

Problem Review Session 1

PHYS 741

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Disclaimer: The problems below are not my own making but are taken from A Guide to Physics Problems: Part 2 (GPP2) and Princeton Problems in Physics (PPP).

Practice Problems

1. (**GPP2 4.12 Hydrogen Rocket**) The reaction chamber of a rocket engine is supplied with a mass flow rate m of hydrogen and sufficient oxygen to allow complete burning of the fuel. The cross section of the chamber A , and the pressure at the cross section is P with temperature T . Calculate the force that the chamber is able to provide.
2. (**PPP 4.4 Phase Coexistence**) The temperature of a long vertical column of a particular substance is T everywhere. Below a certain height $h(T)$ the substance is solid, whereas above $h(T)$ it is in a liquid phase. Calculate the density difference $\Delta\rho = \rho_s - \rho_l$ between the solid and liquid ($|\Delta\rho| \ll \rho_s$) in terms of L (the latent heat of fusion per unit mass), dh/dT , T , ρ_l and g , the acceleration due to gravity.
3. (**GPP2 4.51 Nonideal Gas Equation**) A gas obeys the equation of state

$$P = Nk \left(\frac{T}{V} + \frac{B(T)}{V^2} \right)$$

where $B(T)$ is a function of the temperature T only. The gas is initially at temperature T and volume V_0 and is expanded isothermally and reversibly to volume $V_1 = 2V_0$.

- (a) Find the work done in the expansion.
- (b) Find the heat absorbed in the expansion.
4. (**PPP 4.5 Otto Cycle**) The operation of a gasoline engine is (roughly) similar to the Otto cycle (Fig 1), where:

- $A \rightarrow B$: Gas compressed adiabatically
- $B \rightarrow C$: Gas heated isochorically (constant volume; corresponds to combustion of gasoline)
- $C \rightarrow D$: Gas expanded adiabatically (power stroke)
- $D \rightarrow A$: Gas cooled isochorically

Compute the efficiency of the Otto cycle for an ideal gas (with temperature-independent heat capacities) as a function of the compression ratio V_A/V_B , and the heat capacity per particle C_V .

5. (**GPP2 4.32 Joule Cycle**) Find the efficiency of the Joule cycle, consisting of two adiabats and two isobars (see Fig 2). Assume that the heat capacities of the gas C_P and C_V are constant.
6. (**GPP2 4.53 Critical Parameters**) Consider a system described by the Dietrichi equation of state

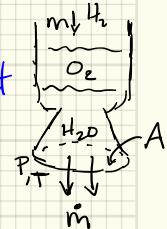
$$P(V - nB) = nN_A k T e^{-nA/N_A k TV}$$

where A , B , k , N_A and R are constants and P , V , T , and n are the pressure, volume, temperature, and number of moles. Calculate the critical parameters, i.e., the values of P , V , and T at the critical point.

Session 1 Problem 1

CPPZ 4.12

Hydrogen Rocket



We begin by considering the forces that act on the chamber

$$F_{\text{tot}} = F_{\text{out}} + F_P$$

where F_{out} refers to the force produced by the mass outflow due to fuel combustion, while F_P is the force due to the pressure differential at the interface of the chamber & the external environment. For a traditional rocket, where the combustion products exit through a nozzle to maximize the outflow velocity, the second term is negligible. For this problem

$$F_P \approx PA$$

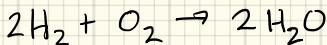
but we will neglect it & focus on the reaction thrust from the mass flow from the chamber, F_{out}

As one may recall from classical mechanics

$$\vec{F} = \frac{d\vec{p}}{dt} = m\vec{v} + \vec{m}\bar{a}$$

In this case \vec{F}_{out} actually arises from the former term rather than the latter

∴ we need to calculate \vec{v} the velocity of material leaving the chamber and \dot{m} , the mass outflow rate. To determine the outflow rate we can first consider the material actually being produced in the combustion chamber. The reaction in the chamber would be



∴ the amount of hydrogen that enters the system is equal to the amount of water vapor that leaves the chamber, i.e.

