Neoclassical Model with Uncertainty

Production: $y_t = z_t F(h_t, k_t)$. $z_{t+1} = z(z_t, \varepsilon_{t+1})$ stationary

Sequential: $V_0 = \max_{\{c_t, h_t, k_{t+1}\}_{z^t}} \sum_{t=0}^{+\infty} \beta^t \sum_{z^t \in Z^t} \pi_0(z^t)\, u(c_t(z^t), h_t(z^t))$

s.t. $c_t(z^t) + k_{t+1}(z^t) = z_t F(h_t(z^t), k_t(z^{t-1})) + (1-\delta) k_t(z^{t-1}) \quad \forall z^t$

DP: $V(k_t, z_t) = \max_{\{c_t, h_t, k_{t+1}\}} u(c_t, 1-h_t) + \beta E_t V(k_{t+1}, z_{t+1})$

s.t. $c_t + k_{t+1} = z_t F(h_t, k_t) + (1-\delta) k_t$

$z_{t+1} = z(z_t, \varepsilon_{t+1})$

FOC: $u_1(c_t, 1-h_t)\, z_t F_h(h_t, k_t) = u_2(c_t, 1-h_t)$

$u_1(c_t, 1-h_t) = \beta E_t V_1(k_{t+1}, z_{t+1})$

Envelope: $V_1(k_t, z_t) = u_1(c_t, 1-h_t)\, z_t F_k(h_t, k_t)$

Euler $u_1(c_t, 1-h_t) = \beta E_t u_1(c_{t+1}, 1-h_{t+1})\, z_{t+1} F_k(h_{t+1}, k_{t+1})$

Example: $F(h,k) = k^\theta h^{1-\theta}$, $\delta = 1$, $u(c, 1-h) = \ln c$, $h = 1$

Euler: $\frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}} z_{t+1} \theta k_{t+1}^{\theta-1} = \beta \frac{\theta}{k_{t+1}} E_t \frac{z_{t+1} k_{t+1}^\theta}{c_{t+1}}$

$\Rightarrow \frac{y_t}{c_t} = 1 + \theta\beta E_t \frac{y_{t+1}}{c_{t+1}}$

Suppose $\frac{c_t}{y_t} = x$ constant. then $c_t = (1-\theta\beta) y_t = (1-\theta\beta) z_t k_t^\theta$

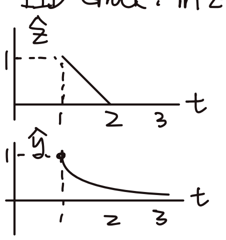
$k_{t+1} = \theta\beta y_t = \theta\beta z_t k_t^\theta$

$\ln k_{t+1} = \ln\theta\beta + \ln z_t + \theta \ln k_t$

$\ln y_{t+1} = \theta \ln y_t + \theta \ln\theta\beta + \ln z_{t+1}$

Brock-Mirman

IID shock: $\ln z_{t+1} = \ln \bar{z} + \varepsilon_{t+1}$. $\varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon)$



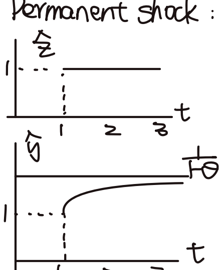
$\varepsilon_{t=1} = 1$, $\varepsilon_{t>1} = 0$ Impulse

$\hat z_t = \varepsilon_t$

$\hat y_{t+1} = \theta \hat y_t + \varepsilon_{t+1} \Rightarrow \hat y_{t+1} = \theta^t$

Impulse shock has permanent impact (shrinking)

Permanent shock: ① $\ln z_{t+1} = \ln z_t + \varepsilon_{t+1}$ $\varepsilon \overset{iid}{\sim} N(0, \sigma_\varepsilon)$

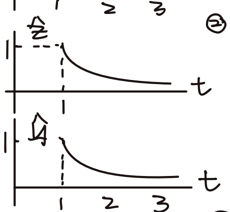


$\varepsilon_{t=1} = 1$, $\varepsilon_{t>1} = 0$

$\hat z_{t+1} = \hat z_t + \varepsilon_{t+1} \Rightarrow \hat z_t = 1$

$\hat y_{t+1} = \theta \hat y_t + \hat z_{t+1} \Rightarrow \hat y_{t+1} = \sum_{t=0}^{t} \theta^t$

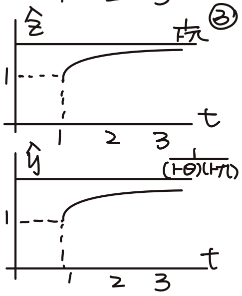
Permanent shock leads to persistent growth



② $\ln z_{t+1} = \pi \ln z_t + \varepsilon_{t+1}$. $\pi \in (0,1)$

$\hat z_{t+1} = \pi \hat z_t + \varepsilon_{t+1} \Rightarrow \hat z_t = \pi^{t-1}$

$\hat y_{t+1} = \theta \hat y_t + \hat z_{t+1} \Rightarrow \hat y_{t+1} = \sum_{\tau=0}^{t} \theta^{t-\tau} \pi^{\tau} = \frac{\theta^{t+1} - \pi^{t+1}}{\theta - \pi}$



③ $\ln\left(\frac{z_{t+1}}{z_t}\right) = \pi \ln\left(\frac{z_t}{z_{t-1}}\right) + \varepsilon_{t+1}$. $\pi \in (0,1)$

$\left(\widehat{\frac{z_{t+1}}{z_t}}\right) = \pi \left(\widehat{\frac{z_t}{z_{t-1}}}\right) + \varepsilon_{t+1} \Rightarrow \left(\widehat{\frac{z_t}{z_{t-1}}}\right) = \pi^{t-1}$

$\Rightarrow \hat z_t = \hat z_{t-1} + \pi^{t-1} \Rightarrow \hat z_t = \sum_{\tau=0}^{t-1} \pi^\tau$

$\left(\widehat{\frac{y_{t+1}}{y_t}}\right) = \theta \left(\widehat{\frac{y_t}{y_{t-1}}}\right) + \left(\widehat{\frac{z_{t+1}}{z_t}}\right) \Rightarrow \left(\widehat{\frac{y_{t+1}}{y_t}}\right) = \frac{\theta^{t+1} - \pi^{t+1}}{\theta - \pi}$

$\Rightarrow \hat y_{t+1} = \hat y_t + \frac{\theta^{t+1} - \pi^{t+1}}{\theta - \pi} \Rightarrow \hat y_t = \frac{1}{\theta - \pi}\left[\frac{\theta(1-\theta^t)}{1-\theta} - \frac{\pi(1-\pi^t)}{1-\pi}\right]$

More fluctuation in y: $\begin{cases} \text{more response in } k \ (\delta\downarrow) \\ \text{response from another input } L \end{cases}$

2 country model with real business cycle RBC

1992

1 good model: Cantor & Mark 1988. Backus & Kehoe & Kydland

Planner: $V_0 = \max \sum_{t=0}^{+\infty} \sum_{s^t \in S^t} \pi(s^t) \beta^t \{\pi_1 u(c_1(s^t)) + \pi_2 u(c_2(s^t))\}$

s.t. $c_1(s^t) + c_2(s^t) + i_1(s^t) + i_2(s^t) = z_1(s^t) F(k_1(s^{t-1})) + z_2(s^t) F(k_2(s^{t-1}))$

$k_1(s^t) = (1-\delta) k_1(s^{t-1}) + i_1(s^t)$

$k_2(s^t) = (1-\delta) k_2(s^{t-1}) + i_2(s^t)$

$[z_1(s^t) \ \ z_2(s^t)]^T = A [z_1(s^{t-1}) \ \ z_2(s^{t-2})]^T + \varepsilon_z$. $\sim N(0, V_\varepsilon)$

DP: $V(X_t) = \max_{a_t} \{\pi_1 u(c_{1t}) + \pi_2 u(c_{2t}) + \beta E_t V(X_{t+1})\}$

s.t. $c_{1t} = z_{1t} F(k_{1t}) + z_{2t} F(k_{2t}) + (1-\delta_1) k_{1t} - k_{1t+1} + (1-\delta_2) k_{2t} - k_{2t+1} - c_{2t}$

$[z_1(s^t) \ \ z_2(s^t)]^T = A [z_1(s^{t-1}) \ \ z_2(s^{t-2})]^T + \varepsilon_z$.

state: $X_t = (k_{1t}, k_{2t}, z_{1t}, z_{2t})$ control: $a_t = (k_{1t+1}, k_{2t+1}, c_{2t})$

FOC: $\pi_1 u'(c_{1t}) = \pi_2 u'(c_{2t})$

$\pi_1 u'(c_{1t}) = \beta E_t \frac{\partial V}{\partial k_1'}$

$\pi_1 u'(c_{1t}) = \beta E_t \frac{\partial V}{\partial k_2'}$

Envelope: $\frac{\partial V}{\partial k_1} = (z_1 f(k_1) + 1 - \delta_1) \pi_1 u'(c_{1t})$

