acoustics. In Section 7 we shall obtain the corresponding eigenfunctions for more general domains.

Let us now determine the phase S of the eigenfunction u. Since the unit vector ∇S is tangent to the caustic, on it S is just equal to arc length σ along the caustic from some point, say from $\theta = 0$. To evaluate S at a point (r,θ) with $r > a_0$ we use (6) taking $t = (r^2 - a_0^2)^{1/2}$ to be the distance from (r,θ) to the caustic and $S_0 = \sigma_1 = a_0[\theta - \cos^{-1}(a_0/r)]$ to be the value of S at the point where the ray through (r,θ) leaves the caustic (see Fig. 6). Thus (6) yields

$$S_1(r,\theta) = a_0\left[\theta - \cos^{-1}\left(\frac{a_0}{r}\right)\right] + (r^2 - a_0^2)^{1/2}. \quad (32)$$

We obtain another value $S_2(r,\theta)$ if we consider the inward traveling ray through (r,θ) (see Fig. 6). In this case the length of the ray from the caustic to the boundary and back to (r,θ) may be employed, and this is just $t = 2(a^2 - a_0^2)^{1/2} - (r^2 - a_0^2)^{1/2}$. The value of S at the point where the ray leaves the caustic is

$$S_0 = \sigma_2 = a_0\left[\theta - \left\{2\cos^{-1}\left(\frac{a_0}{a}\right) - \cos^{-1}\left(\frac{a_0}{r}\right)\right\}\right]$$

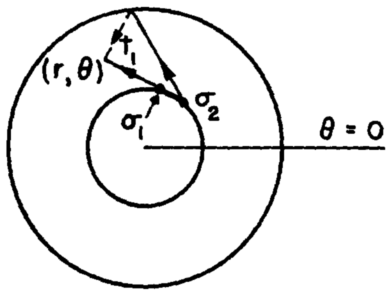

FIG. 6. The two ray paths from the caustic to the point (r,θ). One ray leaves the caustic at $\sigma_1 = a_0[\theta - \cos^{-1}(a_0/r)]$ and travels a distance $t_1 = (r^2 - a_0^2)^{1/2}$ to the point. The other ray leaves the caustic at $\sigma_2 = a_0[\theta + \cos^{-1}(a_0/r) - 2\cos^{-1}(a_0/a)]$, is reflected from the boundary and reaches the same point on the second sheet after traversing a distance $t_2 = 2(a^2 - a_0^2)^{1/2} - (r^2 - a_0^2)^{1/2}$.

With these values inserted, (6) becomes

$$\begin{aligned} S_2(r,\theta) &= a_0\left[\theta + \cos^{-1}\left(\frac{a_0}{r}\right) - 2\cos^{-1}\left(\frac{a_0}{a}\right)\right] + 2(a^2 - a_0^2)^{1/2} \\ &\quad - (r^2 - a_0^2)^{1/2} \qquad (33) \\ &= a_0\left[\theta + \cos^{-1}\left(\frac{a_0}{r}\right)\right] - (r^2 - a_0^2)^{1/2} + \frac{2\pi}{k}\left(n + \frac{1}{4}\right). \end{aligned}$$

The second form of S_2 is obtained by using (24) so it applies if $\partial u/\partial\nu = 0$ on B; in case $u = 0$ on B the final $\frac{1}{4}$ should be replaced by $\frac{3}{4}$.

To determine the amplitude $A(r,\theta)$ we use (9). Since we are considering a two-dimensional case ρ_2 is infinite and (9) becomes

$$A = A_0\left[\frac{\rho_1}{\rho_1 + t}\right]^{1/2}. \qquad (34)$$

If we let the point on the ray from which t is measured tend to the caustic then ρ_1 tends to zero and A_0 becomes infinite but, as (34) shows, the product $A_0\rho_1^{1/2}$ has a finite limit. If we denote this limit by $A_0'(\sigma)$ then (34) may be written simply as

$$A = \frac{A_0'(\sigma)}{t^{1/2}}. \qquad (35)$$

On an outgoing ray (35) becomes

$$A_1(r,\theta) = \frac{A_0'}{(r^2 - a_0^2)^{1/4}}. \qquad (36)$$

To obtain $A_2(r,\theta)$, the value of A on an incoming ray, we again use (35) which yields

$$A_2(r,\theta) = \frac{A_0'}{(r^2 - a_0^2)^{1/4}}. \tag{37}$$

The function A_0' in (37) is equal to A_0' in (36) on corresponding rays, according to (15), if $\partial u/\partial \nu = 0$ on B. Equations (36) and (37) show that, upon traversing a closed path, the function $A(r,\theta)$ will return to its original value, except for a phase factor, provided that A_0' is taken to be a constant.

Now that k, S_1, S_2, A_1, and A_2 have been found, we can combine them in (3) to yield the eigenfunction u. If we set $A_0' = e^{-i\pi/4}/2k^{1/2}$ we obtain

$$u = [(kr)^2 - m^2]^{-1/4} \cos\left\{[(kr)^2 - m^2]^{1/2} - m \cos^{-1}\frac{m}{kr} - \frac{\pi}{4}\right\} . e^{im\theta} \tag{38}$$
$$\partial u/\partial \nu = 0 \text{ on } B.$$

Proceeding similarly for the case $u = 0$ on B, we find from (17) that $A_2 = -A_1$. This minus sign and the extra π in kS_2 cancel to yield the same result (38) for u. These results both hold for

$$r > a_0 = m/k. \tag{39}$$

In order to obtain u for $r < a_0$ we must consider the complex or imaginary rays (*5*).[2] These rays are complex straight lines which are tangent to the caustic. The ray through a point (r,θ) with $r < a_0$ is thus a complex line through that point tangent to the caustic. In Ref. *5* these rays are examined and the two values of the function S are determined by means of them. They are given by

$$S(r,\theta) = a_0\theta \mp i\left[a_0 \cosh^{-1}\left(\frac{a_0}{r}\right) - (a_0^2 - r^2)^{1/2}\right] \qquad r \leq a_0. \tag{40}$$

It should be noticed that (40) can also be obtained from (32) by analytic continuation, merely by permitting r to be less than a_0. The corresponding expression for A can be obtained similarly from (36). It is

$$A(r,\theta) = \frac{A_0' e^{+i\pi/4}}{(a_0^2 - r^2)^{1/4}}. \tag{41}$$

The choice of the branch of the radical in (41) is based on considerations similar to those of Section 2. We now observe that the wave for which the upper sign is chosen in (40) decreases with increasing distance from the caustic while the

[2] Several signs are misprinted in Ref. *5*. On p. 47 in eq. (29) the sign of $i\pi/2$ should be plus. In (30) the signs of all terms in the exponent except $imka\theta$ should be changed. On page 48 in (31) the sign before $H^{(2)}$ should be plus.

other wave increases. We therefore assume that the increasing term must be omitted. Then (3) yields for either boundary condition, when $r < a_0$,

$$u(r,\theta) = \frac{1}{2}[m^2 - (kr)^2]^{-1/4} \exp\left\{im\theta - m \cosh^{-1}\left(\frac{m}{kr}\right) + [m^2 - (kr)^2]^{1/2}\right\}. \tag{42}$$

Equation (42) shows that for large m, u is exponentially small inside the caustic $r = a_0$. Thus the solution differs from zero only in the annular region between the caustic and the boundary.

If, from the beginning, we had considered the clockwise-traveling rays, all our results would have been the same with θ replaced by $-\theta$.

The exact eigenfunctions of (1) for the circle are

$$u = CJ_m(kr)e^{im\theta} \qquad m = 0, \pm 1, \pm 2, \cdots. \tag{43}$$

Here C is a constant and k is determined by either of the conditions

$$J_m'(ka) = 0 \qquad \partial u/\partial \nu = 0 \text{ on } B, \tag{44}$$

$$J_m(ka) = 0 \qquad u = 0 \text{ on } B. \tag{45}$$

If the dominant term of the Debye asymptotic expansion of $J_m(kr)$ for $kr > m$ is used in (44) and (45) these equations become exactly (25) and (26), respectively. When the same expansion is used in (43), it becomes (38) provided that we set $C = (\pi/2)^{1/2}$. If the corresponding form of the Debye expansion of $J_m(kr)$ for $kr < m$ is used in (43) it coincides with (42) when the same value of C is used. These comparisons show that the foregoing results are all asymptotically correct. In particular we note that our geometrical method yields the Debye expansion of the Bessel function, aside from a constant factor.

Since the present problem is separable, all our results could have been obtained by applying the usual WKB method to the ordinary differential equations resulting from separation. Alternatively, the eiconal equation could have been solved by separation into the sum of a function of r and a function of θ. The parameter a_0 could then have been introduced by requiring S to vanish at $\theta = 0$, $r = a_0$. There are four such solutions

$$S(r,\theta) = \pm r(\sin \tau - \tau \cos \tau) \pm a_0\theta \qquad r > a_0. \tag{46}$$

Here τ is defined by

$$\tau = \cos^{-1}\left(\frac{a_0}{r}\right) \qquad r > a_0. \tag{47}$$

For $r < a_0$ the solution S becomes

$$S(r,\theta) = \pm ir(\sinh \tau - \tau \cosh \tau) \pm a_0\theta \qquad r < a_0. \tag{48}$$

Now τ is defined by

$$\tau = \cosh^{-1}\left(\frac{a_0}{r}\right) \qquad r < a_0. \tag{49}$$

The corresponding θ independent solutions of (5) for $A(r,\theta)$ are the same as those found before. By using these solutions, all of our results could have been derived. Of course, these considerations apply only to separable problems, whereas the previous method is not restricted to such cases.

5. THE ELLIPSE

As a second example let us consider a plane domain D bounded by an ellipse with foci on the x axis at $x = \pm c/2$. In elliptic coordinates the equation of the ellipse is $\mu = R_0$ (see Fig. 7). The elliptic coordinates μ and θ are related to Cartesian coordinates by

$$x = \frac{c}{2} \cosh \mu \cos \theta, \tag{50}$$

$$y = \frac{c}{2} \sinh \mu \sin \theta. \tag{51}$$

The curves μ = constant and θ = constant are respectively confocal ellipses and hyperbolas.

If we consider a ray tangent to the ellipse $\mu = \mu_0$, $0 < \mu_0 < R_0$, then all the rays resulting from it by successive reflection at the boundary are also tangent

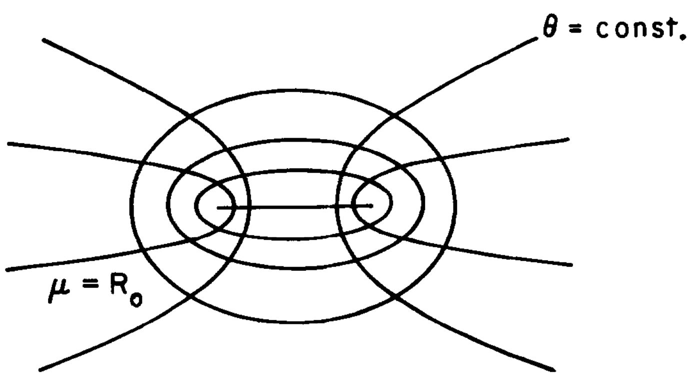

FIG. 7. The elliptic coordinate system. The lines μ = constant are confocal ellipses and the lines θ = constant are arms of confocal hyperbolas. The ellipse $\mu = R_0$ is the boundary of the domain.

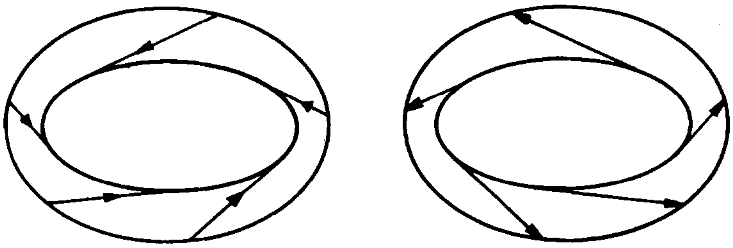

FIG. 8. Two congruences of rays in an elliptic domain. One consists of the outward directed, counterclockwise traveling tangents to the caustic, a confocal ellipse $\mu = \mu_0$. The other consists of the corresponding inward directed tangents. Each congruence fills out the annular region between the caustic and the boundary.

to the same ellipse. This suggests that we consider all the counter-clockwise directed tangents to the ellipse $\mu = \mu_0$ as rays. As before we consider separately the inward and outward traveling rays and thus obtain two normal congruences of rays, each filling out the annulus $\mu_0 \leqq \mu \leqq R_0$ (see Fig. 8). Then $N = 2$ and the two annular regions are joined together at their edges to yield a covering space which is again topologically a torus. Therefore we may apply (21) or (22) to two independent closed curves on this torus and obtain two equations for k and μ_0.

We choose as the first curve the caustic $\mu = \mu_0$ itself and then (21) or (22) becomes

$$4k \frac{c}{2} \cosh \mu_0 E\left(\frac{\pi}{2}, \operatorname{sech} \mu_0\right) = 2\pi m \qquad m = 0, 1, \cdots. \tag{52}$$

In (52) m has been used instead of the n of (21) or (22); the m' in those equations is zero. The elliptic integral of the second kind in (52) is defined by

$$E(x,\kappa) = \int_0^x (1 - \kappa^2 \sin^2 t)^{1/2}\, dt. \tag{53}$$

As the second curve we choose the path of Fig. 9. This consists in part of the

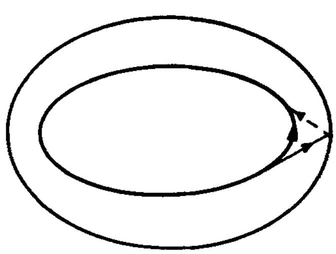

FIG. 9. A closed curve on the toroidal covering space associated with the two ray congruences of Fig. 8. This curve consists of a ray from the caustic to the boundary lying on one sheet, a reflected ray from the boundary to the caustic on the other sheet, and an arc of the caustic between the two points of tangency.

two tangents from the point $\mu = R_0$, $\theta = 0$ on the boundary, to the caustic at $\mu = \mu_0$,

$$\theta = \pm\cos^{-1}\left(\frac{\cosh \mu_0}{\cosh R_0}\right). \tag{54}$$

The path is completed by an arc of the caustic. It is a property of the ellipse that the length of such a path is independent of the point on the boundary from which the tangents are drawn. A straightforward calculation of the length of this path yields the value of the integral in (21). Then (21) becomes

$$\begin{aligned} 2k \Bigg\{\frac{c}{2} \sinh R_0(1 - \cosh^2 \mu_0 \operatorname{sech}^2 R_0)^{1/2} - \frac{c}{2} \cosh u_0 \Bigg[E\left(\frac{\pi}{2}, \operatorname{sech} \mu_0\right) \\ - E\left(\sin^{-1}\left[\frac{\cosh \mu_0}{\cosh R_0}\right], \operatorname{sech} \mu_0\right)\Bigg]\Bigg\} = 2\pi\left(n + \frac{1}{4}\right). \end{aligned} \tag{55}$$

For the case $u = 0$ on B, (22) yields (55) with an additional π on the right side. The first term in the brackets on the left side of (55) is the length of a ray from the boundary to the point of tangency; the remaining terms give the length of the caustic from $\theta = 0$ to one of the points of tangency determined by (54).

