Complete information game: $(S_i, u_i)_{i=1}^n$

Pure strategy: $s_i \in S_i$. $\quad s = \times_i s_i \in S = \times_i S_i$

Mixed strategy: $\sigma_i \in \Delta(S_i)$ $\quad \sigma = \otimes_i \sigma_i \in \Sigma = \otimes_i \Delta(S_i)$

$$u_i(\sigma) = \int_S u_i(s)\, \sigma(ds)$$

Dominant $s_i^* \in S_i$: $\forall s_i \neq s_i^*\ \forall s_{-i} \in S_{-i}\ u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$

Weakly dominant: $s_i^* \in S_i$: $\forall s_i \neq s_i^*$.

$\quad \forall s_{-i} \in S_{-i},\ u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$

$\quad \exists s_{-i} \in S_{-i},\ u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$

Dominated $s_i \in S_i$: $\exists \sigma_i \in \Delta(S_i)\ \forall s_{-i} \in S_{-i},\ u_i(\sigma_i, s_{-i}) > u_i(\sigma_i, s_{-i})$

(Uniformly dominated)

Weakly dominated.

Rational $s_i \in S_i$: $\exists \mu_i \in \Delta(S_{-i})$ (subjective belief) s.t. $s_i \in BR_i(\mu_i)$

Admissible $s_i \in S_i$: $\exists \mu_i \in \Delta(S_{-i})$ $\mu_i \gg 0$ (full support.) s.t. $s_i \in BR_i(\mu_i)$

s_i^* is rational $\Leftrightarrow$ s_i^* is not dominated by a mixed strategy

s_i^* is admissible $\Leftrightarrow$ s_i^* is not weakly dominated by a mixed strategy

$$A = \begin{array}{c} \xrightarrow{\quad s_i \quad} \\ \left[\begin{array}{ccc} u_i(s_i^*, s_{-i}) - u_i(s_i', s_{-i}) & \cdots & u_i(s_i^*, s_{-i}) - u_i(\hat{s}_i, s_{-i}) \\ \vdots & & \vdots \\ u_i(s_i^*, \hat{s}_{-i}) - u_i(s_i', \hat{s}_{-i}) & \cdots & u_i(s_i^*, s_{-i}) - u_i(\hat{s}_i, s_{-i}) \end{array}\right] \end{array} \ \downarrow s_{-i}$$

s_i^* is rational

$\Leftrightarrow \exists \mu_i \in \Delta(S_{-i}),\ \mu_i \geq 0,\ \mu_i \neq 0$ s.t. $\mu_i^T A \geq 0$.

$\Leftrightarrow \exists x \geq 0$ s.t. $x^T A \geq 0$. $x^T b < 0$ for some $b < 0$ ①

s_i^* is dominated by mixed strategy

$\Leftrightarrow \exists \sigma_i \in \Delta(S_i)$, $\sigma_i \geq 0$. $\sigma_i \neq 0$ s.t. $A\sigma_i < 0$

$\Leftrightarrow \exists y \geq 0$ s.t. $Ay \leq b$ for some $b < 0$ ②

Farkas' Lemma: ① $\Leftrightarrow$ $\overline{②}$

s_i^* is admissible.

$\Leftrightarrow \exists \mu_i \in \Delta(S_{-i})$, $\mu_i \gg 0$, s.t. $\mu_i^T A \geq 0$

$\Leftrightarrow \exists x \gg 0$ s.t. $x^T A \geq 0$. $x^T b < 0$. for some $b \leq 0$ ①'

s_i^* is weakly dominated by mixed strategy

$\Leftrightarrow \exists \sigma_i \in \Delta(S_i)$, $\sigma_i \geq 0$ $\sigma_i \neq 0$ s.t. $A\sigma_i \leq 0$, $A\sigma_i \neq 0$

$\Leftrightarrow \exists y \geq 0$ s.t. $Ay \leq b$. for some $b \leq 0$ ②'

Farkas' Lemma: ①' $\Leftrightarrow$ ②'

Rationalizable strategies: $R_i^\infty = \bigcap_{k=0}^{+\infty} R_i^k$. where $R_i^0 = S_i$

$\quad R_i^k = \{ s_i \in S_i : \exists \mu_i \in \Delta(R_{-i}^{k-1})$ s.t. $s_i \in BR_i(\mu_i)\}$

Iterated elimination of dominated strategies (IEDS): $S_i^\infty = \bigcap_{k=0}^{+\infty} S_i^k$

$S_i^k = \{ s_i \in S_i : \nexists \sigma_i \in \Delta(S_i^{k-1})$ s.t. $\forall s_{-i} \in S_{-i}^{k-1},\ u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})\}$

Iterated admissibility (IA): $A_i^\infty = \bigcap_{k=0}^{+\infty} A_i^k$

$\quad A_i^k = \{ s_i \in S_i : \exists \mu_i \in \Delta^0(A_{-i}^{k-1})$, s.t. $s_i \in BR_i(\mu_i)\}$

IEWDS: $W_i^\infty = \bigcap_{k=0}^{+\infty} W_i^k$

$\quad W_i^k = \{ s_i \in S_i : \nexists \sigma_i \in \Delta(W_i^{k-1})$ s.t. σ_i weakly dominates $s_i\}$

$R^\infty = S^\infty$; $\quad A^\infty = W^\infty$ $\quad A^\infty \subsetneq R^\infty$

Sealed-Bid Auction

$$u_i(s) = \begin{cases} \frac{1}{m}(v_i - p(s)) & \text{if } i \in W(s) \\ 0 & \text{o.w.} \end{cases}$$

$p(s)$: price. $\quad W(s) = \{ i \in \{1, \cdots, n\} \mid s_i \geq s_j,\ \forall j \}$

$m = |W(s)|$

2nd-Price Sealed-Bid: $s_i = v_i$ is weakly dominant

1st-Price Sealed-Bid: $s_i = v_i$ is weakly dominated

$\frac{2}{3}$ of Average: $\quad H(s) = \{ i : s_i \in \underset{s \in S}{\arg\min} |s_j - \frac{2}{3} \Sigma s_j| \}$

$S_i = \{1, \cdots, 100\}$ $\quad u_i = \begin{cases} \frac{1}{|H(s)|} & \text{if } s_i \in H(s). \\ 0 & \text{o.w.} \end{cases}$

$R_i^0 = S_i = \{1, \cdots, 100\}$

$R_i^1 = \{1, \cdots, 99\}$: $\nexists \mu_i \notin \Delta(R_{-i}^0)$ s.t. $100 \in BR_i(\mu_i)$

$\quad$ candidate $\mu_i = \{$all the others choose $100\}$

$\quad |\frac{2}{3}100 - 100| > |\frac{2}{3}100 - 99|$ $\quad$ i.e. $100 \notin BR_i(\mu_i)$

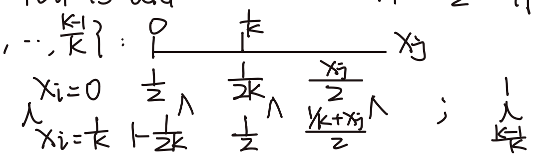
$R_i^\infty = \{1\}$

Hotelling Location Game: $u_i(x_i, x_j) = \begin{cases} \frac{x_i + x_j}{2} & \text{if } x_i < x_j \\ \frac{1}{2} & \text{if } x_i = x_j \\ 1 - \frac{x_i + x_j}{2} & \text{if } x_i > x_j \end{cases}$

$S_i = \{0, \frac{1}{k}, \frac{2}{k}, \cdots, 1\}$. k is even

$R_i^0 = S_i$ $\quad |S_i|$ is odd

$R_i^1 = \{\frac{1}{k}, \frac{2}{k}, \cdots, \frac{k-1}{k}\}$:

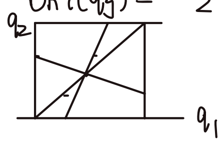


$R_i^\infty = \{\frac{1}{2}\}$

Cournot Duopoly:

$q_i \in [0, \bar{q}]$ $\quad \pi_i(q_i, q_j) = (a - q_i - q_j) q_i - c q_i$

$BR_i(q_j) = \frac{a - c - q_j}{2}$ $\quad R_i^1 = [\frac{a-c-\bar{q}}{2}, \frac{a-c}{2}]$



$R_i^2 = [\frac{a-c}{4}, \frac{a-c+\bar{q}}{4}]$

$R_i^\infty = \frac{a-c}{3}$

Duopoly with Different Products:

$p_i \in [0, \bar{p}]$ $\quad \pi_i(p_i, p_j) = (p_i - c)(a - p_i + p_j)$

$BR_i(p_j) = \frac{a + c + p_j}{2}$ $\quad R_i^1 = [\frac{a+c}{2}, \frac{a+c+\bar{p}}{2}]$

$\quad R_i^\infty = a + c$

Bertrand Duopoly

$p_i \in \{0, 1, \cdots, \bar{p}\}$ $\quad \pi_i(p_i, p_j) = \begin{cases} (a - p_i)(p_i - c_i) & p_i < p_j \\ \frac{1}{2}(a - p_i)(p_i - c_i) & p_i = p_j \\ 0 & p_i > p_j \end{cases}$

