Aggregation (Representative Agent)

Clones: (Homogeneity)

Firm: $y^i = zF(k^i,n^i)$ $\quad i=1,2,\dots,N^f$

F: $F_k, F_n>0$, $F_{kk},F_{nn}<0$, $F\in C^2$, CRS

$$F(k^i,n^i) = n^i F(\tfrac{k^i}{n^i},1) = n^i f(\tfrac{k^i}{n^i}) = k^i F(1,\tfrac{n^i}{k^i}) = k^i g(\tfrac{n^i}{k^i})$$

$F_k(k^i,n^i)=f'(\frac{k^i}{n^i})$ $\quad f'>0$ $\quad f''<0$ $\quad f(\theta)=\theta g(\frac{1}{\theta})$

$F_n(k^i,n^i) = g'(\frac{n^i}{k^i})$ $\quad g'>0$ $\quad g''<0$

Individual: $\pi^i = \max_{k^i,n^i} zF(k^i,n^i) - wn^i - (r+\delta)k^i$

FOC: $\begin{cases} r+\delta = zF_k(k^i,n^i) \\ w = zF_n(k^i,n^i)\end{cases}$ ⇒ $\frac{r+\delta}{w} = \frac{f'(\frac{k^i}{n^i})}{g'(\frac{n^i}{k^i})}$

$\frac{k}{n}\uparrow$ $\begin{cases} f'(\cdot)\downarrow \\ \frac{n}{k}\downarrow \quad g'(\cdot)\uparrow\end{cases}$ ⇒ $\frac{f'(\cdot)}{g'(\cdot)}\downarrow$

⇒ $\frac{r+\delta}{w} = \psi(\frac{k^i}{n^i})$ $\quad \psi'>0$ ⇒ $\frac{k^i}{n^i} = \psi^{-1}(\frac{r+\delta}{w})$ $\quad \forall i=1,\dots,N^f$

Aggregate: $\frac{k^i}{n^i}=\frac{K}{N}$

① $Y=\sum_{i=1}^{N^f} zF(k^i,n^i) = \sum_{i=1}^{N^f} zn^iF(\frac{k^i}{n^i},1) = zNF(\frac{K}{N},1) = zF(K,N)$

② $(K,N) = \arg\max zF(K,N) - wN - (r+\delta)K$

$$\begin{cases}(r+\delta) = zF_k(k^i,n^i) = zf'(\frac{k^i}{n^i}) = zf'(\frac{K}{N}) = zF_K(K,N) \\ w = zF_n(k^i,n^i) = zg'(\frac{n^i}{k^i}) = zg'(\frac{N}{K}) = zF_N(K,N)\end{cases}$$

Household: utility: $U(C^i) = \sum_{t=0}^{+\infty}\beta^t u(c_t^i)$ $\quad i=1,2,\dots,N^h$

u: $u'>0$, $u''<0$

wealth: $a_0^i = k_0>0$ $\quad$ labor: $(l_t)_{t\geq 0}$

Individual: $c^i(a_0^i) = \arg\max_{c^i}\sum_{t=0}^{+\infty}\beta^t u(c_t^i)$

s.t. $a_{t+1}^i = (1+r)(a_t^i - c_t^i + w_tl_t)$

No Ponzi $\quad a_t^i \geqslant \sum_{\tau=0}^{+\infty}\frac{1}{(1+r)^\tau} w_{t+\tau}l_{t+\tau}$

Aggregate: $C = N^hc^i$ $\quad A = N^ha^i$ $\quad L=N^hl$

$C = \arg\max_C \sum_{t=0}^{+\infty}\beta^t u(\frac{C_t}{N^h})$

s.t. $A_{t+1} = (1+r)(A_t - C_t + w_tL_t)$

No Ponzi

Gorman Aggregation (Heterogeneity on wealth & preference)

Aggregate C is a function on W

⇔ Affine Engel Curve: $c^i(p,w^i) = a^i(p)+b(p)w^i$

⇔ Indirect utility: $V^i(p,w^i) = \alpha^i(p)+\beta(p)w^i$

"⇒" $C = \sum_{i=1}^I c^i(p,w^i)$ $\quad \forall w=[w^1\cdots w^I]^T$ $\quad \forall dw = [dw^1\cdots dw^I]^T$

$\sum_i dw^i = 0$ ⇒ $dC_j = \sum_i \frac{dc_j^i(p,w^i)}{dw^i} = 0$ $\quad \forall j=1,\dots,M$

$\forall i,k$, $\frac{dc^i(p,w^i)}{dw^i} = \frac{dc^k(p,w^k)}{dw^k}$ $\quad (dw^i=\Delta,\ dw^k=-\Delta)$

$\begin{cases}\text{Fix } i: \forall w^i, \frac{dc_j^i(p,w^i)}{dw^i} = \text{Const}^i & \Rightarrow \text{linear } c^i \text{ on } w^i \\ \forall i,k: \text{Const}^i = \text{Const}^k & \Rightarrow \text{common coefficient}\end{cases}$

⇒ $c^i(p,w^i) = a^i(p)+b(p)w^i$ $\quad$ Affine Engel Curves

PDE: $0 = \frac{\partial V^i(p,w^i)}{\partial p} + \frac{\partial V^i(p,w^i)}{\partial w^i} c^i(p,w^i)$

Characteristic curve: $\frac{dV^i}{0} = \frac{dp}{1} = \frac{dw^i}{c^i}$

⇒ $V^i(p,w^i) = F(\alpha^i(p)+\beta(p)w^i)$

"⇐" Roy's Identity: $c^i(p,w^i) = \beta(p)^{-1}[\nabla\alpha^i(p)+\nabla\beta(p)w^i]$

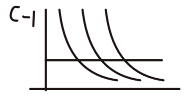
$C = \sum_i c^i(p,w^i) = \sum_i\beta(p)^{-1}\nabla\alpha^i(p) + \beta(p)^{-1}\nabla\beta(p)W$

Aggregate: $C(p,W) = \sum_i a^i(p) + b(p)W$ $\quad$ SARP

⇒ Rationalizable by complete & transitive $\succsim$

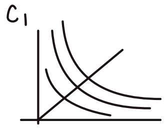
2 Special Cases:

① Quasi-linear utility: $u^i(c_1,c_{-1}) = \alpha c_1 + h^i(c_{-1})$



② Homothetic utility:

e.g. CES: $u^i(c) = [\sum_{j=1}^M \alpha_j c_j^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$



Rubinstein's Model

Individual: $\max_{\{c_t,\alpha_t\}_{t=0}^{+\infty}} \mathbb{E}_0\sum_{t=0}^T\beta^t u(c_t)$

s.t. $w_{t+1} = (w_t-c_t)[\alpha_tR_t^f + (1-\alpha_t)R_t^s]$

$w_{T+1}\geqslant 0$

Hyperbolic absolute risk aversion (HARA) u:

