Domains:

$\mathcal{U} := \{u : \mathbb{R}^L_+ \to \mathbb{R}_+ :$ continuous, increasing, concave, $u(0)=0\}$

$M \in \mathbb{R}^L_+$ $\quad X := \{(x_i)_{i=1}^n : \sum_{i=1}^n x_i = M\}$ Allocations

$L=1$: $\forall x \in X$, x is Pareto optimal

$L \geq 2$: $PO \subsetneq X$

Bargaining Problem: $\quad B \subseteq \mathbb{R}^N_+$ feasible set

(B, d) $\quad d \in B$ reference (disagreement) point

$\mathcal{B}^N_d := \{(B,d) :$ B is convex, bounded & closed,

$\exists x \in B$ s.t. $x > d$

*d-comprehensive: $x \in B$, $x \geq y \geq d \Rightarrow y \in B$

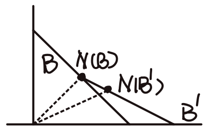
$B \subseteq \mathbb{R}^N_+ \}$

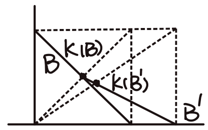
*strictly d-comprehensive: $x \in B$, $x \geq y \geq d \Rightarrow y \in int(B)$

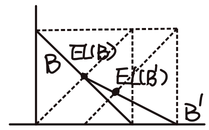
Rule: $(B,d) \to B$

con.comp$\{A\}$: convex & comprehensive hull of A

$\Delta^{|N|-1} := \{x \in \mathbb{R}^N : \sum x_i = 1\}$ unit simplex







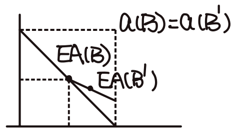
Solution: $B \rightrightarrows B$, $B \in \mathcal{B}^N_0$ $\quad N := \{1, 2, \cdots, n\}$

$P := \{x \in B : \nexists y \in B,\ \forall i \in N,\ y_i \geq x_i,\ \exists i \in N,\ y_i > x_i\}$

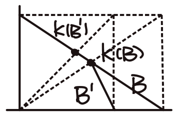
$WP := \{x \in B : \nexists y \in B,\ \forall i \in N,\ y_i > x_i\}$

$L_d := \{x \in B : \forall i \in N,\ x_i \geq d_i\}$ disagreement-point bound solution

$L_{mid} := \{x \in B : x \geq \frac{\sum D^i(B)}{|N|},\ D^i(B) = \max\{u_i, u_j = d_j\}\}$ midpoint



EACM



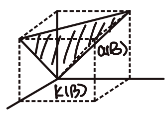
Rule: $B \to B$

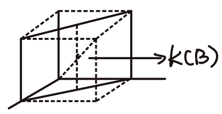
$E := \max\{x \in B : \forall i,j \in N,\ x_i - d_i = x_j - d_j\}$ Egalitarian Rule

Comp. $\Rightarrow E \in WPO$ $\qquad$ SC $\Rightarrow E \in PO$

$K := \partial B \sqcap [d, a(B)]$, aspiration $a_i(B) = \max\{x_i \mid x \in B, x \geq d\}$

Comp. $\Rightarrow K \in WPO$ $\qquad$ SC $\Rightarrow K \in PO$

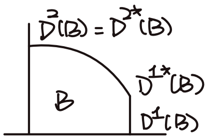




$D^i := \{x \in B : \max x_i,\ x_j = d_j\}$ $\quad$ Dictatorial Rule

$D^{*i} := \{x \in B : \max x_j,\ x_i = D^i_i\}$ $\quad$ Benevolent dictatorial

Comp. $\Rightarrow D^i, D^{*i}$ meaningful

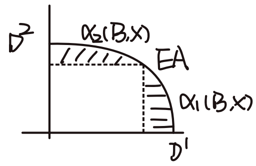


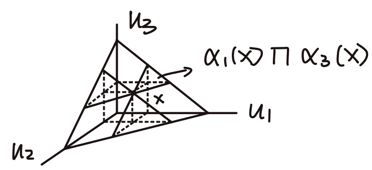
$R^d := \lim_{t \to +\infty} z^t$, $z^t = \frac{\sum D^i(B^{t-1})}{|N|}$, $B^{t-1} := \{x \in B : x \geq z^{t-1}\}$

$R^c := \sum_{t=0}^{+\infty} \Delta z^t$, $\Delta z^t = \varepsilon[z^t - z^{t-1}]$ $\qquad$ Raiffa

$EA := \{x : \alpha_1(B,x) = \alpha_2(B,x),\ x \in PO(B)\}$ Equal-area

$U := \underset{x \in L_d}{\operatorname{argmax}} \sum_i x_i$ $\qquad$ Utilitarian

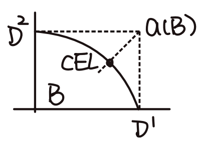


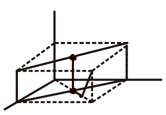


$CEL := \partial B \sqcap [a(B), 45°]$ $\qquad$ Constrained Equal-Losses

$Y^P := \underset{x \in B}{\operatorname{argmin}} \left(\sum_i |a_i(B) - x_i|^P\right)^{\frac{1}{P}}$ $\qquad$ Yu

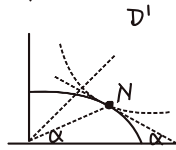
$\lim_{p \to 1} Y^P = U$ $\qquad \lim_{p \to +\infty} Y^P = CEL$ $\qquad \lim_{p \to 0} Y^P = Nash$

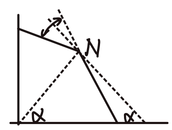




$|N| \geq 3$, CEL might be not PO

$N := \underset{x \in L_d}{\operatorname{argmax}} \prod_i (x_i - d)$ $\qquad$ $F(x) \leq D$

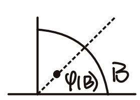


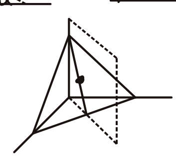


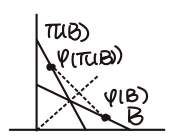
45° line symmetry operator: $\pi(x_1, x_2) = \pi(x_2, x_1)$

Respect of symmetry: If $\pi(B) = B$, then $\varphi(B) \in 45°$ line

Respect of anonymity: $\varphi(\pi(B)) = \pi(\varphi(B))$







Scaling operator: $\pi(x) = (a_1 x_1, \cdots, a_n x_n)$

Scale invariance: $\forall B,\ \forall \pi \in \Lambda^N_0$: $\varphi(\pi(B)) = \pi(\varphi(B))$

Strong monotonicity: If $B' \supseteq B$, then $\varphi(B') \geq \varphi(B)$

Thm 1: Comp. $\mathcal{B}_0^N$: WP, Sym., SM $\Leftrightarrow$ E

"$\Rightarrow$" Take any B. $B' = \text{con.comp.}\{\{D^i(B)\}_i, E(B)\}$

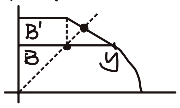
$WP(B') = \partial B'$ Sym. $\Rightarrow \varphi(B) = \partial B' \cap 45^\circ = E(B)$

Comp $B \Rightarrow B' \sqsubseteq B$. SM $\Rightarrow \varphi(B) \geq \varphi(B') = E(B)$

Comp $B \Rightarrow E(B) \in WP(B)$

If $E(B) \in P(B)$: $\nexists y \in B$ s.t. $y \geq E(B) \Rightarrow \varphi(B) = E(B)$

O.W.: BWOC suppose $\varphi(B) = y \neq E(B)$



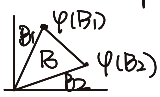
$B' \sqsupseteq B$. $\varphi(B') = E(B') \not\geq \varphi(B)$

"$\Leftarrow$" BWOC: $E \notin WP(B)$. $\exists y \in B$ s.t. $\forall i$. $y_i > E_i(B)$

$\exists \varepsilon > 0$. s.t. $\forall i$. $y_i > E_i(B) + \varepsilon > E_i(B)$.

Comp $\Rightarrow E(B) + \varepsilon 1_N \in B \Rightarrow$ A larger box in B Contrad!

Without Comp. B : WP & SM incompatible!



WP $\Rightarrow \varphi(B_1)$ $\varphi(B_2)$

SM $\Rightarrow \varphi(B) = \varphi(B_1)$ $\varphi(B) = \varphi(B_2)$ X

Individual monotonicity: Expansion in favor of agent i:

$B' \sqsupseteq B$. $\forall j \neq i$. $a_j(B') = a_j(B) \Rightarrow \varphi_i(B') \geq \varphi_i(B)$

Equal-aspirations-conditional monotonicity:

$B' \sqsupseteq B$, $a(B') = a(B) \Rightarrow \varphi(B') \geq \varphi(B)$

Thm 2: $|N| = 2$, Comp. B : P, Sym, SI, EACM $\Leftrightarrow$ K

"$\Rightarrow$" Scale Invariance $\Rightarrow a_i(B) = a_j(B)$

$B' := \text{Con.Comp}\{a_i(B), K(B)\}$. $B' \sqsubseteq B$. $P(B') = K(B)$

$P \Rightarrow \varphi(B') = K(B)$ EACM $\Rightarrow \varphi(B) \geq \varphi(B') = K(B)$

Comp. & sym $B \Rightarrow K(B) = E(B) \in P(B) \Rightarrow \varphi(B) = K(B)$

"$\Leftarrow$" $|N| = 2$, Comp $B \Rightarrow K(B) \in P(B)$

"$\Rightarrow$" Holds for $|N| \geq 3$

Contraction Independence : $B' \sqsubseteq B$. $\varphi(B) \in B' \Rightarrow \varphi(B') = \varphi(B)$

Thm 3. $\mathcal{B}_0^N$: P. Sym. SI, CI $\Leftrightarrow$ N

"$\Rightarrow$" Take B. Scale transform $\lambda : B \to B'$ s.t. $\lambda(N(B)) \in 45^\circ\angle$

N is Scale Invariance $\Rightarrow N(B') = \lambda(N(B)) \in 45^\circ\angle$

$B'' := \{x : \sum_i x_i \leq \sum_i N_i(B')\}$, $P(B'') = \{x : \sum_i x_i = \sum_i N_i(B')\}$

P & Sym $\Rightarrow \varphi(B'') = N(B')$

$B' \sqsubseteq B''$ (Supporting Hyperplane)

Contraction Independence $\Rightarrow \varphi(B') = \varphi(B'') = N(B')$

Scale Invariance $\Rightarrow \varphi(B) = \lambda(\varphi(B')) = \lambda(N(B')) = N(B)$

Thm 4: $\mathcal{B}_0^N$: Lmin . CI $\Leftrightarrow$ N

"$\Rightarrow$" Take B. WTS $\varphi(B) = N(B)$

$B' := \{x : \sum_i x_i \leq \sum_i N_i(B)\}$. $L_{min}(B') = N(B)$ $\varphi(B') = N(B)$

$B \sqsubseteq B'$ (Supporting Hyperplane)

Contraction Independence $\Rightarrow \varphi(B) = \varphi(B') = N(B)$

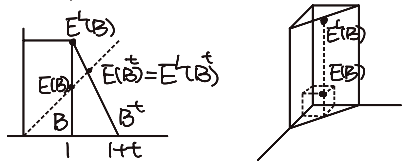
"$\Leftarrow$" Graph : If $N_i(B) < \frac{1}{2} D^i(B)$, then $D^i(B) < \tilde{D}(B)$

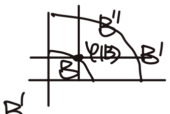
Lexicographic order: $x, y \in \mathbb{R}^N$. $\tilde{x}, \tilde{y}$: increasing order

$x > y$ if $\exists k \in \{1, \cdots, n-1\}$. $\forall i \leq k$: $\tilde{x}_i = \tilde{y}_i$. $\tilde{x}_{k+1} > \tilde{y}_{k+1}$

Lexicographic order is complete & transitive.

