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PONYTAIL MOTION*

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Abstract. A jogger's ponytail sways from side to side as the jogger runs, although her head does not move from side to side. The jogger's head just moves up and down, forcing the ponytail to do so also. We show in two ways that this vertical motion is unstable to lateral perturbations. First we treat the ponytail as a rigid pendulum, and then we treat it as a flexible string; in each case, it is hanging from a support which is moving up and down periodically, and we solve the linear equation for small lateral oscillation. The angular displacement of the pendulum and the amplitude of each mode of the string satisfy Hill's equation. This equation has solutions which grow exponentially in time when the natural frequency of the pendulum, or that of a mode of the string, is close to an integer multiple of half the frequency of oscillation of the support. Then the vertical motion is unstable, and the ponytail sways.

Key words. instability, parametric resonance, Hill's equation

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1. Introduction.

The ponytail of a running jogger sways from side to side, but the jogger's head generally does not move from side to side. The head just moves up and down, so the ponytail also moves up and down with it. But, as we shall show, this vertical motion of the hanging ponytail is unstable to lateral perturbations. The resulting lateral motion, the swaying, is an example of parametric excitation, a phenomenon which is common in oscillating mechanical and electrical systems.

We shall demonstrate this instability, and analyze the resulting motion, in two different ways. First, in section 2, we shall represent the ponytail as a rigid pendulum hanging from a support which is moving up and down periodically. The pendulum also moves up and down periodically. Any small angular deviation $\theta(t)$ from the vertical position satisfies Hill's equation, a linear second order ordinary differential equation with a periodic coefficient (Stoker [1]). This equation has one solution, which grows exponentially in time if the natural frequency of the pendulum is close to an integer multiple of half the frequency of oscillation of the support (Magnus and Winkler [2]). Then the purely vertical motion of the pendulum is unstable, and it sways.

Next, more realistically, in section 3 we represent the ponytail as a flexible string hanging from a vertically oscillating support. Again a purely vertical motion of the string is possible. As was shown by Belmonte et al. [3], the linear equation for small lateral perturbations of this motion has an infinite number of modes of periodic vibration. Each mode amplitude satisfies Hill's equation. Therefore, just like the pendulum, a mode is unstable when its natural frequency is close to an integer multiple of half the frequency of oscillation of the support.

A still more realistic model of a ponytail is an inextensible rod with small bending stiffness, described in section 5.

2. Ponytail as a rigid rod.

Suppose that a runner moves with constant speed U along the positive z-axis, and that her head moves up and down with the periodic

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vertical displacement $a(t)$ along the y-axis. One end of her ponytail is attached to her head, at the position

$$x = 0, \quad y = a(t), \quad z = Ut. \tag{2.1}$$

We assume that the ponytail can move in the transverse plane $z = Ut$ but not normal to that plane, because it would hit the runner's head or neck.

Let the ponytail be represented as a uniform rigid rod of length L, so that its center of mass is at its midpoint. Then its position in the plane $z = Ut$ is determined by the angle $\theta(t)$ which it makes with the downward pointing vertical line through the point (2.1). If it is free to rotate about the point (2.1), it is a pendulum, and the angle $\theta(t)$ satisfies the following equation of motion:

$$\theta_{tt} + \frac{2}{L}(g + a_{tt})\sin\theta = 0. \tag{2.2}$$

This is the usual equation for a pendulum with a fixed endpoint, but with the vertical acceleration a_{tt} of the endpoint added to the acceleration of gravity g.

