1) a) The thermal resistance due to heat flow can be described through the mode of heat transfer s.t.:

$$R_{cond} = \frac{\Delta x}{K A_{cross}}$$ R = thermal resistance per cm of length = ?

$$\Delta x = \frac{l_{rod}}{M} = \frac{11\,cm}{\#\,lumps} = \frac{11\,cm}{11} = 1\,cm$$

$$K = \frac{200\,W}{°C \cdot m}\left(\frac{1\,m}{100\,cm}\right) = \frac{2\,W}{°C \cdot cm}$$

$$A_{cross} = 128.6 \cdot 10^{-6}\,m^2\left(\frac{10000\,cm^2}{1\,m^2}\right) = 1.286\,cm^2$$

$$R = \frac{1\,cm}{\left(2\,\frac{W}{°C \cdot cm}\right)\left(1.286\,cm^2\right)} = 0.389\,°C/W \text{ per cm}$$

b) The capacitance per cm of length of the rod is:

$$C = \rho A_{cross} \Delta x\, c_p$$

$$\rho = \frac{2700\,kg}{m^3}\left(\frac{1\,m^3}{1000000\,cm^3}\right) = 2.7 \cdot 10^{-3}\,kg/cm^3$$

$$A_{cross} = 1.286\,cm^2$$

$$\Delta x = 1\,cm$$

$$c_p = \frac{896\,J}{°C \cdot kg}$$

$$C = \left(\frac{2.7 \cdot 10^{-3}\,kg}{cm^3}\right)\left(1.286\,cm^2\right)\left(1\,cm\right)\left(\frac{896\,J}{°C \cdot kg}\right)$$

$$= 3.111\,J/°C \text{ per cm}$$

2) a) We assume half the distance because heat flow is 3-D, spreading in all directions from a source such that by only accounting for half the distance, the heat flow path is a straight line and evenly distributed. This is based on the assumption that heat flow is one dimensional along the rod and the temperature gradient is linear as well as representative of the entire rod (assuming that the heat flow from the middle to each end of the rod is symmetric and flows equally in both directions). If the rate of heat transfer is roughly proportional to the inverse of the middle of the rod to the point of contact with the ice water, then a first-order approximation can be utilized, thereby reducing the time and energy for exact calculations. Thus, heat at the top of the rod and right above the water would average to be half the full length since heat travels anywhere from 0-8 cm such that if equally distributed (assumed), all points along the rod would sum to average about this halfway point (midpoint approximation for heat movement within rod).

b) $\tau = R_{cond} C$

$= (0.389\ °C/W \text{ per cm})(4\text{cm})(3.111\ J/°C \text{ per cm})(8\text{cm})$

$= (1.556\ °C/W)(24.888\ J/°C)$

$= 38.73$ seconds

c) $T_R(0) = \text{room temperature} = T_i = 22.5°C$

$T_R(\infty) = \text{water temperature} = T_f = 1.0°C$ (measured in lab) $\approx 0°C$ (ideally)

d) $C\dot{T}_R = \sum Q_{in} - \sum Q_{out}$ $\quad T_R = \text{temp. of rod}$

$Q_{in} = 0$ (no input source)

$Q_{out} = (T_i - T_f) R_{eq}$ $\quad R_{eq} = R_{cond} + R_{conv} = R_{cond} \quad C = m c_p$

$$\dot{T}_R = -\frac{T_i - T_f}{C_{eq} R_{eq}}$$

$$\frac{dT_R}{dt} = -\frac{T_i - T_f}{\tau}$$

$$\int \frac{dT_R}{T_i - T_f} = -\int \frac{dt}{\tau}$$

$$e^{\cancel{\ln}\left(\frac{T_i - T_f}{T_R - T_f}\right)} = e^{-\frac{t}{\tau} + C_1}$$

$$\frac{T_i - T_f}{T_R - T_f} = C_1 e^{-\frac{t}{\tau}} (T_R - T_f)$$

$$T_R(t) = T_f + (T_i - T_f)\left(e^{-\frac{t}{\tau}}\right)$$

$$= 1.0 + (22.5 - 1.0)\left(e^{-\frac{t}{38.73}}\right)$$

$$= 1.0 + 21.5 e^{-\frac{t}{38.73}}$$

3) a) $Q = \frac{\Delta T}{R} = -\frac{T_3 - T_1}{R_{cond} \cdot \Delta x} = -\frac{T_3 - T_1}{(0.389\ °C/W \text{ per cm})(2\text{ cm})}$ $\qquad \Delta T = T_3 - T_1 = 3°C$

$= -(T_3 - T_1)\left(\frac{1}{0.778}\right) = -1.286(T_3 - T_1) = 1.286(3) = 3.858\text{ W}$

$T_1 > T_3$

b) $\frac{dQ}{dt} = -KA\frac{dT}{dx}$

$= -\left(\frac{2\text{W}}{°C \cdot \text{cm}}\right)(1.286\text{ cm}^2)(T_3 - T_1)\left(\frac{1}{2\text{ cm}}\right)$ $\qquad \frac{dT}{dx} = \frac{\Delta T}{\Delta x} = \Delta T \cdot \left(\frac{1}{\Delta x}\right)$

