Part I

Lecture 1: Teichmüller Theory

1 Bordered Surfaces

We begin with a classic geometric problem: Given a surface S describe all hyperbolic structures on the surface. For more details see [Pen06] and [FST08, FT18]

Question 1.1. *What kinds of surfaces are we considering?*

We will focus on **bordered surfaces with marked points**. Such a surface is classified by its genus g and number of boundary components b. We divide the marked points into two classes depending on if the marked points are in the interior or boundary of S. Marked points on the interior of S are **punctures**. Let p be the number of punctures and c be the number of marked points on the boundary. We require that at least one marked point be chosen on each boundary component.

Question 1.2. *What is a hyperbolic structure?*

Definition 1.3. *A **finite area hyperbolic structure** on S will be a choice of constant curvature -1 metric on S with geodesic boundary arcs and every marked point (puncture or boundary) taken to a cusp.*

We consider such structures up to diffeomorphisms fixing the marked points homotopic to the identity to obtain Teichmüller space.

Question 1.4. *For what values of g, b, p, c is this space nontrivial?*

One version of the answer comes from Gauss-Bonnet:

$$\int_S K \mathrm{d}A + \int_{\partial S} k_g \mathrm{d}s = 2\pi\chi(S) \tag{1}$$

When each boundary segment is geodesic $\int_{\partial S} k_g \mathrm{d}s$ reduces to the sum of "turning angles" along the boundary. We are taking each marked point on the boundary to a cusp, so the "turning angle" is π. Thus $\int_{\partial S} k_g \mathrm{d}s = \pi c$. Rearranging Equation 1 we obtain

$$\int_S K \mathrm{d}A = 2\pi\chi(S) - \pi c = 2\pi\left(2 - 2g - b - p - \frac{1}{2}c\right) \tag{2}$$

Definition 1.5. *The **total curvature** of a bordered surface with marked points is $2 - 2g - b - p - \frac{1}{2}c$.*

We write $\chi(S)$ for total curvature of a bordered surface as a slight abuse of notation so the following theorem matches the closed surface case.

Theorem 1.6. *A surface S admits a hyperbolic structure iff $\chi(S) < 0$.*

For a closed surface ($p = 0$,$b = 0$ $c = 0$) this implies $g \geq 2$ as usual. For $g = 1$ we need $p \geq 1$ or $b, c \geq 1$. Similarly for $g = 0$ we obtain the following table:

Corollary 1.7. *The number of triangles in an ideal triangulation of S is $-2\chi(S) = 4g - 4 + 2b + 2p + c$*

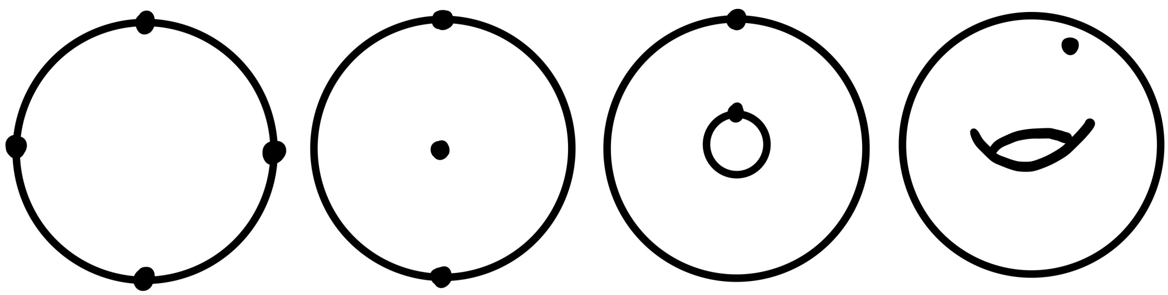
Proof. This follows as the total curvature of an ideal triangle is $-\frac{1}{2}$ (all ideal hyperbolic triangles have area π). ∎

We will focus on non-closed surfaces. Both the triangle and punctured monogon have exactly one hyperbolic structure (there is only one ideal hyperbolic triangle up to isometry). This leads the four smallest nontrivial examples are the square (disk with 4 marked points), the punctured digon, the annulus with 2 marked points, and the once punctured torus. See Figure 2