$\frac{\partial V}{\partial k_2} = (z_2 f(k_2) + 1 - \delta_2) \pi_1 u'(c_{2t})$

Euler: $u'(c_1) = \beta E[z_1' f(k_1') + 1 - \delta_1] u'(c_1')$

$u'(c_2) = \beta E[z_2' f(k_2') + 1 - \delta_2] u'(c_2')$

$\Rightarrow \pi_1 u'(c_{1t}) = \pi_2 u'(c_{2t})$ Consumption smoothing

$E[z_1' f'(k_1') + 1 - \delta_1] \pi_1 u'(c_1') = E[z_2' f'(k_2') + 1 - \delta_2] \pi_2 u'(c_2')$ production shifting

trade balance $tb_{it} = y_{it} - c_{it} - i_{it} = 0$ in closed economies

Countercyclical: $corr(tb_{it}, y_{it}) < 0$ if CS > PS

Procyclical: $corr(tb_{it}, y_{it}) > 0$ if CS < PS

Example: $F(k) = k^\theta$. $\delta = 1$. $\pi_1 = \pi_2 = 1$, $u(c) = \ln c$, z_1, z_2 symmetric

$\Rightarrow c_1 = c_2$. Euler: $\frac{k_1'}{c_1} = \theta\beta E \frac{z_1' k_1'^\theta}{c_1'}$, $\frac{k_2'}{c_2} = \theta\beta E \frac{z_2' k_2'^\theta}{c_2'}$ $\Rightarrow c_1 = \frac{1}{2}(1-\theta\beta)[z_1 k_1^\theta + z_2 k_2^\theta]$

Symmetric z_1 & $z_2 \Rightarrow k_1' = k_2' \Rightarrow k_1' = \frac{1}{2}\theta\beta[z_1 k_1^\theta + z_2 k_2^\theta]$

$tb_{it} = y_{it} - c_{it} - k_{it}' = \frac{1}{2}(z_{it} k_{it}^\theta - z_{jt} k_{jt}^\theta)$

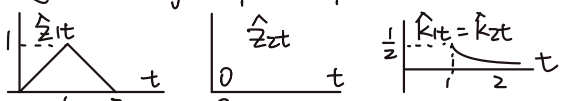
$\hat k_{it+1} = \frac{1}{2}[\hat z_{it} + \theta \hat k_{it} + \hat z_{jt} + \theta \hat k_{jt}]$

$1 + \hat{tb}_{it} = \frac{1}{2}(\theta\beta)^{\frac{\theta}{1-\theta}} [\hat z_{it} - \hat z_{jt} + \theta(\hat k_{it} - \hat k_{jt})]$

$\hat y_{it} = \hat z_{it} + \theta \hat k_{it}$

log-linearization at $k_1 = k_2 = k^* = (\theta\beta)^{\frac{1}{1-\theta}}$, $z_1 = z_2 = z^* = 1$

Purely transitory, surprise 1-period increase in z in home country.

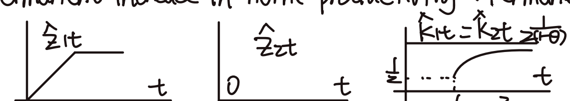


$\hat k_{1t} = \frac{1}{2}\theta^{t-1}$

$\hat k_{1t} + \hat k_{2t} = \theta^{t-1}$

$1 + \hat{tb}_{it} = \begin{cases} \frac{1}{2}(\theta\beta)^{\frac{\theta}{1-\theta}} > 0 & t = 1 \\ 0 & t \geq 2 \end{cases}$ Only 1 period trade surplus

Permanent increase in home productivity: Permanent trade surplus



$\hat k_{1t} = \frac{1}{2}\left\{\frac{1-\theta^{t-1}}{1-\theta} + \theta^{t-1}\right\}$ $1 + \hat{tb}_{it} = \frac{1}{2}(\theta\beta)^{\frac{\theta}{1-\theta}} > 0$. $\forall t \geq 1$

Decentralized Equi. Complete set of contingent claims.

bond $b(s^{t+1})$: get 1 if s^{t+1}, 0 o.w.

DP: $V(k_1,b_1,X) = \max_{\{k_1', b_1'(X')\}} u(c_1) + \beta \mathbb{E} V(k_1', b_1'(X'), X')$

$= \max_{\{k_1', b_1'(X')\}} u(c_1) + \beta \sum_{X'|X} \pi(X'|X) V(k_1', b_1'(X'), X')$

s.t. $c_1 + k_1' + \sum_{X'|X} q(X'|X) b_1'(X') = w_1 h_1 + r_1 k_1 + b_1(X) + (1-\delta) k_1$

$b_1' \ge \underline{b}$ $\forall X'|X$. Rule out Ponzi scheme

aggregate state $X = (z_1, K_1, B_1, z_2, K_2, B_2)$

FOC: $u'(c_1) = \beta \mathbb{E} \frac{\partial V}{\partial k_1'}$

$q(X'|X) u'(c_1) = \beta \pi(X'|X) \frac{\partial V}{\partial b_1'}(k_1', b_1'(X'), X')$

Envelope: $\frac{\partial V}{\partial k_1} = (r_1 + 1-\delta) u'(c_1)$

$\frac{\partial V}{\partial b_1} = u'(c_1)$

Euler: $u'(c_1) = \beta \mathbb{E}(r_1' + 1-\delta) u'(c_1')$

$q(X'|X) u'(c_1) = \beta \pi(X'|X) u'(c_1'(k_1', b_1(X'), X'))$

No arbitrage condition: $q(X'|X) = \pi(X'|X) \frac{\beta u'(c_1')}{u'(c_1)}$

If $B_{10}=B_{20}=0$, then NAC ⇔ $u'(c_1)=u'(c_2)$ in Planner's problem

Recursive Competitive Equi.: $X=(K_1,K_2,z_1,z_2,B_1,B_2)$

$\{K_i(X), H_i(X), B_i(X'|X)\}$ $\{w_i(X), r_i(X), q(X'|X)\}$

$\{k_i(k,b,X), h_i(k,b,X), b_i(k,b,X'|X), V_i(k,b,X)\}$

① Firm's FOC: $w_i(X) = F_H(H_i,K_i,z_i)$; $r_i(X) = F_K(H_i,K_i,z_i)$

② Agent's problem

(Domestic market clearing)

③ Consistency between aggregate & individual level

Capital: $K_i(X) = \int k_i(k_i,b_i,X)\,di = k_i(K_i,B_i,X)$

Labor: $H_i(X) = h_i(K_i,B_i,X)$

Bond: $B_i(X'|X) = b_i(K_i,B_i,X'|X)$

④ International market clearing. $\sum_{i=1}^{2} B_i(X'|X)=0$

2-good model: imperfect substitutions

Resource constraint: $a_{1t}+a_{2t}=y_{1t}=z_{1t}F(h_{1t},k_{1t})$

$x_{1t}+x_{2t}=y_{2t}=z_{2t}F(h_{2t},k_{2t})$

Budget constraint: $c_{1t}+i_{1t}=D(a_{1t},x_{1t})$

Armington (1969) $c_{2t}+i_{2t}=D(x_{2t},a_{2t})$

CES aggregator: $D(a,x)=[\omega a^{1-\alpha}+(1-\omega)x^{1-\alpha}]^{\frac{1}{1-\alpha}}$, $\alpha\in(0,+\infty)$

Elasticity of substitution $|\varepsilon| = \left|\frac{d\ln(a/x)}{d\ln(p_a/p_x)}\right| = \frac{1}{\alpha}$

$\alpha=0$, $|\varepsilon|=+\infty$. Complete substitution $D=\omega a+(1-\omega)x$

$\alpha=1$ $|\varepsilon|=1$. Cobb-Douglass $D=a^{\omega} x^{1-\omega}$

$\alpha=+\infty$ $|\varepsilon|=0$. Complete complement $D=\min\{a,x\}$

home bias parameter: ω

$\omega=1$ closed economy

$\omega=\frac12$ No home bias in consumption

Love of variety! ⇔ $0=\max p_1 D(a_1,x_1) - (p_a a + p_x x)$

Price aggregator: $P_1 = \min_{a_1,x_1} p_a a_1 + p_x x_1$ s.t. $D(a_1,x_1)\ge 1$

$P_2 = \min_{a_2,x_2} p_a a_2 + p_x x_2$ s.t. $D(x_2,a_2)\ge 1$

$P_1 = [\omega^{\frac1\alpha} p_a^{\frac{\alpha-1}{\alpha}} + (1-\omega)^{\frac1\alpha} p_x^{\frac{\alpha-1}{\alpha}}]^{\frac{\alpha}{\alpha-1}}$

$P_2 = [\omega^{\frac1\alpha} p_x^{\frac{\alpha-1}{\alpha}} + (1-\omega)^{\frac1\alpha} p_a^{\frac{\alpha-1}{\alpha}}]^{\frac{\alpha}{\alpha-1}}$

$\frac{P_1}{P_2} = \frac{D_2(x_2,a_2)}{D_1(a_1,x_1)} = \frac{D_1(x_2,a_2)}{D_2(a_1,x_1)}$