$$\dot{n}_{\text{H}_2} = \dot{n}_{\text{H}_2\text{O}}$$

$$\text{Now } m = \dot{n}_{\text{H}_2} M_{\text{H}_2} = (2g) \dot{n}_{\text{H}_2} = (2g) \dot{n}_{\text{H}_2\text{O}}$$

$$\text{while } \dot{m} = \dot{n}_{\text{H}_2\text{O}} M_{\text{H}_2\text{O}} = (18g) \dot{n}_{\text{H}_2\text{O}} = 9m$$

So now we know m & just need to solve for V . Imagine that at a given moment, the H_2O produced in the chamber takes up a volume V . Then the volume decreases by dV given by

$$dV = A dx$$



where A is the cross-section of the chamber & x is the distance along the direction of mass flow. Therefore we can express the mass outflow rate as

$$\dot{m} = \frac{d}{dt}(pV) = p \frac{dV}{dt} = pA \frac{dx}{dt} = pAV$$

where p is the density of H_2O & we recognize $\frac{dx}{dt} = v$

$$\Rightarrow v = \frac{q_m}{pA}$$

So now we must solve for p

$$\Rightarrow p = \frac{n_{H_2O} M_{H_2O}}{V} = \frac{PM_{H_2O}}{RT}$$

where we assumed H_2O product behaves like an ideal gas w/ equation of state

$$\frac{P}{RT} = \frac{n}{V}$$

$$\Rightarrow V = \frac{q_m RT}{PM_{H_2O} A}$$

$$\therefore F = \frac{81 m^2 RT}{M_{H_2O} PA}$$

Notice that this is maximized by reducing PA , which is related to F_p . This is why rocket nozzles are designed to reduce PA & as a result, why F_p is negligible compared to F_{out} .

Session 1 Problem 2

We begin with the Clausius-Clapeyron equation, since this defines the phase transition or phase coexistence curve on a P-T diagram & we have a substance existing in two distinct phases in this problem.

$$\frac{dP}{dT} = \frac{L}{T\Delta v}$$

Recall that v is the volume per a unit mass (i.e. inverse density), Δv describes the specific volume change of the phase transition, and $L = T\Delta s$ is the latent heat w/ s as the entropy per unit mass & Δs is the specific entropy change from the phase transition.

For this problem we are given $L = \text{latent heat of fusion}$ which means we are concerned about the transition from a solid to liquid

$$\Rightarrow \Delta s = s_f - s_s \quad \Delta v = v_f - v_s = \frac{1}{P_e} - \frac{1}{P_s}$$

Fig 1

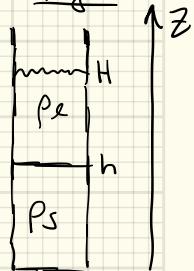
Rearranging the second relation, we see that

$$\Delta v = \frac{P_s - P_e}{P_s P_e} = \frac{\Delta p}{P_s P_e}$$

$$\therefore \Delta p = \frac{L}{T \frac{dP}{dT}} P_s P_e$$

Expanding dP/dT in terms of the height function $h(T)$

$$\Rightarrow \frac{dP}{dT} = \frac{dh}{dT} \frac{dP}{dh}$$



Therefore we need to determine dP/dh . I'll demonstrate a longer derivation first. We know that the pressure at the surface of a liquid in a tube is merely

$$P = P_e g \Delta z \quad \text{where } \Delta z \text{ is the height of the liquid}$$

From Fig 1, we see $\Delta z = H - h$

$$\Rightarrow \frac{dP}{dh} = P_e g \left(\frac{dH}{dh} - 1 \right)$$

where H also depends on h by non-trivially depending on T , just like h .

