MSc Computational Methods in Ecology and Evolution: Maths for Biology

Solutions: Matrices
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[Notation: matrices are upper case bold (e.g. $\boldsymbol{A}$), and vectors have a single underline (e.g. $\underline{a}$)]

QUESTION 1

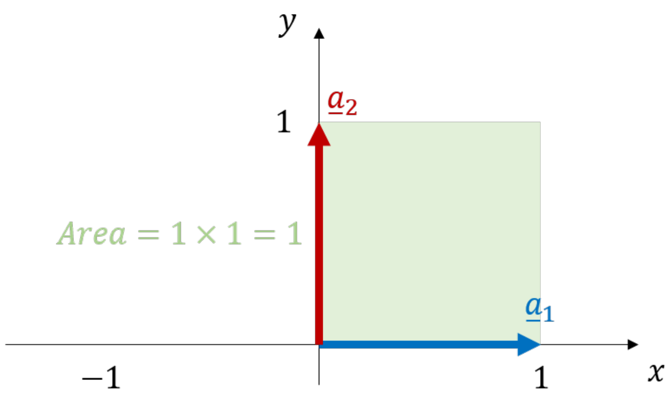
Images of basis vectors, and determinants

A matrix is a "table" of numbers, which are defined to act on vectors and transform them in arbitrary ways. We are going to focus on 2×2 matrices, which act on 2D vectors in a plane.

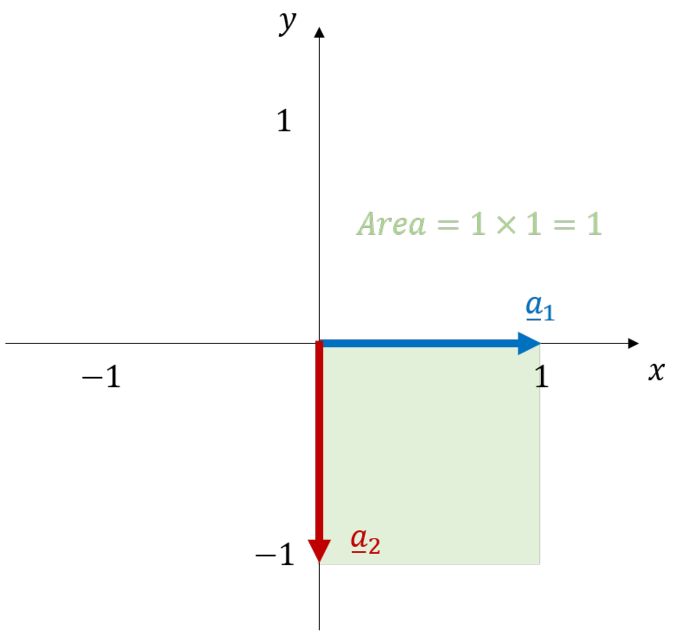
a) For each of the following matrices $\boldsymbol{A}$

$$\text{i)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \text{ii)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \text{iii)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{iv)} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \text{v)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{vi)} \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \text{vi)} \begin{pmatrix} 1 & 1 \\ 0 & 0.5 \end{pmatrix}$$

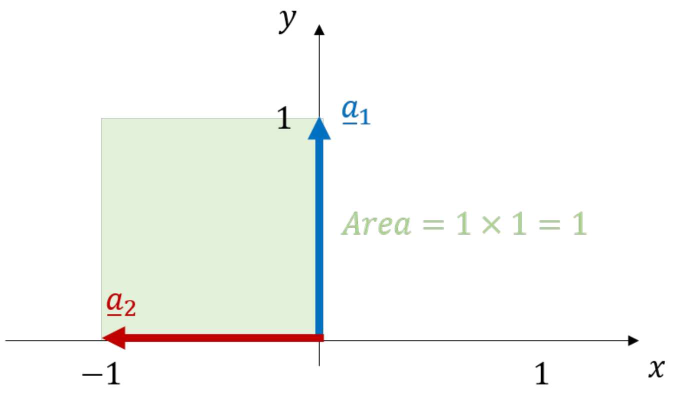
calculate the resultant vectors $\underline{a}_1 = \boldsymbol{A}\underline{e}_1$ & $\underline{a}_2 = \boldsymbol{A}\underline{e}_2$, where $\underline{e}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ & $\underline{e}_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and plot them on an $x - y$ plane. What do you notice about the transformation of vectors $\underline{e}_1$ & $\underline{e}_2$, with respect to the entries in $\boldsymbol{A}$?



