Continuation of Janus chimeras

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I. COMPLEX REPRESENTATION OF JANUS OSCILLATORS

We aim to study the bifurcations in the ring of Janus oscillators defined by the equations

$$\dot{\theta}_n = \omega/2 + \beta\sin(\phi_n - \theta_n) + \sigma\sin(\phi_{n+1} - \theta_n), \tag{1}$$
$$\dot{\phi}_n = -\omega/2 + \beta\sin(\theta_n - \phi_n) + \sigma\sin(\theta_{n-1} - \phi_n), \tag{2}$$

with $n = N + 1$ identified with $n = 0$ for periodic boundary conditions. These equations are known to support localized and attractive traveling chimera solutions [1], and we ask whether these solutions emerge from universal a snaking bifurcation process. Consider the complex equations

$$\dot{z}_n = z_n\left(i\omega/2 + \beta/2\left(w_n z_n^* - z_n w_n^*\right) + \sigma/2\left(w_{n+1}z_n^* - z_n w_{n+1}^*\right)\right) + \gamma\left(1 - z_n z_n^*\right)z_n, \tag{3}$$
$$\dot{w}_n = w_n\left(-i\omega/2 + \beta/2\left(z_n w_n^* - w_n z_n^*\right) + \sigma/2\left(z_{n-1}w_n^* - w_n z_{n-1}^*\right)\right) + \gamma\left(1 - w_n w_n^*\right)w_n. \tag{4}$$

Denote the polar coordinates as $z_n = \rho_n e^{i\theta_n}$ and $w_n = \eta_n e^{i\phi_n}$ and the Cartesian coordinates as $z_n = x_n + iy_n$ and $w_n = u_n + iv_n$. Straighforward change of variables leads to the polar equations of motion

$$\dot{\rho}_n = \gamma\rho_n\left(1 - \rho_n^2\right), \tag{5}$$
$$\dot{\eta}_n = \gamma\eta_n\left(1 - \eta_n^2\right), \tag{6}$$
$$\dot{\theta}_n = \omega/2 + \beta\rho_n\eta_n\sin\left(\phi_n - \theta_n\right) + \sigma\rho_n\eta_{n+1}\sin\left(\phi_{n+1} - \theta_n\right) \tag{7}$$
$$\dot{\phi}_n = -\omega/2 + \beta\rho_n\eta_n\sin\left(\theta_n - \phi_n\right) + \sigma\rho_{n-1}\eta_n\sin\left(\theta_{n-1} - \phi_n\right). \tag{8}$$

Note that the amplitude dynamics decouples from the phases and are attracted to the fixed points $\rho_n = 1$ and $\eta_n = 1$. We can simply initialize the equations with these amplitudes, and we do not have to worry about amplitude dynamics. The phase equations then reduce to the ring of Janus oscillators. The major advantages of the complex representation for numerical continuation is that, unlike the phases, the variables remain bounded in time and the limit-cycle attractors are periodic in the complex variables. To integrate these bounded variables numerically, we express the evolution in Cartesian coordindates,

$$\dot{x}_n = -y_n\left(\omega/2 - \beta\left(-x_n v_n + u_n y_n\right) - \sigma\left(-x_n v_{n+1} + u_{n+1}y_n\right)\right) + \gamma\left(1 - x_n^2 - y_n^2\right)x_n, \tag{9}$$
$$\dot{y}_n = x_n\left(\omega/2 - \beta\left(-x_n v_n + u_n y_n\right) - \sigma\left(-x_n v_{n+1} + u_{n+1}y_n\right)\right) + \gamma\left(1 - x_n^2 - y_n^2\right)y_n, \tag{10}$$
$$\dot{u}_n = -v_n\left(-\omega/2 - \beta\left(-u_n y_n + x_n v_n\right) - \sigma\left(-u_n y_{n-1} + x_{n-1}v_n\right)\right) + \gamma\left(1 - u_n^2 - v_n^2\right)u_n, \tag{11}$$
$$\dot{v}_n = u_n\left(-\omega/2 - \beta\left(-u_n y_n + x_n v_n\right) - \sigma\left(-u_n y_{n-1} + x_{n-1}v_n\right)\right) + \gamma\left(1 - u_n^2 - v_n^2\right)v_n. \tag{12}$$

