Capital Recovery Under Lockdown: A Continuous Model of Reinvestment and Capacity

AMATH 383 Final Project

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1 Abstract

The COVID-19 pandemic caused significant economic disruptions worldwide, with the regional lockdowns impacting productivity and reinvestment in the period. As governments responded with a mix of short-term and long-term investment strategies, understanding how capital evolves under these dynamic circumstances with constrained resources became increasingly important. To explore this, we study a simplified dynamic model of logistic capital growth that captures the effects of reinvestment under growth limitations, based on an optimal economic control framework. We simplified the original model to a nonlinear differential equation with a capital capacity to investigate the influence of the government's policies on a single region with varying reinvestment behaviors over time. Our findings show that reinvestment interacts significantly with determining how long regions can approach full economic recovery. This work demonstrates how simplified models can offer insight into investment effectiveness during constrained recovery periods.

2 Introduction

Pandemic regional lockdowns and temporary situations from natural disasters can significantly impact regional economies by limiting productivity and halting capital investment. Policymakers must make timely decisions about how to reinvest resources to support economic recovery. Modeling with real-life dynamics offers a quantitative lens to evaluate the effectiveness of different strategies.

This project addresses the problem of regional recovery by developing a simplified capital growth model adapted from a broader economic control framework by Tonnoir et al. (2021). While the original formulation involves multiple regions and optimal control theory to maximize total income, our focus is on a single-region scenario under different policy controls and reinvestment behaviors.

Our goal is to understand how basic policy parameters such as reinvestment rate, investment effort, and capital capacity influence the long-term growth trajectory of a region. Using numerical simulations of the nonlinear ODE, we examine a variety of scenarios to assess their outcomes. This model tests recovery strategies and might motivates future extensions involving time-varying policies or multi-region interactions.

3 Materials and Methods

3.1 Mathematical Model

3.1.1 Model Derivation

Our model is adapted from the paper by Tonnoir et al. (2021), which presents the model using optimal control systems. The original formulation generalized from Pražák (2012). It involves multiple regions, interregional capital transfers, and an optimal control problem.

To align with the mathematical level of this course, we simplified the model by:

- Focusing on a single region.
- Removing interregional dynamics and policy inequality parameters.

- Replacing linear capital growth with a logistic function that includes a carrying capacity $K_{\max}$.
- Excluding adjoint equations and the Hamiltonian formalism.

The original model from Tonnoir et al. (2021) formulates an optimal investment problem over N regions. The objective is to maximize the total output at final time T:

$$y(T) = \sum_{i=1}^{N} b_i k_i(T) c_i(T)$$

subject to the following constraints:

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$_i(t) = \left(\frac{\alpha}{N} + u_i(t)\right) \left(\sum_{j=1}^{N} s_j b_j k_j(t) c_j(t)\right) + \tilde{s}_i b_i k_i(t) c_i(t), \quad \forall t \in [0, T]$
$\sum_{i=1}^{N} v_i(t) = 1, \quad \text{where } v_i(t) := \frac{\alpha}{N} + u_i(t), \quad \forall t \in [0, T]$
$u_i(t) \in [0, 1-\alpha], \quad \forall t \in [0, T]$

In this system, $k_i(t)$ denotes the capital in region i at time t, b_i is the output–capital ratio, s_j is the regional saving ratio which assumed to be positive, and $c_i(t)$ reflects piece wise time-dependent productivity effects due to lockdowns. The term $\tilde{s}_i$ denotes the fraction of capital directly reinvested within the same region, while the remaining savings can be distributed across regions.

3.1.2 Logistic Growth Extension

As suggested in the original paper, a more realistic model of regional growth can be achieved by introducing a capacity limit via a logistic growth function. The authors propose the following form:

$$\dot{k}_i(t) = \left(\frac{\alpha}{N} + u_i(t)\right) \sum_{j=1}^{N} s_j b_j \left(k_j(t) - \frac{k_j(t)^2}{K_{\max}}\right) c_j(t)$$

This version introduces a nonlinear saturation term, $\frac{k_j(t)^2}{K_{\max}}$, which bounds capital growth and reflects economic carrying capacity.