Lexicographic egalitarian: $E^L(B) := \{x \in B : \nexists y \in B.\ y > x\}$





Composition up: $\varphi(B') = \varphi(B) + \varphi(B'')$ $\forall B \sqsubseteq B'$

$B'' := \{x'' \in \mathbb{R}^N_+ : \exists x' \in B' \text{ s.t. } x' = \varphi(B) + x''\}$

Weak Pareto-optimality, Symmetric, Composition $\Leftrightarrow$ E.

Monotonicity:

① Adding: If $B' \sqsupseteq B$, $x \in B' \setminus B \Rightarrow x_i \leq \varphi_i(B)$, $x_j \geq \varphi_j(B)$

then $\varphi_i(B') \leq \varphi_i(B)$, $\varphi_j(B') \geq \varphi_j(B)$

② Cutting: If $B' \sqsubseteq B$, $x \in B \setminus B' \Rightarrow x_i \leq \varphi_i(B)$, $x_j \geq \varphi_j(B)$

then $\varphi_i(B') \geq \varphi_i(B)$, $\varphi_j(B') \leq \varphi_j(B)$

③ Twisting: If $x \in B' \setminus B \Rightarrow x_i \geq \varphi_i(B)$, $x_j \leq \varphi_j(B)$

$x \in B \setminus B' \Rightarrow x_i \leq \varphi_i(B)$, $x_j \geq \varphi_j(B)$

then $\varphi_i(B') \geq \varphi_i(B)$, $\varphi_j(B') \leq \varphi_j(B)$

④ Domination: $\forall B, B'$. Either $\varphi(B') \geq \varphi(B)$ or $\varphi(B) \geq \varphi(B')$

Weak Pareto-optimality & sym & Domination $\Leftrightarrow$ E

Dual to contraction independence:

If $B' \sqsupseteq B$, $\varphi(B) \in \partial B'$ then $\varphi(B') = \varphi(B)$

Smooth problems: Nash, Utilitarian.

Distance on $\mathcal{B}_0^N$: $d(B, B') = \max\{\sup_{x \in B} d(x, B'), \sup_{x' \in B'} d(B, x')\}$

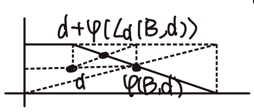
where $d(x, B') = \inf_{x' \in B'} d(x, x')$

If $B \sqsubseteq B'$: $d(B, B') = \sup_{x' \in \partial B'} d(B, x')$

Continuity: If $B^k \to B$ then $\varphi(B^k) \to \varphi(B)$ on $\mathcal{B}_0^N$

If $B^k \to B$, $d^k \to d$ then $\varphi(B^k, d^k) \to \varphi(B, d)$ $\mathcal{B}_d^N$

Independence of alternatives that do not meet the disagreement point lower bound: $\varphi(B, d) = \varphi(L_d(B, d), 0) + d$



Kalai-Smorodinsky rule violates it.

Disagreement utility monotonicity: For (B, d), (B, d')

If $d'_i \geq d_i$, $\forall j \neq i$. $d'_j = d_j$. then $\varphi_i(B, d') \geq \varphi_i(B, d)$

Nash V, KS V, E V, monotone path V.

Strong disagreement point monotonicity: ....

$\varphi_i(B, d') \geq \varphi_i(B, d)$, $\forall j \neq i$. $\varphi_j(B, d') \leq \varphi_j(B, d)$

Nash X, KS X, E V, Monotone path V.

Population: index for all potential agents $= \mathbb{N}_+ = \{1, 2, \cdots\}$

$N \in \mathcal{N}$ class for all finite subset of $\mathbb{N}_+$

Anonymity: $\forall N, \tilde{N} \in \mathcal{N}$: $card(N) = card(\tilde{N})$

$\forall B \in \mathcal{B}_0^N$, $\forall \tilde{B} \in \mathcal{B}_0^{\tilde{N}}$. if $\exists$ bijection $b: \tilde{N} \to N$ s.t.

$\tilde{B} = \{\tilde{x} \in \mathbb{R}^{\tilde{N}} : \exists x \in B$, s.t. $\tilde{x}_i = x_{b(i)}\ \forall i \in \tilde{N}\}$

then $\forall i \in \tilde{N}$. $\varphi_i(\tilde{B}) = \varphi_{b(i)}(B)$

$N' \subseteq N$: $x \in \mathbb{R}^N$, $x_{N'} = Proj(x, \mathbb{R}^{N'})$

$x \in \mathbb{R}^N$. $y \in \mathbb{R}^{N'}$. $(y, x_{N \setminus N'})$

Population monotonicity: $\forall N, N' \in \mathcal{N}$. $N' \subseteq N$

If $B_0^{N'} = {B_0^N}_{N'}$ then $\varphi(B_0^{N'}) \geq \varphi_{N'}(B_0^N)$

Weak Pareto, Sym, CI, Cont, PM $\Leftrightarrow$ E

"$\Rightarrow$" Pick any $B \subseteq \mathcal{B}_0^N$. denote $a := E_i(B)$ $\forall i \in N$

$E(B) \in WP(B)$. WTS $\varphi(B) = E(B)$.

Construct a "simplex" in $\mathbb{R}_+^{N'}$ s.t. $B \subseteq \{x \in \mathbb{R}_+^N : \sum x \leq a \cdot |N'|\}$

WP & Sym $\Rightarrow$ $\varphi(\text{Simplex}) = 1_{N'} \cdot a$

Construct population increase B': $B' = \text{con.Comp}\{B, 1_{N'} \cdot a\}$

$B' \subseteq$ Simplex. Contraction Independen $\Rightarrow$ $\varphi(B') = 1_{N'} \cdot a$

Population monotone: $\varphi(B) \geq \varphi_N(B') = E(B)$.

If $E(B) \in P(B)$, then $\varphi(B) = E(B)$

Otherwise, apply continuity.

WP, Anonymous, Scale Invariance, Cont, PM $\Leftrightarrow$ KS

$\Rightarrow$ $\forall A \in \mathcal{B}_0^N$. By scale Invariance, construct $B \in \mathcal{B}_0^N$

s.t. $\forall i, j$. $D^i(B) = D^j(B)$ WTS: $\varphi(A) = KS(A) \Leftrightarrow \varphi(B) = E(B)$

Construct an anonymous population increase $B' \in \mathcal{B}_0^{N+1}$:

$B' := \text{Con.comp}(B, 1_{N+1} \cdot a)$ s.t. $\forall M : |M| = |N|$. $B'_M = B'_N$

WP & Anonymous $\Rightarrow$ $\varphi(B') = 1_{N+1} \cdot a$

Population monotone $\Rightarrow$ $\varphi(B) \geq \varphi_N(B') = E(B)$

If $E(B) \in P(B)$, done. Otherwise, apply continuity

Reduced problem of $B \in \mathcal{B}_0^N$ w.r.t. N' & x: "on utility"

$r_{N'}^x(B) := \{y \in \mathbb{R}_+^{N'} : (y, x_{N \setminus N'}) \in B\}$ "slice"

Consistency: $\forall N, N' \in \mathcal{N}$. $N' \subseteq N$. $\forall B \in \mathcal{B}_0^N$

$\varphi_{N'}(B) = \varphi(r_{N'}^{\varphi_N(B)})$

Weak consistency: $\varphi_{N'}(B) \leq \varphi(r_{N'}^{\varphi_N(B)})$. E V.

Pareto & Anonymous & Scale invarience & Consistent $\Leftrightarrow$ N

"$\Rightarrow$" $\forall A \in \mathcal{B}_0^N$. scale invariance $\Rightarrow$ $B \in \mathcal{B}_0^N$ $E(B) = N(B)$

WTS: $\varphi(A) = N(A)$ $\Leftrightarrow$ $\varphi(B) = N(B) = E(B)$ $a = E_i(B)$

Push B forward by a along $\mathbb{R}_+^{\Delta N}$, get $(B, 1_{\Delta N} \cdot a)$

Construct anonymous $B^{N+\Delta N} = \text{Con.Comp}\{(B, a)\}$.

Pareto & Anonymous $\Rightarrow$ $\varphi(B^{N+\Delta N}) = 1_{N+\Delta N} \cdot a$

$r_N^a(B^{N+\Delta N}) = B$ Consistency $\Rightarrow$ $\varphi(B) = \varphi_N(B^{N+\Delta N}) = E(B)$

WPareto & Sym & Cont. & Population M & W Consistent $\Leftrightarrow$ E

Pareto & Sym & Individual M & Consistent $\Leftrightarrow$ Lexico -- E

Converse consistency: $\forall N \in \mathcal{N}$. $B \in \mathcal{B}_0^N$. $x \in B$.

If $\forall N' \subseteq N$. $|N'| = 2$. $r_{N'}^{x_{N-N'}}(B) \in \mathcal{B}_0^{N'}$. $x_{N'} = \varphi(r)$

then $x = \varphi(B)$

$\mathcal{B}_0$: WPareto & Sym & Cont. & WConsist. & CCon $\Leftrightarrow$ E

For transferable utility: "slope = -1"

All rules satisfying WPareto & Sym & SI agree.

