THAYER SCHOOL OF ENGINEERING • DARTMOUTH

ENGS 23

HOMEWORK #2

April 10, 2023

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1 About homework # 2

The second homework focuses, again, on lumped thermal systems, specifically the system you will perform measurements on in lab #2. Matlab will helping you to solve the full $4x4$ systems matrix generated by the 4 thermal sensors. Conceptually, what we are trying to teach you with Lab #2 and associated HWs is when a system can no longer be modeled as lumped but instead we have to transition over to a distributed model which means we have to solve a partial differential equation (pde).

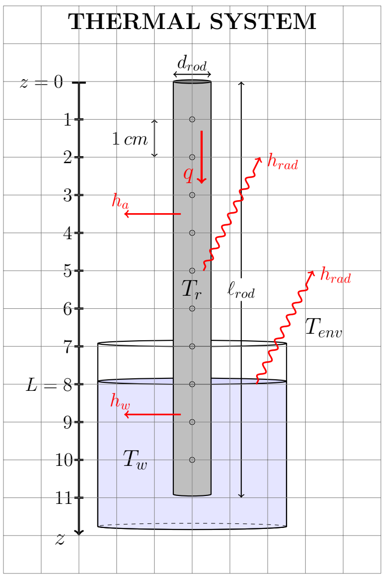

Figure 1: An Aluminum rod inserted into water.

In lab #2 you will measure how the thermal energy, stored in the Aluminum rod, will dissipate as the end of the rod is plunged into ice water. Some of the thermal processes occurring during the lab are depicted in FIg. 1. Note: the directions of the arrows are not necessarily in the correct direction. The arrows are merely representing a heat transfer pathway. The direction of heat flow depends on the temperature gradient (ie is the rod hotter or cooler than the surrounding environment)

System (Aluminum Rod) Properties			
Property	Value	Symbol	Comments
Aluminum type	Z=11	Al	Alloy 6061, temper T6.
Length	$11\,cm = 0.11\,m$	ℓ_{rod}	$3\,cms$ in the water.
Effective Length	$8\,cm = 0.08\,m$	L	the part above the water.
Diameter	$0.5'' = 12.7 \times 10^{-3}\,m$	d_{rod}	$a = \frac{d_{rod}}{2} \approx 6.4 \times 10^{-3} m$
Cross sectional area	$128.6 \times 10^{-6}\,m^2$	A_{cross}	$\pi \cdot a^2$
Volume rod 8cm	$10.3 \times 10^{-6}\,m^3$	V_8	$L \cdot A_{cross}$
Density	$2700\,kg\,/\,m^3$	ϱ	intensive property
Mass 8cm	$27.8 \times 10^{-3}\,kg$	m_8	$V_8 \cdot \varrho$
Mass 11cm	$38.1 \times 10^{-3}\,kg$	m_{11}	$V_{11} \cdot \varrho$
Specific heat	$896\,J\,/\,°\mathrm{C}\,kg$	c_p	constant pressure
Thermal conductivity	$200\,W\,/\,°\mathrm{C}\,m$	κ	167 also quoted
Boundary (Air and Water) Properties			
Area (air)	$3350 \times 10^{-6}\,m^2$	A_{air}	$8\,cm$ rod plus top area
Area (water)	$1330 \times 10^{-6}\,m^2$	A_w	$3\,cm$ rod plus bottom area
Volume of H_2O	$700 \times 10^{-6}\,m^3$	V_w	N.B. lab 2 with ice
Density of H_2O	$997.5\,kg\,/\,m^3$	ϱ_w	at 23 °C.
Specific heat of H_2O	$4180\,J\,/\,°\mathrm{C}\,kg$	c_{pw}	at 23 °C.
Mass of H_2O	$0.7\,kg$	m_w	
Specific heat of Pyrex	$840\,J\,/\,°\mathrm{C}\,kg$	c_{pp}	
Mass of Pyrex bowl	$443 \times 10^{-3}\,kg$	m_p	$6'' \times 2''$(diam x height)
Resistance H_2O to still air	$1.1\,°\mathrm{C}/W$	R_{wa}	ESTIMATE
Capacitance	$3296\,J/°\mathrm{C}$	C_{wp}	water + Pyrex

2 Problems - Due on Monday April 17 at midnight

PROBLEM 1: Aluminum Rod Properties

The thermal system of interest is the temperature of the Al cylinder. Particularly the temperature response when the bottom 3cm of the rod are submerged into an ice-water bath. The environment is everything around the cylinder (the air surrounding the sides and the water the bottom end is submerged in). In lab 1 the resistance to thermal conduction through an Al plate was considered negligible because the plate was so thin it's thermal resistance to conduction was 1000x smaller than the convection away from the surface. In lab 2 the aluminum rod has a much smaller cross sectional area and a much greater length resulting in a non-negligible conductive resistance. In fact the conductive resistance within the Al is considered so large relative to the convective resistance out the end of the rod into the water that the systems first approximation is to estimate convection resistance as zero (ie not include it in the model)

1a Calculate the thermal resistance to heat flow down the road per cm of length. Reference values in the System Properties table. **[5 pts]**

1b Also calculate the capacitance per cm of length of the rod. Again using values in the System Properties table. **[5 pts]**

PROBLEM 2: Preliminary System Response Estimates

Use a crude approximation of the system as a first-order lumped system to estimate its behavior, assuming the temperature at the middle of the rod best represents the average Al temperature. Assume 3 cm is immersed in ice-water, leaving 8 cm out of the water. Use the full thermal mass of the 8 cm length, but only use the thermal resistance of half that length.

