Quasientrainment and Slow Relaxation in a Population of Oscillators with Random and Frustrated Interactions

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It is numerically shown that there may be a new type of ordered state (in some sense glassy) in far-from-equilibrium systems which can be identified with a large population of coupled limit-cycle oscillators, provided couplings are not only random but also frustrated. It is characterized by quasientrainment and algebraic relaxation.

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Large assemblies of coupled limit-cycle oscillators play an important role in many fields of science. Their most remarkable feature is that they exhibit macroscopic mutual entrainment for coupling strength greater than a certain threshold. The resulting coherent oscillations model a variety of rhythmic behaviors observed in diverse far-from-equilibrium systems, such as biological clocks, many physiological organisms, chemical reactors, and so on [1-3]. Quite a few investigations have been carried out analytically as well as numerically for a number of model systems in order to elucidate the nature of such a transition [2].

A common feature of the models used in those studies is that the natural frequency varies from one oscillator to another, obeying a distribution law as a whole, which may be an example of quenched disorder being unavoidable in nature. Besides this frequency variation, however, randomness in the manner of interactions should exist unavoidably as well in real coupled-oscillator systems, making it important to clarify the behavior of oscillator populations with random interactions. The first investigation along this line was undertaken in a previous paper [4] for some models where interactions are random, but not frustrated. In this paper frustration is taken into account since, as we shall see later, it seems fairly common with disordered interactions. As a result, we discover a phase transition to a new type of ordered state which is in some sense analogous to glasses, which have been a central topic in broad areas of science, including the fields of liquids, magnets, and even information processing [5].

Let us begin by considering the following category of models:

$$\dot{\theta}_j=\Omega_j+(2\pi)^{-1}\sum_{i=1}^{N}\tilde{J}_{ij}\sin 2\pi(\theta_i-\theta_j+\alpha_{ij}),\quad j=1,\ldots,N, \tag{1}$$

where $2\pi\theta_j$ is the phase of the jth oscillator, $\dot{\theta}\equiv d\theta/dt$, Ω_j the natural frequency assumed to be distributed over the whole population with a density denoted by $f(\Omega)$, and N the system size. $\tilde{J}_{ij}\geq 0$ are random constants, and the phase constants α_{ij} are assumed to produce frustration and can also be random.

Frustration [6] was previously discussed in the context of coupled oscillators [4,7]. Suppose a pair of oscillators evolve as $\dot{\theta}_1=\tilde{J}\sin 2\pi(\theta_2-\theta_1+\alpha_{21})$, $\dot{\theta}_2=\tilde{J}\sin 2\pi(\theta_1-\theta_2+\alpha_{12})$, whose phase difference asymptotically becomes $\theta_1-\theta_2=\beta_{12}+\sigma_{12}$, where $\beta_{12}\equiv(\alpha_{21}-\alpha_{12})/2$ and $\sigma_{12}=0$ or $\frac{1}{2}$ depending on the value of $\alpha_{12}+\alpha_{21}$. Then, imagine three oscillators, every pair of which has a similar coupling as above. It is easy to see that unless $\sum_{\text{pair}}(\beta_{ij}+\sigma_{ij})=0(\text{mod}\,1)$, the phase difference favored by the coupling cannot come true for *all* of the pairs, leading to a competition among interactions, or frustration. Extension is straightforward to the case of more than three oscillators.

Frustration may be a fairly common feature of real coupled-oscillator systems: Both the sum of β_{ij} and that of σ_{ij} determine whether it exists or not. When the latter typically deviates from 0(mod1), we have strong frustration. Even when this is not the case, the former is expected generically not to be 0(mod1), since unavoidable disorder ought to generate more or less an asymmetry of the couplings (i.e., $\alpha_{ij}\neq\alpha_{ji}$). Of course, couplings may also be intrinsically asymmetric, as is the case with diverse neural systems. In this case frustration should be prevalent.