If preferences are strictly monotone: $P = WP$

1 strictly convex, & 2 convex $\Rightarrow$ PE is a singleton

Linear pref: $u = a_1x_1 + a_2x_2$, $a_1, a_2 \geq 0$, $a_1 + a_2 > 0$

Linear indifference curves: might not parallel or monotonic

Leontief pref: $u = \min\{\frac{x_1}{a_1}, \frac{x_2}{a_2}\}$

Cobb-Douglas: $u = x_1^{\alpha_1} x_2^{\alpha_2}$ $\alpha_1, \alpha_2 > 0$

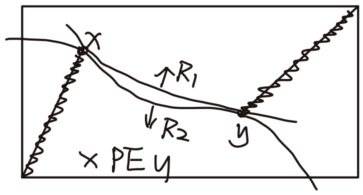
IC: rectangular hyperbolas, except for the lowest one.

Strict monotone except along the axes.

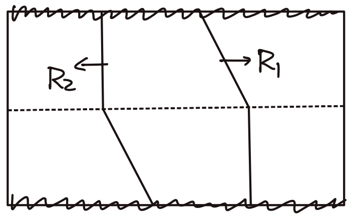
Almost strictly convex except the bottom IC

Smooth except at the origin.

When prefs are not convex, P may be the union of disconnected pieces



When prefs are not convex, P may be 2 opposite sides of the box.



Both prefs: strictly convex, smooth, homothetic

$\Leftrightarrow$ P is either doubly visible or the segment $[O_1, O_2]$

Both prefs: normal for both goods

$\Leftrightarrow$ P is monotonic

Both classic prefs: continuous, str. mon, str. conv.

$\Rightarrow$ P is a continuous curve connecting (O_1, O_2)

R_i' is obtained from R_i by a Gevers-monotonic transformation at x_i if: $L(x_i) \sqsubseteq L'(x_i)$ "lower contour set ↑"

Solution φ is Gevers-invariant if $\forall (R, M)$, $\forall x \in \varphi(R, M)$

$\forall (R', M)$ s.t. $\forall i$, $L^{R_i}(x_i) \sqsubseteq L^{R_i'}(x_i)$: $x \in \varphi(R', M)$

Both smooth, convex, monotonic prefs $\Rightarrow P(R, M)$ G.I

Truncated weak lower contour set: $L(x) \sqcap \{x \leq M\}$

R_i' is obtained from R_i by a Maskin-transformation at x_i if: $TL(x_i) \sqsubseteq TL'(x_i)$

Axiom Types

Pareto efficiency

Unanimity: If x is the most preferred for all $i \in N$, then $x \in \varphi$

Full range: $\forall$ allocation x, $\exists$ problem E s.t. $x \in \varphi(E)$

Equal treatment of equals = respect to symmetry

Monotonicity: Resource, Technology, Population,

Link (graphic case) Claim/Endowment

Invariance: claims truncation: $C_i >$ total endowment, $C_i \uparrow$, $x_i -$

Contraction independence:

replication invariance: $\varphi(k \cdot E) = k\varphi(E)$ ← k-replica

Post-application: consistency

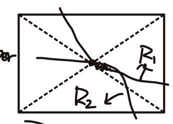
Order preservation: claim problem, $C_i \geq C_j \Rightarrow \varphi_i \geq \varphi_j$

Group order preservation: $\sum_{i \in S} C_i \geq \sum_{j \in T} C_j \Rightarrow \sum_{i \in S} \varphi_i \geq \sum_{j \in T} \varphi_j$

Individual-endowment lower bound

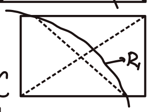
Diversity dividend: Convex: DD $\Leftrightarrow$ Equal division lower

$\frac{\Omega}{|N|} \in$ critical indifference curve

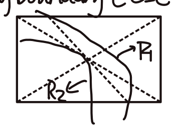


Anti-convex:

ending point of envy boundary $\in$ CIC



Diversity burden: welfare loss from diversity



Anti-convex + Convex

Constant RTS lower bound:

Solidarity: Either all agents are better off or all are worse off.

→ Resource monotonicity: e.g. refugee camp.

φ is efficient $\not\Rightarrow$ φ is R.M.

Robust to the choice of perspective: φ is independent of perspective.

→ Additivity: $cost(E_1) + cost(E_2) = cost(E_1 + E_2)$

→ Composition: claim problem $\varphi(C, M)$.

$\varphi(C, M - \varepsilon) = \varphi(\varphi(C, M), M - \varepsilon)$

Robust to strategic behavior

strategy proof: misrepresenting info. (pref.)

destruction proof: destroying own endowment

withholding proof: withholding own endowment

augmentation proof: borrowing from others

Strategy-proofness: $\forall R \in \mathcal{R}^N.\ \forall i \in N.\ \forall R_i' \in \mathcal{R}^1.$
$\varphi(R)\ R_i\ \varphi(R_i', R_{-i})$

R_0 is adjacent to R_0': $R_0: \{a_1, a_2, \cdots, a_i, a_{i+1}, \cdots, a_n\}$
R_0'

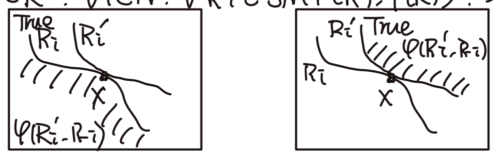
Adjacent strategy-p.: $\forall R \in \mathcal{R}^N.\ \forall i \in N.\ \forall R_i' \in Adj(R_i)$
$\varphi(R)\ R_i\ \varphi(R_i', R_{-i})$ credibility profile

Credibility-constrained SP: $\forall$ - - - $\forall R_i' \in C_i(R_i)$
$\varphi(R)\ R_i\ \varphi(R_i', R_{-i})$

Non-bossiness: $\forall R \in \mathcal{R}^N.\ \forall i \in N.\ \forall R_i' \in \mathcal{R}.$
if $\varphi_i(R) = \varphi_i(R_i', R_{-i})$ then $\varphi(R) = \varphi(R_i', R_{-i})$
Group non-bossiness: if $\varphi_S = \varphi_S'$ then $\varphi = \varphi'$

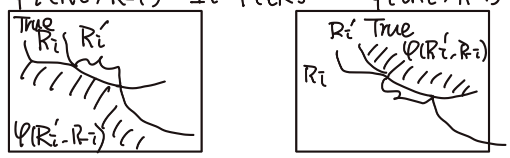
Maskin: Invariance under monotonic transformations
$\forall R \in \mathcal{R}^N.\ \forall x \in \varphi(R).\ \forall R_i' \in MT(R_i, x): L(R_i', x) \supseteq L(R_i, x)$
then $x \in \varphi(R')$
Weaker Invariance under strict monotonic transformations
$\forall R \in \mathcal{R}^N.\ \forall x \in \varphi(R).\ \forall R_i' \in SMT(R_i, x): I_i(x) \cap I_i'(x) = \{x\}$
then $x \in \varphi(R')$

Invariance Lemma: φ: S.P.
$\forall R \in \mathcal{R}^N.\ \forall i \in N.\ \forall R_i' \in SMT(R_i, \varphi(R)). \Rightarrow \varphi_i(R_i', R_{-i}) = \varphi_i(R)$

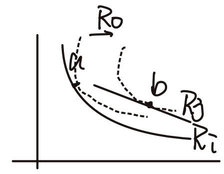


2 regions have only 1 intersection

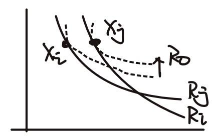
Alternative Invariance Lemma:
$\varphi_i(R_i', R_{-i})\ I_i\ \varphi_i(R)$ $\varphi_i(R_i', R_{-i})\ I_i'\ \varphi_i(R)$

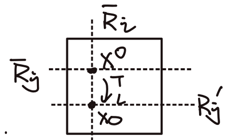


Inclusion Assumption 1: If $b\ R_i\ a$.
$\exists R_0 \in SMT(R_i, a) \cap SMT(R_i, b)$
s.t. $a\ I_0\ b$ does not hold



Inclusion Assumption 2: $R_0 \in MT(R_i, a) \cap MT(R_j, b)$
No-Envy Lemma: ① IA2:
Equal treatment of equals & MI $\Rightarrow$ F
② IA1:
ETE & SP & non-bossiness $\Rightarrow$ F
③ IA1: ET & group S.P $\Rightarrow$ F





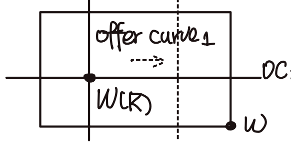
Contamination Lemma: $|N| = 2$.
Domain $= \{R \in \mathcal{R}^2 : \exists i \in N.\ \exists x^0 \in X.$ s.t. $\forall R \in \mathcal{R}^N.\ \forall x \in X$
$x \neq x^0$: $x^0\ P_i\ x$ and $x\ P_j\ x^0\}$
$\varphi = x^0$ on a rectangular subdomain of $\mathcal{R}^2$.
$\Rightarrow \varphi = x^0$ on the entire domain
$|N| = 2$. s.monot. $\mathcal{R}$. If φ is S.P. $O_1 \in \varphi$ or $O_2 \in \varphi$
then φ is a dictatorial rule.

Gibbard-Satterthwaite Theorem
Finite set A of alternatives.
n agents with strict preference on A
A rule has a range containing ≥ 2 alternatives & S.P.
$\Leftrightarrow$ it is a range dictatorship.

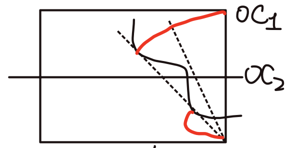
Reallocating individual endowment.

Walrasian: $W(\omega,R)=\{x\in X(\omega): \exists p\in\Delta^{L-1}: \forall i.\ px_i\le p\omega_i,\ \forall x_i': px_i'\le p\omega_i.\ x_i R_i x_i'\}$

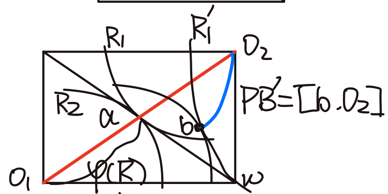
On Cobb-Douglas domain R_{CD}: Walrasian rule is not S.P.