Monopoly price: $p^m = \frac{a + c}{2}$

$c_1 = c_2 = c$; $A^\infty = (c+1, c+1)$

$c_1 < c_2$: $A^\infty = (c_2, c_2 + 1)$

Partially ordered space $(X,\geq)$: reflexive, transitive, anti-symmetric
Lattice $(X,\geq)$: $\forall x,y\in X,\ x\vee y,\ x\wedge y\in X$
Complete lattice $(X,\geq)$: $\forall Y\subseteq X,\ Y\neq\emptyset,\ \sup(Y),\ \inf(Y)\in X$
Sublattice $(Y,\geq)\sqsubseteq(X,\geq)$: $\forall x,y\in Y,\ x\vee y,\ x\wedge y\in Y$
Complete sublattice $(Y,\geq)\sqsubseteq(X,\geq)$: $\forall Y'\subseteq Y,\ Y'\neq\emptyset\ \sup(Y'),\ \inf(Y')\in Y$
Supermodular function : $f:(X,\geq)\to(\mathbb{R},\geq)$
$\forall x,y\in X\quad f(x)+f(y)\leq f(x\vee y)+f(x\wedge y)$
Increasing difference function : $f:(X\times Y,\ \geq_X\times\geq_Y)\to(\mathbb{R},\geq)$
$x,x'\in X,\ x'\geq_X x,\quad y,y'\in Y,\ y'\geq_Y y$
$f(x',y)-f(x,y)\leq f(x',y')-f(x,y')$
$\Leftrightarrow f(x',y)+f(x,y')\leq f(x',y')+f(x,y)$
$f:X\times Y\to\mathbb{R}$: supermodular $\overset{\not\Leftarrow}{\Rightarrow}$ ID   X: partial order
$X\subseteq\mathbb{R}^k$ $f:X\to\mathbb{R}$: supermodular $\Leftrightarrow$ ID $\mathbb{R}^k$: chain

Topkis monotonicity thm: $f:X\times Y\to\mathbb{R}$ S.M. & I.D
$b(y)=\arg\max_x f(x,y)$ $y'\geq y,\ x\in b(y),\ x'\in b(y')$
$\Rightarrow x\vee x'\in b(y'),\ x\wedge x'\in b(y)$
$\Rightarrow b(y)$ is a sublattice of X
If f is order-upper semicontinuous, bounded above, X complet then $b(y)$ is a non empty sublattice of X
$\bar{b}(y),\ \underline{b}(y)$ are non-decreasing functions
Supermodular game : $(S_i,u_i)_{i=1}^n$: $\forall i$
S_i : Complete lattice
u_i : order-continuous & bounded above
$\Rightarrow BR_i(s_{-i})\sqsubseteq S_i$ complete lattice
$\bar{b}_i(s),\ \underline{b}_i(s)\ \uparrow$
u_i : SM in S_i, I.D. in (s_i,s_{-i}) ← strategic complements
D.D. ← strategic substitutes
Serially undominated strategies. $D^\infty=\bigcap_{k=0}^{\infty}D^k$. $D^0=S$. $D^k=U(D^{k-1})$
$\hat{S}=\times_{i=1}^n\hat{S}_i\quad U_i(\hat{S})=\{s_i\in S_i:\forall s_i'\in\hat{S}_i,\ \exists\hat{s}\in\hat{S}\ \text{s.t.}\ u_i(s_i,\hat{s}_{-i})\geq u_i(s_i',\hat{s}_{-i})\}$
$R^\infty=S^\infty\sqsubseteq D^\infty$ (since $S^k\sqsubseteq D^k$)
In supermodular game $(S_i,u_i)_{i=1}^n$: $D^\infty\sqsubseteq[\underline{s},\bar{s}]$. $\underline{s},\bar{s}$ are FP of BR.
$\bar{s}^0=\sup(S)$, $\underline{s}^0=\inf(S)$. $\bar{s}^k=\bar{b}(\bar{s}^{k-1})$ $\underline{s}^k=\underline{b}(\underline{s}^{k-1})$
Induction : $D^{k+1}=U(D^k)\sqsubseteq U([\underline{s}^k,\bar{s}^k])\overset{*}{\sqsubseteq}[\underline{s}^{k+1},\bar{s}^{k+1}]$ * BWOC
$\bar{s}=\lim\bar{s}^k=\inf\bar{s}^k$ exists (Complete S) $\Rightarrow D\sqsubseteq[\underline{s},\bar{s}]$
$\bar{s}$ is FP : BWOC: $\bar{s}\neq BR(\bar{s})$: $\exists i,s_i.\ u_i(s_i,\bar{s}_{-i})>u_i(\bar{s}_i,\bar{s}_{-i})$
Order continuity: $u_i(s_i,\bar{s}_{-i}^k)>u_i(\bar{s}_i^{k+1},\bar{s}_{-i}^k)\Rightarrow\bar{s}_i^{k+1}\notin BR_i(\bar{s}_{-i}^k)$
For strategic substitutes. $D^\infty\sqsubseteq[\underline{s},\bar{s}]$, $\underline{s},\bar{s}$ are FP of BR^2

Quasisupermodular $f:X\to\mathbb{R}$: $f(x)\geq f(x\wedge y)\Rightarrow f(x\vee y)\geq f(y)$
S.M. $\Rightarrow$ Q.S.M. $f(x)>f(x\wedge y)\Rightarrow f(x\vee y)>f(y)$
Single crossing property with increasing (decreasing) returns : SCP-IR
$x'\geq x,\ t'\geq t$. $f(x',t)>f(x,t)\Rightarrow f(x',t')>f(x,t')$
IR $\Rightarrow$ SCP-IR $f(x',t)\geq f(x,t)\Rightarrow f(x',t')\geq f(x,t')$

BR (or BR^2) $\exists!$ FP $\Rightarrow|R^\infty|=1$

Bayesian Game (incomplete information) $(\Theta,(T_i,\pi_i,A_i,u_i)_{i=1}^n)$
$\pi_i:T_i\to\Delta(\Theta\times T_{-i})\qquad u_i:A\times\Theta\to\mathbb{R}$
Hierarchy of beliefs : $T_i\to\Theta$
1st order belief: $\eta_{1,i}:T_i\to B_i^1\sqsubseteq\Delta(\Theta)\quad\eta_{1i}(t_i)=\text{marg}_\Theta\,\pi_i(t_i)$
2nd order belief: $\eta_{2,i}:T_i\to B_i^2\sqsubseteq\Delta(\Theta)\quad\eta_{2i}(t_i)=\text{marg}_{T_{-i}}\pi_i(t_i)\circ\eta_{1,-i}^{-1}$
Kth order belief: $\eta_{k,i}:T_i\to B_i^k\sqsubseteq\Delta(\Theta)\quad\eta_{ki}(t_i)=\text{marg}_{T_{-i}}\pi_i(t_i)\circ\eta_{k-1,-i}^{-1}$
Common prior assumption : $\exists P\in\Delta(\Theta\times T)$ s.t. $\pi_i(t_i)(\cdot)=P(\cdot\mid t_i)$
Expected utility:
Interim : $U_i(a_i,s_{-i};t_i)=\int_{\Theta\times T_{-i}}u_i(a_i,s_{-i}(t_{-i}),\theta)\,\pi_i(t_i)(d(\theta,t_{-i}))$
Ex-ante : $U_i(s)=\int_{\Theta\times T}u_i(s(t),\theta)\,P(d(\theta,t))$
Usual mixed strategy: $\sigma_i\in\Delta(S_i)$, $S_i=\{s_i:T_i\to A_i\}$
Behavioral strategy: $\beta_i:T_i\to\Delta(A_i)$
Distributional strategy: $\mu_i\in\Delta(T_i\times A_i)$ with $\text{marg}_{T_i}\mu_i=\text{marg}_{T_i}P$

Interim Rationalizability (type-by-type) $R_i^\infty(t_i)=\bigcap_{k=0}^{\infty}R_i^k(t_i)$
$\forall i,\ t_i\in T_i,\ R_i^0(t_i)=A_i$
$R_i^k(t_i)=\{a_i\in A_i:\exists\mu\in\Delta(\Theta\times T_{-i}\times A_{-i})\ \text{s.t.}\ a_i\in BR_i(\mu)$
$\text{marg}_{\Theta\times T_{-i}}\mu=\pi_i(t_i)$ and $\mu(\Theta\times Gr(R_{-i}^{k-1}))=1\}$
where $BR_i(\mu)=\arg\max_{a_i\in A_i}\int_{\Theta\times T_{-i}\times A_{-i}}u_i(a_i,a_{-i},\theta)\,\mu(d(\theta,t_{-i},a_{-i}))$
$Gr(R_{-i}^{k-1})=\{(t_{-i},a_{-i})\in T_{-i}\times A_{-i}:a_{-i}\in R_{-i}^{k-1}(t_{-i})\}$
Interim rationalizable strategy $s_i(t_i)\in R_i^\infty(t_i)$