Equation (2.2) has two constant solutions, $\theta_0 = 0$ and $\theta_0 = \pi$, in the interval $0 \leq \theta < 2\pi$. The first, $\theta_0 = 0$, represents the pendulum hanging straight down, and the second, $\theta_0 = \pi$, represents it balanced pointing upward. The stability or instability of either solution is determined by the equation for the perturbation $\dot{\theta}(t)$ obtained by linearizing (2.2) about θ_0. The linearized equation is

$$\dot{\theta}_{tt} \pm \frac{2}{L}(g + a_{tt})\,\dot{\theta} = 0, \qquad + \text{ for } \theta_0 = 0, \quad - \text{ for } \theta_0 = \pi. \tag{2.3}$$

When $a_{tt} = 0$, the solution for $\dot{\theta}$ is sinusoidal for $\theta_0 = 0$ and exponentially growing or decaying for $\theta_0 = \pi$. Then the hanging pendulum is stable (a small perturbation stays small), and the balanced pendulum is unstable (a small perturbation can grow to be large).

When a_{tt} is not zero but is a periodic function of t, (2.2) is called Hill's equation after the American astronomer G.W. Hill. He derived and studied it to determine whether the periodic motion of the Moon about Earth is stable. The results for that equation can be expressed in terms of the dimensionless parameter $2g/L\omega^2$, where ω is the angular frequency of $a(t)$. For $a(t)$ given, there are infinitely many intervals of $2g/L\omega^2$ throughout which Hill's equation has solutions that grow exponentially with t [2]. When $2g/L\omega^2$ lies in one of these intervals, the hanging pendulum is unstable. The resulting large amplitude motion is the observed swaying of the ponytail.

For $|a_{tt}|$ small, each instability interval contains a point $2g/L\omega^2 = k^2/4$, with k an integer. At this point $(2g/L)^{1/2} = k\omega/2$, so there the natural frequency of the pendulum $(2g/L)^{1/2}$ is an integer k times $\omega/2$, which is half the frequency of the vertical motion of the head. For a ponytail of length $L = 25$ cm, the natural frequency is $(2 \times 980/25)^{1/2} \approx 8.85$ radians/sec. ≈ 1.41 cycles/sec. If the frequency of the head motion is twice this ($\omega = 17.71$ radians/sec. ≈ 2.82 cycles/sec.), then the condition given above for instability will hold with $k = 1$.

A cycle corresponds to a step with one leg, so 2.82 cycles/sec. corresponds to $(60)(2.82) = 169$ cycles/minute $= 169$ steps/minute. According to the Web site RunGearRun.com, elite runners' cadence is between 85 and 95 right-foot strikes per minute at all distances from 800 meters to 26 miles, corresponding to 170 to 190 steps/minute. They vary their step length to change their speed. Joggers report 140 to 160 steps/minute. These values indicate that a ponytail of length 25 cm can be expected to sway at a typical running cadence.

Stability of the vertical position of the pendulum $\theta_0 = \pi$ was studied by Stephenson [4] when $a(t) = A\cos\omega t$. (See also Stoker [1].) He found conditions on A, L, g, and ω for which the position is stable. This result was extended to an N-link pendulum by Acheson [5], and verified experimentally by Acheson and Mullin [6] for $N = 1, 2, 3$.

3. Ponytail as a flexible string. We now model the ponytail as an inextensible flexible string of length L and constant density ρ hanging in the plane $z = Ut$. As in section 2, the y-axis points vertically upward, and the x-axis is horizontal and is normal to the direction of running. The top end of the string is attached to the head at the point given in (2.1).

Let $\underline{x}(s,t) = (x(s,t),\, y(s,t))$ be the position at time t of the point at arclength distance s from the top of the string in the plane $z = Ut$. It satisfies the equation of motion

$$\rho \underline{x}_{tt} = (T\underline{x}_s)_s + \rho \underline{g}, \qquad 0 < s < L. \tag{3.1}$$