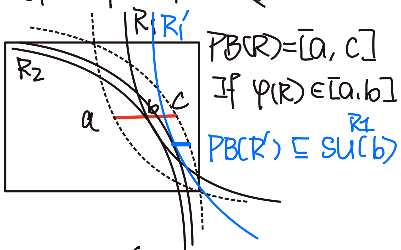
Without convex Pref.
$W(\omega,R)=\emptyset$



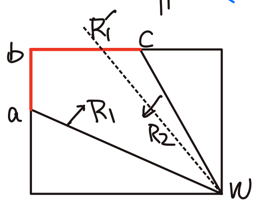
① $|N|=2.\ |L|=2$ $R\in R_{CD}$:
$P\cap S.P.\cap Bend=\emptyset$



② $|N|=2$, $|L|=2$. R: cont. convex. monot. quasi-linear.
$P\cap S.P.\cap Bend=\emptyset$

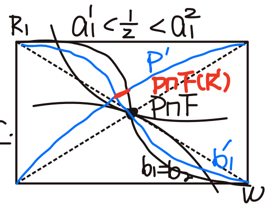


③ $|N|=2.\ |L|=2.$
R: strict monotone & linear.
$P\cap S.P.\cap Bend=\emptyset$



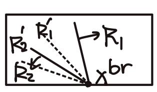
If $\varphi(R)\in[a,b]$. 1 has incentive to deviate to R_1'

Allocating social endowment:
$|N|=2, |L|=2$ $R\in R_{CD}$ or S.m. & linear (R_{Lin*})
$P\cap S.P.\cap F=\emptyset$

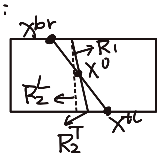


R_{Lin*}: $P\cap S.P.$ = Dictator $\notin F$

(2 agent) If R: R_1 stepper. $P\cap S.P.=x^{br}\in seg[O_1, br. O_2]$
WTS: $\forall R.R'$: 1 stepper: $\varphi(R)=\varphi(R')=x^{br}$
If R_1' flatter R_1: $(R_1,R_2)\to(R_1,R_2')\to(R_1',R_2')$

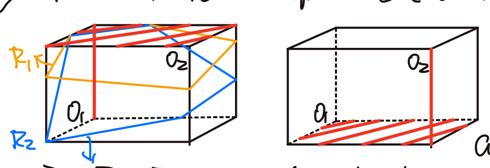


If R: R_2 stepper $P\cap S.P.=x^{tl}\in seg[O_1, tl. O_2]$
If R: $R_1 \parallel R_2$ $P\cap S.P.=x_0\in[x^{br}, x^{tl}]$
Consider different slope. $x^{br}=x^{tl}$:
BWOC: if $x^{br}\neq x^{tl}$.
WTS: $\forall x^0\in[x^{br}, x^{tl}]$
$x^0\notin S.P.$



True: x^0
Lie: x^{br}
$x^{br}\ R_2^T\ x^0$!

(3 agent) Given MRS13 are equal. 3 case of Pareto set



the whole box.

$\Rightarrow D=P\cap SP.$ in a rectangular domain $\xRightarrow{\text{Contamination Lemma}}$ $D=P\cap SP.$

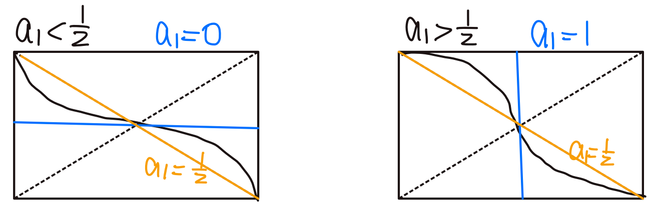
Pareto.	S.P.	Bend	
V	V	X	Dictator
V	X	V	Walrasian
X	V	V	Endowment

$|N|=3$: $\exists\varphi\in P\cap S.P.,\ \varphi\neq D$
$R_1=A\cup B.\ (A\cap B=\emptyset)$ If $R_1\in A$. $\varphi(R)=D_2$
If $R_1\in B$: $\varphi(R)=D_3$

Envy boundary for Cobb-Douglas preference

b: $x_1^{a_1}x_2^{1-a_1}=(\Omega_1-x_1)^{a_1}(\Omega_2-x_2)^{1-a_1}$

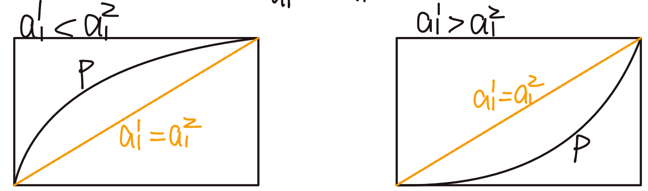
$$\frac{dx_2}{dx_1}=-\frac{\Omega_1}{\Omega_2}\frac{a_1}{1-a_1}\frac{(\Omega_2-x_2)x_2}{(\Omega_1-x_1)x_1}<0$$



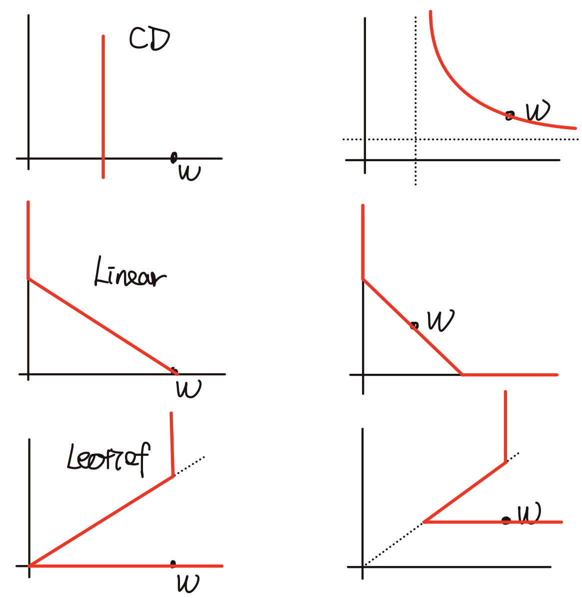
Pareto set for Cobb-Douglas preference

$(x_1,x_2)=\arg\max_{x_1,x_2} x_1^{a_1^1}x_2^{(1-a_1^1)}$ s.t. $(\Omega_1-x_1)^{a_1^2}(\Omega_2-x_2)^{1-a_1^2}\ge u_2$

P: $x_1(\Omega_2-x_2)=\frac{a_1^1}{a_1^2}\frac{1-a_1^2}{1-a_1^1}(\Omega_1-x_1)x_2$



Offer curve



Economy: $R=(R_i)_{i\in N}$, $\Omega\in\mathbb{R}^L_+$

Allocation: $x=(x_i\in\mathbb{R}^L_+)_{i\in N}$, $\sum_{i\in N} x_i=\Omega$

No-domination: $D(R)=\{x\in X : \nexists\{i,j\}\subseteq N \text{ s.t. } x_i \geq x_j\}$

No-envy $\Rightarrow$ No-domination

$|L|=1$ $D(R)=\frac{\Omega}{|N|}$

Compar. equal division:

$Ded(R)=\{x\in X : \nexists i\in N.\ \text{either } x_i\geq\frac{\Omega}{|N|} \text{ or } x_i\leq\frac{\Omega}{|N|}\}$

$|L|>1$: $|N|=2$. $D(R)=Ded(R)$

$|N|\geq 3$ $D(R)\not\supseteq Ded(R)$ $\nearrow$ $\begin{cases}(\frac{1}{3}+\varepsilon, \frac{1}{3}-\varepsilon)\\ (\frac{1}{3}, \frac{1}{3})\\ (\frac{1}{3}-\varepsilon, \frac{1}{3}+\varepsilon)\end{cases}$

Equal-division lower bound:

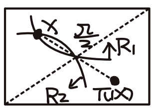
$Bed(R)=\{x\in X : \forall i\in N.\ x_i R_i \frac{\Omega}{|N|}\}$

$P(R)\sqcap Bed(R)\neq\emptyset$ $\Leftarrow$ Feasible set X is compact

No-envy $F(R)=\{x\in X : \forall\{i,j\}\subseteq N.\ x_i R_i x_j\}$

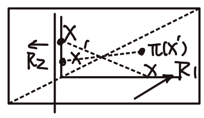
$\frac{\Omega}{|N|}\in F(R)$

$F\not\supseteq P$.



$P\not\supseteq F$: Dictator

F does not satisfies Pareto-indifference



$x\ PI\ x'$
$x\in F$
but $x'\notin F$

At an efficient allocation ① $\exists i\in N$. i envies no one

② $\exists i\in N$ i is envied by no one.

Equal-division Walrasian Wed

$Wed=\{x\in X : \exists p\in\mathbb{R}^L_+.\ \sum_l p_l=1 : \forall i\in N.\ px_i\leq p\frac{\Omega}{|N|}$ (Budget) $\forall x_i' : px_i'\leq p\frac{\Omega}{|N|}.\ x_i R_i x_i'\}$

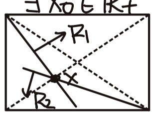
price   utility maximum

$Wed(R)\subsetneq F(R)$

Egalitarian-equivalence E x_0: reference bundle

$E(R)=\{x\in X : \exists x_0\in\mathbb{R}^L_+ \text{ s.t. } \forall i\in N.\ x_i I_i x_0\}$

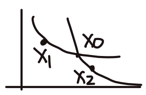
$E\not\supseteq P$



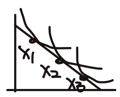
$P\not\supseteq E$ Dictator

E satisfies Pareto-indifference & Equal treatment of equals

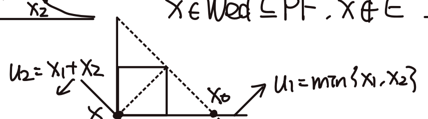
$|N|=2$: $F\subsetneq E$



$|N|\geq 3$: $x\in Wed\subseteq PF$, $x\notin E$

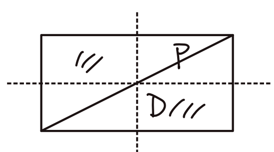


$PE\not\supseteq D$.



$x\in PE$.
$x\notin D$

$\exists$ economy with cont. s.monotone (not convex) preference. s.t. $P\sqcap D=\emptyset$

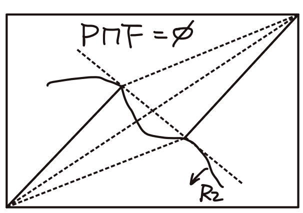


Continuous pref. R

$\forall x,x'\in P(R).\ \forall i\in N.\ x_i I_i x_i'$

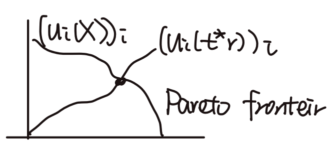
$\Rightarrow \forall\alpha\in[0,1].\ \alpha x+(1-\alpha)x'\in P(R)$

$\Rightarrow P\sqcap F\neq\emptyset$



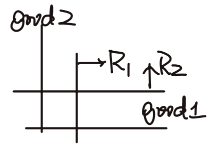
Cont & s.monot. R:

$\Rightarrow P\sqcap E\neq\emptyset$, $P\sqcap E^r\neq\emptyset$



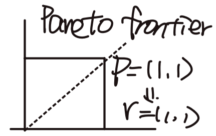
Egalitarian-equivalence solution with reference bundle in direction $r\in\mathbb{R}^L_+\setminus\{0\}$: $E^r(R):=\{x\in E(R) : \text{reference bundle} \propto r\}$

Pref. is not s. monot.