Global Game   Type space: $t_i=\theta+\varepsilon_i$, $\varepsilon_i,\varepsilon_j\overset{iid}{\sim}N(0,\sigma^2)$
Type structure: $\pi_i:\begin{pmatrix}\theta\\t_j\end{pmatrix}\Big|\,t_i\sim N\left(\begin{bmatrix}t_i\\t_i\end{bmatrix},\begin{bmatrix}\sigma^2&\sigma^2\\\sigma^2&2\sigma^2\end{bmatrix}\right)$

	A	B
A	θ,θ	$\theta-1,0$
B	$0,\theta-1$	$0,0$

$R_i^\infty(t_i)=\{B\}$, $t_i<\frac{1}{2}$
$R_i^\infty(t_i)=\{A\}$, $t_i>\frac{1}{2}$

Email Game $\Theta=\{-\frac{1}{2},\frac{1}{2}\}$ $Pr(\theta=-\frac{1}{2})=p$
Pr(A sent email is lost) $=\varepsilon\in(0,\frac{p}{1-p})$ type t_i : # of emails i sent

$\theta=-\frac{1}{2}$

$t_1\backslash t_2$	0	1
0	1, a	
1		
2		

$\theta=\frac{1}{2}$

$t_1\backslash t_2$	0	1	2
0			
1	b, 1-a	1-b, b	
2		b, 1-b	1-b, b

$a=P(\theta=-\frac{1}{2},t_1=0\mid t_2=0)=\dfrac{P(\theta=-\frac{1}{2})}{P(\theta=-\frac{1}{2})+P(\text{lost})P(\theta=\frac{1}{2})}=\dfrac{p}{p+(1-p)\varepsilon}$
$b=P(t_2=0\mid t_1=1)=\dfrac{P(t_1=1,t_2=0)}{P(t_1=1,t_2=0)+P(t_1=1,t_2=1)}=\dfrac{\varepsilon}{\varepsilon+\varepsilon(1-\varepsilon)}=\dfrac{1}{2-\varepsilon}$
$R_i^\infty(t_i)=\{B\}\ \forall t_i\in\mathbb{N}$
If $\theta=\frac{1}{2}$ is common knowledge, $R_i^\infty(t_i)=\{A,B\}$
Rationalizable strategy is discontinuous at the limit to CK.

Another communication protocol: $\underset{\text{\# of emails 2 sents}}{t_2\leq k-1}$ s_2 # of emails 2 receive
$P(t_1=k\mid t_2=k-1)=1$ $P(s_2=k-1\mid t_1=k)=\varepsilon$
$R_i^\infty(t_i)=\{B\}$ $t_i<k$, $R_i^\infty(t_i)=\{A,B\}$, $t_i=k$

Extensive Game

Imperfect information: information state Ω_i. X_w info set

Strategy: complete contingent plan: $S_i = \times_{w\in\Omega_i} A_w$

$\forall s\in S$: $P(x|s) = Pr(x \text{ is reached} \mid \text{playing } s) = \prod$ change Probs.

$$u(s) = \sum_{x\in T} P(x|s)\, u_i(x) \qquad T\text{: Terminal nodes}$$

Perfect recall: $a_w \neq b_w \in A_w$, $w^a, w^b \in \Omega_i \Rightarrow w^a \neq w^b$

With perfect recall.

Behavioral strategy: $\beta_i: \Omega_i \to \Delta(A_i)$ s.t. $\beta_i(w)\in\Delta(A_w)$ $\forall w\in\Omega_i$

⇕

Mixed strategy: $\sigma_i(s_i) = \prod_{w\in\Omega_i} \beta_i(w)(s_i(w))$

s_i is compatible with $w\in\Omega_i$: $\exists s_{-i}\in S_{-i}$ s.t. $\sum_{x\in X_w} P(x|s_i,s_{-i})>0$

σ_i - - - - - - - - - - : $\exists s_i \in S_i^*(w)$ s.t. $\sigma_i(s_i)>0$

$S_i^*(w) = \{s_i\in S_i : s_i \text{ is compatible with } w\}$

$S_i^{**}(a_w,w) = \{s_i\in S_i^*(w) : s_i(w)=a_w\}$

β_i is a behavioral representation of σ_i: $\forall w\in\Omega_i$, $\forall a_w\in A_w$

$$\beta_i(w)(a_w) \sum_{s_i\in S_i^*(w)} \sigma_i(s_i) = \sum_{s_i\in S_i^{**}(a_w,w)} \sigma_i(s_i)$$

σ_i & σ_i' are behaviorally equivalent: represented by β_i

payoff equivalent: $\forall \sigma_{-i}$, $u_i(\sigma_i,\sigma_{-i}) = u_i(\sigma_i',\sigma_{-i})$

Kuhn: In an extensive form game with perfect recall, any 2 behaviorally equivalent mixed strategies are payoff equivalent

Backward Induction

BI & IA: Finite extensive form game of perfect info.

No 2 outcomes are indifferent for a player

$\exists$ an order of EWDS: $s\in A^\infty$ s.t. s BI

We cannot have common knowledge of rationality when rationality includes admissibility.

One-Shot Deviation Principle

Finite game: BI ⇔ No profitable 1-shot deviation

u_i is consistent: 2 paths τ,τ' diverging at w.

$$u_i(\tau)\geq u_i(\tau') \Leftrightarrow u_i(\tau_w)\geq u_i(\tau_w')$$

u_i is continuous at $+\infty$: $\forall\varepsilon>0$. $\exists k>0$ s.t.

$\forall k\geq K$, $\forall\tau'$ sharing same 1st-kth w with τ: $|u_i(\tau')-u_i(\tau)|<\varepsilon$

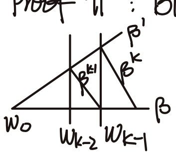
* Continuity at $+\infty$ & Consistent u_i

β can't be improved upon: $\nexists i$, $w\in\Omega_i$, $\hat\beta_i\neq\beta_i$ s.t.

⇓ ⇑* $u_i(\beta_i', \beta_{-i}|w) > u_i(\beta|w)$

No profitable 1-shot deviation: β can't be improved by 1-shot deviation.

Proof ⇑: BWOC



$u(\beta_i', \beta_{-i}|w_0) > u_i(\beta|w_0)$ whole path deviation

continuity at $+\infty$

$\exists k$. $u(\beta_i^k,\beta_{-i}|w_0)>u_i(\beta|w_0)$ K-shot deviation

$\Rightarrow u(\beta_i^{k-1},\beta_{-i}|w_0)>u_i(\beta|w_0)$ (k-1)-shot deviation

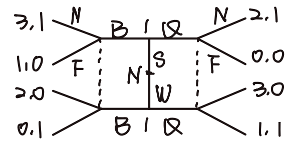
Suppose $\nexists$ 1-shot deviation: $u(\beta_i^{k-1},\beta_{-i}|w_{k-1}) \geq u(\beta_i^k,\beta_{-i}|w_{k-1})$

Consistency: $u(\beta_i^{k-1},\beta_{-i}|w_0)\geq u(\beta_i^k,\beta_{-i}|w_0)$

$\Rightarrow u(\beta_i^1,\beta_{-i}|w_0)>u_i(\beta|w_0)$ 1-shot deviation

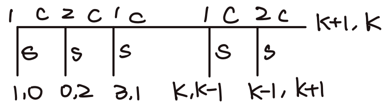
Beer-Quiche Game

$P(\text{strong}) = 0.9$



NE: (BB, NF) (QQ, FN)

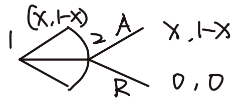
Centipede Game



BI: S

Ultimatum Game.

BI: $\{(1,0), A\}$



Alternating Offers Bargaining Game.

Discounts δ_1, δ_2

# of rounds	outcome
1	x_1 $(1,0)$
2	$(1-\delta_2, \delta_2)$ y_2
3	x_3 $(1-\delta_2+\delta_1\delta_2, \delta_2-\delta_1\delta_2)$
4	$(1-y_4, y_4)$

		Ending at Alice: t is odd.	Bob: t is even
$t+2$	A	$(1-\delta_2(1-\delta_1 x_t), \delta_2(1-\delta_1 x_t))$	$(1- \cdots, \delta_2(1-\delta_1(1-y_t)))$
$t+1$	B	$(\delta_1 x_t, 1-\delta_1 x_t)$	$(\delta_1(1-y_t), 1-\delta_1(1-y_t))$
t	A	$(x_t, 1-x_t)$	$(1-y_t, y_t)$

$$\begin{cases} x_{t+2} = 1-\delta_2(1-\delta_1 x_t) \\ y_{t+2} = \delta_2[1-\delta_1(1-y_t)] \end{cases} \Rightarrow \begin{cases} x^* = \frac{1-\delta_2}{1-\delta_1\delta_2} \\ y^* = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \end{cases}$$

Parametrizing: $\delta_i = \lim_{\Delta t\to 0}(1-\alpha_i\Delta t)^{\frac{\Delta}{\Delta t}} = e^{-\alpha_i\Delta}$

$$\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right) = \left(\frac{\alpha_2\Delta+o(\Delta)}{(\alpha_1+\alpha_2)\Delta+o(\Delta)}, \frac{\alpha_1\Delta+o(\Delta)}{(\alpha_1+\alpha_2)\Delta+o(\Delta)}\right) \to \left(\frac{\alpha_2}{\alpha_1+\alpha_2}, \frac{\alpha_1}{\alpha_1+\alpha_2}\right)$$