II. TIME-SHIFT REDUCTIONS FOR CHIMERA STATES

The chimera state solutions in the ring of Janus oscillators are traveling waves. We cannot change to moving spatial coordinates since the lattice is discrete, but we can instead consider the time-delayed coordinate $\tau_n = t - \nu n$ for oscillator n, where $1/\nu$ is the velocity. We can enact a reduction if we assume that the oscillator dynamics are identical in their respective time-delayed coordinates (i.e., the variables depend on time only through the invariant coordinate τ_n). Additionally, the space translational invariance and the invariance under global phase rotations enables a reduction which we call the cluster-twisted traveling wave ansatz: $z_n(t) = e^{i\eta n}\left(X_{n \bmod q}(t - \nu n) + iY_{n \bmod q}(t - \nu n)\right)$ and $w_n(t) = e^{i\eta n}\left(U_{n \bmod q}(t - \nu n) + iV_{n \bmod q}(t - \nu n)\right)$, where q denotes the number of clusters and η is twist parameter, and ν is, again, the velocity. Inserting this ansatz into Eqs. (3)-(4), taking $X_m + iY_m = e^{i\Theta_m}$, $U_m + iV_m = e^{i\Phi_m}$, and ignoring the (transient) amplitude dynamics, we find a set of $2q$ time-shift equations

$$\dot{\Theta}_m(\tau) = \omega/2 + \beta\sin(\Phi_m(\tau) - \Theta_m(\tau)) + \sigma\sin(\Phi_{m+1 \bmod q}(\tau - \nu) - \Theta_m(\tau) - \eta), \tag{13}$$
$$\dot{\Phi}_m(\tau) = -\omega/2 + \beta\sin(\Theta_m(\tau) - \Phi_m(\tau)) + \sigma\sin(\Theta_{m-1 \bmod q}(\tau + \nu) - \Phi_m(\tau) + \eta). \tag{14}$$

Given a chimera state from numerical simulations, we can attempt to fit η, ν, and q in order to reduce the solution to a limit-cycle solution of Eqs. (13)-(14) with period T. On a ring of N Janus oscillators, periodicity in space implies that $N\nu = pT$ for some integer p, so that $\nu = pT/N$. In the limit of large N, the chimera solutions correspond to a localized disturbance in $\Theta_m(\tau)$ and $\Phi_m(\tau)$, with asymptotic $\tau \to \pm\infty$ behavior corresponding to a steady state of q-clustered synchrony with twisting rate determined by η. In principle, these localized solutions may undergo snaking bifurcations, leading to chimera states with groups of co-traveling asynchronous regions. The time delay here complicates this analysis somewhat both theoretically and numerically, however.

III. DISCRETE SYMMETRIES IN THE RING

The ring of Janus oscillators is not reflection symmetric for $\sigma \neq \beta$: we follow a sequence $+\sigma - \beta$. Instead, Eqs. (1)-(2) posses two discrete parity-reversing symmetries. First, the ring is invariant under the time/parity reversal given by

$$\theta_n(t) \to \pi + \phi_{N-n}(-t) \tag{15}$$
$$\phi_n(t) \to \theta_{N-n}(-t) \tag{16}$$

which maps stable chimera solutions to unstable chimera solutions. Second, the ring is invariant under the parity reversal given by

$$\theta_n(t) \to -\phi_{N-n}(t) \tag{17}$$
$$\phi_n(t) \to -\theta_{N-n}(t) \tag{18}$$

which maps left-travelling solutions to right-travelling solutions. Note that the second parity reversal symmetry leaves the Kuramoto order parameter invariant as well (so there are two branches of solutions corresponding to each line). Even though the right- and left- travelling solutions map to each other under the parity reversal symmetry, they are not equally likely to be observed from random initial conditions.

