Self Insurance: $y_t \in \{\bar{y}_1, \cdots, \bar{y}_N\}$

Dynamic: $V(y,a) = \max_{\{c, a'\}} u(c) + \beta \mathbb{E} V(y', a')$

$\text{s.t. } c + a' = y' + (1+r)a$ Budget

$a' \geq 0$ No borrowing

$$V(y,a) = \max_{a'} u(y' + (1+r)a - a') + \beta \mathbb{E} V(y', a')$$

$\text{s.t. } a' \geq 0 \quad [\pi]$

Cash in hand: $x := y + (1+r)a$

$$V(x) = \max_{a'} u(x - a') + \beta \mathbb{E} V(x')$$

$\text{s.t. } x' = y' + (1+r)a' . \quad a' \geq 0$

$$L = u(x - a') + \beta \mathbb{E} V(x') + \pi a'$$

Euler: $u'(x - a') = \beta(1+r) \mathbb{E} u'(x' - a'') + \pi, \quad a'\pi = 0$

$V'(x) = \beta(1+r) \mathbb{E} V'(x') + \pi, \quad a'\pi = 0$

Case 1: $\beta(1+r) > 1$: $a_t \to +\infty$. $c_t \to +\infty$. $x_t \to +\infty$

Case 2: $\beta(1+r) = 1$: $a_t \to +\infty$ $c_t \to +\infty$ $x_t \to +\infty$

2.1: $a_t \to +\infty$

Proof. BWOC suppose $\exists \bar{a} < +\infty$. s.t. $\forall t$. $a_t \leq \bar{a}$

$x_t = y_t + (1+r)a_t \leq \bar{y}_N + (1+r)\bar{a} = \bar{x}$

$V'(\bar{x}) \geq \mathbb{E} V'(x') = \sum_{i=1}^{N} \pi_i V'(\bar{y}_i + (1+r)a') > V'(\bar{x})$

Then $a_t \to +\infty$. $x_t = y_t + (1+r)a_t \to +\infty$

2.2: $c_t \to +\infty$

Proof. BWOC suppose $\exists \bar{c} < +\infty$. s.t. $\forall t$. $c_t \leq \bar{c}$

$a_t \to +\infty \Rightarrow \pi_t = 0$ Euler: $u'(c_t) = \mathbb{E}_t u'(c_{t+1})$

$c(x) = x - a(x) \in [0, \bar{c}] \Rightarrow a(x) \in [x - \bar{c}, x]$

$\forall x$: $x' = y' + (1+r)a(x) \geq y' + (1+r)(x - \bar{c})$

Take x is large enough. $x' > x \overset{\text{Monotone}}{\Rightarrow} c(x') > c(x)$

$\Rightarrow u'(c(x)) = \mathbb{E} u'(c(x')) < u'(c(x))$

Case 3: $\beta(1+r) < 1$: $c_t \leq \bar{c}$. $a_t \leq \bar{a}$

3.1 $\exists \hat{x} \in (y_1, y_N)$ s.t. $a(x) \begin{cases} = 0 & x < \hat{x} \quad \text{3.1.1} \\ > 0 & x > \hat{x} \quad \text{3.1.2} \end{cases}$

3.1.1 BWOC: suppose $\forall x \geq y_1$, $a(x) > 0$

$\pi = 0 \Rightarrow$ Euler: $u'(c(x)) = \mathbb{E} u'(c(x')) \beta(1+r)$

Take $x = y_1$, $x' = y' + (1+r)a(x) \geq y_1 + (1+r)a(x) > y_1 = x$

$\Rightarrow c(x) < c(x') \Rightarrow u'(c(x)) > \mathbb{E} u'(c(x'))$

3.1.2. ① $\exists x \geq y_1$ s.t. $a(x) > 0$ iff. $\beta(1+r) = 1 - \varepsilon$

BWOC: suppose $\forall x \geq y_1$. $a(x) = 0$. i.e. $x_t = y_t$

$\pi > 0 \Rightarrow$ Euler: $u'(c(x)) > \mathbb{E} u'(c(x')) \beta(1+r)$

Take $x = y_N$. $x' = y' \leq y_N = x$

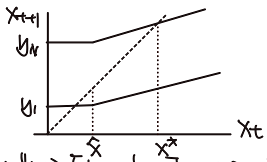
$\Rightarrow c(x) \geq c(x') \Rightarrow u'(c(x)) < \mathbb{E} u'(c(x'))$

② If $\exists x_0 \geq y_1$. $a(x_0) > 0$. then $\forall x \geq x_0$. $a(x) > a(x_0)$

$\begin{cases} u'(c(x_0)) = \beta(1+r) \mathbb{E} V'(y' + (1+r)a(x_0)) \\ u'(c(x)) \geq \beta(1+r) \mathbb{E} V'(y' + (1+r)a(x)) \end{cases}$

$\Rightarrow \forall x \geq x_0$. $a(x) > 0$.