If we want to take the next step beyond existence we will need another view of a hyperbolic structure.

g	b	p	c	$\chi(S)$	Name
0	0	3	0	-1	Thrice-punctured sphere
0	1	0	3	$-\frac{1}{2}$	Triangle
0	1	1	1	$-\frac{1}{2}$	Punctured monogon
0	2	0	2	-1	Annulus with 2 marked points
1	0	1	0	-1	Once punctured Torus
1	1	0	1	$-\frac{3}{2}$	Torus with boundary
2	0	0	0	-2	Two holed torus

Figure 1: Minimal possible negatively curved bordered surfaces with marked points



(a) Disk With 4 MP (b) Punctured Digon (c) Annulus With 2 MP (d) Punctured Torus

Figure 2: The smallest surfaces with a nontrivial amount of hyperbolic structures

1.1 Developing Map

Definition 1.8. *An alternate definition of a hyperbolic structure on a surface S is the existence of the following pair of maps* $\mathrm{dev}\colon \widetilde{S} \to \mathbb{H}^2$ *and* $\mathrm{hol}\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$*. We call such a pair a* ***developing pair****. The functions are related by the following equation for any $\gamma \in \pi_1(S)$ and $p \in \widetilde{S}$*

$$\mathrm{dev}(\gamma.p) = \mathrm{hol}(\gamma) \cdot \mathrm{dev}(p)$$

For our bordered surfaces we require that each puncture and marked point is sent to an ideal point on $\partial\mathbb{H}^2$.

Note that developing pairs are usually considered up the action of $\mathrm{PSL}(2,\mathbb{R})$ by $g(\mathrm{dev},\mathrm{hol}) \mapsto (g \circ \mathrm{dev}, g\,\mathrm{hol}\,g^{-1})$.

Remark 1.9. *When the image of the developing map is all of $\mathbb{H}^2$ we recover S as $\mathbb{H}^2/\,\mathrm{hol}(\pi_1(S))$ and say the hyperbolic structure is complete.*

Remark 1.10. *In general the notion of a developing pair works for any target geometry X and G a group of isometries by replacing $\mathbb{H}^2$ with X and $\mathrm{PSL}(2,\mathbb{R})$ with G.*

Remark 1.11. *The essential data of the developing map is the choice of boundary images of the marked points. We can define a* ***framed representation*** *to be a representation $\rho\colon \pi_1(S) \to \mathbb{H}^2$ with a ρ equivariant choice of ideal points for each marked point of S.*

Example 1.12. *First we consider the disk with 4 marked points (Figure 3a). Here $\pi_1(S)$ is trivial and the holonomy representation is trivial, so all data is in configuration of ideal points.*

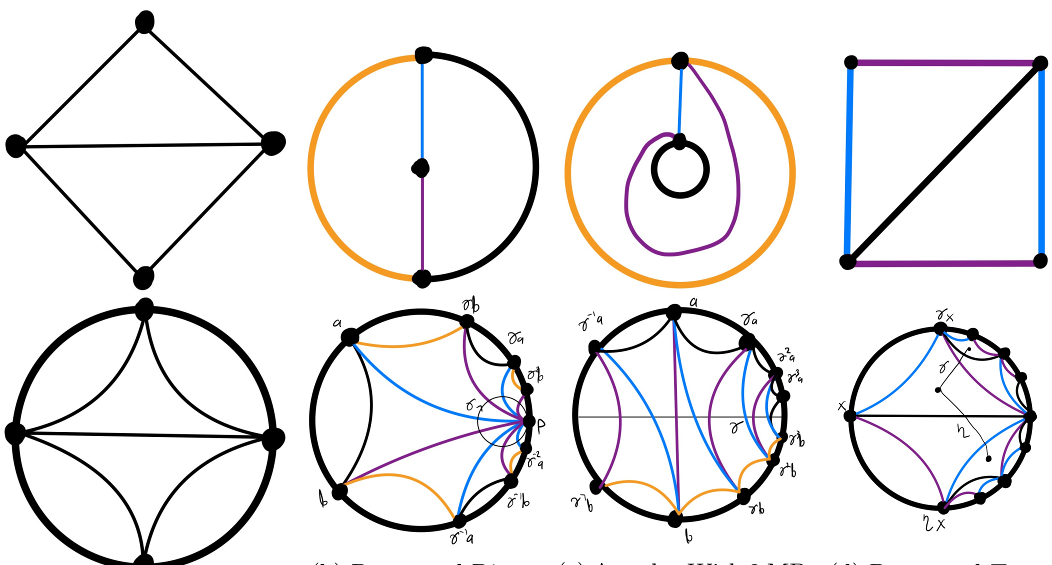
Recall that up to the action of $\mathrm{PSL}(2,\mathbb{R})$ we can send 3 points to $0, 1, \infty$. The last point is sent to x which we call the cross ratio of the four points.

