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ON THE SMALL VIBRATIONS OF A SLIGHTLY STIFF PENDULUM

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SUMMARY

The singular perturbation problem of a slightly stiff pendulum is solved by a method of undetermined coefficients

0. INTRODUCTION

In this paper the singular perturbation problem of small vibrations of a slightly stiff pendulum is solved by a method of undetermined coefficients. G. H. HANDELMAN, and J. B.KELLER, [1] have solved this problem by using a modified version of the method devised by J. MOSER [2]. In that method one has to find the fundamental system of solutions and then by the help of the characteristic determinant one finds the eigenvalues and eigenfunctions. For higher order differential equations the approximation of eigenvalues and eigenfunctions by this method may be quite involved. In our method, we expand the eigenfunctions and eigenvalues in terms of the small parameter η (which is proportional to the square root of the stiffness of the wire) and add the boundary layer expansions to the expansion of eigenfunctions. Since in our method we do not require the fundamental system of solutions, we do not have to solve the higher order equations and at every stage we solve a sequence of lower order problems. This way we get a uniformly valid expansion for the eigenfunctions through out the domain of definition of the problem. (cf. HANDELMAN etc. (3), MIRANKER (4)).

1. FORMULATION OF THE PROBLEM

We consider a wire pendulum of length L which moves in the xy plane. ρ is the mass per unit length of the wire, EI the flexural rigidity of the wire, $y(x, t)$ denotes its deflection from a vertical axis and M is the mass of the bob. The deflections of the wire are assumed to be small so that the ordinary, linear, EULER-BERNOULLI theory can be applied; in addition shear deformation and rotary inertia are neglected. Following HANDELMAN and KELLER[1] we take the equation of motion of the wire as

$$\mathrm{EI}\,\frac{\partial^4 y}{\partial x^4}-\frac{\partial}{\partial x}\left(p\,\frac{\partial y}{\partial x}\right)=-\,\rho\,\frac{\partial^2 y}{\partial t^2}. \tag{1.1}$$

Here $p(x)$, the total tensile force at x, is given by

$$p(x)=\rho g(L-x)+Mg$$

where g is the acceleration of gravity. The pendulum wire is clamped at $x=0$, and hence satisfies the boundary conditions

$$y(0,\ t)=0, \quad \frac{\partial y}{\partial x}(0,\ t)=0. \tag{1.2}$$

At the free end of the pendulum (i.e. the bob) the bending moment vanishes. Therefore

$$\mathrm{EI}\,\frac{\partial^2 y}{\partial x^2}\,(L,\ t)=0. \tag{1.3}$$

The equation of motion of the bob gives us the condition

$$\mathrm{EI}\,\frac{\partial^3 y}{\partial x^3}\,(L,\ t)-Mg\,\frac{\partial y}{\partial x}\,(L,\ t)=M\,\frac{\partial^2 y}{\partial t^2}\,(L,\ t). \tag{1.4}$$

We look for periodic solutions of (1.1) to (1.4). To this end we make the substitution $y(x,\ t)=z(x)e^{i\lambda t}$ in the above equations and introduce the following dimensionless variables and parameters for convenience; $\alpha = \rho L/M$, $\nu=\lambda(L/g)^{1/2}$, $\eta^2=\mathrm{EI}/MgL^2$, $s=\dfrac{x}{L}$, and $w(s)=z(x)/L$. We thus obtain the following system of equations:

$$\eta^2 w^{iv}-[\{\alpha(1-s)+1\}w']'=\alpha\nu^2 w, \qquad 0<s<1 \tag{1.5}$$

$$w(0)=w'(0)=0 \tag{1.6}$$

$$w''(1)=0 \tag{1.7}$$

$$\eta^2 w'''(1)=w'(1)-\nu^2 w(1) \tag{1.8}$$

where' denotes the derivative with respect to s. We wish to find the eigenvalues ν^2 and eigenfunctions $w(s)$ of the foregoing system of equations for prescribed values of η^2 and α.

