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A Pendulum Theorem

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A pendulum theorem†

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We consider N linked pendulums which are inverted and balanced on top of one another, and establish a general theorem which shows how they may be stabilized by small vertical oscillations of the support.

1. Introduction

Stephenson (1908*a*, *b*) showed that it is possible to stabilize a single rigid pendulum in its inverted, or upside-down, equilibrium position by subjecting the pivot to small vertical oscillations of suitably high frequency. He confirmed his theoretical predictions by a practical demonstration of the phenomenon.

While this is a well-known curiosity of classical mechanics it does not seem to be generally known that an inverted *double*, or even *triple* pendulum can be stabilized in the same way. This was, again, first predicted theoretically by Stephenson, in a comparatively overlooked paper of 1909, though the idea has reappeared in a number of subsequent studies (Lowenstern 1932; Hsu 1961; Kalmus 1970; Otterbein 1982; Leiber & Risken 1988).

Here we present a simple but general theorem on the linear stability of an inverted N-pendulum *of any kind*. The generality of the theorem is achieved by relating the stability question to just two elementary properties of the system as a whole when it is in its non-inverted, or downward-hanging, state.

2. The stability of upside-down pendulums

Theorem. *Let N pendulums hang down, one from another, under gravity g, each having one degree of freedom, the uppermost being suspended from a pivot point* O. *Let ω_{max} and ω_{min} denote the largest and the smallest of the natural frequencies of small oscillation about this equilibrium state.*

Now turn the whole system upside-down. The resulting configuration of the pendulums can be stabilized (according to linear theory, at least) if we subject the pivot point O *to vertical oscillations of suitable amplitude ϵ and frequency ω_0. When $\omega_0^2 \gg \omega_{\text{max}}^2$ the stability criterion is*

$$\sqrt{2}g/\omega_0\,\omega_{\text{min}} < \epsilon < 0.450g/\omega_{\text{max}}^2. \tag{2.1}$$

Note. When several pendulums are involved, ω_{max}^2 is typically much greater than ω_{min}^2. The condition $\omega_0^2 \gg \omega_{\text{max}}^2$ is then *necessary* for the stability of the inverted state, so (2.1) then gives the whole stability region in the ϵ–ω_0 plane, as in figure 1.

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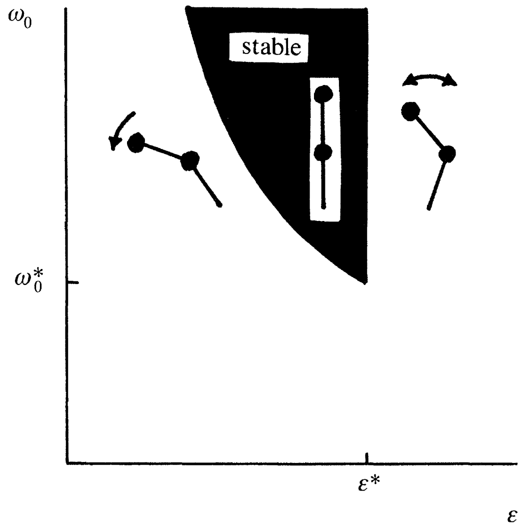

Figure 1. Typical region of linear stability for an upside-down N-pendulum, as given by (2.1). Here $\epsilon^* = 0.450\, g/\omega_{\max}^2$ and $\omega_0^* = 3.143\omega_{\max}^2/\omega_{\min}$. The sketches indicate theoretical predictions of the behaviour just outside this stability region. Thus, if at a given frequency ω_0 the amplitude ϵ is just too large, the system is unstable to rapidly growing buckling oscillations at frequency $\frac{1}{2}\omega_0$. If, on the other hand, ϵ is a little too small, the pendulums fall over comparatively slowly, keeping to the same side of the vertical.

Proof. Consider first small disturbances to the system about the original, i.e. non-inverted equilibrium state, the pivot point O being fixed. Let the natural frequencies be ω_i, and let X_i be the corresponding normal coordinates, each of which will be some linear combination of the (small) angles which the pendulums make with the downward vertical. We then have

$$\ddot{X}_i + \omega_i^2 X_i = 0, \quad i = 1, \ldots, N. \tag{2.2}$$

Now, each of the quantities ω_i^2 will be simply proportional to g; this follows from the standard theory of small oscillations based on Lagrange's equations of motion (see, for example, Landau & Lifschitz 1976), given that the uniform gravitational field g is the sole source of potential energy in this problem. If we tackle the stability of the inverted state by changing the sign of g we then find that small disturbances are governed instead by