3.2 $\exists x^* > \hat{x}$ s.t. $\forall x_t > x^*$. $x_{t+1} \leq x_t$ $\forall y_{t+1} \in \{y_1, \cdots, y_N\}$



If $\lim_{x \to +\infty} \dfrac{\mathbb{E}_t u'(c_{t+1})}{u'(c_{t+1}(x_{t+1}^{max}(x_t)))} = 0$

For $x > \hat{x}$: $a(x) > 0$. $\pi = 0$

Euler: $u'(c(x)) = \beta(1+r) \mathbb{E} V'(y' + (1+r)a(x))$

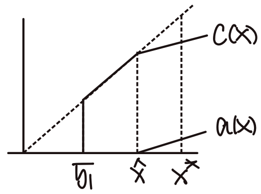
$$u''(c)[1 - a'(x)] = \beta(1+r) \mathbb{E} V''(x')[1 + (1+r)a'(x)]$$

$\Rightarrow a'(x) = \dfrac{u''(c(x))}{\beta(1+r)^2 \mathbb{E} V''(x') + u''(c(x))}$ $|a'(x)| < 1$ (Condition: $\lim_{x \to +\infty} V''(x') < 0$)

Euler: $u'(c_t(x_t)) = \beta(1+r) \dfrac{\mathbb{E} V'(x_{t+1}(x_t))}{u'(c_{t+1}^{max})}$ $u'(c_{t+1}^{max})$.

$\lim_{x_t \to +\infty} \dfrac{u'(c_t(x_t))}{u'(c_{t+1}^{max})} = 0 \Leftrightarrow \lim_{x_t \to +\infty} \dfrac{\mathbb{E} V'(x_{t+1})}{u'(c_{t+1}^{max})} = 0$

$P(\lim_{t \to +\infty} x_t \leq x^*) = 1 \Rightarrow P(\lim_{t \to +\infty} c_t \leq c^*) = 1$



Aiyagari

Stationary Recursive Competitive Equi.

Value $V: Z\times\Lambda \to \mathbb{R}_+$, policy: $\hat{a}': Z\times\Lambda \to \mathbb{R}_+$

$c: Z\times\Lambda \to \mathbb{R}_+$

wage: $w: \Lambda \to \mathbb{R}_+$. interest rate $r: \Lambda \to \mathbb{R}$

stationary distribution of z: $\pi^* \in \Lambda$

Law of motion of π: $H: \Lambda \to \Lambda$ s.t.

① Given $\{w, r\}$, $\{V, \hat{a}', c\}$ solves HH's problem:

$$V(l,a;\pi) = \max_{c,a'} u(c,l) + \beta \mathbb{E} V(l',a';\pi')$$

s.t. $c + a' \le wl + (1+r)a$

$a' \ge -\phi$, $\phi = \min\{b, \frac{w l_{max}}{r}\}$

$\pi' = H\pi$,

Cash-in-hand $z := wl + (1+r)\hat{a} - r\phi$, $\hat{a} = a + \phi \ge 0$

$$V(z;\pi) = \max_{c,\hat{a}'} u(c,l) + \beta \mathbb{E} V(z';\pi')$$

s.t. $c + \hat{a}' \le z$; $z' = wl' + (1+r)\hat{a}' - r\phi$

$\hat{a}' \ge 0$; $\pi' = H\pi$

② $\{w, r\}$ satisfies firm's FOC:

Firm: $\max_{K,L} F(K,L) - wL - (r+\delta)K$

FOC: $F_K(K,L) = r + \delta$

$F_L(K,L) = w$

③ Markets clearing:

Capital: $K = \int \hat{a}'(z)\, d\pi^*(z) - \phi$

Labor: $L = \int_{l_{min}}^{l_{max}} l \, d\pi(l)$

Good: $C + K = F(K,L) + (1-\delta)K$

$\Rightarrow \int c(z)\, d\pi^*(z) + \delta K = F(K,L)$

④ Consistency: $T^*\pi = H\pi \quad \forall \pi \in \Lambda$

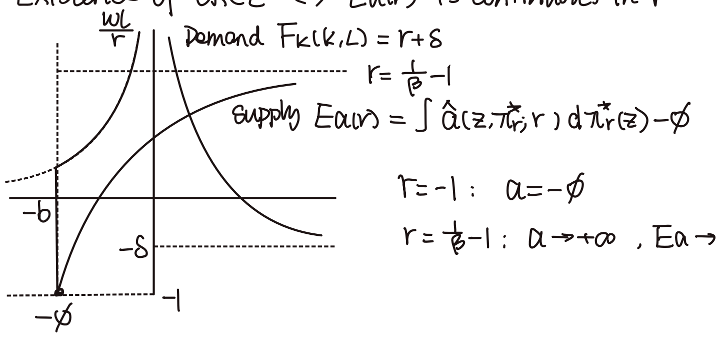
$Q(z,S;\pi) := \int \mathbb{1}\{z'(z,l';\pi) \in S\}\, d\pi(l)$

where $z'(z,l';\pi) = wl' + (1+r)\hat{a}'(z,\pi) - r\phi$

$(T^*\pi)(S) := \int Q(z,S;\pi)\, d\pi(z)$

⑤ Stationary: $\pi^* = T^*\pi^*$

Existence of SRCE $\Leftrightarrow$ $Ea(r)$ is continuous in r



$r = -1$: $a = -\phi$

$r = \frac{1}{\beta} - 1$: $a \to +\infty$, $Ea \to +\infty$

① $\hat{a}(z, \pi_r^*; r)$ is continuous on r.

(1.1) Berge's thm of maximum: $F(x) = \max_{y \in \Gamma(x)} f(x,y)$

$f(x,y)$ is continuous, Γ is continuous & compact

then $F(x)$ is continuous.

$y^*(x)$ is non-empty, compact-valued & u.s.c.

(1.2) $\hat{a}$ is single valued:

feasible set of $\hat{a}$ is convex & $u'' < 0$

② π_r is continuous on r: $r_n \to r$, $\pi_{r_n} \to \pi_r$ weakly.

i. Z is compact

ii. $(z_n, r_n) \to (z_0, r_0) \Rightarrow Q_{r_n}(z_n, \cdot) \xrightarrow{w} Q_{r_0}(z_0, \cdot)$

iii. $\forall r \in \mathbb{R}$, T_r^* has a unique FP $\pi_r^* \in \Lambda(Z, \mathcal{Z})$

2 operators: $(Tf)(z) := \int f(z')\, Q(z, dz') = \mathbb{E}(f(z') | z)$

$(T^*\pi)(S) := \int Q(z,S)\, d\pi(z)$

(2.1) $\exists \pi^*$ s.t. $\pi^* = T^*\pi^*$ under transform function Q

$C(Z)$ is compact & Q has Feller Property

$Z = [z_{min}, z_{max}]$ $\quad T: C(Z) \to C(Z)$ continuous bounded

$C(Z) = Z'$ Dual Space $\quad \forall f \in C(Z)$, $E(f(z')|z) \in C(Z)$

$\Rightarrow T$ has a FP $\Rightarrow T^*$ has a FP

(2.2) π^* is unique: $\forall \pi_0$, $T_r^{*n} \pi_0 \xrightarrow{w} \pi^*$ $\quad \forall \pi_0 \in \Lambda$

Q is monotonic & Feller & monotone mixing condition

f non-decreasing $\quad \exists c \in Z$, $\varepsilon > 0$, $N \ge 1$ s.t.

Tf non-decreasing $\quad Q^N(a, [c,b]) \ge \varepsilon$, $Q^N(b, [a,c]) \ge \varepsilon$

Comparative Static analysis

① CRRA: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, risk aversion $\sigma \uparrow$

$Ea(r) \uparrow \quad r \downarrow \quad K \uparrow$

② Borrowing limit $b \uparrow \quad Ea(r) \downarrow \quad r \uparrow \quad K \downarrow$

③ Persistency of shock $\uparrow \quad \ln l_{t+1} = \rho \ln l_t + \varepsilon$

risk $= Var(\ln l_t) = \frac{\sigma^2}{1-\rho^2} \uparrow \quad Ea(r) \uparrow \quad r \downarrow \quad K \uparrow$

Algorithm to compute SRCE:

① Guess $r^0 \in (-\delta, \frac{1}{\beta} - 1)$

② $\begin{cases} r^0 + \delta = F_K(K_{demand}, L) & \Rightarrow K_{demand} \\ w(r^0) = F_L(K_{demand}, L) & \Rightarrow w(r^0) \end{cases}$