In what follows, we investigate a particular model [4] belonging to the category of (1):

$$\dot{\theta}_j=\Omega_j+(2\pi)^{-1}\sum_{i=1}^{N}J_{ij}\sin 2\pi(\theta_i-\theta_j),\quad j=1,\ldots,N, \tag{2}$$

where the constants $J_{ij}=J_{ji}$ are independent random variables obeying a common distribution denoted by $g(J_{ij})$. In this paper we put $g(J_{ij})=(2\pi J^2/N)^{-1/2}\times\exp(-NJ_{ij}^2/2J^2)$, where J is the control parameter, and $f(\Omega)=(2\pi)^{-1/2}\exp(-\Omega^2/2)$. (Note that the average of Ω_j may be set to zero without loss of generality.) In this model α_{ij} is either 0 or $\frac{1}{2}$ with an even probability, leading to strong frustration. This model of infinite-range couplings may be analogous to the Sherrington-Kirkpatrick model [8] of spin glasses. In the following we examine the behavior of this model through numerical results obtained by a Euler scheme with a time increment of 0.01.

Let us first view the behavior of the local fields (LF's)

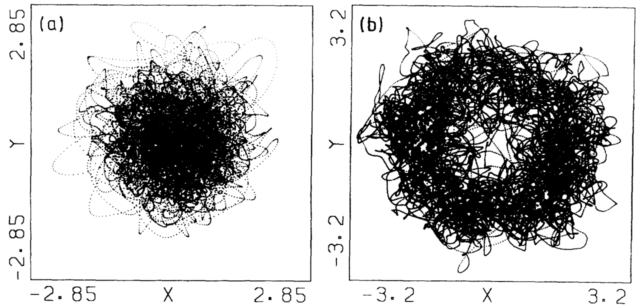

FIG. 1. Phase portraits of a local field ($N=500$): (a) $J=4$ (20000 points); (b) $J=14$ (50000 points).

defined by $p_j = J^{-1}\sum_{i=1}^{N} J_{ij}\exp(2\pi i'\theta_i)$ $[i' \equiv (-1)^{1/2}]$ which are shown by rewriting Eq. (2) to govern the dynamics of the oscillators. These LF's exhibit persistent variations as time passes, the distribution function of which we denote by $P(x,y)$, where $p=x+i'y$. For $J=0$ and N large we have $P(x,y)=\pi^{-1}\exp[-(x^2+y^2)]$ for every LF. As J is increased, the distribution keeps a Gaussian-like form with its peak at the origin until it attains a volcanolike shape for J larger than a certain value (Fig. 1). Although the distribution itself varies from one LF to another, qualitatively the same change seems to happen for every LF. Averaging all the distributions leads to the results in Fig. 2. A natural order parameter of this transition is the value of $|p|$ for which P is maximum, whose behavior seems to suggest a sharp transition at $J=J_c \sim 8$ in the limit $N \rightarrow \infty$ [Fig. 2(b)].

The critical phenomenon discovered above seems to reflect the onset of mutual "entrainment." In fact, the distribution of the average frequency (the time average of $\dot{\theta}_j$) over the population is found to be sharply peaked at the origin (conveniently chosen as the average of natural frequencies) as J is increased beyond a neighborhood of J_c (Fig. 3), indicating the presence of a large cluster of mutually entrained oscillators. No conclusive evidence has yet been obtained, however, to establish that such a cluster is macroscopic with size of $O(N)$ because of a difficulty, due to the very slow dynamics in the ordered state, discussed later.

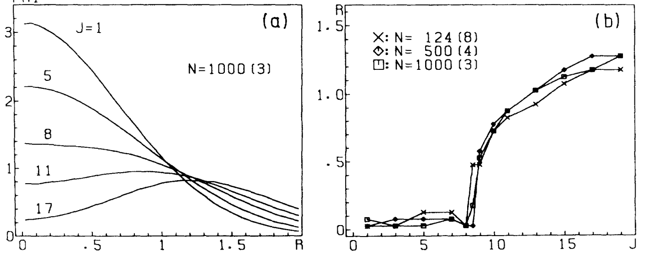

FIG. 2. Behavior of the distribution of a local field averaged over all j: (a) $\mathrm{PHI}(R) \equiv P(x,y)$ vs $R=(x^2+y^2)^{1/2}$, where P is averaged over three samples of J_{ij} as is expressed by (3) in the figure (this convention will be used hereafter); (b) the peak point of $\mathrm{PHI}(R)$ vs J.

