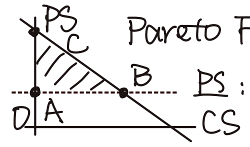
Adverse Selection

Ex. Price Discrimination



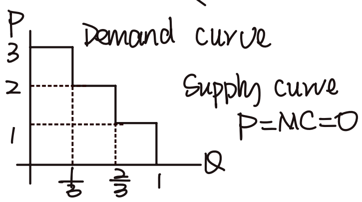
Pareto Frontier

PS:

uniform price: A

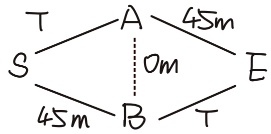
1st PD: C

△ABC: achievable PD



Braess' Paradox

$N=40$ Road ↑ ET ↑



Without AB: (20A, 20B)

$ET_{SE}=65m$

With AB: (40 SABE)

$ET_{SE}=80m$

MLRP: $\frac{f(\pi|e_H)}{f(\pi|e_L)}$ increases in π.

① $\iff \frac{\partial^2}{\partial e\partial\pi}\ln f(\pi|e)>0$ i.e. $\frac{\partial \ln f(\pi|e)}{\partial e}$ increases in π

② $\Rightarrow$ FOSD $F(\pi|e_H)\le F(\pi|e_L)$

①: $\frac{\partial}{\partial\pi}\frac{f(\pi|e+\Delta)}{f(\pi|e)}>0 \iff \frac{\partial}{\partial\pi}\left[\frac{f(\pi|e+\Delta)}{f(\pi|e)}-1\right]>0$

$\iff \frac{\partial}{\partial\pi}\frac{1}{f(\pi|e)}\frac{f(\pi|e+\Delta)-f(\pi|e)}{\Delta}>0$

②: $\frac{f(\pi_L|e_H)}{f(\pi_L|e_L)}\le\frac{f(\pi_H|e_H)}{f(\pi_H|e_L)} \Rightarrow \frac{f(\pi_L|e_H)}{f(\pi_H|e_H)}\le\frac{f(\pi_L|e_L)}{f(\pi_H|e_L)}$

$\Rightarrow \int_{-\infty}^{\pi_H}\frac{f(\pi_L|e_H)}{f(\pi_H|e_H)}d\pi_L \le \int_{-\infty}^{\pi_H}\frac{f(\pi_L|e_L)}{f(\pi_H|e_L)}d\pi_L$

$\Rightarrow \frac{F(\pi_H|e_H)}{f(\pi_H|e_H)}\le\frac{F(\pi_H|e_L)}{f(\pi_H|e_L)} \Rightarrow \frac{f(\pi_H|e_L)}{F(\pi_H|e_L)}\le\frac{f(\pi_H|e_H)}{F(\pi_H|e_H)}$

$\Rightarrow \int_{\pi}^{+\infty}\frac{f(\pi_H|e_L)}{F(\pi_H|e_L)}d\pi_H \le \int_{\pi}^{+\infty}\frac{f(\pi_H|e_H)}{F(\pi_H|e_H)}d\pi_H$

$\Rightarrow \ln F(\pi|e_H)\le \ln F(\pi|e_L) \Rightarrow F(\pi|e_H)\le F(\pi|e_L)$

Signaling (Job Market)

Worker $\theta\in\{L,H\}$ Cost $=\frac{e}{\theta}$ Prior $P(\theta=H)=\pi$

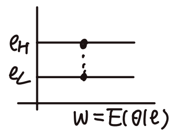
Competitive firm: wage $=E(\theta|e)$

1. Separating Equilibrium

ICH: $E(\theta|e_H)-\frac{e_H}{H}\ge E(\theta|e_L)-\frac{e_L}{H}$

ICL: $E(\theta|e_L)-\frac{e_L}{L}\ge E(\theta|e_H)-\frac{e_H}{L}$

$\iff \frac{e_H-e_L}{L}\ge E(\theta|e_H)-E(\theta|e_L)\ge\frac{e_H-e_L}{H}>0$



$\Rightarrow$ ① Single Crossing: $E(\theta|e_H)-E(\theta|e_L)>0$.

② $e_H\ge e_L$

PBE: $\{(e_H^*, e_L^*, E(\theta|e)), P(\theta=H|e)\}$ ← Belief: Not unique!

