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Introduction

1. Background & Literature Review

This chapter reviews (i) the institutional setup of cryptocurrency inverse options on Deribit, (ii) the main theoretical approaches for pricing and hedging inverse options, (iii) empirical evidence that stochastic volatility and jumps are central for crypto option pricing, and (v) calibration of model parameters together with the Fourier/FFT pricing methodology that enables fast valuation of large option panels.

1.1 Crypto derivatives and Deribit inverse options

Crypto derivatives are primarily listed on specialised exchanges and are typically collateralised in crypto rather than fiat. Although direct options on cryptocurrencies denominated in fiat do exist, this market structure motivates the dominance of *inverse* contracts: instruments where the payoff and premium are denominated in the underlying coin, while the strike is typically quoted in USD. Since the empirical part of this thesis is based on Deribit option quotes, which is the most popular exchange for cryptocurrency options, we first clarify the mechanics and payoff conventions of inverse options.

Deribit lists European-style inverse options on BTC and ETH, with contract sizes equal to 1 BTC for BTC options and 1 ETH for ETH options. Option strikes are quoted in USD, while option prices (premia) and settlement amounts are denominated in the underlying coin. The underlying of a listed option is a termed futures contract (possibly synthetic) with the same expiry. At expiry, Deribit computes a *delivery price* as the time-weighted average of the Deribit index over a specified window (e.g., 07:30–08:00 UTC), and then settles options in coin units based on intrinsic value divided by the delivery price (Deribit, 2025).

Let F_T denote the USD delivery price of the underlying futures contract at maturity T and K denote the USD strike. The (cash) USD intrinsic values are $(F_T - K)^+$ for calls and $(K - F_T)^+$ for puts. The corresponding inverse option payoffs in coin units are therefore

$$r_{\text{call}}(F_T) = \frac{(F_T - K)^+}{F_T}, \quad r_{\text{put}}(F_T) = \frac{(K - F_T)^+}{F_T}. \quad (1.1.1)$$

1.2 Pricing and hedging inverse options

Inverse payoffs naturally arise when the numeraire is changed from USD cash (or a money-market account) to the underlying futures measure. This viewpoint yields clean relationships between (i) the price of a regular USD-settled option and (ii) the price of the corresponding inverse option. It also implies that the *delta* used for hedging inverse options differs from the textbook Black–Scholes/Black delta.

1.2.1 Lucic and Sepp (2024): numeraire invariance, valuation, and the “net delta”

The central inverse option pricing reference for this thesis is Lucic and Sepp (2024). They formalise inverse options using a martingale valuation approach and the *numeraire invariance principle* (Geman et al., 1995), which states that the value of a payoff does not depend on which asset you use as the numeraire, as long as prices are computed under the corresponding martingale measure. In their notation, F_t is the USD price of the underlying futures.

Their main result is a no-arbitrage identity linking direct (USD-settled) European option values and inverse (coin-settled) values. Denoting by $V(t, \cdot)$ the USD value of the regular option and by $\tilde{V}(t, \cdot)$ the coin value of the inverse option, they show that

$$V(t, \cdot) = F_t \tilde{V}(t, \cdot), \quad (1.2.1)$$

under suitable conditions ensuring the existence of both the USD and futures martingale measures (Lucic and Sepp, 2024). Equation (1.2.1) provides a direct route for inverse pricing: any model and numerical method that yields $V(t, \cdot)$ for a standard European option can be repurposed for inverse options by scaling with $1/F_t$.

A second key contribution of Lucic and Sepp (2024) is the model-independent delta relationship implied by (1.2.1). Differentiating with respect to S yields that the hedge ratio for the inverse option involves an adjustment by the option premium itself. In their notation, the hedging position in inverse futures contracts is

$$\tilde{\Delta}(t, \cdot) = \Delta(t, \cdot) - \frac{V(t, \cdot)}{F_t}, \quad (1.2.2)$$

where $\Delta(t, \cdot)$ is the delta of the corresponding regular option, and $\tilde{\Delta}(t, \cdot)$ is the premium-adjusted or net delta of the inverse option.

Finally, Lucic and Sepp (2024) emphasise that delta hedging of listed inverse options is most naturally implemented using inverse futures contracts. Although the underlying

of a listed option is a termed futures contract with the same expiry, such term futures are often substantially less liquid than perpetual futures, particularly at short maturities. In contrast, perpetual futures are highly liquid and constitute the dominant delta-one instrument on major cryptocurrency exchanges and therefore serve as the primary vehicle for dynamic hedging.

1.2.2 Alexander et al. (2023): inverse options as currency-pair/quanto contracts

An alternative interpretation of inverse options is to view them through an FX/currency-pair lens: holding BTC while measuring value in USD naturally resembles an exchange-rate setting. In Alexander et al. (2023), the authors provide a systematic exposition of direct and inverse crypto options, discuss market incompleteness in these markets, and frame inverse options alongside quanto structures that are relevant for fiat-based investors.

While Lucic and Sepp (2024) focus on numéraire invariance and exchange conventions for valuation and hedging, Alexander et al. (2023) emphasise that inverse options can be analysed using the same conceptual toolkit as currency options, including quanto adjustments in the presence of exchange-rate risk and non-trivial collateralisation.

1.3 Why stochastic volatility and jumps matter in crypto option pricing

Empirically, cryptocurrency return distributions exhibit heavy tails, sharp discontinuities, and strong time-variation in volatility. In option markets, these features manifest as pronounced implied volatility smiles/skews and term structures that are difficult to reconcile with constant-volatility diffusion structures. This section focuses on empirical evidence supporting the use of stochastic volatility and jump components when modelling crypto option prices.

1.3.1 Scaillet et al. (2020): high-frequency evidence of jumps

Scaillet et al. (2020) study high-frequency Bitcoin data and document economically significant jump behaviour. While their focus is not limited to option prices, their results provide foundational empirical motivation for jump components in option pricing models: if jumps are a structural feature of the underlying, option-implied distributions and

tails are naturally affected. This type of evidence supports modelling choices that allow discontinuities rather than relying on pure diffusion volatility to match option-implied tail risk.

1.3.2 Chen et al. (2024): option-implied evidence linking IV slopes to jumps

Chen et al. (2024) provide option-market evidence that connects the shape of the implied volatility surface to jump risk. They develop a theoretical relation between implied volatility slopes and positive/negative price jumps, and empirically study this mechanism using tick-by-tick Bitcoin options data from Deribit. Their results support the view that jump components are not merely a convenient modelling choice but are reflected in the cross-sectional structure of implied volatility.

1.3.3 Chaim and Laurini (2018) and Shen et al. (2020): evidence of jumps in volatility

Evidence for jumps in crypto markets extends beyond discontinuities in returns: volatility itself also exhibits abrupt, jump-like behavior. Chaim and Laurini (2018) model Bitcoin with stochastic volatility and jumps and find that volatility jumps are economically meaningful; in particular, they report that volatility shocks associated with jumps have persistent effects, consistent with the idea that volatility dynamics are not well captured by purely continuous diffusions.

Complementary high-frequency evidence comes from Shen et al. (2020), who study Bitcoin volatility forecasting using realized volatility decompositions that explicitly separate continuous and jump components. Their results indicate that jump-related components of realized variance are important for explaining and forecasting volatility, reinforcing that volatility contains discontinuous movements that are relevant for option pricing.

1.3.4 Hou et al. (2020): SVCJ modelling and evidence from crypto options

The core modelling reference for this thesis is Hou et al. (2020), who propose pricing cryptocurrency options with a stochastic volatility model that includes correlated jumps in both the price and volatility processes (SVCJ model). Their motivation is that price

jumps and volatility jumps co-occur in crypto markets and can materially affect option prices, especially for short maturities and far-from-the-money strikes.

A key feature of their specification is a joint jump structure that allows dependence between return jumps and volatility jumps. In simplified form (suppressing some risk-neutral drift details), the SVCJ dynamics are

$$d \log F_t = (\cdot) dt + \sqrt{V_t} dW_t^S + Z_t^y dN_t, \quad (1.3.1)$$

$$dV_t = \kappa(\theta - V_t) dt + \sigma_v \sqrt{V_t} dW_t^V + Z_t^v dN_t, \quad (1.3.2)$$

where N_t is a Poisson process with intensity λ , and (W^S, W^V) are correlated Brownian motions (Hou et al., 2020). The jump sizes are specified to allow co-jumps and dependence through a conditional distribution:

$$Z_t^v \sim \text{Exponential}(\ell_v), \quad Z_t^y \mid Z_t^v \sim \mathcal{N}(\ell_y + q_j Z_t^v, r_y^2), \quad (1.3.3)$$

so that the price jump distribution depends on the volatility jump size through q_j . Empirically, Hou et al. (2020) report that incorporating this structure is important for fitting crypto option prices, and they document economically meaningful co-jump features in their estimation results.

For the purposes of this thesis, the significance of Hou et al. (2020) is twofold. First, it provides a concrete affine SV-with-jumps specification that is empirically supported in crypto option data. Second, it offers a modelling backbone that can be combined with the inverse-option valuation identity in Lucic and Sepp (2024), thereby enabling inverse option pricing under SVCJ dynamics.

1.4 Research gap and contribution

Sections 1.1–1.3 highlight that (i) the most liquid listed crypto options are inverse contracts, (ii) inverse pricing and delta conventions are theoretically well understood under numeraire invariance, and (iii) stochastic volatility and jumps are empirically important for crypto option surfaces. However, these strands have not yet been fully integrated in a single empirical study focused on inverse options.