Planner: $V_0 = \max \sum_{t=0}^{+\infty}\beta^t \sum_{s^t\in S^t}\pi(s^t)[u(c_{1t})+u(c_{2t})]$

s.t. $a_{1t}+a_{2t}=y_{1t}=z_{1t}F(H_{1t},k_{1t})$

$x_{1t}+x_{2t}=y_{2t}=z_{2t}F(H_{2t},k_{2t})$

$c_{1t}+k_{1t+1}=D(a_{1t},x_{1t})+(1-\delta)k_{1t}$

$c_{2t}+k_{2t+1}=D(x_{2t},a_{2t})+(1-\delta)k_{2t}$

DP: $V(k_1,k_2,z_1,z_2)=\max_{\{a_1,x_1,k_1',k_2'\}} u(c_1)+u(c_2)+\beta\mathbb{E}V(k_1',k_2',z_1',z_2')$

s.t. $c_1 = D(a_1,x_1)+(1-\delta)k_1-k_1'$

$c_2 = D(z_2F(H_2,k_2)-x_1,\ z_1F(H_1,k_1)-a_1)+(1-\delta)k_2-k_2'$

FOC: $\left.\begin{aligned}\frac{\partial V}{\partial a_1} &= u'(c_1)D_1(a_1,x_1) - u'(c_2)D_2(x_2,a_2)=0\\ \frac{\partial V}{\partial x_1} &= u'(c_1)D_2(a_1,x_2) - u'(c_2)D_1(x_2,a_2)=0\end{aligned}\right\}$ $\frac{u'(c_1)}{u'(c_2)} = \frac{D_2(x_2,a_2)}{D_1(a_1,x_1)} = \frac{D_1(x_2,a_2)}{D_2(a_1,x_1)}$

⇒ $\frac{u'(c_1)}{u'(c_2)} = \frac{P_1}{P_2} =: q$ Real exchange rate

$\frac{\partial V}{\partial k_1'} = -u'(c_1)+\beta\mathbb{E}V_1(k_1',k_2',z_1',z_2')=0$

$\frac{\partial V}{\partial k_2'} = -u'(c_2)+\beta\mathbb{E}V_2(k_1',k_2',z_1',z_2')=0$

Envelope: $V_1 = u'(c_1)(1-\delta)+u'(c_2)D_2(x_2,a_2)\underline{z_1F_k(H_1,k_1)}$ MPk_1

$V_2 = u'(c_2)(1-\delta)+u'(c_2)D_1(x_2,a_2)\underline{z_2F_k(H_2,k_2)}$ MPk_2

Euler: $u'(c_1) = \beta\mathbb{E}u'(c_1')[D_1(a_1',x_1')MPk_1'+1-\delta]$

$u'(c_2) = \beta\mathbb{E}u'(c_2')[D_1(x_2',a_2')MPk_2'+1-\delta]$

Decentralization: Bond is priced in units of the home final good

Home: $V_1(k_1,b_1;K_1,K_2,B_1,B_2,z_1,z_2)=\max_{k_1',b_1} u(c_1,1-h_1)+\beta\mathbb{E}V_1(k_1',b_1';K_1'\cdots z_2')$

s.t. $P_1[c_1+k_1'+\sum_{s'|s} q(s'|s)b_1(s')] = w_1h_1+r_1k_1+P_1[b_1+(1-\delta)k_1]$

Foreign: $V_2(k_2,b_2;\ ---\ ) = \max_{k_2',b_2} u(c_2,1-h_2)+\beta\mathbb{E}V_2(k_2',b_2';---)$

s.t. $P_2[c_2+k_2']+P_1\sum_{s'|s}q(s'|s)b_2(s') = w_2h_2+r_2k_2+P_1b_2+P_2(1-\delta)k_2$

Normalization: $\forall s^t$: $P_1(s^t)=1$. $P_2\ne P_1$ if $\omega\ne\frac12$

No-arbitrage: $q(s^{t+1}|s^t) = \beta\pi(s^{t+1}|s^t)\frac{u'(c_1(s^{t+1}))}{u'(c_1(s^t))} = \beta\pi(s^{t+1}|s^t)\frac{u'(c_2')}{u'(c_2)}\frac{P_1(s^{t+1})/P_2(s^{t+1})}{P_1(s^t)/P_2(s^t)}$

Backus-Smith-Kollman condition: $\frac{u_c(c_1')}{u_c(c_1)} = \frac{u_c(c_2')}{u_c(c_2)}\frac{q'}{q}$

$\frac{q'}{q}\uparrow$ $\frac{c_1'}{c_1}\downarrow$

Firm (Home): $\max p_a F(h_1,k_1,z_1)-w_1h_1-r_1k_1$ ⇒ $\begin{cases} w_1=p_a MPL_1\\ r_1=p_a MPK_1\end{cases}$

(Foreign): $\max p_x F(h_1,k_1,z_1)-w_2h_2-r_2k_2$ ⇒ $\begin{cases} w_2=p_x MPL_2\\ r_2=p_x MPK_2\end{cases}$

⇒ $\begin{cases}\frac{u_c(c_1,1-h_1)}{P_1} = \frac{u_l(c_1,1-h_1)}{w_1} = \frac{u_l(c_1,1-h_1)}{p_a MPL_1}\\ \frac{u_c(c_2,1-h_2)}{P_2} = \frac{u_l(c_2,1-h_2)}{p_x MPL_2}\end{cases}$

Euler: $u_{c_1} = \beta\mathbb{E}u_{c_1'}(\frac{p_a'}{P_1'}MPK_1'+1-\delta)$

$u_{c_2} = \beta\mathbb{E}u_{c_2'}(\frac{p_x'}{P_2'}MPK_2'+1-\delta)$

Notice that $\begin{cases}P_1D_1(a,x)=p_a\\ P_2D_2(x,a)=p_x\end{cases}$, then Euler^{CE} ⇔ Euler^{SP}

Efficient!

Endogenous Growth

AK model with taxes "No diminishing return"

$u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$; $y = AK$; $c_t = Bk_t - k_{t+1}$, $B = A+1-\delta$

Planner:

DP: $V(k) = \max_{c,k'} u(c) + \beta V(k')$ s.t. $c + k' = Bk$

Euler: $u'(c) = \beta B u'(c')$: $\frac{c_{t+1}}{c_t} = (\beta B)^{\frac{1}{\sigma}}$

TVC: $\lim_{T\to+\infty} \beta^T u'(c_T) k_{T+1} = 0 \Leftrightarrow \lim_{T\to+\infty} B(\beta^{\frac{1}{\sigma}} B^{\frac{1-\sigma}{\sigma}})^T = 0$

$\exists$ BGP: $\gamma^{SP} = (\beta B)^{\frac{1}{\sigma}}$ iff. $B \in (\beta^{-1}, \beta^{\frac{-1}{1-\sigma}})$

Decentralization with capital tax τ_k:

DP: $V(k) = \max_{c,k'} u(c) + \beta V(k')$ s.t. $c + k' = (1-\tau_k) r k + (1-\delta)k$

Firm: $0 = \max AK - rK \Rightarrow r = A$

Euler: $u'(c) = \beta[A(1-\tau_k) + 1-\delta] u'(c')$

$\gamma^{CE} = (\beta[A(1-\tau_k)+1-\delta])^{\frac{1}{\sigma}} < \gamma^{SP} = (\beta[A+1-\delta])^{\frac{1}{\sigma}}$

Externality model "Externality of capital stock"

① $y = F(k,h)\phi(K) = Ak^\theta h^{1-\theta} K^\eta$ $H = h = 1$

Planner:

DP: $V(k) = \max_{c,k'} u(c) + \beta V(k')$ s.t. $c + k' = Ak^{\theta+\eta} + (1-\delta)k$

Euler: $u'(c) = \beta u'(c')[g_k(k) + 1-\delta]$. $g_k(k) = (\theta+\eta)Ak^{\theta+\eta-1}$

Decentralization:

DP: $V(k;K) = \max_{c,k'} u(c) + \beta V(k';K)$ s.t. $c + k' = rk + (1-\delta)k$

Firm: $\max_k Ak^\theta \phi(K) - rk \Rightarrow r = \phi(K) F_k(k) = \theta A k^{\theta+\eta-1}$

Euler: $u'(c) = \beta u'(c')[\phi(K) F_k(k) + 1-\delta]$

Individual return on k < Social return on k

$\Rightarrow$ investment in decentralized equi. < Inv. in SP.

② $y = \phi(K) + F(k,h) = AK + k^\theta l^{1-\theta}$ $l = L = 1$

Planner:

DP: $V(k) = \max_{c,k'} u(c) + \beta V(k')$ s.t. $c + k' = Ak + k^\theta + (1-\delta)k$

Euler: $u'(c) = \beta u'(c')[A + F_k(k) + 1-\delta]$

Decentralization:

DP: $V(k;K) = \max_{c,k'} u(c) + \beta V(k';K')$ s.t. $c + k' = rk + (1-\delta)k$

Firm: $\max_k AK + k^\theta - rk \Rightarrow r = F_k(k) = \theta k^{\theta-1}$

Euler: $u'(c) = \beta u'(c')[F_k(k) + 1-\delta]$

2-sector model "Accumulation of human capital"

① $Y = K^\theta H^{1-\theta}$. CE is efficient.