Risk tolerance: $T(c) := -\frac{u'(c)}{u''(c)} = \rho+\gamma c$, $\gamma\leq 1$

$u(c) = \zeta(\rho+\gamma c)^{\frac{\gamma-1}{\gamma}}$ $\quad (\zeta<0)$

CARA: $\gamma=0$, $\rho\neq 0$ $\quad u(c) = -\rho\exp\{-\frac{c}{\rho}\}$

CRRA: $\rho=0$, $\gamma\neq 0,1$ $\quad u(c) = \zeta\gamma^{\frac{\gamma-1}{\gamma}}c^{\frac{\gamma-1}{\gamma}}$

$\rho=0$, $\gamma=1$ $\quad u(c)=\ln c$

Euler: $u'(c_t) = \beta\max\{R_t^f,R_t^s\}\,\mathbb{E}_tu'(c_{t+1})$

⇒ $[\beta\max\{R_t^f,R_t^s\}]^{-1} = \mathbb{E}_t(\frac{\rho+\gamma c_{t+1}}{\rho+\gamma c_t})^{\frac{-1}{\gamma}}$

Guess & verify: $c_t = a(R_t)+b(R_t)w_t$

If all individuals have same:

❶ *β and taste parameters $\gamma\neq 0$.*

❷ *taste parameters $\gamma = 0$.*

❸ *resources w_0, taste parameters $\rho=0$ and $\gamma=1$.*

Then, aggregate consumption and asset demands are determined as if there is a composite RA with:

❶ *resources: $w_0 = \sum_{i=1}^I w_0^i/I$;*

❷ *tastes: $\sigma = \prod_{i=1}^I(\sigma^i)^{\frac{\rho^i}{\sum_{i=1}^I\rho_i}}$, $\sigma = 1/\beta - 1$ if $\rho\neq 0$ or $\beta=\sum_{i=1}^I\beta^i/I$ if $\rho=0$*

❸ *preference parameters: $\rho = \sum_{i=1}^I\rho^i/I$ and γ.*

Discrete Choice

Individual: $\max_{i\in I} u(c(i)) + \varepsilon(i)$ $\quad \varepsilon = [\varepsilon(1)\cdots\varepsilon(I)]\sim F$

Aggregate: $P_c(i) = \text{Prob}(u(c(i))+\varepsilon(i) \geqslant \max_j u(c(j))+\varepsilon(j))$

$=\int \frac{\partial F}{\partial\varepsilon_i}(u(c_i)-u(c_1)+\varepsilon_i,\dots,\varepsilon_i,\dots,u(c_i)-u(c_I)+\varepsilon_i)\,d\varepsilon_i$

F: independent Gumbel $F_i(\varepsilon_i) = \exp\{-e^{-\varepsilon_i}\}$

$P_c(i) = \int_{-\infty}^{+\infty}\exp\{-e^{-\sum_{j\neq i}(u_i-u_j)}e^{-\varepsilon_i}\}\,d\exp\{-e^{-\varepsilon_i}\}$

$= \int_0^1 x^A dx = \frac{1}{A+1} = \frac{\exp\{u_i\}}{\sum_j\exp\{u_j\}}$

$P_c = \arg\max_{P\in\Delta}\sum_i(u_i - \ln P_i)P_i$

Entropy fn: $-\sum_i \ln P_i\cdot P_i$ $\quad$ "Love of variety"

Social surplus: $W(\vec u) = \ln\sum_i\exp\{u_i\}$

If $u(c_i) = a\ln c$: $P_c(i) = \frac{c_i^a}{\sum_j c_j^a}$ $\quad$ "CES"

Negishi: ∃RA in complete market.

$U_{RA} = \sum_i \mu^i u^i$. Pareto weight μ^i coincides with w^i

CE: $V(p, a_0^i) = \max_{\{c_t^i\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t^i)$ s.t. $\sum_{t=0}^{+\infty} p_t c_t^i \le p_0 a_0^i$

FOC ⇒ $\frac{u'(c_t^i)}{u'(c_t^j)} = \frac{\pi^i}{\pi^j}$

$A_0 = \max_{\{k_{t+1}\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \frac{p_t}{p_0}[f(k_t)+(1-\delta)k_t - k_{t+1}]$

FOC: $1 = \frac{p_{t+1}}{p_t}[f'(k_{t+1})+1-\delta]$

Market clearing: $\sum_i c_t^i = f(k_t)+(1-\delta)k_t - k_{t+1}$

SP: $V^{NP} = \max_{\{c_t,k_{t+1}\}} \sum_i \mu^i \sum_{t=0}^{+\infty} \beta^t u(c_t^i)$ s.t. $\sum_i c_t^i = f(k_t)+(1-\delta)k_t-k_{t+1}$

FOC ⇒ $\frac{u'(c_t^i)}{u'(c_t^j)} = \frac{\mu^i}{\mu^j}$

Constantinides: 2 stage:

$V^{SP} = \max_{\{c_t,k_{t+1}\}} \sum_{t=0}^{+\infty}\beta^t \sum_i \mu^i \pi^i u(c_t^i)$ s.t. $\sum_i \pi^i c_t^i = f(k_t)+(1-\delta)k_t - k_{t+1}$

Intra-temporal: $U(C_t) = \max_{\{c_t^i\}} \sum_i \mu^i\pi^i u(c_t^i)$ s.t. $\sum_i \pi^i c_t^i \le C_t$

Inter-temporal: $V^{SP} = \max_{\{C_t,k_{t+1}\}} \sum_{t=0}^{+\infty}\beta^t U(C_t)$ s.t. $C_t = f(k_t)+(1-\delta)k_t-k_{t+1}$

Maliar: Idiosyncratic labor productivity shock $E\varepsilon_t^i = 1$

CE: HH: $\max_{\{c_t^i,h_t^i,k_{t+1}^i,a_{t+1}^i(\varepsilon)\}} E_0 \sum_{t=0}^{+\infty}\beta^t\left[\frac{(c_t^i)^{1-\sigma}}{1-\sigma} + \psi\frac{(1-h_t^i)^{1-\gamma}}{1-\gamma}\right]$

s.t. $c_t^i + k_{t+1}^i + \int_\varepsilon p_t(\varepsilon) a_{t+1}^i(\varepsilon) d\varepsilon \le w_t\varepsilon_t^i h_t^i + (1+r_t)k_t^i + a_t^i(\varepsilon_t^i)$