1.4.1 What the inverse-option literature delivers (and what it does not)

Lucic and Sepp (2024) provide the key valuation identity (1.2.1) and the premium-adjusted delta formula (1.2.2), together with an empirical study of systematic strategies

and the role of perpetual futures for hedging. Their focus, however, is not on developing a crypto-specific stochastic volatility with jumps model for inverse option prices. Similarly, Alexander et al. (2023) clarify inverse and quanto structures and relate them to FX pricing intuition, but do not develop an empirical pricing and calibration framework for inverse option surfaces that includes stochastic volatility and jumps.

1.4.2 What the crypto SV+jumps literature delivers (and what it does not)

Empirical and option-market studies provide strong evidence that stochastic volatility and jumps in returns and volatility matter in crypto contexts (Scaillet et al., 2020; Chen et al., 2024; Chaim and Laurini, 2018; Shen et al., 2020). In particular, Hou et al. (2020) develop and empirically support an SVCJ framework for pricing cryptocurrency options. Yet, their analysis is primarily aligned with “direct” (USD-denominated) options on the spot asset as the underlying, and does not explicitly target the inverse payoff structure of options on termed futures listed on Deribit.

1.4.3 This thesis: contribution and empirical evaluation

This thesis aims to fill the above gap by combining the inverse-option valuation framework with a crypto-specific affine stochastic volatility model with jumps, calibrating the SVCJ model to Deribit options data, and evaluating it against a set of baseline models:

- **Theory:** Use the numeraire invariance identity of Lucic and Sepp (2024) to price inverse options on futures via standard (USD-like) option prices, and derive the corresponding hedging delta using the premium-adjusted formula (1.2.2).
- **Model:** Adopt the SVCJ model in Hou et al. (2020) (equation (1.3.3)) to price inverse options on futures.
- **Empirics:** Calibrate the competing models (Black, Heston, SVCJ) to Deribit BTC and ETH inverse option panels and produce time series of parameter estimates by repeating calibration on successive option-chain snapshots.
- **Evaluation:** Compare models using out-of-sample pricing errors and, additionally, assess risk-management relevance via delta-hedging performance.

The next sections briefly review the core empirical tools (calibration and Fourier pricing) that make this programme feasible.

1.5 Calibration overview: cross-sectional fitting by MSE

In applied option pricing, model parameters are typically estimated by *cross-sectional* calibration: at each observation time t one fits a parameter vector θ_t to the set of contemporaneous option quotes across strikes and maturities. The calibration objective is commonly a mean squared error (MSE) criterion based on either prices or implied volatilities.

Let $\{(K_i, T_i)\}_{i=1}^{N_t}$ denote the observed option contracts at time t , and let $C_{t,i}^{\text{mkt}}$ denote market prices (or implied vols). A standard calibration problem is

$$\widehat{\theta}_t \in \arg \min_{\theta \in \Theta} \sum_{i=1}^{N_t} w_{t,i} \left(C^{\text{model}}(t; K_i, T_i, \theta) - C_{t,i}^{\text{mkt}} \right)^2, \quad (1.5.1)$$

where $w_{t,i} \geq 0$ are weights (e.g., uniform weights, or weights that down-weight illiquid options), and Θ is the admissible parameter set (e.g., enforcing positivity of variance processes, Feller-type constraints, stationarity restrictions, etc.). In crypto option applications, the large number of options per snapshot and the need to repeat (1.5.1) over time makes fast pricing methods essential (Madan et al., 2019).

In this thesis, “time-varying calibration results” correspond to the sequence $\{\widehat{\theta}_{t,j}\}_{j=1}^J$ obtained by applying (1.5.1) at each snapshot time t_j (e.g., twice per day), separately for BTC and ETH panels. This produces parameter time series that can be analysed for stability, regime shifts, and their relation to market conditions.

1.6 Fourier-based pricing: characteristic functions and FFT

Cross-sectional calibration requires pricing hundreds to thousands of options repeatedly. For affine stochastic volatility models, a standard approach is to price European options via the characteristic function of the log-price under the relevant pricing measure.

If $X_T = \log F_T$, and if the model yields a closed-form characteristic function $\phi_T(u) = \mathbb{E}[\exp(iuX_T) | \mathcal{F}_t]$, then option prices can be obtained using Fourier inversion techniques. In particular, the FFT method of Carr and Madan (1999) converts option valuation across a grid of strikes into fast convolution-type computations, making it highly suitable for calibration loops. The reason affine models are especially convenient is that their characteristic functions are exponential-affine in the state variables, as formalised for

affine jump diffusions in Duffie et al. (2000) and exemplified by the Heston model (Heston, 1993).

For inverse options, the valuation identity (1.2.1) implies a practical workflow:

1. compute the corresponding regular (USD-like) European option value $V(t, \cdot)$ under the chosen affine model using characteristic functions and FFT;
2. obtain the inverse option value by $\tilde{V}(t, \cdot) = V(t, \cdot)/F_t$.

This is the computational strategy adopted in Chapter 2.

2. Pricing Deribit Inverse Options under the SVCJ Model

This chapter develops a pricing and hedging framework for cryptocurrency *inverse European* options traded on Deribit, using the stochastic volatility model with correlated jumps in returns and volatility (SVCJ). We proceed in five steps: (i) specify the SVCJ dynamics as in Hou et al. (2020) and adapt it to options written on *term futures*; (ii) derive the *characteristic function* from first principles; (iii) price inverse options under the *futures numeraire* following the logic of Lucic and Sepp (2024); (iv) derive model deltas and the premium-adjusted *net delta* used for inverse option hedging; (v) show how an inverse option on a term futures contract can be hedged using the more liquid inverse perpetual futures and how to evaluate hedging performance via hedged P&L.

Throughout, we focus on European options on a *term futures* contract with the same expiry as the option, which is the standard convention on Deribit (Lucic and Sepp, 2024).

2.1 The SVCJ model for term-futures underlyings

2.1.1 Baseline SVCJ specification (Hou et al.)

Let $F_t \equiv F_t^{(T)}$ denote the *term futures* price (quoted in USD per coin) for maturity T , observed at time $t \in [0, T]$. Define the log-futures price

$$X_t := \log F_t. \quad (2.1.1)$$

Following Hou et al. (2020), we consider an SVCJ model in which both the log-price and the variance process jump together through a common Poisson process. Let N_t be a Poisson process with constant intensity $\lambda > 0$ and let (W_{1t}, W_{2t}) be two Brownian motions with instantaneous correlation $\rho \in (-1, 1)$:

$$\text{corr}(\mathrm{d}W_{1t}, \mathrm{d}W_{2t}) = \rho. \quad (2.1.2)$$

The SVCJ dynamics under an *arbitrary pricing measure* take the form

$$\mathrm{d}X_t = b_t \mathrm{d}t + \sqrt{V_t} \mathrm{d}W_{1t} + Z^y \mathrm{d}N_t, \quad (2.1.3)$$

$$\mathrm{d}V_t = \kappa(\theta - V_t) \mathrm{d}t + \sigma_v \sqrt{V_t} \mathrm{d}W_{2t} + Z^v \mathrm{d}N_t, \quad (2.1.4)$$

with parameters $\kappa > 0$ (mean reversion), $\theta > 0$ (long-run variance), $\sigma_v > 0$ (vol-of-vol), and $V_t \geq 0$. The drift b_t depends on the measure under which we price, and is derived below.

The jump sizes (Z^y, Z^v) are specified as in Hou et al. (2020): the volatility jump Z^v is exponentially distributed, and the return jump Z^y is Gaussian conditional on Z^v :

$$Z^v \sim \text{Exp}(\text{mean} = \ell_v), \quad \ell_v > 0, \quad (2.1.5)$$

$$Z^y | Z^v \sim \mathcal{N}(\ell_y + \rho_j Z^v, \sigma_y^2), \quad \sigma_y > 0. \quad (2.1.6)$$

Here $\ell_y \in \mathbb{R}$ controls the average log-price jump, σ_y controls the variance of log-price jumps, and $\rho_j \in \mathbb{R}$ controls the dependence between price-jump and volatility-jump sizes (cojump dependence).

2.1.2 SVCJ model under the futures price numeraire

For options written on a futures contract with the same maturity T and with deterministic discounting, it is natural to work under a measure under which F_t is a martingale.¹

We therefore choose the drift b_t in (2.1.3) so that the term-futures price

$$F_t := e^{X_t}$$

is a (local) martingale under the chosen pricing measure. Since F_t may jump when X_t jumps, it is convenient to use the standard left-limit notation

$$F_{t-} := \lim_{s \uparrow t} F_s,$$

i.e., the value of the process immediately before time t (equivalently, F_{t-} is the pre-jump value, while F_t is the post-jump value). The martingale restriction can then be written as

$$\mathbb{E}\left[\frac{dF_t}{F_{t-}} \middle| \mathcal{F}_{t-}\right] = 0. \quad (2.1.7)$$

Theorem (Itô's formula for jump diffusions, Øksendal and Sulem (2007)). *Let $(Y_t)_{t \geq 0}$ be a one-dimensional jump-diffusion process of the form*

$$dY_t = dY_t^{(c)} + \Delta Y_t dN_t, \quad dY_t^{(c)} := a_t dt + b_t dW_t, \quad (2.1.8)$$

where $(W_t)_{t \geq 0}$ is a Brownian motion, $(N_t)_{t \geq 0}$ is a Poisson process, and ΔY_t denotes the jump size at a jump time. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be twice continuously differentiable. Then,

$$df(Y_t) = f'(Y_{t-}) dY_t^{(c)} + \frac{1}{2} f''(Y_{t-}) d[Y^{(c)}]_t + \left(f(Y_{t-} + \Delta Y_t) - f(Y_{t-})\right) dN_t, \quad (2.1.9)$$

where $Y_{t-} = \lim_{s \uparrow t} Y_s$ and $d[Y^{(c)}]_t = b_t^2 dt$ is the quadratic variation of the continuous part of the process.