Planner:

DP: $V(K,H) = \max_{c,K',H'} u(c) + \beta V(K',H')$

s.t. $c + K' + H' = Y + (1-\delta_K)K + (1-\delta_H)H$

Euler: $u'(c) = \beta u'(c')(F_K + 1-\delta_K)$ $\Rightarrow F_K + 1-\delta_K = F_H + 1-\delta_H$

$u'(c) = \beta u'(c')(F_H + 1-\delta_H)$ $x = \frac{K}{H}$

HD1: $F(K,H) = HF(\frac{K}{H},1) = HF(x,1) \Rightarrow F_K(K,H) = F_K(x,1) = \theta x^{\theta-1}$

$F(K,H) = KF(1,\frac{H}{K}) = KF(1,\frac{1}{x}) \Rightarrow F_H(K,H) = F_H(1,\frac{1}{x}) = (1-\theta)x^\theta$

$\delta_K = \delta_H \Rightarrow x = \frac{K}{H} = \frac{\theta}{1-\theta}$

Back to AK: $y_t = AK_t^\theta H_t^{1-\theta} = Ax^\theta H_t = Ax^{\theta-1}K_t$

CRRA utility: $\frac{c_{t+1}}{c_t} = \{\beta[A\theta(\frac{\theta}{1-\theta})^{\theta-1} + 1-\delta]\}^{\frac{1}{\sigma}}$ BGP

② $y = K^\theta H^\alpha L^{1-\theta-\alpha}$ human capital does not magnify hours 1 for 1

Planner:

DP: $V(K,H) = \max_{c,L} u(c, 1-L) + \beta V(K',H')$

s.t. $c + K' + H' = K^\theta H^\alpha L^{1-\theta-\alpha} + (1-\delta_K)K + (1-\delta_H)H$

Euler: $u_c = \beta u_{c'}[F_K(K,H,L) + 1-\delta_K]$ $\Rightarrow F_K = F_H$ $(\delta_K = \delta_H)$

$u_c = \beta u_{c'}[F_H(K,H,L) + 1-\delta_H]$

Intratemporal $u_c F_L(K,H,L) = u_L$

HD1: $F(K,H,L) = HF(\frac{K}{H}, 1, \frac{L}{H}) \Rightarrow F_K(K,H,L) = F_K(x_K, 1, x_L) = \theta x_K^{\theta-1} x_L^{1-\theta-\alpha}$

$\Rightarrow F_L(K,H,L) = F_L(x_K, 1, x_L)$

$= KF(1, \frac{H}{K}, \frac{L}{K}) \Rightarrow F_H(K,H,L) = F_H(1, \frac{1}{x_K}, \frac{x_L}{x_K}) = \alpha x_K^\theta x_L^{1-\theta-\alpha}$

$\Rightarrow x_K = \frac{K}{H} = \frac{\theta}{\alpha}$

$y_t = K_t^\theta H_t^\alpha L_t^{1-\theta-\alpha} = (\frac{\theta}{\alpha})^\theta H_t^{\theta+\alpha} L_t^{1-\theta-\alpha} = (\frac{\alpha}{\theta})^\alpha K_t^{\theta+\alpha} L_t^{1-\theta-\alpha}$

No BGP!

Linear Technology in Capital Production (Rebelo 1991) <u>Efficient</u>

Capital K $\begin{cases} K_C \to C = K_C^\theta L^{1-\theta} \\ K_I \to K' = (1-\delta)K + \underline{AK_I} \text{ linear tech.} \end{cases}$

DP: $V(K) = \max_{c,K_C,K_I,K'} u(c) + \beta V(K')$

s.t. $C = K_C^\theta L^{1-\theta}$; $K' = (1-\delta)K + AK_I$; $K_C + K_I = K$

$\mathcal{L}(K_C, K', \lambda; K) = u(K_C^\theta L^{1-\theta}) + \beta V(K') + \lambda[A(K-K_C) + (1-\delta)K - K']$

FOC: $\left.\begin{array}{l} \frac{\partial \mathcal{L}}{\partial K_C} = u'(c) F_{K_C}(K_C, L) - A\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial K'} = \beta V'(K') - \lambda = 0 \end{array}\right\} \Rightarrow u'(c) F_{K_C} = A\beta V'(K')$

Envelope: $V'(K) = \mathcal{L}'(K) = \lambda(A+1-\delta)$

Euler: $u'(c) F_{K_C}(K_C, L) = \beta A \lambda'(A+1-\delta) = \beta(A+1-\delta) u'(c') F_{K_C}(K_C', L)$

$\left.\begin{array}{l} \text{CRRA } u: (\frac{C_{t+1}}{C_t})^\sigma = \beta(A+1-\delta)(\frac{K_C'}{K_C})^{\theta-1} \\ C = K_C^\theta L^{1-\theta} \Rightarrow \frac{C_{t+1}}{C_t} = (\frac{K_{C,t+1}}{K_C})^\theta \end{array}\right\} \Rightarrow \gamma_{K_C}^{1-\theta(1-\sigma)} = \beta(A+1-\delta)$

Suppose $K_C = \alpha K$. Budget $\Rightarrow \frac{K'}{K} = \gamma_{K_C} = (1-\delta) + A(1-\alpha) \Rightarrow \alpha$.

Endogenous Innovation Romer 1990

Non-rival ideas & positive externality

Final good producer: $y_t = L_{p,t}^{1-\theta} \int_0^{A_t} x_t(i)^\theta di$ imperfect substitute

$\max_{L_{p,t}, x_t(i)} L_{p,t}^{1-\theta} \int_0^{A_t} x_t(i)^\theta di - \int_0^{A_t} p_t(i) x_t(i) di - w_t L_{p,t}$

FOC: $w_t = (1-\theta) L_{p,t}^{-\theta} \int_0^{A_t} x_t(i)^\theta di$

$p_t(i) = \theta L_{p,t}^{1-\theta} x_t(i)^{\theta-1}$

Intermediate good (monopoly): $\pi(i) = \max_{x_t(i)} p_t(i) x_t(i) - x_t(i)$

s.t. $p_t(i) = \theta L_{p,t}^{1-\theta} x_t(i)^{\theta-1}$

FOC: $\frac{\partial p}{\partial x} x + p - 1 = 0 \Rightarrow p = \underbrace{\frac{1}{1+\varepsilon^{-1}}}_{\text{markup}} \overset{MC}{1}$, $\varepsilon = \frac{\partial x/x}{\partial p/p} = (\theta-1)^{-1}$

$\Rightarrow p_t(i) = \frac{1}{\theta}$; $x_t(i) = \theta^{\frac{2}{1-\theta}} L_{pt}$; $\pi_t(i) = (1-\theta)\theta^{\frac{1+\theta}{1-\theta}} L_{p,t}$

Symmetry in $x_t(i)$: $w_t = (1-\theta) L_{p,t}^{-\theta} A_t x_t(i)^\theta = (1-\theta)\theta^{\frac{2\theta}{1-\theta}} A_t$

$y_t = \theta^{\frac{2\theta}{1-\theta}} L_{p,t} A_t$ (AK Model)

R&D: $\pi_{RD,t} = \max_{A_{t+1}, L_{RD,t}} P_t^P (A_{t+1} - A_t) - w_t L_{RD,t}$ Non-rival ideas

s.t. $A_{t+1} = A_t + A_t \delta L_{RD,t}$ Linear tech.