IV. ACCOMMODATING CONTINUOUS SYMMETRIES IN AUTO

Since the phase equations depend only on phase differences, the equations are invariant under global phase rotations $\theta_i \to \theta_i + \psi$ and $\phi_i \to \phi_i + \psi$. This means that limit cycle attractors and fixed point attractors will have a neutrally stable perturbation direction corresponding to phase rotations. For limit cycle attractors, with the additional neutral perturbation corresponding to time shifts $\theta_i \to \theta_i + \epsilon\dot{\theta}_i$ and $\phi_i \to \phi_i + \epsilon\dot{\phi}_i$, there will be two unit Floquet multipliers for all parameter values. This makes it slightly more difficult to identify bifurcation points numerically, since every point appears neutrally stable according to linear stability analysis. To fix this problem in AUTO, we can change variables to include any conserved quantities, such as $\Theta = \sum_n (\theta_n + \phi_n)$ in the Janus ring. In such coordinates, the conserved quantities will decouple from a reduced system, and we can study stability in the reduced system instead. Alternatively, if we create our own continuation code, we can simply disregard the zero Floquet exponents corresponding to each symmetry. The easiest way to implement a reduction in the ring of Janus oscillators is to move into a reference frame that rotates at the speed of, say, oscillator z_0. Define quantities $\tilde{z}_n = z_n/z_0$ and $\tilde{w}_n = w_n/z_0$ (whose phases are, respectively, $\theta_n - \theta_0$ and $\phi_n - \theta_0$). Then $\dot{\tilde{z}}_n = \dot{z}_n/z_0 - (z_n/z_0)(\dot{z}_0/z_0)$ and $\dot{\tilde{w}}_n = \dot{w}_n/z_0 - (w_n/z_0)(\dot{z}_0/z_0)$. Assuming (wlog) that z_0 is initialized with $z_0 = 1$, we then have upon substituting into Eqs. 3-4

$$\dot{\tilde{z}}_n = i\tilde{z}_n\left(\beta/2\left(\tilde{w}_n\tilde{z}_n^* - \tilde{z}_n\tilde{w}_n^* - \tilde{w}_0 + \tilde{w}_0^*\right) + \sigma/2\left(\tilde{w}_{n+1}\tilde{z}_n^* - \tilde{z}_n\tilde{w}_{n+1}^* - \tilde{w}_1 + \tilde{w}_1^*\right)\right) + \gamma\left(1 - \tilde{z}_n\tilde{z}_n^*\right)\tilde{z}_n, \tag{19}$$
$$\dot{\tilde{w}}_n = i\tilde{w}_n\left(-\omega + \beta/2\left(\tilde{z}_n\tilde{w}_n^* - \tilde{w}_n\tilde{z}_n^* - \tilde{w}_0^* + \tilde{w}_0\right) + \sigma/2\left(\tilde{z}_{n-1}\tilde{w}_n^* - \tilde{w}_n\tilde{z}_{n-1}^* - \tilde{w}_1 + \tilde{w}_1^*\right)\right) + \gamma\left(1 - \tilde{w}_n\tilde{w}_n^*\right)\tilde{w}_n. \tag{20}$$

V. DISCRETE SYMMETRIES IN THE RING

Note also that the ring of Janus oscillators is not reflection symmetric for $\sigma \neq \beta$. The stable chimera solutions propagate counter clockwise around the ring, and there also exists a branch of unstable chimera solutions which propagate clockwise around the ring. Instead, Eqs. (1)-(2) are invariant under the time/parity reversal given by $\theta_n(t) \to \pi + \phi_{N-n}(-t)$, and $\phi_n(t) \to \theta_{N-n}(-t)$, which maps the the unstable and stable chimera solutions to each other.