Here $T(s,t)$ is the tension in the string, and $\underline{g} = (0, -g)$ is the acceleration of gravity. The condition that s is arclength requires

$$\underline{x}_s^2 = 1, \qquad 0 < s < L. \tag{3.2}$$

At the end $s = 0$, the position in the plane $z = Ut$ is given by (2.1):

$$\underline{x}(0,t) = (0, a(t)). \tag{3.3}$$

At the end $s = L$, the tension vanishes:

$$T(L,t) = 0. \tag{3.4}$$

One solution of (3.1)–(3.4), representing a vertically hanging string moving up and down, is

$$\underline{x}^0(s,t) = [0, a(t) - s]. \tag{3.5}$$

The corresponding tension $T^0(s,t)$ is obtained by using (3.5) in the y-component of (3.1), integrating with respect to s, and using (3.4) to eliminate the constant of integration. The result is

$$T^0(s,t) = \rho(g + a_{tt})(L - s). \tag{3.6}$$

The stability of this solution is governed by the linearized problem for the perturbation $\dot{x}, \dot{T}$. That problem, obtained by linearizing (3.1)–(3.4) around the solution $\underline{x}^0$, T^0, is

$$\rho \dot{\underline{x}}_{tt} = \left(\dot{T}\underline{x}_s^0 + T^0\dot{\underline{x}}_s\right)_s = -\dot{T}_s\hat{y} + (T^0\dot{\underline{x}}_s)_s, \tag{3.7}$$

$$\underline{x}_s^0 \cdot \dot{\underline{x}}_s = -\dot{y}_s = 0, \tag{3.8}$$

$$\dot{\underline{x}}(0,t) = 0, \tag{3.9}$$

$$\dot{T}(L,t) = 0. \tag{3.10}$$

In (3.7), $\hat{y}$ is a unit vector in the positive y direction.

Upon integrating (3.8) with respect to s, and using the y-component of (3.9), we obtain

$$\dot{y}(s,t) = 0. \tag{3.11}$$

When (3.11) is used in the y-component of (3.7), it yields $\dot{T}_s = 0$. Integration of this equation using (3.10) yields

$$\dot{T}(s,t) = 0. \tag{3.12}$$

We must still determine the lateral displacement $\dot{x}(s,t)$, which satisfies the x-components of (3.7) and (3.9).

4. Solution for the lateral displacement. The x-component of (3.7) becomes, when (3.6) is used for T^0,

$$\rho \dot{x}_{tt} = \rho(g + a_{tt}) \left[(L-s)\, \dot{x}_s\right]_s . \tag{4.1}$$

We seek a solution of the product form

$$\dot{x}(s,t) = u(t)v(s). \tag{4.2}$$

Substitution of (4.2) into (4.1), and separation of variables, yields

$$u_{tt}(g + a_{tt})^{-1} u^{-1} = \left[(L-s)\, v_s\right]_s v^{-1} = -\lambda. \tag{4.3}$$

Here λ is a constant.

From (4.3) we get two equations:

$$\left[(L-s)\, v_s\right]_s + \lambda v = 0, \qquad 0 < s < L, \tag{4.4}$$

$$u_{tt} + \lambda(g + a_{tt})u = 0. \tag{4.5}$$

The boundary condition (3.9) requires that

$$v(0) = 0. \tag{4.6}$$

The only solution of (4.4) which is regular at $s = L$ is a constant multiple of the Bessel function J_0:

$$v(s) = J_0\left[2\lambda^{1/2}\,(L-s)^{1/2}\right]. \tag{4.7}$$

Requiring this solution to satisfy (4.6) yields

$$J_0\left(2\lambda^{1/2} L^{1/2}\right) = 0. \tag{4.8}$$

The function J_0 has an infinite increasing sequence of positive roots j_n, $n = 1, 2, \ldots$. Thus (4.8) shows that λ has one of the values λ_n defined by

$$\lambda_n = j_n^2/4L, \qquad n = 1, 2, \ldots. \tag{4.9}$$

We call the solution (4.7) with $\lambda = \lambda_n$ the nth mode $v_n(s)$:

$$v_n(s) = J_0\left[2\lambda_n^{1/2}\,(L-s)^{1/2}\right] = J_0\left[\left(1 - \frac{s}{L}\right)^{1/2} j_n\right], \quad n = 1, 2, \ldots. \tag{4.10}$$

Then (4.2) becomes

$$\dot{x}(s,t) = u(t,\lambda_n)\ J_0\left[(1-s/L)^{1/2}\ j_n\right]. \tag{4.11}$$

This result was obtained by Belmonte et al. [3] for $a(t)$ a cosine function, in which case (4.5) is a Mathieu equation.