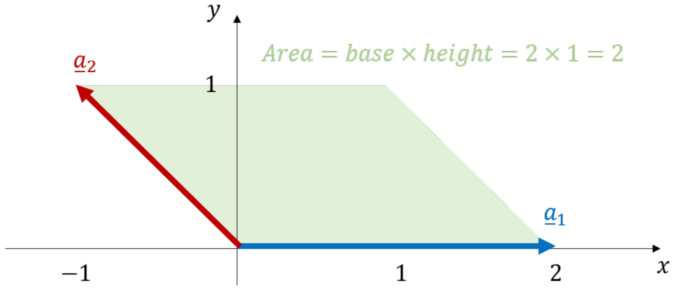
i) $\underline{a}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right); \underline{a}_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$



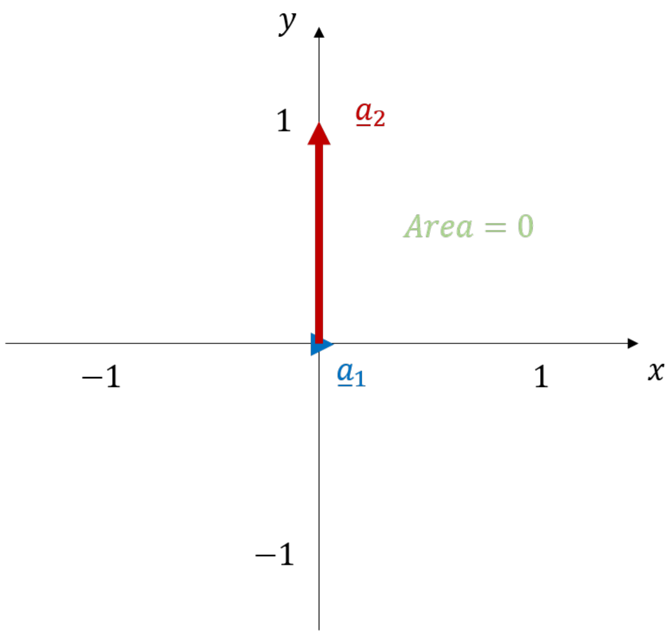
ii) $\underline{a}_1 = \binom{1}{0}; \underline{a}_2 = \binom{0}{-1}$



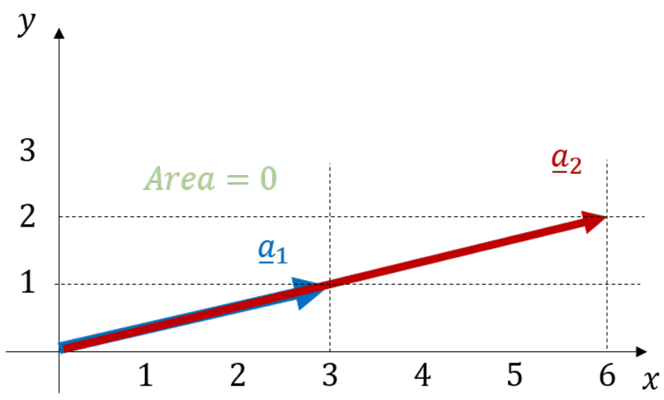
iii) $\underline{a}_1 = \binom{0}{1}; \underline{a}_2 = \binom{-1}{0}$