Equations (52) and (55) determine one eigenvalue k and the corresponding value of the parameter μ_0 for each pair of integers m and n. Upon using (52) to simplify (55) and then taking the ratio of the simplified (55) to (52) we obtain the following equation for μ_0

$$\begin{aligned} \frac{\sinh R_0 \,(\operatorname{sech}^2 \mu_0 - \operatorname{sech}^2 R_0)^{1/2} + E\left(\sin^{-1}\left[\frac{\cosh \mu_0}{\cosh R_0}\right], \operatorname{sech} \mu_0\right)}{E(\pi/2, \operatorname{sech} \mu_0)} \\ = 1 + \frac{2(n + 1/4)}{m}. \end{aligned} \tag{55'}$$

The left side of (55′) increases monotonically as μ_0 decreases from R_0 to 0. Thus (55′) has a solution for μ_0 provided that the right side lies between the extreme values of the function on the left. This occurs only when m and n satisfy the inequalities

$$0 \leqq n \leqq \frac{m}{2}\left[\frac{\sinh^2 R_0}{\cosh R_0} + E\,(\sin^{-1}(\operatorname{sech} R_0), 1) - 1\right] - \frac{1}{4}. \tag{55''}$$

When (55″) is satisfied, (55′) can be solved for μ_0 and then k can be found from (52). In this way we have calculated a table of eigenvalues of an elliptic domain for which $\cosh R_0 = 2$. The values of $kc/2$, rather than those of k itself, are shown in Table II for values of m from 1 to 10. The range of n for each value of m is given by (55″). The eigenvalues determined in this way lie to the lower left of the heavy zig-zag line in the table. The table also includes the eigenvalues

TABLE II

EIGENVALUES $(kc/2)_{mn}$ AND $(kc/2)'_{mn}$ FOR AN ELLIPTIC DOMAIN OF ECCENTRICITY $\cosh R_0 = 2$ WITH BOUNDARY CONDITIONS $u = 0$, AND $\partial u/\partial n = 0$, RESPECTIVELY[a]

$(kc/2)_{mn}$

m	$n = 0$	1	2	3	4
1	1.127	2.787	3.674	4.569	5.468
2	1.774	4.355	5.233	6.123	7.018
3	2.372	4.283	5.924	6.798	7.681
4	2.967	4.969	7.494	8.362	9.244
5	3.557	5.633	7.429	9.063	9.929
6	4.137	6.286	8.127	10.632	11.497
7	4.713	6.927	8.807	10.573	12.205
8	5.287	7.561	9.478	11.276	13.773
9	5.861	8.188	10.139	11.968	13.714
10	6.427	8.872	10.796	12.647	14.425

$(kc)/2'_{mn}$

m	$n = 0$	1	2	3	4
1	2.356	3.229	4.121	5.019	5.919
2	2.709	3.927	4.792	5.677	6.569
3	3.380	5.498	6.358	7.238	8.126
4	4.029	5.856	7.069	7.926	8.802
5	4.663	6.549	8.639	9.494	10.367
6	5.323	7.224	9.001	10.210	11.063
7	5.902	7.886	9.702	11.781	12.632
8	6.510	8.542	10.388	12.142	13.352
9	7.115	9.186	11.065	12.850	14.923
10	7.716	9.827	11.733	13.545	15.286

[a] The entries below the heavy zig-zag lines correspond to solutions with elliptic caustics and were computed from (52) and (55). Those above the lines correspond to solutions with hyperbolic caustics and were computed from (90) and (91). For the boundary condition $u = 0$, n is replaced by $n + \frac{1}{2}$ in (55) and (91). The eigenvalues $(kc/2)_{mn}$ and $(kc/2)'_{mn}$ approximate the $(n + 1)$st zeroes of Je_m or Jo_m and of Je_m' or Jo_m', respectively.

of the same domain when the eigenfunction, rather than its normal derivative, vanishes on the boundary. In this case (55′) and (55″) still hold provided the $\frac{1}{4}$ is changed to $\frac{3}{4}$ on the right side of each. No comparison is made with the exact eigenvalues because they do not appear to have been tabulated.

Equations (52) and (55) can be simplified and solved explicitly in various limiting cases. The first is that in which $\operatorname{sech} R_0$, the eccentricity of the bounding ellipse, and sech μ_0, the eccentricity of the caustic, are small. In this case we have

$$E\left(\frac{\pi}{2}, \operatorname{sech} \mu_0\right) = \frac{\pi}{2}\left[1 - \frac{1}{4}\operatorname{sech}^2 \mu_0 + \cdots\right], \tag{56}$$

$$\begin{aligned} E\left(\sin^{-1}\left(\frac{\cosh \mu_0}{\cosh R_0}\right), \operatorname{sech} \mu_0\right) &= \sin^{-1}\left(\frac{\cosh \mu_0}{\cosh R_0}\right) - \frac{\operatorname{sech}^2 \mu_0}{4} \\ &\cdot\left[\sin^{-1}\left(\frac{\cosh \mu_0}{\cosh R_0}\right) - \frac{\cosh \mu_0}{\cosh R_0}\sqrt{1 - \frac{\cosh \mu_0}{\cosh R_0}} + \cdots\right]. \end{aligned} \tag{57}$$

By using (56) we find that (52) becomes

$$\frac{kc}{2}\cosh \mu_0 = m\left[1 + \frac{1}{4}\operatorname{sech}^2 \mu_0 + \cdots\right]. \tag{58}$$

We now solve (58) for cosh μ_0 by iteration and obtain

$$\frac{kc}{2}\cosh \mu_0 = m\left[1 + \frac{1}{4}\left(\frac{kc}{2m}\right)^2 + \cdots\right]. \tag{59}$$

Next we use (56), (57), and (59) in (55). In doing so we also eliminate c by the relation $c \cosh R_0 = 2a$ where $2a$ is the major axis of the boundary ellipse. In this way we obtain the following equation for the eigenvalue k:

$$\begin{aligned} ka \Bigg\{&\left(1 - \frac{1}{2}\operatorname{sech}^2 R_0 + \cdots\right)\left(1 - \frac{m^2}{k^2a^2} - \frac{1}{2}\operatorname{sech}^2 R_0 + \cdots\right)^{1/2} \\ &- \frac{m}{ka}\cos^{-1}\left(\frac{m}{ka}\right) + \left[\left(1 - \frac{m^2}{k^2a^2}\right)^{1/2} + \left(1 - \frac{m^2}{k^2a^2}\right)^{-1/2}\right] \\ &\cdot\frac{\operatorname{sech}^2 R_0}{4} + \cdots\Bigg\} = \pi\left(n + \frac{1}{4}\right). \end{aligned} \tag{60}$$

To solve (60) we assume that $1 - m^2/k^2a^2 \gg \operatorname{sech}^2 R_0$. Then (60) becomes

$$\begin{aligned} [(ka)^2 - m^2]^{1/2} - m\cos^{-1}\left(\frac{m}{ka}\right) - \frac{\operatorname{sech}^2 R_0}{4}[(ka)^2 - m^2]^{1/2} + \cdots \\ = \pi(n + 1/4). \end{aligned} \tag{61}$$

Equation (61) coincides with (25), the eigenvalue equation for the circle, if the eccentricity sech R_0 is zero. Now (61) is valid for ellipses of small eccentricity provided that $\operatorname{sech}^2 R_0 \ll m^2/k^2a^2$ and that $\operatorname{sech}^2 R_0 \ll 1 - m^2/k^2a^2$. Therefore it will yield eigenvalues differing from those of the circle by small eccentricity corrections.

If $m \ll ka$, (61) can be solved with the result

$$ka = \pi\left(n + \frac{m}{2} + \frac{1}{4}\right)\left(1 + \frac{\operatorname{sech}^2 R_0}{4}\right) + \cdots. \tag{62}$$

In this case, the major axis of the caustic $c \cosh \mu_0$ is equal to $(m/ka)2a$ and is thus small compared to the major axis of the boundary ellipse.

On the other hand, if m is nearly equal to ka we define ϵ as in (29) and then (61) becomes an equation for ϵ. Once it is solved, (29) yields for k the result

$$ka = m + \frac{m^{1/3}}{2}\left[3\pi\left(n + \frac{1}{4}\right)\right]^{2/3} + \frac{m}{4}\operatorname{sech}^2 R_0 + \cdots. \tag{63}$$

In this case the caustic is very close to the bounding ellipse. The result (63) holds if the third term on the right side of (63) is small compared to the second term but large compared to the next omitted term, which is proportional to m^{-1} times the square of the second term. These conditions can be fulfilled only if $\operatorname{sech} R_0 \ll 1$ and therefore only if m is large.

Another case in which (52) and (55) can be solved approximately is that in which μ_0 is nearly zero, when the caustic practically coincides with the interfocal line. In this case, with the aid of expansions of the elliptic integrals *(6)* (52) becomes

$$kc = m\pi + kc\frac{\mu_0^2}{2}\log \mu_0 + \cdots. \tag{64}$$

Before expanding (55) it is convenient to first eliminate $E(\pi/2, \operatorname{sech} \mu_0)$ by means of (52). Then, upon expanding the resulting equation for small μ_0, we obtain

$$kc \cosh R_0 = 2\pi\left(n + \frac{m}{2} + \frac{1}{4}\right) + O(\mu_0^2). \tag{65}$$

The last equation yields for the eigenvalue

$$kc = 2\pi\left(n + \frac{m}{2} + \frac{1}{4}\right)\operatorname{sech} R_0 + \cdots. \tag{66}$$

Now (64) becomes the following equation for μ_0

$$\mu_0^2 \log \mu_0 = 2 - \frac{m}{n + \frac{m}{2} + \frac{1}{4}}\cosh R_0 + \cdots. \tag{67}$$

In order that (66) and (67) be valid, the values of n and m must be such that μ_0, determined by (67), is small. This requires that n and m be related by

$$n = \frac{m}{2}(\cosh R_0 - 1) - \frac{1}{4} + \cdots. \tag{68}$$

For the boundary condition $u = 0$ on B, the results (60)–(67) hold when $n + \frac{1}{4}$ is replaced by $n + \frac{3}{4}$.

Now that we have seen how to determine μ_0 and the eigenvalue k, let us construct the phase function S and the amplitude A of the eigenfunctions. Since ∇S is tangent to the caustic, S is equal to arc length along the caustic from some point on it, say from $\theta = 0$. Then we may construct S at any point P outside the caustic by using (6) with S_0 equal to the value of S at the point of tangency of the tangent from the caustic to P and t equal to the length of this tangent. If θ' denotes the value of θ at the point of tangency, then S_0 and t are given by

$$t = \frac{c}{2} \{(\cosh \mu \cos \theta - \cosh \mu_0 \cos \theta')^2 + (\sinh \mu \sin \theta - \sinh \mu_0 \sin \theta')^2\}^{1/2} \quad (69)$$

$$S_0 = \frac{c}{2} \int_0^{\theta'} (\cosh^2 \mu_0 - \cos^2 \theta)^{1/2} \, d\theta. \quad (70)$$

By the use of an addition formula for elliptic integrals it may be shown (7) that t can also be written in the form

$$t = \frac{c}{2} \left\{ \int_{\mu_0}^{\mu} (\cosh^2 \mu - \cosh^2 \mu_0)^{1/2} \, d\mu \pm \int_{\theta'}^{\theta} (\cosh^2 \mu_0 - \cos^2 \theta)^{1/2} \, d\theta \right\}. \quad (71)$$

The upper sign in (71) applies if $\theta > \theta'$ and the lower sign applies if $\theta < \theta'$.