Example 1.13. *Now consider the annulus with two marked points (Figure 3c).*

Here $\pi_1(S) = \mathbb{Z}[\gamma]$ where γ is the loop around the puncture. If we fix an ideal triangulation of the annulus we can develop one triangle at a time. After two triangles we return to the same edges "up to the action of γ". As such we see that actually choice of one ideal point is forced by the representation. The next triangle has one new endpoint again determined by γ. From the last example we saw that one square "corresponded" to one parameter of data. In this case we have two distinct squares and remark that our triangulation has two interior edges.

Example 1.14. *Finally we look at the punctured torus (Figure 3d.*

Here $\pi_1(S) = \mathbb{Z}[\gamma] * \mathbb{Z}[\eta]$. If we develop triangle by triangle we observe that ρ determines the value of flags. Starting from a flag x we see the flag the other flags on the triangle should be $\gamma.x$, $\eta.x$ respectively. When we had the next triangle we have a potential problem, as this flag should both be $\gamma\eta.x$ and $\eta\gamma.x$. This condition is equivalent to $\eta^{-1}\gamma^{-1}\eta\gamma.x = x$. The path $\eta^{-1}\gamma^{-1}\eta\gamma$ is the path around the puncture and so we see that part of the data of a framed representation is that the paths around punctures must fix the flag there.
We can count distinct squares here as well and see there are 3 choices (each square has 2 of the 3 arcs in the triangulation). Once again note this is the number of non-boundary edges of the triangulation.



(a) Disk With 4 MP

(b) Punctured Digon

(c) Annulus With 2 MP

(d) Punctured Torus

Figure 3: Developing Images Of Basic Examples

2 Coordinate System

The first coordinate system we consider is the "X-space". To our surface with an ideal triangulation we can assign a set of cross ratios to each quadrilateral.

2.1 Cross Ratios

Recall that $\partial\mathbb{H}^2$ can be identified with the projective line S^1. So given 4 points $x, y, z, w \in \partial\mathbb{H}^2$ we define

$$\operatorname{cr}(x_1, x_2, x_3, x_3) = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_2)(x_4 - x_1)} \tag{3}$$

Remark 2.1. *The usual cross ratio in projective geometry is* $\operatorname{cr}(x_1, x_3, x_2, x_4) = 1 - \operatorname{cr}(x_1, x_2, x_3, x_4)$. *One reason for this change is our choice behaves especially nicely under cyclic rotation of the points:*

$$\operatorname{cr}(x_1, x_2, x_3, x_4) = \frac{1}{\operatorname{cr}(x_2, x_3, x_4, x_1)} = \operatorname{cr}(x_3, x_4, x_1, x_2) = \frac{1}{\operatorname{cr}(x_4, x_1, x_2, x_3)}$$

Consider a square in an ideal triangulation. If we read the vertices clockwise starting from either end of the diagonal we obtain the same cross ratio $\operatorname{cr}(x_1, x_2, x_3, x_4)$. As such we associate this cross ratio to the diagonal of the square.

Lemma 2.2. *The cross ratio is invariant under isometries of* $\mathbb{H}^2$.