2. SOLUTION OF THE PROBLEM

We seek solutions of (1.5)—(1.8) of the form

$$w(s) = exp[\eta^{-1}\psi(s)] \sum_{k=0}^{\infty} u_k(s)\eta^k + \sum_{k=0}^{\infty} v_k(s)\eta^k \tag{2.1}$$

and

$$\nu^2 = \sum_{k=0}^{\infty} \eta^k \nu_k^2 \tag{2.2}$$

If we substitute these expansions in (1.5) and equate to zero the coefficients of $\eta^0, \eta, \eta^2, \eta^{-2}\exp(\eta^{-1}\psi(s)), \eta^{-1}\exp(\eta^{-1}\psi(s)), \exp(\eta^{-1}\psi(s)),\ldots$ we obtain the following set of equations :
From the coefficient of η^j,

$$-[\alpha(1-s)+1]v_j'' + \alpha v_j' = \sum_{k+l=j} \alpha \nu_k^2 v_l - v_{j-2}^{iv}, \; j = 0, 1, 2\ldots \tag{2.3}$$

where $v_{-1} = v_{-2} = 0$.
From the coefficient of $\eta^{-2}\exp(\eta^{-1}\psi(s))$

$$(\psi')^4 - [\alpha(1-s)+1](\psi')^2 = 0. \tag{2.4}$$

From the coefficient of $\eta^{-1}\exp(\eta^{-1}\psi(s))$

$$6(\psi')^2\psi u_0 + 4(\psi')^3 u_0' - [\alpha(1-s)+1](2\psi' u_0' + \psi' u_0) + \alpha\psi' u_0 = 0 \tag{2.5}$$

and from the coefficient of $\exp(\eta^{-1}\psi(s))$

$$\begin{aligned}&6(\psi')^2\psi' u_1 + 4(\psi')^3 u_1' - [\alpha(1-s)+1](2\psi' u_1' + \psi'' u_1) + \alpha\psi' u_1 + \\ &+ 6(\psi')^2 u_0'' + 6\psi''\psi' u_0' + 4\psi''\psi' u_0 + 9\psi'\psi'' u_0' - [\alpha(1-s)+1]u_0'' \\ &+ \alpha u_0' - \alpha\nu_0^2 u_0 = 0.\end{aligned} \tag{2.6}$$

By substituting the expansions (2.1) and (2.2) into the boundary conditions (1.6)—(1.8) and equating different powers of η to zero we obtain the following sets of boundary conditions :
From $w(0) = 0$,

$$u_j(0) + v_j(0) = 0, \quad j = 0, 1, 2,\ldots \tag{2.7}$$

From $w'(0) = 0$,

$$u_j(0) = (\alpha+1)^{-1/2}[u_{j-1}'(0) + v_{j-1}'(0)], \; j = 0, 1, 2, \ldots \tag{2.8}$$

where $u_{-1} = v_{-1} \equiv 0$.

From $w'(1) = 0$,

$$u_j(1) = \frac{\alpha}{2} u_{j-1}(1) - 2u''_{j-2}(1) - v''_{j-2}(1), \quad j = 0, 1, 2, \ldots \tag{2.9}$$

where $u_{-1} = u_{-2} = v_{-1} = v_{-2} \equiv 0$
and from $\eta^2 w'''(1) = w'(1) - \nu^2 w(1)$,

$$v_0'(1) - \nu_0^2 v_0(1) = 2u_0'(1) - \frac{3\alpha}{2} u_0(1) + \nu_0^2 u_0(1) \tag{2.10}$$

$$v_1'(1) = \nu_0^2 v_1(1) = 3u''_0(1) - \frac{3\alpha}{2} u_0'(1) - \frac{\alpha^2}{4} u_0(1) + 2u_0'(1) - \\ - \frac{3\alpha}{2} u_1(1) + \nu_0^2 u_1(1) + \nu_1^2[v_0(1) + u_0(1)] \tag{2.11}$$

etc. . . .
From (2.4) we get two solutions for $\psi(s)$

$$\psi^- = -\int_0^s [\alpha(1-t) + 1]^{1/2} dt \tag{2.12}$$

and

$$\psi^+ = \int_1^s [\alpha(1-t) + 1]^{1/2} dt \tag{2.13}$$

such that ψ^- vanishes at $s = 0$ and ψ^+ vanishes at $s = 1$.