¹For crypto inverse options on Deribit, discounting is assumed to be negligible (Lucic and Sepp, 2024).

Now, apply Itô's formula for jump diffusions to $F_t = f(X_t) = e^{X_t}$. Separating the continuous and jump parts of X_t :

$$dX_t = dX_t^{(c)} + Z^y dN_t, \quad dX_t^{(c)} := b_t dt + \sqrt{V_t} dW_{1t}.$$

The continuous quadratic variation is

$$d[X^{(c)}]_t = (dX_t^{(c)})^2 = V_t dt.$$

Since $f'(x) = f''(x) = e^x$, Itô's formula gives

$$\begin{aligned} dF_t &= f'(X_{t-}) dX_t^{(c)} + \frac{1}{2} f''(X_{t-}) d[X^{(c)}]_t + \left(f(X_{t-} + Z^y) - f(X_{t-}) \right) dN_t \\ &= F_{t-} \left(b_t dt + \sqrt{V_t} dW_{1t} \right) + \frac{1}{2} F_{t-} V_t dt + F_{t-} (e^{Z^y} - 1) dN_t \\ &= F_{t-} \left(b_t + \frac{1}{2} V_t \right) dt + F_{t-} \sqrt{V_t} dW_{1t} + F_{t-} (e^{Z^y} - 1) dN_t. \end{aligned} \quad (2.1.10)$$

Dividing by F_{t-} yields

$$\frac{dF_t}{F_{t-}} = \left(b_t + \frac{1}{2} V_t \right) dt + \sqrt{V_t} dW_{1t} + (e^{Z^y} - 1) dN_t. \quad (2.1.11)$$

which is the expression used to impose the martingale condition (2.1.7).

Taking conditional expectations in (2.1.11), we compute

$$\mathbb{E}\left[\frac{dF_t}{F_{t-}} \mid \mathcal{F}_{t-}\right] = \mathbb{E}\left[\left(b_t + \frac{1}{2} V_t\right) dt + \sqrt{V_t} dW_{1t} + (e^{Z^y} - 1) dN_t \mid \mathcal{F}_{t-}\right]. \quad (2.1.12)$$

Since dW_{1t} has zero conditional mean and is independent of \mathcal{F}_{t-} , and since dN_t is independent of \mathcal{F}_{t-} with $\mathbb{E}[dN_t \mid \mathcal{F}_{t-}] = \lambda dt$, this reduces to

$$\begin{aligned} \mathbb{E}\left[\frac{dF_t}{F_{t-}} \mid \mathcal{F}_{t-}\right] &= \left(b_t + \frac{1}{2} V_t\right) dt + \mathbb{E}[e^{Z^y} - 1] \mathbb{E}[dN_t \mid \mathcal{F}_{t-}] \\ &= \left[\left(b_t + \frac{1}{2} V_t\right) + \lambda \mathbb{E}[e^{Z^y} - 1]\right] dt. \end{aligned} \quad (2.1.13)$$

Imposing the martingale condition $\mathbb{E}[dF_t/F_{t-} \mid \mathcal{F}_{t-}] = 0$ therefore yields

$$\left(b_t + \frac{1}{2} V_t\right) + \lambda \mathbb{E}[e^{Z^y} - 1] = 0, \quad (2.1.14)$$

and hence the drift of the log-futures process must be chosen as

$$b_t = -\frac{1}{2} V_t - \lambda \kappa_F, \quad \kappa_F := \mathbb{E}[e^{Z^y} - 1]. \quad (2.1.15)$$

Under the Hou-type jump specification (2.1.5)–(2.1.6), κ_F is available in closed form:

$$\mathbb{E}[e^{Z^y} \mid Z^v] = \exp\left(\ell_y + \rho_j Z^v + \frac{1}{2} \sigma_y^2\right), \quad (2.1.16)$$

$$\mathbb{E}[e^{Z^y}] = \exp\left(\ell_y + \frac{1}{2} \sigma_y^2\right) \mathbb{E}[e^{\rho_j Z^v}]. \quad (2.1.17)$$

If Z^v is exponential with mean ℓ_v , then for any complex s with $\text{Re}(s) < 1/\ell_v$,

$$\mathbb{E}[e^{sZ^v}] = \frac{1}{1 - \ell_v s}. \quad (2.1.18)$$

Therefore, provided $1 - \ell_v \rho_j > 0$ (when $\rho_j > 0$),

$$\kappa_F = \exp\left(\ell_y + \frac{1}{2}\sigma_y^2\right) \frac{1}{1 - \ell_v \rho_j} - 1. \quad (2.1.19)$$

Then, the SVCJ dynamics under the futures price numeraire are as follows.

SVCJ dynamics under the futures price numeraire

$$dX_t = \left(-\frac{1}{2}V_t - \lambda\kappa_F\right) dt + \sqrt{V_t} dW_{1t} + Z^y dN_t, \quad (2.1.20)$$

$$dV_t = \kappa(\theta - V_t) dt + \sigma_v \sqrt{V_t} dW_{2t} + Z^v dN_t, \quad (2.1.21)$$

with $\text{corr}(dW_{1t}, dW_{2t}) = \rho$, where N_t is a Poisson process with intensity λ , and the jump sizes satisfy

$$Z^v \sim \text{Exponential}(\text{mean } \ell_v), \quad (2.1.22)$$

$$Z^y \mid Z^v \sim \mathcal{N}(\ell_y + \rho_j Z^v, \sigma_y^2). \quad (2.1.23)$$

The jump compensator $\kappa_F = \mathbb{E}[e^{Z^y} - 1]$ is given in closed form by

$$\kappa_F = \exp\left(\ell_y + \frac{1}{2}\sigma_y^2\right) \frac{1}{1 - \ell_v \rho_j} - 1, \quad 1 - \ell_v \rho_j > 0. \quad (2.1.24)$$

The parameters of the model are:

$$\Theta_{\text{SVCJ}} = (\kappa, \theta, \sigma_v, \rho, \lambda, \ell_y, \sigma_y, \ell_v, \rho_j). \quad (2.1.25)$$

To ensure non-negativity of the variance process, we additionally impose the standard Feller condition

$$2\kappa\theta \geq \sigma_v^2, \quad (2.1.26)$$

which is sufficient (though not necessary) for V_t to remain strictly positive (Heston, 1993). Additional admissibility conditions include $\lambda > 0$, $\ell_v > 0$, and $1 - \ell_v \rho_j > 0$, ensuring finite jump moments and well-defined characteristic functions.

2.2 Deriving the characteristic function for the SVCJ model

This section derives the characteristic function of $X_T = \log F_T$ conditional on (X_t, V_t) under the martingale drift choice (2.1.15). The derivation follows the affine-jump-diffusion logic underlying SVCJ-type models (Duffie et al., 2000; Hou et al., 2020).

2.2.1 Martingale approach to the characteristic function

We derive the characteristic function using a martingale argument based on Itô's formula for jump diffusions.

Fix $u \in \mathbb{C}$ and define the *conditional characteristic function*

$$\phi(u; t, T) := \mathbb{E}[e^{iuX_T} \mid X_t = x, V_t = v]. \quad (2.2.1)$$

Since (X_t, V_t) solves a system of stochastic differential equations whose coefficients depend only on the current state and are driven by Brownian motions and a Poisson process with independent increments, the process (X_t, V_t) is a time-homogeneous Markov process (Øksendal and Sulem, 2007). Then, $\phi(u; t, T)$ depends on the current state only through (t, x, v) . Our objective is to determine ϕ explicitly.

Motivated by the affine structure of the SVCJ model, we conjecture that the characteristic function admits an exponential-affine representation of the form

$$\phi(u; t, T) = \exp(A(\tau; u) + B(\tau; u)V_t + iuX_t), \quad \tau := T - t, \quad (2.2.2)$$

with terminal conditions

$$A(0; u) = 0, \quad B(0; u) = 0,$$

so that $\phi(u; T, T) = e^{iuX_T}$ (Duffie et al., 2000).

Define the process

$$\Phi_t := \exp(A(\tau; u) + B(\tau; u)V_t + iuX_t). \quad (2.2.3)$$

If the functions $A(\cdot; u)$ and $B(\cdot; u)$ are chosen such that $(\Phi_t)_{t \in [0, T]}$ is a martingale, then by construction

$$\Phi_t = \mathbb{E}[\Phi_T \mid \mathcal{F}_t] = \mathbb{E}[e^{iuX_T} \mid \mathcal{F}_t],$$

and hence $\Phi_t = \phi(u; t, T)$ is the characteristic function.