$P\sqcap E^r=\emptyset$

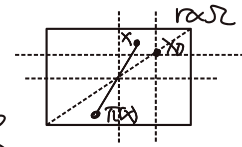
For $r\neq(1,1)$. $P\sqcap E^r=\emptyset$



Equal division lower bound:

$\bar{E}(R):=\{x\in X : \exists x_0\in\mathbb{R}^L_+ \text{ s.t. } x_0\geq\frac{\Omega}{|N|}.\ \forall i\in N,\ x_i I_i x_0\}$

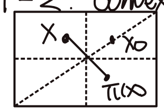
$P\bar{E}(R)=\bigcup_{r\in\Delta^{L-1}} P\bar{E}_r(R)$



$|N|=2$. $P\bar{E}_\Omega\subseteq D$

BWOC. $x\notin D$. $x\sim_1 x_0$. $\pi(x)\prec_2 x_0$

$|N|=2$. Convex R: $P\bar{E}_\Omega\subseteq F\subseteq D$

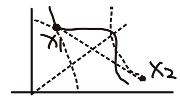


BWOC: $x\prec_1\pi(x)$

Convex $\Rightarrow x\prec_1\frac{\Omega}{2}$ S. monot $\Rightarrow \frac{\Omega}{2}\prec_1 x_0$

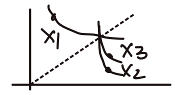
$|N|=2$. No convex R:

$P\bar{E}_\Omega\not\subseteq F$



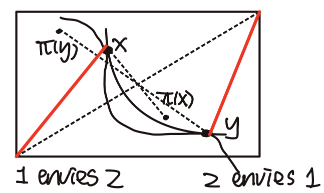
Envy-free is not good.

$|N|\geq 3$. $P\bar{E}_\Omega\not\subseteq D$:



$\exists$ classical economy s.t. $P\sqcap E\sqcap D=\emptyset$

$\exists$ non-convex preference. s.t. $P\sqcap F=\emptyset$



$P\sqcap F\sqcap$ Resource monotone $=\emptyset$

$P\sqcap F\sqcap$ Population monotone $=\emptyset$

Smooth Economy:

$P\sqcap\bar{E}\sqcap$ Replication invariance $\sqcap$ Consistency $\subseteq W$

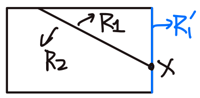
Nash implementation

Equal-division lower bound solution: $\bar{E} \Rightarrow$ M.I.

Envy-free solution: $F \Rightarrow$ M.I.

Strictly monotonic pref. Pareto $P \Rightarrow$ M.I.

Weakly monotonic: $P \not\Rightarrow$ M.I.



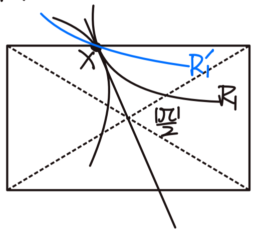
$x \in P(R)$

$x \notin P(R_1', R_2)$

On classical domain, at interior allocation (equal division) Walrassian solution $\Rightarrow$ M.I.

At boundary allocations:

$W \not\Rightarrow$ M.I.



If R: continuous, convex, s. monotonic

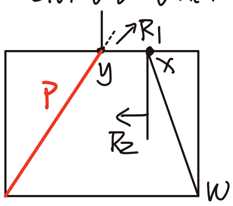
$\exists$ good k: $x_{ik} = 0 \Rightarrow x_i I_i 0$

then $W \in \mathrm{Int}(\mathrm{Box})$. $W \Rightarrow$ M.I.

Constrained Walrasian solution: W^C

$\exists$ price $p \in \Delta^{L-1}$ s.t. $\forall i$, $\forall x_i'$: $px_i' \le pw_i$, $\underset{\text{truncation}}{\underline{x_i' \in \Omega}}$, $x_i R_i x_i'$

$W^C \not\Rightarrow P$.

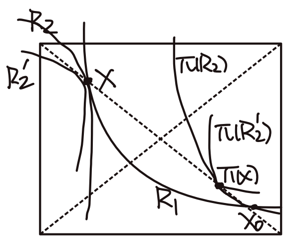


$x \in W^C(w, R)$, $x \notin P$

y Pareto dominates x

Pareto-and-egalitarian-equivalence solution $\not\Rightarrow$ M.I.

$x \in PE(x_0)$, $x \notin PE(x_0, R_2')$



$\varphi : R^N \to X$ (R contains R_L (linear pref)).

① $\varphi \in$ M.I.

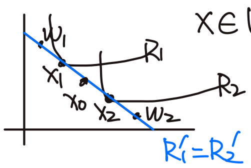
② If $\exists x_0 \in \mathbb{R}^L_+$ s.t. $\forall i \in N$, $x_i I_i x_0$, $(x_0, \cdots, x_0) \in P(R)$ $\quad x_0 = \frac{\Omega}{|N|}$?

then $x \in \varphi(R)$

then $W_{ei} \sqsubseteq \varphi$ (Equal income Walrasian rule)

②': If $(\frac{\Omega}{|N|}, \cdots, \frac{\Omega}{|N|}) \in P(R)$, $\forall i \in N$, $x_i I_i \frac{\Omega}{|N|}$ then $x \in P(R)$

then $W_{ed} \sqsubseteq \varphi$



$x \in W_{ei}$ $\quad x_0 := \frac{\Omega}{|N|}$ $\quad x_i R_i' x_0$ $\quad x_0 \in P(R')$

$\overset{②}{\Rightarrow} x \in \varphi(R')$

$R \in MT(R', x) \overset{①}{\Rightarrow} x \in \varphi(R)$

No veto power: $\forall R \in \mathcal{R}^N$, $a \in A$, $i \in N$.

If $\forall j \in N \setminus \{i\}$, $\forall b \in A$, $a R_j b$, then $a \in \varphi(R)$

Implementability of φ in equilibrium E:

$\exists$ a game form $\Gamma := (S, h)$ $\quad S$: strategy space (preference)

h: allocation $\quad h: S \to A$

s.t. $\forall R \in \mathcal{R}^N$ $\quad h(E(S, h; R)) = \varphi(R)$

φ is implementable $\Rightarrow \varphi \in$ M.I.

$|N| > 2$: $\varphi \in$ M.I., No veto power $\Rightarrow \varphi$ is implementable

On the subdomain of $\mathcal{R}_{cl}^N$ of strictly monotonic prefs.

Divide-and-Permute implements the no-envy solution.

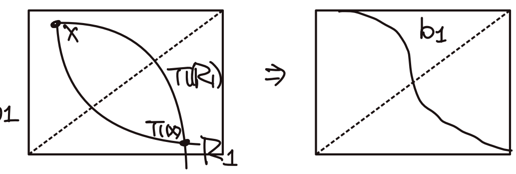
Γ^{DP}: $\quad S_1 = S_2 = X \times \Pi^N \quad S_3 = \cdots = S_n = \Pi^N$

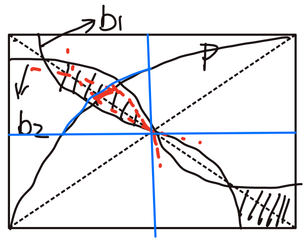
$s = ((x^1, \pi_1), (x^2, \pi_2), \pi_3, \cdots, \pi_n) \in S$

$$h(s) = \begin{cases} (0, 0, \cdots, 0) & \text{if } x^1 \neq x^2 \\ \pi_n \circ \pi_{n-1} \circ \cdots \circ \pi_1(x^1) & \text{if } x^1 = x^2 \end{cases}$$

Envy boundary

$x, \pi(x) \in b_1$





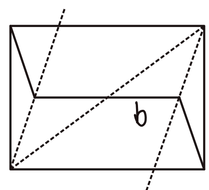
Equal-division Walrasian

$\subsetneq P \sqcap$ Equal division lower bound

$\subsetneq P \sqcap$ Envy-free

$\subsetneq P \sqcap D$

Leontief.

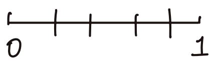


No-envy Lemma:

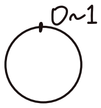
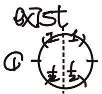
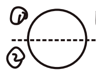
Classical domain: ETE + M.I. $\Rightarrow F$

Homothetic domain: ETE + M.I. $\not\Rightarrow F$.