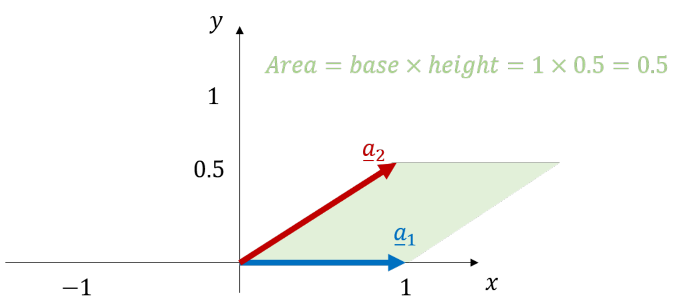
iv) $\underline{a}_1 = \binom{2}{0}; \underline{a}_2 = \binom{-1}{1}$



v) $\underline{a}_1 = \binom{0}{0}; \underline{a}_2 = \binom{0}{1}$



vi) $\underline{a}_1 = \binom{3}{1}; \underline{a}_2 = \binom{6}{2}$



vii) $\underline{a}_1 = \binom{1}{0}; \underline{a}_2 = \binom{1}{0.5}$

b) i) Calculate $|\boldsymbol{A}| = \det \boldsymbol{A}$ for each matrix above and ii) calculate the area of the parallelogram made by two vectors $\underset{\sim}{a}_1$ & $\underset{\sim}{a}_2$? What do you notice? iii) Comment on the determinant for matrix v, vi & vii and the directions of $\underset{\sim}{a}_1$ & $\underset{\sim}{a}_2$. iv) Why is the sign of determinant of matrix ii) negative?

i)

i) $|\boldsymbol{A}| = 1$
ii) $|\boldsymbol{A}| = -1$
iii) $|\boldsymbol{A}| = 1$
iv) $|\boldsymbol{A}| = 2$
v) $|\boldsymbol{A}| = 0$
vi) $|\boldsymbol{A}| = 0$
vii) $|\boldsymbol{A}| = 0.5$

ii)See figures above; your answers for the areas of the parallelogram should be the same as in part i) up to a sign (use the base $\times$ height formula).

iii) for matrices $\boldsymbol{A}$ in v, vi, the columns are pointing in exactly the same direction, and hence the area of the transformation represented by $\boldsymbol{A}$ is zero. For vii) the columns are pointing in a similar direction and so the determinant or area of the transformation is reduced.

iv) The determinant is a *signed* area and the sign represents whether the relative orientation of the images of the basis vectors is the same as those of the basis vectors. Hence, the determinant of ii) is negative since the columns of matrix $\boldsymbol{A}$ have the reverse orientation to the basis vectors $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$.

c) for the matrix

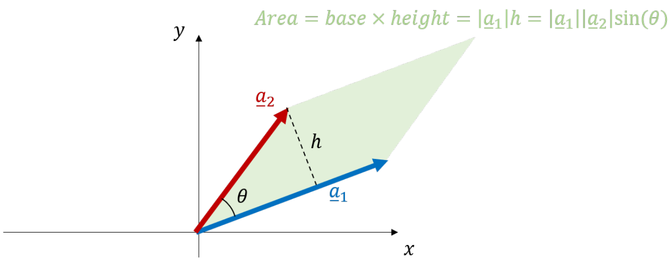
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

calculate the resultant vectors $\underset{\sim}{a}_1 = \boldsymbol{A}\underset{\sim}{e}_1$ & $\underset{\sim}{a}_2 = \boldsymbol{A}\underset{\sim}{e}_2$. What do you notice about the transformation of vectors $\underset{\sim}{e}_1$ & $\underset{\sim}{e}_2$, with respect to the entries in $\boldsymbol{A}$?

$\underset{\sim}{a}_1 = \left(\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right); \underset{\sim}{a}_2 = \left(\begin{smallmatrix} a_{12} \\ a_{22} \end{smallmatrix}\right)$

d) i) Calculate $|\boldsymbol{A}| = \det \boldsymbol{A}$ and *ii) show that the parallelogram made by $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ has area $=|\boldsymbol{A}|$

i) $|\boldsymbol{A}| = a_{11}a_{22} - a_{12}a_{21}$
*ii) The area of the parallelogram formed by vectors $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ is given by the base $\times$ height.