Firm: $\Pi_t = \max_{\{k_t,L_t\}} z_t F(k_t,L_t) - (r_t+\delta)k_t - w_t L_t$

MC: $k_t = \int k_t^i d\pi^i$

$L_t = \int \varepsilon_t^i h_t^i d\pi^i$

$\int c_t^i d\pi^i + k_{t+1} = z_t F(k_t,L_t)+(1-\delta)k_t$

$\int a_{t+1}^i(\varepsilon) d\pi^i = 0$

SP: $U(C_t,L_t) = \max_{\{c_t^i,h_t^i\}_{i\in I}} \int \mu^i\left[\frac{(c_t^i)^{1-\sigma}}{1-\sigma}+\psi\frac{(1-h_t^i)^{1-\gamma}}{1-\gamma}\right]d\pi^i$

s.t. $\int c_t^i d\pi^i \le C_t$. $\int \varepsilon_t^i h_t^i d\pi^i \le L$

FOC: $\mu^i (c_t^i)^{-\sigma} = \pi_t^c$ ⇒ $c_t^i = \left(\frac{\mu^i}{\pi_t^c}\right)^{\frac1\sigma}$

$-\mu^i\psi(1-h_t^i)^{-\gamma} = \pi_t^L \varepsilon_t^i$ ⇒ $h_t^i = 1 + \left(\frac{\psi\mu^i}{\pi_t^L\varepsilon_t^i}\right)^{\frac1\gamma}$

⇒ $C_t = \int c_t^i d\pi^i = (\pi_t^c)^{-\frac1\sigma}\int(\mu^i)^{\frac1\sigma}d\pi^i$

$L_t = \int \varepsilon_t^i h_t^i d\pi^i = 1 + \left(\frac{\psi}{\pi_t^c}\right)^{\frac1\gamma}\int(\varepsilon_t^i)^{1-\frac1\gamma}(\mu^i)^{\frac1\gamma}d\pi^i$

⇒ $c_t^i = A^i C_t$, $A^i = (\mu^i)^{\frac1\sigma}\left[\int(\mu^i)^{\frac1\sigma}d\pi^i\right]^{-1}$

$1-h_t^i = B_t^i(1-L_t)$, $B_t^i = (\mu^i)^{\frac1\gamma}(\varepsilon_t^i)^{-\frac1\gamma}\left[\int(\mu^i)^{\frac1\gamma}(\varepsilon_t^i)^{1-\frac1\gamma}d\pi^i\right]^{-1}$

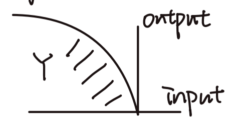
⇒ $U(C_t,L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} + \Psi_t\frac{(1-L_t)^{1-\gamma}}{1-\gamma}$

$\Psi_t := \psi\left[\int(\mu^i)^{\frac1\gamma}(\varepsilon_t^i)^{1-\frac1\gamma}d\pi^i\right]^\gamma$

If $\mu^i = 1$ $\forall i$, $\gamma>1$: $\Psi_t > \psi$ RA taste for h < Ind.

Production Aggregation (Competitive market)

Production possibility set $Y = \{y: F(y) < 0\} \subseteq \mathbb{R}^L$



Non-emptyness $Y \ne \emptyset$

Closedness $Y = cl(Y)$

Possibility of shut down $0\in Y$

No free lunch $Y\cap\mathbb{R}_+^L = 0$

Free disposal $Y - \mathbb{R}_+^L \subseteq Y$

Convexity/Concavity: De/Increasing Return to Scale

Single output: $Y = \{(q,-z_1,\dots,-z_{L-1}): q\le f(z_1,\dots,z_{L-1})\}$

Profit maximization: $\pi(p) = \max_{y\in Y} p\cdot y$ ⇒ $p = \lambda\nabla F(y^*)$

$\pi(p,w) = \max_z p f(z) - w\cdot z$ s.t. $z\ge0$

⇒ $p\frac{\partial f(z^*)}{\partial z_l} \le w_l$ "=" if $z_l^* > 0$

$y(p) = \sum_j y_j(p)$: $y(p) = \arg\max_{y\in Y} p\cdot y$. $Y = \sum_j Y_j$

$y_i(p) = \arg\max_{y\in Y_i} p\cdot y$

$\Pi(w) = \max_{\{\{k_{jn}\}_{n=1}^N\}_{j=1}^J} \sum_{j=1}^J f_j(\{k_{jn}\}_{n=1}^N) - \sum_{n=1}^N w_n\sum_{j=1}^J k_{jn}$

$= \max_{\{k_n\}_{n=1}^N} F(\{k_n\}_{n=1}^N) - \sum_{n=1}^N w_n k_n$

where $F(\{k_n\}_{n=1}^N) = \max_{k_{jn}}\sum_{j=1}^J f_j$ s.t. $\sum_{j=1}^J k_{jn}\le k_n$

Houthakker's Example: $f_j(x_1,x_2) = \min\left\{\frac{x_1}{a_{1j}},\frac{x_2}{a_{2j}}\right\}$

$p_0 - a_{1j}p_1 - a_{2j}p_2 > 0 \Leftrightarrow$ produce φ_j, $[x_{1j}\ x_{2j}] = [a_{1j}\ a_{2j}]\varphi_j$

Total output: $X_0 = \iint\{a_1p_1+a_2p_2<p_0\}\varphi(a_1,a_2)\,d(a_1,a_2)$

$= \int_0^{p_0/p_1}\int_0^{(p_0-a_1p_1)/p_2}\varphi(a_1,a_2)\,da_2\,da_1$

Total input: $X_i = \int_0^{p_0/p_1}\int_0^{(p_0-a_1p_1)/p_2} a_i\varphi(a_1,a_2)\,da_2\,da_1$

Pareto form: $\varphi(a_1,a_2) = Aa_1^{\alpha_1-1}a_2^{\alpha_2-1}$ $\alpha_1,\alpha_2\ge1$

$X_1 = \frac{Ap_0^{\alpha_1+\alpha_2+1}}{\alpha_2 p_1^{\alpha_1+1}p_2^{\alpha_2}}B(\alpha_1+1,\alpha_2+1)$

$X_0 = \frac{Ap_0^{\alpha_1+\alpha_2}}{\alpha_2p_1^{\alpha_1}p_2^{\alpha_2}}B(\alpha_1,\alpha_2+1) = \frac{\alpha_1+\alpha_2+1}{\alpha_1\alpha_2}\frac{Ap_0^{\alpha_1+\alpha_2}}{p_1^{\alpha_1}p_2^{\alpha_2}}B(\alpha_1+1,\alpha_2+1)$

$X_2 = \frac{Ap_0^{\alpha_1+\alpha_2+1}}{(\alpha_2+1)p_1^{\alpha_1}p_2^{\alpha_2+1}}B(\alpha_1,\alpha_2+2) = \frac{1}{\alpha_1+\alpha_2+1}\frac{Ap_0^{\alpha_1+\alpha_2}}{p_1^{\alpha_1}p_2^{\alpha_2+1}}B(\alpha_1+1,\alpha_2+1)$