$E(\theta|e)=H\,1\{e\ge e^*\}+L\,1\{e<e^*\} \Rightarrow e_L^*=0$

$P(\theta=H|e)=1\{e\ge e^*\}$ $\quad e_H^*=e^*$

SC: $\begin{cases} H-\frac{e^*}{H}\ge L \\ L\ge H-\frac{e^*}{L}\end{cases} \Rightarrow e^*\in[(H-L)L, (H-L)H]$

2. Pooling Equilibrium. $e_H=e_L=e^*$

PBE: $\{(e_H^*=e^*, e_L^*=e^*, E(\theta|e)), P(\theta=H|e)\}$

$E(\theta|e)=P(\theta=H|e)H+[1-P(\theta=H|e)]L$

$P(\theta=H|e)=\pi\,1\{e=e^*\}$

$\begin{cases}\pi H+(1-\pi)L-\frac{e^*}{H}\ge L-\frac{e'}{H} & \forall e'\ne e^* \\ \pi H+(1-\pi)L-\frac{e^*}{L}\ge L-\frac{e'}{L} & \forall e'\ne e^*\end{cases}$

$\Rightarrow \pi H+(1-\pi)L-\frac{e^*}{L}\ge L \Rightarrow e^*\in[0,\pi(H-L)L]$

Reasonable PBE: Consider $e^*=0$.

Let $e'=e$ s.t. $\underline{H-\frac{e}{L}}<\pi H+(1-\pi)L<\underline{H-\frac{e}{H}}$

maximum value off path

Only H has incentive to deviate to e'.

$P(\theta=H|e')=0$ is unreasonable belief.

Intuitive Criterion: Belief $P(\theta=L|e)=0$ if e is only EDS for H type.

Equilibrium-dominated Signals: Take $e\ne e^*$ consider the best situation of one type.

If this type has incentive to deviate, then e is EDS

IC has no restriction on non-EDS nor both-EDS.

No pooling equi. satisfies IC:

$\forall e^*\in[0,\pi(H-L)L]$: $\pi H+(1-\pi)L-\frac{e^*}{L}\ge L$

Take $e'\ne e^*$: $\pi H+(1-\pi)L-\frac{e^*}{L}=H-\frac{e'}{L}$ $\quad (e'>e^*)$

$\Rightarrow \pi H+(1-\pi)L-\frac{e^*}{H}<H-\frac{e'}{H}$

$e'+\varepsilon$ is only EDS for H.

3. Semi-separating Equilibrium.

$\exists\,\theta\in\{H,L\}$. Type θ has mixed strategy.

No semi-separating equi. satisfies IC!

Optimal Insurance Contract

Risk averse agent: $u,\ (w.\ I)$ $\quad p$

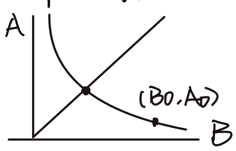
Risk neutral insurance company

$A = w - I + m.\quad B = w - c$

Firm $\max_{\{c,m\}} (1-p)c - pm \Leftrightarrow \min_{\{A,B\}} pA + (1-p)B.$

$\Leftrightarrow \min_{\{a,b\}} p u^{-1}(a) + (1-p) u^{-1}(b)$

1. Homogeneity agent $\Rightarrow$ Full insurance is optimal.



$\min_{\{A,B\}} pA + (1-p)B$

s.t. $pu(A) + (1-p)u(B) \geq u_0$ $\quad$ IR

$\Rightarrow$ FOC: $u'(A) = u'(B)$

2. Heterogeneity $p_L < p_H$. Prior $P(H) = t$

2.1 Optimal Separating contract

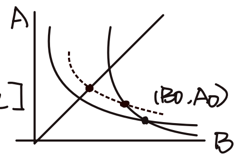
$\min t[p_H A_H + (1-p_H)B_H] + (1-t)[p_L A_L + (1-p_L)B_L]$

s.t. $p_i u(A_i) + (1-p_i)u(B_i) \geq u_{0i}$ $\quad$ IR

$p_i u(A_i) + (1-p_i)u(B_i) \geq p_i u(A_j) + (1-p_i)u(B_j)$ $\quad$ IC

IRL + ICH $\Rightarrow$ IRH

ICH binds. ICL does not bind.