2.2.2 Application of Itô's formula to Φ_t

We derive the *Riccati equations* for $A(\tau; u)$ and $B(\tau; u)$, i.e., the ODEs that can be solved to get the characteristic function, by applying Ito's formula for jump diffusions to the exponential-affine process

$$\Phi_t := \exp(A(\tau; u) + B(\tau; u)V_t + iuX_t), \quad \tau := T - t, \quad (2.2.4)$$

where $u \in \mathbb{C}$ is fixed. The functions $A(\tau; u)$ and $B(\tau; u)$ are deterministic in τ and will be chosen so that $(\Phi_t)_{t \in [0, T]}$ is a martingale. Note that $\tau = T - t$ implies $d\tau = -dt$.

Theorem (Itô's formula for two-dimensional jump–diffusions, (Øksendal and Sulem, 2007)). *Let $(X_t, V_t)_{t \geq 0}$ be a two-dimensional jump–diffusion of the form*

$$dX_t = dX_t^{(c)} + Z^y dN_t, \quad (2.2.5)$$

$$dV_t = dV_t^{(c)} + Z^v dN_t, \quad (2.2.6)$$

where $(X_t^{(c)}, V_t^{(c)})$ denotes the continuous part, N_t is a Poisson process, and (Z^y, Z^v) denotes the jump sizes (possibly random, and independent of \mathcal{F}_{t-}). Let $\Phi : [0, \infty) \times \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be once continuously differentiable in t and twice continuously differentiable in (x, v) . Define $\Phi_t := \Phi(t, X_t, V_t)$ and denote left limits by $X_{t-} := \lim_{s \uparrow t} X_s$ and $V_{t-} := \lim_{s \uparrow t} V_s$. Then Φ_t admits the decomposition

$$d\Phi_t = d\Phi_t^{(c)} + d\Phi_t^{(J)}, \quad (2.2.7)$$

where the continuous part is

$$\begin{aligned} d\Phi_t^{(c)} &= \frac{\partial \Phi}{\partial t}(t, X_{t-}, V_{t-}) dt + \frac{\partial \Phi}{\partial x}(t, X_{t-}, V_{t-}) dX_t^{(c)} + \frac{\partial \Phi}{\partial v}(t, X_{t-}, V_{t-}) dV_t^{(c)} \\ &\quad + \frac{1}{2} \frac{\partial^2 \Phi}{\partial x^2}(t, X_{t-}, V_{t-}) (dX_t^{(c)})^2 + \frac{1}{2} \frac{\partial^2 \Phi}{\partial v^2}(t, X_{t-}, V_{t-}) (dV_t^{(c)})^2 \\ &\quad + \frac{\partial^2 \Phi}{\partial x \partial v}(t, X_{t-}, V_{t-}) dX_t^{(c)} dV_t^{(c)}, \end{aligned} \quad (2.2.8)$$

and the jump part is

$$d\Phi_t^{(J)} = \left(\Phi(t, X_{t-} + Z^y, V_{t-} + Z^v) - \Phi(t, X_{t-}, V_{t-}) \right) dN_t. \quad (2.2.9)$$

Now, applying Itô's formula for two-dimensional jump–diffusions to the process Φ_t .

Step 1: derivatives of $\Phi(t, x, v)$. For $\Phi(t, x, v) = \exp(A(\tau; u) + B(\tau; u)v + iux)$ with $\tau = T - t$, the spatial derivatives are

$$\frac{\partial \Phi}{\partial x} = iu \Phi, \quad \frac{\partial \Phi}{\partial v} = B(\tau; u) \Phi,$$

$$\frac{\partial^2 \Phi}{\partial x^2} = -u^2 \Phi, \quad \frac{\partial^2 \Phi}{\partial v^2} = B(\tau; u)^2 \Phi, \quad \frac{\partial^2 \Phi}{\partial x \partial v} = iu B(\tau; u) \Phi.$$

The time derivative must account for $\tau = T - t$:

$$\frac{\partial \Phi}{\partial t} = \Phi \left(\frac{\partial}{\partial t} (A(\tau; u) + B(\tau; u)v) \right) = \Phi (-A_\tau(\tau; u) - B_\tau(\tau; u)v),$$

where $A_\tau(\tau; u) := \partial A(\tau; u)/\partial \tau$ and $B_\tau(\tau; u) := \partial B(\tau; u)/\partial \tau$.

Step 2: dynamics and quadratic variations. Under the futures price measure, split each state variable into its continuous and jump parts:

$$dX_t = dX_t^{(c)} + Z^y dN_t, \quad dX_t^{(c)} = \left(-\frac{1}{2}V_t - \lambda\kappa_F \right) dt + \sqrt{V_t} dW_{1t}, \quad (2.2.10)$$

$$dV_t = dV_t^{(c)} + Z^v dN_t, \quad dV_t^{(c)} = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_{2t}, \quad (2.2.11)$$

with $\text{corr}(dW_{1t}, dW_{2t}) = \rho$. The *continuous* quadratic covariations are

$$(dX_t^{(c)})^2 = V_t dt, \quad (dV_t^{(c)})^2 = \sigma_v^2 V_t dt, \quad dX_t^{(c)} dV_t^{(c)} = \rho \sigma_v V_t dt.$$

Step 3: continuous part of $d\Phi_t$. Substituting the derivatives into (2.2.8) and using the quadratic variations yields

$$\begin{aligned} d\Phi_t^{(c)} = \Phi_t & \left\{ \left[-A_\tau(\tau; u) - B_\tau(\tau; u)V_t + iu \left(-\frac{1}{2}V_t - \lambda\kappa_F \right) + B(\tau; u)\kappa(\theta - V_t) \right. \right. \\ & - \frac{1}{2}u^2 V_t + \frac{1}{2}\sigma_v^2 B(\tau; u)^2 V_t + \rho \sigma_v iu B(\tau; u)V_t \Big] dt \\ & \left. \left. + iu \sqrt{V_t} dW_{1t} + \sigma_v B(\tau; u) \sqrt{V_t} dW_{2t} \right\}. \right. \end{aligned} \quad (2.2.12)$$

Step 4: jump part of $d\Phi_t$. At a jump time, (X_t, V_t) moves from (X_{t-}, V_{t-}) to $(X_{t-} + Z^y, V_{t-} + Z^v)$, so

$$\Phi(t, X_{t-} + Z^y, V_{t-} + Z^v) - \Phi(t, X_{t-}, V_{t-}) = \Phi_{t-} (e^{iuZ^y + B(\tau; u)Z^v} - 1), \quad (2.2.13)$$

and therefore

$$d\Phi_t^{(J)} = \Phi_{t-} (e^{iuZ^y + B(\tau; u)Z^v} - 1) dN_t. \quad (2.2.14)$$

Step 5: full dynamics and predictable drift. Combining (2.2.12) and (2.2.14), it is convenient to rewrite the jump term using the *compensated Poisson process*

$$\tilde{N}_t := N_t - \lambda t, \quad \text{so that} \quad dN_t = d\tilde{N}_t + \lambda dt,$$

and $\mathbb{E}[\mathrm{d}\tilde{N}_t \mid \mathcal{F}_{t-}] = 0$. Substituting this decomposition into the jump contribution yields the semimartingale dynamics

$$\mathrm{d}\Phi_t = \Phi_{t-} \left(\tilde{\mathcal{D}}_t \mathrm{d}t + iu\sqrt{V_t} \mathrm{d}W_{1t} + \sigma_v B(\tau; u) \sqrt{V_t} \mathrm{d}W_{2t} \right) + \Phi_{t-} (e^{iuZ^y + B(\tau; u)Z^v} - 1) \mathrm{d}\tilde{N}_t, \quad (2.2.15)$$

where the *predictable drift coefficient* $\tilde{\mathcal{D}}_t$ is

$$\begin{aligned} \tilde{\mathcal{D}}_t &= -A_\tau(\tau; u) - B_\tau(\tau; u)V_t + \kappa(\theta - V_t)B(\tau; u) - \frac{1}{2}(u^2 + iu)V_t + \frac{1}{2}\sigma_v^2 B(\tau; u)^2 V_t \\ &\quad + \rho\sigma_v iu B(\tau; u)V_t - iu\lambda\kappa_F + \lambda(M(u, B(\tau; u)) - 1), \end{aligned} \quad (2.2.16)$$

with the *jump transform*

$$M(u, B) := \mathbb{E}[e^{iuZ^y + BZ^v}]. \quad (2.2.17)$$

Equation (2.2.15) separates $\mathrm{d}\Phi_t$ into a predictable drift term, a continuous local martingale part driven by (W_{1t}, W_{2t}) , and a purely discontinuous martingale part driven by the compensated jump process \tilde{N}_t .

