4.1 解

$p_1(x)$ 过 $(4, \frac{1}{2})$ ，$p_2(x)$ 过 $(2, \frac{1}{2})$

则

$p_1(x) = \frac{1}{8}x$ ，$x \in (0, 4)$

$p_2(x) = \frac{1}{2}$ ，$x \in (0, 2)$

$$h(X_1) = -\int_0^4 p_1(x)\log p_1(x)\,dx$$

$$= -\int_0^4 \frac{1}{8}x \log \frac{1}{8}x \,dx$$

$$= -\int_0^4 \log\frac{x}{8}\, d\frac{x^2}{16}$$

$$= \left(\frac{1}{32\ln 2}x^2 - \frac{x^2}{16}\log\frac{x}{8}\right)_0^4$$

$$= \frac{1}{2\ln 2} + 1 \quad \text{bit/自由度}$$

$$h(X_2) = -\int_0^2 \frac{1}{2}\log\frac{1}{2}\,dx = 1 \quad \text{bit/自由度}$$

所以 $p_1(x)$ 的熵大.

4.3 解:

(1) $$p(xy) = \frac{1}{\sqrt{2\pi\cdot 4}}e^{-\frac{(x-1)^2}{8}} \cdot \frac{1}{\sqrt{2\pi\cdot 4}}e^{-\frac{(y+2)^2}{8}}$$

$$= p(x)\cdot p(y)$$

则 $X \sim N(1,4)$, $Y \sim N(-2,4)$

且 X, Y 相互独立.

则 $$h(X) = \frac{1}{2}\log 2\pi e\sigma^2 = \frac{1}{2}\log 8\pi e = h(Y)$$

$$= h(X|Y)$$

$$h(XY) = h(Y) + h(X|Y) = \log 8\pi e$$

$$I(X;Y) = h(X) - h(X|Y) = 0$$

(2) $Z \sim (-1, 8)$

则 $$h(Z) = \frac{1}{2}\log 2\pi e\cdot 8 = \frac{1}{2}\log 16\pi e$$

4.6 解：

$h(X) = -\int_0^\infty p(x)\log p(x)\,dx$ 为极值时限定条件

为：$\begin{cases} \int_0^\infty p(x)\,dx = 1 & ① \\ \int_0^\infty x\,p(x)\,dx = A & ② \end{cases}$

$F(x) = -p(x)\log p(x)$

$\varphi_1(x, p) = p(x)$

$\varphi_2(x, p) = x\,p(x)$

$\frac{\partial F}{\partial p} = -[1 + \ln p(x)]$，$\frac{\partial \varphi_1}{\partial p} = 1$，$\frac{\partial \varphi_2}{\partial p} = x$

$-[1 + \ln p(x)] + \lambda_1 + \lambda_2 x = 0$

$p(x) = e^{\lambda_1 - 1} \cdot e^{\lambda_2 x}$ ③

③ 代入 ① 得：

$e^{\lambda_1 - 1}\int_0^\infty e^{\lambda_2 x}\,dx = 1$

$\Rightarrow e^{\lambda_1 - 1} = -\lambda_2$ ④

④ ③代入②

$$\int_0^\infty x(-\lambda_2 e^{\lambda_2 x})\,dx = A$$

$$\left(\frac{e^{\lambda_2 x}}{\lambda_2} - x e^{\lambda_2 x}\right)_0^\infty = A$$