$$\ddot{X}_i - \omega_i^2 X_i = 0, \quad i = 1, \ldots, N. \tag{2.3}$$

Suppose now that the pivot point O oscillates up and down so that its coordinate in the direction of the upward vertical is $h = -\epsilon \cos \omega_0 t$. We may allow for this simply by replacing g in the above argument by apparent gravity $g + \ddot{h} = g + \epsilon\omega_0^2 \cos \omega_0 t$. In this way we find that

$$\ddot{X}_i - \omega_i^2 (1 + (\epsilon\omega_0^2/g) \cos \omega_0 t) X_i = 0, \quad i = 1, \ldots, N.$$

On introducing the scaling $T = \omega_0 t$ we therefore obtain N uncoupled Mathieu equations:

$$\mathrm{d}^2 X_i/\mathrm{d}T^2 + (\alpha_i + \beta_i \cos T) X_i = 0, \quad i = 1, \ldots, N, \tag{2.4}$$

where

$$\alpha_i = -\omega_i^2/\omega_0^2, \quad \beta_i = -\omega_i^2 \epsilon/g. \tag{2.5}$$

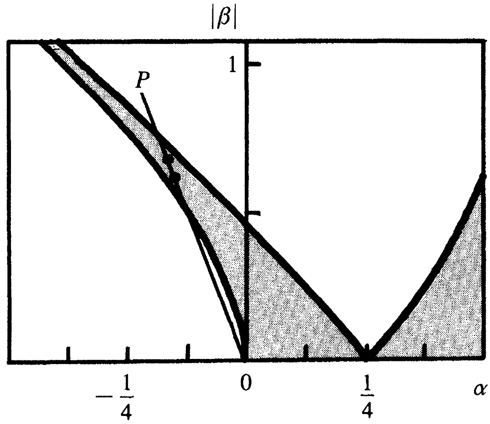

Figure 2. Stable (shaded) and unstable regions for the $X = 0$ solution of Mathieu's equation $\mathrm{d}^2X/\mathrm{d}T^2 + (\alpha + \beta \cos T)X = 0$. The points (2.6) lie on the straight line OP, and for the inverted state to be stable they must all lie within the shaded region with $\alpha < 0$, in the manner indicated (for $N = 2$).

Now, the stability regions of Mathieu's equation are well known and indicated in figure 2. We achieve stability of our inverted system if, by choosing ϵ and ω_0 appropriately, we can place all the points

$$\omega_i^2(-1/\omega_0^2, \epsilon/g), \quad i = 1, \ldots, N \tag{2.6}$$

in the shaded part of the region $\alpha < 0$.

To see that this can indeed be done it is simplest to pass directly to the high-frequency limit, i.e. $\omega_0^2 \gg \omega_{\max}^2$, so that ω_i^2/ω_0^2 is small for all $i = 1, \ldots, N$. Now, when $|\alpha|$ is small the stability boundary which passes through the origin is given asymptotically by $\alpha = -\frac{1}{2}\beta^2$ (see, for example, Jordan & Smith 1987, p. 257). Moreover, the upper boundary to the shaded region in $\alpha < 0$ is known to meet the $\alpha = 0$ axis at $|\beta| = 0.450$. Thus all the points (2.6) will be in the stable region if

$$(2\omega_i^2/\omega_0^2)^{\frac{1}{2}} < \omega_i^2\,\epsilon/g < 0.450, \quad i = 1, \ldots, N,$$

i.e. if

$$\sqrt{2}g/\omega_i\,\omega_0 < \epsilon < 0.450g/\omega_i^2 \tag{2.7}$$

for all $i = 1, \ldots, N$. This is ensured by (2.1).

While this completes the proof, it is worth considering a more general and geometric approach to the matter. All the points (2.6) lie on a line through the origin with slope $-\epsilon\omega_0^2/g$. Imagine, then, that we attach an elastic band OP to the origin in figure 2, stretch it out, and mark points along it so that their distances from the origin are in the proportion $\omega_1^2 : \omega_2^2 : \ldots : \omega_N^2$. By choosing ϵ and ω_0 we may vary the length of the band, and its slope $-\epsilon\omega_0^2/g$, at will.