3.1.3 Simplified Model for a Single Region

To make the model align with the scope of this course, we consider a single-region setting ($N = 1$) and simplify the model by:

- Fixing all control variables as constants: $u_i(t) = u$
- Setting $c_j(t) = 1$, $b_j = b$, and $s_j = s$
- Removing inter-regional redistribution $\tilde{s}_i$

This yields the simplified logistic growth model:

$$\dot{k}(t) = \left(\frac{\alpha}{N} + u\right) sb \left(k(t) - \frac{k(t)^2}{K_{\max}}\right)$$

We denote the combined growth factor as (larger than 0):

$$r = \left(\frac{\alpha}{N} + u\right) sb$$

So the final model becomes:

$$\dot{k}(t) = r \left(k(t) - \frac{k(t)^2}{K_{\max}}\right)$$

- $k(t)$: Capital at time t.
- $K_{\max}$: Maximum capital capacity.
- u: Policy-driven investment effort.

- s: Saving rate (fraction of output reinvested).
- α: Equity parameter (ensures every region gets minimum share of total investment).
- N: Number of regions.
- b: Output-to-capital proportionality constant.

This nonlinear logical capital growth model focuses on a single region with diminishing returns, allowing us to analyze the impact of fixed investment and reinvestment strategies under resource constraints.

3.2 Analytical Behavior

We analyze the logistic capital growth model given by:

$$\frac{dk}{dt} = r\left(k - \frac{k^2}{K_{\max}}\right)$$

where $r = \left(\frac{\alpha}{N} + u\right) sb$ is the combined growth factor and $K_{\max}$ is the capital carrying capacity.

3.2.1 Equilibrium Points

Setting $\frac{dk}{dt} = 0$, we solve:

$$r\left(k - \frac{k^2}{K_{\max}}\right) = 0 \Rightarrow k\left(1 - \frac{k}{K_{\max}}\right) = 0$$

Yields two equilibria:

$$k = 0 \quad \text{and} \quad k = K_{\max}$$

3.2.2 Stability Analysis

Examine the sign of $\frac{dk}{dt}$ between the equilibria.

- When $k < 0$:
  Try $k = -1$.
  Then $f(k) = r(-1 - \frac{1}{K_{\max}}) < 0$.
  So, k decreases.
- When $0 < k < K_{\max}$:
  Try $k = \frac{K_{\max}}{2}$.
  Then
  $$f(k) = r\left(\frac{K_{\max}}{2} - \frac{K_{\max}^2}{4K_{\max}}\right) = r \cdot \frac{K_{\max}}{4} > 0$$
  So, k increases.
- When $k > K_{\max}$:
  Let $k = K_{\max} + \varepsilon$, where $\varepsilon >> 0$, then:
  $$f(k) = r\left(K_{\max} + \varepsilon - \frac{(K_{\max} + \varepsilon)^2}{K_{\max}}\right)$$
  Expanding the square:
  $$= r\left(K_{\max} + \varepsilon - \left(K_{\max} + 2\varepsilon + \frac{\varepsilon^2}{K_{\max}}\right)\right) = -r\left(2\varepsilon + \frac{\varepsilon^2}{K_{\max}}\right) < 0$$
  So, $k > K_{\max} \Rightarrow$ decreases.

3.2.3 Stability Result

- $k = 0$: unstable equilibrium - $k = K_{\max}$: stable equilibrium

This behavior confirms that under positive growth conditions $r > 0$, capital will grow toward the saturation level $K_{\max}$, regardless of initial condition $k_0 > 0$.

3.3 Numerical Simulation

We numerically simulate the logistic capital growth model to explore how changes in reinvestment rate s, investment effort u, and carrying capacity $K_{\max}$ affect the time evolution of capital $k(t)$.

The simulations are run using MATLAB's `ode45` solver with an initial capital $k(0) = 1$ over a time horizon of 800 units.

Simulation Parameters

Unless otherwise specified, the following parameter values are used consistently across simulations:

- $\alpha = 0.5$: Equity parameter, ensuring a moderate share of investment to each region.
- $N = 1$: Number of regions, simplified to a single-region model.
- $b = 0.1$: Output–capital ratio, representing how productive capital is in generating output.
- $k_0 = 1$: Initial capital value at time $t = 0$.
- $t \in [0, 800]$: Simulation time interpreted as days for post-lockdown analysis.

The effective growth rate is computed as:

$$r = \left(\frac{\alpha}{N} + u\right) sb$$

where s is the reinvestment rate and u is the investment effort or policy control variable. In the simulation that follows, we will adjust their values for testing purposes.