If we take the base to be the vector $\underset{\sim}{a}_1$ and the angle between vectors as θ, then $area = |\underset{\sim}{a}_1| \times height = |\underset{\sim}{a}_1||\underset{\sim}{a}_2| \sin\theta$. We can get information about θ from the dot product between two vectors: $\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_2 = |\underset{\sim}{a}_1||\underset{\sim}{a}_2| \cos\theta$ and the fact that $\sin^2\theta + \cos^2\theta = 1$. Plugging this in and then simplifying the algebra we get the area as $\pm|a_{11}a_{22} - a_{12}a_{21}|$, which is the determinant up to a sign, which can be determined by the relative orientation of the images of the basis vectors as discussed above.

The vectors $\underset{\sim}{e}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ & $\underset{\sim}{e}_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ are known as the basis vectors and the columns of $\boldsymbol{A}$ represent the *images* of these basis vectors, i.e. $\boldsymbol{A} = (\underset{\sim}{a}_1, \underset{\sim}{a}_2)$. The determinant is the area of the transformation, of the unit square, represented by $\boldsymbol{A}$

QUESTION 2

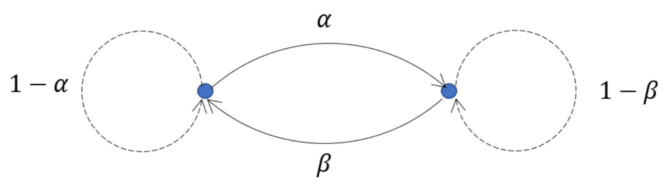


FIG. 1. 2-state Markov process

If we have some system that has only 2-states A and B (e.g. 2-alleles in a population, people are infected with a virus or not, is it raining today or it is dry), and the probability of making a transition only depends on the current state (A or B), then we have a Markov stochastic process. We can then fully describe the stochastic dynamics for $t \to t+1$ with just two transition probabilities

$$Prob(A \to B) = \alpha$$
$$Prob(B \to A) = \beta$$

and the matrix

$$M = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix}$$

describing the change in the probability vector $\underline{p}_{t+1} = \boldsymbol{M}\underline{p}_t$ is called is Markov matrix (also called a stochastic matrix or a transition matrix). The solution for an initial condition $\underline{p}_0$ is

$$\underline{p}_t = \boldsymbol{M}^t \underline{p}_0$$

Consider the Markov matrix

$$\boldsymbol{M} = \begin{pmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{pmatrix}$$

a) What is the sum of the columns of this matrix and why must this be so?

If we take the first column, the first element is

$$M_{11} = \mathrm{Prob}(A|A)$$

i.e. what is the probability of the state being A in the next time step given that it as an A in the current time step, and the second element is

$$M_{21} = \mathrm{Prob}(B|A)$$

i.e. what is the probability of the state being B in the next time step given that it as an A in the current time step. There are only two states, hence these probabiities must sum to 1. The same is true for the second column which represents transitions given that in the current time step we have a B.

b) Let $\underline{p}_0 = \binom{1}{0}$ and calculate in your favourite numerical software $\underline{p}_t$ for $t = 1, 2, 3, 100$

$\underline{p}_1 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$ $\underline{p}_2 = \binom{0.611}{0.389}$ $\underline{p}_3 = \binom{0.602}{0.398}$ $\underline{p}_{100} = \binom{0.600}{0.400}$

c) Repeat for $\underline{p}_0 = \binom{0}{1}$. What do you notice in both cases?

$\underline{p}_1 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ $\underline{p}_2 = \binom{0.583}{0.417}$ $\underline{p}_3 = \binom{0.597}{0.403}$ $\underline{p}_{100} = \binom{0.600}{0.400}$

We see that as t becomes large, irrespective of the initial condition that the vector tends to a constant vector $\binom{0.6}{0.4}$. This vector is the *steady-state* probability vector of the Markov process.

d) Explain your result for b) and c) in terms of the fact that the transition probability for $A \to B$ is less than the reverse transition $B \to A$

Since the transition rate for $A \to B$ is less than the reverse transition $B \to A$, this would suggest that the system spends more time on average in state A than B and so we would expect the steady-state probability vector to have elements such that $p(A) > p(B)$.

e) Let $\underline{p}_0 = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$ and calculate $\underline{p}_t$ for $t = 1, 2, 3, 100$. What do you notice? You should find $\underline{p}_0 = \underline{p}_1 = \underline{p}_2 = \underline{p}_3 = \underline{p}_{100}$.