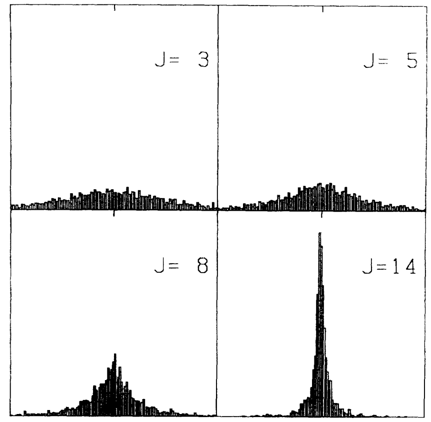

FIG. 3. Distribution of $\omega_j \equiv \langle \dot{\theta}_j \rangle$ averaged over four samples ($N=500$). Each box covers $|\omega| < 1.6335$, the density of distribution is between 0 and 6, and $\omega=0$ is marked by a small vertical bar.

Let us check the behavior of the phase variables in more detail. The entrainment is actually not of the ordinary type observed in many models of coupled-oscillator systems, including the one with uniform couplings [9] and the random but frustration-free one [4]. In other words, the entrainment occurring in the present model is frequency entrainment, but *not* phase locking. As Fig. 4 shows, the entrained oscillators perform diffusive motion in such a way that their mutual distances *diverge* for $t \rightarrow \infty$. We call such a behavior *quasientrainment* (QE): More precisely, two oscillators (say, i and j) are defined to be in (mutual) QE provided they satisfy not only

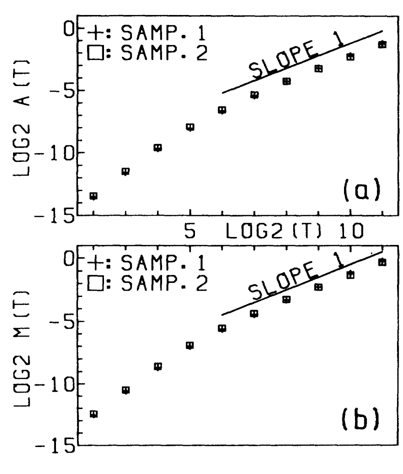

FIG. 4. Evidence of QE ($J=15$, $N=500$): In (a) $A(\tau) \equiv \langle [\theta_j(t+\tau)-\theta_j(t)]^2 \rangle$ averaged over all entrained oscillators (EO's), and in (b) $M(\tau) \equiv M_{ij}(\tau)$ averaged over all pairs of EO's, where the criterion of an EO adopted here is that the average frequency is less than the minimum of $|\Omega_j|$ ($=2.5 \times 10^{-3}$) for 24000 iterations. The unit of time is 0.01 (one step of iteration) (likewise in Fig. 5).

$\langle\dot{\theta}_i\rangle=\langle\dot{\theta}_j\rangle$ but also $M_{ij}(\tau)\equiv\langle[\delta\theta_{ij}(t+\tau)-\delta\theta_{ij}(t)]^2\rangle \to\infty$ for $\tau\to\infty$, where $\delta\theta_{ij}\equiv\theta_i-\theta_j$ and the angular brackets stand for a time average. It is obvious how to define the QE of a single oscillator to an external periodic force. Although no stochastic perturbations are applied to the present system, QE may also occur in a variety of coupled-oscillator systems driven by external noise. It may be conjectured, then, that the critical value J_c marks the onset of macroscopic mutual QE such that $M_{ij}(t)\propto t$ for t large [Fig. 4(b)].