To solve for $\frac{dT}{dh}$ we can consider that, while the height of the mixture is not constant as a function of temperature, the total mass of the system is constant, i.e.

$$m = \rho_s A h + \rho_e A (H - h) \quad \text{where } A \text{ is the cross-section of the tube}$$

$$\Rightarrow \frac{dm}{dh} = 0 = \rho_s A + \rho_e A \left(\frac{dT}{dh} - 1 \right)$$

$$\therefore \frac{dP}{dh} = -\rho_s g$$

which gives us the final result

$$\Delta P = -\frac{\rho_e L}{g T \left(\frac{dh}{dT} \right)}$$

Session 1 Problem 3

GPPZ 4.57 Nonideal Gas Equation

We have the equation of state $P = Nk \left(\frac{1}{V} + \frac{B(T)}{V^2} \right)$

a)

Work done in the expansion: $W = - \int_{V_i}^{V_f} P dV$ w/ $V_i = V_0$ & $V_f = 2V_0$ for this problem

Note we pick up a negative sign because we are concerned about the work done in the system, not by the system on its environment

$$\Rightarrow W = - \int_{V_0}^{2V_0} Nk \left(\frac{1}{V} + \frac{B(T)}{V^2} \right) dV$$

$$W = -NkT \ln 2 - \frac{NkB(T)}{2V_0}$$

b) So now we are looking for the heat absorbed, which is given by

$$Q_{in} = \int_{V_0}^{2V_0} T dS$$

since we are dealing w/ a reversible process. This is an isothermal expansion so $T = \text{const}$ & independent of S

$$\Rightarrow Q_{in} = T [S(2V_0) - S(V_0)]$$

To relate S & V we must first relate S & P . We can do this by considering the Helmholtz free energy

$$dA = -SdT - PdV$$

which gives us the Maxwell relation

$$\left(\frac{\partial S}{\partial V} \right)_T = \left(\frac{\partial P}{\partial T} \right)_V$$

$$\Rightarrow S = Nk \int \left(\frac{1}{V} + \frac{B'(T)}{V^2} \right) dV = Nk \ln \left(\frac{V}{V_c} \right) - \frac{NkB'(T)}{V} + C(T)$$

where V_c is a constant of integration as well as $C(T)$ due to holding T const and $B'(T) = \frac{\partial B}{\partial T}$

Evaluating $S(V)$ for the initial & final volumes, we see that

$$Q_{in} = NkT \left(\ln 2 + \frac{B'(T)}{2V_0} \right)$$

Session 1 Problem 4

We want to compute the efficiency, which is defined by

$$\eta = \frac{W_{\text{tot}}}{Q_{\text{in}}}.$$

We are dealing w/ a cyclic process, so $dU=0$

$$\Rightarrow dW = dQ$$

The process is also reversible

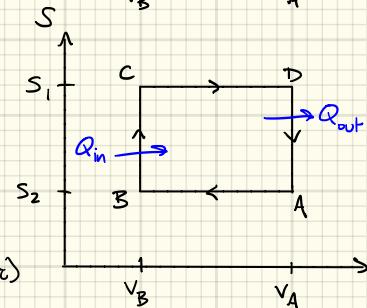
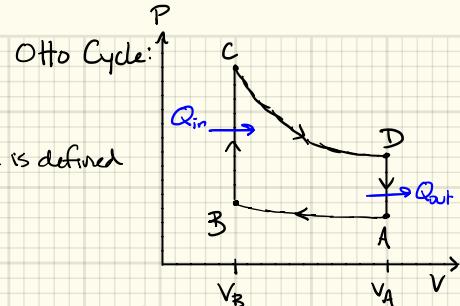
$$\Rightarrow dQ = T dS$$

From the diagrams, we see that heat is absorbed/rejected in the transformations from $B \rightarrow C$ & $D \rightarrow A$ respectively (since $C \rightarrow D$ & $A \rightarrow B$ are adiabatic)

$$\therefore W_{\text{tot}} = Q_{B \rightarrow C} + Q_{D \rightarrow A}$$

$$Q_{\text{in}} = Q_{B \rightarrow C}$$

$$\Rightarrow \eta = 1 + \frac{Q_{D \rightarrow A}}{Q_{B \rightarrow C}}$$



* Note in the diagrams we use $Q_{\text{out}} = -Q_{D \rightarrow A} > 0$

To calculate the heat for these two transformations, we must determine T as a function of S . So we begin with the first law

$$dU = T dS - P dV = T dS - \frac{NkT}{V} dV \quad \text{assuming an ideal gas}$$