For the outgoing rays we will denote the S function by S_1. Then $\theta > \theta'$ and (69), (70) yield for $S_1 = S_0 + t$ the result, when $\mu > \mu_0$,

$$S_1 = \frac{c}{2} \left\{ \int_{\mu_0}^{\mu} (\cosh^2 \mu - \cosh^2 \mu_0)^{1/2} \, d\mu + \int_0^{\theta} (\cosh^2 \mu_0 - \cos^2 \theta)^{1/2} \, d\theta \right\}. \quad (72)$$

For the incoming rays we denote S by S_2. Then $\theta < \theta'$ and we obtain for $S_2 = S_0 - t$ the result, when $\mu > \mu_0$,

$$S_2 = \frac{c}{2} \left\{ -\int_{\mu_0}^{\mu} (\cosh^2 \mu - \cosh^2 \mu_0)^{1/2} \, d\mu + \int_0^{\theta} (\cosh^2 \mu_0 - \cos^2 \theta)^{1/2} \, d\theta \right\}. \quad (73)$$

To determine the amplitude functions $A_1(\mu,\theta)$ and $A_2(\mu,\theta)$, we make use of (35) which shows that A_1 and A_2 are proportional to $t^{-1/2}$. By using the expression (69) for t it is possible to show (7) that

$$\begin{aligned} t^2 = {} & (\sinh \mu_0 \cosh \mu_0)^{-2} (\cosh^2 \mu_0 - \cos^2 \theta) \\ & \cdot (\cosh^2 \mu - \cosh^2 \mu_0)(\cosh^2 \mu_0 \sin^2 \theta' + \sinh^2 \mu_0 \cos^2 \theta'). \end{aligned} \quad (74)$$

Consequently, (35) may be written in the form

$$A_1(\mu,\theta) = B_1(\theta')(\cosh^2 \mu_0 - \cos^2 \theta)^{-1/4} (\cosh^2 \mu - \cosh^2 \mu_0)^{-1/4}, \quad (75)$$

$$A_2(\mu,\theta) = B_2(\theta'')(\cosh^2 \mu_0 - \cos^2 \theta)^{-1/4} (\cosh^2 \mu - \cosh^2 \mu_0)^{-1/4}. \quad (76)$$

In (75) and (76), θ' and θ'' denote the points of tangency with the caustic of the outgoing and incoming rays through the point (μ,θ). Since the outgoing rays are the continuations past the caustic of the incoming rays, it follows from the discussion following (17) that $B_1(\theta') = e^{-i\pi/2}B_2(\theta')$. Furthermore for points on the boundary, (15) shows that $B_2(\theta'') = B_1(\theta')$. These two relations may be combined to yield

$$|B_2(\theta')| = |B_2(\theta'')|. \tag{77}$$

Absolute values are used in (77) because the phase variation of the amplitude has already been taken into account in the consideration of the single-valuedness condition. In (77) θ'' is related to θ' by the condition that both θ' and θ'' denote points of tangency of rays from a common point on the boundary. The simplest solution of the functional Eq. (77) is B_2 = constant. Therefore we choose

$$B_2(\theta') = \tfrac{1}{2}e^{i\pi/4}. \tag{78}$$

Let us now collect our results for the eigenfunction u. We must insert the amplitudes given by (75) and (76), and the phases given by (72) and (73) into the expression (3) for u. Since S_2 was computed by following a ray backward from the caustic, the relation $B_1 = e^{-i\pi/2}B_2$ appropriate to the caustic must also be used. In this way we obtain for $\mu > \mu_0$ the result

$$\begin{aligned} u(\mu,\theta) = {} & (\cosh^2\mu_0 - \cos^2\theta)^{-1/4}(\cosh^2\mu - \cosh^2\mu_0)^{-1/4} \\ & \cdot \exp\left[i\frac{kc}{2}\int_0^\theta (\cosh^2\mu_0 - \cos^2\theta)^{1/2}\,d\theta\right] \\ & \cdot \cos\left[\frac{kc}{2}\int_{\mu_0}^{\mu}(\cosh^2\mu - \cosh^2\mu_0)^{1/2}\,d\mu - \frac{\pi}{4}\right]. \end{aligned} \tag{79}$$

To obtain an expression for u inside the caustic, where $\mu < \mu_0$, we must redetermine S and A by utilizing imaginary rays as described in Ref. *5*. By the method of that reference we obtain instead of (79) the following result, which is valid for $\mu < \mu_0$:

$$\begin{aligned} u(\mu,\theta) = {} & \frac{1}{2}(\cosh^2\mu_0 - \cos^2\theta)^{-1/4}(\cosh^2\mu_0 - \cosh^2\mu)^{-1/4} \\ & \cdot \exp\left[i\frac{kc}{2}\int_0^\theta (\cosh^2\mu_0 - \cos^2\theta)^{1/2}\,d\theta\right] \\ & \cdot \exp\left[-\frac{kc}{2}\int_{\mu}^{\mu_0}(\cosh^2\mu_0 - \cosh^2\mu)^{1/2}\,d\mu\right]. \end{aligned} \tag{80}$$

The integrals in (79) and (80) are expressed in terms of standard elliptic integrals by Eqs. (A8), (A35) and (A38) of the Appendix.

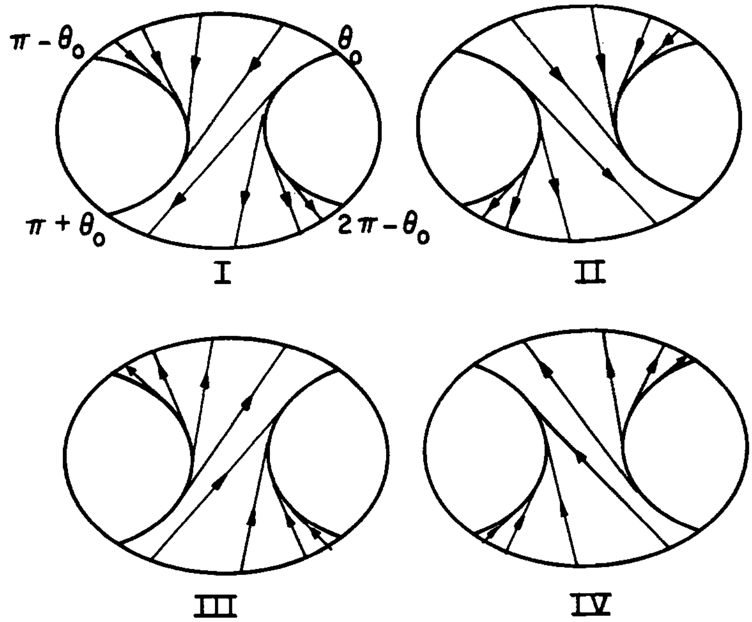

FIG. 10. The four ray congruences with hyperbolic caustics in an elliptic region. The caustics are the four hyperbolic arms $\theta = \theta_0$, $\pi - \theta_0$, $\pi + \theta_0$, $2\pi - \theta_0$.

For the boundary condition $u = 0$ on B, exactly the same results, (79) and (80), are obtained for u.

The preceding construction began with the consideration of a ray tangent to the ellipse $\mu = \mu_0 < R_0$ and the observation that all the rays resulting from it by successive reflection are tangent to the same ellipse. However if we had chosen a ray which crosses the line segment joining the focal points of the bounding ellipse, this ray would not have been tangent to any confocal ellipse. But it would be tangent to a confocal hyperbola, and all the rays resulting from it by successive reflections would also be tangent to the same hyperbola. Therefore we may obtain additional normal congruences of rays by considering all the tangents to a confocal hyperbola $\theta = \theta_0$ (see Fig. 10). Then we can construct additional eigenfunctions by proceeding with these rays just as we did with the other rays. However, rather than repeat that analysis, we will obtain these solutions by the alternative method, described in Section 3, which is based upon the phase functions S. To apply that method we must obtain a family of phase functions depending upon a number of parameters. For this purpose we must consider the eiconal equation (4).

In elliptic coordinates (4) becomes

$$\left(\frac{\partial S}{\partial \mu}\right)^2 + \left(\frac{\partial S}{\partial \theta}\right)^2 = \frac{c^2}{4} (\cosh^2 \mu - \cos^2 \theta). \tag{81}$$

This equation can be solved by separation of variables if we set

$$S = U(\mu) + T(\theta). \tag{82}$$

The resulting equations for U and T are

$$(U')^2 + b^2 - \frac{c^2}{4}\cosh^2 \mu = 0, \tag{83}$$

$$(T')^2 - b^2 + \frac{c^2}{4}\cos^2 \theta = 0. \tag{84}$$

In these equations the constant b^2 is the separation constant. The case of the elliptic caustics, considered previously, can be shown to correspond to the case $b^2 > c^2/4$. Therefore we now suppose that $b^2 < c^2/4$ and define θ_0 by

$$b^2 = \frac{c^2}{4}\cos^2 \theta_0 . \tag{85}$$

When (85) is used in (83) and (84), the only solutions of these equations are found to be the following, within additive constants.

$$U(\mu) = \pm\frac{c}{2}\int_{\mu}^{R_0} (\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu \tag{86}$$

$$\begin{aligned} T(\theta) &= \pm\frac{c}{2}\int_{\theta_0}^{\theta} (\cos^2 \theta_0 - \cos^2 \theta)^{1/2}\, d\theta \qquad \theta_0 \leqq \theta \leqq \pi - \theta_0 , \\ &= \pm\frac{ic}{2}\int_{\theta_0}^{\theta} (\cos^2 \theta - \cos^2 \theta_0)^{1/2}\, d\theta \qquad 0 \leqq \theta \leqq \theta_0 . \end{aligned} \tag{87}$$

By using (86) and (87) in (82) with the various choices of sign, we can construct four phase functions S_i. In the region $\theta_0 \leqq \theta \leqq \pi - \theta_0$ they are

$$\begin{aligned} S_1(\mu,\theta) = -S_3(\mu,\theta) = \frac{c}{2}\int_{\mu}^{R_0} &(\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu \\ &+ \frac{c}{2}\int_{\theta_0}^{\theta} (\cos^2 \theta_0 - \cos^2 \theta)^{1/2}\, d\theta \end{aligned} \tag{88}$$

$$\begin{aligned} S_2(\mu,\theta) = -S_4(\mu,\theta) = \frac{c}{2}\int_{\mu}^{R_0} &(\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu \\ &- \frac{c}{2}\int_{\theta_0}^{\theta} (\cos^2 \theta_0 - \cos^2 \theta)^{1/2}\, d\theta. \end{aligned} \tag{89}$$

In the region $\pi + \theta_0 \leqq \theta \leqq 2\pi - \theta_0$ we find by continuity that

$$\begin{aligned} S_1(\mu,\theta) &= -S_3(\mu,\theta) = -\frac{c}{2}\int_\mu^{R_0} (\cosh^2\mu - \cos^2\theta_0)^{1/2}\,d\mu \\ &+ 2\frac{c}{2}\int_0^{R_0} (\cosh^2\mu - \cos^2\theta_0)^{1/2}\,d\mu + \frac{c}{2}\int_\theta^{2\pi-\theta_0} (\cos^2\theta_0 - \cos^2\theta)^{1/2}\,d\theta, \end{aligned} \tag{88'}$$

$$\begin{aligned} S_2(\mu,\theta) &= -S_4(\mu,\theta) = -\frac{c}{2}\int_\mu^{R_0} (\cosh^2\mu - \cos^2\theta_0)^{1/2}\,d\mu \\ &+ 2\frac{c}{2}\int_0^{R_0} (\cosh^2\mu - \cos^2\theta_0)^{1/2}\,d\mu - \frac{c}{2}\int_\theta^{2\pi-\theta_0} (\cos^2\theta_0 - \cos^2\theta)^{1/2}\,d\theta. \end{aligned} \tag{89'}$$

From these equations we see that $S_1 = S_2$ and $S_3 = S_4$ at $\theta = \theta_0$ and $\theta = 2\pi - \theta_0$. Also $S_1 = S_4$ and $S_2 = S_3$ at $\mu = R_0$ for $\theta_0 \leqq \theta \leqq \pi - \theta_0$. Now we consider four replicas of the region $0 \leqq \mu \leqq R_0$, $\theta_0 \leqq \theta \leqq \pi - \theta_0$, $\pi + \theta_0 < \theta < 2\pi - \theta_0$, and define a function S which is equal to S_i on replica i. We join the edges of these replicas in such a way that ∇S is continuous at the caustics and that S is continuous at the boundary $\mu = R$, $\theta_0 \leqq \theta \leqq \pi - \theta_0$. Thus sheet one is joined to sheet two and sheet three to sheet four at $\theta = \theta_0$, $\pi - \theta_0$, $\pi + \theta_0$ and $2\pi - \theta_0$. At $\mu = R_0$ sheet one is joined to sheet four and sheet two to sheet three. The resulting surface, on which ∇S is single valued, is topologically a torus. Since there are two independent closed curves on the torus, and since we have a one parameter (θ_0) family of S functions, we can impose the two conditions (21) or (22) to determine k and θ_0.

As the first curve to be used in (21) or (22) we choose one having μ = constant and on which θ increases from θ_0 to $\pi - \theta_0$ on sheet one and then decreases from $\pi - \theta_0$ to θ_0 on sheet two. For the second curve we set $\theta = \theta_1$ and let μ decrease from R_0 to 0 and then increase from 0 to R_0 on sheet one, along $\theta = 2\pi - \theta_1$. Then on sheet four we follow the same path in the reverse direction. The first curve crosses two caustics so for it $m' = 2$. The second does not cross any caustic so for it $m' = 0$. When (21) is applied to these two curves, with S given by (88) and (89), the results are easily seen to be

$$4\frac{kc}{2}\int_{\theta_0}^{\pi/2} (\cos^2\theta_0 - \cos^2\theta)^{1/2}\,d\theta = 2\pi\left(m + \frac{1}{2}\right) \qquad m = 0, 1, \cdots, \tag{90}$$

$$4\frac{kc}{2}\int_0^{R_0} (\cosh^2\mu - \cos^2\theta_0)^{1/2}\,d\mu = 2\pi n \qquad n = 1, 2, \cdots. \tag{91}$$

In (90) we have denoted the integer n of (21) by m. Equations (90) and (91) determine the eigenvalue k and the constant θ_0 for each pair of integers m and n.

The integrals in (90) and (91) can be expressed in terms of elliptic integrals as is shown in (A15) and (A26). The elliptic integral B in these equations is defined below (A12). When (90) and (91) are expressed in terms of B, the ratio of these two equations yields the following equation for θ_0 :

$$\frac{\sec\theta_0 \sinh R_0 (\sec^2\theta_0 - \operatorname{sech}^2 R_0)^{1/2} + B(\sin^{-1}(\operatorname{sech} R_0), \cos\theta_0)}{B(\pi/2, \cos\theta_0)} = 1 + \frac{n}{m + \frac{1}{2}}. \tag{91'}$$

As θ_0 increases from 0 to $\pi/2$, the left side of (91′) increases monotonically from its minimum to infinity. Therefore (91′) has a solution for θ_0 only when the right side exceeds the minimum of the left side. This occurs only if m and n satisfy the inequality

$$n \geqq \left(m + \frac{1}{2}\right)\left[\frac{\sinh^2 R_0}{\cosh R_0} + B(\sin^{-1}(\operatorname{sech} R_0), 1) - 1\right]. \tag{91''}$$

When (91″) is satisfied, (91′) can be solved for θ_0 and then k can be determined from (90). This is the way in which we calculated the eigenvalues which are shown to the upper right of the zig-zag line in Table II. The values of n shown in the table are not the values which were used in (91′) to calculate the eigenvalues. The calculations were made with the first few values of n satisfying (91″). Then the eigenvalues were entered in the table immediately following the largest eigenvalue determined, for the same m, from solutions with elliptic caustics. This is not unexpected since there is no necessary relation between the integers n in (90) and in (55).