⇒ $X_0\propto\tilde p_1^{-\alpha_1}\tilde p_2^{-\alpha_2}$

$X_1 \propto \tilde p_1^{-(\alpha_1+1)}\tilde p_2^{-\alpha_2}$

$X_2\propto\tilde p_1^{-\alpha_1}\tilde p_2^{-(\alpha_2+1)}$

• Beta: $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$

• $B(p+1,q) = \frac{p}{p+q}B(p,q)$

• $B(p,q+1) = \frac{q}{p+q}B(p,q)$

⇒ $X_0 \propto X_1^{\gamma_1}X_2^{\gamma_2}$

$$\begin{cases}\gamma_1(\alpha_1+1)+\gamma_2\alpha_1 = \alpha_1\\ \gamma_1\alpha_2+\gamma_2(\alpha_2+1) = \alpha_2\end{cases} \Rightarrow \begin{cases}\gamma_1 = \frac{\alpha_1}{\alpha_1+\alpha_2+1}\\ \gamma_2 = \frac{\alpha_2}{\alpha_1+\alpha_2+1}\end{cases}$$

Take care when calibrating macro model with micro data!

Hulten's Theorem: Perfectly Competitive Economies

Final: $\pi_0 = \max_{\{c_i\}_{i=1}^I} D(c_1,\dots,c_I) - \sum_{i=1}^I p_ic_i$

Interm: $\pi_i = \max_{\{l_{ij},x_{ij}\}} p_iA_iF(\{l_{ij}\}_j,\{x_{ij}\}_j) - \sum_{j=1}^I p_jx_{ij} - \sum_{j=1}^N w_jl_{ij}$

Grand: $\Pi = \max_{\{c_i,l_{ij},x_{ij}\}}\sum_{i=0}^I\pi_i$

$= \max\{D(c_1,\dots,c_I) + \sum_{i=1}^I p_i\{A_iF_i - c_i - \sum_{j=1}^I x_{ji}\} - \sum_{i=1}^I\sum_{j=1}^N w_jl_{ij}$

Envelope: $\frac{\partial\Pi}{\partial A_i} = p_iF_i - \sum_{i=1}^I\sum_{j=1}^N\frac{\partial w_j}{\partial A_i}l_{ij} + \sum_{i=1}^N\frac{\partial p_i}{\partial A_i}\underbrace{(A_iF_i - c_i - \sum x_{ji})}_{=0\ \text{Equilibrium}}$

Total income: $Y = \Pi + \sum_{i=1}^{I}\sum_{j=1}^{N} w_j l_{ij} \Rightarrow \frac{\partial Y}{\partial A_i} = \frac{\partial \Pi}{\partial A_i} + \sum_{i=1}^{I}\sum_{j=1}^{N}\frac{\partial w_j}{\partial A_i} l_{ij}$

$\Rightarrow \varepsilon = \frac{\partial \ln Y}{\partial \ln A_i} = \frac{P_i A_i F_i}{Y}$ = firm i's sale/GDP "Domar weight"

Production Aggregation with Distortion (Hseih & Klenow)

Final good: $\max_{y_i} P(\sum_{i=1}^N y_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} - \sum_{i\in I} P_i y_i$

FOC: $P Y^{\frac{1}{\sigma}} y_i^{-\frac{1}{\sigma}} = P_i \Rightarrow y_i = (\frac{P_i}{P})^{-\sigma} Y$

Cost: $C(Y) = \min_{y_i} \sum_{i\in I} P_i y_i$ s.t. $(\sum_{i=1}^N y_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} \ge Y$

FOC: $P_i = \lambda Y^{\frac{1}{\sigma}} y_i^{-\frac{1}{\sigma}}$

① $\Rightarrow P_i y_i = \lambda Y^{\frac{1}{\sigma}} y_i^{\frac{\sigma-1}{\sigma}} \Rightarrow C(Y) = \lambda Y$

② $\Rightarrow P_i^{1-\sigma} = \lambda^{1-\sigma} Y^{\frac{1-\sigma}{\sigma}} y_i^{\frac{\sigma-1}{\sigma}} \Rightarrow \sum_{i\in I} P_i^{1-\sigma} = \lambda^{1-\sigma}$

$\Rightarrow C(Y) = (\sum_{i\in I} P_i^{1-\sigma})^{\frac{1}{1-\sigma}} Y$

Aggregate input price index: $P = (\sum_{i\in I} P_i^{1-\sigma})^{\frac{1}{1-\sigma}}$

Interm: $\pi_i = \max_{k_i,l_i} (1-\tau_i^y) P_i z_i k_i^\alpha l_i^{1-\alpha} - (1+\tau_i^l) w l_i - (1+\tau_i^k) r k_i$

Cost: $C_i(y_i) = \min_{k_i,l_i} (1+\tau_i^l) w l_i + (1+\tau_i^k) r k_i$ s.t. $z_i k_i^\alpha l_i^{1-\alpha} \ge y_i$

$(1+\tau_i^k) r k_i = \alpha C_i(y_i)$

$(1+\tau_i^l) w l_i = (1-\alpha) C_i(y_i)$

$C_i(y_i) = m_i y_i$, $\frac{m_i}{1-\tau_i^y} = \frac{1}{z_i} \underbrace{(\frac{r}{\alpha})^\alpha (\frac{w}{1-\alpha})^{1-\alpha}}_{\psi} \underbrace{\frac{(1+\tau_i^k)^\alpha (1+\tau_i^l)^{1-\alpha}}{1-\tau_i^y}}_{w_i}$

Monopoly: $p(y) y = P Y^{\frac{1}{\sigma}} y^{\frac{\sigma-1}{\sigma}}$

Profit: $\pi_i = \max_{y_i} (1-\tau_i^y) P Y^{\frac{1}{\sigma}} y_i^{\frac{\sigma-1}{\sigma}} - m_i y_i$

Markup: $(1-\tau_i^y) P_i = \frac{\sigma}{\sigma-1} m_i \Rightarrow \pi_i = \frac{1}{\sigma} (1-\tau_i^y) P_i y_i$

$y_i = (\frac{P_i}{P})^{-\sigma} Y = P^\sigma Y (\frac{\sigma}{\sigma-1})^{-\sigma} (\frac{m_i}{1-\tau_i^y})^{-\sigma}$

$\Rightarrow \begin{cases} k_i = \frac{1-\tau_i^y}{1+\tau_i^k} (\frac{\psi w_i}{z_i})^{1-\sigma} \frac{\alpha}{r} P^\sigma Y \mu^{-\sigma} \\ l_i = \frac{1-\tau_i^y}{1+\tau_i^l} (\frac{\psi w_i}{z_i})^{1-\sigma} \frac{1-\alpha}{w} P^\sigma Y \mu^{-\sigma} \end{cases}$ $\mu := \frac{\sigma}{\sigma-1}$