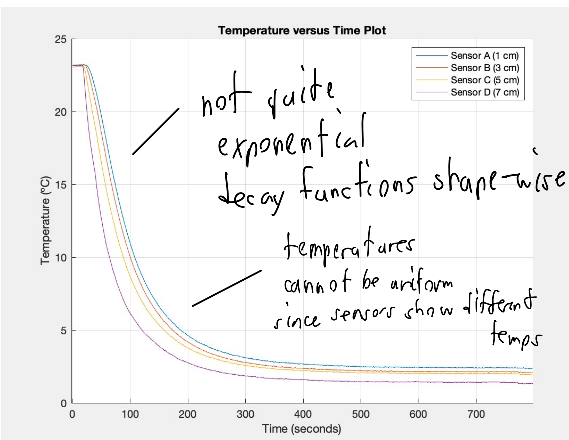
$\Delta x = \frac{L}{M} = \frac{4\text{cm}}{2} = 2\text{ cm}$

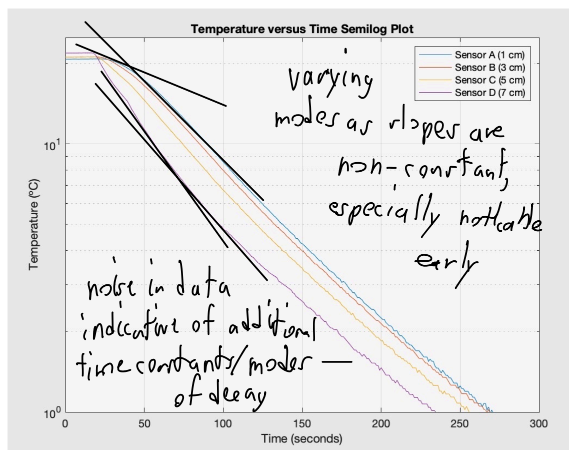
$= -1.286(T_3 - T_1) = 1.286(3)$

$= 3.858\text{ W}$

4) b) By comparing the plots to straight lines, we can see there is an evident nonlinear relationship between temperature and time due to the plots having a non-constant slope despite the utilization of a semilog plot, which should theoretically appear to have the same rate of change (even though it actually doesn't). Ideally, the temperature decreases monotonically and smoothly with minimal deviations, yet there is identifiable noise as the time approaches 150 seconds, thereby also suggesting the exponential decay model is not an exceptional fit due to variability.

c)





First-Order Lumped System	Distributed System
- System is homogenous and temp. is uniformly distributed throughout rod - Simple exponential decay function - Assumed dominant method of heat transfer - Time constant calculated due to thermal properties of material and rate of heat transfer	- System is not homogenous and temp. isn't uniformly distributed throughout rod (temp. gradients) - Multiple stages with different rates of decay - Many modes as shown by slope towards beginning not being constant as well as changes throughout - Many methods of heat transfer occurring simultaneously (conduction, convection, radiation)

We would expect there to be additional modes, possibly due to convection currents of the water and temperature gradients (differences in $C_{thermal}$ and heat capacities between rod and water). We see these show up and become less pronounced, though the noise does increase despite a relatively linear trend in the semilog plot. The initial modes show up at the beginning due to the initial establishment of temp. gradients and convection currents with the rod.

e) $Q_{leakage} = \dfrac{\Delta T}{R_{cond} \cdot \Delta x} = \dfrac{0.334}{(0.389)(2)}$

$= 0.429 \text{ W}$

$\Delta T =$ change between two sensors

$= T_{A\_ss} - T_{B\_ss}$

$= 2.381 - 2.047$

$= 0.334$

$A_{\_ss}$ = Sensor A steady state

$B_{\_ss}$ = Sensor B steady state

f) $Q_{\_ss} = 0.711$ (estimated from plot)

$\Delta T = T_{7ss} - T_w$

$= 2.381 - 1.0$

$= 1.381$

$Q = \dfrac{\Delta T}{R_{tot}} = \dfrac{T_7 - T_w}{R_{cond} \cdot \Delta x + R_{conv}}$