$\min_{\{a,b\}} t[p_H A_H + (1-p_H)B_H] + (1-t)[p_L A_L + (1-p_L)B_L]$

s.t. $p_L u(A_L) + (1-p_L)u(B_L) \geq u_{0L}$

$p_H u(A_H) + (1-p_H)u(B_H) \geq p_H u(A_L) + (1-p_H)u(B_L)$

$L = t[p_H A_H + (1-p_H)B_H] + (1-t)[p_L A_L + (1-p_L)B_L]$

$+ \lambda[u_{0L} - p_L u(A_L) - (1-p_L)u(B_L)]$ $\quad (\lambda \geq 0)$

$+ \mu[p_H u(A_L) + (1-p_H)u(B_L) - p_H u(A_H) - (1-p_H)u(B_H)]$ $\quad (\mu \geq 0)$

FOC: $\frac{\partial L}{\partial A_H} = tp_H - \mu p_H u'(A_H) = 0$

$\frac{\partial L}{\partial B_H} = t(1-p_H) - \mu(1-p_H)u'(B_H) = 0$

$\frac{\partial L}{\partial A_L} = (1-t)p_L - \lambda p_L u'(A_L) + \mu p_H u'(A_L) = 0$

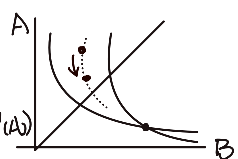
$\frac{\partial L}{\partial B_L} = (1-t)(1-p_L) - \lambda(1-p_L)u'(B_L) + \mu(1-p_H)u'(B_L) = 0$

$\Rightarrow \begin{cases} \mu > 0. & \text{H type is indifferent between } (A_H, B_H) \ \&\ (A_L, B_L) \\ \lambda > 0. & \text{L type is indifferent between } (A_L, B_L) \ \&\ \text{no} \end{cases}$

$\Rightarrow \begin{cases} A_H = B_H & \text{Full insurance for H type} \\ \frac{u'(A_L)}{u'(B_L)} = \frac{\lambda + \mu\frac{1-p_H}{1-p_L}}{\lambda + \mu\frac{1-p_H}{1-p_L}} < 1,\ A_L > B_L & \text{No full ins. L} \end{cases}$

2.2 Pooling contract is not optimal $\{A, B\}$

Case 1: $A \neq B$ $\quad$ WLOG $A > B$



$(A + dA, B + dB) \sim_L (A, B)$: $\quad (dB > 0)$

$p_L u'(A)dA + (1-p_L)u'(B)dB = 0$ $\quad (u'(B) > u'(A))$

H chooses (A, B): $d\,\text{utility}_H = p_H u'(A)dA + (1-p_H)u'(B)dB < 0$

$\pi_L \downarrow$: $d\pi_L = (1-t)[p_L dA + (1-p_L)dB] < 0$

Firm has incentive to deviate to $\{(A, B), (A+dA, B+dB)\}$

Case 2: $A = B$

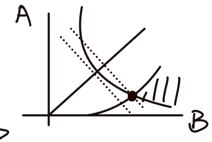
Firm: $\min_{A,B} \bar{p}A + (1-\bar{p})B$ $\quad$ s.t. $p_L u(A) + (1-p_L)u(B) \geq u_{0L}$

$\bar{p} = tp_H + (1-t)p_L$ $\quad$ IRH does not bind

FOC: $\frac{\bar{p}}{1-\bar{p}} = \frac{p_L u'(A)}{(1-p_L)u'(B)}$ $\quad \Rightarrow A \neq B$ since $\bar{p} \neq p_L$

Moral Hazard

1. RA indiv.: $u(\cdot).\ w.\ I.$ $\quad p_s > p_e \leftarrow$ effort, cost C



Not fully insured

$\min_{(A,B)} p_e A + (1-p_e)B$

s.t. $p_e u(A) + (1-p_e)u(B) - C \geq u_0$ $\quad$ IR

$p_e u(A) + (1-p_e)u(B) - C \geq p_s u(A) + (1-p_s)u(B)$ $\quad$ IC

$\Leftrightarrow u(B) \geq u(A) + \frac{C}{p_s - p_e}$

2. Wage $\begin{cases} P: & w(y) \text{ wage contract on output } y \\ A: & u(w),\ y \sim F(\cdot|e) \text{ on } [0, \bar{\pi}].\ \text{cost } g(e) \\ & \text{reservation utility } \bar{u} \end{cases}$