Proof. One can check that Möbius transformations fix the cross ratio (suffices to check translation $x \mapsto x + a$, homothety $x \mapsto bx$ and inversion $x \mapsto 1/x$). □

This is leads to an alternate phrasing of the cross ratio. Given four points (x_1, x_2, x_3, x_4) chose an isometry f sending (x_2, x_3, x_4) to $(0, 1, \infty)$. Such an f is unique as $\mathrm{PSL}(2, \mathbb{R})$ acts simply transitively on triples (there is only one ideal triangle!). Then

$$\operatorname{cr}(x_1, x_2, x_3, x_4) = \operatorname{cr}(f(x_1), 0, 1, \infty) = \frac{(0 - f(x_1))(\infty - 1)}{(1 - 0)(\infty - x_1)} = -f(x_1).$$

Remark 2.3. *As the action of* $\mathrm{PSL}(2, \mathbb{R})$ *preserves orientation, from this description we see that* $f(x_1) < 0$ *and so the cross ratio is positive.*

Theorem 2.4. *An ideal triangulation on a surface determines a list of* $6g - 6 + 3p + 3b + c$ *cross ratios. The values of these cross ratios for a particular hyperbolic structure, uniquely identify the structure.*

Proof. Recall the developing pair assigns points on the boundary of hyperbolic space to each marked point on S. As described above there is a well defined cross ratio associated to each square with diagonal. The number of these squares correspond to the number of interior arcs in the triangulation. We recall from Corollary 1.7 that the triangulation contains $4g - 4 + 2p + 2b + c$ triangles. Note that each interior edge belongs to 2 triangles and each boundary edge belong to one triangle. We note that the number of boundary edges is c. So if e is the number of interior edges we have

$$\begin{aligned} 3t &= 2e + c \\ 12g - 12 + 6p + 6b + 3c &= 2e + c \\ 6g - 6 + 3p + 3b + c &= e \end{aligned}$$

We postpone the proof of reconstructing a representation given a list of positive real numbers associated to each interior edge to a future lecture. □

2.2 Changing Triangulation

One natural question is how the X coordinates change if we chose a new triangulation. Analyzing the change between two arbitrary triangulations is difficult. However there is a theorem that any two triangulations are connected by sequence of "geometric flips".

Definition 2.5. *A* ***geometric flip*** *at an interior edge e of a triangulation T produces a new triangulation $\mu_e(T)$ given by removing e and replacing it with the other diagonal of the square containing e.*

Theorem 2.6. *Any two triangulations of a bordered surface can be reached by a sequence of geometric flips.*

Proof. See [FT18] Proposition 3.8 for sources of a proof. □

Thus to understand how coordinates change we only need to understand a single geometric flip.

Lemma 2.7. *If the edge e is replaced with the edge f in a geometric flip then the new cross ratio $X_f = X_e^{-1}$.*

Proof. This follows from the analysis of the cyclic rotation of a cross ratio in Remark 2.1. □

However we see that this geometric flip changes what squares appear in the triangulation. Analyzing the way these neighboring cross ratios change is difficult. Instead we will add some information to the hyperbolic structure. This will allow us to define a related coordinate system which we can use to understand the X coordinates.

3 Decorated Teichmüller Space

Definition 3.1. *A* ***decoration*** *of a hyperbolic structure on a bordered surface S is a choice of horocycle for each marked point of S.*

Recall that informally a horocycle is the set points "equidistant" from an ideal point called the center of the horocycle. Formally a horocycle is a curve whose normal directions all limit to the same ideal point. In the Poincare Disk model horocycles correspond to circles tangent to the boundary at the ideal point. Similarly in the upper half plane model horocycles are circles tangent to the x-axis or horizontal lines for horocycles centered at infinity.
It is instructive to think about horocycles in the hyperboloid model as well. Here ideal points correspond to lines in the asymptotic cone. Given such an ideal point $\lambda(x, y, \sqrt{x^2 + y^2})$ a horocycle is the intersection the hyperboloid with a plane whose normal vector is $\lambda(x, y, \sqrt{x^2 + y^2}$. Note that the family of horocycles centered at a given point can be parameterized by the magnitude of the translation from the plane through the origin. Through this lens a choice of ideal point corresponds to a choice of line through the origin in $\mathbb{R}^2$ and the horocycle corresponds to a choice of point on this line (specifying the plane translation).