$$\Rightarrow -\frac{1}{\lambda_2}=A \quad \therefore \lambda_2=-\frac{1}{A}$$

$$p(x)=\frac{1}{A}e^{-\frac{x}{A}}, \quad x>0$$

即为X的最佳分布，

且最大熵为：

$$h(X)=-\int_0^\infty p(x)\log\left(\frac{1}{A}e^{-\frac{x}{A}}\right)dx$$

$$=-\int_0^\infty p(x)(-\log A)dx+(\log e)\int_0^\infty \frac{x}{A}p(x)dx$$

$$=\log A+\frac{\log e}{A}\cdot A=\log Ae$$

4.9解：

$$h(U)+h(Y)-h(UY)=I(U;Y)$$

因为X,Y,Z为独立的正态变量，

则(U,Y)为二维正态矢量，

$$Cov(U,Y)=E\{[U-E(U)][Y-E(Y)]\}$$

$$=E(UY)-E(U)E(Y)$$

$$=E[(X+kY)Y]-E(X+kY)E(Y)$$

$$=E(X)E(Y)+kQ-E(X)E(Y)$$

$$=kQ$$

$$\sigma_U^2=E[(U-E(U))^2]$$

$$=E(U^2)-E^2(U)$$

$$= E(X^2 + k^2 Y^2 + 2kXY) - 0$$

$= P + k^2 Q$

所以(U, Y)协方差矩阵

$$B = \begin{pmatrix} P + k^2 Q & kQ \\ kQ & Q \end{pmatrix}$$

$\therefore |B| = PQ$

$\therefore h(U, Y) = \frac{1}{2} \log [(2\pi e)^2 \cdot PQ]$

$= \log 2\pi e + \log \sqrt{PQ}$

$h(U) = \frac{1}{2} \log 2\pi e (P + k^2 Q)$

$h(Y) = \frac{1}{2} \log 2\pi e Q$

$\therefore I(U; Y) = h(U) + h(Y) - h(U, Y)$

$= \frac{1}{2} \log \frac{P + k^2 Q}{P}$

② (U, V)也是二维正态矢量

$\text{cov}(U, V) = E(UV) - E(U)E(V)$

$= E(X^2 + XY + XZ + kXY + kY^2 + kYZ)$

$= P + kQ$

$\sigma_V^2 = P + Q + N$

$$B = \begin{pmatrix} P + k^2 Q & P + kQ \\ P + kQ & P + Q + N \end{pmatrix}$$

$|B| = (1-k)^2 PQ + PN + k^2 QN$

$h(U, V) = \frac{1}{2} \log [(2\pi e)^2 |B|]$

$$I(U;V) = h(U) + h(V) - h(U,V)$$

$$= \frac{1}{2}\log 2\pi e(P+k^2Q) + \frac{1}{2}\log 2\pi e(P+Q+N)$$

$$- \frac{1}{2}\log[(2\pi e)^2|B|]$$

$$= \frac{1}{2}\log \frac{(P+k^2Q)(P+Q+N)}{(1-k)^2PQ+PN+k^2QN}$$

4-11 解：①

$$h(X)=\int_0^a -p(x)\log p(x)\,dx$$

$$=\int_0^a -p(x)(\log b+2\log x)\,dx$$

$$=-\log b-\int_0^a bx^2\cdot 2\log x\,dx$$

$$=-\ln b+\frac{2a^3b}{3}\left(\frac{1}{3}-\ln a\right)$$ （取自然对数）

又 $\int_0^a bx^2\,dx=1$

$$\Rightarrow ba^3=3$$

$\therefore h(X)=-2\ln a-\ln b+\frac{2}{3}$ 奈特.

② $p(y)=\begin{cases} b(y-K)^2, & K\le y\le a+K \\ 0, & \text{other} \end{cases}$

$$h(Y)=\int_K^{a+K}-p(y)(\log b+2\log(y-K))\,dy$$

$$=-\ln b-\int_K^{a+K} b(y-K)^2\cdot 2\ln(y-K)\,dy$$

$$=-2\ln a-\ln b+\frac{2}{3}$$

③ X，Z 的分布函数为 $F_X(x)$，$F_Z(z)$

$$F_Z(z)=P\{Z\le z\}=P\{2X\le z\}$$

$$=P\{X\le \frac{z}{2}\}=F_X(\frac{z}{2})$$

$$p(z) = F_Z'(z) = p_X\left(\frac{z}{2}\right)\cdot\frac{1}{2}$$

$$= \begin{cases} \frac{b}{8}z^2, & 0 \le z \le 2a \\ 0, & 其它 \end{cases}$$

同样 $ba^3 = 3$

$$h(Z) = \int_0^{2a} -p(z)\ln\frac{b}{8}z^2\,dz$$

$$= -\ln\frac{b}{8} - \int_0^{2a}\frac{b}{8}z^2\cdot 2\ln z\,dz$$

$$= -\ln\frac{b}{8} - ba^3\left(\frac{1}{3}\ln 2a - \frac{1}{9}\right)$$

$$= -\ln 2a - \ln\frac{b}{8} + \frac{1}{3}$$

4.18 解:

(1) $C = B\log(1+\frac{S}{N}) = \log(1+10) = 3.46\ \text{Mbit/s}$

(2) 在信道容量不变时，信噪比变为5，带宽为

$$B = \frac{C}{\log(1+\frac{S}{N})} = \frac{3.46}{\log(1+5)} = 1.34\ \text{MHz}$$

(3) 在信道容量不变条件下，带宽变为0.5MHz

信噪比变为： $C = 0.5\times\log(1+\frac{S}{N}) = 3.46$

$$\Rightarrow \frac{S}{N} = 120$$

4.20 解:

(1) $W_1 = 3\,MHz$, $W_2 = 6-4 = 2\,MHz$

$W = 3+2 = 5\,MHz$

$$C = W\log\left(1+\frac{P}{N_0W}\right)$$

$$= 5\times10^6\times\log\left(1+\frac{10}{10^{-8}\times5\times10^6}\right)$$

$$= 38.225\,Mbps$$

(2) 设B_1和B_2分配信号功率P_1，P_2，

根据功率分配注水原理

$P_1 + N_0W_1 = pW_1$, $P_2 + 50N_0W_2 = pW_2$

又 $P_1 + P_2 = P = 10W$

所以 $p = \dfrac{P+N_0W_1+50N_0W_2}{W_1+W_2} = 2.206\times10^{-6}$

$$C = 3\times10^6\times\log\frac{2.206\times10^{-6}}{10^{-8}} + 2\times10^6\times\log\frac{2.206\times10^{-6}}{50\times10^{-8}}$$

$$= 27.639\,Mbps$$

4.21 解：

$$I = -\log\frac{1}{162.25\times10^6} = 9\times10^6\,bit$$

$$C = \frac{9\times10^6}{3\times60} = 5\times10^4\,bit/s$$

$$10\lg\frac{S}{N} = 30 \Rightarrow \frac{S}{N} = 10^3$$

$$\therefore C = B\log\left(1+\frac{S}{N}\right)$$

$$\Rightarrow B = \frac{5\times10^4}{\log1001} = 5016.4\,Hz$$

7.22

7.22

$$(1) C = B\log(1+SNR) = 6000\log 1024 = 60000\text{比特}/s$$

$$(2) H(X) = H\left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right) = 1.5\text{比特}/\text{符号}$$

$$\therefore R = \frac{C}{h(X)} = 40000\text{符号}/s$$