A: $\max_e \int u(w(\pi))dF(\pi|e) - g(e)$

P: $\max_{w(\cdot)} \int [\pi - w(\pi)]dF(\pi|e^*)$

s.t. $e^* \in \arg\max_e \int u(w(\pi))dF(\pi|e) - g(e)$ $\quad$ IC

$\int u(w(\pi))dF(\pi|e^*) - g(e^*) \geq \bar{u}$ $\quad$ IR

2.1 $e \in \{e_L, e_H\}$. $\quad e^* = e_L$: constant w.

$e^* = e_H$: increasing w

$L = \int [\pi - w(\pi)]dF(\pi|e_H)$ $\quad (\lambda \geq 0)$

$+ \lambda[\int u(w(\pi))dF(\pi|e_H) - g(e_H) - \int u(w(\pi))dF(\pi|e_L) + g(e_L)]$

$+ \mu[\int u(w(\pi))dF(\pi|e_H) - g(e_H) - \bar{u}]$ $\quad (\mu \geq 0)$

FOC: $\frac{\partial L}{\partial w} = -f(\pi|e_H) + \lambda[u'(w)f(\pi|e_H) - u'(w)f(\pi|e_L)]$

$(\forall \pi)$ $\quad + \mu u'(w)f(\pi|e_H) = 0$

$\Leftrightarrow \frac{1}{u'(w)} = \lambda[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}] + \mu$

Claim: both IR & IC bind: $\quad (\lambda > 0, \mu > 0)$

Proof: $\lambda = \mu = 0 \Rightarrow u'(w) = +\infty$ $\quad \times$

$\lambda = 0 \Rightarrow u'(w(\pi)) = \text{constant} \Rightarrow w(\pi) = \bar{w} \Rightarrow e^* = e_L$

$\mu = 0 \Rightarrow \frac{f(\pi|e_L)}{f(\pi|e_H)} < 1 \Rightarrow \int f(\pi|e_L)d\pi < \int f(\pi|e_H)d\pi = 1$

Claim: $F(\cdot|e)$ MLRP $\Rightarrow w'(\pi) > 0$

2.2 $e \in [e_{min}, e_{max}]$

A: $\max_e \int u(w(\pi))dF(\pi|e) - g(e)$

FOC: $\int u(w(\pi))\frac{df(\pi|e)}{de}d\pi = g'(e)$

P: $\max_w \int [\pi - w(\pi)]dF(\pi|e^*)$

s.t. $\int u(w(\pi))\frac{df(\pi|e^*)}{de^*}d\pi = g'(e^*)$ $\quad$ IC

$\int u(w(\pi))dF(\pi|e^*) - g(e^*) \geq \bar{u}$ $\quad$ IR

FOC: $\frac{1}{u'(w)} = \lambda\frac{1}{f(\pi|e^*)}\frac{df(\pi|e^*)}{de^*} + \mu$

Claim: MLRP $F(\cdot|e) \Rightarrow w'(\pi) > 0$

3. Risk Neutral agent: $u'(w) = \text{Const.}$ $\quad u(w) = w.$

P can fully delegate the choice of e by $w(\pi) = \pi - \alpha$

$\rightarrow$ Awaya & Do 2022

Mechanism Design

Auction: N risk-neutrel buyers $\overset{\text{true value}}{X_i} \sim F_i[0, w_i]$ iid

Revealation principle:

$\forall$ mechanism (B, π, μ). $\forall$ equi. $\beta^*: X \to B$

$\exists$ direct mech (Q, M): $Q(x) = \pi(\beta(x))$ $M(x) = \mu(\beta(x))$

s.t. $\exists$ truthful equi & same outcome

Direct (Q, M) is IC $\Leftrightarrow$ q_i is non-decreasing.