Remark 3.2. *Isometries of hyperbolic space act on horocycles so choosing one horocycle for each marked point of S gives a well defined choice of horocycle at every developed lift by the action of the holonomy representation. See Figure 4*

Definition 3.3. *The* ***decorated Teichmüller space*** *$\mathcal{T}'(S)$ associated to a surface S is the space of decorated hyperbolic structures on S up to diffeomorphisms fixing the marked points.*

There is a natural map from decorated Teichmüller space to Teichmüller space ($\mathcal{T}'(S) \to \mathcal{T}(S)$) given by forgetting the decoration. Each fiber of this map is $\mathbb{R}_+^{c+p}$.

3.1 Lambda Lengths

The decoration allows us to assign a value to each edge of triangulation independently from the other choices. This means when we perform a geometric flip only one of these values would change.

Definition 3.4. *The* ***lambda length*** *between two ideal points with chosen horocycles is* $e^{d/2}$ *where* d *is the length of the geodesic between the horocycles. We take* d *to be negative if the two horocycles intersect.*

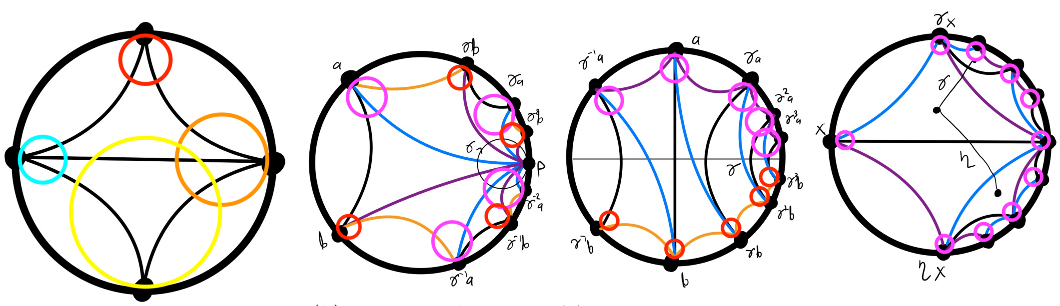
This definition is a little odd at first with two glaring questions: "why exponentiate?" and "why divide by 2?". One small justification for exponentiating is that the lambda length is always positive. As we will see positivity will be an important theme going forward.
A stronger reason for these operations is that mutation and the relation to the cross ratio are much cleaner this way. To begin we will prove the "magic formula" for computing the length of the piece of a horocycle inside a triangle.

Definition 3.5. *The* ***lambda angle*** *at vertex* v_1 *of a decorated ideal triangle is* $T_1^{23} = \frac{\lambda_{23}}{\lambda_{12}\lambda_{23}}$.

Remark 3.6. *It is clear that* $T_1^{32} = T_1^{23}$ *as lambda lengths are symmetric.*

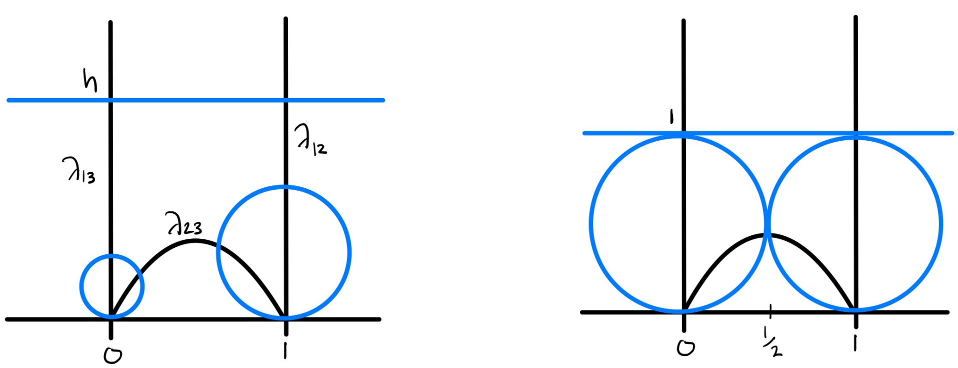
Theorem 3.7. *Given a decorated ideal triangle* v_1, v_2, v_3 *the length of the horocycle centered at* v_1 *between the geodesics connecting to* v_2 *and* v_3 *is* T_1^{23}

Proof. This is easiest to see in the upper half plane where without loss of generality we can take (v_1, v_2, v_3) to be $(\infty, 0, 1)$ (Figure 5a). Then the length we are computing is the length of the horizontal line at height