Equations (90) and (91) for k and θ_0 can be simplified and solved approximately if θ_0 is nearly equal to $\pi/2$. In this case, by making use of expansions of the elliptic integrals (*6*), we can simplify these equations to

$$kc\left[\cos^2\theta_0 + \frac{1}{8}\cos^4\theta_0 + \cdots\right] = 4\left(m + \frac{1}{2}\right), \tag{92}$$

$$kc\left[\sinh R_0 - \cos^2\theta_0\left(\tan^{-1} e^{R_0} - \frac{\pi}{4}\right) + \cdots\right] = n\pi. \tag{93}$$

These equations yield

$$kc = \frac{n\pi}{\sinh R_0} + \frac{(4m+2)}{\sinh R_0}\left[\tan^{-1} e^{R_0} - \frac{\pi}{4}\right] + \cdots, \tag{94}$$

$$\cos^2\theta_0 = \frac{4m+2}{n\pi}\sinh R_0 + \cdots. \tag{95}$$

These results are valid only if n and m are such that the right side of (95) is small. The result (94) for k can be rewritten in terms of $b = (c/2) \sinh R_0$, the semiminor axis of the bounding ellipse, in the form

$$k \cdot 2b = n\pi + (4m + 2)\left[\tan^{-1} e^{R_0} - \frac{\pi}{4}\right] + \cdots. \tag{96}$$

If only the first term on the right is retained, this equation requires the wavelength $\lambda = 2\pi/k$ to be equal to $4b/n$. Thus an integral number of half wavelengths must fit into the minor axis. This is to be expected for a wave which is bouncing back and forth between two parallel surfaces a distance $2b$ apart. This is the case here since the rays are confined to a narrow strip around the minor axis.

It is also possible to solve Eqs. (90) and (91) approximately for k and θ_0 when θ_0 is nearly equal to zero. By again making use of the expansions of elliptic integrals (*6, 8*) we may reduce these equations to the simpler forms

$$kc\left\{1 - \frac{1}{2}\sin^2\theta_0\left(\log\frac{4}{\sin\theta_0} + \frac{1}{2}\right) + \cdots\right\} = \pi\left(m + \frac{1}{2}\right), \tag{97}$$

$$kc\left\{\cosh R_0 - 1 + \frac{\sin^2\theta_0}{2}\left(\log\frac{4}{\sin\theta_0} - \frac{1}{2}\log\frac{\cosh R_0 + 1}{\cosh R_0 - 1}\right) + \cdots\right\} = \pi n. \tag{98}$$

Now we can combine these equations to obtain

$$kc \cosh R_0 = \pi\left(n + m + \frac{1}{2}\right) + \frac{\pi\left(n + m + \frac{1}{2}\right)}{4\cosh R_0}\sin^2\theta_0 \cdot\left[1 + \log\frac{\cosh R_0 + 1}{\cosh R_0 - 1}\right] + \cdots, \tag{99}$$

$$\sin^2\theta_0\left(\log\frac{4}{\sin\theta_0} + \frac{1}{2}\right) = 2\left[1 - \cosh R_0\frac{\left(m + \frac{1}{2}\right)}{\left(n + m + \frac{1}{2}\right)}\right] + \cdots. \tag{100}$$

To first order in $\sin^2\theta_0$, the result for the eigenvalue k may be written in terms of the semimajor axis of the ellipse $a = (c/2)\cosh R_0$ as

$$k \cdot 2a = \pi(n + m + \tfrac{1}{2}) + O(\sin^2\theta_0). \tag{101}$$

For (99)–(101) to be valid, n and m must be such that $\sin^2\theta_0$ as determined by (100) is small. This requires that n and m be related by

$$n = (\cosh R_0 - 1)(m + \tfrac{1}{2}) + \cdots. \tag{102}$$

The rays in this case fill up the entire ellipse except for two thin strips which extend along the major axis from the foci to the boundary. In the case of the elliptic caustics with $\mu_0 \sim 0$, the rays fill up the entire ellipse except for a thin strip surrounding the interfocal line.

Now that k and θ_0 have been determined, let us determine the amplitudes A_i by using (5). When any one of the four functions S_i given by (88) and (89) is inserted into (5), the same equation is obtained for A_i, namely

$$\begin{aligned}2\,(\cos^2\theta_0 - \cos^2\theta)^{1/2}\frac{\partial A}{\partial\theta} &+ 2\,(\cosh^2\mu - \cos^2\theta_0)^{1/2}\frac{\partial A}{\partial\mu} \\ &+ A[(\cos^2\theta_0 - \cos^2\theta)^{-1/2}\sin\theta\cos\theta \\ &\quad + (\cosh^2\mu - \cos^2\theta_0)^{-1/2}\sinh\mu\cosh\mu] = 0.\end{aligned} \tag{103}$$

This equation can be solved by separation of variables if we set $A = B(\theta)D(\mu)$. The separated equations are

$$2B'/B + \sin\theta\cos\theta\,(\cos^2\theta_0 - \cos^2\theta)^{-1} = \beta, \tag{104}$$

$$2D'/D - \sinh\mu\cosh\mu(\cosh^2\mu - \cos^2\theta_0)^{-1} = -\beta. \tag{105}$$

In these equations β is the separation constant. The solutions of (104) and (105) are

$$B_i(\theta) = B_i'\,(\cos^2\theta_0 - \cos^2\theta)^{-1/4}\exp\left[\frac{\beta}{2}\int^{\theta}(\cos^2\theta_0 - \cos^2\theta)^{-1/2}\,d\theta\right], \tag{106}$$

$$D_i(\mu) = D_i'\,(\cosh^2\mu - \cos^2\theta_0)^{-1/4}\exp\left[\frac{\beta}{2}\int^{\mu}(\cosh^2\mu - \cos^2\theta_0)^{-1/2}\,d\mu\right]. \tag{107}$$

The four solutions for A_i are thus found to be

$$\begin{aligned}A_i(\mu,\theta) &= A_i'\,(\cos^2\theta_0 - \cos^2\theta)^{-1/4}\,(\cosh^2\mu - \cos^2\theta_0)^{-1/4} \\ &\cdot\exp\left[\frac{\beta}{2}\int^{\mu}(\cosh^2\mu - \cos^2\theta_0)^{-1/2}\,d\mu\right]\exp\left[\frac{\beta}{2}\int^{\theta}(\cos^2\theta_0 - \cos^2\theta)^{-1/2}\,d\mu\right].\end{aligned} \tag{108}$$

We find that the A_i differ from each other only by the constant factors A_i'. From (15) it follows that $A_1' = A_4'$ and $A_2' = A_3'$. Now from the relation between the A's at a caustic, it follows that $|A_1'| = |A_2'| = |A_3'| = |A_4'|$. For simplicity, we will set $|A_i'| = \frac{1}{4}$.

To determine the constant β in (108) we could employ the method explained in Section 3. However, instead we will make use of the following simpler method which is convenient in separable problems such as the present one. This method, based upon flux conservation, was given by Landauer (*2*, *3*). In the present case conservation requires that the total flux carried by the outgoing wave across any curve μ = constant must be independent of μ:

$$\oint A_i^2(\mu,\theta)\nabla\mathbf{S}_i\cdot\mathbf{n}\,d\sigma = \text{constant}. \tag{109}$$

In (109) **n** denotes the unit normal to the curve μ = constant and $d\sigma$ denotes the element of arc length along this curve. Upon using (108) for A_i and (88) for S_i, we find from (109)

$$-\frac{c}{2}(D_i')^2 \exp\left[\beta \int^{\mu} (\cosh^2 \mu - \cos^2 \theta_0)^{-1/2}\, d\mu\right] \int_0^{2\pi} B_i^2(\theta)\, d\theta = \text{constant}. \quad (110)$$

From (110) we see at once that $\beta = 0$. Therefore (108) becomes

$$A_i(\mu,\theta) = A_i'(\cos^2 \theta_0 - \cos^2 \theta)^{-1/4} (\cosh^2 \mu - \cos^2 \theta_0)^{-1/4}. \quad (111)$$

We must now determine the phases of the A_i'. To do so we examine the four normal congruences of rays and see how they match up at the boundary and on the caustics. This examination shows that the A' must satisfy the relations $A_2' = A_3'$, $A_1' = A_4'$, $A_1' = e^{-i\pi/2}A_2'$ and $A_4' = e^{-i\pi/2}A_3'$. If we now set $A_2' = \frac{1}{4}e^{i\pi/4}$ we have

$$A_1' = A_4' = \tfrac{1}{4}e^{-i\pi/4}, \qquad A_2' = A_3' = \tfrac{1}{4}e^{i\pi/4}. \quad (112)$$

We may now combine our results to find u. Using (3) for u, with the S_i given by (88) and (89) and the A_i given by (111) and (112), we obtain for $\theta_0 < \theta < \pi - \theta_0$

$$\begin{aligned} u(\mu,\theta) = {} & (\cos^2 \theta_0 - \cos^2 \theta)^{-1/4} (\cosh^2 \mu - \cos^2 \theta_0)^{-1/4} \\ & \cdot \cos\left[\frac{kc}{2} \int_{\mu}^{R_0} (\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu\right] \\ & \cdot \cos\left[\frac{kc}{2} \int_{\theta_0}^{\theta} (\cos^2 \theta_0 - \cos^2 \theta)^{1/2}\, d\theta - \frac{\pi}{4}\right]. \end{aligned} \quad (113)$$

By making use of (91), this result may be rewritten in the form

$$\begin{aligned} u(\mu,\theta) = {} & (\cos^2 \theta_0 - \cos^2 \theta)^{-1/4} (\cosh^2 \mu - \cos^2 \theta_0)^{-1/4} \\ & \cdot \cos\left[\frac{kc}{2} \int_0^{\mu} (\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu - \frac{n\pi}{2}\right] \\ & \cdot \cos\left[\frac{kc}{2} \int_{\theta_0}^{\theta} (\cos^2 \theta_0 - \cos^2 \theta)^{1/2}\, d\theta - \frac{\pi}{4}\right]. \end{aligned} \quad (114)$$

For θ in the interval $-\theta_0 < \theta < \theta_0$, we can construct u by making use of imaginary rays. Since this construction is the same as in the previous cases, we will just state the result, which is

$$\begin{aligned} u(\mu,\theta) = {} & \frac{1}{2}(\cos^2 \theta - \cos^2 \theta_0)^{-1/4} (\cosh^2 \mu - \cos^2 \theta_0)^{-1/4} \\ & \cdot \cos\left[\frac{kc}{2} \int_0^{\mu} (\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu - \frac{n\pi}{2}\right] \\ & \cdot \exp\left[-\frac{kc}{2} \int_{\theta}^{\theta_0} (\cos^2 \theta - \cos^2 \theta_0)^{1/2}\, d\theta\right]. \end{aligned} \quad (115)$$

We have now completed the determination of the eigenvalues and eigenfunctions for the elliptic domain by the present method. For comparison, let us now examine the exact eigenfunctions for this domain. In the notation of Morse and Feshbach (*9*), which is also explained in the Appendix, these eigenfunctions are

$$u = CeJe_m\left(\frac{kc}{2}, \cosh \mu\right) Se_m\left(\frac{kc}{2}, \cos \theta\right), \tag{116}$$

$$u = CoJo_m\left(\frac{kc}{2}, \cosh \mu\right) So_m\left(\frac{kc}{2}, \cos \theta\right). \tag{117}$$

In these equations, Ce and Co are constants, Se_m and So_m are the even and odd Mathieu functions, respectively, and Je_m, Jo_m are the radial even and odd Mathieu functions of the first kind. For each integer m the eigenvalue k is determined by the boundary condition (2) which yields

$$Je_m'\left(\frac{kc}{2}, \cosh R_0\right) = 0 \qquad \text{for (116)}, \frac{\partial u}{\partial \nu} = 0 \text{ on } B, \tag{118}$$

$$Jo_m'\left(\frac{kc}{2}, \cosh R_0\right) = 0 \qquad \text{for (117)}, \frac{\partial u}{\partial \nu} = 0 \text{ on } B. \tag{119}$$

The asymptotic forms of (116)–(119) for $kc/2$ large can be obtained by utilizing the asymptotic formulas for the S and J functions, which are derived in the Appendix. If we use (A6) for Se_m and (A30), (A33) for Je_m with $h = \frac{1}{2}kc$, (116) becomes

$$\begin{aligned} u \sim CeAe \sinh \mu_0 \, (\cosh^2 \mu_0 - \cos^2 \theta)^{-1/4} (\cosh^2 \mu_0 - \cosh^2 \mu)^{-1/4} \\ \cdot \cos\left[\frac{kc}{2}\int_0^\theta (\cosh^2 \mu_0 - \cos^2 \theta)^{1/2}\, d\theta\right] \\ \cdot \cosh\left[\frac{kc}{2}\int_0^\mu (\cosh^2 \mu_0 - \cosh^2 \mu)^{1/2}\, d\mu\right], \quad 0 \leqq \mu < \mu_0. \end{aligned} \tag{120}$$

$$\begin{aligned} u \sim CeAe \sinh \mu_0 \exp\left[\frac{kc}{2}\int_0^{\mu_0} (\cosh^2 \mu_0 - \cosh^2 \mu)^{1/2}\, d\mu\right] \\ \cdot (\cosh^2 \mu_0 - \cos^2 \theta)^{-1/4} (\cosh^2 \mu - \cosh^2 \mu_0)^{-1/4} \\ \cdot \cos\left[\frac{kc}{2}\int_0^\theta (\cosh^2 \mu_0 - \cos^2 \theta)^{1/2}\, d\theta\right] \\ \cdot \cos\left[\frac{kc}{2}\int_{\mu_0}^\mu (\cosh^2 \mu - \cosh^2 \mu_0)^{1/2}\, d\mu - \frac{\pi}{4}\right], \quad \mu_0 < \mu \leqq R_0. \end{aligned} \tag{121}$$

The constant μ_0 in (120) and (121) is related to $kc/2$ and to m by (A9)

$$kc \cosh \mu_0 E\left(\frac{\pi}{2}, \operatorname{sech} \mu_0\right) = m\pi \qquad m = 0, 1, 2, \cdots. \tag{122}$$

From (A33) and (A35), the boundary condition (118) becomes

$$kc\left[\sinh R_0\left(1-\frac{\cosh^2\mu_0}{\cosh^2 R_0}\right)^{1/2}-\cosh\mu_0\left\{E\left(\frac{\pi}{2},\operatorname{sech}\mu_0\right)\right.\right.$$
$$\left.\left.-E\left(\sin^{-1}\frac{\cosh\mu_0}{\cosh R_0},\operatorname{sech}\mu_0\right)\right\}\right]=\left(n+\frac{1}{4}\right)2\pi \qquad n=0,1,2,\cdots. \tag{123}$$

The eigenvalue equations (122) and (123) coincide precisely with (52) and (55).