$u_i(x_i) = E_{-i}[Q_i(x_i, x_{-i})x_i - M_i(x_i, x_{-i})] = q_i(x_i)x_i - m_i(x_i)$

where $q_i(x_i) = E_{-i} Q_i(x_i, x_{-i}) = \int Q_i(x_i, x_{-i}) dF(x_{-i})$

$m_i(x_i) = E_{-i} M_i(x_i, x_{-i}) = \int M_i(x_i, x_{-i}) dF(x_{-i})$

"$\Rightarrow$" $u_i(x_i) = \max_{z_i \in X_i} \{q_i(z_i)x_i - m_i(z_i)\} = q_i(x_i)x_i - m_i(x_i)$

u_i is convex: maximum of affine functions

$u_i'(x_i) = q_i(x_i) \geq 0$ u_i is non-decreasing

$u_i''(x_i) = q_i'(x_i) \geq 0$ q_i is non-decreasing

"$\Leftarrow$" WTS: $q_i(x_i)x_i - m_i(x_i) \geq q_i(z_i)x_i - m_i(z_i)$

$\Leftrightarrow q_i(x_i)x_i - m_i(x_i) \geq q_i(z_i)z_i - m_i(z_i) + q_i(z_i)(x_i - z_i)$

$\Leftrightarrow u_i(x_i) - u_i(z_i) = \int_{z_i}^{x_i} q_i(t)dt \geq q_i(z_i)(x_i - z_i)$

Revenue Equivalence:

Expected payment $m_i(x_i) = m_i(0) + q_i(x_i)x_i - \int_0^{x_i} q_i(t_i)dt_i$

of (Q, M) (Q, M') are same up to a constant.

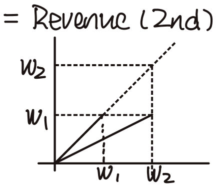
Proof: $u_i(x_i) = u_i(0) + \int_0^{x_i} q_i(t_i)dt_i$

Ex. symmetric X_i: Revenue(1st price) = Revenue(2nd)

Asymmetric X_i: Always ">"

1st price: $q_1(w_1) = 1$

2nd price: $q_1(w_1) < 1$



Direct (Q, M) is IR $\Leftrightarrow$ $m_i(0) \leq 0$

Optimal Mechanism: (Q^*, M)

$\max_{(Q,M)} \sum_i E m_i(X) = \sum_i m_i(0) + \sum_i \int_X \psi_i(x_i) Q_i(x) f(x) dx$

s.t. IC: q_i non-decreasing

IR: $m_i(0) \leq 0$

Virtual valuation: $\psi_i(x_i) = x_i - \frac{1 - F(x_i)}{f(x_i)}$. $E\psi_i(X_i) = 0$

MR of monopolist: $q = 1 - F(p) \Rightarrow p = F^{-1}(1-q)$

$R = pq = qF^{-1}(1-q)$ $MR = \frac{dR}{dq} = p - \frac{q}{f(p)} = p - \frac{1-F(p)}{f(p)}$

Regular problem: $\forall i$. $\psi_i(x_i)$ increases on x_i

$\Leftarrow$ hazard rate $\lambda_i(x_i) = \frac{1 - F(x_i)}{f(x_i)}$ decreases on x_i

Optimal (Q^*, M) (Regular)

$Q_i^*(x) = 1\{\psi_i(x_i) \geq \max_{j \neq i}\{\psi_j(x_j), 0\}\} = 1\{x_i \geq y_i(x_{-i})\}$

$M_i(x) = Q_i(x)x_i - \int_0^{x_i} Q_i(t_i, x_{-i}) dt_i = y_i(x_{-i}) 1\{x_i \geq y_i(x_{-i})\}$

Revenue $= \mathbb{E} \max_i \{\psi_i(X_i), 0\}$

Optimal (Q^*, M) (Regular & Symmetric)

$Q_i^*(x) = 1\{x_i \geq \max\{x_{-i}, \psi^{-1}(0)\}\}$

2nd price auction with reserve price $\psi^{-1}(0)$

# of bidders does not affect reserve price $\psi^{-1}(0)$

Optimal (Q^*, M) is inefficient:

① Good may not be allocated: discriminatory $r_i^* = \psi_i^{-1}(0)$

② Good is allocated based on ψ:

MLRP: $\frac{f_1(x)}{f_2(x)} \uparrow x \Rightarrow \lambda_1(x) > \lambda_2(x)$ 2 wins more often

$\Rightarrow F_1(x) < F_2(x)$ 2 is weaker

Optimal (Q, M) is not

① Universal (depending on F_i)

② Anonymous (identity matters) $Q_i(b_i, b_j) \neq Q_j(b_j, b_i)$

No revenue loss for anonymous mechanism!