$\underline{p}_1 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$ $\underline{p}_2 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$ $\underline{p}_3 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$ $\underline{p}_{100} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$

We see that as with $\underline{p}_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$ the probability vector does not change with time; if we start in the steady-state then we would not expect any change from the steady-state.

f) 2x2 square matrices each have at most 2 special vectors that do not change direction under action of the matrix, $\boldsymbol{M}\underline{v} = \lambda\underline{v}$, but can be compressed or stretched along this direction by a factor λ. These vectors are called an eigenvectors, and each has an associated eigenvalue, which is the stretch factor. It is clear $\begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$ is an eigenvector what is the associated eigenvalue? Let this eigenvalue be λ_1.

The eigenvalue is $\lambda_1 = 1$, since each application of $\boldsymbol{M}$ returns the same vector unscaled.

g) Let $\underline{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and calculate $\boldsymbol{M}^t\underline{v}$ for $t = 1, 2, 3, 100$. What do you notice? What is the ratio of values of the elements of each vector between successive time points? What is the eigenvalue for this eigenvector? Let this eigenvalue be λ_2.

$\underline{p}_1 = \begin{pmatrix} -1/6 \\ 1/6 \end{pmatrix}$ $\underline{p}_2 = \begin{pmatrix} -(1/6)^2 \\ (1/6)^2 \end{pmatrix}$ $\underline{p}_3 = \begin{pmatrix} -(1/6)^3 \\ (1/6)^3 \end{pmatrix}$ $\underline{p}_{100} = \begin{pmatrix} -(1/6)^{100} \\ (1/6)^{100} \end{pmatrix}$

We see that as with $\underline{p}_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ the elements of the probability vector diminish by a factor $1/6$ for each multiplication of $\boldsymbol{M}$, hence the eigenvalue is $\lambda = 1/6$.

h) With the eigenvalues and eigenvectors the solution $\underline{p}_t$ can be expressed in a more convenient form

$$\underline{p}_t = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} \lambda_1^t + \gamma \begin{pmatrix} -1 \\ 1 \end{pmatrix} \lambda_2^t$$

where λ_1 is often called the leading eigenvalue as it is largest in value. By setting $t = 0$ and with initial condition $\underline{p}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ calculate γ and verify this equation for $\underline{p}_t$ gives the same answer as in part b).

$$\underline{p}_0 = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} \lambda_1^0 + \gamma \begin{pmatrix} -1 \\ 1 \end{pmatrix} \lambda_2^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence, we have $3/5 - \gamma = 1$, which gives $\gamma = 3/5 - 1 = -2/5$ and $2/5 + \gamma = 0$, which also gives $\gamma = -2/5$ (as it should).

This gives the equation for $\underline{p}_t$ given an initial condition $\underline{p}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as

$$\underline{p}_t = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} \lambda_1^t - \frac{2}{5} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \lambda_2^t$$

You should verify this gives the same answer for $\underline{p}_t$ as for part b).

i) Using this equation, explain intuitively why as t becomes large $\underline{p}_t \to \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$.

Since the leading eigenvalue is $\lambda_1 = 1$ (this is always the case for Markov matrices), $\lambda_1^t = 1 \quad \forall t$. On the other hand as $\lambda_2 < 1$, λ_2^t decreases exponentially for increasing t, which means the contribution of the non-leading eigenvector tends to zero, which leaves only the contribution of the leading eigenvector, which is the steady-state probability vector.

j) Using the methods taught in the lectures calculate the eigenvectors and eigenvalues of $\boldsymbol{M}$ and verify you get the same answer. (Hint: eigenvectors can be normalised arbitrarily: here we want to normalise the first eigenvector, such that it is a probability vector, as it corresponds to the long-time steady state probability.)

