Subgroups of $GL_2(\mathbb{C})$ Acting on $\mathfrak{gl}_2(\mathbb{C})$

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Abstract

The purpose of this paper is to give an analysis of the orbits of the group action of $GL_2(\mathbb{C})$ and its subgroups on the lie algebra $\mathfrak{gl}_2(\mathbb{C}) = M_{2\times 2}(\mathbb{C})$ under conjugation. To do this, we will review various properties of group orbits and Jordan canonical form. The Jordan similarity theorem makes the cases of $GL_2(\mathbb{C}) \curvearrowright \mathfrak{gl}_2(\mathbb{C})$ and $SL_2(\mathbb{C}) \curvearrowright \mathfrak{gl}_2(\mathbb{C})$ trivial, but does not always apply for the other subgroups. We endeavor to classify the orbits of the group action of any subgroup $H \subset GL_2(\mathbb{C})$.

1 Group Actions and Orbits (and other definitions)

Definition 1.1. *Given a group G and a set S, $G \times S \to S$ is a* group action *if it satisfies the following:*

1. *$e \cdot x = x$ where e is the identity element of G.*
2. *$g \cdot (h \cdot x) = (gh) \cdot x$ where $g, h \in G$.*

We denote the group action of G on S by $G \curvearrowright S$

Definition 1.2. *Given a group action G on S, the* orbit *$G \cdot x$ of some element $x \in S$ is the following set: $\{g \cdot x : g \in G\}$, where $\cdot$ is the action operation. The* stabilizer *of x is $\{g \in G : g \cdot x = x\}$.*

Lemma 1.1. *Given a group action of a group G on a set S, the orbits of G partition S.*

Proof. First, we must show that orbits are in fact equivalence classes, and that elements in the same orbit can be related by an equivalence relation. Let G be a group, S be a set, and let G act on S by some operation $\cdot$. The orbits of this action are the sets $\{g \cdot x : g \in G\}$ for every $x \in S$. Let two elements u, v be equivalent if there exists $g \in G$ such that $u = g \cdot v$.

1. Suppose a, b in the orbit of $x \in S$. Then, $a = g \cdot x$ and $b = h \cdot x$ for $g, h \in G$. Then, $a \sim a$ since the identity element must be in $\mathfrak{gl}_2(\mathbb{C})$ by the group axioms, and so $a = e \cdot a$.

2. Suppose there exists some $j, k \in G$ such that $a = j \cdot b$ and $b = k \cdot c$, and so $a \sim b$ and $b \sim c$. Then, $a = j \cdot k \cdot c = (jk) \cdot c$ and so $a \sim c$.

3. Finally, take j to be the same as in 2., so $a = j \cdot b$. Then, $j^{-1} \in G$ by the group axioms, and $j^{-1} \cdot a = j^{-1} \cdot j \cdot b = j^{-1} \cdot a = (j^{-1} j) \cdot b = e \cdot b = b$. We see that the relation is reflexive, transitive and symmetric, and so it is an equivalence relation, and orbits are equivalence classes.

We see that equivalence classes must be equal or disjoint. Suppose this were not the case; then, given two equivalence classes G_x, G_y there would exist some $z \in G_x \cap G_y$. But then, $z \sim x$ and $z \sim y$ by definition of equivalence class, and by transitivity and symmetry, we have that $x \sim z \sim y \implies x \sim y$, and so x, y are in the same equivalence class for every $x, y \in G_x, G_y$, so it must be the case that $G_x = G_y$. Therefore, since we can repeat this process for every $x, y \in S$, the orbits of G on S form a partition of S. $\square$

2 Jordan canonical Form and Algebraic Closure

Definition 2.1. *A $k \times k$* Jordan Block *with eigenvalue λ is the $k \times k$ matrix with λ along the main diagonal and* 1 *along the first superdiagonal. A matrix is in* Jordan canonical form *if it is a block diagonal matrix with Jordan blocks along the diagonal.*

For any vector space V over a field F, the existence of the basis in which any matrix A has the canonical form is guaranteed if and only if F is algebraically closed, i.e if and only if all roots of the characteristic polynomial are contained in F. Therefore, we see that such a basis is guaranteed in vector spaces of the form $\mathbb{C}^n$, but not in $\mathbb{R}^n$.