Now, if $\omega_{\max}$ and $\omega_{\min}$ are almost equal, so that the marked points are tightly clustered along the band, we may evidently steer those points into the stable region in figure 2 even if $|\alpha|$ is not particularly small, so that ω_0^2 is not large compared with $\omega_{\max}^2$. If, on the other hand, $\omega_{\max}$ is substantially greater than $\omega_{\min}$ – as will typically be the case – we may still steer all the points into the stable region, but only by moving the band very close to the $\alpha = 0$ axis in figure 2 and by shrinking it accordingly. The stability criterion (2.1) will then apply, and the region of stability in the ϵ–ω_0 plane will be as indicated in figure 1.

3. Examples

(a) *Stephenson's system inverted*

Stephenson's original double-pendulum analysis was for two identical rods of uniform density and equal length l. An elementary exercise shows that the natural frequencies are then

$$\omega_{\max,\min} = (3 \pm 6/\sqrt{7})^{\frac{1}{2}}(g/l)^{\frac{1}{2}}, \tag{3.1}$$

and (2.1) becomes

$$(1.653/\omega_0)(g/l)^{\frac{1}{2}} < \epsilon/l < 0.085. \tag{3.2}$$

The first of these inequalities may be written as

$$(\tfrac{1}{2}\epsilon\omega_0)^2 > 0.683gl, \tag{3.3}$$

which is Stephenson's original criterion, derived by a different method.

His formula for the inverted triple pendulum appears, unfortunately, to be in error; the theorem of §2 gives

$$(1.738/\omega_0)(g/l)^{\frac{1}{2}} < \epsilon/l < 0.03 \tag{3.4}$$

as the high-frequency stability criterion for an inverted triple pendulum consisting of three equal heavy rods.

(b) *The point-mass double pendulum*

Consider now a double pendulum consisting of two point masses m_1, m_2 and two light rods of equal length l (as sketched in figure 1). The normal modes of small oscillation about the lower, inherently stable equilibrium position (with O fixed) have frequencies

$$\omega_{\max} = \frac{(g/l)^{\frac{1}{2}}}{(1-m^{\frac{1}{2}})^{\frac{1}{2}}} \quad \text{and} \quad \omega_{\min} = \frac{(g/l)^{\frac{1}{2}}}{(1+m^{\frac{1}{2}})^{\frac{1}{2}}}, \tag{3.5}$$

where

$$m = m_2/(m_1+m_2), \tag{3.6}$$

m_1 denoting the point mass which is attached directly to O. The high-frequency criterion (2.1) for the stability of the inverted equilibrium position then reduces to

$$\frac{\sqrt{2}(1+m^{\frac{1}{2}})^{\frac{1}{2}}}{\omega_0}\left(\frac{g}{l}\right)^{\frac{1}{2}} < \frac{\epsilon}{l} < 0.450(1-m^{\frac{1}{2}}). \tag{3.7}$$

For comparison, we show in figure 3 the stability boundaries for this inverted double pendulum obtained by direct numerical integration of the equations of motion, linearized about the upward vertical. At high frequencies ω_0 the stable range of ϵ/l is given well by (3.7), but if the upper mass m_2 is much smaller than the lower mass m_1 it is possible to stabilize the system with a comparatively small driving frequency ω_0 and (3.7) does not then apply. This case corresponds, of course, to $\omega_{\max}$ and $\omega_{\min}$ being almost equal (see (3.5) and the end of §2); indeed, when $m = 0$ the two coincide and (3.7) reduces to the stability criterion for an inverted single pendulum.

More typically, rather high frequencies ω_0 are needed to give stability, and (2.1) then provides a good approximation to the whole stability region in the ϵ–ω_0 plane. The case of equal masses ($m = 0.5$) is a good example:

$$(1.85/\omega_0)(g/l)^{\frac{1}{2}} < \epsilon/l < 0.132, \tag{3.8}$$

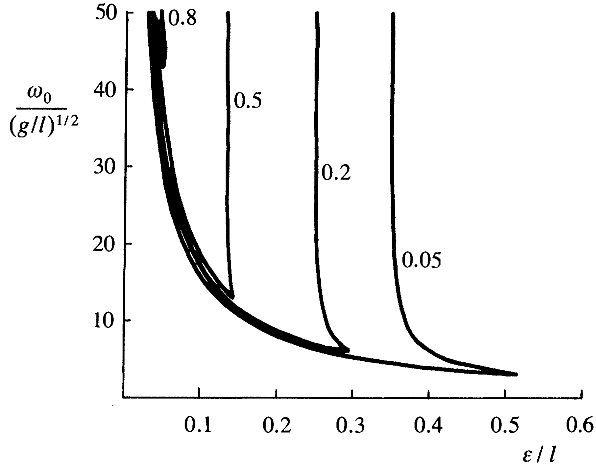

Figure 3. Computed linear stability boundaries for the inverted point-mass double pendulum for four values of $m = m_2/(m_1+m_2)$.

for the right-hand stability boundary in figure 3 is indeed almost a straight line parallel to the ω_0-axis, save at the very lowest frequencies giving stability.