To determine the eigenvalues solve $|\boldsymbol{M} - \lambda \boldsymbol{I}| = 0$:

$$\boldsymbol{M} = \begin{vmatrix} 2/3-\lambda & 1/2 \\ 1/3 & 1/2-\lambda \end{vmatrix} = (2/3-\lambda)(1/2-\lambda) - 1/6 = 0$$

This gives the characteristic equation $\lambda^2 - 7\lambda/6 + 1/6 = 0$, which has solution

$$\lambda = \frac{7 \pm \sqrt{49-24}}{12} = \{1, 1/6\}.$$

Let $\underline{v} = \binom{v_1}{v_2}$ be the to-be-determined eigenvectors for each eigenvalue, then plugging each eigenvalue and solving gives the associated eigenvector:

$$\boldsymbol{M} = \begin{pmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

For $\lambda = \lambda_1 = 1$

$$\boldsymbol{M} = \begin{pmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hence, $(2/3)v_1 + (1/2)v_2 = v_1 \Rightarrow v_1/v_2 = (1/2)/(1/3) = 3/2$ and so our unnormalised eigenvector $\underline{v} = \binom{3}{2}$. We want this eigenvector to be normalised as a probability vector, so the sum of it's elements must equal 1, and hence $\underline{v}_1 = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$.

For $\lambda = \lambda_2 = 1/6$:

$$\boldsymbol{M} = \begin{pmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Hence, $(2/3)v_1 + (1/2)v_2 = v_1/6 \Rightarrow v_1/v_2 = -1$ and so our eigenvector $\underline{v}_2 = \binom{-1}{1}$.

k) If our two states are two alleles/variants in a population which mutate into each other with the same probability of μ per generation, what is the Markov matrix for this stochastic process? What are its eigenvalues and eigenvectors? Identify the non-leading eigenvalue and the steady state probability vector and comment on their significance? Discuss whether this system corresponds to evolution in an individual or a population, or both.

$$\boldsymbol{M} = \begin{pmatrix} 1-\mu & \mu \\ \mu & 1-\mu \end{pmatrix}$$

This gives the characteristic equation $\lambda^2 - 2(1-\mu)\lambda + 1 - 2\mu = 0$, with roots $\lambda = \{1, 1-2\mu\}$. Calculating the eigenvalues, for $\lambda = 1$, $(1-\mu)v_1 + \mu v_2 = v_1 \Rightarrow v_1/v_2 = 1$, hence $\underline{v}_1 = \binom{1}{1}$; further, normalising as a probability vector gives $\underline{v}_1 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$. Repeating this for $\lambda = 1-2\mu$, gives $(1-\mu)v_1 + \mu v_2 = (1-2\mu)v_1 \Rightarrow v_1/v_2 = -1$ and $\underline{v}_1 = \binom{-1}{1}$.

The steady-state probability vector is the eigenvector associated with $\lambda = 1$ and shows that both alleles are equally likely in the long run, which is as we would expect as the rate of mutation is the same between both alleles. The non-leading eigenvalue is $1-2\mu$ which is roughly the probability of only a single transition occuring in a single time-step: the probability of no transitions is $(1-\mu)^2$, while the probability of ....

(**For the last two parts, if you have not been taught how to calculate eigenvalues and eigenvectors yet, they are optional and can be done at a later date.**)