If we consider (117) and use (A7) for So_m and (A31), (A34) for Jo_m, we obtain

$$u\sim C_0A_0\left(\frac{2}{kc}\right)^2(\sinh\mu_0)^{-1}(\cosh^2\mu_0-\cos^2\theta)^{-1/4}(\cosh^2\mu_0-\cosh^2\mu)^{-1/4}$$
$$\cdot\sin\left[\frac{kc}{2}\int_0^\theta(\cosh^2\mu_0-\cos^2\theta)^{1/2}\,d\theta\right] \tag{124}$$
$$\cdot\sinh\left[\frac{kc}{2}\int_0^\mu(\cosh^2\mu_0-\cosh^2\mu)^{1/2}\,d\mu\right],$$

$$u\sim C_0A_0\left(\frac{2}{kc}\right)^2(\sinh\mu_0)^{-1}\exp\left[\frac{kc}{2}\int_0^{\mu_0}(\cosh^2\mu_0-\cosh^2\mu)^{1/2}\,d\mu\right]$$
$$\cdot(\cosh^2\mu_0-\cos^2\theta)^{-1/4}(\cosh^2\mu-\cosh^2\mu_0)^{-1/4}$$
$$\cdot\sin\left[\frac{kc}{2}\int_0^\theta(\cosh^2\mu_0-\cos^2\theta)^{1/2}\,d\theta\right] \tag{125}$$
$$\cdot\cos\left[\frac{kc}{2}\int_{\mu_0}^\mu(\cosh^2\mu-\cosh^2\mu_0)^{1/2}\,d\mu-\frac{\pi}{4}\right] \qquad \mu_0<\mu\leqq R_0.$$

The boundary condition (119) again yields (123). Thus, μ_0 and k are determined by (122) and (123), as for the case of solution (116).

The asymptotic form of the sum of the two solutions, (115) and (116), is given by (120) plus (124) for $0\leqq\mu<\mu_0$, and by (121) plus (125) for $\mu_0<\mu\leqq R_0$. These can be made to coincide with (80) and (79), respectively, provided Ce and Co are chosen such that

$$CeAe\sinh\mu_0\exp\left[\frac{kc}{2}\int_0^{\mu_0}(\cosh^2\mu_0-\cosh^2\mu)^{1/2}\,d\mu\right]=1 \tag{126}$$

$$CoAo(\sinh\mu_0)^{-1}\left(\frac{2}{kc}\right)^2\exp\left[\frac{kc}{2}\int_0^{\mu_0}(\cosh^2\mu_0-\cosh^2\mu)^{1/2}\,d\mu\right]=+i. \tag{127}$$

In addition, the exponentially small terms in (120) and (124) must be neglected. With this same choice of constants, the difference between the asymptotic forms of the two solutions also coincides with (79) and (80) with θ replaced by $-\theta$. Thus in the case of the elliptic caustic, the two solutions given by our method, corresponding to clockwise and counterclockwise traveling rays, respectively, coincide precisely with the asymptotic forms of the exact solutions. It is to be noted that each eigenvalue is doubly degenerate within the accuracy of our results.

Now let us expand (116) again, using (A10) or (A13) for Se_m and (A24) for Je_m. This yields

$$\begin{aligned} u \sim CeAe \sin \theta_0(\cos^2 \theta - \cos^2 \theta_0)^{-1/4}(\cosh^2 \mu - \cos^2 \theta_0)^{-1/4} \\ \cdot \cosh\left[\frac{kc}{2}\int_0^{\theta} (\cos^2 \theta - \cos^2 \theta_0)^{1/2}\, d\theta\right] \\ \cdot \cos\left[\frac{kc}{2}\int_0^{\mu} (\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu\right], \qquad 0 \leqq \theta < \theta_0. \end{aligned} \tag{128}$$

$$\begin{aligned} u \sim CeAe \sin \theta_0(\cos^2 \theta_0 - \cos^2 \theta)^{-1/4}(\cosh^2 \mu - \cos^2 \theta_0)^{-1/4} \\ \cdot \exp\left[\frac{kc}{2}\int_0^{\theta_0} (\cos^2 \theta - \cos^2 \theta_0)^{1/2}\, d\theta\right] \\ \cdot \cos\left[\frac{kc}{2}\int_{\theta_0}^{\theta} (\cos^2 \theta_0 - \cos^2 \theta)^{1/2}\, d\theta - \pi/4\right] \\ \cdot \cos\left[\frac{kc}{2}\int_0^{\mu} (\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu\right], \qquad \theta_0 < \theta \leqq \pi/2. \end{aligned} \tag{129}$$

For other values of θ the asymptotic form of u can be found from (128) and (129) by the evenness and periodicity of the Se function.

The constant θ_0 in (128) and (129) is related to $kc/2$ and m by (A20), which is

$$kc \cos^2 \theta_0\, B(\pi/2, \cos \theta_0) = (m + \tfrac{1}{2})\pi \qquad m = 0, 1, \cdots. \tag{130}$$

The boundary condition (118) is, from (A24) and (A26),

$$\begin{aligned} kc\{\sinh R_0(1 - \cos^2 \theta_0 \operatorname{sech}^2 R_0)^{1/2} - \cos^2 \theta_0[B(\pi/2, \cos \theta_0) \\ - B(\sin^{-1}(\operatorname{sech} R_0), \cos \theta_0)]\} = 2n'\pi \qquad n' = 1, 2, \cdots. \end{aligned} \tag{131}$$

The equations (130) and (131) which determine θ_0 and the eigenvalue k coincide with (90) and (91) when in (91) we set $n = 2n'$. When n is even, the asymptotic forms of the solution given by (128) and (129) coincide with (115) and (114), respectively, provided we choose Ce such that

$$CeAe \sin \theta_0 \exp\left[\frac{kc}{2}\int_0^{\theta_0} (\cos^2 \theta - \cos^2 \theta_0)^{1/2}\, d\theta\right] = (-1)^{n/2} \qquad n \text{ even}. \tag{132}$$

In obtaining (132), the exponentially small term in (128) was neglected.

The solution (117) becomes, when (A11) and (A14) are used for So_m and (A25) for Jo_m,

$$u \sim CoAo\left(\frac{2}{kc}\right)^2 (\sin\theta_0)^{-1}(\cos^2\theta - \cos^2\theta_0)^{-1/4}(\cosh^2\mu - \cos^2\theta_0)^{-1/4} \cdot\sinh\left[\frac{kc}{2}\int_0^\theta (\cos^2\theta - \cos^2\theta_0)^{1/2}\,d\theta\right] \cdot\sin\left[\frac{kc}{2}\int_0^\mu (\cosh^2\mu - \cos^2\theta_0)^{1/2}\,d\mu\right], \qquad 0 \leqq \theta < \theta_0. \tag{133}$$

$$u \sim CoAo\left(\frac{2}{kc}\right)^2 (\sin\theta_0)^{-1}(\cos^2\theta_0 - \cos^2\theta)^{-1/4}(\cosh^2\mu - \cos^2\theta_0)^{-1/4} \cdot\exp\left[\frac{kc}{2}\int_0^{\theta_0} (\cos^2\theta - \cos^2\theta_0)^{1/2}\,d\theta\right] \cdot\cos\left[\frac{kc}{2}\int_{\theta_0}^{\theta} (\cos^2\theta_0 - \cos^2\theta)^{1/2}\,d\theta - \pi/4\right] \cdot\sin\left[\frac{kc}{2}\int_0^\mu (\cosh^2\mu - \cos^2\theta_0)^{1/2}\,d\mu\right] \qquad \theta_0 < \theta < \pi/2. \tag{134}$$

The constant θ_0 is still given by (130). The boundary condition (119) now becomes, with the aid of (A25) and (A26), the same as (131) with $2n' + 1$ in place of $2n'$ on the right side. This equation coincides with (91) when in (91) we set $n = 2n' + 1$. When n is odd, the asymptotic forms (133) and (134) of the solution coincide with (115) and (114), respectively, if we neglect the exponentially small terms and choose for Co the value given by

$$CoAo\left(\frac{2}{kc}\right)^2 (\sin\theta_0)^{-1} \exp\left[\frac{kc}{2}\int_0^{\theta_0} (\cos^2\theta - \cos^2\theta_0)^{1/2}\,d\theta\right] = (-1)^{(n-1)/2}, \qquad n \text{ odd}. \tag{135}$$

6. THE RECTANGLE AND EQUILATERAL TRIANGLE

Let the domain D be a rectangle. If we consider a ray which makes an angle $\alpha \neq 0, \pi/2$ with one of the sides, we find that it and all the rays which arise from it by successive reflections are of one of four types. All the rays of each type are parallel to each other. This suggests that we introduce four normal congruences of rays, each consisting of all lines parallel to one of these four directions. One of these directions makes the angle α with one of the sides, and the other three directions are determined by the law of reflection. Thus we have a one parameter family of sets of normal congruences, α being the parameter. To each congruence we assign a replica of the domain D (Fig. 11) and from these replicas we construct

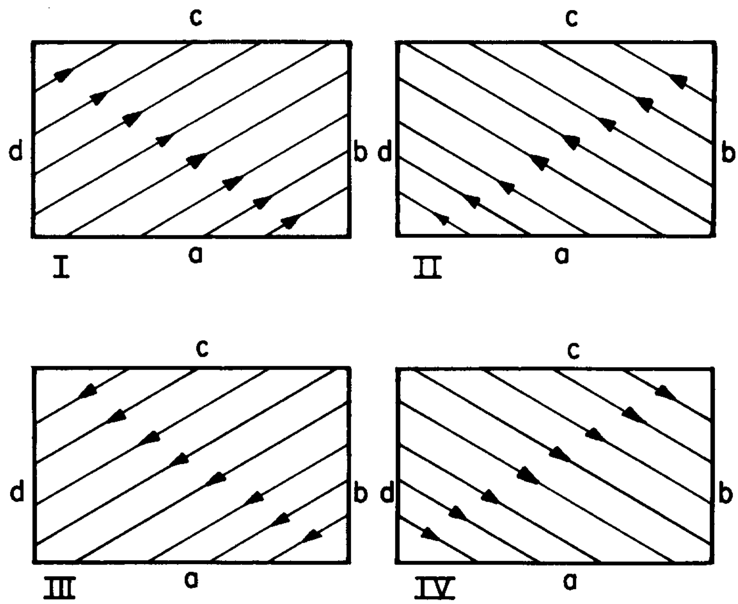

FIG. 11. Four replicas of the rectangular domain and four congruences of rays

the covering space by matching edges in accordance with the law of reflection. Thus when a ray of congruence I hits side b, a ray of congruence II is produced. Therefore side I b must be joined to side II b. Likewise I c must be joined to IV c. Proceeding in this way, we find that the resulting covering space is topologically a torus: replica pairs (I, II) and (III, IV) each join at edges b and d while pairs (I, IV) and (II, III) each join at edges a and c. (See Fig. 12).

When we apply (21) to two nonequivalent paths on this space, we obtain equations for the eigenvalues and the corresponding values of the parameter α.

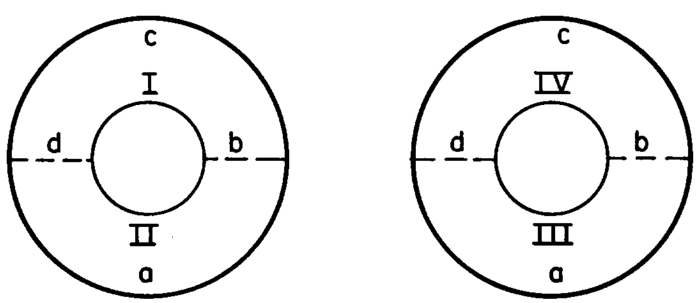

FIG. 12. The covering space for the rays in the rectangular domain is equivalent to that obtained by joining together at their edges these two annular regions. Each annulus is obtained by joining together two replicas of the rectangular domain.

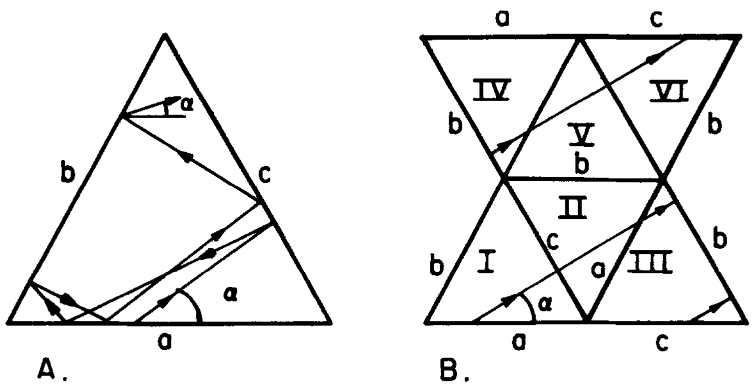

FIG. 13. (A) A ray which makes the angle α with side a of an equilateral triangle yields, upon successive reflections, rays having just six different directions. (B) The six replicas of the equilateral triangle I appropriately joined at their edges. One ray congruence corresponds to each triangle. The directions of these congruences are indicated by the sample ray shown. The covering space is obtained by joining the edge pairs (Ia,IVa), (IIIc,VIc), (Ib,IIIb), and (IVb,VIb). This space is topologically a torus.

The four phase functions are then found to be linear functions of the coordinates x and y, and the four amplitudes are found to be equal and constant. The resulting eigenfunctions, as well as the eigenvalues, are the exact solutions in this case. The special cases $\alpha = 0$; and $\alpha = \pi/2$ correspond to solutions consisting of just two normal congruences each, and each covering space is topologically a cylinder. There is just one independent closed path on this space, and when (21) is applied to it, it yields an equation for k, since α is already known. Again the exact solution of the problem is obtained. Since the calculations are simple and similar to those already presented, and since the results are well known, there is no point in describing this example any further.