③ VFI, policy: $\{a'(a,l;r^0), c(a,l;r^0)\}$

④ TFI. $\pi^*(a',l';r^0) = \int_{A\times L} Q((a,l),(a',l'))\, d\pi^*(a,l;r^0)$

$Q((a,l),(a',l')) = \mathbb{1}\{a'(a,l';r^0) = a'\}\, \pi(l'|l)$

i.e. $\pi^0(a,l;r^0) \xrightarrow{Markov} P(a,l';r^0) \xrightarrow{Policy} \pi^1(a',l';r^0)$

⑤ $A_{supply}(r^0) = \int_{A\times S} a'(a,l;r^0)\, d\pi^*(a,l;r^0)$

⑥ Update $r^0 = \frac{1}{2}(r^0 + F_K(A_{supply}(r^0), L) - \delta)$

Transversality / Terminal condition: $\lim_{t\to+\infty} \frac{a_{t+1}}{(1+r)^t} \le 0$

Natural borrowing limit:

$c_t + a_{t+1} = y_t + (1+r)a_t \Rightarrow a_t \ge \frac{1}{1+r}(a_{t+1} - y_t)$

Iteration: $a_t \ge \left(\frac{1}{1+r}\right)^T a_{t+T} - \sum_{\tau=1}^{T} \left(\frac{1}{1+r}\right)^\tau y_{t+\tau-1}$

$\ge \lim_{T\to+\infty} \frac{a_{T+t}}{(1+r)^T} - \frac{1}{r} y_{min}$

$a_t \ge -\frac{1}{r} y_{min} \Leftrightarrow \lim_{t\to+\infty} \frac{a_{t+1}}{(1+r)^t} \ge 0$ Non-ponzi Condition

Government determine (τ, B)

⓪: Balance budget: $T + (1+r)B = \tau w L + B'$ (issue bond)

①: Given $\{r, w, \tau, T\}$

$$V(a,b,l;P(a,b)) = \max_{c,a',b'} u(c,l) + \beta \mathbb{E} V(a',b',l';P(a',b'))$$

$$\text{s.t. } c + a' + b' \le (1-\tau)wl + (1+r)(a+b) + T$$

$$c \ge 0, \quad a' \ge -\phi, \quad b' \ge 0$$

$$P(a',b') = H P(a,b)$$

Cash-in-hand: $z = (1-\tau)wl + T + (1+r)(\hat{a}+b) - r\phi$

$$\hat{a} = a + \phi,$$

$$V(z;\pi) = \max_{c,\hat{a}',b'} u(c,l) + \beta \mathbb{E} V(z';\pi')$$

$$\text{s.t. } c + \hat{a}' + b' \le z, \quad c \ge 0, \ \hat{a}' \ge 0, \ b \ge 0$$

$$\pi'(S) = H\pi(S) \quad \forall S \in (Z, \mathcal{Z})$$

②: $\{r, w\}$ solves firm's FOC: $\begin{cases} r + \delta = F_K(K,L) \\ w = F_L(K,L) \end{cases}$

③ Markets clear:

Capital: $K = \int \hat{a}'(z;\vec{\pi})\, d\pi^*(z) - \phi$

Bond: $B = \int b'(z;\vec{\pi})\, d\pi^*(z)$

Labor: $L = \int_{l_{min}}^{l_{max}} l\, d\pi(l)$

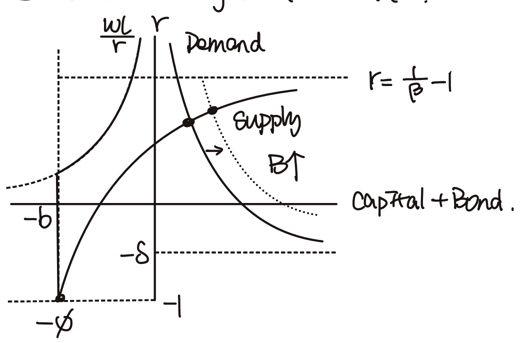
Good: $F(K,L) = \int c(z;\vec{\pi})\, d\pi^*(z) + \delta K$