Step 6: drift cancellation and Riccati equations. Requiring (Φ_t) to be a martingale implies $\tilde{\mathcal{D}}_t \equiv 0$ for all t and all states. Since (2.2.16) is affine in V_t , we set separately the coefficient of V_t and the constant term to zero. Some algebra yields the *Riccati system*

$$B_\tau(\tau; u) = \frac{1}{2}\sigma_v^2 B(\tau; u)^2 + (\rho\sigma_v iu - \kappa)B(\tau; u) - \frac{1}{2}(u^2 + iu), \quad B(0; u) = 0, \quad (2.2.18)$$

$$A_\tau(\tau; u) = \kappa\theta B(\tau; u) + \lambda(M(u, B(\tau; u)) - 1 - iu\kappa_F), \quad A(0; u) = 0. \quad (2.2.19)$$

Solving (2.2.18)–(2.2.19) yields the conditional characteristic function

$$\phi(u; t, T) = \mathbb{E}[e^{iuX_T} \mid X_t, V_t] = \exp(A(\tau; u) + B(\tau; u)V_t + iuX_t), \quad \tau = T - t.$$

Step 7: closed form of the jump transform Under (2.1.5)–(2.1.6), conditional on $Z^v = z$,

$$\mathbb{E}[e^{iuZ^y} \mid Z^v = z] = \exp\left(iu(\ell_y + \rho_j z) - \frac{1}{2}u^2\sigma_y^2\right). \quad (2.2.20)$$

Thus

$$\begin{aligned} M(u, B) &= \mathbb{E}\left[\exp\left(iu(\ell_y + \rho_j Z^v) - \frac{1}{2}u^2\sigma_y^2\right) e^{BZ^v}\right] \\ &= \exp\left(iu\ell_y - \frac{1}{2}u^2\sigma_y^2\right) \mathbb{E}[e^{(B+iu\rho_j)Z^v}]. \end{aligned} \quad (2.2.21)$$

If Z^v is exponential with mean ℓ_v , then

$$\mathbb{E}[e^{(B+iu\rho_j)Z^v}] = \frac{1}{1 - \ell_v(B + iu\rho_j)}, \quad \operatorname{Re}(B + iu\rho_j) < \frac{1}{\ell_v}, \quad (2.2.22)$$

so the jump transform is

$$M(u, B) = \exp\left(iu\ell_y - \frac{1}{2}u^2\sigma_y^2\right) \frac{1}{1 - \ell_v(B + iu\rho_j)}. \quad (2.2.23)$$

2.2.3 The characteristic function

This subsection solves the Riccati system (2.2.18)–(2.2.19) and collects a semi-closed-form representation of the conditional characteristic function

$$\phi(u; t, T) = \mathbb{E}[e^{iuX_T} | X_t, V_t] = \exp(A(\tau; u) + B(\tau; u)V_t + iuX_t), \quad \tau := T - t,$$

Closed-form solution for $B(\tau; u)$. The Riccati equation for $B(\tau; u)$,

$$B_\tau = \frac{1}{2}\sigma_v^2 B^2 + (\rho\sigma_v iu - \kappa)B - \frac{1}{2}(u^2 + iu), \quad B(0; u) = 0,$$

is identical to the diffusion (Heston) Riccati equation, because jumps enter only the A -equation in (2.2.19). Hence B admits the standard Heston closed form (Heston, 1993). Define, for each complex $u \in \mathbb{C}$,

$$d(u) := \sqrt{(\kappa - \rho\sigma_v iu)^2 + \sigma_v^2(u^2 + iu)}, \quad (2.2.24)$$

$$g(u) := \frac{\kappa - \rho\sigma_v iu - d(u)}{\kappa - \rho\sigma_v iu + d(u)}, \quad (2.2.25)$$

where the square-root branch is chosen such that $\Re(d(u)) > 0$. This branch choice is important for numerical stability: it ensures $e^{-d(u)\tau}$ decays as τ increases and prevents spurious explosions from an incorrect branch selection.

With (2.2.24)–(2.2.25), the closed-form solution is

$$B(\tau; u) = \frac{\kappa - \rho\sigma_v iu - d(u)}{\sigma_v^2} \frac{1 - e^{-d(u)\tau}}{1 - g(u)e^{-d(u)\tau}}. \quad (2.2.26)$$

Given $B(\tau; u)$, the function $A(\tau; u)$ solves

$$A_\tau(\tau; u) = \kappa\theta B(\tau; u) + \lambda(M(u, B(\tau; u)) - 1 - iu\kappa_F), \quad A(0; u) = 0.$$

It is convenient to decompose A into its diffusion and jump contributions,

$$A(\tau; u) = A_{\text{diff}}(\tau; u) + A_{\text{jump}}(\tau; u), \quad (2.2.27)$$

where A_{diff} collects the terms that remain when $\lambda = 0$.

Closed-form diffusion component $A_{\text{diff}}(\tau; u)$. Using (2.2.26), the diffusion component has the standard closed form (Heston, 1993)

$$A_{\text{diff}}(\tau; u) = \frac{\kappa\theta}{\sigma_v^2} \left[(\kappa - \rho\sigma_v iu - d(u))\tau - 2\log\left(\frac{1 - g(u)e^{-d(u)\tau}}{1 - g(u)}\right) \right]. \quad (2.2.28)$$

Jump component $A_{\text{jump}}(\tau; u)$. The jump contribution in (2.2.19) depends on τ only through $B(\tau; u)$. Since $A(0; u) = 0$, integrating the A -ODE over $[0, \tau]$ yields directly

$$A_{\text{jump}}(\tau; u) = \lambda \int_0^\tau (M(u, B(s; u)) - 1 - iu \kappa_F) ds, \quad (2.2.29)$$

where

$$M(u, B) := \mathbb{E}[e^{iuZ^y + BZ^v}], \quad \kappa_F := \mathbb{E}[e^{Z^y} - 1].$$

In general, because $M(u, B(s; u))$ is a nonlinear function of $B(s; u)$, this integral does not simplify to an elementary closed form and is computed numerically.

Summarizing:

Characteristic function (SVCJ under the futures measure)

Let $\tau := T - t$ and $X_t = \log F_t$. The conditional characteristic function is exponential-affine:

$$\phi(u; t, T) := \mathbb{E}[e^{iuX_T} | X_t, V_t] = \exp(A(\tau; u) + B(\tau; u)V_t + iuX_t). \quad (2.2.30)$$

Define

$$\begin{aligned} d(u) &:= \sqrt{(\kappa - \rho\sigma_v iu)^2 + \sigma_v^2(u^2 + iu)}, \\ g(u) &:= \frac{\kappa - \rho\sigma_v iu - d(u)}{\kappa - \rho\sigma_v iu + d(u)}, \quad \Re(d(u)) > 0. \end{aligned}$$

Then

$$B(\tau; u) = \frac{\kappa - \rho\sigma_v iu - d(u)}{\sigma_v^2} \frac{1 - e^{-d(u)\tau}}{1 - g(u)e^{-d(u)\tau}}. \quad (2.2.31)$$

$A(\tau; u) = A_{\text{diff}}(\tau; u) + A_{\text{jump}}(\tau; u)$, with

$$A_{\text{diff}}(\tau; u) = \frac{\kappa\theta}{\sigma_v^2} \left[(\kappa - \rho\sigma_v iu - d(u))\tau - 2 \log\left(\frac{1 - g(u)e^{-d(u)\tau}}{1 - g(u)}\right) \right], \quad (2.2.32)$$

$$A_{\text{jump}}(\tau; u) = \lambda \int_0^\tau (M(u, B(s; u)) - 1 - iu\kappa_F) ds, \quad (2.2.33)$$

where,

$$M(u, B) := \mathbb{E}[e^{iuZ^y + BZ^v}] = \exp\left(iu\ell_y - \frac{1}{2}u^2\sigma_y^2\right) \frac{1}{1 - \ell_v(B + iu\rho_j)}, \quad (2.2.34)$$

$$\kappa_F := \mathbb{E}[e^{Z^y} - 1] = \exp\left(\ell_y + \frac{1}{2}\sigma_y^2\right) \frac{1}{1 - \ell_v\rho_j} - 1. \quad (2.2.35)$$

The characteristic functions for the Black and Heston models under the futures price numeraire are derived in Appendix B.

2.3 Pricing inverse options under the futures numeraire

2.3.1 Inverse option payoffs and change of numeraire

Let $K > 0$ be the strike in USD per coin. The standard (“regular”) USD-settled European payoffs are

$$u_{\text{call}}(F_T) = (F_T - K)^+, \quad u_{\text{put}}(F_T) = (K - F_T)^+. \quad (2.3.1)$$

Inverse options on Deribit are quoted and settled in units of the underlying coin; they correspond to dividing the USD payoff by the terminal underlying price. Thus the inverse payoffs are (Lucic and Sepp, 2024)

$$r_{\text{call}}(F_T) = \frac{(F_T - K)^+}{F_T}, \quad r_{\text{put}}(F_T) = \frac{(K - F_T)^+}{F_T}. \quad (2.3.2)$$

The key observation (used throughout Lucic and Sepp (2024)) is that inverse option valuation is most naturally expressed under a numeraire equal to the underlying forward/futures price. Let F_t be the numeraire and let \mathbb{Q}^F be the corresponding martingale measure (“futures measure”). Then for any USD payoff $u(F_T)$, the time- t USD price satisfies the standard change-of-numeraire identity (Geman et al., 1995; Lucic and Sepp, 2024)

$$V_t^{\text{USD}} = F_t \mathbb{E}^{\mathbb{Q}^F} \left[\frac{u(F_T)}{F_T} \mid \mathcal{F}_t \right]. \quad (2.3.3)$$

Therefore the *coin*-denominated price of the inverse payoff $r(F_T) = u(F_T)/F_T$ is simply

$$V_t^{\text{coin}} = \mathbb{E}^{\mathbb{Q}^F} [r(F_T) \mid \mathcal{F}_t] = \frac{V_t^{\text{USD}}}{F_t}. \quad (2.3.4)$$

In words: to price an inverse option in coin units, we can price the corresponding regular USD option and divide by the current underlying futures price.