$0.711 = \dfrac{1.381}{0.389(1) + R_{conv}}$

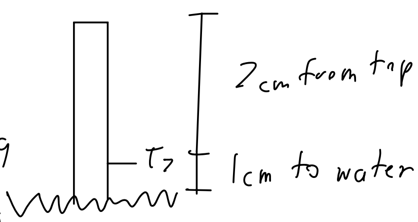
$R_{cond} \cdot \Delta x = (0.389)(1)$

$= 0.389$

$0.711(0.389 + R_{conv}) = 1.381$

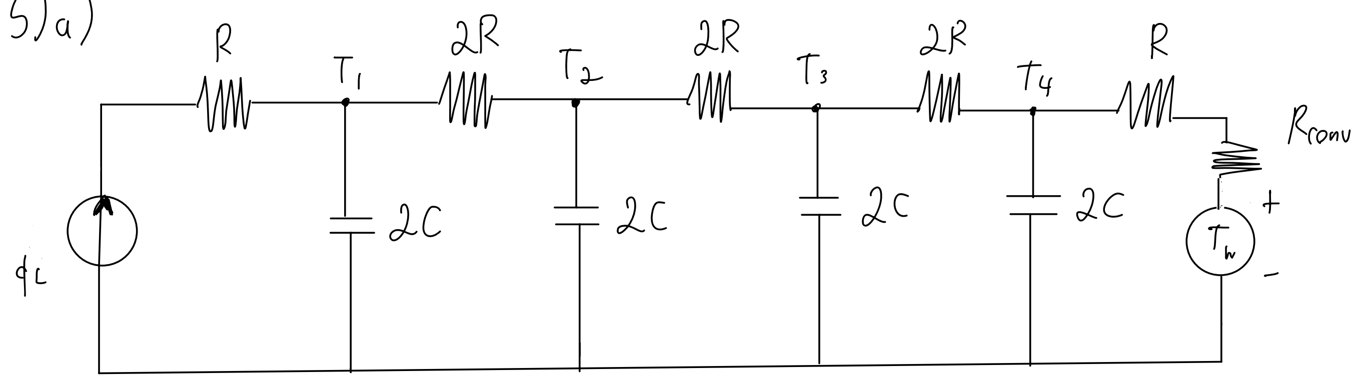
$R_{conv} = 1.553$

d) $Q = \dfrac{\Delta T}{R_{cond} \Delta x} = \dfrac{T_5 - T_7}{(0.389)(2)} = \dfrac{1.925 - 1.309}{(.389)(2)}$



$= 0.792 \text{ W}$

5)a)



$$q = \frac{\Delta T}{R}$$
resistor

$$q = C\frac{dT}{dt} \Rightarrow \frac{dT}{dt} = \frac{q}{C}$$
capacitor

$$2C\dot{T}_1 = \frac{T_0 - T_1}{R} - \frac{T_1 - T_2}{2R}$$

$$2C\dot{T}_1 = q_L - \frac{T_1 - T_2}{2R}$$

$$\dot{T}_1 = \frac{q_L}{2C} - \frac{T_1 - T_2}{4RC}$$

$$2C\dot{T}_2 = \frac{T_1 - T_2}{2R} - \frac{T_2 - T_3}{2R}$$

$$\dot{T}_2 = \frac{T_1 - T_2}{4RC} - \frac{T_2 - T_3}{4RC}$$

$$2C\dot{T}_3 = \frac{T_2 - T_3}{2R} - \frac{T_3 - T_4}{2R}$$

$$\dot{T}_3 = \frac{T_2 - T_3}{4RC} - \frac{T_3 - T_4}{4RC}$$

$$2C\dot{T}_4 = \frac{T_3 - T_4}{2R} - \frac{T_4 - T_w}{R + R_{conv}}$$

$$\dot{T}_4 = \frac{T_3 - T_4}{4RC} - \frac{T_4 - T_w}{2C(R + R_{conv})}$$

$$\dot{T}_1 = \frac{1}{2C}\left[-\frac{T_1}{2R} + \frac{T_2}{2R} + q_L\right]$$

$$\dot{T}_2 = \frac{1}{2C}\left[\frac{T_1}{2R} - \frac{T_2}{R} + \frac{T_3}{2R}\right]$$

$$\dot{T}_3 = \frac{1}{2C}\left[\frac{T_2}{2R} - \frac{T_3}{R} + \frac{T_4}{2R}\right]$$

$$\dot{T}_4 = \frac{1}{2C}\left[\frac{T_3}{2R} - \frac{T_4}{2R} - \frac{T_4}{R + R_{conv}} + \frac{T_w}{R + R_{conv}}\right]$$

$$\underbrace{\begin{bmatrix}\dot{T}_1\\ \dot{T}_2\\ \dot{T}_3\\ \dot{T}_4\end{bmatrix}}_{\dot{x}(t)} = \underbrace{\frac{1}{2C}\begin{bmatrix}-\frac{1}{2R} & \frac{1}{2R} & 0 & 0\\ \frac{1}{2R} & -\frac{1}{R} & \frac{1}{2R} & 0\\ 0 & \frac{1}{2R} & -\frac{1}{R} & \frac{1}{2R}\\ 0 & 0 & \frac{1}{2R} & \left(-\frac{1}{2R} - \frac{1}{R+R_{conv}}\right)\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}T_1\\ T_2\\ T_3\\ T_4\end{bmatrix}}_{x} + \underbrace{\frac{1}{2C}\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & \frac{1}{R+R_{conv}}\end{bmatrix}}_{B} \underbrace{\begin{bmatrix}Q_L\\ T_w\end{bmatrix}}_{f}$$

$$y(t) = \underbrace{\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}_{C} \underbrace{\begin{bmatrix}T_1\\ T_2\\ T_3\\ T_4\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}}_{D} \underbrace{\begin{bmatrix}Q_L\\ T_w\end{bmatrix}}_{f}$$