④ Consistency (Ration belief) $H\pi = T^*\pi \quad \forall \pi.$

$$T^*\pi(S) := \int Q(z,S)\, d\pi(z)$$

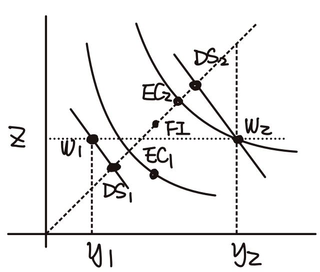
$$Q(z,S) := \int_{l_{min}}^{l_{max}} \mathbb{1}\{(1-\tau)wl' + T + (1+r)(\hat{a}+b) - r\phi \in S\}\, d\pi(l)$$

⑤ Stationarity: $\pi^* = T^*\pi^*$. $B = B'$



The effecs of government debt:

- (+): If $B > 0$, we have an equilibrium with higher interest rate and more total wealth.
- (+): Higher interest rates make self-insurance cheaper for the households, so higher debt improves consumption smoothing.
- (-): Now that r increases, there is less demand for capital by the firms, so there is a crowding out effect of government debt against capital stock.
- (-): If $r > 0$ is paid by taxes, we are introducing an additional distortion.



$W_{1\&2}$: Endowment Autarky

$W_1 . DS_2$: R Storage Autarky

$DS_{1\&2}$: Debt-sequrity

$EC_{1\&2}$: Efficient contract

FI: Full insurance

EC_2 is expost full insurance. EC_1 is not.

Endogenously Incomplete Markets

Asymmetric private information (Green & Oh 1991)

Set-up: Exchange economy $t = 0, 1, 2$

Endowment: $y \in \{y_1, \dots, y_N\}$ on $\{\pi_i\}$. $\quad z$

Identical preference: $u(c_1, c_2) = w(c_1) + v(c_2)$, $v = \frac{1}{R} w$

$$\max_{\{c_{1i}, c_{2i}\}_{i=1}^N} \sum_{i=1}^N \pi_i u(c_{1i}, c_{2i})$$

Linear tech: x good 1 $\to Rx$ good 2

4 Allocations:
1. Endowment Autarky
2. Full-insurance
3. expost efficient: securities traded at $t=1$
4. exante efficient: contracts binded at $t=0$

2. Full-insurance: $\max_{\{c_{1i}, c_{2i}\}_{i=1}^N} \sum_{i=1}^N \pi_i [w(c_{1i}) + v(c_{2i})]$

$$\text{s.t. } \sum_{i=1}^N \pi_i c_{1i} + \frac{1}{R} \sum_{i=1}^N \pi_i c_{2i} \le \sum_{i=1}^N \pi_i y_i + \frac{1}{R} z \qquad \lambda$$

$$\sum_{i=1}^N \pi_i c_{2i} \ge z \qquad \mu$$

$$c_{1i} \ge 0, \quad c_{2i} \ge 0 \qquad \gamma_{1i}, \gamma_{2i}$$

FOC: $\frac{\partial L}{\partial c_{1i}} = \pi_i w'(c_{1i}) - \pi_i \lambda + \gamma_{1i} = 0$

$$\frac{\partial L}{\partial c_{2i}} = \pi_i v'(c_{2i}) - \pi_i \lambda + \pi_i \mu + \gamma_{2i} = 0$$

Inada: $w'(0) = v'(0) = +\infty \Rightarrow \gamma_{1i} = \gamma_{2i} = 0$

$w'(+\infty) = v'(+\infty) = 0 \Rightarrow \lambda > 0 \quad \lambda \ge \mu$ Resource binding, Borrowing $\mu \ge 0$

If Borrowing constraint is bounded: $\mu > 0$ $\left(\sum_{i=1}^N \pi_i y_i \le z\right)$

$$c_{1i} = c_1 = \sum_{i=1}^N \pi_i y_i \qquad c_{2i} = c_2 = z$$

Otherwise: $c_{1i} = c_{2i} = \frac{R \sum_{i=1}^N \pi_i y_i + z}{R+1}$ Full insurance $\left(\sum_{i=1}^N \pi_i y_i > z\right)$

Net trade: $\gamma_{1i}^* = \frac{R\bar{y} + z}{R+1} - y_i$; $\gamma_{2i}^* = \frac{R(\bar{y} - z)}{R+1} = \gamma_2^* > 0$

Expost IC ✗: $w(y_i + \gamma_{1i}^*) + v(z + \gamma_{2i}^*) < w(y_i + \gamma_{1l}^*) + v(z + \gamma_{2l}^*)$