∞ # of rounds: Using OSDP

Stationary game ⇒ Stationary strategy (s_1^*, s_2^*)

$s_1^* = \{$propose $(x^*,1-x^*)$, accept $(y,1-y)$ iff $y\geq y^*\}$

$s_2^* = \{$propose $(y^*,1-y^*)$, accept $(x,1-x)$ iff $x\leq x^*\}$

1 facing $(y^*,1-y^*)$ an offer at cutoff:

indifference condition: $u_1(\text{accept}) = u_1(\text{reject})$: $y^* = \delta_1 x^*$

1's turn to propose an offer:

OSDP: $u_1((x^*,1-x^*))\geq u_1((x,1-x))$: $x^* \geq \begin{cases} x & \text{if } x<x^* \\ \delta_1 y^* & \text{if } x>x^* \end{cases}$

2 facing $(x^*,1-x^*)$ an offer at cutoff:

Indifference condition $u_2(\text{accept}) = u_2(\text{reject})$: $1-x^* = \delta_2(1-y^*)$

2's turn to propose an offer:

OSDP: $u_2((y^*,1-y^*))\geq u_2((y,1-y))$: $1-y^* \geq \begin{cases} 1-y & \text{if } y>y^* \\ \delta_2(1-x^*) & \text{if } y<y^* \end{cases}$

Thus. $\begin{cases} y^* = \delta_1 x^* \\ 1-x^* = \delta_2(1-y^*) \end{cases} \Rightarrow \begin{cases} x^* = \frac{1-\delta_2}{1-\delta_1\delta_2} \\ y^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \end{cases}$

Nash Equilibrium: Self-supporting

$(S_i, u_i)_{i=1}^n$: $s_i^* \in BR_i(s_{-i}^*)$, $\sigma_i^* \in BR_i(\sigma_{-i}^*)$ Rationality + Perfect foresight

NE can be in inadmissible strategies.

A^∞(IEWDS) $\subsetneq$ NE $\subsetneq$ R^∞ (IEDS)

2-player 0-sum Game:

Security strategy: $\sigma_i^* = \arg\max_{\sigma_i} \min_{\sigma_{-i}} u_i(\sigma_i, \sigma_{-i})$

N players $(N>2)$: $u_i(\sigma_i^*, \sigma_{-i}^*) < u_i(NE)$

2 players: $u_i(\sigma_i^*, \sigma_j^*) = u_i(NE)$, NE $\Leftrightarrow$ security strategy

$\sigma_1^* = \arg\max_{\sigma_1} \min_{\sigma_2} u_1(\sigma_1, \sigma_2)$ $\sigma_2^* = \arg\min_{\sigma_2} \max_{\sigma_1} u_1(\sigma_1, \sigma_2)$

"$\Rightarrow$" $u_1(\sigma_1^*, \sigma_2^*) = \max_{\sigma_1} u_1(\sigma_1, \sigma_2^*) \geq \max_{\sigma_1} \min_{\sigma_2} u_1(\sigma_1, \sigma_2) \geq \min_{\sigma_2} u_1(\sigma_1^*, \sigma_2)$

$u_1(\sigma_1^*, \sigma_2^*) = \min_{\sigma_2} u_1(\sigma_1^*, \sigma_2) \leq \min_{\sigma_2} \max_{\sigma_1} u_1(\sigma_1, \sigma_2) \leq \max_{\sigma_1} u_1(\sigma_1, \sigma_2^*)$

$\Rightarrow \sigma_1^* = \arg\max_{\sigma_1} \min_{\sigma_2} u_1(\sigma_1, \sigma_2)$. $\sigma_2^* = \arg\min_{\sigma_2} \max_{\sigma_1} u_1(\sigma_1, \sigma_2)$

"$\Leftarrow$" $\max_{\sigma_1} u_1(\sigma_1, \sigma_2^*) \geq \max_{\sigma_1} \min_{\sigma_2} u_1(\sigma_1, \sigma_2) = \min_{\sigma_2} u_1(\sigma_1^*, \sigma_2)$

$\min_{\sigma_2} u_1(\sigma_1^*, \sigma_2) \leq \min_{\sigma_2} \max_{\sigma_1} u_1(\sigma_1, \sigma_2) \leq \max_{\sigma_1} u_1(\sigma_1, \sigma_2^*)$

$\Rightarrow u_1(\sigma_1^*, \sigma_2^*) \leq \max_{\sigma_1} u_1(\sigma_1, \sigma_2^*) = \min_{\sigma_2} u_1(\sigma_1^*, \sigma_2) \leq u_1(\sigma_1^*, \sigma_2^*)$

$\Rightarrow u_1(\sigma_1^*, \sigma_2^*) = \max_{\sigma_1} u_1(\sigma_1, \sigma_2^*)$. i.e. $\sigma_1^* = \arg\max_{\sigma_1} u_1(\sigma_1, \sigma_2^*)$

Linear progamming: $z^* = \max z$

s.t. $\sum_{s_1} \sigma_1(s_1) u_1(s_1, s_2) \geq z$. $\forall s_2 \in S_2$

$\sum_{s_1} \sigma_1(s_1) = 1$. $\sigma_1(s_1) \geq 0$.

*Entire families of NE: $\forall a \in (c_1, c_2]$. $\varepsilon > 0$. $a > c_1 + \varepsilon$

Firm 2: $p_2 \sim U(a, a+\varepsilon]$

Firm 1: $u_1(p) = \begin{cases} p - c_1 & \text{if } p \leq a \\ \frac{a+\varepsilon-p}{\varepsilon}(p - c_1) & \text{if } p \in (a, a+\varepsilon] \\ 0 & \text{if } p > a+\varepsilon \end{cases}$ $BR_1 = a$

War of Attrition

$t_i \in [0, +\infty)$ $u_i(t_i, t_j) = \begin{cases} V_i - t_j & \text{if } t_i > t_j \\ \frac{1}{2}V_i - t_j & \text{if } t_i = t_j \\ -t_i & \text{if } t_i < t_j \end{cases}$

Pure NE: $\{t_i = 0 . t_j \geq V_i\}$

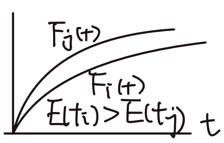
$BR_i(t_j) = \begin{cases} (t_j, +\infty) & t_j < V_i \\ (t_j, +\infty) \cup \{0\} & t_j = V_i \\ 0 & t_j > V_i \end{cases}$

Mixed NE: (F_i, F_j) differentiable. strictly increasing on $\mathbb{R}_+$

$\forall t_i \in \mathbb{R}_+$. $u_i(t_i, F_j) = (1 - F(t_i))(-t_i) + \int_0^{t_i} (V_i - t_j) \, dF_j(t_j) = u_i$

ODE: $\frac{1}{V_i} F_j(t_i) + f_j(t_i) = \frac{1}{V_i}$. $\Rightarrow F_j(t) = 1 - e^{-\frac{t}{V_i}}$

$\Rightarrow u_i = 0$



$V_i < V_j$. $F_j(t) > F_i(t)$ $\forall t > 0$.

t Valuation ↓ likelihood to stay longer in game ↑

Legislating Bargaining (3 players voting) δ.

$\forall$ round: $P(A \text{ proposes}) = p_A$. (p_A, p_B, p_C)

Stationary strategy: (s_1^*, s_2^*, s_3^*)

$s_1^* = \{\text{propose } (x^*, 1-x^*, 0)$; accept (x', y', z') if $x' \geq 1 - z^*\}$

$s_2^* = \{\text{propose } (0, y^*, 1-y^*)$; accept (x', y', z') if $y' \geq 1 - x^*\}$

$s_3^* = \{\text{propose } (1-z^*, 0, z^*)$; accept (x', y', z') if $z' \geq 1 - y^*\}$

1 facing $(1-z^*, 0, z^*)$ an offer at cutoff.

indifference condition: $u_1(A) = u_1(R)$: $1 - z^* = \delta_A(p_A x^* + p_C(1-z^*))$

1 propose $(x, 1-x)$:

OSDP: $u_1(x^*, 1-x^*) = x^* \geq u_1(x, 1-x) = \begin{cases} x & x < x^* \\ \delta_A[p_A x^* + p_C(1-z^*)] & x > x^* \end{cases}$

$\begin{cases} 1 - z^* = \delta_A[p_A x^* + p_C(1-z^*)] \\ 1 - x^* = \delta_B[p_A(1-x^*) + p_B y^*] \\ 1 - y^* = \delta_C[p_B(1-y^*) + p_C z^*] \end{cases}$ If $\delta_A = \delta_B = \delta_C$. $p_A = p_B = p_C$

"Symmetric game":

propose $(1 - \frac{\delta}{3}, \frac{\delta}{3}, 0)$. accept $\frac{\delta}{3}$

$N = 2k+1$ players: $s^* = \{\text{propose } (1-x^*, \underbrace{\tfrac{x^*}{k}, \cdots, \tfrac{x^*}{k}}_{k}, \underbrace{0, \cdots, 0}_{k}) ; \frac{x^*}{k}\}$

$\frac{x^*}{k} = \delta\left[\frac{k}{N}\frac{x^*}{k} + \frac{1}{N}(1-x^*)\right] \Rightarrow x^* = \frac{\delta k}{N}$.