Aggregation: $P = (\sum_{i\in I} P_i^{1-\sigma})^{\frac{1}{1-\sigma}} = \mu\psi (\sum_{i\in I} (\frac{w_i}{z_i})^{1-\sigma})^{\frac{1}{1-\sigma}}$

$\Rightarrow \begin{cases} K = \sum_i k_i = \frac{\alpha}{r} P^\sigma Y \psi^{1-\sigma}\mu^{-\sigma}[\sum_{i\in I} \frac{1-\tau_i^y}{1+\tau_i^k}(\frac{w_i}{z_i})^{1-\sigma}] \\ L = \sum_i l_i = \frac{1-\alpha}{w} P^\sigma Y \psi^{1-\sigma} \mu^{-\sigma} [\sum_{i\in I} \frac{1-\tau_i^y}{1+\tau_i^l}(\frac{w_i}{z_i})^{1-\sigma}] \end{cases}$

Shares $\frac{P_i Y_i}{PY} = (\frac{P_i}{P})^{1-\sigma} = \frac{(w_i/z_i)^{1-\sigma}}{\sum_{i\in I} (w_i/z_i)^{1-\sigma}}$

$\Rightarrow \frac{Y_i}{Y} = (\frac{P_iY_i}{PY})^{\frac{-\sigma}{1-\sigma}}$, $\frac{P_i}{P} = (\frac{P_iY_i}{PY})^{\frac{1}{1-\sigma}}$

$\Rightarrow \frac{K_i}{K} = \frac{\frac{1-\tau_i^y}{1+\tau_i^k}(\frac{w_i}{z_i})^{1-\sigma}}{\sum_{i\in I} \cdots}$ $\frac{L_i}{L} = \frac{\frac{1-\tau_i^y}{1+\tau_i^l}(\frac{w_i}{z_i})^{1-\sigma}}{\sum_{i\in I}\cdots}$

$\Rightarrow Y_i = z_i K_i^\alpha L_i^{1-\alpha} = \hat\Omega (\frac{w_i}{z_i})^{-\sigma} K^\alpha L^{1-\alpha}$

where $\hat\Omega^{-1} = [\sum_{i\in I} \frac{1-\tau_i^y}{1+\tau_i^k}(\frac{w_i}{z_i})^{1-\sigma}]^\alpha [\sum_{i\in I} \frac{1-\tau_i^y}{1+\tau_i^l}(\frac{w_i}{z_i})^{1-\sigma}]^{1-\alpha}$

$\Rightarrow Y = (\sum_{i\in I} Y_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} = Z K^\alpha L^{1-\alpha}$, $Z = \hat\Omega(\sum_{i\in I} (\frac{z_i}{w_i})^{\sigma-1})^{\frac{\sigma}{\sigma-1}}$

If $\Omega = \hat\Omega[\sum_i (\frac{w_i}{z_i})^{1-\sigma}]$ then $Y = \Omega(\sum_i (\frac{z_i}{w_i})^{\sigma-1})^{\frac{1}{\sigma-1}} K^\alpha L^{1-\alpha}$

TFP: $TFPQ_i = \frac{Y_i}{K_i^\alpha L_i^{1-\alpha}} = z_i$ Quantity-based

$TFPR_i = \frac{P_iY_i}{K_i^\alpha L_i^{1-\alpha}} = \mu\psi\frac{(1+\tau_i^k)^\alpha(1+\tau_i^l)^{1-\alpha}}{1-\tau_i^y}$ Revenue-based

No distortion $\Rightarrow \forall i,j.$ $TFPR_i = TFPR_j$

i distortion $\Rightarrow \tau_i > 0 \Rightarrow TFPR_i \uparrow$

CES Aggregation: $Y = (\sum_{i=1}^N y_i^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}$ $\sigma\in[0,+\infty]$

$\frac{y_i}{y_j} = (\frac{P_i}{P_j})^{-\sigma} \Rightarrow \varepsilon = \frac{d\ln(y_i/y_j)}{d\ln(P_i/P_j)} = -\sigma$

$\lim_{\sigma\to0} Y = \exp\{\lim_{\sigma\to0}\frac{\sigma}{\sigma-1}\ln(\sum_{i=1}^N y_i^{\frac{\sigma-1}{\sigma}})\} = \prod_{i=1}^N y_i^{\{\frac{y_i}{\min}\}^{-\infty}}$ = Min Leontief

$\lim_{\sigma\to1} Y = \prod_{i=1}^N y_i^{\{\frac{y_i^0}{\sum y_i^0}\}} = \prod_{i=1}^N y_i$ Cobb-Douglas

$\lim_{\sigma\to+\infty} Y = \sum_{i=1}^N y_i$ Linear

Homogenous of degree k: $G(tx) = t^k G(x)$

① $\sum_n x_n \frac{\partial G(x)}{\partial x_n} = kG(x)$

② $\frac{\partial G(tx)}{\partial x_n} = t^{k-1}\frac{\partial G(x)}{\partial x_n}$

Consumption

GDP accounting: $Y_t = C_t + I_t + G_t + NX_t$

Long run: C & GDP tracked each other closely.

Keynesian Model: $\begin{cases} Y_t = C_t + G_t \\ C_t = \alpha + \gamma(Y_t - T_t) \end{cases} \Rightarrow Y_t = \frac{1}{1-\gamma}(\alpha + G_t - \gamma T_t)$ $\quad \gamma$: MPC

Short run: C is smoother than GDP.

Permanent income hypothesis (PIH)

unexpected transitory Δ income $\xrightarrow{\text{weak}}$ C_t

unexpected persistent Δ income $\xrightarrow{\text{strong}}$

Buffer stock models ① borrowing constraint

② precautionary motive for saving

Preference: $U^i(\{c_t^i(s^t)\}) = \sum_{t=0}^{+\infty}\beta^t \sum_{s^t\in S^t}\pi(s^t)\, u^i(c_t^i(s^t))$

Autarky: $c_t^i(s^t) = y_t^i(s^t)$

Complete Market:

① Arrow-Debreu contingent claim "time-0 market"

$\sum_{t=0}^{+\infty}\sum_{s^t\in S^t} P_t(s^t)\, c_t^i(s^t) \le \sum_{t=0}^{+\infty}\sum_{s^t\in S^t} P_t(s^t)\, y_t^i(s^t)$

② Arrow securities "sequential markets"

$c_t^i(s^t) + \sum_{s_{t+1}\in S} q_t(s^t, s_{t+1})\, a_{t+1}^i(s^t, s_{t+1}) \le y_t^i(s^t) + a_t^i(s^t)$