QE is "fuzzy" in its character since it keeps a property of nonentrainment [i.e., $M_{ij}(\infty)=\infty$] in spite of being a kind of entrainment. Consequently, phase order does not appear even for $J>J_c$, which means that $Z(t)\equiv N^{-1}\sum_{j=1}^N \exp[2\pi i'\theta_j(t)]$ always vanishes as $t\to\infty$. Let us focus on the manner of its relaxation from an initial finite value. Figure 5 is devoted to results obtained by averaging ten samples with a common initial condition such that $|Z(0)|=1$. As is demonstrated for $N=1000$ in Fig. 5(a), a transition was found to occur from exponential relaxation to algebraic decay for $J=9.3\ldots$ ($N=500$), $6.5\ldots$ ($N=1000$), and $6.0\ldots$ ($N=2000$), which are somewhat smaller than J_c inferred from Fig. 2(b). At the moment, it is not clear what causes the discrepancy. Actually, $Z(t)$ does not completely vanish as long as N is finite [10,11], as is evident from the same figure (near the bottom). Figures 5(b) and 5(c) display the behavior of Γ and α defined by $|Z(t)|\sim e^{-\Gamma t}$ and $|Z(t)|\sim t^{-\alpha}$, respectively. Like the border value of J, the data show fairly good convergence with respect to N, suggesting that the characteristic time of exponential damping, Γ^{-1}, is not divergent at the threshold, and that the index α approaches unity for increasing J. In view of the relaxation of Z as well, the present system is in clear contrast to the models dealt with in Refs. [4,9], in which the ordinary exponential relaxation alone was found [11]. This fact may reveal the importance of frustration for the occurrence of such anomalous relaxation. Besides Z, correlation functions of the form $\langle\exp\{2\pi i'[\theta_j(t+\tau)-\theta_j(t)]\}\rangle$ and their average with respect to j were also found to exhibit slow nonexponential decay for $J>J_c$ [11].

There exists an analogy between the phenomena reported above and those observed in a variety of glassy systems [5]. In fact, slower than exponential relaxation over a *global* range of the control parameter just as found above is one of the important features of those systems. They are also characterized by the "freezing" of relevant degrees of freedom. For instance, in liquid glasses, the atomic motion is frozen into a state of vanishing diffusion constant, and in spin glasses, every spin is trapped by a random LF acting on it, which is the freezing of spin. As to the present system, we may regard quasientrained oscillators as frozen since their average frequencies are zero, meaning that their phases perform slow diffusive motion alone [Fig. 4(a)]. These similarities, however, may not be adequate to call the population in the super-

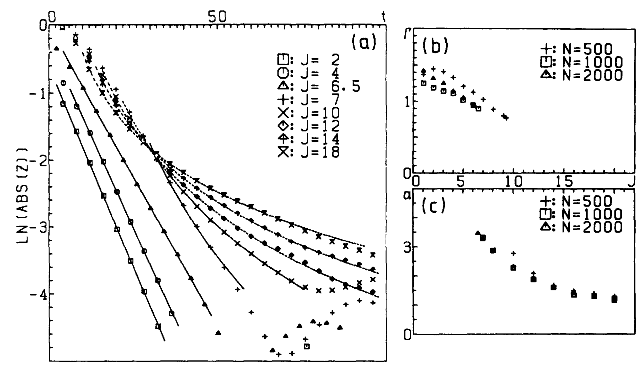

FIG. 5. Relaxation of sample-averaged Z. In (a) $N=1000$ and the data for $J=2,4,6.5$ are shifted along the t axis by $-19.7, -15.7, -9.8$, respectively. The dashed curves show fitted power-law behaviors.

critical regime an *oscillator glass* (OG) [4]. Further careful analyses are necessary to examine whether it deserves to be called a glass or not.

As mentioned earlier, there are many examples of coupled-oscillator systems among diverse biological and physiological organisms. What will follow if any of them are found to become glassy or at least to be in the same state as reported above for $J>J_c$? It may be the case that such a state can only appear either as a disease or in laboratory experiments under some pathological conditions. If otherwise, a new insight may be gained into the nature of living organisms because the state is only *marginally* stable as the algebraic relaxation reveals [5], contradicting the often quoted hypothesis that any living state must be stable enough for the sake of homeostasis [12]. Recently it has been reported that in a certain neural information-processing system, neurons behave as oscillators [13]. In such a system it may be possible that the OG state, if any, has some positive role.