Partitioning 1-dimensional & non-homogenous & infinitely divisible continuum

Linear:  $\begin{cases} F \neq \emptyset, & F \Rightarrow P. \\ F \Rightarrow \text{Group envy free} \end{cases}$

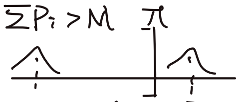
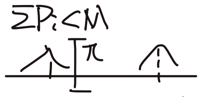
$|N|=2$: $F \sqcap S.P. = \emptyset$, $P \sqcap S.P. = D$

Circular:  $\begin{cases} F \text{ may not exist} \\ F \not\Rightarrow P \end{cases}$  $u_1=\frac{3}{2}$, $u_2=\frac{3}{2}$  $u_1=2$, $u_2=2$

$|N|=2$: $F \sqcap S.P. \sqcap \text{Bend} = \emptyset$. $P \sqcap S.P. = D$

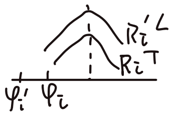
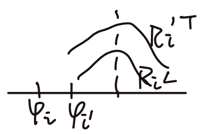
Multi-dimensional: $\begin{cases} P \sqcap F \text{ may not exist} \\ P \sqcap E \neq \emptyset \end{cases}$

Full allocation with single-peaked preference.*

$\Sigma P_i > M$  $\Sigma P_i < M$ 

$u_i = \min\{P_i, \pi\}$ s.t. $\Sigma_i u_i = M$; $u_i = \max\{P_i, \pi\}$ s.t. $\Sigma_i u_i = M$

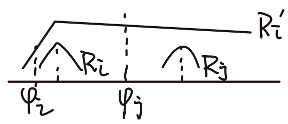
$U \in P \sqcap F \sqcap S.P. \sqcap \text{NonB}$
$U \in$ Weakly group F
$U \in \bar{E}$ equal-division lower
$U \in$ Weak equal-division core

$P \sqcap F \sqcap S.P. \Leftrightarrow U$  

① $P_i = P_i' \overset{S.P.}{\Rightarrow} \varphi_i = \varphi_i'$

② $\varphi \neq U \Rightarrow \varphi \notin F$

$\exists i.\ \varphi_i < u_i$ $\exists j.\ \varphi_j > u_j \overset{P.}{\Rightarrow} \varphi_i < u_i \leq u_j = \pi < \varphi_j$

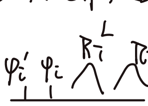
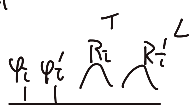


$\varphi_i' = \varphi_i < \varphi_j = \varphi_j'$

$\varphi_j\ R_i'\ \varphi_i \Rightarrow \varphi \notin F$

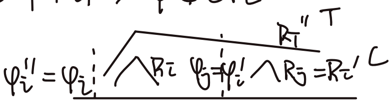
$P \sqcap$ Equal treatment of equals $\sqcap S.P. \Leftrightarrow U$

$(P \sqcap F \Rightarrow ETE)$

① $P_i, P_i' > \varphi_i, \varphi_i' \overset{S.P.}{\Rightarrow} \varphi_i = \varphi_i'$  

② $\varphi \neq U \overset{S.P.}{\Rightarrow} \varphi \notin ETE$

ETE: $\varphi_i' = \varphi_j > \varphi_i = \varphi_i''$



$\underset{\text{Lie}}{\varphi_i'}\ \underset{\text{Truth}}{R_i''}\ \underset{\text{Truth}}{\varphi_i''}$, $\varphi \notin S.P.$

$P \sqcap$ Anonymity $\sqcap S.P. \Leftrightarrow U$. (Anonymous $\Rightarrow$ ETE)

P.	F.	S.P.	
		X	Equal division. Walrasian. slacks
	X		Sequential rationality
X			Equal rule. proportional rule

Uniform rule is Nash implementable.

Manipulation game: Strategy $S = R_{\text{single-peaked}}$

$CEL_i(P) = \max\{P_i - \Delta, 0\}$ st. $\sum_{i \in N} \max\{P_i - \Delta, 0\} = M$

Nash Equilibrium $E(S, CEL, R) = U(R)$ $\forall R \in R_{SP}^N$

Public choice with single-peaked preference

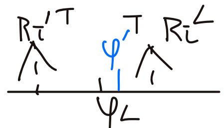
Generalized median rule with signature $a \in [0, M]^{N-1}$, gm^a:

$gm^a(R) = \text{med}\{(P_i)_{i=1}^N, (a_j)_{j=1}^{N-1}\}$

$P \sqcap \text{Anonymity} \sqcap \text{Peak-only}^* (\varphi(R)=\varphi(P)) \sqcap S.P. = gm^a$

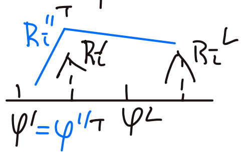
X ✓ ✓ ✓ $(n+1)$ a

* can be replaced with continuity or uncompromisingness. $P_i < \varphi_i \Rightarrow \varphi_i = \varphi_i(P_i', R_{-i})$, $P_i' < P_i$

① $P_i > \varphi$, $P_i' < \varphi$: $R_i' \leq \varphi' \leq \varphi$ 

S.P. $\Rightarrow \varphi' \leq \varphi$

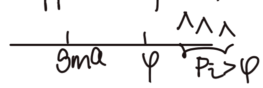
BWOC: $\varphi^{Lie}\ R_i'^{\,Truth}\ \varphi'^{\,Truth}$

Peak-only & S.P. $\Rightarrow \varphi' \geq R_i'$ 

Peak-only: $\varphi'' = \varphi'$

S.P.: $\varphi^{Lie}\ R_i''^{\,Truth}\ \varphi''^{\,Truth}$

② Suppose $\varphi \in P \sqcap A \sqcap \text{Peak} \sqcap S.P.$ $\varphi \neq gm^a$

 move $P_i : (P_i > \varphi)$ to $P_i' = gm^a$.

$gm^{a'} = gm^a$. $\varphi' \in [gm^a, \varphi]$

Object allocation

F may not exist: same preference

$F \Rightarrow P$. $F \Rightarrow$ Group F

Sequential priority rule: $SP^{\pi \leftarrow \text{order}}$

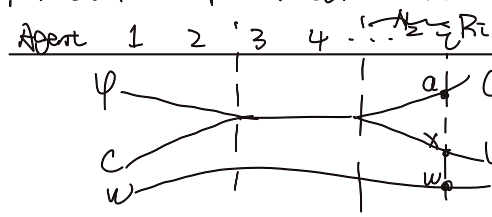
$SP^\pi \in P \sqcap S.P.$

$\forall \varphi \in P.\ \exists \pi_\varphi$ s.t. $\varphi = SP^{\pi_\varphi}$

Random arrival rule $\in P \sqcap S.P.$

Object allocation with endowment. (w, R)

$P \sqcap \text{Bond} \sqcap S.P. \Leftrightarrow \text{Core} = TTC$

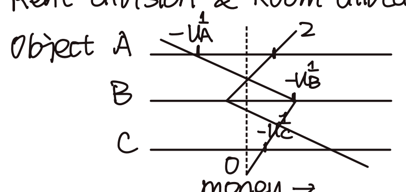


R_i': $\alpha = C_i'$, $w = \varphi_i'$

Iteration: $\forall i \in N_2$ $C_i > \varphi_i' = w_i'$

$N \setminus N_2$: $C \neq$ Core

Rent-division & Room allocation



$u_i(m, x) = m + u_i^x$. $x \in \{A, B, C\}$

Indiff_i^0: $m + u_i^x = 0$

Quasilinear $\Rightarrow$ Parallel Indiff_i

$\begin{cases} F \Leftrightarrow \text{Equal endowment } I = \frac{M}{N}. \text{ Walrasian rule} \\ F \Rightarrow P \quad F \Rightarrow \text{Group F} \end{cases}$

* Full allocation of multi-dimensional good with SP pref in each.

Apply uniform rule in each dim: $U \not\Rightarrow P$. $U \Rightarrow S.P.$ $U \not\Rightarrow$ GroupSP.

Coalitional game: $v = (v(S))_{S\subseteq N} \in \mathbb{R}^{2^n-1}$ Domain $\mathcal{V}^N$

Strategic game $\Gamma = (M, h)$

$\forall S\subseteq N$. $\bar\Gamma = (M = (\prod_S M_i, \prod_{N\setminus S} M_i),\ h = (\sum_{i\in S} h_i, \sum_{i\in N\setminus S} h_i))$

Nash equilibrium $M^S(\bar\Gamma) \subseteq M$

$v(S) := \sup/\inf \{\bar h_S(m) : m\in M^S(\bar\Gamma)\}$

Claim problem: $(C, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$. $\sum C_i \geq E$

$\forall S\subseteq N$ $v(S) := \max\{0, E - \sum_{i\notin S} C_i\}$

or $v(S) := \min\{\sum_{i\in S} C_i, E\}$

Oligopoly: $(w_i, f_i : \mathbb{R}_+\to\mathbb{R}_+)_{i\in N}$

$v(S) := \max\{p \sum_{i\in S} f_i(z_i) : z\in\mathbb{R}^S_+,\ \sum_{i\in S} z_i = \sum_{i\in S} w_i\}$

Exchange economy with cardinal utility: $(R_i, w_i)_{i\in N}$

$v(S) := \max\{\sum_{i\in S} u_i(x_i) : \sum_{i\in S} x_i = \sum_{i\in S} w_i\}$

Queueing problem: $C = (C_i)_{i\in N} \in \mathbb{R}^N_+$ unit cost for waiting

$\forall S\subseteq N$. $-v(S) := \min\{\sum_{i\in S} t_i C_i : (t_i)_{i\in S} \leftrightarrow (0,1,\cdots,|S|-1)\}$

$-v(S) := \min\{\sum_{i\in S} t_i C_i : (t_i)_{i\in S} \leftrightarrow (N-|S|, \cdots, N-1)\}$

$-v(S) = \frac{1}{2}(-v(S)^{\text{optimistic}} + -v(S)^{\text{pessimistic}})$

Voting game (0-1 game): $v(S) = 0$ or 1

① T-unanimity game: $u^T(S) = \mathbb{1}\{T\subseteq S\}$. $u^T = (u^T(S))_{S\subseteq N}$

e.g. $N=\{1,2\}$. $u^{\{1\}} = (1,0,1)$ $u^{\{2\}} = (0,1,1)$ $u^{\{1,2\}} = (0,0,1)$

span$\{u^{\{1\}}, u^{\{2\}}, u^{\{1,2\}}\} = \mathbb{R}^{2^{|N|}-1} = \mathcal{V}^N$

② $v(S)=0 \Leftrightarrow v(N\setminus S) = 1$

③ (2k+1)-person majority game $v(S) = \mathbb{1}\{|S|\geq k+1\}$ quota

④ n-person weighted majority game. $v(S) = \mathbb{1}\{\sum_{i\in S} w_i \geq q\}$

Axioms

1. Additivity: $\forall S,T\subseteq N$. $S\cap T=\emptyset$. $v(S\cup T) = v(S)+v(T)$
2. Constant sum: $\exists C\in\mathbb{R}$ st. $\forall S\subseteq N$. $v(S)+v(N\setminus S) = C$
3. Symmetry: $\forall S, S'\subseteq N$. $|S|=|S'|$ $v(S)=v(S')$
4. Monotone: $\forall S,T\subseteq N$. $S\subseteq T$ $v(S)\le v(T)$
5. Super-additivity: $\forall S,T\subseteq N$. $S\cap T=\emptyset$. $v(S\cup T)\ge v(S)+v(T)$
6. Convexity: $\forall i\in N$. $\forall S,T\subseteq N\setminus\{i\}$. $S\subseteq T$

$v(S\cup\{i\}) - v(S) \le v(T\cup\{i\}) - v(T)$

$\Leftrightarrow \forall S,T\subseteq N$. $v(S)+v(T)\le v(S\cup T)+v(S\cap T)$

Super-additive + non-negative $\begin{matrix}\not\Leftarrow\ v(S)=1\ \forall S\subseteq N\\ \Rightarrow\end{matrix}$ Monotonic