Theorem 2.1. ***Jordan Similarity*** *For every $n \times n$ complex matrix A, there exists P such that $A = PJP^{-1}$, where J is the Jordan canonical form of A. In other words, every matrix is similar to a matrix in Jordan canonical form. If A is similar to a diagonal matrix D, then D is A's Jordan canonical form.*

For example, consider the 2×2 case $A = \begin{bmatrix} 2 & 4 \\ 9 & 2 \end{bmatrix}$. Then, we see that $\begin{bmatrix} 2 & 4 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix}$ and so the Jordan canonical form of A is $J = \begin{bmatrix} -4 & 0 \\ 0 & 8 \end{bmatrix}$.

3 Example: $SL_2(\mathbb{C}) \curvearrowright sl_2(\mathbb{C})$ under conjugation

Lemma 3.1. *Let $A, B \in \mathfrak{gl}_2(\mathbb{C})$. A, B are conjugate by some matrix in $GL_2(\mathbb{C})$ if and only if A, B are conjugate by some matrix in $SL_2(\mathbb{C})$.*

Proof. Suppose that there is some matrix $A \in \mathfrak{gl}_2(\mathbb{C})$ and $P \in GL_2(\mathbb{C})$, so that $PAP^{-1} = B$. Then, we see that $\frac{1}{\sqrt{\det A}} PA\sqrt{\det P}P^{-1} = \frac{\sqrt{\det P}}{\sqrt{\det P}}(PAP^{-1}) = PAP^{-1}$ and so $\frac{1}{\sqrt{\det P}}P$ also conjugates A to B. Since the determinant is multilinear, and so the determinant of $\frac{1}{\sqrt{\det P}}P$ is $\frac{\det P}{(\sqrt{\det P})^2} = 1$. $\square$

First, note that the lie algebra $sl_2(\mathbb{C})$ is the set of 2×2 trace 0 matrices. Consider the action of $SL_2(\mathbb{C})$ on this set. For matrices with distinct eigenvalues, the following proof that eigenvalues determine conjugacy holds:

Proposition 1. *Matrices X and Y in $sl_2(\mathbb{C})$ with distinct eigenvalues are are conjugate by matrices in $SL_2(\mathbb{C})$ if and only if they have the same eigenvalues*

Proof. First let $Y = AXA^{-1}$, and let (λ, v) be an eigenpair of X. Then, $Xv = \lambda v$, and so $AxA^{-1}(Av) = A\lambda v$. Let $v' = Av$; then, we have $AXA^{-1}(Av) = AXA^{-1}v' = \lambda v'$, and so $AXA^{-1} = Y$ have the same eigenvalue but a different eigenvector. We can repeat this process for every such eigenpair.

Conversely, consider matrices with the same eigenvalues. Since all matrices in $sl_2(\mathbb{C})$ have trace 0, unless an eigen value of a matrix is 0, the eigenvalues for all matrices in $sl_2(\mathbb{C})$ are distinct. Therefore, all matrices in $sl_2(\mathbb{C})$ with non-zero eigenvalues can be conjugated to a diagonal matrix. Then, for any two matrices (X, Y) which have the same eigenvalues, X, Y will conjugate to the same diagonal matrix D. Since conjugation is an equivalence relation, $X \sim D$ and $D \sim Y \implies X \sim Y$ (where $\sim$ denotes conjugate elements), and so X, Y are in the same conjugacy class. $\square$

However, the proof above fails for matrices with one repeated eigenvalue, for which we need to use Jordan Similarity.

Proposition 2. *Matrices X and Y in $sl_2(\mathbb{C})$ are conjugate by matrices in $SL)2(\mathbb{C})$ if and only if they have the same Jordan Cannonical Form.*

Proof. First let X, Y have the same JCF J. Then, $X = AJA^{-1}$ and $Y = BJB^{-1}$ Then, $J = A^{-1}XA$ and so $Y = BA^{-1}XAB^{-1}$, and so $X \sim Y$.