No Ponzi: $\lim_{t\to+\infty} \prod_{j=0}^{t} q_j(s^{j+1})\, a_{t+1}^i(s^{t+1}) \ge 0$

Allocation: $\max U^i(\{c_t^i(s^t)\})$ s.t. "time-0 market"

FOC: $\pi(s^t)\, u^{i\prime}(c_t^i(s^t)) = \lambda^i P_t(s^t) \Rightarrow \frac{u^{i\prime}(c_t^i)}{u^{j\prime}(c_t^j)} = \frac{\lambda^i}{\lambda^j}$

$\Rightarrow c_t^j = u^{j\prime -1}\left[\frac{\lambda^j}{\lambda^i}\, u^{i\prime}(c_t^i)\right]$

Good market clear $Y_t(s^t) = \sum_{j\in I} u^{j\prime -1}\left[\frac{\lambda^j}{\lambda^i}\, u^{i\prime}(c_t^i)\right]$

$u^i(c) = \frac{c^{1-\sigma}}{1-\sigma} \Rightarrow c_t^i(s^t) = \frac{\lambda^{i\frac{1}{\sigma}}}{\sum_j \lambda^{j\frac{1}{\sigma}}}\, C_t(s^t) = \frac{\lambda^{i\frac{1}{\sigma}}}{\sum_j \lambda^{j\frac{1}{\sigma}}}\, Y_t(s^t)$

Empirical Result: $\ln\left(\frac{c_t^i}{c_{t-1}^i}\right) = b_1\ln\left(\frac{C_t}{C_{t-1}}\right) + b_2\ln\left(\frac{Y_t^i}{Y_{t-1}^i}\right) + \varepsilon_t$

Autarky: $b_1 = 0$. $b_2 = 1$ ✗ If common β is true

Complete Market: $b_1 = 1$. $b_2 = 0$ ✗ Partial risk sharing!

Asset Market Restriction

1. Non-contingent asset: $c_t^i(s^t) + q_t(s^t)\, a_{t+1}^i(s^t) \le y_t^i(s^t) + a_t^i(s^{t-1})$

or $c_t + \frac{1}{1+r_t}a_{t+1} \le y_t + a_t$

$a_t(s^{t-1}) \ge -y_t + \frac{1}{1+r_t}a_{t+1} \ge -y_t - \frac{1}{1+r_t}y_{t+1} + \frac{1}{(1+r_t)(1+r_{t+1})}a_{t+2}$

$\ge -\sum_{\Delta t=0}^{+\infty}\prod_{s=1}^{\Delta t}\frac{1}{1+r_{t+s}}\, y_{t+\Delta t} + \lim_{T\to+\infty}\prod_{s=1}^{T}\frac{1}{1+r_{t+s}}\, a_{t+T+1}$

$\Downarrow$ No Ponzi $\Leftrightarrow$ sequence of borrowing constraint

$a_t(s^{t-1}) \ge \underline{a}(s^{t-1}) := -\min_{s^\infty \succ s^{t-1}} \sum_{\Delta t=0}^{+\infty}\prod_{s=1}^{\Delta t}\frac{1}{1+r_{t+s}}\, y_{t+\Delta t}$

$\Downarrow a_t(s^{t-1}) \ge -\sum_{\Delta t=0}^{+\infty}\left(\frac{1}{1+r_{max}}\right)^{\Delta t} y_{min} = -\frac{1+r_{max}}{r_{max}}\, y_{min}$ Natural Debt Limit

$\Downarrow a_t(s^{t-1}) \ge -\phi$ "ad hoc limit"

Canonical Consumption Savings Problem

Sequential: $V_0^i(a_0^i, y_0^i) = \max_{\{c_t^i, a_{t+1}^i\}_{t=0}^{+\infty}} \mathbb{E}_0\left[\sum_{t=0}^{+\infty}\beta^t u(c_t^i)\right]$

s.t. $c_t^i + \frac{1}{1+r_t}a_{t+1}^i \le y_t^i + a_t^i$

$a_{t+1}^i \ge \underline{a}_{t+1}^i$, $\quad c_t^i \ge 0$

Inada $\Rightarrow c_t^i \ge 0$. Quadratic, CARA violates Inada!

Recursive: $V_t(a_t, y_t) = \max_{c_t, a_{t+1}} u(c_t) + \beta\,\mathbb{E}_t V_{t+1}(a_{t+1}, y_{t+1})$

↑ (Markov y_t) $\quad$ s.t. $c_t + \frac{1}{1+r_t}a_{t+1} \le y_t + a_t$

$a_{t+1} \ge \underline{a}_{t+1}$, $\quad c_t \ge 0$

Euler: $u'(c_t) \ge \beta(1+r_t)\,\mathbb{E}_t u'(c_{t+1})$. "=" if $a_{t+1} > \underline{a}_{t+1}$

Strict Permanent Income Hypothesis (PIH)

Assume ① $u(c) = -\frac{\alpha}{2}(c-\bar{c})^2$ $\quad\bar{c}$: bliss point

② $r_t = r = \frac{1-\beta}{\beta}$ i.e. $\beta(1+r) = 1$

Submartingale $c_t \le \mathbb{E}_t c_{t+1}$

Without borrowing constraint: $c_t = \mathbb{E}_t c_{t+1} = \mathbb{E}_t c_{t+\Delta t}$

Aggregated budget: $\sum_{\Delta t=0}^{+\infty}\left(\frac{1}{1+r}\right)^{\Delta t} c_{t+\Delta t} = \sum_{\Delta t=0}^{+\infty}\left(\frac{1}{1+r}\right)^{\Delta t} y_{t+\Delta t} + a_t$

$\Rightarrow \frac{1+r}{r}c_t = \sum_{\Delta t=0}^{+\infty}\left(\frac{1}{1+r}\right)^{\Delta t}\mathbb{E}_t(c_{t+\Delta t}) = \underbrace{\sum_{\Delta t=0}^{+\infty}\left(\frac{1}{1+r}\right)^{\Delta t}\mathbb{E}_t y_{t+\Delta t} + a_t}_{\text{total wealth } a_t+h_t}$

$\Rightarrow c_t = \frac{r}{1+r}(a_t + h_t)$ permanent income

Only expectation matters. No need for higher moments y_t

Dynamics:

$\Delta c_t = c_t - c_{t-1} = c_t - \mathbb{E}_{t-1}(c_t) = \frac{r}{1+r}[h_t - \mathbb{E}_{t-1}h_t]$

$= \frac{r}{1+r}\sum_{\Delta t=0}^{+\infty}\left(\frac{1}{1+r}\right)^{\Delta t}[\mathbb{E}_t y_{t+\Delta t} - \mathbb{E}_{t-1} y_{t+\Delta t}]$

$\Delta a_{t+1} = a_{t+1} - a_t = r a_t + (1+r)(y_t - c_t) = r a_t + (1+r)y_t - r(a_t + h_t)$

$= (1+r)\left[y_t - \frac{r}{1+r}h_t\right] = -r\sum_{\Delta t=0}^{+\infty}\left(\frac{1}{1+r}\right)^{\Delta t}\mathbb{E}_t(y_{t+\Delta t} - y_t)$

unexpected transitory $y_t > y$: $\quad y_t = y + \varepsilon_t$, ε_t iid.