QUESTION 3

Circular motion with matrices

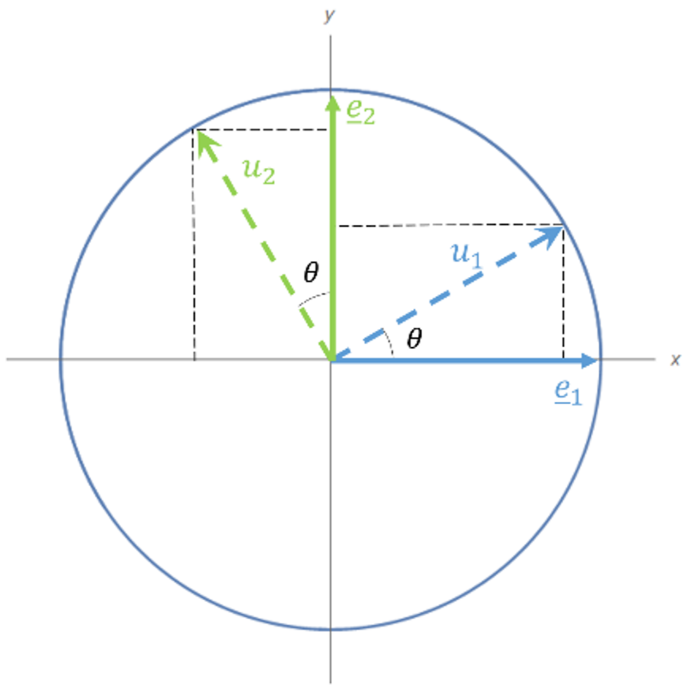


FIG. 2. Unit circle

a) Above is a figure of the unit circle with the basis vectors each rotated an angle θ; write down the images of the basis vectors $\underset{\sim}{u}_1$ & $\underset{\sim}{u}_2$ in terms θ.

$$\underset{\sim}{u}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} ; \underset{\sim}{u}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

b) Given this result show that the rotation matrix $\boldsymbol{U}(\theta)$, which rotates vectors by angle θ is:

$$\boldsymbol{U}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

if $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$ are the images of the basis vectors $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$, under rotation, then if we form a matrix whose columns are these images, then this defines our rotation matrix:

$$\boldsymbol{U}(\theta) = (\underset{\sim}{u}_1, \underset{\sim}{u}_2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

c) Evaluate the determinant $|\boldsymbol{U}|$. Explain your result in terms of the answer to Q1.

$$|\boldsymbol{U}| = \det(\boldsymbol{U}) = \cos^2(\theta) + \sin^2(\theta) = 1$$

This means that the transformation of rotation is area preserving; 1) since both basis vectors are rotated the same amount they keep the same relative orientation and angle ($\pi/2$ radians) & 2) rotation does not shrink or stretch the basis vectors, the area made by the images is always 1.

d) Evaluate $\boldsymbol{U}(\theta)$ for $\theta = 0, \pm\pi/2, \pm\pi, \pm 3\pi/2, 2\pi$. For each θ, what is the corresponding complex number that describes the same rotation? (Hint: refer back to tutorial from Tuesday, where $u(\theta) = \cos\theta + i\sin\theta$ is discussed)

$$\boldsymbol{U}(\theta = 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The corresponding complex number is $u(\theta) = \cos\theta + i\sin\theta \Rightarrow u(0) = 1$.

$$\boldsymbol{U}(\theta = \pm\pi/2) = \begin{pmatrix} \cos(\pm\pi/2) & -\sin(\pm\pi/2) \\ \sin(\pm\pi/2) & \cos(\pm\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The corresponding complex number is $u(\theta = \pm\pi/2) = \pm i$.

$$\boldsymbol{U}(\theta = \pm\pi) = \begin{pmatrix} \cos(\pm\pi) & -\sin(\pm\pi) \\ \sin(\pm\pi) & \cos(\pm\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The corresponding complex number is $u(\theta = \pm\pi) = -1$.

$$\boldsymbol{U}(\theta = \pm 3\pi/2) = \begin{pmatrix} \cos(\pm 3\pi/2) & -\sin(\pm 3\pi/2) \\ \sin(\pm 3\pi/2) & \cos(\pm 2\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix} = \mp \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The corresponding complex number is $u(\theta = \pm 3\pi/2) = \mp i$.

$$\boldsymbol{U}(\theta = 2\pi) = \begin{pmatrix} \cos(2\pi) & -\sin(2\pi) \\ \sin(2\pi) & \cos(2\pi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The corresponding complex number is $u(\theta = 2\pi) = 1$.

Comparing to $e^{i\theta}$, we see that there is a mathematical correspondence (known as an *isomorphism* $\cong$) to $\boldsymbol{U}(\theta)$ (i.e. $e^{i\theta} \cong \boldsymbol{U}(\theta)$).

e) What effect does the diagonal matrix

$$\boldsymbol{R} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

have on the basis vectors $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$, if $R < 1$ and if $R > 1$? What is the determinant of this matrix?