The amplitude $u(t,\lambda_n)$ in (4.11) satisfies (4.5), which is Hill's equation, with $\lambda = \lambda_n = j_n^2/4L$. The product $\lambda_n g = j_n^2 g/4L$ is the square of the frequency of the nth mode. As was stated in section 2, for any periodic function $a_{tt}(t)$ with frequency ω there are ranges of $\lambda_n g/\omega^2$ for which Hill's equation has solutions which grow exponentially with t. When $\lambda_n g/\omega^2$ lies in one of these instability intervals, the vertical motion (3.5) is unstable to the lateral perturbation (4.11).

Just as in the last paragraph of section 2, if $|a_{tt}|$ is small, each instability interval contains a point $\lambda_n g/\omega^2 = k^2/4$ with k an integer. There $(\lambda_n g)^{1/2} = k(\omega/2)$, so at this point the frequency of the nth mode is an integer multiple of $\omega/2$. For the lowest mode $n = 1$ and $j_1 \approx 2.4$, the mode frequency is $j_1/2\ (g/L)^{1/2} \approx 1.2\,(g/L)^{1/2}$. This is slightly smaller than the pendulum frequency of a rigid rod of length L discussed in section 2, which is $(2g/L)^{1/2} \approx 1.4\,(g/L)^{1/2}$. The lowest mode will become unstable for a ponytail with $L = 25$ cm when ω is around twice the lowest mode frequency, i.e., $2(1.2)(980/25)^{1/2} \approx 15.0$ radians/sec. $= 2.39$ cycles/sec. $= 143.5$ steps/minute. This is slightly less than the cadence required for swaying of the rigid pendulum of the same length, but still within the range of joggers' cadences.

5. Ponytail as flexible rod. A still more realistic model of a ponytail is an inextensible flexible rod with small bending stiffness B. This model is intermediate between the rigid rod of section 2, which has infinite stiffness, and the string of section 3, which has zero stiffness.

One advantage of this model is that its slope at the point of attachment can be specified to have any value. If it is horizontal there, then when the runner is not moving, the ponytail will extend away from the head and hang downward in its characteristic shape. This is just the well-known shape of a cantilever beam, which is obtained by solving the equilibrium equations governing a thin rod.

When the runner is moving, and her head is bobbing up and down, the cantilever beam shape is no longer a solution, except when it hangs straight down. In any other case, the ponytail will oscillate in the vertical y, z plane. This motion has not been calculated. The instability of this motion to lateral perturbation would determine when swaying occurs, and would determine the swaying mode shape.

In the special case in which the rod hangs straight down, that shape remains a solution when the runner is moving. The linear partial differential equation governing the small lateral displacement $\dot{x}(s,t)$ is (4.1) with the bending term $-B\dot{x}_{ssss}$ added to the right-hand side. Since it is of fourth order in s, it requires four boundary conditions, which are

$$\dot{x}(0,t) = 0, \quad \dot{x}_s(0,t) = 0, \quad \dot{x}_{ss}(L,t) = 0, \quad \dot{x}_{sss}(L,t) = 0. \tag{5.1}$$

The first two of these conditions denote that the ponytail is clamped at the top, and the second two denote that it is free at the bottom.

The presence of $a_{tt}(t)$ as a coefficient prevents the use of separation of variables to solve the modified (4.1). Instead (4.1) and (5.1) could be solved by treating the stiffness B as small. This would lead to a singular perturbation problem, such as was treated by Handelman and Keller [7] using matched asymptotic expansions, and by Champneys and Fraser [8] using both two-timing and a numerical implementation of Floquet theory. It would be interesting to apply these methods to this problem.

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