Indirect: $\pi_i(z) = 1\{z_i > \max_{j \neq i}\{z_j, 0\}\}$

$\exists \mu: B \to \mathbb{R}_+$ s.t. $\beta_i(x_i) = \psi_i(x_i)$ is an Equi.

Example: $B_i = [\psi_i(\alpha_i), \psi_i(w_i)] \sim G_i(z_i)$

Optimal mech: $m_i^*(x_i) = m_i(0) + q_i(x_i)x_i - \int_0^{x_i} q_i(t_i)dt_i$

For $b_i \in B_i \setminus B_j$: $M_i^A(b_i, b_j) = m_i^*(\psi_i^{-1}(b_i))$

$b_i \in B_i \cap B_j$: $M_i^A(b_i, b_j) = \begin{cases} \overline{M}(b_i) & b_j \geq \hat{b} \\ \underline{M}(b_i) & b_j < \hat{b} \end{cases}$

$\hat{b}: G_1(\hat{b}) \neq G_2(\hat{b})$

s.t. $m_1^*(\psi_1^{-1}(b_i)) = [1 - G_2(\hat{b})]\overline{M}(b_i) + G_2(\hat{b})\underline{M}(b_i)$

$m_2^*(\psi_2^{-1}(b_i)) = [1 - G_1(\hat{b})]\overline{M}(b_i) + G_1(\hat{b})\underline{M}(b_i)$

Auction & Negotiation

Negotiation: reserve price $\psi^{-1}(0)$. $EN = E\max\{\psi(X_1), 0\}$

2nd price Auction: invite another one $EA = E\max\{\psi(X_1), \psi(X_2)\}$

$E\max\{\psi(X_1), \psi(X_2)\} \geq E\max\{\psi(X_1), 0\}$

Efficient Allocation: $Q^*(x) = \arg\max_{Q \in \Delta} \sum_j Q_j x_j$

Social welfar: $W(x) := \sum_j Q_j(x) x_j$

Leave-1-out: $W_{-i}(x) := \sum_{j \neq i} Q_j(x) x_j$

VCG mechanism: (Q^*, M^V): $M_i^V(x) := W(\underline{x_i}, x_{-i}) - W_{-i}(x)$

Externality due to the existence of i. ($\underline{x_i}$ lowest)

For single private good:

$Q_i^*(x) = 1\{x_i > \max\{x_{-i}, 0\}\} = Q_i^{\text{optimal}}(x)$ for symmetric F.

$M_i^V(x) = 1\{x_i > \max\{x_{-i}, 0\}\} \max x_{-i} = M_i^{\text{optimal}}(x)$

For public good: Cost C. $X_i \sim [0, w_i]$

$Q_i^*(x) = 1\{\sum x_i \geq C\}$

i is pivotal

$M_i^V(x) = 1\{C - x_i \leq \sum x_{-i} < C\}(C - \sum x_{-i})$

$<C$

Deficits: $\sum_i M_i^V(x) \leq NC - (N-1)\sum x_i = C - (N-1)(\sum x_i - C)$

VCG (Q^*, M^V) is IC $\Rightarrow \beta_i(x_i) = x_i$ WD $\Rightarrow$ Truthful Equi.

$u_i(z_i, x_i, x_{-i}) = Q_i^*(z_i, x_{-i})x_i - M_i^V(z_i, x_{-i})$

$= \sum_j Q_j^*(z_i, x_{-i})x_j - W(\underline{x_i}, x_{-i})$

$\Rightarrow u_i^V(x_i, x_{-i}) = W(x) - W(\underline{x_i}, x_{-i})$ increases in x_i

VCG (Q^*, M^V) is IR:

$u_i^V(x_i) \geq u_i^V(\underline{x_i}) = 0$

Among all Efficient & IR & IC mechanisms,

VCG (Q^*, M^V) maximizes expected payment of each agent

Efficient $\Rightarrow$ same Q : $(Q^*, \bar{M})$

Revenue Equivalence: $m_i^V(x_i) = \bar{m}_i(x_i) + \text{Const.}$

$m_i^V(\underline{x}_i) = 0$. IC & IR $\Rightarrow \bar{m}_i(\underline{x}_i) \le 0 \Rightarrow \text{Const.} \ge 0$