(a) Disk With 4 MP (b) Punctured Digon (c) Annulus With 2 MP (d) Punctured Torus

Figure 4: Decorated Developing Images Of Basic Examples



(a) Generic (b) Tangent Horocycles

Figure 5: Ideal Decorated Triangle In Upper Half Plane Model

h between 0 and 1. This is a simple computation

$$\int_\gamma \frac{\mathrm{d}s}{y} = \int_0^1 \mathrm{d}t \frac{1}{h} = \frac{1}{h}(1-0)$$

We then check how this length and the ratio of lambda lengths change as we change the size of the horocycles. We see that changing a horocycles at one end of the curve so the distance increases by ε changes the associated lambda length by $e^{\varepsilon/2}$.

$$\lambda'_{12} = e^{(d(v_1,v_2)+\varepsilon)/2} = \lambda_{12}e^{\varepsilon/2}$$

Changing the horocycle at $v_2 = 0$ sends $\lambda_{12} \mapsto \lambda_{12}e^{\varepsilon/2}$ and $\lambda_{23} \mapsto \lambda_{23}e^{\varepsilon/2}$. The $e^{\varepsilon/2}$ cancels in T_1^{23} which matches the fact the horocycle length we are measuring hasn't changed. Similarly changing the horocycle at $v_3 = 1$ fixes both sides.
It remains to see what happens when we move the horocycle at $v_1 = \infty$ from h to he^{ε}. This is a shift by hyperbolic distance ε so $\lambda_{12} \mapsto \lambda_{12}e^{\varepsilon/2}$ and $\lambda_{13} \mapsto e^{\varepsilon/2}$. Thus $T_1^{23} = \frac{\lambda_{23}}{\lambda_{12}\lambda_{13}}$ is transformed by a factor of $e^{-\varepsilon}$. Similarly $\frac{1}{h} \mapsto \frac{1}{he^{\varepsilon}} = \frac{1}{h}e^{-\varepsilon}$.
Finally we check that these functions agree for some choice of horocycle. Take the horocycles that all meet at a single point (See Figure 5b). Then $\lambda_{12} = \lambda_{23} = \lambda_{13} = 1$ and T_1^{23} is also 1. In this configuration the horocycle centered at infinity is at height 1 and so the length of the horocyclic arc is also 1. □

Remark 3.8. *The $\frac{1}{2}$ in the definition of lambda length was necessary for the ratio to transform correctly as the horocycle we were measuring varied.*

Claim 3.9. *The lengths of adjacent horocyclic arcs is additive. Symbolically $T_1^{23} + T_1^{34} = T_1^{24}$*

We will use this additivity to derive a mutation rule for the geometric flip.

Corollary 3.10. *The lambda lengths corresponding to four decorated ideal points v_1, v_2, v_3, v_4 satisfy the relation $\lambda_{13}\lambda_{24} = \lambda_{12}\lambda_{34} + \lambda_{14}\lambda_{23}$*

Proof. We start with the additivity formula around v_1:

$$\begin{aligned} T_1^{24} =& \ T_1^{23} + T_1^{34} \\ \frac{\lambda_{24}}{\lambda_{12}\lambda_{14}} =& \ \frac{\lambda_{23}}{\lambda_{12}\lambda_{13}} + \frac{\lambda_{34}}{\lambda_{13}\lambda_{14}} \\ \lambda_{24}\lambda_{13} =& \ \lambda_{23}\lambda\lambda_{14} + \lambda_{34}\lambda_{12} \end{aligned}$$

□

Explicitly this means that performing the geometric flip replacing the edge γ_{13} with γ_{24} correspond to replacing the coordinate λ_{13} with $\lambda_{24} = \frac{\lambda_{12}\lambda_{34}+\lambda_{14}\lambda_{23}}{\lambda_{13}}$. In the next lecture we will see this mutation relation will generalize to the A type cluster mutation rule.

Claim 3.11. *Given a triangulation the set of lambda lengths corresponding to the edges uniquely specifies the hyperbolic structure up the action of* $\mathrm{PSL}(2,\mathbb{R})$.