6) a) According to Euler's formula: $e^{ix} = \cos x + i\sin x$

$$c_1 e^{-i w_1 t} + c_2 e^{i w_1 t} = c_1(\cos(-w_1 t) + i\sin(-w_1 t)) + c_2(\cos(w_1 t) + i\sin(w_1 t))$$

$$= c_1\cos(w_1 t) - c_1 i\sin(w_1 t) + c_2\cos(w_1 t) + c_2 i\sin(w_1 t)$$

$$= (c_1 + c_2)\cos(w_1 t) + i(c_2 - c_1)\sin(w_1 t)$$

$$= A\cos(w_1 t) + B\sin(w_1 t) \text{ if } \Rightarrow \quad A = c_1 + c_2$$

$$B = i(c_2 - c_1)$$

b) If $c_1 e^{-i w_1 t} + c_2 e^{i w_1 t}$ is real, then its imaginary term $= 0$. Thus:

$$i(c_2 - c_1) = 0$$

$$(c_1 + c_2)\cos(w_1 t) = 2c_1\cos(w_1 t)$$

$$= 2c_2\cos(w_1 t)$$

We also know the imaginary part of $c_1 e^{-i w_1 t}$ is $-c_1\sin(w_1 t)$ and for $c_2 e^{i w_1 t}$ is $c_2\sin(w_1 t)$ s.t.:

$$-c_1\sin(w_1 t) = c_2\sin(w_1 t)$$

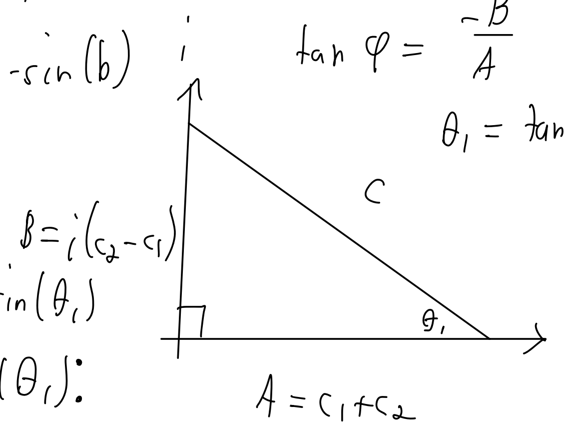
Thus, c_1 and c_2 have the same magnitude and phase angle yet opposite signs for the imaginary part. Therefore, the expression is real when c_1 and c_2 are complex congugates of one another such that the i terms cancel with one another.

c) We know the following identity to be true:

$$\cos(a+b) = \cos(a)\cdot\cos(b) - \sin(a)\cdot\sin(b)$$

$$\tan\varphi = \frac{-B}{A}$$

$$\theta_1 = \tan^{-1}\left(\frac{-B}{A}\right)$$



$$C\cos(w_1 t + \theta_1)$$

$$= C\cos(w_1 t)\cdot\cos(\theta_1) - C\sin(w_1 t)\cdot\sin(\theta_1)$$

If $A = C\cos(\theta_1)$ and $B = -C\sin(\theta_1)$:

$C\cos(w_1 t+\theta) = A\cos(w_1 t) + B\sin(w_1 t)$ and we know $A = c_1 + c_2$, $B = i(c_2 - c_1)$ from before.

If we solve for θ_1 and C, we can prove that the following expression is true. We can set up a system of linear equations and solve in terms of c_1 and c_2:

$$C\cos(\theta_1) = c_1 + c_2$$

$$-C\sin(\theta_1) = i(c_2 - c_1)$$

$$C = \frac{c_1+c_2}{\cos(\theta_1)} = \frac{i(c_1-c_2)}{\sin(\theta_1)}$$

$$\frac{\sin\theta_1}{\cos\theta_1} = \frac{i(c_1-c_2)}{c_1+c_2} = \tan\theta_1$$

$$\theta_1 = \tan^{-1}\left(\frac{i(c_1-c_2)}{c_1+c_2}\right)$$

$$C\cos(\theta_1) = c_1 + c_2$$

$$C\cos\left(\tan^{-1}\left(\frac{i(c_1-c_2)}{c_1+c_2}\right)\right) = c_1 + c_2$$

According to the trig identity:

$$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{C}{\sqrt{1+\left(\frac{i(c_1-c_2)}{c_1+c_2}\right)^2}} = c_1 + c_2$$

$i^2 = -1$

$$\frac{i^2(c_1-c_2)^2}{(c_1+c_2)^2} = \frac{-(c_1-c_2)^2}{(c_1+c_2)^2}$$

$$C = (c_1+c_2)\cdot\sqrt{1 - \frac{(c_1-c_2)^2}{(c_1+c_2)^2}}$$

We can check by triangle trigonometry:

$$\tan\theta_1 = -\frac{B}{A} = \frac{-i(c_2-c_1)}{c_1+c_2} = \frac{i(c_1-c_2)}{c_1+c_2}$$