2.3.2 Fourier pricing of the regular futures call option

We price the regular USD option using the characteristic function derived in 2.2.3 and a Fourier method. A computationally convenient approach is the FFT-based method of Carr and Madan (1999). Let $\tau = T - t$ be the time to maturity and $k = \log K$ be the log-strike. Let $C(K)$ denote the regular call price in USD. Under some regularity conditions, Carr and Madan (1999) show that

$$C(K) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \text{Re}(e^{-iuk} \Psi(u)) du, \quad (2.3.5)$$

where

$$\Psi(u) = \frac{\phi(u - (\alpha + 1)i; t, T)}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u}, \quad (2.3.6)$$

and $\phi(\cdot; t, T)$ is the characteristic function. In practice, the call price can be evaluated using FFT on a grid of strikes (Carr and Madan, 1999).

The Carr–Madan method prices options by transforming the call price as a function of log-strike $k = \log K$ into the frequency domain, where it can be expressed directly in terms of the characteristic function of $X_T = \log F_T$. Because $C(e^k)$ typically does not decay fast enough for a Fourier transform to exist, we introduce the exponentially damped call

$$\tilde{C}(k) := e^{\alpha k} C(e^k), \quad \alpha > 0,$$

chosen so that $\tilde{C} \in L^2(\mathbb{R})$.

We then take the Fourier transform of \tilde{C} (with respect to k),

$$\hat{\tilde{C}}(u) := \int_{-\infty}^{\infty} e^{iuk} \tilde{C}(k) dk, \quad u \in \mathbb{R}. \quad (2.3.7)$$

Starting from $C(e^k) = \mathbb{E}[(F_T - e^k)^+ | \mathcal{F}_t]$ and substituting into (2.3.7), one obtains an expression of the form

$$\hat{\tilde{C}}(u) = \mathbb{E} \left[\int_{-\infty}^{\infty} e^{iuk} e^{\alpha k} (F_T - e^k)^+ dk \mid \mathcal{F}_t \right].$$

Under the same integrability conditions that justify the damping, the order of expectation and integration can be exchanged. The resulting inner integral can be evaluated in closed form and reduces the transform $\hat{\tilde{C}}(u)$ to a rational factor times an exponential moment of X_T , which is precisely the characteristic function evaluated at a complex argument. This yields the closed-form representation

$$\hat{\tilde{C}}(u) = \Psi(u), \quad (2.3.8)$$

where $\Psi(u)$ is given in (2.3.6).

Finally, the damped call is recovered by the inverse Fourier transform,

$$\tilde{C}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \hat{\tilde{C}}(u) du = \frac{1}{\pi} \int_0^{\infty} \operatorname{Re}(e^{-iuk} \Psi(u)) du, \quad (2.3.9)$$

and undoing the damping gives the desired call price

$$C(K) = C(e^k) = e^{-\alpha k} \tilde{C}(k),$$

which is (2.3.5).

In practice, the integral in (2.3.5) is approximated on an evenly spaced frequency grid $u_j = j\Delta u$, $j = 0, \dots, N - 1$. Using the trapezoidal rule, the inverse transform becomes a

discrete Fourier transform mapping the sampled values $\Psi(u_j)$ to call prices on an evenly spaced log-strike grid $k_m = k_0 + m\Delta k$, $m = 0, \dots, N - 1$, with the grid spacings linked by $\Delta k \Delta u = 2\pi/N$. This is precisely the structure computed efficiently by the FFT, allowing simultaneous pricing of a large set of strikes in $O(N \log N)$ time.

For a more detailed discussion of the Carr-Madan procedure, see Appendix A.

2.3.3 Inverse call and put option price

Given $C(K)$ in USD units, the inverse call option price in coin unit is

$$C^{\text{coin}}(K) = \frac{C(K)}{F_t}, \quad (2.3.10)$$

which is the direct application of (2.3.4).

The put inverse option on the futures can be calculated using the following put–call parity for inverse options²:

Theorem (Put–call parity for inverse options). *Let F_t be the time- t price of a futures contract maturing at T , and let $C_t(K, T)$ and $P_t(K, T)$ denote the time- t prices of European call and put options with strike K and maturity T written on this futures. Define the futures-numeraire prices*

$$\widehat{C}_t(K, T) := \frac{C_t(K, T)}{F_t}, \quad \widehat{P}_t(K, T) := \frac{P_t(K, T)}{F_t}.$$

Then, under the pricing measure associated with the futures numeraire, the following put–call parity holds:

$$\widehat{C}_t(K, T) - \widehat{P}_t(K, T) = 1 - \frac{K}{F_t}. \quad (2.3.11)$$

For the proof of the above theorem, see Appendix C.

2.4 Model deltas and net deltas

Inverse option hedging requires careful attention to what “delta” means when the option is quoted in coin units. The main takeaway from Lucic and Sepp (2024) is that inverse-option hedging requires a *premium adjustment*: the hedging ratio must account for the fact that the option price is itself denominated in coin.

²Assuming zero discounting, following (Lucic and Sepp, 2024)

2.4.1 Regular delta under SVCJ

Let $C(K)$ denote the regular USD call price. The regular futures delta is

$$\Delta_t^{\text{reg}} := \frac{\partial C(K)}{\partial F_t}. \quad (2.4.1)$$

For numerical work, Δ_t^{reg} can be computed by differentiating the Fourier representation (2.3.5) with respect to F_t (noting $X_t = \log F_t$ enters the characteristic function affinely), or equivalently via a probability representation (common in affine models):

$$C(K) = F_t \Pi_1 - K \Pi_2, \quad (2.4.2)$$

where Π_2 is the risk-neutral probability of finishing in the money and Π_1 is the corresponding probability under the “share” (or forward-numeraire) measure. Under standard regularity, this implies $\Delta_t^{\text{reg}} = \Pi_1$.

In affine models, Π_1 and Π_2 can be computed by Fourier inversion using the characteristic function ϕ (Gil-Pelaez, 1951; Heston, 1993).

2.4.2 Inverse delta and premium-adjusted net delta

The inverse call price in coin units is $C^{\text{coin}}(K) = C(K)/F_t$ from (2.3.10). Its sensitivity with respect to F_t is

$$\frac{\partial C^{\text{coin}}(K)}{\partial F_t} = \frac{1}{F_t} \left(\Delta_t^{\text{reg}} - \frac{C(K)}{F_t} \right) = \frac{1}{F_t} (\Delta_t^{\text{reg}} - C^{\text{coin}}(K)). \quad (2.4.3)$$

However, Lucic and Sepp (2024) emphasize that the *hedge ratio* for inverse options (when hedging with inverse futures) is the *premium-adjusted* or *net delta*, defined model-independently by differentiating the numeraire relation $V_t^{\text{USD}} = F_t V_t^{\text{coin}}$:

Net delta for inverse options

$$\Delta_t^{\text{net}} := \Delta_t^{\text{reg}} - \frac{C(K)}{F_t} = \Delta_t^{\text{reg}} - C^{\text{coin}}(K). \quad (2.4.4)$$

The same logic applies to puts, with C replaced by P .

2.5 Hedging inverse options with perpetual futures and measuring hedging performance

2.5.1 Why hedge with the perpetual rather than the term future?

On Deribit, the underlying of an option expiring at T is the *term* futures contract $F_t^{(T)}$ with the same settlement date (Lucic and Sepp, 2024). Empirically, the perpetual futures market is typically more liquid than individual term futures. As a result, delta hedging is commonly implemented using the inverse perpetual future (“perp”) rather than the less liquid term future (Lucic and Sepp, 2024).

Let F_t^\perp denote the (mark) price of the inverse perpetual future. Because $F_t^{(T)}$ and F_t^\perp are not identical, and the deltas of an option on a term futures and a perpetual futures are not identical, this introduces *basis risk* driven by funding/carry effects (discussed explicitly in Lucic and Sepp (2024)).

2.5.2 Perpetual-futures P&L and hedge sizing

Lucic and Sepp (2024) provide a convenient convention for computing the coin P&L of an inverse perpetual futures position with fixed USD notional N_{USD} . Over a small interval $[t, t + \delta t]$, the coin P&L is

$$\text{P\&L}^{\text{coin}}(t, t + \delta t) = N_{\text{USD}} \left(\frac{1}{F_t^\perp} - \frac{1}{F_{t+\delta t}^\perp} \right). \quad (2.5.1)$$

This formula is the discrete-time building block for hedging simulations.

Now consider a *single* inverse option written on the term future $F_t^{(T)}$ with net delta Δ_t^{net} computed with respect to the term futures price (via (2.4.4)). A natural hedge is to take an inverse perpetual position with USD notional

$$N_{\text{USD},t} = -F_t^{(T)} \Delta_t^{\text{net}}. \quad (2.5.2)$$

This is the single-contract specialization of the portfolio aggregation rule in Lucic and Sepp (2024). Intuitively, multiplying Δ_t^{net} (a coin-denominated hedge ratio) by $F_t^{(T)}$ converts it into USD notional exposure, which is exactly how inverse perpetual positions are parameterized in practice.

2.5.3 Dynamic delta-hedge P&L and hedging performance metrics

Fix a hedging grid $t_0 = t < t_1 < \dots < t_n = T$ (e.g., hourly or twice-daily). Consider the P&L of a *short* inverse option position (this is the standard backtesting convention in option risk-premia studies). Let the inverse option premium at inception be $V_{t_0}^{\text{coin}}$ and the terminal inverse payoff be $r(F_T^{(T)})$ from (2.3.2).