$\Delta c = \frac{r}{1+r}[y_t - y]$ $\qquad \Delta a_{t+1} = y_t - y$

unexpected persistent $y_t > y$: $\quad y_{t+1} = y_t + \varepsilon_{t+1}$. ε_{t+1} iid

$\Delta c = y_t - y$ $\qquad \Delta a_{t+1} = 0$

unexpected persistent increase: $y_{t+j} - y_t = (1+g)^{j-1}$ $\quad g < r$

$\Delta c = \frac{r}{1+r}\frac{1}{r-g}$ $\qquad \Delta a_{t+1} = -\frac{r}{r-g}$

Without $\beta(1+r) = 1$: Euler: $\mathbb{E}_t c_{t+1} = \frac{c_t}{\beta(1+r)} + \left[1 - \frac{1}{\beta(1+r)}\right]\bar{c}$

$\Rightarrow \mathbb{E}_t c_{t+\Delta t} = \left[\frac{1}{\beta(1+r)}\right]^{\Delta t} c_t + Const.$

$\Rightarrow c_t = \left[1 - \frac{1}{\beta(1+r)^2}\right](a_t + h_t) + Const(r)$

$r\uparrow \quad \uparrow \quad \downarrow$

Empirical Result:

1. Excess sensitivity: $c_{t+1} = c_t + \varepsilon_{t+1}$ $\quad E(\varepsilon_{t+1}|\mathcal{F}_t) = 0$

Hall: $c_{t+1} = \gamma_0 + \gamma_1 c_t + \gamma_2 z_t + \varepsilon_{t+1}$. $\quad \gamma_1 = 1$ but $\gamma_2 \ne 0$

Flavin: $\Delta c_t = \mu_0 + \mu_1 z_{t-1}$ $\quad \mu_1 \ne 0$

Deaton: $\Delta c_t = 11.39 + 0.121^* \Delta y_{t-1}$ lagged income growth

2. Time aggregation: $\overbrace{\tau-4 \quad \tau-3}^{t-2}\ \overbrace{\tau-2 \quad \tau-1}^{t-1}\ \overbrace{\tau \quad \tau+1}^{t}$

$\Delta c_t^A = (c_\tau + c_{\tau+1}) - (c_{\tau-1} + c_{\tau-2}) = \Delta c_{\tau+1} + 2\Delta c_\tau + \Delta c_{\tau-1}$

Even though $Corr(\Delta c_{\tau-1}, \Delta c_\tau) = 0$. $Corr(\Delta c_t^A, \Delta c_{t+1}^A) \ne 0$

$\Delta y_{t-1}^A = \Delta y_{\tau-1} + 2\Delta y_{\tau-2} + \Delta y_{\tau-3} \Rightarrow Corr(\Delta c_t^A, \Delta y_{t-1}^A) \ne 0$

$Corr(\Delta c_\tau, \Delta y_{\tau-j}) = 0$ but $Corr(\Delta c_\tau, \Delta y_\tau) \ne 0$

Instrument $(\Delta c_{t-2}, \Delta y_{t-2})$: $\Delta c_t = 10.63 + 0.74^* \Delta y_{t-1}$ Sensitivity!

3. Predictable Income Change $\quad$ Households $\begin{cases} \lambda & \text{hand-to-mouth} \\ 1-\lambda & \text{PIH} \end{cases}$

$\Delta c_t = \lambda \Delta y_t + (1-\lambda)\varepsilon_t$

Instrument $(\Delta y_{t-2-j})_{j=1}^{6}$ $\quad \Delta c_t = \mu + 0.506^* \Delta y_t$

$\Downarrow$

$\lambda \approx \frac{1}{2}$

4. Excess Smoothness

Transitory income shock: $y_t = \varepsilon_t + \gamma\varepsilon_{t-1}$

$$\Delta C_t = \frac{r}{1+r}\sum_{j=0}^{+\infty}\left(\frac{1}{1+r}\right)^j (E_t y_{t+j} - E_{t-1} y_{t+j})$$
$$= \frac{r}{1+r}\left[\varepsilon_t + \frac{1}{1+r}\gamma\varepsilon_t\right] = \frac{r}{1+r}\left(1+\frac{\gamma}{1+r}\right)\varepsilon_t$$

Permanent income shock: $\Delta y_t = \alpha + \gamma\Delta y_{t-1} + \varepsilon_t$

$$\Delta C_t = \frac{r}{1+r}\sum_{j=0}^{+\infty}\left(\frac{1}{1+r}\right)^j\left(\sum_{s=0}^{j}\gamma^s\right)\varepsilon_t = \frac{1+r}{1+r-\gamma}\varepsilon_t$$

Empirical: $\sigma(\Delta C_t) < \sigma(\Delta y_t)$. Excess smoothness puzzle.

Marginal propensity to consume: Empirical

① predicted. MPC $=0$. MPC:

② unpredicted transitory. MPC >0 .15 ~ .25

③ unpredicted permanent MPC $>>0$

2.1 Quasi-experimental evid.: episodes. government shut-down

2.2 survey 2.3 semistructural methods

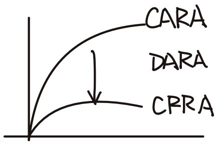
Arrow-Pratt measure: $A(c) = -\frac{u''(c)}{u'(c)}$ (ARA)

Pratt's gamble: ① $u(C-\pi) = E_Y u(C+Y)$. $EY=0$

$$\pi(Y,C) = -\frac{u''(C)}{u'(C)}\frac{Var(Y)}{2} = ARA(C)\frac{Var(Y)}{2}$$

② $u((1-\pi)C) = E_Y u(C+Y)$

$$\pi(Y,C) = -\frac{u''(C)C}{u'(C)}\frac{Var(Y)}{2} = RRA(C)\frac{Var(Y)}{2}$$



$u(c) = -\frac{1}{\alpha}\exp\{-\alpha c\}$. $ARA(c) = \alpha$

$A'(c) = -\frac{u'''(c)}{u'(c)} + \left(\frac{u''(c)}{u'(c)}\right)^2 < 0 \Rightarrow \boxed{u''' > 0}$ prudence