Conversely, suppose $X \sim Y$. Then, $X = AYA^{-1}$. Let J be the JCF of X and K be the JCF of Y. Then, $X = BJB^{-1}$, $Y = CKC^{-1}$, and so $X = ACKC^{-1}A^{-1} \implies BJB^{-1} = ACKC^{-1}A^{-1} \implies J =$

$B^{-1}ACKC^{-1}A^{-1}$. Let $D + B^{-1}AC$; then, $J = DKD^{-1}$ and so $J \sim K$. However, since X, Y conjugate, they must have the same eigenvalues, and so not only are J, K conjugate, they are in fact equal. $\square$

We see from these two proofs the orbits of this action are determined by Jordan canonical form. Recall from theorem 2.1 that if a matrix is similar to a diagonal matrix D, then D is the Jordan canonical form of that matrix. Since each matrix M in $sl_2(\mathbb{C})$ with eigenvalues $a \neq b$ is similar to a diagonal matrix determined by a and b, transitivity implies that every such matrix is in the same conjugacy class.

4 Orbits for subgroups of $GL_2(\mathbb{R})$

Definition 4.1 Center of a group**.** *Given a group G with binary operation $\cdot$, the set of all elements $\{g \in G : g \cdot x = x \cdot g \forall x \in G\}$ is called the center of G, denoted $Z(G)$.*

Theorem 4.1 Center of $GL_2(\mathbb{R})$**.** *The center of $GL_2(\mathbb{R})$ is the set*

$$aI_2 = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R} \right\}$$

Proof. First, we show that all such matrices commute with any matrix in $\mathfrak{gl}_2(\mathbb{C})$: Let $M = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw & ax \\ ay & az \end{bmatrix}$$

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} aw & ax \\ ay & az \end{bmatrix}$$

Next, we show that only matrices in aI_2 commute with all elements of $\mathfrak{gl}_2(\mathbb{C})$:
We will do this manually, although there are more elegant proofs. Use the same M as above:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} aw + xc & wb + xd \\ ya + zc & yb + zd \end{bmatrix}$$

Then, we are left with the following system of equations:

$$1) \;\; aw + by = aw + xc \implies by = xc$$

$$2) \;\; ax + bz = wb + xd \implies d = a, (z = w \text{ or } b = 0)$$

$$3) \;\; cw + dy = ya + zc \implies d = a, (z = w \text{ or } c = 0)$$

$$4) \;\; cx + dz = yb + zd \implies yb = cx$$

First, consider that in equations 2) and 3), $z = w$ will only hold for some matrices in $\mathfrak{gl}_2(\mathbb{C})$. However, $b = 0$ and $c = 0$ will hold for any choice of M, and will also preserve the implications from equations 1) and 4). We also see from 2) and 3) that $d = a$ Therefore, we see that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ will commute with all $M \in \mathfrak{gl}_2(\mathbb{C})$ if and only if $b = c = 0$ and $a = d$, and so the center of $GL_2(\mathbb{R})$ contains exactly the set aI_2. □

Therefore, we see that each matrix in aI_2 must be its own orbit, since for every $d \in aI_2 \subset \mathfrak{gl}_2(\mathbb{C})$ and every $g \in GL_2(\mathbb{R})$, we have $gdg^{-1} = gg^{-1}d = d$. Since $GL_2(\mathbb{R}) \subset \mathfrak{gl}_2(\mathbb{C})$, aI_2 commutes with $GL_2(\mathbb{R})$ as well.

Consider the eigenvalues of matrices in $M_{2\times 2}\mathbb{R}$. There are three possibilities for any matrix $A \in \mathfrak{gl}_2(\mathbb{C})$: Either A has 1 real eigenvalue with multiplicity 2, 2 distinct real eigenvalues, or 2 distinct complex eigenvalues.