Because C_V is temperature independent $C_V = \left(\frac{\partial U}{\partial T}\right)_V \Rightarrow dU = C_V dT$

$$\Rightarrow dS = C_V \frac{dT}{T} + Nk \frac{dV}{V}$$

Integrating this equation we find that

$$S = C_V \ln T + Nk \ln V + \alpha$$

where α is an integration constant

Inverting to get T as a function of T

$$\Rightarrow T = \beta e^{\frac{S/C_V}{Nk} V^{-Nk/C_V}} \quad \text{where } \beta = e^{-\alpha/C_V}$$

Now we can insert our temperature function into the integrals over entropy

$$\Rightarrow Q_{B \rightarrow C} = \int_{S_2}^{S_1} T(S, V_B) dS = \beta V_B^{-Nk/c_V} \int_{S_2}^{S_1} e^{S/c_V} dS \\ = \beta V_B^{-Nk/c_V} C_V (e^{S_1/c_V} - e^{S_2/c_V})$$

$$Q_{D \rightarrow A} = \int_{S_1}^{S_2} T(S, V_A) dS = \beta V_A^{-Nk/c_V} C_V (e^{S_2/c_V} - e^{S_1/c_V})$$

Consequently, plugging into our expression for the efficiency we find that

$$\eta = 1 + \frac{Q_{D \rightarrow A}}{Q_{B \rightarrow C}}$$

$$\boxed{\eta = 1 - \left(\frac{V_A}{V_B} \right)^{-Nk/c_V}}$$

As a useful exercise you can also calculate the efficiency from

$$\eta = \frac{W_{\text{tot}}}{Q_{\text{in}}}$$

by considering $W_{\text{tot}} = W_{A \rightarrow B} + W_{C \rightarrow D}$ and calculating

$$W_{i \rightarrow j} = \int_{V_i}^{V_f} P dV = Nk \int_{V_i}^{V_f} \frac{T(V, S_{i,j})}{V} dV \text{ for each transformation}$$

This should give you the same value of η as above.

Session 1 Problem 5

GPPZ 4.32 Joule Cycle

Recall that the efficiency is given by

$$\eta = \frac{W_{\text{tot}}}{Q_{\text{in}}}$$

where Q_{in} is the heat absorbed & W_{tot} is the total work done by the system

From the diagram, we see that $Q_{\text{in}} = Q_{2 \rightarrow 3}$
where $Q_{2 \rightarrow 3}$ is the heat absorbed from $2 \rightarrow 3$.

This is a cyclic process $\therefore dU = 0 \Rightarrow dQ = dW$

$$\Rightarrow W_{\text{tot}} = Q_{2 \rightarrow 3} + Q_{4 \rightarrow 1} \quad \text{since } 3 \rightarrow 4 \text{ & } 1 \rightarrow 2 \text{ are adiabatic}$$

Because these are constant pressure processes & C_p is temperature independent
we can take advantage of the relation

$$C_p = \left(\frac{\partial Q}{\partial T} \right)_P$$

by rewriting it as $dQ = C_p dT \Rightarrow Q_{i \rightarrow j} = C_p(T_j - T_i)$

where T_i & T_j are the temperatures at points i & j on the PV plot where $i, j = 1, 2, 3, 4$.

$$\therefore \eta = \frac{W_{\text{tot}}}{Q_{\text{in}}} = 1 + \frac{Q_{4 \rightarrow 1}}{Q_{2 \rightarrow 3}} = 1 + \frac{T_1 - T_4}{T_3 - T_2}$$

But we want our answer in terms of P_1 & $P_2 \therefore$ we first use the ideal gas law