Now let us consider a domain D which is an equilateral triangle. If we follow a ray which makes the angle α with one of the sides, we find that successive reflections generate rays which have only six different directions[3] (See Fig. 13A.) This suggests that we introduce six normal congruences of rays, each consisting of all the lines parallel to one of these six directions. When we join the edges of the six corresponding replicas in accordance with the law of reflection, we again obtain a torus for the covering space. (See Fig. 13B.) Application of (21) to two paths on this space yields equations for k and α. The six phase functions are

[3] This may be seen by covering the plane with successive reflections of the original triangle. A ray plus its successive reflections is then just a straight line in the plane. There are only five kinds of differently oriented triangles in the plane which cannot be obtained from the original triangle by a translation, but which require an additional reflection and/or rotation. The six ray segments in the six essentially nonequivalent triangles give rise to the six different ray directions when they are brought back to the original triangle.

again found to be linear in x and y, and the six amplitudes are equal and constant. As for the rectangle, the eigenfunctions and eigenvalues constructed by this method are the exact solutions of the problem. For the same reasons given above, it is unnecessary to describe these results any further.

7. AN ARBITRARY CONVEX REGION

Let us now consider a convex plane domain D bounded by a smooth closed curve B. Let us suppose that within D we can find a one parameter family of smooth closed curves $C(\alpha)$ depending upon a parameter α and having the following property: Any tangent to C goes into another tangent to C upon reflection at the boundary B. Then we can consider all the outward and counterclockwise directed tangents to C as a normal congruence of rays, and the inward counterclockwise directed tangents as another normal congruence. The outward traveling rays go into the inward traveling ones upon reflection at B, and the inward travelling rays go into the outward traveling ones at C, which is a caustic for both congruences. The annular region between C and B is doubly covered by rays, and the corresponding covering space for ∇S is the torus obtained by joining two replicas of this annulus at their edges. Since there are two independent closed paths on the torus, and we have a one parameter family of curves $C(\alpha)$, and therefore of congruences of rays, we can apply (21) to each of two independent curves. The two resulting equations will determine the eigenvalues k and the corresponding values of the parameter α.

As one of the two paths we choose the caustic C. For this path $m' = 0$, and if L denotes the length of C, then (21) becomes, with m in place of n,

$$kL = 2\pi m \qquad \mathrm{m} = 1, 2, \cdots . \tag{136}$$

For the other path we choose a ray from C to B, together with the corresponding reflected ray from B to C, and the shorter arc of C between the two points of tangency of these two rays. (See Fig. 9.) Let σ denote the length of this arc and t_1 and t_2 the lengths of the rays. For this path $m' = 1$ so (21) becomes

$$k(t_1 + t_2 - \sigma) = 2\pi(n + \tfrac{1}{4}) \qquad n = 0, 1, \cdots . \tag{137}$$

The length of this path, $t_1 + t_2 - \sigma$, is independent of the particular rays considered (*10*).[4]

The preceding considerations are similar to those we employed in the cases of the circular and elliptical domains. In those cases the curves $C(\alpha)$ were concentric circles and confocal ellipses, respectively. For an arbitrary convex do-

[4] This fact enables us to draw B by a string construction once C is given (Ref. *10*, pp. 453, 458). For this purpose we require a closed loop of string of length $t_1 + t_2 + L - \sigma$ which we wrap around C and draw out taut. Then the string consists of two tangents to C and the longer arc of C. The point at which the tangents meet lies on B. If we place a pencil point at this place and move it around C, we can draw B.

main no method is known for finding the curves $C(\alpha)$. In fact, there is some doubt about whether any such curves exist for nonelliptical domains (*10*). Nevertheless if we assume that they do exist, we can determine approximately those which lie close to the boundary B. This construction is essentially that given by Birkhoff (*11*) in connection with the "billiard-ball problem", which is mathematically equivalent to that of finding normal congruences of rays.

Let τ denote arclength along B, let $a(\tau)$ denote the radius of curvature of B, and let $\rho(\tau)$ denote the distance from B to C along the normal to B. We must determine $\rho(\tau)$ in such a way that C has the properties described above. When C is sufficiently close to B, the radius of curvature of C at any point is nearly equal to that of B at the corresponding point. Then we find that $t_1 \approx t_2 \approx (2a\rho)^{1/2}$ and $\sigma \approx 2a \tan^{-1}(t_1/a)$. The constancy of $t_1 + t_2 - \sigma$ now yields

$$2(2a\rho)^{1/2} - 2a \tan^{-1}(2\rho/a)^{1/2} = 4\alpha. \tag{138}$$

This is the equation of the curve $C(\alpha)$ since, for each value of the constant α, it determines the function $\rho(\tau)$. Upon solving for $\rho(\tau)$, we obtain

$$\rho(\tau) = \alpha^{2/3}a^{1/3}(\tau). \tag{139}$$

Now that the curves $C(\alpha)$ have been determined, we can apply (136) and (137) to determine k and α. Upon making use of (138), (137) yields at once

$$k\alpha = \pi(n + 1/4) \qquad n = 0, 1, \cdots. \tag{140}$$

To apply (136) we must compute L which is given by

$$L = \int_0^{L_0} \frac{a - \rho}{a}\, d\tau = L_0 - \alpha^{2/3} \int_0^{L_0} a^{-2/3}(\tau)\, d\tau + \cdots. \tag{141}$$

Here L_0 is the length of B. When this value of L and the above value of α are used in (136), it yields

$$2\pi m = kL_0 - k^{1/3}\left[\pi\left(n + \frac{1}{4}\right)\right]^{2/3} \int_0^{L_0} a^{-2/3}(\tau)\, d\tau + \cdots, m = 1, 2, \cdots. \tag{142}$$

After solving for k, we find

$$k = \frac{2\pi m}{L_0} + \left(\frac{2\pi m}{L_0}\right)^{1/3}\left[\pi\left(n + \frac{1}{4}\right)\right]^{2/3} L_0^{-1} \int_0^{L_0} a^{-2/3}(\tau)\, d\tau + \cdots. \tag{143}$$

If only the first term in this expression for k is retained, then this condition requires that $\lambda = L_0/m$, where $\lambda = 2\pi/k$ is the wavelength of the wave motion.

The eigenfunctions whose eigenvalues have just been found are practically zero everywhere inside the caustic $C(\alpha)$. This could be seen by utilizing complex rays to construct them. They would be found to decay exponentially with k and with distance from the caustic. Therefore these eigenfunctions are essentially different from zero only in the thin strip between C and B. Consequently they account for the "whispering gallery" phenomenon of acoustics in which a person who speaks near the wall of a convex room can be heard across the room near the wall, but not in the interior of the room. Therefore we call these solutions the whispering gallery modes (see Fig. 14). They generalize the corresponding results for a circular room (see Section 4) and for an elliptic room (see Section 6). Mechanically these solutions describe a particle sliding along the wall, or bouncing along it and always staying very close to it. Although we could now construct the eigenfunctions in the thin strip near the boundary, we will not do so.

Another set of eigenfunctions can be found for a convex domain. These are analogous to those solutions for the ellipse which have hyperbolic caustics. Mechanically they describe a particle bouncing back and forth between two points on the wall, along a diameter which is perpendicular to the boundary at both its ends. In order that such a motion be stable, the diameter must be the minimum diameter of the domain. The eigenfunctions differ essentially from zero only in a thin strip around this diameter, and the strip is bounded by two caustic curves C_1 and C_2 (see Fig. 15). To find these solutions we must find a one-parameter

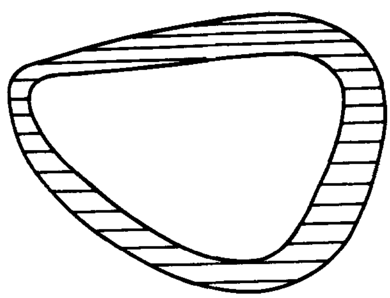

FIG. 14. The "whispering gallery" modes of an arbitrary convex region are essentially different from zero only in the shaded thin strip lying next to the boundary.

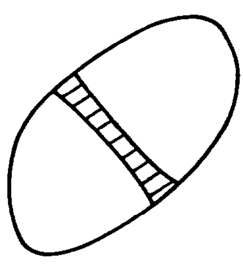

FIG. 15. The "bouncing ball" modes of an arbitrary convex region are essentially different from zero only in the shaded thin strip surrounding the minimum diameter of the domain.

set of pairs of curves $C_1(\alpha)$ and $C_2(\alpha)$, such that every tangent to C_1 goes into a tangent to C_2 after one or more reflections at B, and vice-versa. Then we can define four normal congruences of rays. These rays are the tangents, or rather half-tangents, to C_1 and C_2. In general, it is necessary to extend C_1 and C_2 outside of D and to include as rays the tangents to these extensions. One congruence consists of the half-tangents which terminate on C_1 or C_2 and are directed from one side of the domain to the other—say upward. Another congruence consists of the half-tangents which originate on C_1 or C_2 and are also directed upward. The other two congruences consist of the corresponding downward directed half-tangents (see Fig. 10). Corresponding to these four congruences of rays, we consider four replicas of the "curvilinear rectangle" bounded by B, C_1, and C_2. Upon joining their edges appropriately we obtain a torus. Then the two conditions (21) yield equations for k and α. Rather than carry out this construction, we will just state the leading term in the result for the eigenvalue. It depends only upon the length L_1 of the minimum diameter of D, and is

$$k = \frac{\pi n}{L_1} + \cdots \qquad n = 1, 2, \cdots. \tag{144}$$

8. THE SPHERE

Let us now take for D the interior of the three-dimensional sphere of radius a. If we consider a ray which is tangent to some concentric sphere of radius a_0, then all the rays which arise from it by successive reflection will be tangent to the same sphere. Furthermore all these rays lie in the plane containing the original ray and the center. This suggests that we introduce normal congruences of rays as follows. We choose a radius a_0, an angle θ_0 and a line through the center of the sphere, which we will call the polar axis. We consider any plane through the center whose normal makes the angle $(\pi/2) - \theta_0$ with the axis. In this plane we consider four congruences of rays, the four sets of half-tangents to the circle $r = a_0$ in which the plane intersects the sphere of radius a_0. These tangents may travel either inward or outward and clockwise or counterclockwise. Since two such planes pass through any point whose θ coordinate satisfies $\theta_0 < \theta < \pi - \theta_0$, there are eight rays through each such point, provided $a_0 < r < a$. Thus we have defined eight normal congruences of rays which depend upon the two parameters a_0 and θ_0, as well as upon the choice of the polar axis. The sphere $r = a_0$ and the cones $\theta = \theta_0$ and $\theta = \pi - \theta_0$ are the caustic surfaces of all these congruences.

Each congruence fills out the region $a_0 \leqq r \leqq a$, $\theta_0 \leqq \theta \leqq \pi - \theta_0$ and $0 \leqq \phi \leqq 2\pi$. When we match up the eight replicas of this region to obtain the covering space, we find that it consists of two disjoint parts. The replicas belonging to the four congruences of rays which travel in the direction of increasing ϕ combine to form one space, and the four replicas associated with the direction of decreasing

ϕ form another. Each of these spaces is topologically equivalent to the (Cartesian) product of a torus and a circle. Consequently there are three nonequivalent closed paths on either of these spaces. We may choose one of them to be the circle formed by the intersection of the sphere $r = a_0$ with the cone $\theta = \theta_0$. The other two may be chosen in one of the planes introduced before, just as the two paths were chosen in the case of the circular domain (see Fig. 5). When (21) is applied to these three paths on either space, it yields

$$k(2\pi a_0 \sin \theta_0) = 2\pi m \qquad m = 0, 1, 2 \cdots, \tag{145}$$

$$k[2(a^2 - a_0^2)^{1/2} - 2a_0 \cos^{-1}(a_0/a)] = 2\pi(n + \tfrac{1}{4}) \quad n = 0, 1, 2 \cdots, \tag{146}$$

$$k(2\pi a_0) = 2\pi(l + \tfrac{1}{2}) \qquad l = 0, 1, 2 \cdots. \tag{147}$$

These three equations determine the eigenvalue k and the two parameters a_0 and θ_0. From (145) and (147) we see that $|m| \leqq l$.

To determine the eight functions S_i we proceed as in the previous examples, and first determine them on the caustic $r = a_0$. Then by using (6) we will find them at points off the caustic. From the way in which the ray congruences were constructed, it follows that on the caustic $r = a_0$, S_i varies as arc length along any great circle which lies in a plane making the angle $(\pi/2) - \theta_0$ with the axis. Therefore along the equator $(\theta = \pi/2)$, $S_i = \pm a_0\phi \sin \theta_0$ provided that $S_i = 0$ at $(a_0, \pi/2, 0)$. The choice of sign depends upon whether S_i corresponds to rays travelling in the direction of increasing or decreasing values of ϕ. To find S_i at any point P_1 which is on the sphere $r = a_0$ but is not on the equator, we consider the two great circles through P_1 lying in planes of the type described above. We determine the value of S_i at the point where either great circle cuts the equator and add or subtract to it the distance along the great circle from the equator to P_1 (see Fig. 16). In this way we obtain four possible values which are

$$S_i(P_1) = \pm a_0 \phi \sin \theta_0 \pm a_0 \left[\cos^{-1} \frac{\cos \theta}{\cos \theta_0} - \sin \theta_0 \cos^{-1} \frac{\cot \theta}{\cot \theta_0} \right]. \tag{148}$$

The two choices of sign in (148) are independent.

To find S_i at a point $P = (r,\theta,\phi)$ where $a_0 \leqq r \leqq a$, we use (6). There are two rays from the caustic to P, and they both have the length $t = (r^2 - a_0^2)^{1/2}$. By considering the location of the point of tangency of the ray, we find from (6) that

$$S_i(P) = S_i(P_1) \pm [(r^2 - a_0^2)^{1/2} - \cos^{-1}(a_0/r)]. \tag{149}$$

Here $P_1 = (a_0, \theta,\phi)$ is the projection of P onto the caustic. The last term in (149)

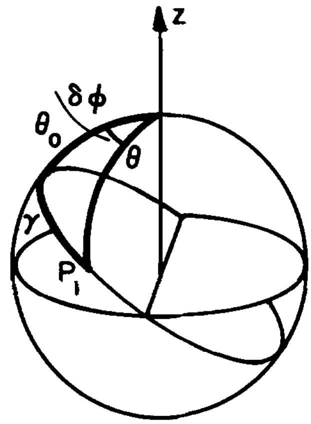

FIG. 16. The spherical angles necessary to determine the eiconal for a spherical domain are given by the relations $\cos\gamma = \cos\theta \sec\theta_0$ and $\cos\delta\phi = \tan\theta_0 \cot\theta$.

accounts for the difference between S_i at P_1 and at the point of tangency. When (148) is inserted into (149) we obtain the eight functions S_i which are given by

$$S_i(r,\theta,\phi) = \pm a_0\phi \sin\theta_0 \pm a_0\left[\cos^{-1}\frac{\cos\theta}{\cos\theta_0} - \sin\theta_0 \cos^{-1}\frac{\cot\theta}{\cot\theta_0}\right] \pm [(r^2 - a_0^2)^{1/2} - \cos^{-1}(a_0/r)]. \tag{150}$$

To determine the amplitudes A_i we consider the tube of rays belonging to any one congruence and tangent to the caustic $r = a_0$ in the strip bounded by the circles θ and $\theta + d\theta$. For simplicity we assume, rather than deduce, that the A_i are independent of ϕ. Upon applying (8) or (9), or the equivalent flux conservation requirement, to such a tube, we obtain

$$A_i = A_i' r^{-1/2}(r^2 - a_0^2)^{-1/4}(\sin^2\theta - \sin^2\theta_0)^{-1/4}. \tag{151}$$

From the conditions (17) at $r = a$ and from the relation between the A_i at the caustic, we see that the constants must be equal in magnitude in those four of (151) which correspond to increasing ϕ. Pairwise, their phases are either equal or differ by $-i\pi/2$, according as the corresponding replicas of the domain join at the boundary or at the caustic. The same remarks apply to the four A_i which correspond to decreasing ϕ.