The larger the value of m, clearly, the smaller we have to take ϵ and the higher we have to take ω_0, but there is in theory no upper limit to the mass ratio m_2/m_1 which can be stabilized by vertical oscillations of the support.

(c) *An inverted N-pendulum*

We now apply the theorem of §2 to a pendulum with N equal point masses connected by N equal light rods of total length $L = Nl$.

The natural frequencies of oscillation about the lower equilibrium position are

$$\omega_i = \xi_i^{\frac{1}{2}}(g/l)^{\frac{1}{2}}, \quad i = 1, \ldots, N, \tag{3.9}$$

where ξ_i denotes the ith zero of the Laguerre polynomial $L_N(x)$; this was shown by Daniel Bernoulli and by Euler in the late 1730s (see Cannon & Dostrovsky 1981).

In the case $N = 3$ this gives

$$\omega_1 = 0.64(g/l)^{\frac{1}{2}}, \quad \omega_2 = 1.51(g/l)^{\frac{1}{2}}, \quad \omega_3 = 2.51(g/l)^{\frac{1}{2}}, \tag{3.10}$$

and on taking the greatest and least of these we find that the criterion (2.1) becomes

$$(2.21/\omega_0)(g/l)^{\frac{1}{2}} < \epsilon/l < 0.07. \tag{3.11}$$

For comparison, figure 4 shows the stability boundary obtained by numerical integration of the linearized equations of motion.

When N is large we can use the asymptotic expression $\xi_i \sim \chi_i^2/4N$, where χ_i denotes the ith zero of the Bessel function $J_0(x)$. With $\chi_1 \doteqdot 2.4$ and $\chi_N \doteqdot (N-\frac{1}{4})\pi$ we then have

$$\omega_{\min} \doteqdot 1.2(g/L)^{\frac{1}{2}}, \quad \omega_{\max} \doteqdot \tfrac{1}{2}(N-\tfrac{1}{4})\,\pi(g/L)^{\frac{1}{2}},$$

and therefore

$$\frac{1.18}{\omega_0}\left(\frac{g}{L}\right)^{\frac{1}{2}} < \frac{\epsilon}{L} < \frac{0.182}{(N-\frac{1}{4})^2} \tag{3.12}$$

as the criterion for the stability of this particular inverted N-pendulum, when N is large.

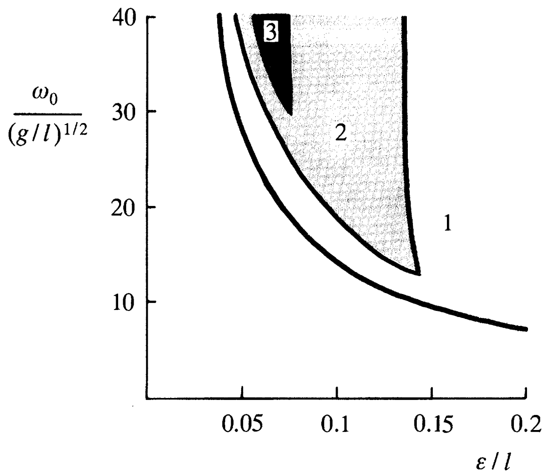

Figure 4. Computed regions of linear stability for the upside-down N-pendulum, for $N = 1, 2, 3$, in the case when the various (point) masses are equal.

Suppose finally that L is fixed. Then, the larger the value of N, the smaller we must take ϵ, and, consequently, the larger we must take ω_0 if stability is to be achieved. The limit $N \to \infty$ for fixed L corresponds to a perfectly flexible string, and was of major interest in the original investigations of Bernoulli and Euler. The region of stability of the *inverted* state vanishes altogether in this limit, so we cannot perform the Indian rope trick.

4. Concluding remarks

While the theorem of this paper concerns linear stability, numerical integrations of the full equations of motion have also been carried out, together with laboratory experiments (Acheson & Mullin 1993). These further studies show the stability of these inverted states to be remarkably robust, at least in the cases $N = 1, 2, 3$. In particular, we have seen an inverted triple pendulum recover from very severe initial disturbances and gradually wobble back to the upward vertical.

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