The unanimity games is a basis of $\mathbb{R}^{2^n-1}$:

Span$\{(u^T)_{T\subseteq N}\} = \mathcal{V}^N = \mathbb{R}^{2^n-1}$

Solution on $\mathcal{V}^N$: $\varphi: \mathcal{V}^N \rightrightarrows \mathbb{R}^N_+$: $\sum_{i\in N}\varphi_i \le v(N)$

- Player-centric solution
- Coalition-centric solution

Efficiency: $\forall v\in\mathcal{V}^N$. $\forall x\in\varphi(v)$. $\sum x_i = v(N)$

↳ pre-imputation of v

Plain equal-division: $ed(v) = (\frac{v(N)}{|N|}, \cdots, \frac{v(N)}{|N|})$

Individual-worths lower bound: $v(N)\ge\sum v(\{i\}) \Rightarrow x_i\ge v(\{i\})$

Equal-division-over-individual-worths:

$\varphi_i(v) = v(\{i\}) + \frac{v(N)-\sum_j v(\{j\})}{|N|}$

Principal-contributions: $PC_i(v) = v(N) - v(N\setminus\{i\})$

Equal-division-over-principal-contributions:

$\varphi_i(v) = v(N)-v(N\setminus\{i\}) + \frac{v(N) - \sum_j [v(N)-v(N\setminus\{j\})]}{|N|}$

Shapley: S i $N-S-1$ $\quad$ S $\downarrow i$ $N-S$

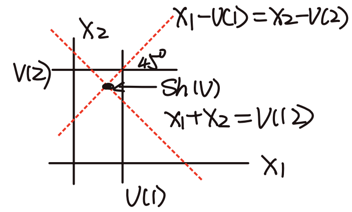
$Sh_i(v) = \sum_{S\subseteq N,\ i\notin S} \frac{|S|!\,|N-S-1|!}{|N|!}\{v(S\cup\{i\}) - v(S)\}$

$= \sum_{S\subseteq N,\ i\in S}\frac{|S-1|!\,|N-S|!}{|N|!}\{v(S)-v(S\setminus\{i\})\}$

2-agent case:

Order	Agent 1	Agent 2
(1, 2)	$v(1)$	$v(12)-v(1)$
(2, 1)	$v(12)-v(2)$	$v(2)$

Shapley $v(1)+\frac{v(12)-v(2)-v(1)}{2}$ $v(2)+\cdots$



$Sh_i(v) = \frac{1}{|N|}\{v(N)-v(N\setminus i) + \sum_{j\ne i} Sh_i(v|N-\{j\})\}$

$\sum_{j\ne i} Sh_i(v|N\setminus j) = \sum_{j\ne i}\sum_{S\subseteq N\setminus j,\ i\in S}\frac{(|S|-1)!\,(|N|-|S|-1)!}{(|N|-1)!}[v(S)-v(S\setminus i)]$

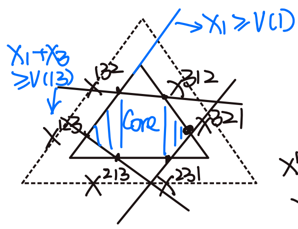
$= \sum_{S\subsetneq N,\ i\in S}\sum_{j\ne i}\mathbb{1}\{S\subseteq N\setminus j\}\frac{(|S|-1)!\,(|N|-|S|-1)!}{(|N|-1)!}[v(S)-v(S\setminus i)]$

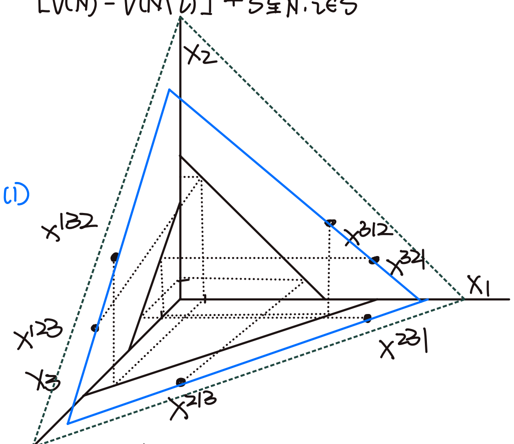
$= \sum_{S\subsetneq N,\ i\in S}\frac{(|S|-1)!\,(|N|-|S|)!}{(|N|-1)!}[v(S)-v(S\setminus i)]$

RHS $= \frac{(|N|-1)!\,0!}{|N|!}[v(N)-v(N\setminus i)] + \sum_{S\subsetneq N,\ i\in S}\cdots$

$= Sh_i(v)$

3-agent case:





$Sh(v) = \underset{x\in\mathbb{R}^N}{\operatorname{argmin}}\sum_{S\subseteq N}\frac{1}{|N|-1}\binom{|N|-2}{|S|-1}^{-1}[v(S)-\sum_{i\in S}x_i]^2$

s.t. $\sum_{i\in N}x_i = v(N)$

$Sh_i(v) = \sum_{S\subseteq N,\ i\in S}\frac{(S-1)!\,(N-S)!}{N!}[v(S)-v(N\setminus S)]$

$= \sum_{S\subseteq N,\ i\notin S}\frac{S!\,(N-S-1)!}{N!}[v(N\setminus S)-v(S)]$

$\min_x \sum_{S\subseteq N} m(S)[v(S)-\sum_{j\in S}x_j]^2$ s.t. $\sum_{j\in N} x_j = v(N)$

$L = \sum_{S\subseteq N} m(S)[v(S)-\sum_{j\in S}x_j]^2 + 2\pi(\sum_{j\in N}x_j - v(N))$

$\frac{\partial L}{\partial x_i}: \sum_{S\subseteq N, i\in S} m(S)[v(S)-\sum_{j\in S}x_j] = \pi$ $\quad (S=N: v(S)-\sum x=0)$

$\Rightarrow \sum_{S\subsetneq N, i\in S} m(S)v(S) - \sum_{j\in N}\sum_{S\subsetneq N, i,j\in S} m(S)x_j = \pi$

Matrix: $\begin{bmatrix} a & & b \\ & a & \\ & \ddots & \\ b & & a \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \alpha_1^m(v) \\ \vdots \\ \alpha_N^m(v) \end{bmatrix} - \pi$

where $a = \sum_{S\subseteq N, i\in S} m(S) = \sum_{s=1}^{N-1} C_{N-1}^{s-1} m(s)$

$b = \sum_{S\subseteq N, i,j\in S} m(S) = \sum_{s=2}^{N-2} C_{N-2}^{s-2} m(s)$

$\alpha_i^m(v) = \sum_{S\subsetneq N, i\in S} m(S)v(S)$

Summation across rows:

$(a+(N-1)b)\,v(N) = \sum \alpha_i^m(v) - N\pi$

$\Rightarrow \begin{bmatrix} a & & b \\ & \ddots & \\ b & & a \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \alpha_1^m(v) - \frac{1}{N}\sum_i \alpha_i^m(v) \\ \vdots \\ \alpha_N^m(v) - \frac{1}{N}\sum_i \alpha_i^m(v) \end{bmatrix} + \frac{(a+(N-1)b)}{N} v(N)$

$\Rightarrow \begin{bmatrix} a-b & & 0 \\ & a-b & \\ & \ddots & \\ 0 & & a-b \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \alpha_1^m(v) - \frac{1}{N}\sum_i \alpha_i^m(v) \\ \vdots \\ \alpha_N^m(v) - \frac{1}{N}\sum_i \alpha_i^m(v) \end{bmatrix} + \frac{a-b}{N} v(N)$

$\Rightarrow x_i = \frac{1}{N}v(N) + \frac{1}{(a-b)N}[N\alpha_i^m(v) - \sum_j \alpha_j^m(v)]$

where $a-b = \sum_{s=2}^{N-1}(C_{N-1}^{s-1} - C_{N-2}^{s-2})m(s) + 1\,m(1)$

$= \sum_{s=2}^{N-1} C_{N-2}^{s-1} m(s) + C_{N-2}^{1-1} m(1)$

$= \sum_{s=1}^{N-1} C_{N-2}^{s-1} m(s)$

and $N\alpha_i^m(v) - \sum_j \alpha_j^m(v)$ $\quad (S\neq N)$

$= \sum_{S: i\in S} N\,m(S)v(S) - \sum_{j\in N}\sum_{S: i\in S} m(S)v(S)$

$= \sum_{S: i\in S} N\,m(S)v(S) - \sum_{S\neq N} |S|\,m(S)v(S)$

$= \sum_{S: i\in S} |N-S|\,m(S)v(S) - \sum_{S: i\notin S} |S|\,m(S)v(S)$

$= \sum_{S: i\in S} |N-S|\,m(S)v(S) - \sum_{T: i\in T} |N-T|\,m(T)v(N\setminus S)$

$= \sum_{S: i\in S} |N-S|\,m(S)[v(S)-v(N\setminus S)]$

Let $m(S) = (C_{N-2}^{s-1})^{-1}\frac{1}{N-1}$

$x_i = \sum_{S\subseteq N, i\in S} \frac{(s-1)!(N-s)!}{N!}[v(S)-v(N\setminus S)] = Sh_i(v)$

Core : $C(v) := \{x\in\mathbb{R}^N : x(N) = v(N).\ \forall S\subseteq N.\ x(S)\geq v(S)\}$

S can improve upon x (S blocks x) : $\exists S: x(S) < v(S)$

$C(u^T) = \{x\in\mathbb{R}^N_+ : x(T) = 1\}$

Balanced pair $(\bar{S}, \overset{\to\text{positive}}{\delta})$: $\forall i\in N\ \sum_{S\in\bar{S}} 1\{i\in S\}\delta_S = 1$

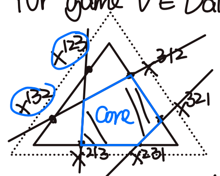
$C(v)\neq\emptyset \Leftrightarrow v(N)\geq \max\{\sum_{S\in\bar{S}}\delta_S v(S) \mid (\bar{S},\delta) \text{ is balanced}\}$

$v\in V_+^N$, $C(v)\neq\emptyset \Rightarrow v = \sum_{i\in N}\alpha_i u^{\{i\}}$ since $C(u^{\{i\}}) = \{1_{\{i\}}\}$

TU game : $v^T(S) = 1\{T\subseteq S\}$ $\forall S\subseteq N$: TU game is convex.