FOC: $P_t^P A_t \delta = w_t$,

patent: $P_t^P = \sum_{\tau=1}^{+\infty}\left(\frac{1}{1+r}\right)^{\tau}\pi_{t+\tau}(i) = \frac{1}{r}(1-\theta)\theta^{\frac{1+\theta}{1-\theta}} L_{p,t}$

$\Rightarrow w_t = \frac{\delta}{r}(1-\theta)\theta^{\frac{1+\theta}{1-\theta}} L_{p,t} A_t = (1-\theta)\theta^{\frac{2\theta}{1-\theta}} A_t$

Labor market has uniform wage: w_t for R&D = w_t for P

$\Rightarrow L_{p,t} = \frac{r}{\theta\delta}$; $\frac{A_{t+1}}{A_t} = 1+\delta(L-\frac{r}{\theta\delta}) = \frac{y_{t+1}}{y_t}$

Consumer's problem: $\max_{\{C_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty}\beta^t u(C_t)$ s.t. $\sum_{t=0}^{+\infty}\left(\frac{1}{1+r}\right)^t C_t \le Y$

Euler: $u'(C_t) = \beta(1+r)u'(C_{t+1}) \Rightarrow \frac{C_{t+1}}{C_t} = 1+\gamma = [\beta(1+r)]^{\frac{1}{\sigma}}$

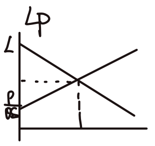
$\frac{y_{t+1}}{y_t} = \frac{C_{t+1}}{C_t} \Rightarrow \begin{cases} 1+\delta(L-\frac{r}{\theta\delta}) = 1+\gamma \\ [\beta(1+r)]^{\frac{1}{\sigma}} = 1+\gamma \xrightarrow{\text{Approx.}} \gamma = \frac{1}{\sigma}(r-\rho) \end{cases}$ $\rho = \frac{1}{\beta}-1$

$\Rightarrow \gamma = \frac{\theta\delta L-\rho}{\theta+\sigma}$ BGP

All workers are in production sector $\Leftrightarrow \gamma<0 \Leftrightarrow \rho>\theta\delta L$

Trade-off: current GDP y_t & future growth γ:

$y_t = \theta^{\frac{2\theta}{1-\theta}}(L-\frac{\gamma}{\delta})A_t$ $\gamma\uparrow$, $y_t\downarrow$, $A_{t+1}\uparrow$, $y_{t+1}\uparrow$



$\begin{cases} L_p = \frac{r}{\theta\delta} = \frac{\sigma\gamma+\rho}{\theta\delta} \\ L_p = L-L_{RD} = L-\frac{\gamma}{\delta} \end{cases}$

Decentralized equi. is inefficient: ① monopoly of interm. firms ② positive externality of ideas $\Rightarrow$ R&D $MR_{individual} < MR_{social}$

Planner: $\max_{\{C_t,x_t,L_{pt},L_{RDt}\}} \sum_{t=0}^{+\infty}\beta^t u(C_t)$

s.t. $C_t + A_t x_t = Y_t = L_{pt}^{1-\theta} A_t x_t^{\theta}$

$A_{t+1} = A_t + A_t\delta L_{RDt}$

$L_{pt} + L_{RDt} = L$

DP: $V(A) = \max_{x,L_p,A'} u(C) + \beta V(A')$

s.t. $C = (L_p^{1-\theta}x^\theta - x)A$; $A' = A(1+\delta(L-L_p))$

$\mathcal{L}(x,L_p,A',\lambda;A) = u(C)+\beta V(A')+\lambda[A(1+\delta(L-L_p))-A']$

s.t. $C = (L_p^{1-\theta}x^\theta - x)A$

FOC: $\frac{\partial\mathcal{L}}{\partial x} = u'(C)[\theta L_p^{1-\theta}x^{\theta-1}-1]A = 0 \Rightarrow \frac{x}{L_p} = \theta^{\frac{1}{1-\theta}}$

$\frac{\partial\mathcal{L}}{\partial L_p} = u'(C)(1-\theta)L_p^{-\theta}x^\theta A - A\delta\lambda = 0 \Rightarrow \lambda = \frac{u'(C)}{\delta}(1-\theta)\theta^{\frac{\theta}{1-\theta}}$

$\frac{\partial\mathcal{L}}{\partial A'} = \beta V'(A') - \lambda = 0$

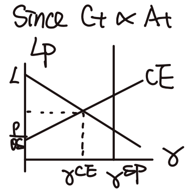
Envelope: $V'(A) = \mathcal{L}'(A) = u'(C)[L_p^{1-\theta}x^\theta - x] + \lambda[1+\delta(L-L_p)]$

$\Rightarrow \lambda = \beta V'(A') = \{u'(C')[L_p'^{1-\theta}x'^{\theta}-x'] + \lambda'[1+\delta(L-L_p')]\}\beta$

$\Rightarrow \frac{u'(C)}{\delta}(1-\theta)\theta^{\frac{\theta}{1-\theta}} = \beta u'(C')(1-\theta)\theta^{\frac{\theta}{1-\theta}}[\frac{1}{\delta}+L]$

$\Rightarrow \left(\frac{C_{t+1}}{C_t}\right)^\sigma = \beta(1+\delta L) = (1+\gamma)^\sigma \xrightarrow{\text{Approx.}} \sigma\gamma = \delta L-\rho$

Since $C_t \propto A_t$, $\frac{C_{t+1}}{C_t} = \frac{A_{t+1}}{A_t} = 1+\gamma_C = 1+\delta(L-L_p) \Rightarrow L_p = L-\frac{\gamma}{\delta}$



$\begin{cases} L_p = L-\frac{\gamma}{\delta} \\ \gamma = [\beta(1+\delta L)]^{\frac{1}{\sigma}}-1 \end{cases}$ (SP) v.s. (CE) $L_p = \frac{r}{\theta\delta} = \frac{\sigma\gamma+\rho}{\theta\delta}$

$\gamma^{SP} = \frac{\delta L-\rho}{\sigma} > \gamma^{CE} = \frac{\theta\delta L-\rho}{\theta+\sigma}$

① 3 Variations of Romer's model

Lab-equipment: $A_{t+1}-A_t = Y_x/\mu$

$C + Ax + Y_x = Y$

Final: $\max_{L,x_t(i)} L^{1-\theta}\int_0^{A_t} x_t^\theta(i)\,di - w_t L - \int_0^{A_t} p_t(i)x_t(i)\,di$

FOC: $w_t = (1-\theta)L^{-\theta}\int_0^{A_t} x_t^\theta(i)\,di$

Intermediate: $p_t(i) = \theta L^{1-\theta}x_t^{\theta-1}(i)$

$p_t(i) = \frac{1}{\theta}$; $x_t(i) = \theta^{\frac{2}{1-\theta}}L$; $\pi_t(i) = (1-\theta)\theta^{\frac{1+\theta}{1-\theta}}L$

R&D: $\pi_{RD,t} = \max_{A_{t+1},Y_x} P_t^P(A_{t+1}-A_t) - Y_{xt} = \max_{Y_x}(\frac{P_t^P}{\mu}-1)Y_{xt}$

s.t. $A_{t+1}-A_t = Y_{xt}/\mu$

FOC: $P_t^P = \mu$ Patent: $P_t^P = \frac{1}{r}\pi(i) = \frac{1}{r}(1-\theta)\theta^{\frac{1+\theta}{1-\theta}}L$

$\Rightarrow r = (1-\theta)\theta^{\frac{1+\theta}{1-\theta}}\frac{L}{\mu}$

Consumer: $\max_{\{C_t\}}\sum_{t=0}^{+\infty}\beta^t u(C_t)$ s.t. $\sum_{t=0}^{+\infty}\left(\frac{1}{1+r}\right)^t C_t = \sum_{t=0}^{+\infty}\left(\frac{1}{1+r}\right)^t w_t$

Euler: $u'(C_t) = \beta(1+r)u'(C_{t+1}) \Rightarrow 1+\gamma = \frac{C_{t+1}}{C_t} = [\beta(1+r)]^{\frac{1}{\sigma}}$

$\Rightarrow \gamma = \frac{r-\rho}{\sigma} = \frac{(1-\theta)\theta^{\frac{1+\theta}{1-\theta}}L/\mu-\rho}{\sigma}$ by approximation $\sigma\gamma = r-\rho$

②

production $Y_t = \int_0^{A_t} x_t^\theta(i)\,di$, $x_t(i) = l_{it}$, $x = L_{p,t}$

Final: $\max_{x_t(i)} \int_0^{A_t} x_t^\theta(i)\,di - \int_0^{A_t} p_t(i)x_t(i)\,di$

FOC: $\theta x_t^{\theta-1}(i) = p_t(i)$

Interm: $\pi_t(i) = \max_{x_t,p_t} p_t(i)x_t(i) - w_t x_t(i)$

s.t. $\theta x_t^{\theta-1}(i) = p_t(i)$

Markup: $p_t(i) = \frac{w_t}{1+\varepsilon^{-1}}$, $\varepsilon = (\theta-1)^{-1}$

$p_t(i) = \frac{w_t}{\theta}$, $x_t(i) = \theta^{\frac{2}{1-\theta}}w_t^{\frac{-1}{1-\theta}}$, $\pi_t(i) = \frac{1-\theta}{\theta}w_t x_t$

R&D: $\pi_{RD,t} = \max_{A_{t+1},L_{RD,t}} P_t^P(A_{t+1}-A_t) - w_t L_{RD,t} = \max_{L_{RD,t}}(P_t^P\delta A_t - w_t)L_{RD,t}$

s.t. $A_{t+1} = A_t + A_t\delta L_{RD,t}$

FOC: $P_t^P\delta A_t = w_t$ Patent: $P_t^P = \frac{1}{r}\pi_t(i) = \frac{1}{r}\frac{1-\theta}{\theta}w_t x_t$

$\Rightarrow r = \delta\frac{1-\theta}{\theta}L_{p,t} = \delta\frac{1-\theta}{\theta}(L-L_{RD,t})$

BGP: $\gamma_A = \frac{A_{t+1}-A_t}{A_t} = \delta L_{RD,t} \Rightarrow r = \delta\frac{1-\theta}{\theta}(L-\frac{\gamma}{\delta})$