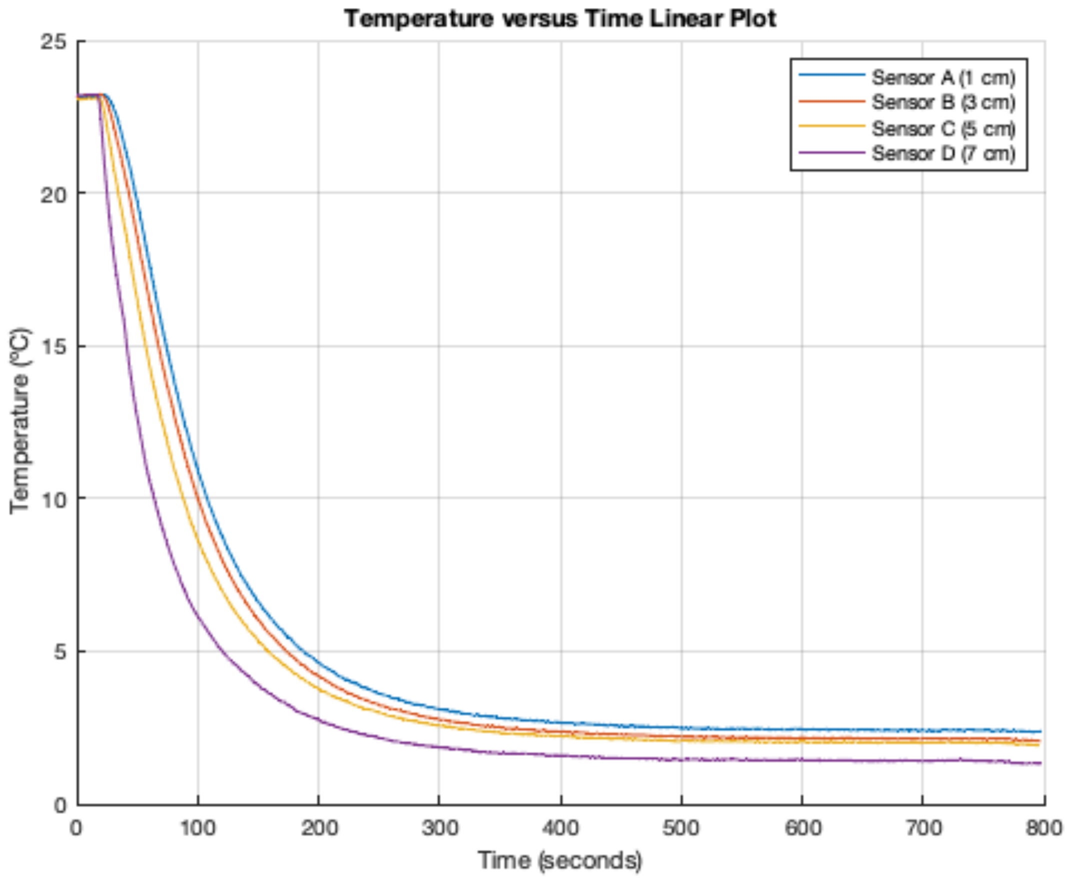
$$\theta_1 = \tan^{-1}\left(\frac{i(c_1-c_2)}{c_1+c_2}\right)$$

```
%Zachary Nelson-Marois
%ENGS23 HW2

clear, close all;

%PART 4A
%Store lab data within matrix
matrix = readmatrix('engs23_lab2.csv');
t_l = matrix(:,1);
%Create plot
figure(1)
grid on, hold on;
plot(t_l, matrix(:,2), t_l, matrix(:,3), t_l, matrix(:,4), t_l, matrix(:,5));
xlim([0 800])
title('Temperature versus Time Linear Plot');
xlabel('Time (seconds)'), ylabel('Temperature (ºC)');
legend('Sensor A (1 cm)', 'Sensor B (3 cm)', 'Sensor C (5 cm)', 'Sensor D (7
 cm)');

%PART 4B
%Define constants for semilog plot
n = 300;
t_s = matrix(1:n,1);
%Define vectors including subtraction from steady state values (determined
%from lab data)
A_s = matrix(1:n,2) - 2.381;
B_s = matrix(1:n,3) - 2.047;
C_s = matrix(1:n,4) - 1.925;
D_s = matrix(1:n,5) - 1.309;
%Create plot
figure(2)
hold off;
semilogy(t_s, A_s, t_s, B_s, t_s, C_s, t_s, D_s);
grid on;
ylim([1 25])
title('Temperature versus Time Semilog Plot');
xlabel('Time (seconds)'), ylabel('Temperature (ºC)');
legend('Sensor A (1 cm)', 'Sensor B (3 cm)', 'Sensor C (5 cm)', 'Sensor D (7
 cm)');

%PART 4D
%Define constants
R_cond = 0.389;
delta_x = 2;
T5 = matrix(:,4);
T7 = matrix(:,5);
Q = (T5 - T7)/(R_cond * delta_x);
%Create plot
figure(3);
hold off;
plot(t_l, Q)
grid on;
```