Bertrend Duopoly: unique NE in inadmissible strategies

$p_i \in [0, 1]$ $u_i(p_i, p_j) = \begin{cases} p_i & p_i < p_j \\ \frac{1}{2}p_i & p_i = p_j \\ 0 & p_i > p_j \end{cases}$

$BR_i(p_j) = \begin{cases} p_j - \varepsilon & \text{if } p_j > 0 \\ 0 & \text{if } p_j = 0 \end{cases}$ $\Rightarrow (0, 0)$ unique NE

$p_i = 0$ is weakly dominated

Reporting a crime: n players

$s_i = \{0, 1\}$ $u_i(s_i, s_{-i}) = \begin{cases} V & \text{if } s_i = 0 . \ s_{-i} = 1 \\ V - C & \text{if } s_i = 1 \\ 0 & \text{if } s_i = 0 \ \ s_{-i} = 0 \end{cases}$

$\sigma^* = \{0 \text{ w.p. } (1-p), 1 \text{ w.p. } p\}$.

Indifference Condition

$u_i(0, \sigma_{-i}^*) = (1-p)^{n-1} \cdot 0 + [1 - (1-p)^{n-1}] V$

$\parallel$

$u_i(1, \sigma_{-i}^*) = V - C$

Bertrend Duopoly: $p \in [0, +\infty)$. $u_i(p_i, p_j) = \begin{cases} D(p_i)(p_i - c_i) & p_i < p_j \\ \frac{1}{2}D(p_i)(p_i - c_i) & p_i = p_j \\ 0 & p_i > p_j \end{cases}$

$NE(F_1, F_2)$

$\forall p \in \text{supp}(F)$: $u_i(p) = [1 - F_j(p)] D(p)(p - c_i) = u_i$

$\Rightarrow F_j(p) = 1 - \frac{u_i}{D(p)(p - c_i)}$

Pick $a \in [\max\{c_1, c_2\}, +\infty)$: $F_j(a) = 1 - \frac{u_i}{D(a)(a - c_i)} = 0 \Rightarrow u_i$

$\forall p' \notin \text{supp}(F)$: $u_i(p') = [1 - F_j(p')] D(p')(p' - c_i) \leq u_i = D(a)(a - c_i)$

$F_i(p) = 1 - \frac{D(a)(a - c_j)}{D(p)(p - c_j)}$ $p \in [a, +\infty)$.

*Bertrend Duopoly: $p \in [0, 1]$ $u_i(p_i, p_j) = \begin{cases} p_i - c_i & p_i < p_j \\ \frac{1}{2}(p_i - c_i) & p_i = p_j \\ 0 & p_i > p_j \end{cases}$

$0 < c_1 < c_2 < 1$ No pure strategy

$BR_1(p_2) = \begin{cases} p_2 - \varepsilon & \text{if } p_2 > c_1 \\ [p_2, 1] & \text{if } p_2 = c_1 \\ (p_2, 1] & \text{if } p_2 < c_1 \end{cases}$ $BR_2(p_1) = \begin{cases} p_1 - \varepsilon & \text{if } p_1 > c_2 \\ [p_1, 1] & \text{if } p_1 = c_2 \\ (p_1, 1] & \text{if } p_1 < c_2 \end{cases}$

Perfect Equi. $\vec{\sigma}$ in $(S_i, u_i)_{i=1}^n$ "Trembling hand"

$\exists \sigma^k \to \vec{\sigma}$. s.t. $\sigma_i^k \in \Delta^0(S_i)$ $\quad \vec{\sigma_i} \in BR_i(\sigma_{-i}^k)$

Perfect Equilibrium $\underset{\Leftarrow_*}{\Rightarrow}$ Admissible NE. $\forall s_i \in supp(\vec{\sigma_i})$. admissible

$\quad\quad$ * 2 player game

If 3 players: $\vec{\sigma_1} \in BR_1(\vec{\sigma_2} \otimes \vec{\sigma_3})$.

$\quad \vec{\sigma_1} \in BR_1(\sigma_{-1}')$, $\sigma_{-1}' \in \Delta^0(S_2 \times S_3)$

$\sigma_{-1}^k = (1-\frac{1}{k})(\vec{\sigma_2} \otimes \vec{\sigma_3}) + \frac{1}{k}\sigma_{-1}' \notin \Delta^0(S_2) \times \Delta^0(S_3)$

ε-Perfect Equi. σ^ε: $\varepsilon > 0$. $\sigma_i^\varepsilon \in \Delta^0(S_i)$

$\quad u_i(s_i, \sigma_{-i}^\varepsilon) < u_i(s_i', \sigma_{-i}^\varepsilon) \Rightarrow \sigma_i^\varepsilon(s_i) < \varepsilon$

Perfect Equi $= \lim_{\varepsilon \downarrow 0} \varepsilon$-perfect Equi.

$\quad \vec{\sigma_i}(s_i) > 0 \Rightarrow u_i(s_i, \sigma_{-i}^\varepsilon) \geq u_i(s_i', \sigma_{-i}^\varepsilon)$. $\forall s_i' \in S_i$

$\quad\quad \Rightarrow s_i \in BR_i(\sigma_{-i}^\varepsilon) \Rightarrow \vec{\sigma_i} \in BR_i(\sigma_{-i}^\varepsilon)$

$\quad \vec{\sigma_i} \in BR_i(\sigma_{-i}^k)$: if $u_i(s_i, \sigma_{-i}^k) < u_i(s_i', \sigma_{-i}^k)$, then $\vec{\sigma_i}(s_i) = 0$

$\quad\quad \sigma_i^k(s_i) \to \vec{\sigma_i}(s_i) = 0$. $\forall \varepsilon > 0$. $\exists K(\varepsilon)$ s.t. $\sigma_i^{K(\varepsilon)}(s_i) < \varepsilon$

$\quad\quad \sigma^{K(\varepsilon)}$ is ε-perfect Equi. $\lim_{\varepsilon \downarrow 0} \sigma^{K(\varepsilon)} = \lim_{K \uparrow \infty} \sigma^K = \sigma^*$

ε-proper Equi. $\varepsilon > 0$. $\sigma_i^\varepsilon \in \Delta^0(S_i)$ " Proper trembles"

$\quad u_i(s_i, \sigma_{-i}^\varepsilon) < u_i(s_i', \sigma_{-i}^\varepsilon) \Rightarrow \sigma_i^\varepsilon(s_i) \leq \varepsilon \sigma_i^\varepsilon(s_i')$

Proper Equi. $\vec{\sigma} = \lim_{\varepsilon \downarrow 0} \sigma^\varepsilon$

Every finite game $(S_i, u_i)_{i=1}^n$ has a proper equilibrium.

$\quad m = \max_i |S_i|$ $\quad \forall \varepsilon \in (0,1)$ $\quad \delta = \frac{\varepsilon^m}{m}$

$\quad \Delta^\delta(S_i) = \{\sigma_i \in \Delta(S_i) : \sigma_i(s_i) \geq \delta.\ \forall s_i \in S_i\}$ convex & compact

$\quad F_i(\sigma_{-i}) = \{\hat{\sigma}_i \in \Delta^\delta(S_i) : u_i(s_i, \sigma_{-i}) < u_i(s_i', \sigma_{-i}) \Rightarrow \hat{\sigma}_i(s_i) \leq \varepsilon \hat{\sigma}_i(s_i')\}$

$\quad\quad F_i : \times_{j \neq i} \Delta^\delta(S_j) \rightrightarrows \Delta^\delta(S_i)$ convex & compact valued

$\quad\quad F_i(\sigma_{-i}) \neq \emptyset$: $\hat{\sigma}_i(s_i) = \frac{\varepsilon^{n(s_i)}}{\sum_{s_i' \in S_i} \varepsilon^{n(s_i')}}$

$\quad\quad\quad n(s_i) = |\{s_i' \in S_i : u(s_i, \sigma_{-i}) < u(s_i', \sigma_{-i})\}|$

$\quad\quad F_i$ usc $\Leftarrow$ closed graph.