3. Securities at $t=1$: $\max_{c_{1i}, c_{2i}} w(c_{1i}) + v(c_{2i})$

$$\text{s.t. } c_{1i} + \frac{1}{R} c_{2i} \le y_i + \frac{1}{R} z$$

$$\Rightarrow c_{1i} = c_{2i} = \frac{R y_i + z}{R+1}$$

Net trade: $\gamma_{1i}^* = c_{1i} - y_i = \frac{z - y_i}{R+1}$; $\gamma_{2i}^* = c_{2i} - z = \frac{R(y_i - z)}{R+1}$

Expost IC: $w(y_i + \gamma_{1i}^*) + v(z + \gamma_{2i}^*) \ge w(y_i + \gamma_{1j}^*) + v(z + \gamma_{2j}^*)$

Expost IR: $w(y_i + \gamma_{1i}^*) + v(z + \gamma_{2i}^*) \ge w(y_i) + v(z)$

$$\Rightarrow \text{Exante IR: } \sum_{i=1}^N \pi_i u_i(y_i + \gamma_{1i}^*, z + \gamma_{2i}^*) \ge u_{Autarky}$$

3'. Storage at $t=1$: Borrowing constraint: $c_{2i} \ge z$

$\Rightarrow$ If $y_i \ge z$: $c_{1i} = c_{2i} = \frac{R y_i + z}{R+1}$

Otherwise: $c_{1i} = y_i$, $c_{2i} = z$

4. Contract at $t=0$: Constrained Efficient Allocation

$$\max_{\{\gamma_{1i}, \gamma_{2i}\}} \sum_{i=1}^N \pi_i [w(y_i + \gamma_{1i}) + v(z + \gamma_{2i})]$$

$$\text{s.t. } \sum_{i=1}^N \pi_i \gamma_{2i} \ge 0 \qquad \sum_{i=1}^N \pi_i \left(\gamma_{1i} + \frac{1}{R} \gamma_{2i}\right) \le 0$$

$$y_i + \gamma_{1i} \ge 0, \quad z + \gamma_{2i} \ge 0$$

IC $w(y_i + \gamma_{1i}) + v(z + \gamma_{2i}) \ge w(y_i + \gamma_{1j}) + v(z + \gamma_{2j}) \quad \forall j$

IR $\sum_{i=1}^N \pi_i [w(y_i + \gamma_{1i}) + v(z + \gamma_{2i})] \ge$ EU storage

$\forall$ contract Γ satisfying IC & IR: $U(\Gamma)$

Γ^* is an equilibrium contract if

$\nexists \Gamma$: $U(\Gamma) \ge U(\Gamma^*)$, $-\sum_{i=1}^N \pi_i \left(\gamma_{1i} + \frac{1}{R}\gamma_{2i}\right) \ge 0$ "$>$" ≥ 1

Γ^* is efficient:

Limited Commitment:

Endowment: $(\bar y_s, 1-\bar y_s)$ $P(y_t=\bar y_s)=\pi_s$, $\bar y_s\in[0,1]$

Recursive Formulation of Constrained Efficient Allocation:

$$Q_s(x) = \max_{c,\{x_j\}_{j=1}^S} u(1-c) - u(1-\bar y_s) + \beta\sum_{j=1}^S \pi_j Q_j(x_j)$$

μ s.t. $u(c)-u(\bar y_s)+\beta\sum_{j=1}^S\pi_j x_j \ge x$ promise keeping

$\beta\pi_j\lambda_j$ $x_j\ge 0$ IR1

$\beta\pi_j\theta_j$ $Q_j(x_j)\ge 0$ IR2

$c\in[0,1]$

$\forall j$: Q_j: decreasing, strictly concave, C^1

$x_j\in[0,\bar x_j]$: $Q_j(\bar x_j)=0$

FOC: $u'(1-c)=\mu u'(c)$

$\mu+\lambda_j+(1+\theta_j)Q_j'(x_j)=0$

Envelope: $Q_s'(x)=-\mu \Rightarrow Q_s'(x) = -\frac{u'(1-c)}{u'(c)} =: g(c)$

$Q_s'(x)=\lambda_j+(1+\theta_j)Q_j'(x_j)$

$x\in[0,\bar x_s]\Rightarrow c\in[\underline c_s,\bar c_s]$:

$Q_s'(0) = -\frac{u'(1-\underline c_s)}{u'(\underline c_s)}$, $Q_s'(\bar x_s) = -\frac{u'(1-\bar c_s)}{u'(\bar c_s)}$

$\frac{u'(1-c)}{u'(c)} = -\lambda_j + (1+\theta_j)\frac{u'(1-c_j)}{u'(c_j)}$:

$\lambda_j=\theta_j=0$: $c=c_j$ Full insurance

$\lambda_j>0$, $\theta_j=0$: $c<c_j$ IR1 $c_j=\underline c_j$

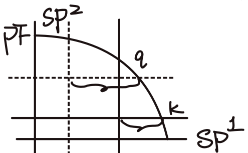
$\lambda_j=0$, $\theta_j>0$: $c>c_j$ IR2 $c_j=\bar c_j$

① $\Rightarrow c_j = \begin{cases}\underline c_j & \text{if } c<\underline c_j\\ c & \text{if } c\in[\underline c_j,\bar c_j]\\ \bar c_j & \text{if } c>\bar c_j\end{cases}$

② $\bar y_k>\bar y_q \Rightarrow \bar c_k>\bar c_q$ & $\underline c_k>\underline c_q$

$Q_k(x-u(\bar y_k)+u(\bar y_q)) = Q_q(x)-u(1-\bar y_k)+u(1-\bar y_q)$

$\Rightarrow Q_k'(x-u(\bar y_k)+u(\bar y_q)) = Q_q'(x)$



$\begin{cases}Q_k(x)=PF(x+u_k^{1Aut})-u_k^{2Aut}\\ Q_q(x)=PF(x+u_q^{1Aut})-u_q^{2Aut}\end{cases}$

$u_k^{1Aut}-u_q^{1Aut}=u(\bar y_k)-u(\bar y_q)$ (iid y_t)

$Q_k(\bar x_q-u(\bar y_k)+u(\bar y_q)) = -u(1-\bar y_k)+u(1-\bar y_q)>0=Q_k(\bar x_k)$

$\Rightarrow \bar x_q-u(\bar y_k)+u(\bar y_q)<\bar x_k$

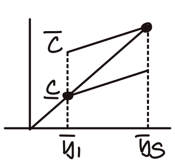
$\Rightarrow Q_k'(\bar x_k)<Q_k'(\bar x_q-u(\bar y_k)+u(\bar y_q))=Q_q'(\bar x_q)$

$\Rightarrow g(\bar c_k)<g(\bar c_q) \Rightarrow \bar c_k>\bar c_q$

$Q_k'(0)=Q_q'(u(\bar y_k)-u(\bar y_q))<Q_q'(0)$

$\Rightarrow g(\underline c_k)<g(\underline c_q) \Rightarrow \underline c_k>\underline c_q$

③ $\bar y_s\in[\underline c_s,\bar c_s]$, $\bar y_1=\underline c_1$, $\bar y_N=\bar c_N$



For $\bar c_s$: let $x=\bar x_s$

$Q_s(\bar x_s)=u(1-\bar c_s)-u(1-\bar y_s)+\beta\sum_{j=1}^N\pi_j Q_j(x_j)=0$

$Q_j(x_j)\ge 0 \Rightarrow u(1-\bar c_s)\le u(1-\bar y_s)$

$Q_N(\bar x_N)=u(1-\bar c_N)-u(1-\bar y_N)+\beta\sum_{j=1}^N\pi_jQ_j(x_j)=0$

$Q_N'(\bar x_N)=Q_j'(\bar x_N+u(\bar y_N)-u(\bar y_j))\le Q_j'(\bar x_j)\le Q_j'(x_j)$

$Q_N'(\bar x_N)=-\lambda_j+(1+\theta_j)Q_j'(x_j)$ ($\lambda_j=0$ $\forall j$)

$\Rightarrow \theta_j>0$ $\forall j<N$. $\Rightarrow x_j=\bar x_j$ $\Rightarrow \bar c_N=\bar y_N$

if $\theta_N=0$: $Q_N'(\bar x_N)=Q_N'(x_N)$. otherwise $Q_N(x_N)=0 \Rightarrow x_N=\bar x_N$

For $\underline c_s$: $x=0$ $u(\underline c_s)-u(y_s)+\beta\sum_{j=1}^N\pi_j x_j=0$

$\forall j$, $x_j\ge 0 \Rightarrow u(\underline c_s)\le u(y_s) \Rightarrow \underline c_s\le y_s$

$x(\bar y_1)=u(\underline c_1)-u(y_1)+\beta\sum_{j=1}^N\pi_j x_j=0$

$Q_1'(0)=-\lambda_j+(1+\theta_j)Q_j'(x_j)$ ($\theta_j=0$ IR2 does not bind)

$\Rightarrow \lambda_j>0$ $\forall j$ $\Rightarrow x_j=0$ $\forall j$ $\Rightarrow \underline c_1=y_1$

④ $\forall s$, $\underline c_s<\bar c_s$ unless Autarky is the only sustainable alloct.