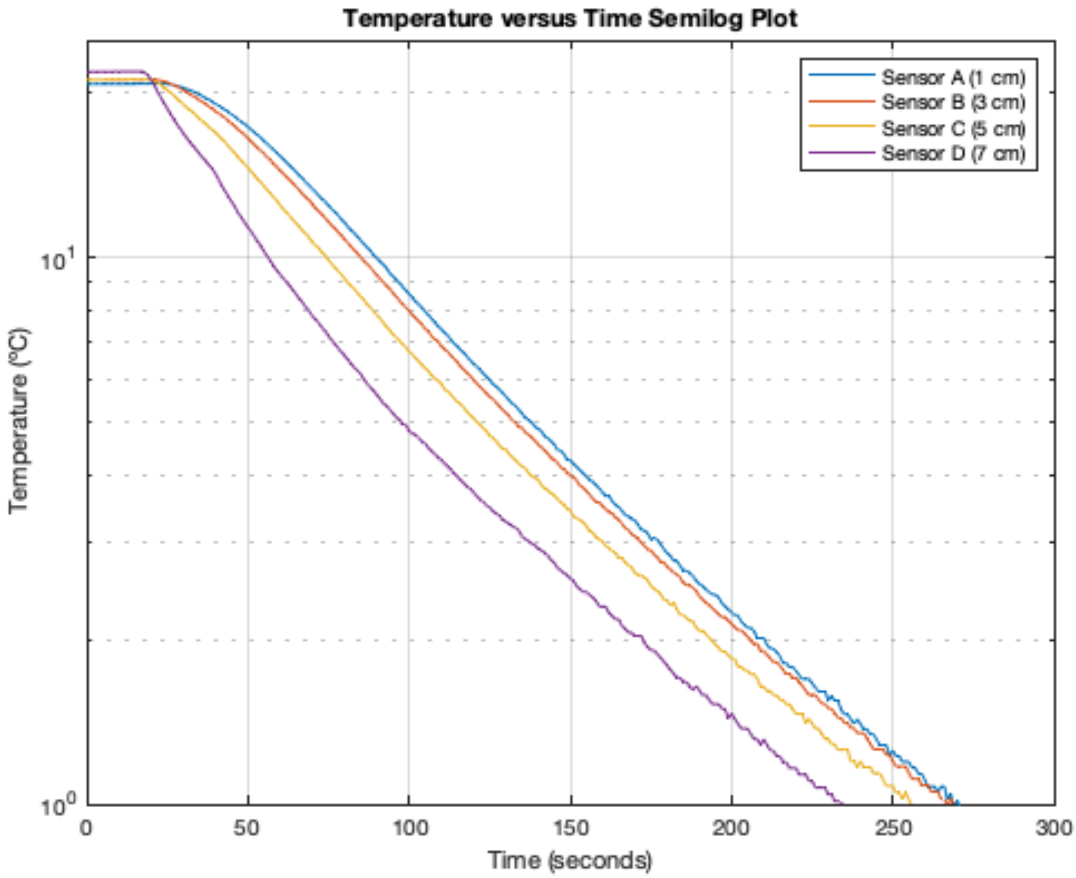
```
title('Heat Flow Between Bottom Two Temperature Sensors Plot');
xlabel('Time (seconds)'),ylabel('Heat Flow (Q)');

%PART 5B
%Define constants
R = 0.389;
C = 3.111;
T_w = 1.0;
T_a = 22.5;
R_conv = 1.553;
Q_l = 0.429;
%Define state-space model parameters
A = (1/(2*C)) * [(-1/(2*R)) (1/(2*R)) 0 0; 1/(2*R) -1/R 1/(2*R) 0; 0 1/(2*R)
 -1/R 1/(2*R); 0 0 1/(2*R) (-1/(2*R) -1/(R+R_conv))];
B = (1/(2*C)) * [1 0; 0 0; 0 0; 0 (1/(R+R_conv))];
C = eye(4);
D = zeros(4,2);
%Time vector for simulated data
t_sim = 0:1:300;
%Time vector for lab data adjusted to fit simulated functions
t_data = t_s - 30;
%Initial temperature of all four sensors is ambient temperature
x_0 = [T_a T_a T_a T_a];
u = [Q_l; T_w];
%Replicate input signal for each time point of the simulation
u = ones(length(t_sim),1) * u';
sys = ss(A, B, C, D);
y = lsim(sys, u, t_sim, x_0);
%Create plot
figure(4)
grid on, hold on;
plot(t_sim, y(:,1), 'r--', t_sim, y(:,2), 'g--', t_sim, y(:,3), 'c--', t_sim,
 y(:,4), 'b--');
plot(t_data, matrix(1:300,2), 'r', t_data, matrix(1:300,3), 'g', t_data,
 matrix(1:300,4), 'c', t_data, matrix(1:300,5), 'b');
title('State Space Model with Respective Experimental Data Plot');
xlabel('Time (seconds)'), ylabel('Temperature (ºC)');
legend('Sensor A Model', 'Sensor B Model', 'Sensor C Model', 'Sensor D
 Model', 'Sensor A Data', 'Sensor B Data', 'Sensor C Data', 'Sensor D Data');
xlim([0 300]);

%PART 5C
%Calculate differences between predicted values (simulation) minus observed
%values (data)
%NOTE - matrix data is shifted to account for proper RMS error
A_d = y(3:272,1) - matrix(32:301,2);
B_d = y(3:272,2) - matrix(32:301,3);
C_d = y(3:272,3) - matrix(32:301,4);
D_d = y(3:272,4) - matrix(32:301,5);
%Define new time vector for only values being compared
t_d = 0:1:269;
%Create plot
figure(5)
hold off;
```