Proposition 3. *If a real matrix A has 2 distinct complex eigenvalues, then the eigenvalues are in fact complex conjugates of each other.*

Proof. The characteristic polynomial of A will have only real coefficients. Note, however, that in the case of

$$2 \times 2$$

matrices the characteristic polynomial for A is given by $t^2 - tr(A) + \det(A)$, and is as such quadratic for every such A. Therefore, by the fundamental theorem of algebra, it will have exactly two roots in $\mathbb{C}$. If some root r is in $\mathbb{C}/\mathbb{R}$, by the complex conjugate root theorem, its complex conjugate must also be a root. The complex conjugate of r must also obviously be in $\mathbb{C}/\mathbb{R}$, and we see from the fundamental theorem of algebra that there cannot be more than two roots. Then, r and its complex conjugate $\bar{r}$ must be the only roots, and as such the only eigenvalues. $\square$

Proposition 4. *Let A be a real matrix with complex eigenvalues $a \pm bi$. Each such A is conjugate to* $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

Proof. Let v be an eigenvector of A, and let $v = Re(v) + iIm(v)$. Then, we have that $Av = \lambda v$, and so it follows that

$$ARe(v) = aRe(v) - bIm(v)$$

$$AIm(v) = bRe(v) + aIm(v)$$

Let $V = [Re(v)|Im(v)]$. Then, we see that we can represent $ARe(v)$ as $V \begin{bmatrix} a \\ -b \end{bmatrix}$ and $AIm(v)$ as $V \begin{bmatrix} b \\ a \end{bmatrix}$. Therefore, we can compute AV:

$$AV = \left[V \begin{bmatrix} a \\ -b \end{bmatrix} \middle| V \begin{bmatrix} b \\ a \end{bmatrix} \right] = V \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

Let the matrix $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ be denoted $\bigwedge$. Then, we have $AV = [ARe(v)|AIm(V)] = V \bigwedge \implies VAV^{-1} = \bigwedge$. Note that we can repeat this process for any A with the eigenvalues $a \pm bi$, and that $A, V, \bigwedge$ are all real matrices, and so for any such A, the respective $V \in GL_2(\mathbb{R})$.

$\square$

Proposition 5. *If some real matrix M has two distinct real eigenvalues, then it is conjugate to its Jordan canonical form* $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ *where $a, d \in \mathbb{R}$ are eigenvalues of M.*

Proof. To prove this, we require an additional theorem:

Theorem 4.2 Theorem 23, Dummit and Foote (1)**.** *Let A be an $n \times n$ matrix with entries in a field F and let F contain all eigenvalues of A. Then, the matrix A is similar to a matrix in Jordan canonical form, i.e.,* *there is an invertible $n \times n$ matrix P with entries in F such that PAP^{-1} is in Jordan canonical form. This is a generalized statement of the Jordan Similarity theorem stated earlier.*

We see from this that if we consider only the set of matrices with two distinct real eigenvalues Jordan similarity holds, since $\mathbb{R}$ contains all eigenvalues of all such matrices. Since we are assuming that the two eigenvalues are distinct, the Jordan canonical form for such matrices is $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ where $a, d \in \mathbb{R}$ are eigenvalues. $\square$

Consider the case of real matrices with one real eigenvalue. We see that we can again apply Jordan similarity, since all relevant eigenvalues are contained in $\mathbb{R}$. Therefore, we see that there are two possible forms.

Theorem 4.3. *All matrices matrices with one real eigenvalue in $M_2(\mathbb{R})$ are either of the form $A \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} A^{-1}$ where A is an invertible matrix or aI_2 where I_2 is the two by two identity matrix.*

Proof. The first case is in Jordan canonical form. Since conjugation preserves eigenvalues, all matrices similar to it also have one real eigenvalue. We see that Jordan Similarity (theorem 2.1) holds, since we are only considering the case of one real eigenvalue, and so for each $\lambda \in \mathbb{R}$, there will be a conjugacy class determined by the Jordan matrix $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. For the second case, we see again upon inspection that a is the only eigenvalue, and in fact, each matrix of the form aI_2 for some $a \in \mathbb{R}$ is in the center of $GL_2(\mathbb{R})$, and so as shown in theorem 4.1, each such matrix is it's own orbit. To see that all 2×2 matrices with one real eigenvalue must be of one of these two forms, consider the following lemma;

Lemma 4.1. *If A is a 2×2 non-diagonal matrix with one real eigenvalue, then it is conjugate to the Jordan matrix $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$*

Proof. The proof follows Theorem 2.1, recalling that if we are only consider matrices with one real eigenvalue that $\mathbb{R}$ contains all eigenvalues of relevance, and also that the Jordan canonical form for matrices with one eigenvalue λ is $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