Proof. We give an informal sketch here. Later we will discuss explicit formulas to reconstruct a decorated representation $\rho \colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$. After picking one triangle to send to $(0, 1, \infty)$ the lambda lengths associated to these edges fix the horocycles centered on the vertices of the triangles. As we saw the lambda lengths in the next triangle specify the horocyclic length between the new geodesics which fixes the new ideal point and horocycle centered there. □

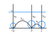


Figure 6: Decorated Ideal Square In The Upper Half Plane Model

3.2 Relationship Between Lambda Lengths and Cross Ratios

Recall that the cross ratio was an invariant we associated to the diagonal edge of a square in the triangulation. If we label the vertices of the square $v_1, \dots, v_4$ so the diagonal goes from v_1 to v_3 we have the following theorem:

Theorem 3.12. *The cross ratio* $X = \mathrm{cr}(v_1, v_2, v_3, v_4)$ *is equal to* $\frac{\lambda_{12}\lambda_{34}}{\lambda_{14}\lambda_{23}}$.

Proof. Using $\mathrm{PSL}(2, \mathbb{R})$ we can send (v_1, v_2, v_3, v_4) to $(-X, 0, 1, \infty)$. Again we compute in the upper half space model (Figure 6). The horocycle centered at infinity is the horizontal line $y = h$. Then the length of this horocycle between $-x$ and 0 is $\frac{1}{h}X$. From our magic formula (Theorem 3.7) we know $\frac{1}{h} = T_4^{12}$. Similarly the length between 0 and 1 is $\frac{1}{h} = T_4^{23}$. Therefore

$$X = T_4^{12}(T_4^{23})^{-1} = \frac{\lambda_{12}}{\lambda_{14}\lambda_{24}}\frac{\lambda_{24}\lambda_{34}}{\lambda_{23}} = \frac{\lambda_{12}\lambda_{34}}{\lambda_{14}\lambda_{23}}$$

□

Corollary 3.13. *We can also write* $1 + X$ *and* $1 + X^{-1}$ *as ratios of lambda lengths. Explicitly:*

$$1 + X = \frac{\lambda_{13}\lambda_{24}}{\lambda_{14}\lambda_{23}} \quad 1 + X^{-1} = \frac{\lambda_{13}\lambda_{24}}{\lambda_{12}\lambda_{34}}$$

Using this relationship we can derive the mutation rule for cross ratios. We focus on a pentagon with initial triangulation γ_{13} and γ_{14}. Here we have two cross ratios which we can write in lambda lengths as

$$X_{13} = \frac{\lambda_{12}\lambda_{34}}{\lambda_{14}\lambda_{23}} \quad X_{14} = \frac{\lambda_{13}\lambda_{45}}{\lambda_{15}\lambda_{34}}$$

If we perform geometric exchange inside the square 1234 we replace γ_{13} with γ_{24}. Writing the new cross ratios here we obtain

$$X_{24} = \frac{\lambda_{14}\lambda_{23}}{\lambda_{12}\lambda_{34}} \quad X_{14} = \frac{\lambda_{12}\lambda_{45}}{\lambda_{15}\lambda_{24}}$$

By comparing with the old formulas we see the cross ratio at the mutated edge changed by inversion. However the neighboring cross ratio X_{14} changes by a factor of $\frac{\lambda_{12}\lambda_{34}}{\lambda_{13}\lambda_{24}} = (1 + X_{13}^{-1})^{-1}$.
We can perform a similar computation performing the exchange in the square 1345 replacing the arc γ_{14} with γ_{35}. The new cross ratios are

$$X_{13} = \frac{\lambda_{12}\lambda_{35}}{\lambda_{23}\lambda_{15}} \quad X_{35} = \frac{\lambda_{15}\lambda_{34}}{\lambda_{13}\lambda_{45}}$$

Now the adjacent cross ratio X_{13} changes by a factor of $\frac{\lambda_{14}\lambda_{35}}{\lambda_{15}\lambda_{34}} = 1 + X_{14}$.
Which rule to chose for the neighbor depends on if the neighbor comes before or after the exchanged arc in the clockwise orientation of the shared triangle. We will see in the next lecture that this generalizes into the X type cluster mutation rule.

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