At each hedging time t_i , we compute $\Delta_{t_i}^{\text{net}}$ and set the perpetual hedge notional to N_{USD,t_i} as in (2.5.2). Ignoring funding and transaction costs for the moment, the hedged coin P&L over $[t_i, t_{i+1}]$ from the perpetual is

$$\text{P&L}_{\perp}^{\text{coin}}(t_i, t_{i+1}) = N_{\text{USD},t_i} \left(\frac{1}{F_{t_i}^{\perp}} - \frac{1}{F_{t_{i+1}}^{\perp}} \right). \quad (2.5.3)$$

The total hedged P&L (coin-denominated) for the short option is then

$$\text{P&L}^{\text{hedged}} = V_{t_0}^{\text{coin}} - r(F_T^{(T)}) + \sum_{i=0}^{n-1} \text{P&L}_{\perp}^{\text{coin}}(t_i, t_{i+1}) - \text{TransactionCosts} - \text{Funding}. \quad (2.5.4)$$

Hedging performance evaluation. Given a collection of inverse option contracts (different strikes and maturities) over an out-of-sample period, we will evaluate hedging performance using:

- the mean and standard deviation of $\text{P&L}^{\text{hedged}}$;
- the root mean squared hedging error (RMSHE), i.e. $\sqrt{\mathbb{E}[(\text{P&L}^{\text{hedged}})^2]}$;
- tail risk measures (e.g. 5% quantile / expected shortfall) of hedged P&L;
- comparisons across models (SVCJ vs benchmark models in later chapters) using the same hedging grid and cost assumptions.

This links the statistical fit of the model (pricing errors) to its trading-relevant implications (hedging errors).

Conclusion

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Appendix A: Detailed Derivation of the Carr–Madan FFT Procedure

This appendix provides a detailed derivation of the Carr–Madan Fast Fourier Transform (FFT) method for pricing European options, following Carr and Madan (1999). The method computes option prices over a grid of strikes efficiently when the characteristic function of the log-underlying is available in (semi-)closed form, as in affine (jump-)diffusion models.

A.1 Foundations and the damping factor

Let S_T be the underlying price at maturity T , and let $s_T := \ln S_T$ denote the log-price. Under a pricing measure \mathbb{Q} with deterministic discounting (constant short rate r for simplicity), the time-0 price of a European call with strike K is

$$C(K, T) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}(S_T - K)^+] = e^{-rT} \int_{-\infty}^{\infty} (e^s - K)^+ f_{s_T}(s) ds,$$

where f_{s_T} is the \mathbb{Q} -density of s_T . (In settings where discounting is negligible, one may set $r = 0$.)

Treat the call price as a function of log-strike $k := \ln K$. Then $\lim_{k \rightarrow -\infty} C(e^k, T) = e^{-rT} \mathbb{E}^{\mathbb{Q}}[S_T] = S_0$ (in standard equity settings), so $C(e^k, T)$ is not square-integrable in k over \mathbb{R} and cannot be Fourier-transformed directly.

Following Carr and Madan (1999), introduce an exponential damping factor $\alpha > 0$ and define the *damped call price*

$$c(k, T) := e^{\alpha k} C(e^k, T). \quad (\text{A.1})$$

For suitable α , $c(\cdot, T)$ is square-integrable. A sufficient condition is existence of the $(\alpha + 1)$ -moment:

$$\mathbb{E}^{\mathbb{Q}}[S_T^{\alpha+1}] < \infty,$$

which ensures $c(k, T) \rightarrow 0$ as $k \rightarrow \pm\infty$ and justifies Fourier inversion.

A.2 Fourier transform of the damped price

Define the Fourier transform of $c(\cdot, T)$ by

$$\widehat{c}(v, T) := \int_{-\infty}^{\infty} e^{ivk} c(k, T) dk, \quad v \in \mathbb{R}.$$

Using (A.1) and writing the call payoff integral in terms of k gives

$$\widehat{c}(v, T) = \int_{-\infty}^{\infty} e^{ivk} e^{\alpha k} \left(e^{-rT} \int_k^{\infty} (e^s - e^k) f_{s_T}(s) ds \right) dk.$$

Under the moment condition above, Fubini's theorem permits exchanging the order of integration:

$$\widehat{c}(v, T) = e^{-rT} \int_{-\infty}^{\infty} f_{s_T}(s) \left(\int_{-\infty}^s e^{(\alpha+iv)k} (e^s - e^k) dk \right) ds.$$

Evaluate the inner integral:

$$\begin{aligned} \int_{-\infty}^s e^{(\alpha+iv)k} (e^s - e^k) dk &= e^s \int_{-\infty}^s e^{(\alpha+iv)k} dk - \int_{-\infty}^s e^{(\alpha+1+iv)k} dk \\ &= e^s \frac{e^{(\alpha+iv)s}}{\alpha + iv} - \frac{e^{(\alpha+1+iv)s}}{\alpha + 1 + iv} \\ &= \frac{e^{(\alpha+1+iv)s}}{(\alpha + iv)(\alpha + 1 + iv)}. \end{aligned}$$

Therefore,

$$\widehat{c}(v, T) = \frac{e^{-rT}}{(\alpha + iv)(\alpha + 1 + iv)} \int_{-\infty}^{\infty} e^{(\alpha+1+iv)s} f_{s_T}(s) ds = \frac{e^{-rT} \mathbb{E}^{\mathbb{Q}}[e^{(\alpha+1+iv)s_T}]}{(\alpha + iv)(\alpha + 1 + iv)}.$$

Let $\phi_T(u) := \mathbb{E}^{\mathbb{Q}}[e^{ius_T}]$ denote the characteristic function of s_T . Since $e^{(\alpha+1+iv)s_T} = e^{i(v-i(\alpha+1))s_T}$, we obtain

$$\widehat{c}(v, T) = \frac{e^{-rT} \phi_T(v - i(\alpha + 1))}{(\alpha + iv)(\alpha + 1 + iv)} = \frac{e^{-rT} \phi_T(v - i(\alpha + 1))}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}, \quad (\text{A.2})$$

where we used $(\alpha + iv)(\alpha + 1 + iv) = \alpha^2 + \alpha - v^2 + i(2\alpha + 1)v$.

A.3 Inversion and the Carr–Madan pricing formula

By Fourier inversion,

$$c(k, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \widehat{c}(v, T) dv.$$

Since $c(k, T)$ is real-valued, the integral can be written in the one-sided form

$$c(k, T) = \frac{1}{\pi} \int_0^{\infty} \Re(e^{-ivk} \widehat{c}(v, T)) dv.$$

Finally, undo the damping $c(k, T) = e^{\alpha k} C(e^k, T)$ to obtain

$$C(K, T) = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} \Re(e^{-ivk} \widehat{c}(v, T)) dv, \quad k = \ln K, \quad (\text{A.3})$$

with $\widehat{c}(v, T)$ given explicitly by (A.2). Equations (A.2)–(A.3) are the Carr–Madan Fourier pricing representation (Carr and Madan, 1999).

A.4 Discretization and the FFT

To compute (A.3) numerically, truncate the integral and discretize the Fourier variable. Choose N (typically a power of 2 for FFT efficiency) and define a grid

$$v_j := j\eta, \quad j = 0, 1, \dots, N - 1,$$

with spacing $\eta > 0$. Approximating the integral by a weighted Riemann/trapezoidal sum yields

$$c(k, T) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} w_j \Re(e^{-iv_j k} \widehat{c}(v_j, T)) \eta,$$

where w_j are numerical quadrature weights (e.g. trapezoidal weights $w_0 = w_{N-1} = 1/2$ and $w_j = 1$ otherwise; Alternatively, Simpson's rule may be used, with weights $w_0 = w_{N-1} = 1$, $w_j = 4$ for odd j , and $w_j = 2$ for even $j \in \{2, \dots, N-2\}$, which yields higher-order accuracy for smooth integrands, reducing the discretization error.).

To exploit the FFT, compute prices on an evenly spaced log-strike grid

$$k_u := b + u\lambda, \quad u = 0, 1, \dots, N - 1,$$

where $\lambda > 0$ is the log-strike spacing and b sets the minimum log-strike. Impose the FFT compatibility condition which synchronizes the strike grid and frequency grid so that the inverse Fourier transform reduces to a discrete Fourier transform:

$$\lambda\eta = \frac{2\pi}{N}.$$

Then

$$c(k_u, T) \approx \frac{1}{\pi} \Re \left(\sum_{j=0}^{N-1} e^{-iv_j(b+u\lambda)} \widehat{c}(v_j, T) w_j \eta \right) = \frac{1}{\pi} \Re \left(\sum_{j=0}^{N-1} e^{-iv_j b} e^{-i\frac{2\pi}{N}ju} \widehat{c}(v_j, T) w_j \eta \right).$$

Define the sequence

$$x_j := e^{-iv_j b} \widehat{c}(v_j, T) w_j \eta, \quad j = 0, \dots, N - 1.$$

Then the bracketed sum is precisely the discrete Fourier transform (DFT) of (x_j) evaluated at index u , which can be computed for all u in $O(N \log N)$ time via the FFT:

$$c(k_u, T) \approx \frac{1}{\pi} \Re(\text{DFT}(x)_u), \quad C(e^{ku}, T) = e^{-\alpha k_u} c(k_u, T).$$

In our application, $\phi_T(\cdot)$ is provided by the model characteristic function derived in the main text (e.g. Section 2.2.3), evaluated at the complex argument $v - i(\alpha + 1)$ as required by (A.2).