$\square$

Every such A can be conjugated by some matrix P to the corresponding jordan matrix, so we have that $PAP^{-1} = J \implies A = PAP^{-1}$. The only such matrices that cannot be conjugated to the Jordan matrix

J are precisely diagonal matrices with one real eigenvalue, which we have already accounted for. Therefore, all 2×2 matrices with one real eigenvalue are of one of the two specified forms. $\square$

We see that we have a complete partition of eigenvalues for real 2×2 matrices; we know by the fundamental theorem of algebra that quadratic polynomials can have at most 2 roots, and since eigenvalues are the roots of the characteristic polynomial, and all characteristic polynomials of 2×2 matrices are quadratic. We see also by proposition 2 that a 2×2 real matrix can only have 1 real root, 2 real roots or 2 complex roots, and since we have accounted for all of these cases, we have found all the orbits of $M_{2\times2}(\mathbb{R})$

Lemma 4.2. *Let $A \in M_{2\times2}(\mathbb{C})$, and let $A = Re(A) + iIm(A)$, $g \in GL_2(\mathbb{R})$. Then, $gAg^{-1} = gRe(A)g^{-1} + igIm(A)g^{-1}$.*

Proof. Recall that the set $M_{n\times n}(\mathbb{C})$ forms a ring under matrix addition and multiplication. Therefore, the distributive property applies, and so for matrices $A, B, C \in M_{n\times n}(\mathbb{C})$ we have

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

Therefore, for $g \in GL_2(\mathbb{R}), A \in M_{2\times2}(\mathbb{C})$ we have

$$A = Re(A) + Im(A)$$

$$gAg^{-1} = g(Re(A) + Im(A))g^{-1} = (gRe(A) + gIm(A))g^{-1} = gRe(A)g^{-1} + gIm(A)g^{-1}$$

$\square$

Recall that a Cartesian product of two sets A, B is the set $\{(a, b) : a \in A, b \in B\}$. Therefore, the Cartesian product of the orbits $G \cdot Re(A)$ and $G \cdot Im(A)$ where $A \in \mathfrak{gl}_{\ncong}(\mathbb{C})$ is

$$\{(gRe(A)g^{-1}, hIm(A)h^{-1}) : g, h \in GL_2(\mathbb{R}),\ gRe(A)g^{-1} \in G \cdot Re(A),\ hIm(A)h^{-1} \in G \cdot Im(A)\}$$

We see by lemma 4.2 that the orbits of $A \in \mathfrak{gl}_2(\mathbb{C})$ is all elements in $G \cdot Re(A) \times G \cdot Im(A)$ such that the real parts and the complex parts are conjugated by the same element, or all pairs $(gRe(A)g^{-1}, gIm(A)g^{-1})$.

5 Orbits of subgroups of $GL_2(\mathbb{C})$

Recall that orbits are equivalence classes in the set they act on, and consider the canonical surjection;

$$\mathfrak{gl}_2(\mathbb{C}) \to \mathfrak{gl}_2(\mathbb{C})/GL_2(\mathbb{C})$$

$$A \mapsto [A]$$

where $A \in \mathfrak{gl}_2(\mathbb{C})$ and $[A]$ is the conjugacy class of A. We see that the preimage of $[A]$ is precisely the orbit $GL_2(\mathbb{C}) \cdot A \in \mathfrak{gl}_2(\mathbb{C})$.

Theorem 5.1 Orbit-Stabilizer Relation**.** *Given a group action $G \curvearrowright X$, there is a bijective set map*

$$\phi : G/G_x \to G \cdot x$$

$$[aG_x] \mapsto ax$$

where G_x is the stabilizer of $x \in X$ and $G \cdot x$ is the orbit of x.