Upon choosing the value for the constant in (151) appropriately, and inserting the four S_i from (150) and the corresponding A_i from (151) into (3), we obtain for u in the range $a_0 < r \leqq a$, $\theta_0 < \theta < \pi - \theta_0$, the result

$$u = r^{-1/2}\left[k^2r^2 - \left(l+\frac{1}{2}\right)^2\right]^{-1/4}\left[\sin^2\theta - \frac{m^2}{\left(l+\frac{1}{2}\right)^2}\right]^{-1/4} e^{\pm im\phi}$$

$$\cdot\cos\left\{\left[k^2r^2 - \left(l+\frac{1}{2}\right)^2\right]^{1/2} - \left(l+\frac{1}{2}\right)\cos^{-1}\frac{l+\frac{1}{2}}{kr} - \frac{\pi}{4}\right\}$$

$$\cdot\cos\left\{\left(l+\frac{1}{2}\right)\cos^{-1}\frac{\left(l+\frac{1}{2}\right)\cos\theta}{\left[\left(l+\frac{1}{2}\right)^2 - m^2\right]^{1/2}} - m\cos^{-1}\frac{m\cot\theta}{\left[\left(l+\frac{1}{2}\right)^2 - m^2\right]^{1/2}} - \frac{\pi}{4}\right\}. \tag{152}$$

In writing (152) we have introduced m from (145). The sign in $e^{\pm im\phi}$ depends upon whether we choose the four waves which travel in the direction of increasing or decreasing ϕ. By using complex rays. corresponding expressions for u can be obtained in the rest of the sphere.

The exact eigenfunctions for this problem are

$$u = j_l(kr)P_l^m(\cos\theta)e^{\pm im\phi}. \tag{153}$$

If the appropriate asymptotic forms of the spherical Bessel function $j_l(kr)$ and the associated Legendre functions $P_l^m(\cos\theta)$ are inserted into (153), the result coincides exactly with (152) within a numerical factor. The asymptotic form of the eigenvalue equation $j_l'(ka) = 0$ is identical with (146) when a_0 is eliminated from (146) by means of (147).

Some of the calculations and results of this section are the same as those previously given by Landauer (*2, 3*). However, he did not consider the eigenvalue problem in a bounded domain but instead considered waves in an unbounded region. Consequently, his construction is somewhat more arbitrary than ours. He relied upon separation of variables to characterize a particular solution. One of his results is the asymptotic formula referred to above for $P_l^m(\cos\theta)$, which can also be obtained from the differential equation for this function, but which apparently had not previously been given.

9. SPHERICALLY SYMMETRIC POTENTIALS

As a final illustration of our method, let us consider the Schrödinger equation for a particle in a spherically symmetric attractive potential $V(r)$. We shall write this equation in the form

$$\Delta^2 u + k^2 n^2(r,k) u = 0. \tag{154}$$

Here the index of refraction $n(r,k)$ is defined by

$$n^2(r,k) = -1 - k^{-2}V(r). \tag{155}$$

We wish to determine the energy levels $-k^2$ and the corresponding eigenfunctions u for large values of k^2. The domain is now all of space. This problem is exactly of the type considered in Ref. *1*. There it was shown that all the procedures which we have been using are applicable to this problem provided we replace the straight line rays by the trajectories of a particle of mass m and energy $\hbar^2 k^2/2m$ in the potential $\hbar^2 V/2m$. We must still find normal congruences of such trajectories, determine the covering space, etc. But now the phase function S, which is the Hamilton–Jacobi function, satisfies the equation $(\nabla S)^2 = n^2(r)$. Thus $|\nabla S| = n(r)$, and this must be taken into account in applying (21).

In order to define normal congruences of rays we assume that $V(r)$ is of such a form that bounded trajectories exist for some range of energies k^2. Then for each k in this range a bounded trajectory will lie between two spheres, $a_0 \leqq r \leqq a$. Each trajectory is a plane curve, due to the symmetry of V. Therefore we may apply exactly the same considerations as we did in the preceding section to define eight normal congruences of rays which depend upon a_0, θ_0, and k. The radius a of the outer sphere is determined by a_0 and k. The covering space is the same as before. We may choose the same paths as before in applying (21), but we must use trajectories instead of straight lines in the second path. Then (21) yields

$$kn(a_0)2\pi a_0 \sin\theta_0 = 2\pi m, \qquad m = 0, 1, \cdots, \tag{156}$$

$$2k \int_{a_0}^{a} n(r)\left[1 + r^2\left(\frac{d\gamma}{dr}\right)^2\right]^{1/2} dr - 2ka_0 n(a_0)\pi = 2\pi\left(n + \frac{1}{2}\right), \qquad n = 0, 1, \cdots, \tag{157}$$

$$kn(a_0)2\pi a_0 = 2\pi(l + ½), \qquad l = 0, 1, \cdots. \tag{158}$$

In (157) the final ½ occurs on the right side because the path is tangent to the inner and outer spheres, both of which are caustics. The angle γ is the polar angle in the plane of the trajectory. The π on the left side occurs because it is the angular separation between points at which $r = a_0$ and $r = a$ on a given trajectory. These equations determine k, a_0, and θ_0. As before, (156) and (158) show that $|m| \leqq l$. To simplify the integral in (157) we use the fact that $d\gamma/dr = \pm r^{-1} a_0 n(a_0)[r^2 n^2(r) - a_0^2 n^2(a_0)]^{-1/2}$. Then (157) becomes

$$k \int_{a_0}^{a} [n^2(r) - r^{-2} a_0^2 n^2(a_0)]^{1/2}\, dr = \pi\left(n + \frac{1}{2}\right), \quad n = 0, 1, \cdots. \tag{159}$$

To determine the S_i we proceed as in the preceding problem. It follows that the angular dependence of the S_i is the same as before. When the correct variation of S_i along a trajectory is included, we obtain

$$S_i(P) = S_i(P_1) \pm \int_{a_0}^{r} [n^2(r) - r^{-2}a_0^2n^2(a_0)]^{1/2}\, dr. \tag{160}$$

To obtain the amplitudes A_i we also proceed as before, noting that the flux is proportional to $(\nabla S)^2 A_i^2$ multiplied by the cross-sectional area of a tube of rays. Then instead of (151) we obtain

$$A_i = A_i' r^{-1/2}[r^2n^2 - a_0^2n^2(a_0)]^{-1/4}(\sin^2\theta - \sin^2\theta_0)^{-1/4}. \tag{161}$$

The constants in this equation are found as before. When the four S_i and A_i corresponding to trajectories travelling in the direction of increasing values of ϕ are inserted into (3), we obtain for u in the range $a_0 < r < a$, $\theta_0 < \theta < \pi - \theta_0$ the result

$$\begin{aligned} u = {} & r^{-1/2}[r^2n^2(r) - a_0^2n^2(a_0)]^{-1/4}\,[\sin^2\theta - m^2(l + \tfrac{1}{2})^{-2}]^{-1/4}e^{\pm im\phi} \\ & \cdot\cos\left\{\int_{a_0}^{r} [n^2(r) - r^{-2}a_0^2n^2(a_0)]^{1/2}\, dr - \pi/4\right\} \\ & \cdot\cos\left\{\left(l + \frac{1}{2}\right)\cos^{-1}\left[\frac{\left(l + \frac{1}{2}\right)\cos\theta}{\left[\left(l + \frac{1}{2}\right)^2 - m^2\right]^{1/2}}\right]\right. \\ & \qquad \left. - m\cos^{-1}\left[\frac{m\cot\theta}{\left[\left(l + \frac{1}{2}\right)^2 - m^2\right]^{1/2}}\right] - \frac{\pi}{4}\right\}. \end{aligned} \tag{162}$$

The negative sign in (162) results if the waves corresponding to decreasing values of ϕ are used. By using imaginary trajectories similar expressions for u can be obtained in the other regions of the sphere.

The exact eigensolutions of (154) are of the form

$$u = f_l(r,k)P_l^m(\cos\theta)e^{\pm im\phi}. \tag{163}$$

When the appropriate asymptotic forms of $f_l(r,k)$ and $P_l^m(\cos\theta)$, obtainable by the WKB method, are inserted into (163), the result coincides with (162). The asymptotic form of the equation for the eigenvalue coincides with (159).

10. CONCLUSION

We have presented a method for finding the asymptotic solutions of eigenvalue problems for certain partial differential equations. This method replaces the problem of solving the partial differential equation by that of finding families of solutions of certain ordinary differential equations. In the case of the Schrödinger equation, the ordinary equations are the classical equations of motion. For the reduced wave equation, they are the ray equations of geometrical optics. To obtain complete results, complex solutions of these ordinary equations must also be employed. In the two cases just referred to, this introduces "imaginary mechanics" and "imaginary optics", respectively. In all the examples treated here which could be solved by other methods, our method gave the same result as the asymptotic form of the other result. This is a partial verification of our method. We have also treated the arbitrary convex two-dimensional region, for which no other method of solution is known, and obtained the "whispering gallery" and "bouncing ball" modes for it. Let us now consider possible additional applications of our method.

First let us consider the reduced wave equation in a three-dimentional convex domain. To obtain the analogs of the bouncing ball modes for it, we again consider a minimum diameter of the domain. We can find eigenfunctions which are essentially different from zero only in a narrow tube around this diameter. For them the eigenvalues are again given by (144). To obtain the "whispering gallery" modes we consider on the boundary the closed geodesic of minimum length L_0. Again we can find eigenfunctions which are essentially different from zero only in a narrow tube around this geodesic. The corresponding eigenvalues are $k = 2\pi m/L_0 + \cdots$.

To obtain solutions of the Schrödinger equation for an arbitrary potential, we begin with a one parameter family of stable periodic classical oribts depending upon the energy k^2. The two cases referred to above are special instances of this procedure in which all the orbits coincide. Then all trajectories of energy k^2 which pass near the periodic one of the same energy with nearly the same direction will remain near it. Therefore for each k we can construct normal congruences of such trajectories, each of which will fill up a narrow tube around the stable orbit. By means of them we can construct eigenfunctions which are essentially different from zero only within such a tube. The first terms of the corresponding eigenvalues are then determined by the condition that a half-integral number of wavelengths must fit around the original orbit. In applying this condition, the optical length or change in "action" around the path must be used. Then the condition on k is just (21) with $m' = 0$. The integral in (21) is the optical length $L(k)$ of the motion in the stable orbit with energy k^2. Thus the result for k becomes $kL(k) = 2\pi m$.

The reason why a stable orbit was required in this construction is that only when the orbit is stable will all nearby and nearly parallel trajectories stay near it. If the original periodic orbit is not stable, the nearby trajectories will not stay near it. From these facts it follows that only the stable classical motions can be approximated arbitrarily closely for all time by quantum mechanical wave motions. This remark, which also applies to nonperiodic motions, accounts for macroscopic manifestations of quantum mechanical effects, such as the limitation of the number of bounces of one ping-pong ball on top of another (see Ref. *12*). In such cases the classical motion is unstable, so that the quantum mechanical fluctuations are capable of changing it.

We could apply our method to problems without stable orbits and to other problems if we could determine the classical trajectories. In the neighborhood of a stable orbit they can be found by perturbation methods, using the stable orbit as the starting point. The difficulty in finding the classical trajectories in general seems to be the main limitation on the use of our method.

APPENDIX I. ASYMPTOTIC FORMULAS FOR MATHIEU FUNCTIONS

The Mathieu functions $Se_m(h, \cos\theta)$ and $So_m(h, \cos\theta)$ are solutions of the equation

$$S'' + (b - h^2 \cos^2\theta)S = 0. \tag{A1}$$

They satisfy the following conditions

$$Se(h,1) = 1, \qquad Se'(h,1) = 0, \tag{A2}$$

$$So(h,1) = 0, \qquad So'(h,1) = 1. \tag{A3}$$

It follows from these conditions that Se is an even function of θ and So an odd function. The constant b in (A1) is determined by the condition that the solution Se or So be periodic in θ with period 2π. In each case this condition determines a countable set of values of b which we denote by $be_m(h)$ and $bo_m(h)$, respectively, where m is a non-negative integer. If $b \geqq h^2$ we define μ_0 by the equation

$$b^{1/2} = h \cosh \mu_0 . \tag{A4}$$

If $b \leqq h^2$ we define θ_0 by the equation

$$b^{1/2} = h \cos \theta_0 . \tag{A5}$$

We now seek asymptotic formulas for Se and So valid for h large. We will also permit b to become large by assuming that b is given by (A4) or (A5) with μ_0 or θ_0 fixed. In the first case, $b \geqq h^2$, when (A4) holds, (A1) has no turning point for any real value of θ. Therefore the usual WKB method is immediately applicable and yields

$$Se(h, \cos\theta) \sim (\sinh \mu_0)^{1/2} (\cosh^2 \mu_0 - \cos^2\theta)^{-1/4} \cdot \cos\left[h \int_0^\theta (\cosh^2 \mu_0 - \cos^2\theta)^{1/2}\, d\theta\right], \tag{A6}$$

$$So(h, \cos\theta) \sim h^{-1} (\sinh \mu_0)^{-1/2} (\cosh^2 \mu_0 - \cos^2 \theta)^{-1/4} \cdot \sin\left[h \int_0^\theta (\cosh^2 \mu_0 - \cos^2 \theta)^{1/2} \, d\theta\right]. \tag{A7}$$

The integral in (A6) and (A7) is given by

$$\int_0^\theta (\cosh^2 \mu_0 - \cos^2 \theta)^{1/2} \, d\theta = \cosh \mu_0 \left[E\left(\frac{\pi}{2}, \operatorname{sech} \mu_0\right) - E\left(\frac{\pi}{2} - \theta, \operatorname{sech} \mu_0\right)\right]. \tag{A8}$$

The constant μ_0, which now plays the role of b, is determined by the requirement that the solution Se or So be periodic in θ with period 2π. Applying this condition to (A6) or (A7) leads to the condition

$$4h \cosh \mu_0 \, E\left(\frac{\pi}{2}, \operatorname{sech} \mu_0\right) = m2\pi \qquad m = 0, 1, \cdots. \tag{A9}$$

Thus we see that for each value of the integer m, one value of μ_0 is determined by (A9). Within the accuracy of (A6) and (A7) the same values of μ_0 are found for both Se and So. It is customary to label the solutions Se_m and So_m with the value of the integer m.