$\quad F = \times_{i=1}^n F_i$ $\quad$ Kakutani Fixed Point $\forall \varepsilon$. $\exists \sigma^\varepsilon \in F(\sigma^\varepsilon)$

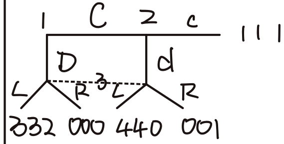
	A	B	C
A	1,1	0,0	-1,-2
B	0,0	0,0	0,-2
~~C~~	~~-2,-1~~	~~-2,0~~	~~-2,-2~~

	Befor EDS	After EDS
NE	AA . BB	AA BB
Perfect	AA	AA BB
Proper	AA	AA

	L	R		L	R
U	111	001	U	000	000
D	001	001	D	000	110
	W			E	

	Before EDS	After EDS
NE	ULW DRW	ULW DRW
Perfect	ULW DRW	ULW
Proper	ULW DRW	ULW

Selten's Horse



	c^q	d^{1-q}
p C	111	440
1-p D	332	332
	L	

	c	d
C	111	001
D	000	000
	R 1-r	

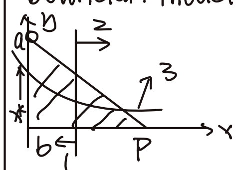
NE : $\{C, c, r \in [0, \frac{1}{4}]\}$ $\quad\quad \{p \in [\frac{2}{3}, 1), d, R\}$

= SPE $\{D, q \in [\frac{1}{3}, 1], L\}$

Perfect: $\{C, c, r \in [0, \frac{1}{4}]\}$

Proper : $\{C, c, r \in [0, \frac{1}{4}]\}$

Downsian model (payoff discontinuity) No NE



2-player 0-sum symmetric game

Assume (σ, σ') is a NE: $\forall x \in P$ $u(x, \sigma') \leq 0$

$\lim_k u(x^k, \sigma') = \sigma'(P \setminus \{a\}) - \sigma'(\{a\}) \leq 0$

$u(b, \sigma') = \sigma'(\{a\}) - \sigma'(P \setminus \{a, b\}) \leq 0$

$\Rightarrow \sigma'(\{a\}) = \sigma'(P \setminus \{a, b\}) = \frac{1}{2}$, $\sigma'(\{b\}) = 0$.

$\Rightarrow u(a, \sigma') = \sigma'(P \setminus \{a, b\}) = \frac{1}{2} > 0$ Contradiction !

Bayes-Nash Equilibrium

Bayesian game $(T_0, (T_i, A_i, u_i, p_i)_{i=1}^n)$

$p_i: T_i \to \Delta(T_{-i})$ $\quad u_i: A\times T \to \mathbb{R}$

$BR_i(t_i, s_{-i}) = \{\varphi_i(t_i,s_{-i}) = \arg\max_{a_i} V_i(a_i, t_i; s_{-i})\}$

$V_i(a_i, t_i; s_{-i}) = \int_{T_{-i}} u_i(a_i, s_{-i}(t_{-i}), t_i, t_{-i})\, p_i(t_i)(dt_{-i})$

$BR_i(s_{-i}) = \{s_i\in S_i : s_i(t_i)\in\varphi_i(t_i,s_{-i}),\ \forall t_i\in T_i\}$

BNE in pure strategies = Fixed point of BR

If $u_i: A_i\times T_i \to \mathbb{R}$, $p_i(t_i)=p_i(t_i')$

u_i: supermodular in A_i & ID in (a_i,t_i) ⇒ $V_i(a_i,t_i;s_{-i})$

Maximum thm ⇒ top & bottom of $\varphi_i(t_i;s_{-i})$ monotone

Single-object Auction (n players)

IPV: independent & private values $X_i \overset{iid}{\sim} F[0,w]$

$Y_1 = \max\{X_2, X_3, \dots, X_n\}$, $G(y) = F^{n-1}(y)$

Second-price: $V_1(b,x;s) = \int_0^{s^{-1}(b)} (x - s(y))\,dG(y)$

FOC: $\frac{\partial V_1}{\partial b} = \frac{1}{s'(s^{-1}(b))}(x - s(s^{-1}(b)))G'(s^{-1}(b)) = 0$

Symmetry: $s(x)=b$: ODE: $\frac{1}{s'(x)}(x-s(x))G'(x)=0$.

⇒ $s(x)=x$ $\quad$ *payoff$(x) = \int_0^x y\,dG(y) = G(x)\mathbb{E}(Y_1|Y_1<x)$

First-price: $V_1(b,x;s)=\int_0^{s^{-1}(b)}(x-b)\,dG(y) = (x-b)G(s^{-1}(b))$

FOC: $\frac{\partial V_1}{\partial b} = -G(s^{-1}(b)) + (x-b)G'(s^{-1}(b))\frac{1}{s'(s^{-1}(b))}$

Symmetry: $s(x)=b$ $\quad$ ODE: $\frac{1}{s'(x)}(x-s(x))G'(x) = G(x)$

⇒ $s(x) = \frac{1}{G(x)}\int_0^x z\,dG(z) = \mathbb{E}(Y_1|Y_1<x) = x - \int_0^x \frac{G(z)}{G(x)}dz$

*payoff$(x) = G(x)s(x) = G(x)\mathbb{E}(Y_1|Y_1<x)$

*Revenue Equivalence

Interdependent value & Affiliated signal $(X_1,\dots,X_n)\sim f$ log-supermodular

$V_i = v_i(X_1,\dots,X_n)$ $\quad \ln f(x\vee x') + \ln f(x\wedge x') \ge \ln f(x) + \ln f(x')$

$\forall y'\ge y,\ z'\ge z.$ $\quad f(y',z')f(y,z) \ge f(y',z)f(y,z')$

⇒ $f(y'|z')f(y|z) \ge f(y'|z)f(y|z')$ $\quad$ Monotone likelihood ratio

⇒ $\frac{f(y|x')}{F(y|x')} \ge \frac{f(y|x)}{F(y|x)}$, $\frac{f(y|x')}{1-F(y|x')} \le \frac{f(y|x)}{1-F(y|x')}$, $\forall x'\ge x$

Second-price: $V_1(b,x;s) = \int_{-\infty}^{s^{-1}(b)} v(x,y) - s(y)\,dG(y|x)$

FOC: $\frac{\partial V_1}{\partial b} = [v(x,s^{-1}(b)) - s(s^{-1}(b))]G'(s^{-1}(b)|x)\frac{1}{s'(s^{-1}(b))}$

Symmetric: $s(x)=b$ $\quad$ ODE: $\frac{1}{s'(x)}[v(x,x)-s(x)]G'(x|x)=0$

⇒ $s(x)=v(x,x)$

payoff$(x) = \int_{-\infty}^x v(y,y)\,dG(y|x) = G(x|x)\mathbb{E}(v(Y,Y)|Y\le x)$

First-price: $V_1(b,x;s) = \int_{-\infty}^{s^{-1}(b)}(v(x,y)-b)\,dG(y|x)$

FOC: $\frac{\partial V_1}{\partial b} = [v(x,s^{-1}(b))-b]G'(s^{-1}(b)|x)\frac{1}{s'(s^{-1}(b))} - \int_{-\infty}^{s^{-1}(b)} 1\,dG(y|x)$

Symmetric: $s(x)=b$ $\quad$ ODE: $[v(x,x)-s(x)]G'(x|x)\frac{1}{s'(x)} - G(x|x)=0$

$s'(x) + \frac{G'(x|x)}{G(x|x)}s(x) = v(x,x)\frac{G'(x|x)}{G(x|x)}$ $\quad \times \exp\{\int_{-\infty}^x \frac{G'(z|z)}{G(z|z)}dz$

⇒ $s(x) = \int_{-\infty}^x v(z,z)\,dL(z|x)$, $L(z|x) = \exp\{\int_x^z \frac{G'(t|t)}{G(t|t)}dt\}$

payoff$(x) = G(x|x)s(x) \le$ payoff(x) 2nd price

Matching Pennies $\quad \varepsilon_i \overset{iid}{\sim} U[-\varepsilon,\varepsilon]$

	H	T
H	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
T	$-1, 1+\varepsilon_2$	$1, -1$

Cutoff strategy: $s_i(\varepsilon_i) = \begin{cases} H & \text{if } \varepsilon_i\ge\varepsilon_i^* \\ T & \text{if } \varepsilon_i<\varepsilon_i^*\end{cases}$

$V_1(H,\varepsilon_1;s_{-1}) = \int_{-\varepsilon}^{\varepsilon} u_1(H,\varepsilon_1,s_{-1}(\varepsilon_{-1}))\,p_1(\varepsilon_1)(\varepsilon_{-1})$

$= \frac{\varepsilon_2^*+\varepsilon}{2\varepsilon}(-1+\varepsilon_1) + \frac{\varepsilon-\varepsilon_2^*}{2\varepsilon}(1+\varepsilon_1) = -\frac{\varepsilon_2^*}{\varepsilon}+\varepsilon_1$

$V_1(T,\varepsilon_1;s_{-1}) = \frac{\varepsilon_2^*+\varepsilon}{2\varepsilon}(1) + \frac{\varepsilon-\varepsilon_2^*}{2\varepsilon}(-1) = \frac{\varepsilon_2^*}{\varepsilon}$

⇒ $\varepsilon_1^* = \frac{2\varepsilon_2^*}{\varepsilon}$

$V_2(H,\varepsilon_2;s_{-2}) = \frac{\varepsilon_1^*+\varepsilon}{2\varepsilon}(1+\varepsilon_2) + \frac{\varepsilon-\varepsilon_1^*}{2\varepsilon}(-1+\varepsilon_2) = \frac{\varepsilon_1^*}{\varepsilon}+\varepsilon_2$

$V_2(T,\varepsilon_2;s_{-2}) = \frac{\varepsilon_1^*+\varepsilon}{2\varepsilon}(-1) + \frac{\varepsilon-\varepsilon_1^*}{2\varepsilon}(1) = -\frac{\varepsilon_1^*}{\varepsilon}$