$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $RRA(c) = \gamma$

DARA: Leland 1968

$V(y_0) = \max_{c_0, c_1, a_1} u(c_0) + \beta E_0(u(c_1))$ s.t. $c_0 + a_1 = y_0$

① 2-Period case $c_1 = y_1 + (1+r)a_1$

$\Rightarrow V(y_0) = \max_{a_1} u(y_0 - a_1) + \beta E_0 u(y_1 + (1+r)a_1)$ $(\beta(1+r)=1)$

Euler: $u'(y_0 - a_1) = \beta(1+r) E u'(y_1 + (1+r)a_1)$

Var $y_1 \uparrow$ $E u'(y_1+(1+r)a_1) \uparrow$

$\Rightarrow a_1 \uparrow$ precautionary saving

② T-period case:

$V_t(X_t) = \max_{a_{t+1}} u(X_t - a_{t+1}) + \beta E V_{t+1}(y_{t+1} + (1+r)a_{t+1})$

Euler: $u'(X_t - a_{t+1}) = \beta(1+r) E V'_{t+1}(y_{t+1} + (1+r)a_{t+1})$

Backward: $V_T(X_T) = u(X_T)$

$u''' > 0 \Rightarrow V_T''' > 0 \overset{Euler}{\Rightarrow} V_t''' > 0$

CRRA: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$

Euler: $C_t^{-\gamma} = \beta E_t(1+r_{t+1})C_{t+1}^{-\gamma} \Leftrightarrow E_t\left[\beta(1+r_{t+1})\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} - 1\right] = 0$

GMM to estimate $\theta = (\beta, \gamma)$: $\max_\theta m(\theta)^T W m(\theta)$

where $m(\theta) = \frac{1}{n}\sum_{t=1}^{n} g(Y_t;\theta)$, $g(Y_t;\theta) = [\quad -1]\cdot z_t$

Euler Equation: $\beta E_t\left\{(1+r_{t+1})\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right\} = 1$

1. Covariance: Asset returns

2. 1st log-linearity on $(\ln(1+r_{t+1}), \ln\frac{C_{t+1}}{C_t})$

$\ln\beta + \ln E_t\left\{\exp\left\{\ln(1+r_{t+1}) - \gamma\ln\frac{C_{t+1}}{C_t}\right\}\right\} = 0$

$\Rightarrow \ln\beta + E_t\ln(1+r_{t+1}) - \gamma E_t\ln\frac{C_{t+1}}{C_t} = 0$

$\Rightarrow E_t\ln\frac{C_{t+1}}{C_t} = \frac{1}{\gamma}(E_t r_{t+1} - \rho)$. $\rho = \frac{1}{\beta} - 1$

γ: $\begin{cases}\text{intertemporal elasticity of substitution} \\ \text{coefficient of relative risk aversion.}\end{cases}$

3. Distributional assumptions. $X \sim \log N \Leftrightarrow \ln X \sim N$

$1 = \beta E_t\left\{\exp^{\ln(1+r_{t+1}) - \gamma\ln\frac{C_{t+1}}{C_t}}\right\}$

$$\begin{bmatrix}\ln(1+r_{t+1}) \\ \ln\frac{C_{t+1}}{C_t}\end{bmatrix} \sim N\left(\begin{bmatrix}E_t\ln(1+r_{t+1}) \\ E_t\ln\frac{C_{t+1}}{C_t}\end{bmatrix}, \begin{bmatrix}\sigma^2_{r,t} & \sigma_{cr,t} \\ \sigma_{cr,t} & \sigma^2_{c,t}\end{bmatrix}\right)$$

$\ln X \sim N(\mu, \sigma^2) \Rightarrow EX = \exp\{\mu + \frac{1}{2}\sigma^2\}$

$\Rightarrow 1 = \beta\exp\left\{E_t\ln(1+r_{t+1}) - \gamma E_t\ln\frac{C_{t+1}}{C_t} + \frac{1}{2}\sigma^2_{r,t} + \frac{\gamma^2}{2}\sigma^2_{c,t} - \gamma\sigma_{cr,t}\right\}$

$\Rightarrow E_t\ln\frac{C_{t+1}}{C_t} = \frac{1}{\gamma}(E_t r_{t+1} - \rho) + \frac{\gamma}{2}\sigma^2_{c,t} + \frac{1}{2\gamma}\sigma^2_{r,t} - \sigma_{cr,t}$

For riskless asset: $E_t\ln\frac{C_{t+1}}{C_t} = \frac{1}{\gamma}(r_{t+1} - \rho) + \frac{\gamma}{2}\sigma^2_{c,t}$

Buffer Stock Model: Borrowing constraints

$\begin{cases}\text{Budget: } C_t = y_t + a_t - \frac{1}{1+r}a_{t+1} \\ \text{Euler: } C_t = E_t C_{t+1} - \pi.\ \pi a_{t+1} = 0.\ \pi \geq 0.\ a_{t+1} \geq 0\end{cases}$ (Quadratic u)

$C_t = \begin{cases} E_t C_{t+1} < y_t + a_t & \text{if } a_{t+1} > 0 \\ y_t + a_t \leq E_t C_{t+1} & \text{if } a_{t+1} = 0\end{cases} = \min\{y_t + a_t, E_t C_{t+1}\}$

Liquidity constraints generates precautionary savings

Life Cycle (Aging) in Buffer Stock Model

born at t: $V_t^0(a_0, y_0) = \max_{\{c_s, a_{s+1}\}_{s=0}^{S}} E_0\left\{\sum_{s=0}^{S}\beta^s u^s(c_s) + \beta^S W(a_S)\right\}$

s.t. $c_s + \frac{1}{1+r_{t+s+1}}a_{s+1} \leq y_s + a_s$

$a_{s+1} \geq \underline{a}^{s+1}_{t+s+1}$ $c_s \geq 0$

Survival probability: δ_s

$V_t^s(a, y) = \max_{c, a'} u^s(c) + \beta\left\{\delta_s E_t V_{t+1}^{s+1}(a', y') + (1-\delta_s)W(a')\right\}$

s.t. $c + \frac{1}{1+r_{t+1}}a' \leq y + a$

$a' \geq \underline{a}^s_{t+1}$ $c \geq 0$

Perpetual youth: $\delta_s = \delta < 1$, $W(a') = 0$

Dynasty: $\delta_s = 1$. $s < S$ $\delta_s = 0$. $s = S$ $W(a') = V_{t+1}^{s+1}(a, y)$

Buffer: Euler: $u'(C_t) = \beta(1+r)E_t u'(C_{t+1}) + \pi$, $\pi a_{t+1} = 0$

Convergence?	$\beta(1+r) > 1$	$\beta(1+r) = 1$	$\beta(1+r) < 1$
Deterministic	X	√	√
Stochastic	X	X	√