In the second case, $b \leqq h^2$, when (A5) applies, (A1) has turning points at $\theta = \pm\theta_0 + 2\pi j$ and $\theta = \pm(\pi - \theta_0) + 2\pi j$ where j is any integer. Let us first consider the asymptotic form of Se and So in the interval $0 \leqq \theta < \theta_0$. In this range the WKB method yields

$$Se(h, \cos\theta) \sim (\sin \theta_0)^{1/2} (\cos^2 \theta - \cos^2 \theta_0)^{-1/4} \cdot \cosh\left[h \int_0^\theta (\cos^2 \theta - \cos^2 \theta_0)^{1/2} \, d\theta\right]. \tag{A10}$$

$$So(h, \cos\theta) \sim h^{-1} (\sin \theta_0)^{-1/2} (\cos^2 \theta - \cos^2 \theta_0)^{-1/4} \cdot \sinh\left[h \int_0^\theta (\cos^2 \theta - \cos^2 \theta_0)^{1/2} \, d\theta\right]. \tag{A11}$$

The integral in (A10) and (A11) can be expressed in terms of the elliptic integral $B(x,k)$ as

$$\int_0^\theta (\cos^2 \theta - \cos^2 \theta_0)^{1/2} \, d\theta = \sin^2 \theta_0 B\left(\sin^{-1}\left[\frac{\sin\theta}{\sin\theta}\right], \sin\theta_0\right), \qquad 0 \leqq \theta \leqq \theta_0. \tag{A12}$$

Here B and F, the elliptic integral of the first kind, are defined by

$$B(x,k) = \int_0^x (1 - k^2 \sin^2 \phi)^{-1/2} \cos^2 \phi \, d\phi = k^{-2}E(x,k) - k^{-2}(1 - k^2)F(x,k),$$

$$F(x,k) = \int_0^x (1 - k^2 \sin^2 \phi)^{-1/2} \, d\phi.$$

Now we apply the WKB connection formulas to obtain the asymptotic formulas in the range $\theta_0 \leqq \theta \leqq \pi - \theta_0$. In this way we obtain

$$\begin{aligned} Se(h, \cos\theta) \sim\ & (\sin\theta_0)^{1/2} (\cos^2\theta_0 - \cos^2\theta)^{-1/4} \\ & \cdot \exp\left[h \int_0^{\theta_0} (\cos^2\theta_0 - \cos^2\theta_0)^{1/2}\, d\theta\right] \\ & \cdot \cos\left[h \int_{\theta_0}^{\theta} (\cos^2\theta_0 - \cos^2\theta)^{1/2}\, d\theta - \frac{\pi}{4}\right], \end{aligned} \tag{A13}$$

$$\begin{aligned} So(h, \cos\theta) \sim\ & h^{-1}(\sin\theta_0)^{-1/2} (\cos^2\theta_0 - \cos^2\theta)^{-1/4} \\ & \cdot \exp\left[h \int_0^{\theta_0} (\cos^2\theta - \cos^2\theta_0)^{1/2}\, d\theta\right] \\ & \cdot \cos\left[h \int_{\theta_0}^{\theta} (\cos^2\theta_0 - \cos^2\theta)^{1/2}\, d\theta - \frac{\pi}{4}\right]. \end{aligned} \tag{A14}$$

The last integral in (A13) and (A14) is given by

$$\begin{aligned} & \int_{\theta_0}^{\theta} (\cos^2\theta_0 - \cos^2\theta)^{1/2}\, d\theta \\ & \quad = \cos^2\theta_0 \left[B\left(\frac{\pi}{2}, \cos\theta_0\right) - B\left(\sin^{-1}\left[\frac{\cos\theta}{\cos\theta_0}\right], \cos\theta_0\right)\right]. \end{aligned} \tag{A15}$$

Let us now impose the condition that Se and So be periodic in θ with period 2π. This condition will determine the possible values of θ_0. To apply this condition it is not necessary to obtain the asymptotic formulas for Se and So in the interval $\pi - \theta_0 \leqq \theta \leqq \pi$. In fact the formulas for the interval $0 \leqq \theta \leqq \pi/2$ suffice for this purpose. This is so because the periodic solutions are either even or odd about the value $\theta = \pi/2$, as follows from (1)–(3). Therefore the periodicity conditions reduce to

$$Se(h,0) = 0, \qquad \text{or} \qquad Se'(h,0) = 0, \tag{A16}$$

$$So(h,0) = 0, \qquad \text{or} \qquad So'(h,0) = 0. \tag{A17}$$

The vanishing of either Se or So yields the same equation, namely

$$h \cos^2\theta_0 B\left(\frac{\pi}{2}, \cos\theta_0\right) - \frac{\pi}{4} = (2m' + 1)\frac{\pi}{2} \qquad m' = 0, 1, \cdots. \tag{A18}$$

The vanishing of either Se' or So' yields the equation

$$h \cos^2\theta_0 B\left(\frac{\pi}{2}, \cos\theta_0\right) - \frac{\pi}{4} = 2m'\frac{\pi}{2} \qquad m' = 0, 1, 2, \cdots. \tag{A19}$$

Both (A18) and (A19) can be written as the single equation

$$h \cos^2 \theta_0 B\left(\frac{\pi}{2}, \cos \theta_0\right) = \left(m + \frac{1}{2}\right)\frac{\pi}{2} \qquad m = 0, 1, 2, \cdots. \tag{A20}$$

This completes our derivation of the asymptotic formulas for *Se* and *So*.

We will now consider the solutions $Je(h, \cosh \mu)$ and $Jo(h, \cosh \mu)$ of the modified Mathieu equation

$$J'' - (b - h^2 \cosh^2 \mu)J = 0. \tag{A21}$$

In terms of certain constants Ae and Ao, to be defined later, the initial conditions can be expressed as

$$Je(h,1) = Ae, \qquad Je'(h,1) = 0, \tag{A22}$$

$$Jo(h,1) = 0, \qquad Jo'(h,1) = Ao. \tag{A23}$$

As before, we express b by (A4) or (A5). In the second case (A5), Eq. (A21) has no turning point. Therefore, for large h, the WKB method yields the following asymptotic formulas

$$Je(h, \cosh \mu) \sim Ae(\sin \theta_0)^{1/2} (\cosh^2 \mu - \cos^2 \theta_0)^{-1/4} \cdot \cos\left[h \int_0^\mu (\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu\right], \tag{A24}$$

$$Jo(h, \cosh \mu) \sim Aoh^{-1} (\sin \theta_0)^{-1/2} (\cosh^2 \mu - \cos^2 \theta_0)^{-1/4} \cdot \sin\left[h \int_0^\mu (\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu\right]. \tag{A25}$$

The integral in (A24) and (A25) is given by

$$\int_0^\mu (\cosh^2 \mu - \cos^2 \theta_0)^{1/2}\, d\mu = \sinh \mu(1 - \cos^2 \theta_0 \operatorname{sech}^2 \mu)^{1/2} - \cos^2 \theta_0 \left[B\left(\frac{\pi}{2}, \cos \theta_0\right) - B\,(\sin^{-1}(\operatorname{sech} \mu), \cos \theta_0)\right]. \tag{A26}$$

For large values of μ, (A24) and (A25) simplify. If, in addition, $\cos \theta_0$ satisfies (A20), as in the case of product solutions of the reduced wave equation, (A24) and (A25) become

$$Je(h, \cosh \mu) \sim [Ae(h \sin \theta_0)^{1/2}](h \cosh \mu)^{-1/2} \cos\left[h \sinh \mu - \left(m + \frac{1}{2}\right)\frac{\pi}{2}\right], \tag{A27}$$

$$Jo(h, \cosh \mu)$$
$$\sim [Ao(h \sin \theta_0)^{-1/2}](h \cosh \mu)^{-1/2} \sin\left[h \sinh \mu - \left(m + \frac{1}{2}\right)\frac{\pi}{2}\right]. \tag{A28}$$

The constants Ae and Ao are defined so that

$$Ae(h \sin \theta_0)^{1/2} = Ao(h \sin \theta_0)^{-1/2} = 1. \tag{A29}$$

Next, let us consider the case in which b is given by (A4). Then (A21) has turning points at $\mu = \pm\mu_0$. First we apply the WKB method when μ is in the interval $0 \leqq \mu < \mu_0$ to obtain

$$Je(h, \cosh \mu) \sim Ae(\sinh \mu_0)^{1/2}$$
$$\cdot(\cosh^2 \mu_0 - \cosh^2 \mu)^{-1/4} \cosh\left[h \int_0^\mu (\cosh^2 \mu_0 - \cosh^2 \mu)^{1/2}\, d\mu\right], \tag{A30}$$

$$Jo(h, \cosh \mu) \sim Aoh^{-1}(\sinh \mu_0)^{-1/2}$$
$$\cdot(\cosh^2 \mu_0 - \cosh^2 \mu)^{-1/4} \sinh\left[h \int_0^\mu (\cosh^2 \mu_0 - \cosh^2 \mu)^{1/2}\, d\mu\right]. \tag{A31}$$

The integral in (A30) and (A31) is given by

$$\int_0^\mu (\cosh^2 \mu_0 - \cosh^2 \mu)^{1/2}\, d\mu = \cosh \mu_0 \left[\sinh \mu(\tanh^2 \mu_0 - \tanh^2 \mu)^{1/2} + \tanh^2 \mu_0 D\left(\sin^{-1} \frac{\tanh \mu}{\tanh \mu_0}, \tanh \mu_0\right)\right]. \tag{A32}$$

Here D is defined by

$$D(x,k) = \int_0^x (1 - k^2 \sin^2 \phi)^{-1/2} \sin^2 \phi\, d\phi = k^{-2}F(x,k) - k^{-2}E(x,k).$$

Now by applying the WKB connection formulas we obtain, for $\mu > \mu_0$, the formulas

$$Je(h, \cosh \mu) \sim Ae(\sinh \mu_0)^{1/2}$$
$$\cdot(\cosh^2 \mu - \cosh^2 \mu_0)^{-1/4} \exp\left[h \int_0^{\mu_0} (\cosh^2 \mu_0 - \cosh^2 \mu)^{1/2}\, d\mu\right] \cdot\cos\left[h \int_{\mu_0}^\mu (\cosh^2 \mu - \cosh^2 \mu_0)^{1/2}\, d\mu - \frac{\pi}{4}\right], \tag{A33}$$

$$Jo(h, \cosh \mu) \sim Aoh^{-1}(\sinh \mu_0)^{1/2}$$
$$\cdot(\cosh^2 \mu - \cosh^2 \mu_0)^{-1/4} \exp\left[h \int_0^{\mu_0} (\cosh^2 \mu_0 - \cosh^2 \mu)^{1/2}\, d\mu\right] \cdot\cos\left[h \int_{\mu_0}^\mu (\cosh^2 \mu - \cosh^2 \mu_0)^{1/2}\, d\mu - \frac{\pi}{4}\right]. \tag{A34}$$

The last integral in (A33) and (A34) is given by

$$\begin{aligned}\int_{\mu_0}^{\mu} (\cosh^2 \mu - \cosh^2 \mu_0)^{1/2}\, d\mu &= \sinh \mu \left(1 - \frac{\cosh^2 \mu_0}{\cosh^2 \mu}\right)^{1/2} \\ &- \cosh \mu_0 \left[E\left(\frac{\pi}{2}, \operatorname{sech} \mu_0\right) - E\left(\sin^{-1} \frac{\cosh \mu_0}{\cosh \mu}, \operatorname{sech} \mu_0\right)\right].\end{aligned} \tag{A35}$$

As before, if μ is large and if μ_0 satisfies (A9), the asymptotic formulas (A33) and (A34) both simplify to

$$J(h, \cosh \mu) \sim (h \cosh \mu)^{-1/2} \cos\left[h \sinh \mu - \left(m + \frac{1}{2}\right)\frac{\pi}{2}\right]. \qquad \text{A(36)}$$

In (A36) we have utilized the definitions of Ae and Ao, which yield the relations

$$\begin{aligned}&Ae(h \sinh \mu_0)^{1/2} \exp\left[h \int_0^{\mu_0} (\cosh^2 \mu_0 - \cosh^2 \mu)^{1/2}\, d\mu\right] \\ &\quad = Ao(h \sinh \mu_0)^{-1/2} \exp\left[h \int_0^{\mu_0} (\cosh^2 \mu_0 - \cosh^2 \mu)^{1/2}\, d\mu\right] = 1.\end{aligned} \tag{A37}$$

For completeness, we include the following integral:

$$\begin{aligned}\int_{\mu}^{\mu_0} (\cosh^2 \mu_0 - \cosh^2 \mu)^{1/2}\, d\mu &= \cosh \mu_0 \left[\tanh^2 \mu_0 D\left(\frac{\pi}{2}, \tanh \mu_0\right)\right. \\ &\left. - \tanh^2 \mu_0 D\left(\sin^{-1} \frac{\tanh \mu}{\tanh \mu_0}, \tanh \mu_0\right) - \sinh \mu (\tanh^2 \mu_0 - \tanh^2 \mu)^{1/2}\right].\end{aligned} \tag{A38}$$

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