A.5 Practical implementation considerations

- **Choice of damping factor α .** The parameter $\alpha > 0$ must be chosen so that $\mathbb{E}^{\mathbb{Q}}[S_T^{\alpha+1}] < \infty$, equivalently, so that $\phi_T(v - i(\alpha + 1))$ exists for the required range of v . In models with jumps, this can impose nontrivial restrictions on α (through exponential moments of the jump component). Practical guidance on admissible damping and numerical stability is discussed in Carr and Madan (1999) and Lee (2004).
- **Truncation/discretization parameters.** The truncation level $v_{\max} = (N - 1)\eta$ should be large enough to control truncation error, and η should be small enough to control discretization error. In practice one selects (N, η) to cover the strike range of interest via $\lambda = 2\pi/(N\eta)$ and checks numerical stability by refinement.
- **Characteristic function evaluation.** Care is required when evaluating $\phi_T(u)$ at complex arguments, in particular in models where ϕ_T involves complex square roots and logarithms (e.g. Heston/SVCJ Riccati solutions). A consistent branch choice (typically enforcing $\Re(d(u)) > 0$ for the Heston discriminant) helps avoid numerical discontinuities.

This FFT-based approach yields an efficient way to compute a full grid of call prices across strikes, which is particularly useful for calibration to cross-sectional option data.

Appendix B: Characteristic functions under the futures price numeraire for the Black and Heston models

This appendix derives the conditional characteristic functions of $X_T = \log F_T$ given (X_t, V_t) (or X_t alone in the Black model) under the *futures price numeraire*. In both cases we work under the pricing measure for which the futures price $(F_t)_{t \geq 0}$ is a martingale, i.e.

$$\mathbb{E}\left[\frac{dF_t}{F_t} \middle| \mathcal{F}_t\right] = 0.$$

Equivalently, for $X_t = \log F_t$ the drift is chosen so that $F_t = e^{X_t}$ has zero drift in expectation.

B.1 Black model

Model. Under the futures price measure, the Black model assumes

$$dF_t = \sigma F_t dW_t, \quad \sigma > 0, \tag{B.1}$$

so that F_t is a martingale. Let $X_t := \log F_t$. By Itô's formula,

$$dX_t = -\frac{1}{2}\sigma^2 dt + \sigma dW_t. \tag{B.2}$$

Conditional characteristic function. Fix $u \in \mathbb{C}$ and define

$$\phi_{BL}(u; t, T) := \mathbb{E}[e^{iuX_T} \mid \mathcal{F}_t].$$

From (B.2),

$$X_T = X_t - \frac{1}{2}\sigma^2(T-t) + \sigma(W_T - W_t),$$

and $W_T - W_t \sim \mathcal{N}(0, T-t)$ is independent of \mathcal{F}_t . Hence

$$\begin{aligned} \phi_{BL}(u; t, T) &= \exp\left(iu\left(X_t - \frac{1}{2}\sigma^2(T-t)\right)\right) \mathbb{E}[e^{iu\sigma(W_T - W_t)}] \\ &= \exp\left(iuX_t - \frac{1}{2}iu\sigma^2\tau\right) \exp\left(-\frac{1}{2}u^2\sigma^2\tau\right), \quad \tau := T-t. \end{aligned} \tag{B.3}$$

Therefore,

$$\phi_{BL}(u; t, T) = \exp\left(iuX_t - \frac{1}{2}\sigma^2\tau(u^2 + iu)\right). \tag{B.4}$$

B.2 Heston model

Model. Under the futures price measure, the Heston model assumes

$$dF_t = F_t \sqrt{V_t} dW_{1t}, \quad (\text{B.5})$$

$$dV_t = \kappa(\theta - V_t) dt + \sigma_v \sqrt{V_t} dW_{2t}, \quad (\text{B.6})$$

with $\kappa > 0$, $\theta > 0$, $\sigma_v > 0$, and $\text{corr}(dW_{1t}, dW_{2t}) = \rho$. Let $X_t := \log F_t$. By Itô's formula applied to $F_t = e^{X_t}$,

$$dX_t = -\frac{1}{2}V_t dt + \sqrt{V_t} dW_{1t}. \quad (\text{B.7})$$

The drift $-\frac{1}{2}V_t$ is the unique choice ensuring that F_t is a martingale under this measure.

Martingale ansatz. Fix $u \in \mathbb{C}$ and define the conditional characteristic function

$$\phi_H(u; t, T) := \mathbb{E}[e^{iuX_T} | X_t, V_t].$$

Motivated by the affine structure of (B.7)–(B.6), we conjecture the exponential-affine form

$$\phi_H(u; t, T) = \exp(A_H(\tau; u) + B_H(\tau; u)V_t + iuX_t), \quad \tau := T - t, \quad (\text{B.8})$$

with terminal conditions $A_H(0; u) = 0$ and $B_H(0; u) = 0$.

Define $\Phi_t := \exp(A_H(\tau; u) + B_H(\tau; u)V_t + iuX_t)$. As in the SVCJ derivation in Section 2.2, we choose A_H and B_H so that $(\Phi_t)_{t \in [0, T]}$ is a martingale. Applying Itô's formula to $\Phi(t, X_t, V_t)$ and collecting the predictable drift terms yields a drift that is affine in V_t . Setting this drift equal to zero for all (X_t, V_t) produces the Riccati system

$$\frac{\partial}{\partial \tau} B_H(\tau; u) = \frac{1}{2}\sigma_v^2 B_H(\tau; u)^2 + (\rho\sigma_v iu - \kappa) B_H(\tau; u) - \frac{1}{2}(u^2 + iu), \quad B_H(0; u) = 0, \quad (\text{B.9})$$

$$\frac{\partial}{\partial \tau} A_H(\tau; u) = \kappa\theta B_H(\tau; u), \quad A_H(0; u) = 0. \quad (\text{B.10})$$

Closed-form solution. Define

$$d(u) := \sqrt{(\kappa - \rho\sigma_v iu)^2 + \sigma_v^2(u^2 + iu)}, \quad (\text{B.11})$$

$$g(u) := \frac{\kappa - \rho\sigma_v iu - d(u)}{\kappa - \rho\sigma_v iu + d(u)}, \quad \Re(d(u)) > 0. \quad (\text{B.12})$$

Then (B.9) admits the closed-form solution (Heston, 1993)

$$B_H(\tau; u) = \frac{\kappa - \rho\sigma_v iu - d(u)}{\sigma_v^2} \frac{1 - e^{-d(u)\tau}}{1 - g(u)e^{-d(u)\tau}}, \quad (\text{B.13})$$

and integrating (B.10) gives (Heston, 1993)

$$A_H(\tau; u) = \frac{\kappa\theta}{\sigma_v^2} \left[(\kappa - \rho\sigma_v iu - d(u))\tau - 2 \log\left(\frac{1 - g(u)e^{-d(u)\tau}}{1 - g(u)}\right) \right]. \quad (\text{B.14})$$

Result. Therefore, the Heston characteristic function under the futures price measure is

$$\boxed{\phi_H(u; t, T) = \exp(A_H(\tau; u) + B_H(\tau; u)V_t + iuX_t), \quad \tau = T - t,} \quad (\text{B.15})$$

with A_H and B_H given by (B.14) and (B.13).

Appendix C: Proof of Put–Call Parity under the Futures Numeraire

This appendix proves the put–call parity stated in Theorem 2.3.11 in the main text. Throughout, we assume zero discounting (equivalently, a zero interest rate) and a frictionless market.

C.1 Payoff identity

Let F_T denote the futures price at maturity T and fix a strike $K > 0$. The European call and put payoffs at T are

$$(F_T - K)^+, \quad (K - F_T)^+.$$

For every real number x , the identity $(x)^+ - (-x)^+ = x$ holds. Applying it with $x = F_T - K$ yields the pointwise payoff equality

$$(F_T - K)^+ - (K - F_T)^+ = F_T - K. \quad (\text{C.1})$$

Therefore, a portfolio that is long one call and short one put (same K, T) replicates the forward-type payoff $F_T - K$.

C.2 Pricing the forward-type payoff with a futures contract (zero discounting)

Let $C_t(K, T)$ and $P_t(K, T)$ be the time- t prices (in USD units) of the call and put. By linearity of pricing,

$$C_t(K, T) - P_t(K, T) \quad \text{is the time-}t \text{ value of a claim paying } F_T - K \text{ at } T. \quad (\text{C.2})$$

Under zero discounting, the payoff $F_T - K$ can be replicated using (i) a long position in one futures contract and (ii) a static cash position:

- Enter a long futures position at time t . With standard mark-to-market accounting and zero interest, the cumulative gain from t to T is $F_T - F_t$.
- Hold a cash amount of $(F_t - K)$ from t to T (no growth under zero discounting).

The terminal value of this self-financing strategy is

$$(F_t - K) + (F_T - F_t) = F_T - K,$$

which matches the desired payoff. Hence the no-arbitrage time- t value of the claim $F_T - K$ equals the initial cost of this replicating strategy, namely $F_t - K$:

$$C_t(K, T) - P_t(K, T) = F_t - K. \quad (\text{C.3})$$

C.3 Parity in futures-numeraire (coin) units

Define the futures-numeraire (coin) prices

$$\widehat{C}_t(K, T) := \frac{C_t(K, T)}{F_t}, \quad \widehat{P}_t(K, T) := \frac{P_t(K, T)}{F_t}.$$

Dividing (C.3) by F_t gives

$$\widehat{C}_t(K, T) - \widehat{P}_t(K, T) = \frac{F_t - K}{F_t} = 1 - \frac{K}{F_t}, \quad (\text{C.4})$$

which is exactly the parity stated in (2.3.11).