⇒ $\varepsilon_2^* = -\frac{2\varepsilon_1^*}{\varepsilon}$

⇒ $\varepsilon_1^*=\varepsilon_2^*=0$ $\quad$ Purification

$P(H)=P(L)=\frac12$. $\quad ((\frac12,\frac12),(\frac12,\frac12))$ is a mixed NE

Bertrend Duopoly

$X_i\sim G$ on $[0,\bar x]$ $\quad u_i(p_i,p_j,x_i) = \begin{cases}\frac23(p_i-x_i) & p_i<p_j\\ \frac12(p_i-x_i) & p_i=p_j\\ \frac13(p_i-x_i) & p_i>p_j\end{cases}$

$\bar x<1$ $\quad p_i\in[0,1]$

$s:[0,\bar x]\to[0,1]$ ① ↑↑ ② differentiable ③ $s(\bar x)=1$

$V(p,x;s) = G(s^{-1}(p))\frac13(p-x) + [1-G(s^{-1}(p))]\frac23(p-x)$

$= \frac13[2(p-x) - G(s^{-1}(p))(p-x)]$

FOC: $\frac{3\partial V}{\partial p} = 2 - G'(s^{-1}(p))\frac{1}{s'(s^{-1}(p))}(p-x) - G(s^{-1}(p)) = 0$

Symmetry: $s(x)=p$ $\quad$ ODE: $[(2-G(x))s(x)]' = -G'(x)x$

⇒ $s(x) = \frac{1}{2-G(x)}[1+\int_x^{\bar x} z\,dG(z)]$

Verify: ① s ↑↑: $\underbrace{(2-G(x))}_{>0}s'(x) = \underbrace{G'(x)}_{>0}(s(x)-x)$ ⇒ $s'(x)>0$

$\forall x\in[0,\bar x]$: $s(x)>x$: $s(0)>0$ $\quad s(\bar x)=1>\bar x$

② FOC: $\frac{\partial V}{\partial p} = \frac{1}{3s'(z)}[(2-G(z))s'(z) - G'(z)(s(z)-x)]$

$p=s(z)$:	$p<s(x)\ (z<x)$	$p=s(x)\ (z=x)$	$p>s(x)\ (z>x)$
$\frac{\partial V}{\partial p}$	+	0	−

$p<s(0)$: p is dominated by $s(0)$

Purification: $H^\varepsilon(p) = \begin{cases}\Pr(s(x)\le p) & \text{if } p\in[\frac{1+E^\varepsilon(x)}{2},1]\\ 0 & \text{if } p\in[0,\frac{1+E^\varepsilon(x)}{2})\end{cases}$

G^ε on $[0,\varepsilon]$

$\frac{1}{2-G(x)} \le \frac{1}{2-G(x)}[1+\int_x^\varepsilon z\,dG(z)] \le \frac{1+[1-G(x)]\varepsilon}{2-G(x)}$

⇒ $\Pr(\frac{1}{2-G(x)}\le p) \ge H^\varepsilon(p) \ge \Pr(\frac{1+[1-G(x)]\varepsilon}{2-G(x)}\le p)$

⇒ $H^\varepsilon(p)\in[2-\frac1p,\ 2-\frac{1-\varepsilon}{p-\varepsilon}]$

Double Auction: 1: seller 2: buyer $p=\frac12(b_1+b_2)$ if $b_2\ge b_1$

$x_i\overset{iid}{\sim}U[0,1]$

Linear strategy: $b_i(x_i) = a_i + c_i x_i$ ⇒ $b_2\sim U[a_2, a_2+c_2]$

$V_1(b_1,x_1;b_2) = P(b_2<b_1)x_1 + P(b_2>b_1)\mathbb{E}(\frac12(b_1+b_2)|b_2>b_1)$

$= x_1\frac{b_1-a_2}{c_2} + \frac{a_2+c_2-b_1}{c_2}\frac12(b_1+\frac{b_1+a_2+c_2}{2})$

FOC ⇒ $b_1 = \frac23x_1+\frac13(a_2+c_2)$ ⇒ $a_1=\frac13(a_2+c_2)$, $c_1=\frac23$

$V_2(b_2,x_2;b_1) = P(b_2>b_1)(x_2-\mathbb{E}(\frac12(b_1+b_2)|b_2>b_1))$

$= \frac{b_2-a_1}{c_1}[x_2-\frac12(\frac{a_1+b_2}{2}+b_2)]$

FOC ⇒ $b_2 = \frac23x_2+\frac13a_1$ ⇒ $a_2=\frac13a_1$, $c_2=\frac23$

Sequentially rational: β^*: $\forall i.\ \forall w \in \Omega_i\ \ \beta_i(w) \in BR_i(\beta_{-i})$

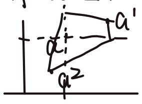
NE: on-path optimality

SPE: BI with perfect info.

Finitely repeated game: stage $(A_i, u_i)_{i=1}^{n} = G$

If $G\ \exists!$ NE: $\exists!$ SPE for finitely repeated game

If $G\ \exists$ NE a^1, a^2: $u_i(a^1) \geqslant u_i(a^2)\ \ \forall i$



$b_i(a_{-i}) = \max_{a_i} u_i(a_i, a_{-i})$

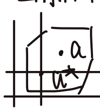
$\forall a \in A$: $b_i(a_{-i}) - u_i(a) \leq u_i(a^1) - u_i(a^2)$

$u_i(a) \geqslant u_i(a^2)\ \ \forall i$

then $\exists$ SPE: a for $t = 1, \cdots, T-1$, a^1 for T

Strategies: $s(h^t) = \begin{cases} a_i & \text{if } h^t = \emptyset \text{ or } h^t = (a, a, \cdots, a), t < T \\ a_i^1 & \text{if } t = T \\ a_i^2 & \text{o.w.} \end{cases}$

Infinitely repeated game: Discounts δ

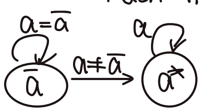


Convex hull $F = co\{(u_1(a), \cdots, u_n(a)) : a \in A\}$

$\forall a \in F$: a dominates a^*.

$\exists$ SPE obtaining $u(a)$ as long as δ is large enough

"Nash-threat" trigger strategy



$V_i(\bar{a}) = \max\{(1-\delta) u_i(\bar{a}) + \delta V_i(\bar{a}),$
$(1-\delta) b_i(\bar{a}_{-i}) + \delta V_i(a^*)\}$

$V_i(a^*) = (1-\delta) u_i(a^*) + \delta V_i(a^*) = u_i(a^*)$

OSDP: $V_i(\bar{a}) = u_i(\bar{a}) > (1-\delta) b_i(\bar{a}_{-i}) + \delta u_i(a^*)$

$\Leftrightarrow \delta(b_i(\bar{a}_{-i}) - u_i(a^*)) > b_i(\bar{a}_{-i}) - u_i(\bar{a})$ high δ.

Folk Thm: $\forall v \in F^* = \{v \in F : v_i > \text{minmax}_i\}$ can be obtained

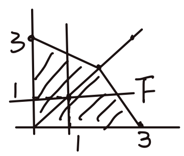
Prisoner's Dilemma (Tit-for-tat strategy)

	C	D
C	2,2	0,3
D	3,0	$\underline{1},\underline{1}$

NE DD $u(D,D) = (1,1)$

CC $u(C,C) = (2,2)$ ↑ dominate

$s_i(h^t) = \begin{cases} C & \text{if } h^t = \emptyset\ \ t=1 \\ s_j(h^{t-1}) \end{cases}$



Recursive / OSDP

$V_{CC} = (1-\delta)2 + \delta V_{CC} = 2 \geqslant (1-\delta)3 + \delta V_{DC}$

$V_{DC} = (1-\delta)0 + \delta V_{CD} = \frac{3\delta}{1+\delta} \geqslant (1-\delta)1 + \delta V_{DD}$

$V_{CD} = (1-\delta)3 + \delta V_{DC} = \frac{3}{1+\delta} \geqslant (1-\delta)2 + \delta V_{CC}$

$\delta = \frac{1}{2}$

$V_{DD} = (1-\delta)1 + \delta V_{DD} = 1 \geqslant (1-\delta)0 + \delta V_{CD}$

	A	B	C
A	$\underline{5},\underline{5}$	$\underline{4},3$	$\underline{2},2$
B	$3,\underline{4}$	3,3	$\underline{2},2$
C	$2,\underline{2}$	$2,\underline{2}$	0,0

Carrot-and-stick strategy

NE AA $u(A,A) = (5,5)$

Carrot BB target

Stick CC punishment (1-period)

$V_C = (1-\delta)3 + \delta V_C = 3 \geqslant (1-\delta)4 + \delta V_S$ $\delta \geqslant \frac{1}{3}$

$V_S = (1-\delta)0 + \delta V_C = 3\delta \geqslant (1-\delta)2 + \delta V_S$ $\delta \geqslant \frac{2}{3}$

minmax $= \min\{5, 4, 2\} = 2$

maxmin $= \max\{2, 2, 0\} = 2$

To obtain $(2+\varepsilon, 2+\varepsilon)$: play (CB, BC, CB, BC) k rounds followed by BB. $k = \frac{1-\varepsilon}{\varepsilon}$

i.e. $(3,3)$ w.p. ε. $(2,2)$ w.p. $1-\varepsilon$

$V_C = (1-\delta)(2+\varepsilon) + \delta V_C = 2+\varepsilon \geqslant (1-\delta)4 + \delta V_S$

$V_S = (1-\delta)0 + \delta V_C = \delta(2+\varepsilon) \geqslant (1-\delta)2 + \delta V_S$ $\delta \geqslant \frac{2}{2+\varepsilon}$

$\delta \geqslant \frac{2}{2+\varepsilon}$.

n-firm Cournot model

$P(Q) = \max\{D(Q), 0\}$. $Q = q_1 + \cdots + q_n$ MC: $c > 0$

$V(q; Q_{-i}) = (P(q + Q_{-i}) - c) q$.