3.3.1 Varying s, Fixed $K_{\max}$ and u

In this case, we fix $u = 0.4$ and $K_{\max} = 10$ to represent a moderate level of investment effort and a medium-scale economy. We then explore how different reinvestment rates $s \in \{0.1, 0.3, 0.5\}$ influence capital growth. As shown in Figure 1, higher values of s lead to faster convergence to the equilibrium, while the final capital level remains bounded by the fixed $K_{\max}$. The shape of the trajectories reflects typical logistic growth behavior, where the rate of expansion is governed by the reinvestment rate s.

This simulation demonstrates that, under the same national policy (fixed u) and similar economic capacity (fixed $K_{\max}$), local populations respond differently in how much of their income they save and reinvest. For instance, one region might have a strong local culture of saving or quick recovery policies encouraging reinvestment, while another may prioritize immediate consumption or face barriers to reinvestment. Regions with higher reinvestment rates recover more rapidly and build up capital faster despite identical policy support.

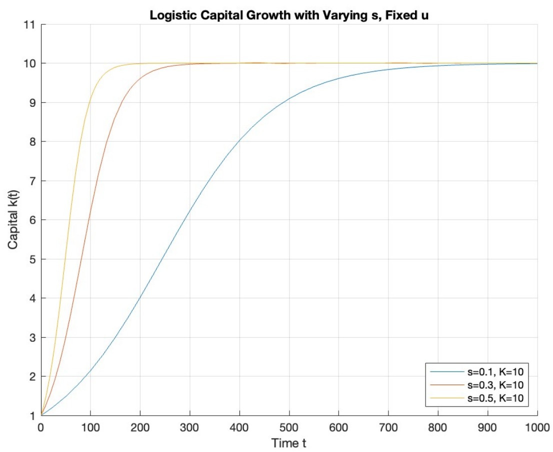

Figure 1: Logistic capital growth under varying reinvestment rate s, with fixed $u = 0.4$, and structural limit $K_{\max} = 10$

3.3.2 Varying $K_{\max}$, Fixed s and u

Here, we fix $s = 0.3$ and $u = 0.4$ to simulate a typical reinvestment rate and moderate government effort. We then vary the structural capacity of the economy $K_{\max} \in \{5, 10, 15\}$ to observe how it influences capital growth. As shown in Figure 2, all trajectories exhibit similar logistic growth patterns, but the final equilibrium capital level increases with higher $K_{\max}$. Since the reinvestment and policy parameters remain constant, the rate of convergence does not change significantly.

This simulation reflects how economies with different structural limitations (e.g., infrastructure, industrial base, or institutional capacity) can grow under identical policy and behavioral settings. With the same reinvestment rate and national-level support (fixed s and u), regions with higher structural capacity ($K_{\max}$) allow the economy to accumulate more capital in the long run, representing systems that are capable of sustaining higher levels of productive capacity. However, it also takes longer for the economy to fully recover to its carrying capacity.

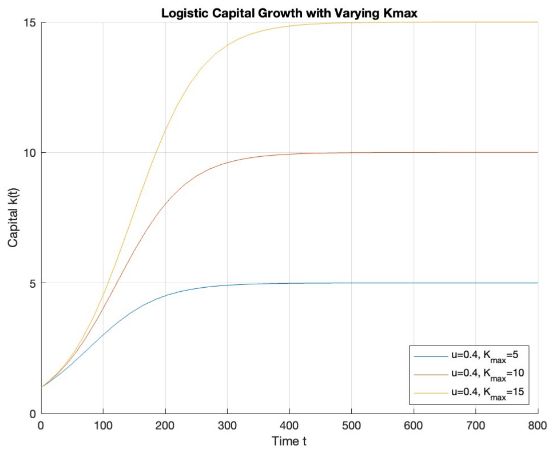

Figure 2: Logistic capital growth with varying $K_{\max}$, fixed $u = 0.4$, $s = 0.3$.

3.3.3 Varying u, Fixed s and $K_{\max}$

In this case, we fix $s = 0.3$ and $K_{\max} = 10$ to represent a stable reinvestment environment and a medium-scale economy. We vary the policy-driven investment effort $u \in \{0.1, 0.3, 0.6\}$ to examine its effect on capital growth. As shown in Figure 3, higher values of u result in faster convergence toward the same carrying capacity.

This simulation illustrates the role of government effort in accelerating economic recovery. Holding reinvestment behavior and economic size constant (fixed s and $K_{\max}$), regions that receive stronger or more timely policy support (e.g., public investment, tax relief) experience faster recovery (higher u). Although the long-term capital level remains determined by $K_{\max}$, the speed of growth improves with stronger policy support.