2a Why do you think we only use half the distance for calculating the thermal resistance? **[5 pts]**

2b Find the time constant of this lumped system (assuming R convection to the water is negligible). **[5 pts]**

2c Identify the expected initial and final temperatures of the thermal capacitor. **[5 pts]**

2d Write an equation for the temperature as a function of time. **[5 pts]**

PROBLEM 3: System Spatial Estimates

During the lab, you will be able to directly measure temperature $T(x,t)$, at several positions x along the rod, but you do not have a direct way to measure heat flow Q. However, you can calculate an approximate value of Q [W] from two adjacent temperature measurements. There are two ways to think about this, a distributed approach and a lumped approach. We will do both to show that the result is the same.

3a *Lumped:* Using the rod's resistance per cm and the thermal equivalent of Ohm's lab show how you can use 2 temperature measurements along the rod that are 2cm apart and determine the heat flow (Q, in watts) through that segment of rod: Write a numerical formula for Q, proportional to the temperature difference. Then apply the equation in a test calculation using an assumed measured temperature difference $(T_3 - T_1)$ of 3 C. **[5 pts]**

3b *Distributed:* With access into the rod at discrete points along its length, you can measure an approximate gradient, $\frac{dT(x)}{dx} \approx \frac{(T_3-T_1)}{(x_3-x_1)}$. From this you can compute an approximate heat flow, Q, from Fourier's law (SFW, p. 62) and temperature measurements. As a check, make sure that this formula arrives at the same result as part 3a. **[5 pts]**

PROBLEM 4: Al Rod System Data Analysis

This final question will be completed after you have collected data from Lab 2.

4a Using the Lab 2 data: Plot Temp vs time for all 4 sensors in matlab. **[5 pts]**

4b Create a new plot of all four temperature curves on a semilog scale. Prior to plotting subtract the final steady-state temperature for each sensor so each sensor decays to a zero value (relative to steady state). Use a vertical scale over just one order of magnitude and of course adjust the horizontal scale and the size on the screen so you can see it clearly. Recall that an exponential decay to zero plotted on a semi-log scale gives you a straight line with a slope related to the time constant. Based on comparing the plots to straight lines, is the behavior a simple exponential decay? If not, where does it differ and in what way? **[5 pts]**

4c Provide an overview of the differences in behavior between the ideal first-order lumped system you analyzed and the distributed behavior you observed during lab. It would be beneficial to annotate aspects of the previous Temp vs time plots (both semi-log and linear scaling) to support your argument. As you do this, think about modes. Does the observed behavior include evidence of additional modes participating in the response? Do they show up in the beginning or the end of the time period you looked at? Is this as you would expect? **[5 pts]**

4d Calculate and plot an approximation of the heat flow between the bottom two temp sensors as a function of time. You can use the formula from prob. 3a with the measured time-varying temperatures to calculate this. **[5 pts]**

4e Model refinement: The initial model assumed perfect insulation. From your experimental data calculate an approximate heat flow into the rod, through the *imperfect* insulation, when the system is in steady state. **[5 pts]**

4f Further model refinement: The initial model also assumed the end of the rod was exactly equal to the ice water temp, thus the convective resistance between the water and the rod was zero. Using the steady state heat flow, the bottom (T7) temp sensor steady state value, and the known resistance per cm of the Al rod make an estimate of Rconv between the rod and water. **[5 pts]**

PROBLEM 5: Al rod Model vs data

5a Draw a 4RC model for the Al rod. Include a heat source at the top to represent the leakage heat flow calculated above, a temp source Twater at the bottom to represent the ice water temperature, and Rconv at the bottom boundary between the rod and Twater. Write the system odes into state space form. **[10 pts]**

5b Use Matlab's SS and lsim to simulate the state space model. Plot the model's temp vs time with the experimental data for all four temp sensors. **[10 pts]**

5c Calculate and plot the difference between model and data for each temperature sensors. Calculate the average model deviation (RMS error) for each measurement location. **[10 pts]**

PROBLEM 6: Complex notation - going between exponentials and trig functions In one of my derivations in class I claimed that,

$$\underline{x}(t) = c_1\underline{\phi_1}\, e^{-i\omega_1 t} + c_2\underline{\phi_1}\, e^{i\omega_1 t} + c_3\underline{\phi_2}\, e^{-i\omega_2 t} + c_4\underline{\phi_2}\, e^{i\omega_2 t} = c_1'\underline{\phi_1}\, \cos(\omega_1 t + \theta_1) + c_2'\underline{\phi_2}\, \cos(\omega_2 t + \theta_2) \quad (1)$$

In this problem you will show that this claim is correct under certain conditions.

3a Let's begin by focusing on everything having to do with $\underline{\phi_1}$.

Show that $c_1\, e^{-i\omega_1 t} + c_2\, e^{i\omega_1 t}$ can be written as $A\, \cos(\omega_1 t) + B\, \sin(\omega_1 t)$. **[5 pts]**

3b $c_1\, e^{-i\omega_1 t} + c_2\, e^{i\omega_1 t}$ has to be a real number. What does this mean for constraints on c_1 and c_2? **[10 pts]**

3c Show that $A\, \cos(\omega_1 t) + B\, \sin(\omega_1 t) = C\cos(\omega_1 t + \theta_1)$ and write C and θ_1 in terms of c_1 and c_2. **[5 pts]**