Euler: $\left(\frac{C_{t+1}}{C_t}\right)^\sigma = \beta(1+r) = (1+\gamma_C)^\sigma \xrightarrow{\text{Appro.}} \sigma\gamma = r-\rho$

$\Rightarrow \gamma = \frac{\frac{1-\theta}{\theta}\delta L-\rho}{\frac{1-\theta}{\theta}+\sigma}$

③ Crowding out/in of innovation Benassy 1998

$Y_t = A_t^{1+v-\frac{1}{\theta}}L_{pt}^{1-\theta}\left(\int_0^{A_t}x_t^\theta(i)\,di\right)^{\frac{1}{\theta}}$

Final: $\max_{L_{pt},x_t} A_t^{1+v-\frac{1}{\theta}}L_{pt}^{1-\theta}\left(\int_0^{A_t}x_t^\theta(i)\,di\right)^{\frac{1}{\theta}} - w_t L_{pt} - \int_0^{A_t}p_t(i)x_i(t)\,di$

FOC: $(1-\theta)A_t^{1+v-\frac{1}{\theta}}L_{pt}^{-\theta}\left(\int_0^{A_t}x_t^\theta(i)\,di\right)^{\frac{1}{\theta}} = w_t$

$A_t^{1+v-\frac{1}{\theta}}L_{pt}^{1-\theta}\left(\int_0^{A_t}x_t^\theta(i)\,di\right)^{\frac{1}{\theta}-1}x_t^{\theta-1}(i) = p_t(i)$

$Y_t = A_t^{1+v}L_{pt}^{1-\theta}x$: FOC: $(1-\theta)A_t^{1+v}L_{pt}^{-\theta}x_t = w_t$

$A_t^v L_{pt}^{1-\theta} = p_t$

Interm: $\pi_t = \max_{x_t,p_t} p_t x_t - x_t$ s.t. $A_t^v L_{pt}^{1-\theta} = p_t$

To ensure $x_t = \frac{1}{1-\theta}A_t^{-(1+v)}L_{pt}^{\theta}w_t$, $p_t = A_t^v L_{pt}^{1-\theta} = 1$

$\begin{cases} v>0: L_{RD,t}\uparrow\ L_{pt}\downarrow\ A_t\uparrow & \text{thick market externality} \\ v<0: L_{RD,t}\uparrow\ L_{pt}\downarrow\ A_t\downarrow & \text{congestion effect} \end{cases}$

Semi-endogenous Growth : Scale effect (population growth)

Jones 1995 : $A_{t+1} = A_t + \delta A_t^{\gamma_A} L_{RD,t}^{1-\gamma_L}$, $\frac{L_{t+1}}{L_t} = 1+n$

Final : $\max_{L_{pt}, x_t} L_{pt}^{1-\theta} \int_0^{A_t} x_t^{\theta}(i)\, di - w_t L_{pt} - \int_0^{A_t} P_t(i) x_t(i)\, di$

FOC: $(1-\theta) L_{pt}^{-\theta} \int_0^{A_t} x_t^{\theta}(i)\, di = w_t \Rightarrow w_t = \frac{(1-\theta) Y_t}{L_{pt}}$

$\theta L_{pt}^{1-\theta} x_t^{\theta-1}(i) = P_t(i)$

Interm: $\pi_t(i) = \max_{P_t, x_t} P_t(i) x_t(i) - x_t(i)$ s.t. $\theta L_{pt}^{1-\theta} x_t^{\theta-1}(i) = P_t(i)$

markup : $P_t = \frac{1}{1+\varepsilon^{-1}}$, $\varepsilon = \frac{d \ln x_t}{d \ln P_t} = (\theta-1)^{-1}$

$P_t = \frac{1}{\theta}$. $x_t = \theta^{\frac{2}{1-\theta}} L_{pt}$ $\pi_t = (\frac{1}{\theta}-1)\theta^{\frac{2}{1-\theta}} L_{pt}$

$\Rightarrow Y_t = A_t L_{pt} \theta^{\frac{2\theta}{1-\theta}}$ $w_t = (1-\theta)\theta^{\frac{2\theta}{1-\theta}} A_t$

R&D : $\pi_{RD,t} = \max_{A_{t+1}, L_{RD,t}} P_t^P (A_{t+1} - A_t) - w_t L_{RD,t}$

s.t. $A_{t+1} = A_t + \delta A_t^{\gamma_A} L_{RD,t}^{1-\gamma_L}$

FOC: $(1-\gamma_L)\delta P_t^P A_t^{\gamma_A} L_{RD,t}^{-\gamma_L} = w_t = (1-\theta)\theta^{\frac{2\theta}{1-\theta}} A_t$

Patent : $P_t^P = \frac{1+n}{r-n} \pi_t = \frac{1+n}{r-n}(1-\theta)\theta^{\frac{1+\theta}{1-\theta}} L_{pt}$

$\Rightarrow \frac{A_t^{1-\gamma_A}}{L_t^{1-\gamma_L}} = \theta\delta(1-\gamma_L)(1-\alpha)\alpha^{-\gamma_L}\frac{1+n}{r-n}$ constant

$\Rightarrow g_A = \frac{A_{t+1}-A_t}{A_t} = \delta\alpha^{1-\gamma_L}\frac{L_t^{1-\gamma_L}}{A_t^{1-\gamma_A}} = \frac{\alpha}{\theta(1-\gamma_L)(1-\alpha)}\frac{r-n}{1+n}$ constant

$g_A(1-\gamma_A) = (1-\gamma_L) g_L = (1-\gamma_L) n$

$g_Y = g_C = g_A + g_L = (\frac{1-\gamma_L}{1-\gamma_A} + 1) n$

Euler : $u'(C) = \beta u'(C')(1+r) \Rightarrow (\frac{C_{t+1}}{C_t})^{\sigma} = (1+g_C)^{\sigma} = \beta(1+r)$

$\rightarrow \sigma g_C = r - \rho$

Final accounting : $g_y = (\frac{K}{Y})^{\alpha} h (RD)^{\gamma} L^{\gamma}$

Open economy Rivera-Batiz & Romer 1991

$Y_t = L_{pt}^{1-\theta}[\int_0^{A_t} x_t^{\theta}(i)\, di + \int_0^{A_t^*} b_t^{*\theta}(i)\, di]$

Final H : $\max_{L_{pt}, x_t, b_t^*} Y_t - w_t L_{pt} - \int_0^{A_t} P_t^x(i) x_t(i)\, di - \int_0^{A_t^*} P_t^b(i) \tau b_t^*(i)\, di$

Demand : $P_t^x(i) = \theta L_{p,t}^{1-\theta} x_t^{\theta-1}(i)$; $\tau P_t^b(i) = \theta L_{p,t}^{1-\theta} b_t^{*\theta-1}(i)$

$w_t = (1-\theta) Y_t / L_{p,t}$

Final F : $\max_{L_{p,t}^*, x_t^*, b_t} Y_t^* - w_t^* L_{pt}^* - \int_0^{A_t} P_t^x(i) \tau x_t^*(i)\, di - \int_0^{A_t^*} P_t^b(i) b_t(i)\, di$

Demand : $\tau P_t^x(i) = \theta L_{p,t}^{*1-\theta} x_t^{*\theta-1}(i)$; $P_t^b(i) = \theta L_{p,t}^{*1-\theta} b_t^{\theta-1}(i)$

$w_t^* = (1-\theta) Y_t^* / L_{pt}^*$

Resource Constraint : $y_t(i) = x_t(i) + x_t^*(i)$

$y_t^*(i) = b_t^*(i) + b_t(i)$

Interm H : $\pi_t(i) = \max_{x, x^*, p^x} P_t^x(i) x_t(i) + P_t^x(i) x_t^*(i) - (x_t(i) + x_t^*(i))$

s.t. $P_t^x(i) = \theta L_{p,t}^{1-\theta} x_t^{\theta-1}(i)$

$P_t^x(i) = \frac{1}{\tau}\theta L_{p,t}^{*1-\theta} x_t^{*\theta-1}$

$\Rightarrow P_t^x = \frac{1}{\theta}$. $x_t = \theta^{\frac{2}{1-\theta}} L_{p,t}$ $x_t^* = (\frac{\theta^2}{\tau})^{\frac{1}{1-\theta}} L_{p,t}^*$

$\pi_t = (\frac{1}{\theta} - 1)(x_t + x_t^*)$

Interm F : $\pi_t^*(i) = \max_{b, b^*, p^b} P_t^b(i) b_t(i) + P_t^b(i) b_t^*(i) - (b_t(i) + b_t^*(i))$

s.t. $P_t^b(i) = \theta L_{p,t}^{*1-\theta} b_t^{\theta-1}(i)$

$\tau P_t^b(i) = \theta L_{p,t}^{1-\theta} b_t^{*\theta-1}(i)$

$\Rightarrow P_t^b = \frac{1}{\theta}$. $b_t = \theta^{\frac{2}{1-\theta}} L_{p,t}^*$ $b_t^* = (\frac{\theta^2}{\tau})^{\frac{1}{1-\theta}} L_{p,t}$

$\pi_t^* = (\frac{1}{\theta} - 1)(b_t + b_t^*)$

International lending market : $r^* = r$

R&D H : $0 = \max_{A_{t+1}, L_{RD,t}} P_t^P (A_{t+1} - A_t) - w_t L_{RD,t}$

s.t. $A_{t+1} - A_t = (A_t + \phi(\eta, \tau) A_t^*)\delta L_{RD,t}$.