$\text{Core}(\{v^T\}) = \{x\in\mathbb{R}^N_+ : \sum_{i\in T} x_i = 1,\ x_j = 0.\ \forall j\notin T\}$

For game $v\in\mathcal{V}^N_{all}$. if $\forall$ order π : $mc^\pi(v)\in C(v)$. then v convex



Not convex !

Total balanced game v : $\forall S\subseteq N.\ C(v_S)\neq\emptyset$

v is T.B. $\Leftrightarrow$ $\exists$ Economy $\mathcal{E} = (X, u, e)$ s.t. $\forall S\subseteq N$

$v(S) := \max \sum_{i\in S} u_i(x_i)$ s.t. $\sum_{i\in S} x_i \leq \sum_{i\in S} e_i$

v : Convex (TU) game : $Sh(v)\in C(v)$

Nucleolus : Nuc(v) : Lexicographically min. $v(S)-x(S)$

On the domain of TU games with a non-empty set of imputations, the nucleolus is ① well-defined.

$v\in V^N_{I\neq\emptyset}$ $\quad I(v)$: $I^1(v) := \arg\min_{x\in I(v)}\{\max_{S\subseteq N}\{v(S)-x(S)\}\}$ $\quad S\neq N.\ S\neq\emptyset$ $|v(S)| = 2^N-2$

$I^2(v) := \arg\min_{x\in I^1(v)}\{2\text{nd}\max_{S\subseteq N}\{v(S)-x(S)\}\}$

...

$Nuc(v) := \arg\min_{x\in I^{2^N-2}(v)}\{\min_{S\subseteq N}\{v(S)-x(S)\}\}$

② Single-valued : by convexity of $I(v)$

③ continuous

If $\text{Core}(v)\neq\emptyset$, then $Nuc(v)\in \text{Core}(v)$

$\sum_i x_i = \sum_i y_i$:

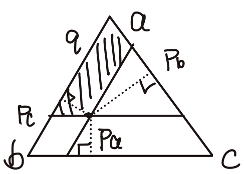
x Lorenz-dominates y : $\sum_{i=1}^k \theta_i(x) \geq \sum_{i=1}^k \theta_i(y)$, $\exists k$ ">"

$\Leftrightarrow$ $\forall$ strictly concave f : $\sum f(x_i) > \sum f(y_i)$

vNM set for a game v : vNM(v)

① $\nexists$ $x, y\in vNM(v)$ s.t. $x\succ y$

② $\forall z\notin vNM(v)$. $\exists x\in vNM(v)$ s.t. $x\succ z$



$q_a\geq p_a$

$q_a+q_b\geq p_a+p_b$

$q_a+q_b+q_c = p_a+p_b+p_c = 1$ $\Rightarrow q_c\leq p_c$

Properties on 1. Core 2. Shapley value 3. Nucleolus

Player $i \in N$ is a dummy player in v if
$\forall S \subseteq N \setminus \{i\} \quad v(S \cup \{i\}) = v(S) + v(\{i\})$

Dummy player axiom: i dummy $\Rightarrow \varphi_i = v(\{i\})$

Player $i \in N$ is a null player in v if
$\forall S \subseteq N \setminus \{i\} \quad v(S \cup \{i\}) = v(S)$. $\Rightarrow v(\{i\}) = 0$

Null player axiom: i null $\Rightarrow \varphi_i = 0$

S is a dummy coalition in v if.
$\forall S' \subseteq N \setminus S \quad v(S \cup S') = v(S') + v(S)$

Dummy coalition axiom: S dummy $\Rightarrow \sum_{i \in S} \varphi_i(v) = v(S)$

2 players i & j are symmetric in v if
$\forall S \subseteq N \setminus \{i, j\}$. $v(S \cup \{i\}) = v(S \cup \{j\})$

Respect to symmetry: i, j symmetric $\Rightarrow \varphi_i = \varphi_j$
$\Uparrow$ Anonymity

Player i is at least as useful than j:
$\forall S \subseteq N \setminus \{i, j\}$. $v(S \cup i) \geq v(S \cup j)$

Order preservation: i more useful than $j \Rightarrow \varphi_i \geq \varphi_j$

Smallest-contribution lower bound: $\varphi_i \geq \min_{S \subseteq N \setminus i} \{v(S \cup i) - v(S)\}$

Largest-contribution upper bound: $\varphi_i \leq \max_{S \subseteq N \setminus i} \{v(S \cup i) - v(S)\}$

Additivity: $\forall v, w \in V^N$. $\varphi(v + w) = \varphi(v) + \varphi(w)$

P & Sym & dummy & add $\Leftrightarrow Sh(\cdot) \Rightarrow$ single-value

"$\Rightarrow$" Unanimity game: $u^T(S) = \mathbb{1}\{T \subseteq S\}$

$\forall i \notin T$: $v(S \cup i) = v(S)$, i is null $\overset{\text{Dummy}}{\Longrightarrow} \varphi_i = v(i) = 0$

$\forall i, j \in T$: $v(S \cup i) = v(S \cup j)$ i, j sym $\overset{\text{Sym.}}{\Longrightarrow} \varphi_i = \varphi_j$

$P \Rightarrow \varphi_i(u^T) = \frac{1}{|T|} \mathbb{1}\{i \in T\}$

$\forall$ TU game $v = \sum_{k=1}^{K} \pi_k u^{S_k}$

Add. $\Rightarrow \varphi_i(v) = \sum_{k: i \in S_k} \frac{\pi_k}{|S_k|}$

For unanimity game. $Sh_i(u^T) = \frac{1}{|T|} \mathbb{1}\{i \in T\}$:

$$
\begin{aligned}
Sh_i(u^T) &= \sum_{S \subseteq N, i \in S} \frac{(S-1)!(N-S)!}{N!} (u^T(S) - u^T(S \setminus i)) \\
&= \sum_{S \subseteq N, i \in S, T \subseteq S, T \not\subseteq S \setminus i} \frac{(S-1)!(N-S)!}{N!} \cdot 1 \\
&= \begin{cases} \sum_{S \subseteq N: T \subseteq S} \frac{(S-1)!(N-S)!}{N!} & \text{if } i \in T \\ 0 \quad (\text{since } T \subseteq S \Leftrightarrow T \subseteq S \setminus i) & \text{if } i \notin T \end{cases} \\
&= \frac{1}{|T|} \mathbb{1}\{i \in T\}
\end{aligned}
$$

Contribution of player i to coalition S:

$$
C_S^i(v) := v(S \cup i) - v(S \setminus i) = \begin{cases} v(S \cup i) - v(S) & \text{if } i \notin S \\ v(S) - v(S \setminus i) & \text{if } i \in S \end{cases}
$$

$C_S^i(v) = C_{S \cup i}^i(v)$, $\forall S$ s.t. $i \notin S$

Contribution-vector monotonicity: For $v, w \in V^N$, $i \in N$
$C^i(v) \geq C^i(w) \Rightarrow \varphi_i(v) \geq \varphi_i(w)$

$C^i(v + w) = C^i(v) + C^i(w)$.

$C^i(u^T) = 0$ if $i \notin T$.

Contribution-vector only: $C^i(v) = C^i(w) \Rightarrow \varphi_i(v) = \varphi_i(w)$

Contribution of coalition S: $C_T^S(v) := v(T \cup S) - v(T \setminus S)$

Coalitional-contribution-vector-only: $C^S(v) = C^S(w) \Rightarrow \sum_{i \in S} \varphi_i(v) = \sum \varphi_i(w)$

$P \sqcap$ Sym $\sqcap$ Contribution-vector-only $\Leftrightarrow$ sh.

"$\Rightarrow$" ① $C^i(v) = \vec{0} \Rightarrow \varphi_i(v) = 0$

$w := \vec{0} \quad \forall i, j$. i sym $j \Rightarrow \varphi_i = \varphi_j$

$P \Rightarrow \sum_i \varphi_i = 0 \Rightarrow \varphi_i(w) = 0$

Contribution-vector-only: $C^i(v) = C^i(w) = \vec{0} \Rightarrow \varphi_i(v) = 0$

② $v = \sum_{k=1}^{K} \pi_k u^{T_k}$. $Sh_i(v) = \sum_{k: i \in T_k} \frac{\pi_k}{|T_k|}$

Induction: $K = 0$. $v = \vec{0} \Rightarrow \varphi_i(v) = 0 = Sh_i(v)$.

$k = 1$: $v = \pi_1 u^{T_1}$. $P \sqcap$ Sym $\Rightarrow \varphi_i(v) = \frac{\pi_1}{|T_1|} \mathbb{1}\{i \in T_1\} = Sh_i$

Suppose $v = \sum_{k=1}^{K} \pi_k u^{T_k}$: $\varphi_i(v) = Sh_i$. For $K+1$:

$\rightarrow$ if $i \notin \bigsqcap_{k=1}^{K+1} T_k$: $w = \sum_{k: i \in T_k} \pi_k u^{T_k}$

$$
\begin{aligned}
C_S^i(v) &= \sum_{T_k \subseteq S \cup i} \pi_k - \sum_{T_k \subseteq S \setminus i} \pi_k \\
&= \sum_{T_k \subseteq S \cup i, i \in T_k} \pi_k - \sum_{T_k \subseteq S \setminus i, i \in T_k} \pi_k = C_S^i(w)
\end{aligned}
$$

$\varphi_i(v) = \varphi_i(w) = Sh_i(w) = \sum_{k: i \in T_k} \frac{\pi_k}{|T_k|} = Sh_i(v) \quad i \notin \bigsqcap_{k=1}^{K+1} T_k$

$\rightarrow \forall i, j \in \bigsqcap_{k=1}^{K+1} T_k$: i, j Sym

$$
\begin{aligned}
\varphi_i(v) &= \frac{1}{|\bigsqcap_{k=1}^{K+1} T_k|} \left(v(N) - \sum_{j \notin \sqcap T_k} Sh_j(v)\right) = \frac{1}{|T|} \left(\sum_k \pi_k - \sum_{j \notin T} \sum_{k: j \in T_k} \frac{\pi_k}{|T_k|}\right) \\
&= \frac{1}{T} \left(\sum_k \pi_k - \sum_k (|T_k| - |T|) \frac{\pi_k}{|T_k|}\right) = \sum_k \frac{\pi_k}{|T_k|}
\end{aligned}
$$

P	Sym	Contribution-vector-only	$\Leftrightarrow$ sh.
V	V	X	$(\frac{v(N)}{\lvert N \rvert}, \dots, \frac{v(N)}{\lvert N \rvert})$
V	X	V	weighted sh.
X	V	V	$\frac{1}{2}$ sh.