In summary, a class of phase models, Eqs. (1), has been proposed to clarify the role of randomness and frustration in the dynamics of coupled limit-cycle oscillators, and numerical evidence has been presented for the existence of a new type of ordered state in a particular example of (1), Eqs. (2), which is characterized by quasientrainment as well as slow relaxation, and, in a sense, is glassy. The discovery of such a state in real coupled-oscillator systems would be very interesting.

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[1] A. T. Winfree, *The Geometry of Biological Time* (Springer, New York, 1980); Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin,

1984).

[2] A. T. Winfree, J. Theor. Biol. **16**, 15 (1967); Y. Kuramoto, in *Proceedings of the International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki (Springer, New York, 1975); Y. Aizawa, Prog. Theor. Phys. **56**, 703 (1976); Y. Yamaguchi, K. Kometani, and H. Shimizu, J. Stat. Phys. **26**, 719 (1981); H. Sakaguchi, S. Shinomoto, and Y. Kuramoto, Prog. Theor. Phys. **77**, 1005 (1987); H. Daido, Phys. Rev. Lett. **61**, 231 (1988); S. H. Strogatz and R. E. Mirollo, Physica (Amsterdam) **31D**, 143 (1988); M. Shiino and M. Frankowicz, Phys. Lett. A **136**, 103 (1989); K. Satoh, J. Phys. Soc. Jpn. **58**, 2010 (1989); H. Daido, J. Stat. Phys. **60**, 753 (1990); Prog. Theor. Phys. Suppl. **99**, 288 (1989); P. C. Matthews and S. H. Strogatz, Phys. Rev. Lett. **65**, 1701 (1990).

[3] A. H. Cohen, P. J. Holmes, and R. H. Rand, J. Math. Biol. **13**, 345 (1982); G. B. Ermentrout and N. Kopell, SIAM J. Math. Anal. **15**, 215 (1984); J. M. Kowalski, A. Ansari, P. S. Prueitt, R. L. Dawes, and G. Gross, Complex Systems **2**, 441 (1988); L. F. Abbott, J. Phys. A **23**, 3835 (1990).

[4] H. Daido, Prog. Theor. Phys. **77**, 622 (1987).

[5] *Heidelberg Colloquium on Glassy Dynamics*, edited by J. L. van Hemmen and I. Morgenstern (Springer, Berlin, 1987); K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).

[6] G. Toulouse, Commun. Phys. **2**, 115 (1977).

[7] S. Omata, Y. Yamaguchi, and H. Shimizu, Physica (Amsterdam) **31D**, 397 (1988).

[8] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).

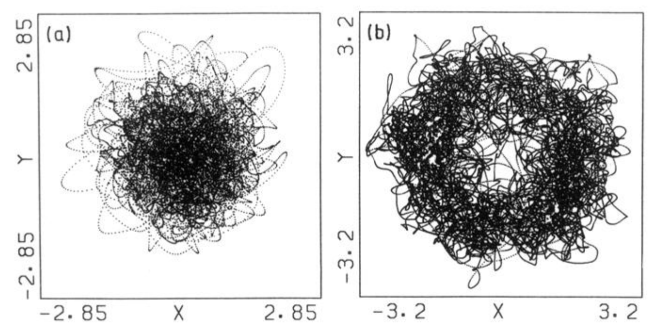
[9] Kuramoto (Ref. [2]).

[10] Daido (1990) (Ref. [2]).

[11] H. Daido (to be published).

[12] A. Garfinkel, Am. J. Physiol. **245**, R455 (1983).

[13] C. M. Gray, P. König, A. K. Engel, and W. Singer, Nature (London) **338**, 334 (1989).



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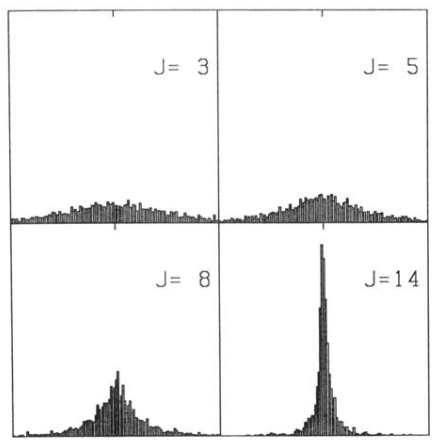

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