FOC : $w_t = \delta(A_t + \phi A_t^*) P_t^P$ with $w_t = (1-\theta) Y_t / L_{p,t}$

Patent : $P_t^P = \frac{1}{r}\pi_t = \frac{1-\theta}{r\theta}(x_t + x_t^*)$

R&D F : $0 = \max_{A_{t+1}^*, L_{RD,t}^*} P_t^{P*}(A_{t+1}^* - A_t^*) - w_t^* L_{RD,t}^*$

s.t. $A_{t+1}^* - A_t^* = (A_t^* + \phi(\eta, \tau) A_t)\delta L_{RD,t}^*$

FOC : $w_t^* = \delta(A_t^* + \phi A_t) P_t^{P*}$ with $w_t^* = (1-\theta) Y_t^* / L_{p,t}^*$

Patent : $P_t^{P*} = \frac{1}{r^*}\pi_t^* = \frac{1-\theta}{r\theta}(b_t + b_t^*)$

Euler : $(\frac{C_{t+1}}{C_t})^{\sigma} = \beta(1+r) = (1+\gamma)^{\sigma} \longrightarrow \sigma\gamma = r - \rho$

$A^* = A$, $L_p^* = L$

$$\begin{cases} r = \theta\delta(1+\phi)(L - L_{RD}) \\ \sigma\gamma = r - \rho \\ \gamma = (1+\phi)\delta L_{RD} \end{cases} \Rightarrow \gamma^{open} = \frac{\theta\delta\frac{1+\phi}{2}(L+L^*) - \rho}{\theta + \sigma}$$

$$\gamma^{closed} = \frac{\theta\delta L - \rho}{\theta + \sigma}$$

$\phi > 0 \Leftrightarrow \gamma^{open} > \gamma^{closed}$

$\phi(\eta, \tau)$: spillover of ideas . $\frac{\partial\phi}{\partial\eta} > 0$: η : knowledge spillovers

$\frac{\partial\phi}{\partial\tau} < 0$: τ : trade cost

$\tau = +\infty \downarrow$ finite ; scale effect : $x_t^* \uparrow$. $y_t(i) \uparrow$ output $\uparrow$

growth effect : $\frac{\partial\phi}{\partial\tau} < 0$. $\phi \uparrow$ $\gamma \uparrow$

$\eta \uparrow$: growth effect : $\phi \uparrow$. $A_{t+1} - A_t \uparrow$ $\gamma \uparrow$

OLG ∄ cross-generation trades

Walrasian CE:

$\forall t\ge1,\ G_t:\ \max_{c_{t1},c_{t2}} u(c_{t1},c_{t2})\quad \text{s.t. } p_t c_{t1} + p_{t+1} c_{t2} = p_t e_1 + p_{t+1} e_2$

FOC: $\frac{p_t}{p_{t+1}} = \frac{u_1(c_{t1},c_{t2})}{u_2(c_{t1},c_{t2})} = \mu(c_{t1},c_{t2})$

$t=0$: G_0: $c_{02} = e_2$

Market clearing: $N_t c_{t,1} + N_{t-1} c_{t-1,2} = N_t e_1 + N_{t-1} e_2$

$\Rightarrow \frac{p_t}{p_{t+1}} = \mu(e_1,e_2)$, $c_{t1} = e_1$, $c_{t2} = e_2$

Recursive CE (Sequential Markets Equi)

$\forall t\ge1,\ G_t:\ \max_{c_{t1},c_{t2}} u(c_{t1},c_{t2})\quad \text{s.t. } c_{t1} = e_1 - s_t$

$c_{t2} = e_2 + R_t s_t$

FOC: $R_t = \frac{u_1(c_{t1},c_{t2})}{u_2(c_{t1},c_{t2})} = \mu(c_{t1},c_{t2})$

$t=0$: G_0: $c_{02} = e_2$

Market clearing: $N_t c_{t,1} + N_{t-1} c_{t-1,2} = N_t e_1 + N_{t-1} e_2$

$s_t = 0$

$\Rightarrow R_t = \mu(e_1,e_2)$, $c_{t1} = e_1$, $c_{t2} = e_2$

$N_t = nN_{t-1}$	$t=1$	$t=2$	∃ Pareto improvement
G_{t-1}, N_{t-1}	$e_2+\varepsilon$	$e_2+\varepsilon$	⇔ $\exists \varepsilon>0$ s.t.
G_t, N_t	$e_1-\frac{\varepsilon}{n}$	$e_1-\frac{\varepsilon}{n}$	$u(e_1-\frac{\varepsilon}{n}, e_2+\varepsilon) > u(e_1,e_2)$
RCE	$\mu(e_1,e_2)<n$	$\mu(e_1,e_2)=n$	$\mu(e_1,e_2)>n$
Efficient	Samuelsonian case (Dynamic Inefficient)		Classical case

Fiat money: intrinsically worthless

WCE:

$\forall t\ge1,\ G_t:\ \max_{c_{t1},c_{t2},m_t} u(c_{t1},c_{t2})$

$\text{s.t. } p_t c_{t1} + p_{t+1} c_{t2} + q_t m_t = p_t e_1 + p_{t+1} e_2 + q_{t+1}(m_t + \tau_t)$

$t=0$: G_0 $c_{02} = e_2 + q_1 M$ (Normalize $p_1 = 1$)

MC: $N_t c_{t1} + N_{t-1} c_{t-1,2} = N_t e_1 + N_{t-1} e_2$

$N_t m_t = M_t$. ($M_{t+1} = zM_t$)

$N_t \tau_t = M_{t+1} - M_t = (z-1)M_t$

$\Rightarrow q_t = q^*$: $\mu(e_1 - \frac{q^* M}{n}, e_2 + z q^* M) = n$

$p_{t+1} = \frac{1}{n} p_t$, $p_1 = 1$

$c_{t1} = e_1 - \frac{q_t M_t}{N_t}$, $c_{t2} = e_2 + \frac{q_{t+1}}{p_{t+1}} \frac{zM_t}{N_t}$

RCE: Unexpected monetary policy:

$\forall t\ge1\ G_t:\ \max_{c_{t1},c_{t2}} u(c_{t1},c_{t2})\quad \text{s.t. } c_{t1} = e_1 - q_t m_t$

$c_{t2} = e_2 + q_{t+1}(m_t + \tau_t)$

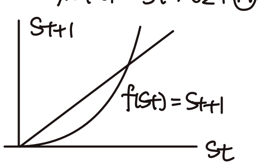
$t=0$ G_0: $c_{02} = e_2 + q_1 M$

MC: $n c_{t,1} + c_{t-1,2} = n e_1 + e_2$

$N_t m_t = M_t$; $\tau_t = \frac{M_{t+1} - M_t}{N_t}$; $M_{t+1} = zM_t$

$\Rightarrow \mu(e_1 - q_t \frac{M_t}{N_t}, e_2 + q_{t+1} \frac{zM_t}{N_t}) = \frac{q_{t+1}}{q_t}$

$\Rightarrow \mu(e_1 - s_t, e_2 + n s_{t+1}) = \frac{s_{t+1}}{s_t} \frac{n}{z}$ Real balance $s_t = q_t m_t$



$f(0) = 0$. $f'(0) = \frac{z}{n} \mu(e_1,e_2)$

∃ stationary monetary Equi.