VI. NUMERICAL CONTINUATION

We have continued one chimera branch in a ring of 10 Janus oscillators using custom python code, which were validated against AUTO when possible. Figure 1 shows the results of thi continuation. The stable chimera branch (labeled SC) undergoes a torus bifurcation (TB) near $\sigma = 0.3728$ and becomes unstable. The two unstable Floquet multipliers then coalesce just above (TB) and one unstable multiplier then coalesces with the neutrally stable multipliers at a saddle-node bifurcation (SN1) at $\sigma = 0.373$

neutrally stable remotely synchronized solution branch undergoes a similar bifurcation at the saddle-node on the invariant circle bifurcation at $\sigma = 0.25$. With increasing σ, the stable chimera limit cycle branch eventually reaches undergoes a torus bifurcation before reaching a limit point and turning around. The continuation fails to converge before we see any snaking (and the system may be too small for snaking to occur in the first place).

Numerically continuing limit cycle solutions is slow and can be difficult. Consequently, we have not been able to track the subsequent snaking of the unstable chimera branch, and we have restricted our analysis to a ring of only 10 Janus oscillators.

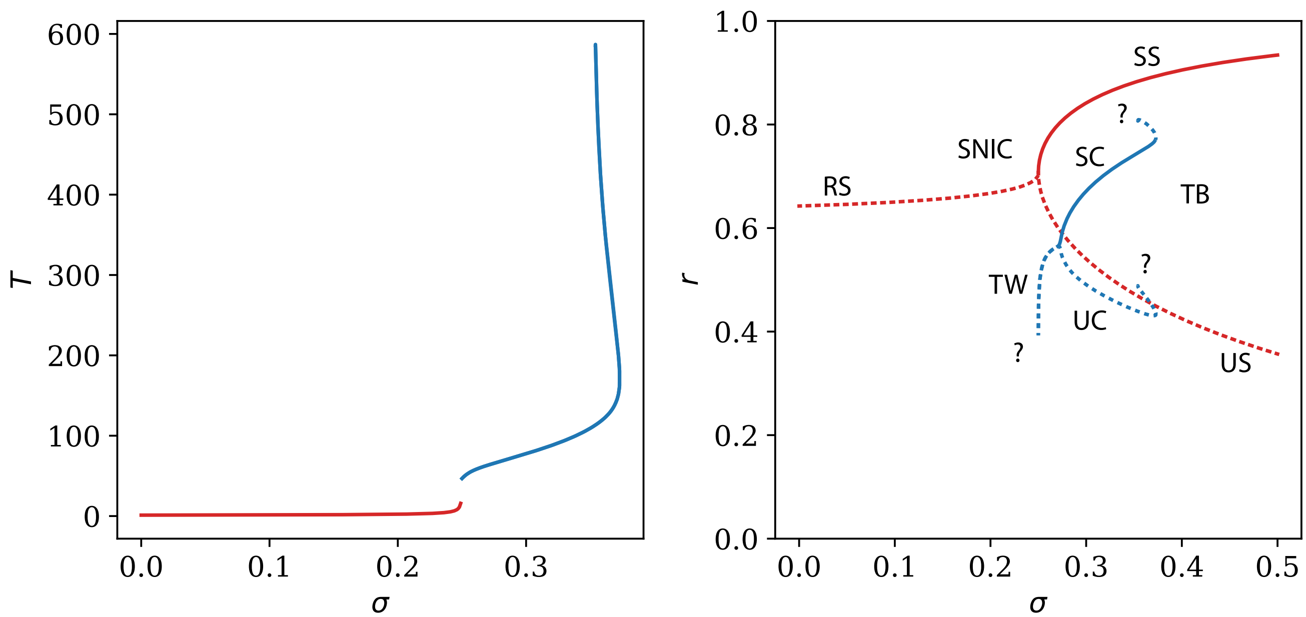

FIG. 1. Bifurcation diagram with the synchronous stable branch (thick red), synchronous unstable branch (thin black), stable limit-cycle chimeras (green dots), and unstable limit-cycle chimeras (blue circles).

[1] Z. G. Nicolaou, D. Eroglu, and A. E. Motter. Multifaceted dynamics of Janus oscillator networks. *Phys. Rev. X* **9**, 011017 (2019).