$E m_i^V(x_i) = E\bar{m}_i(x_i) + \text{Const.} \Rightarrow E m_i^V(x_i) \ge E\bar{m}_i(x_i)$

$\exists$ Efficient & IR & IC & BB $(Q^*, \bar{M})$

$\Leftrightarrow$ VCG (Q^*, M^V) runs an exante surplus: $\sum_i E m_i^V \ge 0$

"$\Rightarrow$" BWOC: $\sum_i E\bar{m}_i \le \sum_i E m_i^V < 0$

"$\Leftarrow$" $\bar{M}_i(x) := m_i^V(x_i) - \frac{1}{N-1}\sum_{j\ne i} m_j^V(x_j) + \frac{1}{N-1}\sum_{j\ne i} E m_j^V - \frac{1}{N}\sum_j E m_j^V$

$\bar{m}_i(x_i) = m_i^V(x_i) - \frac{1}{N}\sum_j E m_j^V$ IC

$\bar{m}_i(\underline{x}_i) = m_i^V(\underline{x}_i) - \frac{1}{N}\sum_j E m_j^V \le m_i^V(\underline{x}_i) = 0$ IR

Exante surplus.

$\sum_i \bar{M}_i(x) = 0$ BB

Bilateral Trade:

Seller: cost c. Buyer: value v. $c, v \overset{iid}{\sim} \text{Unif}[0,1]$

$\nexists$ Efficient & IR & IC & BB mechanism.

$Q^*(c,v) = 1\{v \ge c\}$

$M_b^V(c,v) = W(c,0) - W_s(c,v) = 1\{v \ge c\}(c-0)$

$M_s^V(c,v) = W(1,v) - W_b(c,v) = 1\{v \ge c\}(0-v)$

$M_b^V + M_s^V = c - v \le 0 \Rightarrow \sum_i E m_i^V = E_{(c,v)} 1\{v > c\}(c-v) < 0$

Interdependent Value $V_i = v_i(x_1, \cdots, x_N)$ $x_i \in [\underline{x}_i, \bar{x}_i]$

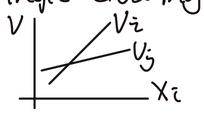
Common value: $x_i = v + \varepsilon_i$. $\varepsilon_i \sim iid(0, \sigma^2)$

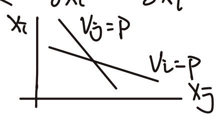
Violates Single-crossing

Winner's curse: $E(\max_{j\ne i} x_j | V = v) > E(x_i | V = v) = v$

Private value: $V_i = v_i(x_1, \cdots, x_N) = x_i$.

Single-crossing $\Leftarrow \frac{\partial v_i}{\partial x_i} > \frac{\partial v_j}{\partial x_i}$ At $v_i = v_j = P$





$\left|\frac{\partial x_i}{\partial x_j}\right| = \left|\frac{\partial v_i / \partial x_j}{\partial v_i / \partial x_i}\right| < 1$

Weaker SC: $\forall x$: $V_i(x) = V_j(x) = \max_k V_k(x)$

$\Updownarrow$

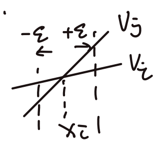
$\partial V_i / \partial x_i > \partial V_j / \partial x_i$

$\exists$ Efficient mechanism with expost equilibrium

"$\Leftarrow$" $\exists$ efficient (Q^*, M) $\beta_i = x_i$ (Revealation Principle)

BWOC: $\exists x$: $V_i(x) = V_j(x) = \max_k V_k(x)$.

$\frac{\partial V_i}{\partial x_i} \le \frac{\partial V_j}{\partial x_i}$



$V_i(x_i - \varepsilon, x_{-i}) > V_j(x_i - \varepsilon, x_{-i})$

$V_i(x_i + \varepsilon, x_{-i}) < V_j(x_i + \varepsilon, x_{-i})$

Efficiency $\Rightarrow$ i wins at $x_i - \varepsilon$, but not $x_i + \varepsilon$

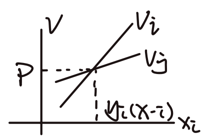
Expost Equi $\Rightarrow$ $x_i - \varepsilon \ge x_i + \varepsilon$ Contradiction!