$\Leftrightarrow \mu(e_1,e_2) < \frac{n}{z}$

Quantity theory of money $PY = VM$

$\mu(e_1 - s_t, e_2 + n s_{t+1}) = \frac{s_{t+1}}{s_t} \frac{n}{z}$

Money is neutral: $M \not\to s$

Money is not superneutral: $z \to s$

Example: $u(c_{t1},c_{t2}) = \ln c_{t1} + \beta \ln c_{t2}$

$\Rightarrow \frac{e_2 + n s_{t+1}}{\beta(e_1 - s_t)} = \frac{n s_{t+1}}{z s_t}$ $\qquad s^* = \frac{n\beta e_1 - z e_2}{n(z+\beta)}$

∃ stationary monetary Equi. $\Leftrightarrow s^* > 0 \Leftrightarrow \mu(e_1,e_2) = \frac{e_2}{\beta e_1} < \frac{n}{z}$

$\Rightarrow c_{t1} = c_1^* = e_1 - s^* = \frac{z(ne_1 + e_2)}{n(z+\beta)}$ $\quad \forall t\ge1$

$c_{t2} = c_2^* = e_2 + n s^* = \frac{\beta(ne_1+e_2)}{z+\beta}$ $\quad \forall t\ge0$

$c_{02} = e_2 + q_1 M = e_2 + n s^* \Rightarrow q_1 = \frac{n s^*}{M} = \frac{n\beta e_1 - z e_2}{(z+\beta)M}$ $\quad \binom{N_1=1}{M_1=M}$

$(PY = VM) \Rightarrow p_1(ne_1 + e_2) = (z+\beta)\frac{ne_1+e_2}{n\beta e_1 - z e_2} M$

$q_t \frac{M_t}{N_t} = s^* = q_1 M \Rightarrow q_t = (\frac{n}{z})^{t-1} M q_1$

$\forall t\ge1,\ G_t:\ u^*(z) = u(c_1^*, c_2^*) = \ln z - (1+\beta)\ln(z+\beta) + \text{Const.}$

$\frac{du^*(z)}{dz} = \frac{1}{z} - \frac{1+\beta}{z+\beta} = 0 \Rightarrow z = 1$

$t=0$ G_0: $u_0(z) = \beta \ln c_2^* = -\beta\ln(z+\beta) + \text{Const.}$ $\quad z\downarrow,\ u_0(z)\uparrow$

The stationary monetary Equ. is PO $\Leftrightarrow z \le 1$

Expected monetary policy (Proportional transfer)

RCE: $\forall t\ge1$ $G_t:\ \max_{c_{t1},c_{t2},m_t} u(c_{t1},c_{t2})\quad \text{s.t. } c_{t1} = e_1 - q_t m_t$

$c_{t2} = e_2 + q_{t+1} z m_t$

$t=0$ G_0: $c_{02} = e_2 + q_1 M$ $\quad (N_1 = 1, M_1 = M)$

MC: $n c_{t1} + c_{t-1,2} = n e_1 + e_2$

$N_t m_t = M_t$

$\Rightarrow \mu(e_1 - q_t m_t, e_2 + q_{t+1} z m_t) = \frac{z q_{t+1}}{q_t}$

$\Rightarrow \mu(e_1 - s_t, e_2 + n s_{t+1}) = n\frac{s_{t+1}}{s_t}$

Money is neutral: $M \not\to s$

Money is superneutral: $z \not\to s$ $\quad \forall z$. Stationary monetary Equi. is efficient.

Seigniorage: Distortion

RCE: $\forall t\ge1$ $G_t:\ \max_{c_{t1},c_{t2},m_t} u(c_{t1},c_{t2})\quad \text{s.t. } c_{t1} = e_1 - q_t m_t$

$c_{t2} = e_2 + q_{t+1} m_t$

$t=0$ G_0: $c_{02} = e_2 + q_1 M$ $\quad (N_1 = 1, M_1 = M)$

MC: $N_t c_{t1} + N_{t-1} c_{t-1,2} + G_t = N_t e_1 + N_{t-1} e_2$

$N_t m_t = M_t$

Government balanced budget: $G_t = q_t(M_t - M_{t-1}) = \frac{z-1}{z} q_t M_t$

$\Rightarrow \mu(e_1 - q_t m_t, e_2 + q_{t+1} m_t) = \frac{q_{t+1}}{q_t}$

$\Rightarrow \mu(e_1 - s_t, e_2 + \frac{n}{z} s_{t+1}) = \frac{s_{t+1}}{s_t} \frac{n}{z}$ $\quad$ Distortion!

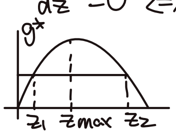
MC: $(e_1 - s_t) + \frac{1}{n}(e_2 + \frac{n}{z} s_t) + g_t = e_1 + \frac{1}{n} e_2$ holds! $g_t = \frac{G_t}{N_t} = \frac{z-1}{z} s_t$

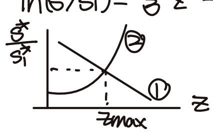
Bailey Curve

SS: $g^* = \frac{z-1}{z} s^*$; $\mu(e_1 - s^*, e_2 + \frac{n}{z} s^*) = \frac{n}{z}$ ①

$\frac{dg^*}{dz} = \frac{1}{z^2} s^* + \frac{z-1}{z} \frac{ds^*}{dz}$ $\qquad z\uparrow\ \frac{dg^*}{dz}\downarrow.$ $\quad (\frac{ds^*}{dz} < 0)$

$\frac{dg^*}{dz} = 0 \Leftrightarrow \ln(s^*/s_1^*) = \frac{1}{3} z^3 - \frac{1}{2} z^2 + \frac{1}{6}$ ② $\quad s_1^* = s^*(z=1)$





z_1 Pareto dominates z_2.

Storage (Safe Asset)

$\forall t \geq 1$ G_t : $u(C_{t1}, C_{t2})$ s.t. $C_{t1} = e_1 - q_t m_t - k_t$

$C_{t2} = e_2 + q_{t+1}(m_t + \tau_t) + x k_t$

$t=0$ G_0 : $C_{02} = e_2 + q_1 M$

MC : $n C_{t1} + C_{t-1,2} = n e_1 + e_2$

$N_t m_t = M_t \Rightarrow N_t \tau_t = M_{t+1} - M_t$, $\tau_t = (z-1) m_t$

① $\exists$ non-monetary Equi. with k $\Leftrightarrow \mu(e_1, e_2) < x$

$\mu(e_1 - k_t, e_2 + x k_t) = x \Rightarrow k^*$

	$t=1$	$t=2$
$G_{t-1} : N_{t-1}$	$C_2 + \varepsilon$	$C_2 + \varepsilon$
$G_t : N_t$	$C_1 - \frac{\varepsilon}{n}$	$C_t - \frac{\varepsilon}{n}$

Equi is not P.O. $\Leftrightarrow$ $\exists \varepsilon > 0$ $u(C_1 - \frac{\varepsilon}{n}, C_2 + \varepsilon) > u(C_1, C_2)$

$\Leftrightarrow \mu(C_1, C_2) = x < n$

② $\exists$ monetary Equi. with k

$\Leftrightarrow \begin{cases} x \leq \frac{n}{z} & \text{better to save by money than storage} \\ \mu(e_1, e_2) \leq x . & \text{people want to save} \end{cases}$

$\mu(e_1 - s_t - k_t, e_2 + n s_{t+1} + x k_t) = x = \frac{n}{z} \frac{s_{t+1}}{s_t}$

Equi is P.O. $\Leftrightarrow \mu(C_1, C_2) = x \geq n$

③ ME with $k \neq$ NME with k.

If $s_t + k_t = k^*$. $n s_{t+1} + x k_t = x(z s_t + k_t) = x(z-1)s_t + x k^*$

$= x k^* \Leftrightarrow z = 1$!

Even though $s_t \to 0$ (Non. stationary ME) :

$s_t + k_t \neq k^*$ when $z \neq 1$. but $k_t \to k^*$

Risky storage

$\forall t \geq 1$. G_t : $\max_{C_{t1}, C_{t2}, m_t, k_t} \frac{C_1^{1-\sigma} - 1}{1-\sigma} + \beta \mathbb{E} \frac{C_2^{1-\sigma} - 1}{1-\sigma}$

s.t. $C_{t1} = e_1 - k_t - q_t m_t$

$C_{t2} = e_2 + x_t k_t + q_{t+1} m_t$ $\quad x_t = \begin{cases} 0 & \frac{1}{2} \\ x & \frac{1}{2} \end{cases}$

FOC : $(e_1 - k_t - q_t m_t)^{-\sigma} = \beta \frac{x}{2} (e_2 + x k_t + q_{t+1} m_t)^{-\sigma}$ ①

$(e_1 - k_t - q_t m_t)^{-\sigma} = \beta \frac{1}{2} \frac{q_{t+1}}{q_t} \{ (e_2 + x k_t + q_{t+1} m_t)^{-\sigma} + (e_2 + q_{t+1} m_t)^{-\sigma} \}$ ②

② $\Rightarrow$ $s_t = q_t m_t = f(k)$

① $\Rightarrow$ $\bar{k} = [(\frac{\beta x}{2})^{\frac{1}{\sigma}} (e_1 - s) - (e_2 + s)] [x + (\frac{\beta x}{2})^{\frac{1}{\sigma}}]^{-1}$

$k^* = \max\{0, \bar{k}\}$

Fiscal Policy in nonmonetary OLG

Policy $\{T_{t1}, T_{t2}, B_t, q_t, g_t\}$:

Budget $N_t T_{t,1} + N_{t-1} T_{t-1,2} + q_t B_t = N_t g_t + B_{t-1}$

$\forall t \geq 1$. G_t : $\max_{C_{t1}, C_{t2}, b_t} u(C_{t1}, C_{t2})$ s.t. $C_{t1} = e_1 - T_1 - q_t b_t$

$C_{t2} = e_2 - T_2 + b_t$

$t=0$ G_0 : $C_{02} = e_2 - T_2 + b_0$

Assume balanced budget : $B_t = 0$. and OG : $g_t = 0$

$C_{t1} = e_1 - T$, $C_{t2} = e_1 + T$. q