If $\exists k$ s.t. $\underline c_k=\bar c_k=\bar y_k$, then $\forall s$, $\underline c_s=\bar c_s=\bar y_s$.

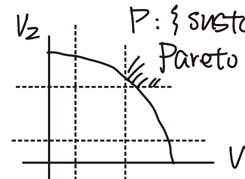
$\bar x_k=0$: $\bar x_k\ge u(\bar c_k=\bar y_k)-u(\bar y_k)+\beta\sum_{j=1}^N\pi_j x_j\Rightarrow x_j=0$ $\forall j$

$Q_k(\bar x_k)=0$: $Q_k(\bar x_k)=u(1-\bar y_k)-u(1-\bar y_k)+\beta\sum_{j=1}^N\pi_j Q_j(x_j)$

$\Rightarrow Q_j(x_j=0)=0$ $\forall j$. $\Rightarrow \bar x_j=0$ $\forall j$

FB allocation is sustainable $\Leftrightarrow \bigcap_s[\underline c_s,\bar c_s]\ne\emptyset$

$\Leftrightarrow \frac{1}{1-\beta}u(\frac12)\ge\max_s V_{aut}(y_s)$ If symmetry



Example: $y\in\{y_L,y_H\}$

$y_H-y_L\uparrow$ $u_H-u_L\uparrow$ $Pr(FB\in\Gamma)\downarrow$

$\pi\uparrow$ $V_H-V_L\uparrow$ $Pr(FB\in\Gamma)\downarrow$

$V_{aut}^1(y_H)=u(y_H)+\beta[\pi V^1_{aut}(y_H)+(1-\pi)V^1_{aut}(y_L)]$

$V_{aut}^1(y_L)=u(y_L)+\beta[(1-\pi)V^1_{aut}(y_H)+\pi V^1_{aut}(y_L)]$

$\Rightarrow \begin{cases}(1-\beta)(V_H+V_L)=u_H+u_L\\(1+\beta-2\beta\pi)(V_H-V_L)=u_H-u_L\end{cases}$

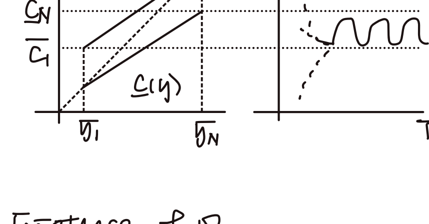
$\Rightarrow(1-\beta)V_H=\frac{1-\beta\pi}{1-\beta\pi+\beta(1-\pi)}u_H+\frac{\beta(1-\pi)}{1-\beta\pi+\beta(1-\pi)}u_L$

Sustainable: $u(c_{FB})\ge(1-\beta)V_H$: strictly decreasing on β.

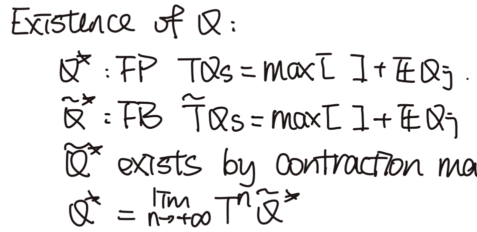
Folk thm: $\exists\beta^*\in(0,1)$, s.t. $\forall\beta\ge\beta^*$. FB is sustainable

With iid shock: $u(c_{FB})\ge(1-\beta)V_H=(1-\beta)\{u_H+\beta Eu\}$

$$\beta^*=\frac{u_H-u(c_{FB})}{u_H-Eu}$$



FB is sustainable $\Leftrightarrow\bigcap_{j=1}^N[\underline c_j,\bar c_j]\ne\emptyset \Leftrightarrow \underline c_N\ge\bar c_1$



FB is not sustainable

Range of fluctuation $=\underline c_N-\bar c_1$

Existence of Q:

Q^*: FP $TQ_s=\max[\ ]+\mathbb{E}Q_j$. s.t. PK, IR1, IR2

$\tilde Q^*$: FB $\tilde T Q_s=\max[\ ]+\mathbb{E}Q_j$ s.t. PK

$\tilde Q^*$ exists by contraction mapping thm.

$Q^*=\lim_{n\to+\infty}T^n\tilde Q^*$

Both SPE

Q is not unique: Constrained efficient & Autarky