Let H be a subgroup of $GL_2(\mathbb{C})$. Since for all $h \in H$, $h \in GL_2(\mathbb{C})$, it must be the case that for each $A \in \mathfrak{gl}_2(\mathbb{C})$, $(H \cdot A) \subset (GL_2(\mathbb{C}) \cdot A)$. Therefore, if we consider the action quotient map

$$\mathfrak{gl}_2(\mathbb{C})/H \to \mathfrak{gl}_2(\mathbb{C})/GL_2(\mathbb{C})$$

we see that the preimage of $[A] \in \mathfrak{gl}_2(\mathbb{C})$ is $GL_2(\mathbb{C})/H$, since the action of H on $\mathfrak{gl}_2(\mathbb{C})$ restricts to the action of H on the orbits of $GL_2(\mathbb{C}) \curvearrowright \mathfrak{gl}_2(\mathbb{C})$. Then, we have:

$$(GL_2(\mathbb{C}) \cdot A)/H \mapsto [A]$$

assuming $H \notin Stab(A)$, in which case the preimage of $[A]$ is $GL_2(\mathbb{C}) \cdot x$ and $Stab(A)$ is non trivial or contained in H, in which case the preimage is $GL_2(\mathbb{C})/H$. To illustrate this, consider the following:

$$(GL_2(\mathbb{C}) \cdot A)/H = \{hgAg^{-1}h^{-1} : gAg^{-1} \in GL_2(\mathbb{C}) \cdot A, h \in H\} = \{hBh^{-1} : B = gAg^{-1} \in GL_2(\mathbb{C}) \cdot A, h \in H\}$$

We see that $H \cdot A$ is contained in $(G \cdot A)/H$. For all $hgAg^{-1}h^{-1}$ where g is also in H, $hgAg^{-1}h^{-1} \in H \cdot A$, since by closure, $g, h \in H \implies gh \in H$. Therefore, we see that $(G \cdot A)/H$ is partitioned into orbits of H.

Since we see that the action of any subgroup $H \subset G$ on $\mathfrak{gl}_2(\mathbb{C})$ restricts to the action of H on the orbits of G, we can go through each of $GL_2(\mathbb{R})$ orbits, and classify the orbits of each subgroup action on each of the sets. Recall our partition from the earlier section:

Matrices in $M_{2\times 2}(\mathbb{C})$ with one real eigenvalue

Theorem 4.3 gives us the two possible forms for all such matrices. If some such matrix A is of the form aI_2, then we see that it commutes with all elements of $GL_2(\mathbb{R})$, and as such is it's own orbit. This holds for all subgroups of $GL_2(\mathbb{C})$. However, if the matrix is non-diagonal, it is conjugate to the Jordan matrix

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

. Therefore, we can write the set of all such matrices as

$$\{MJM^{-1} : M \in GL_2(\mathbb{C})\}$$

where J is the Jordan matrix denoted above, and we see that they are all in the same conjugacy class. Since we can conjugate J by every $M \in GL_2(\mathbb{R})$ and get a matrix with one real eigenvalue, there exists a subset $\{hJh^{-1} : h \in H \subset GL_2(\mathbb{C})\}$. This subset will be the orbit of all matrices $A = MJM^{-1} \in M_{2\times 2}(\mathbb{C})$ under the action of some subgroup $H \in GL_2(\mathbb{C})$.

Matrices in $M_{2\times 2}(\mathbb{C})$ with two distinct real eigenvalues

By propositin 3, we see that matrices with two distinct real eigenvalues a, d are conjugate to the matrix

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

We see that this implies that all such matrices are in fact conjugate to each other, and in fact, can be represented by the set

$$\left\{ M \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} M^{-1} : M \in GL_2(\mathbb{C}) \right\}$$

Therefore, for some matrix A with eigenvalues $a, d \in \mathbb{R}$, the orbit of A under the action of some subgroup $H \subset GL_2(\mathbb{R})$ is

$$\left\{ h \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} h^{-1} : h \in H \right\}$$

Matrices in $M_{2\times 2}(\mathbb{C})$ with two distinct complex eigenvalues

By Proposition 2, we see that all matrices with eigenvalues $a \pm bi$ are in the set

$$\left\{ M \begin{bmatrix} a & b \\ -b & a \end{bmatrix} M^{-1} : M \in GL_2(\mathbb{C}) \right\}$$

where $M \in GL_2(\mathbb{C})$. We can adopt a similar approach as above; the set

$$\left\{ h \begin{bmatrix} a & b \\ -b & a \end{bmatrix} h^{-1} : h \in H \subset GL_2(\mathbb{C} \right\}$$

will be the orbit of h on some matrix A with eigenvalues $a \pm bi$.

6 References

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