If $R < 1$ then this shrinks or contracts the basis vectors, such that their images have lengths $R < 1$. If $R > 1$ then this stretches or expands the basis vectors, such that their images have lengths $R > 1$. The determinant of this diagonal matrix is $|\boldsymbol{R}| = R^2$, which represents the fact that the area after transformation by $\boldsymbol{R}$ has changed by factor R^2, or each dimension by factor R.

f) What would be the corresponding matrix to represent the complex number $Re^{i\theta}$, which represents an expansion/contraction by R of vectors as well as rotation by angle θ?

The diagonal matrix $\boldsymbol{R}$ expands/contracts vectors by factor R, while $\boldsymbol{U}$ rotates them, hence the matrix that corresponds to $Re^{i\theta}$ is

$$\boldsymbol{RU} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} R\cos\theta & -R\sin\theta \\ R\sin\theta & R\cos\theta \end{pmatrix}$$

g) We want to use the rotation matrix $\boldsymbol{U}(\omega t)$ to represent the motion of our predator prey species about some fixed point with time-dependent vector given initial values x_0 & y_0 with some ω already given. Verify that the following time-dependent vector

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

does just this, by showing that $x^2(t) + y^2(t) = x_0^2 + y_0^2 = R^2$.

$$x(t) = x_0 \cos(\omega t) - y_0 \sin(\omega t)$$

$$y(t) = x_0 \sin(\omega t) + y_0 \cos(\omega t)$$

$$\begin{aligned} \Rightarrow x^2 + y^2 &= (x_0 \cos(\omega t) - y_0 \sin(\omega t))(x_0 \sin(\omega t) + y_0 \cos(\omega t)) \\ &= (x_0^2 + y_0^2)(\cos^2(\omega t) + \sin^2(\omega t)) \\ &= x_0^2 + y_0^2 \\ &= R^2 \end{aligned}$$

h) Calculate $\boldsymbol{U}^{-1}(\omega t)$ and show that it is given by $\boldsymbol{U}(-\omega t)$ explain this result.

The inverse of a 2×2 matrix $\boldsymbol{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is given by

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Hence, the inverse of the rotation matrix is

$$\boldsymbol{U}^{-1}(\omega t) = \frac{1}{|\boldsymbol{U}|} \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} = \begin{pmatrix} \cos(-\omega t) & -\sin(-\omega t) \\ \sin(-\omega t) & \cos(-\omega t) \end{pmatrix} = \boldsymbol{U}(-\omega t)$$

i) Hence, show that

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \boldsymbol{U} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ \Rightarrow \boldsymbol{U}^{-1} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \boldsymbol{U}^{-1}\boldsymbol{U} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \end{aligned}$$

j) Using your result for the prediction of the number of excess hares and lynxes from question Q1h of the Tuesday tutorial, use this equation to show that 15 years earlier the number of hares and lynxes was $x_0 = 80$ and $y_0 = 60$.

From the previous tutorial we know $R = \sqrt{x_0^2 + y_0^2} = 100$, $\omega = \pi/5$ and $x(15) = -80$ and $y(15) = -60$, and so using the above equation

$$x_0 = x(t)\cos(\omega t) + y(t)\sin(\omega t) = -80\cos(\pi/5 \times 15) - 60\sin(\pi/5 \times 15) = 80$$

$$y_0 = -x(t)\sin(\omega t) + y(t)\cos(\omega t) = 80\sin(\pi/5 \times 15) - 60\cos(\pi/5 \times 15) = 60$$

Summary of different equivalent ways to describe circular motion with a constant angular velocity ω: Initial condition specified by constant radius $R^2 = x_0^2 + y_0^2$ and phase delay $\phi(x_0, y_0) = -\tan^{-1}(y_0/x_0)$

Separate components	Complex separate components	Complex polar form
$x(t) = R\cos(\omega t - \phi)$ $y(t) = R\sin(\omega t - \phi)$	$z(t) = R(\cos(\omega t - \phi) + i\sin(\omega t - \phi))$	$z(t) = Re^{i(\omega t - \phi)}$

Initial condition specified directly by x_0, y_0 ($z_0 = x_0 + iy_0$)

Complex separate components	Complex Polar	Matrix form
$z(t) = (x_0 + iy_0)(\cos(\omega t) + i\sin(\omega t))$	$z(t) = z_0 e^{i\omega t}$	$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$