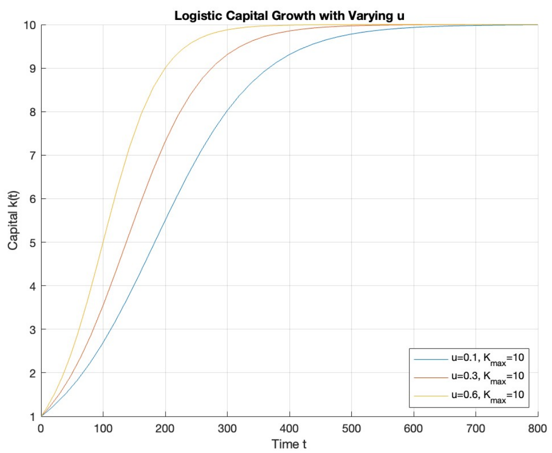

Figure 3: Logistic capital growth under varying government investment effort u, with fixed $s = 0.3$, $K_{\max} = 10$.

4 Discussion

This study explores how capital accumulation evolves during a pandemic lockdown using a simplified logistic growth model derived from a more complex optimal investment framework. By isolating the effects of reinvestment behavior (s), policy intervention (u), and structural economic capacity ($K_{\max}$), we analyze their individual and combined impacts on economic recovery.

The simulations reveal several key insights:

- **Higher reinvestment rates** (s) result in faster recovery. This reflects how communities or regions with stronger savings and reinvestment cultures recover more quickly. In other words, during the lockdowns, regions with more community-driven reinvestment and local spending are more likely to resilience. Strategies such as prioritizing investment in local industries, protecting domestic businesses from external shocks, and promoting targeted exports have been identified as key enablers of recovery in the post-pandemic world Shannon & Carlson (2021).

- **Increased policy support** (u) accelerates capital growth. This can be seen as countries with strong fiscal stimulus programs and direct support to healthcare corperators, businesses and individuals. For instance, the U.S. Senate (2020) CARES Act helped many regions recover faster by preserving capital through direct transfers and subsidies.

- **Greater structural capacity** ($K_{\max}$) raises the upper limit of capital recovery. The larger the limit, the bigger size the economy has. It illustrates that under the same policy control, larger region with larger infrastructures and populations takes longer time to recovery compared to the smaller regions.

Together, these observations suggest that recovery from economic lockdowns depends not only on government policies but also on local reinvestment behavior and local structural capabilities. Our model highlights the relationship of governance, economic structure, and reinvestment dynamics during crisis periods. A uniform policy may yield uneven results if regional capacity or behavioral responses vary widely.

Limitations and Extensions

Our model simplifies reality by focusing on a single region with constant parameters and deterministic behavior. In practice, capital growth is influenced by policy changes over time, population responses to risk, and stochastic disruptions like supply chain shocks or recurring lockdowns. In addition, our approach omits the optimal control formulation in the original paper involving adjoint variables and Hamiltonian dynamics, instead focusing on the qualitative behavior of the system under logistic assumptions.

Future extensions could include:

- Modeling time-varying policy effort $u(t)$ to reflect evolving government strategies.

- Adding a decay term to capture long-term scarring effects of prolonged lockdowns and resources cutdowns, such as permanent business closures and supply chain shocks.

- Introducing multiple regions using summations and inter-regional dynamics to explore inequalities between areas with different s, u, or $K_{\max}$.

- Conducting bifurcation analysis to identify points where small changes in parameters produce large shifts in outcomes.

5 Conclusion

In this project, we developed a simplified continuous-time model to study capital growth during pandemic lockdowns. We focused on a logistic formulation of capital evolvement over time to explore how reinvestment behavior (s), policy intervention (u), and structural capacity ($K_{\max}$) individually influence economic recovery.

Through numerical simulations, we demonstrated that higher reinvestment rates and stronger government intervention lead to faster recovery. Larger economic capacity determines the long-term ceiling of capital growth. These findings highlight that both behavioral and policy-driven factors must align to ensure recovery during lockdowns in pandemics or natural disasters.

While our simplified model omits inter-regional dynamics and optimal control theory, it provides a straightforward framework for understanding fundamental recovery mechanisms. Future research could introduce multi-region coupling, time-varying policy decisions, and decay terms with bifurcation analysis, to deepen insights.

Regarding the limitations to simulate more complexed real-life large scale situations, our study emphasizes that policy designs with localized economic behavior, and structural investment, both play a critical role in building sustainable post-crisis recovery.

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