"$\Rightarrow$" Generalized VCG:

$Q_i^*(x) = 1\{x_i \ge y_i(x_{-i})\}$

$M_i^G(x) = V_i(y_i(x_{-i}), x_{-i}) 1\{\ \} = P$

$y_i(x_{-i}) := \inf\{z_i : V_i(z_i, x_{-i}) \ge \max_{j\ne i}\{V_j(z_i, x_{-i}), 0\}\}$



GVCG (Q^*, M^G) has Expost Equi $\beta_i(x_i) = x_i$.

$U_i(x) = 1\{x_i \ge y_i(x_{-i})\}[V_i(x_i, x_{-i}) - V_i(y_i(x_{-i}), x_{-i})]$

① Expost Equi. cannot be replaced with weakly dominance

② GVCG is not detail-free (V) nor anomynous.

Opfimal Mechanism: Seller obtains all surplus!

Discrete X_i. SC v. $\Pi_i = \{\pi_i(x_{-i}|x_i)\}_{t_i \times t_{-i}}$ full row rank

$\Rightarrow \exists (Q^*, M^C)$ st. $\beta^*(x) = x$ is Efficient & Expost Equi.

$U_i^C(x_i) = 0$ $\forall i$ $\forall x_i$

$U_i^G(x_i) = \sum_{x_{-i}} \pi_i(x_{-i}|x_i)[Q_i^*(x) x_i - M_i^G(x)]$

$U_i^G(t_i \times 1) = \Pi_i(t_i \times t_{-i}) \cdot C_i(t_{-i} \times 1) \Leftarrow \Pi_i$ full row rank

$\forall x_i$: $U_i^G(x_i) = \sum_{x_{-i}} \pi_i(x_{-i}|x_i) C_i(x_{-i})$

CM mech: (Q^*, M^C) : $M_i^C(x) := M_i^G(x) + C_i(x_{-i})$

Expost IC: $U_i^C(x) = Q_i^*(x) x_i - M_i^G(x) - C_i(x_{-i})$

IR: $U_i^C(x_i) = U_i^G(x_i) - \sum_{x_{-i}} \pi_i(x_{-i}|x_i) C_i(x_{-i}) = 0$

When Π_i is close to independent belief, eignvalues $\to 0$,

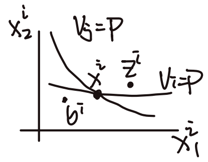
$C_i \to +\infty$, payment $M_i^C(x)$ is large.

Multidimensional Signals.

No SC. $\nexists$ Efficient & Expost IC Equi.

$y^i, z^i \in \mathbb{R}^k$: $V_i(y^i, x^{-i}) > \max_{j\ne i} V_j(y^i, x^{-i})$

$V_i(z^i, x^{-i}) < \max_{j\ne i} V_j(y^i, x^{-i})$



Efficiency: i wins at (y^i, x^{-i}) but not (z^i, x^{-i})

Expost IC: $V_i(y^i, x^{-i}) \ge V_i(z^i, x^{-i})$

But $V_i(y^i, x^{-i}) = p - \varepsilon < p + \varepsilon = V_i(z^i, x^{-i})$

Iso-value curves intersects & have difference slopes.

Bundling: k goods. 2 buyers. VCG mechanism.

It is optimal to bundle all! (≥ 3 buyers X)

If divide k into σ^1, σ^2 :

VCG: efficient: $x^1(\sigma^1) + x^2(\sigma^2) \ge \max\{x^1(k), x^2(k)\}$

VCG externality: $M_1(x) = x^2(k) - x^2(\sigma^2)$

$M_2(x) = x^1(k) - x^1(\sigma^1)$

Revenue: $R = M_1(x) + M_2(x) \le x^1(k) + x^2(k) - \max\{x^1(k), x^2(k)\}$

If bundle k (2nd price) $R = \min\{x^1(k), x^2(k)\}$

Notice: $\min\{X, Y\} = X + Y - \max\{X, Y\}$.

Dynamic Mechanism Design Endowment e.

Agent $\in [0,1]$ utility $= \theta u(c)$ $\theta \overset{iid}{\sim} \{\theta_H, \theta_L\}$ $\pi(\cdot)$