FOC: $\frac{\partial V}{\partial q} = [P(q + Q_{-i}) - c] + P'(q + Q_{-i}) q = 0$

Symmetric: $Q_{-i} = (n-1) q$. ODE: $p(nq^c) + p'(nq^c) q^c = c$ $q^c > \frac{1}{n} Q^m = q^m$

Monopoly: $Q^m = \arg\max_q (P(q) - c) q$: $P(Q^m) + P'(Q^m) Q^m = c$

① Trigger strategy: $s(h^t) = \begin{cases} \frac{1}{n} Q^m & \text{if } h^t = \emptyset \text{ or } h^t \in \text{Good} \\ q^c & \text{o.w.} \end{cases}$

$V_G = (1-\delta) u(q^m) + \delta V_G = u(q^m) \overset{\text{OSDP}}{\geqslant} (1-\delta) b(q^m) + \delta V_B$

$V_B = (1-\delta) u(q^c) + \delta V_B = u(q^c) \geqslant (1-\delta) b(q^c) + \delta V_B$ NE

$\delta(b(q^m) - u(q^c)) \geqslant b(q^m) - u(q^m)$

If δ is not such high: achive q^*: $\delta = \frac{b(q^*) - u(q^*)}{b(q^m) - u(q^c)}$

② Carrot-and-stick: $s(h^t) = \begin{cases} \underline{q} & \text{if } h^t \in C \\ \bar{q} & \text{if } h^t \in S \end{cases}$

$V_C = (1-\delta) u(\underline{q}) + \delta V_C = u(\underline{q})$ $\geqslant (1-\delta) b(\underline{q}) + \delta V_S$

$V_S = (1-\delta) u(\bar{q}) + \delta V_C = (1-\delta) u(\bar{q}) + \delta u(\underline{q}) \geqslant (1-\delta) b(\bar{q}) + \delta V_S$

Perfect Bayesian Equilibrium PBE (β, μ)

β is sequentially rational given belief μ:

$$\beta_i(\omega) \in \underset{\beta_i'}{\operatorname{argmax}} \sum_{y \in x_\omega} \mu_\omega(y) u_i((\beta_i', \beta_{-i}) \omega \mid y) \quad \forall \omega \in \Omega_i$$

$$u_i((\beta)\omega \mid y) = \sum_{x \in T} P(x \mid (\beta)\omega, y)\, u_i(x)$$

μ is consistent with β

1st info. set: $\mu_\omega(x) = \dfrac{P(x \mid \beta)}{\sum_{y \in x_\omega} P(y \mid \beta)}$

2nd info. set: $\mu_{\omega'}(x') = \dfrac{\mu_\omega(x)\, \beta_i(\omega)(a_\omega)}{\sum_{x \in x_\omega} \mu_\omega(x) \sum_{a \in A_\omega^{\omega'}(x)} \beta_i(\omega)(a)}$

...

Weak PBE: free μ_ω for off-path info. sets

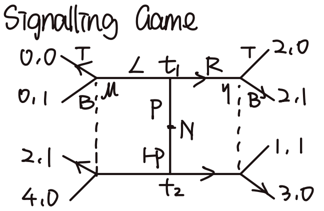
Sequential Equilibrium SE (β, μ)

β is sequentially rational given μ.

μ is fully consistent with β

$\exists \beta^k : \beta_i^k \in \times_{\omega \in \Omega_i} \Delta^o(A_\omega) \quad \beta_i^k \to \beta_i \quad \mu_\omega(x) = \lim_{k \to +\infty} \dfrac{P(x \mid \beta^k)}{\sum_{y \in x_\omega} P(y \mid \beta^k)}$

Signalling Game



$\binom{p}{1-p}$	TT	TB	BT	BB
LL	0,0 / 2,1	0,0 / 2,1	0,1 / 4,0	0,1 / 4,0
LR	0,0 / 2,0	0,0 / 2,1	0,1 / 1,1	0,1 / 3,0
RL	2,0			
RR				

PBE = SE = $\{(RR, TB), (\mu \le \frac{1}{2}, \eta = p)\}$

$= \{(R, R\frac{p}{1-p}, T\,T\frac{1}{2}), (\mu = 0, \eta = \frac{1}{2})\}$

Bargaining with One-Sided Asymmetric Information

\# of rounds = 1:

Bob: facing x: accept if $\pi - x \ge 0$

Alice: $\max_x x P(\pi - x \ge 0) = \max_x x(1-x) \quad \Rightarrow x = \frac{1}{2}$

\# of rounds = 2:

Alice

Signaling Game: 1: $\theta \in \Theta$, $a \in A$ $\quad$ 2: $b \in B$ $\quad \mu \in \Delta(\Theta)$

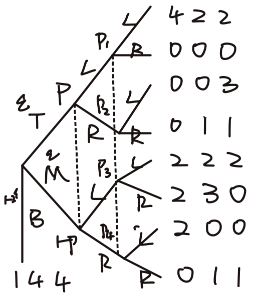
Some SE might have unreasonable beliefs.

SE (α^*, β^*). $V_1^*(\theta)$

$\forall a$ off-path action: $D(a, \theta) = \{b \in B : u_1(a, b, \theta) \ge V_1^*(\theta)\}$.

$\mu(\cdot \mid a)$ is a reasonable belief if "Intuitive condition"

$\mu(\theta \mid a) = 0$ when $D(a, \theta) = \emptyset$. $D(a, \theta') \ne \emptyset$.



	L	R
T	422	003
M	222	200
B	144	144

L

	L	R
T	000	011
M	230	011
B	144	144

R

NE: TLL $\quad$ BRR

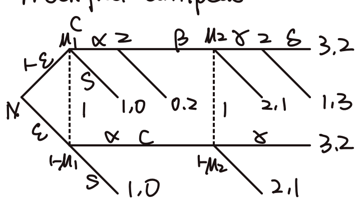
SPE: TLL $\quad$ BRR

Weak PBE: $\{TLL, p, p_4 \le \frac{2}{3}\}$ $\{BRR, p \ge \frac{2}{3}, p_4 \le \frac{2}{3}\}$

PBE: $\{TLL, (p, 1-p), (p, 0, 1-p, 0) : p \in [0,1]\}$

SE: $\{TLL, (p, 1-p), (p, 0, 1-p, 0) : p \in [0,1]\}$

Modified Centipede $\quad \varepsilon \in (\frac{1}{4}, \frac{1}{2})$



N $\langle^{1-\varepsilon}_{\varepsilon}$

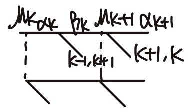
	CC	CS	S·
CC	3,2 / 3,2	1,3 / 3,2	0,2 / 3,2
CS	2,1 / 2,1	2,1 / 2,1	0,2 / 2,1
S·	1,0 / 1,0	1,0 / 1,0	1,0 / 1,0

SE: $\{(\alpha = 1, \beta = \frac{\varepsilon}{1-\varepsilon}, \gamma = \frac{1}{2}, \delta = 0), (\mu_1 = 1-\varepsilon, \mu_2 = \frac{1}{2})\}$

PBE: SE + $\{(\alpha = 0, \beta = 0, \gamma = 0, \delta = 0), (\mu_1 = 1-\varepsilon, \mu_2 \ge \frac{1}{2})\}$

\# of rounds = n. $\quad \varepsilon \in (\frac{1}{2^n}, \frac{1}{2^{n-1}})$

SE: $\mu_n = \frac{1}{2}$ $\quad$ given μ_{k+1}, find μ_k, α_k, β_k



②: $u(C) = \alpha_{k+1}(k+2) + (1-\alpha_{k+1})k = k + 2\alpha_{k+1}$

$u(S) = k+1 \quad \Rightarrow \alpha_{k+1} = \frac{1}{2}$

①: $u(C) = \mu_k \beta_k (k+1) + \mu_k(1-\beta_k)(k-1) + (1-\mu_k)(k+1)$

$u(S) = k$

Bayes: $\mu_{k+1} = \dfrac{\mu_k \beta_k}{\mu_k \beta_k + 1 - \mu_k}$

$\Rightarrow \mu_k = 1 - \dfrac{1}{2^{n-k+1}}$, $\quad \mu_1 = 1-\varepsilon$

$\beta_k = \dfrac{2^{n-k} - 1}{2^{n-k+1} - 1}$, $\quad \beta_1 = (2^{n-1} - 1)\dfrac{\varepsilon}{1-\varepsilon}$

$\alpha_k = \frac{1}{2}$ $\quad \alpha_1 = 1$

Job Market Signaling