If we substitute the value of ψ^- and its derivatives in (2.5) we obtain the differential equation for u_0^-,

$$4[\alpha(1-s) + 1]u_0^{-\prime} - 3\alpha u_0^- = 0. \tag{2.14}$$

This differential equation together with the boundary condition (2.8) for $j = 0$ (i.e. $u_0(0) = 0$) yields the solution

$$u_0^-(s) \equiv 0. \tag{2.15}$$

Similarly by substituting the value of ψ^+ from (2.13) into (2.5) we obtain exactly the same differential equation for u_0^+ as that for u_0^-. From this differential equation and from the boundary condition (2.9) for $j = 0$ (*i.e.* $u_0^+(1) = 0$)
We obtain

$$u_0^+(s) = 0 \tag{2.16}$$

Now we solve the equation (2.3) for $j = 0$, i.e.,

$$[\alpha(1-s)+1]v_0'' - \alpha v' + \alpha \nu_0^2 = 0 \tag{2.17}$$

with the boundary conditions

$$v_0(0) = 0, \; v_0'(1) - \nu_0^2 v_0(1) = 0, \qquad 2.18, \text{ a, b})$$

Equation (2.17) can be solved in terms of Bessel functions and the eigenvalues and eigenfunctions can be found with the help of the boundary conditions (2.18).

For small values of α, we let

$$v_0 = v_{00} + \alpha v_{01} + \alpha^2 v_{02} + \dots \tag{2.19}$$

and

$$\alpha \nu_0^2 = \nu_{00}^2 + \alpha \nu_{01}^2 + \alpha^2 \nu_{02}^2 + \dots \tag{2.20}$$

Substituting these expansions in (2.17) and solving the resulting differential equations for v_{00}, v_{01} etc., we obtain

$$v_{00}(s) = \sqrt{2} \sin n \pi s, \; \nu_{00}^2 = n^2\pi^2 \qquad (2.21 \text{ a, b})$$

and

$$v_{01}(s) = \frac{\sqrt{2}}{4}\left(\frac{2}{n^2\pi^2} - 1\right) \sin n \pi s + \frac{\sqrt{2}}{2} s \sin n \pi s$$

$$+ \sqrt{2}\left(\frac{1}{n\pi} - \frac{n\pi}{4}\right) s \cos n \pi s + \sqrt{2}\, \frac{n\pi}{4} s^2 \cos n\pi s \tag{2.22}$$

$$\nu_{01}^2 = \frac{n^2\pi^2}{2} + 2. \tag{2.23}$$

Next we find u_1^+ and u_1^-. In equation (2.6) we substitute the values of ψ^- and its derivatives and obtain the equation

$$4[\alpha(1-s)+1]u_1^{-\prime} - 3\alpha u_1^- = 0 \tag{2.24}$$

From this equation and the boundary condition (2.8) for $j = 1$, we obtain

$$u_1^-(s) = (\alpha+1)^{\frac{1}{4}}\left\{\sqrt{2}n\pi + \alpha\left(\frac{3\sqrt{2}}{2n\pi} - \frac{\sqrt{2}\,n\pi}{2}\right)\right\}(\alpha(1-s)+1)^{-\frac{3}{4}} \tag{2.25}$$

A similar equation for u_1^+ and the boundary condition (2.9) for $j = 1$ gives us

$$u_1^+(s) = 0 \tag{2.25}$$

For finding ν_1^2, we consider (2.3), (2.7) for $j = 1$, and (2.11), i.e.,

$$[\alpha(1 - s) + 1]v_1'' - \alpha v_1' + \alpha v_0^2 \nu_1 + \alpha \nu_1^2 v_0 = 0 \qquad (2.26)$$

$$v_1(0) = -u_1(0), \; v_1'(1) - \nu_0^2 v_1(1) = v_1^2 \nu_0(1) \qquad (2.27\text{ a, b})$$

We multiply equation (2.26) by $v_0 (= v_{00} + \alpha v_{01})$, and integrate it from 0 to 1. After doing some integration by parts and using the boundary conditions for v_0 and v_1, we obtain

$$\nu_1^2 \sim (1 + \alpha)^{\frac{1}{2}} \left\{\sqrt{2} n\pi + \alpha \left(\frac{3\sqrt{2}}{2n\pi} - \frac{\sqrt{2}}{2} n\pi\right)\right\}^2 \left(\alpha + \frac{2\alpha^2}{n^2\pi^2}\right)^{-1} \qquad (2.28)$$

If we expand this in power of α we get

$$\nu_1^2 = \frac{1}{\alpha}\,[2n^2\pi^2 + \alpha(2 - n^2\pi^2) + 0(\alpha^2)]$$

which agrees with the expansion of equation (5.11) of reference [1], up to terms of order α.

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