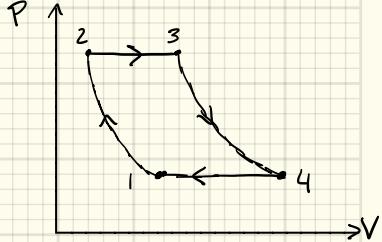
$$\Rightarrow \eta = 1 + \frac{P_1}{P_2} \left(\frac{V_1 - V_4}{V_3 - V_2} \right) \quad \text{when we used the fact that} \\ P_1 = P_3 \quad \& \quad P_2 = P_3$$

We can eliminate V by considering the adiabatic transformations, along which

$$PV^\gamma = \text{const} \quad \text{where } \gamma \equiv C_p/C_v$$

$$\Rightarrow V \sim P^{-1/\gamma} \Rightarrow \frac{V_4}{V_3} = \left(\frac{P_2}{P_1} \right)^{1/\gamma} \quad \frac{V_1}{V_2} = \left(\frac{P_2}{P_1} \right)^{1/\gamma}$$

$$\therefore \text{we find that } \eta = 1 + \frac{P_1}{P_2} \left[\frac{(V_2 - V_3)(P_2/P_1)^{1/\gamma}}{V_3 - V_2} \right] \quad \text{or} \quad \boxed{\eta = 1 - \left(\frac{P_1}{P_2} \right)^{(r-1)/r}}$$



Session 1 Problem 6 GPPZ 4.53 Critical Parameters

Recall that a critical point meets the conditions

$$\left(\frac{\partial P}{\partial V}\right)_{T=T_c} = 0 \quad \& \quad \left(\frac{\partial^2 P}{\partial V^2}\right)_{T=T_c} = 0$$

$$\therefore P = nN_A kT e^{-nA/(N_A kT)} (V - nB)^{-1}$$

$$= \alpha(T) e^{-\beta(T)V} (V - nB)^{-1} \quad \text{with } \alpha(T) \equiv nN_A kT$$

$$\Rightarrow \left(\frac{\partial P}{\partial V}\right)_{T_c} = -\alpha_c e^{-\beta_c V} (V_c - nB)^{-2} + \alpha_c e^{-\beta_c V_c} \beta_c V_c^{-2} (V_c - nB)^{-1} = 0$$

$$\text{which gives us } \beta_c V_c^{-2} (V_c - nB)^{-1} = (V_c - nB)^{-2} \quad \text{with } \alpha_c \equiv \alpha(T_c)$$

$$\Rightarrow \beta_c^{-1} = (V_c - nB) V_c^{-2} \quad \beta_c \equiv \beta(T_c)$$

Likewise we have

$$\begin{aligned} \left(\frac{\partial^2 P}{\partial V^2}\right)_{T=T_c} &= 2\alpha_c e^{-\beta_c V_c} (V_c - nB)^{-2} - \alpha_c e^{-\beta_c V_c} \beta_c V_c^{-2} (V_c - nB)^{-2} - \alpha_c e^{-\beta_c V_c} \beta_c (V_c - nB)^{-2} V_c^{-2} \\ &\quad - 2\alpha_c e^{-\beta_c V_c} \beta_c V_c^{-3} (V_c - nB)^{-1} + \alpha_c e^{-\beta_c V_c} \beta_c^2 V_c^{-4} (V_c - nB)^{-1} = 0 \end{aligned}$$

Multiplying by $V_c (V_c - nB)^3$ & recall that $\beta_c = V_c^2 (V_c - nB)^{-1}$

$$\Rightarrow 2V_c^2 V_c - 2(V_c - nB) + V_c = 0$$

$$V_c = 2nB$$

$$\Rightarrow \beta_c^{-1} = N_A kT_c / nA = nB / (4n^2 B^2) = (4nB)^{-1}$$

$$\Rightarrow T_c = \frac{A}{4N_A kB}$$

Plugging into our equation for pressure we find

$$P_c = nN_A \left(\frac{A}{4N_A kB}\right) e^{-(nA/nA)(\frac{4N_A kB}{A})(\frac{1}{2nB})} (nB)^{-1} = \frac{A}{4B^2} e^{-2} = P_c$$