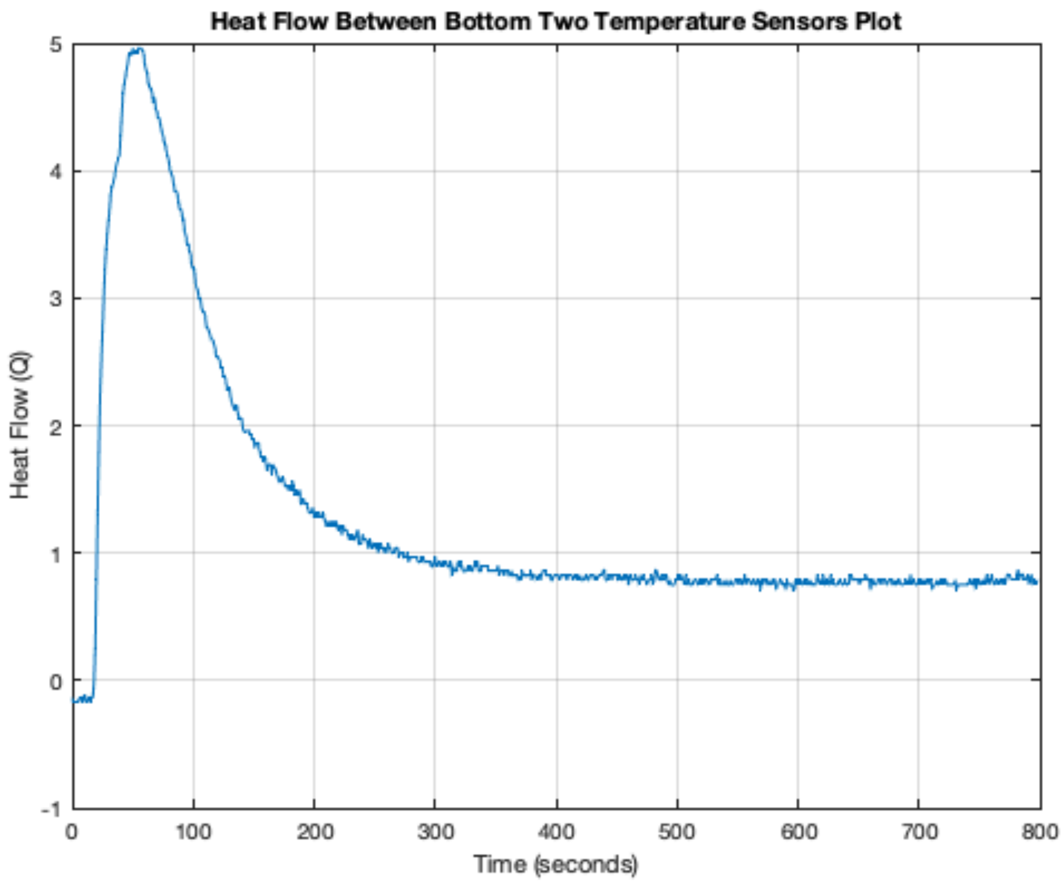
```
plot(t_d, A_d, t_d, B_d, t_d, C_d, t_d, D_d);
grid on;
title('Difference Between Predicted (Simulation) and Observed (Lab Data)
 Values Plot');
xlabel('Time (seconds)'), ylabel('Temperature (ºC)');
legend('Sensor A (1 cm)', 'Sensor B (3 cm)', 'Sensor C (5 cm)', 'Sensor D (7
 cm)');
%RMS values
A_rms = rms(A_d)
B_rms = rms(B_d)
C_rms = rms(C_d)
D_rms = rms(D_d)
```

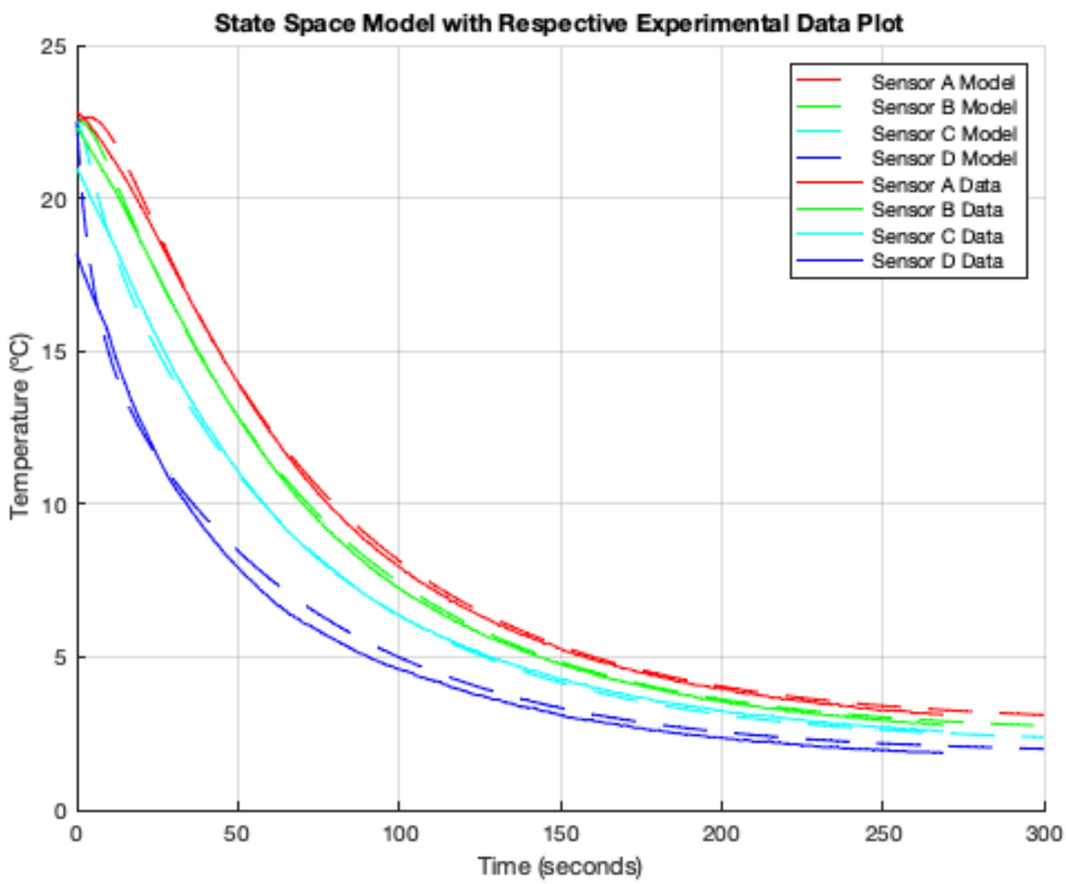
```
A_rms =

    0.1297


B_rms =

    0.1279


C_rms =

    0.2560


D_rms =

    0.3565
```

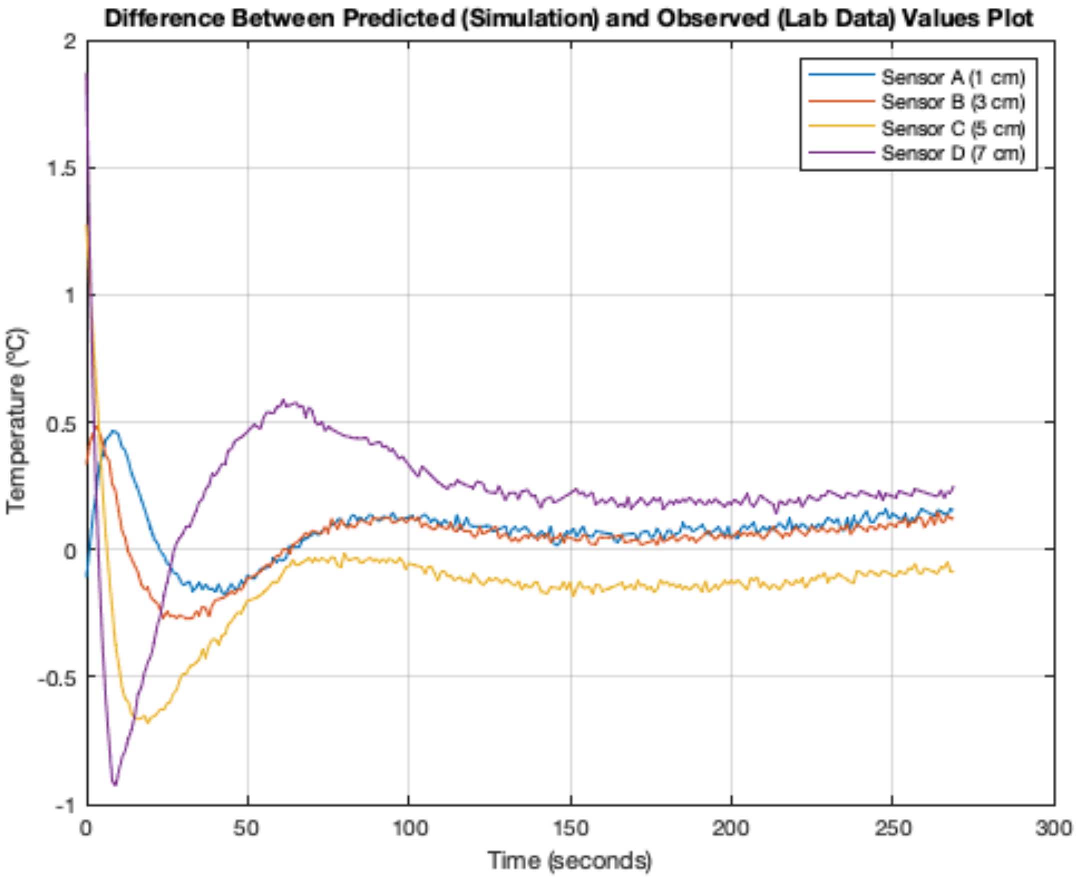


Temperature versus Time Linear Plot
Sensor A (1 cm)
Sensor B (3 cm)
Sensor C (5 cm)
Sensor D (7 cm)
Temperature (ºC)
Time (seconds)
0
5
10
15
20
25
0
100
200
300
400
500
600
700
800


Temperature versus Time Semilog Plot
Sensor A (1 cm)
Sensor B (3 cm)
Sensor C (5 cm)
Sensor D (7 cm)
Temperature (°C)
10^1
10^0
0
50
100
150
200
250
300
Time (seconds)


Heat Flow Between Bottom Two Temperature Sensors Plot
Heat Flow (Q)
Time (seconds)
5
4
3
2
1
0
-1
0
100
200
300
400
500
600
700
800


State Space Model with Respective Experimental Data Plot
Sensor A Model
Sensor B Model
Sensor C Model
Sensor D Model
Sensor A Data
Sensor B Data
Sensor C Data
Sensor D Data
Temperature (ºC)
Time (seconds)
0
5
10
15
20
25
0
50
100
150
200
250
300



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