Maths4Bio - Tutorial 01

1 a) Expanding the factored expression:

$$R(x) = Kx(a-x) = Kxa - Kx^2 = (-K)x^2 + (aK)x$$

which shows that $R(x)$ is a polynomial of degree 2.

b) The function $R(x) = 2x(6-x)$ have two real roots: $x=0$ and $x=6$ (since $R(0) = R(6) = 0$).

Expanding the polynomial we have:

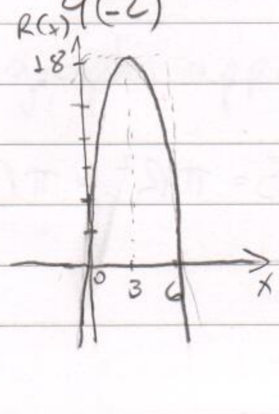
$$R(x) = -2x^2 + 12x.$$

Since the leading term is negative, this represents a concave-down parabola. The value of x for which the reaction rate is maximal corresponds to the vertix, for which (considering a polynomial of the form $R(x) = Ax^2 + Bx + C$):

$$x_V = \frac{-B}{2A} = \frac{-12}{2(-2)} = 3$$

$$y_V = \frac{-\Delta}{4A} = \frac{-(B^2 - 4AC)}{4A} = \frac{-12^2}{4(-2)} = 18.$$

The graph is then:



2

Find the maximum/minimum value of the following polynomials
(a) $p(x) = x^2 - 2x - 3$
(b) $p(x) = -x^2 - 6x + 8$
(*Hint:* Calculate the roots and consider the symmetry of the parabola)

The expression to calculate the maximum/minimum of a parabola with real roots is

$$x_{ext} = \frac{x_2 - x_1}{2} + x_1 \tag{1}$$

Where $x_1 < x_2$ In order to find the roots of polynomials of the form $ax^2 + bx + c$ I apply the formula to solve quadratic equations

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

In part (a) the polynomial is concave because $a > 0$. Therefore, it surely has a minimum

$$x_1 = -1 \quad , \quad x_2 = 3$$

Using equation 1 I find that

$$x_{max} = 1$$

Substituting this value of x in the polynomial I obtain that the minimum value of the function is

$$y_{min} = -4$$

In part (b) the polynomial is convex because $a < 0$. Therefore, it surely has a maximum

$$x_1 = -4 \quad , \quad x_2 = -2$$

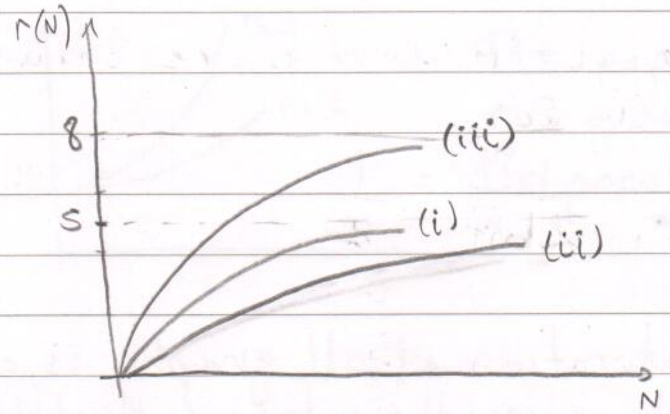
Using equation 1 I find that

$$x_{max} = -3$$

Substituting this value of x in the polynomial I obtain that the minimum value of the function is

$$y_{min} = 17$$

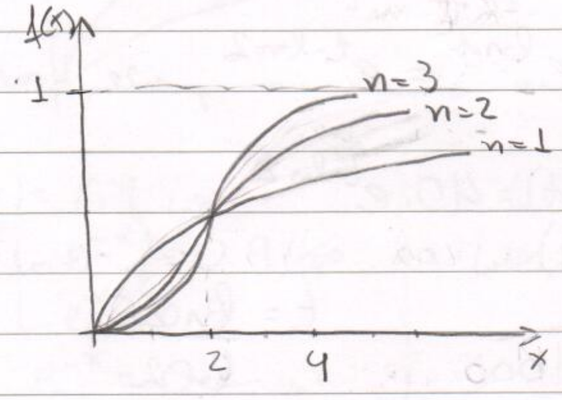
3 a)



b) When you increase a the function tends to saturate at a higher value. In order words, a seems to control the value approached by the function as N increases (compare graphs (i) and (iii)).

c) When you change K (Keeping the same value of a) you change how fast the function approaches the saturation value (compare graphs (i) and (ii)). Smaller K leads to faster approach.

4 a)



b) The graphs intersect at $(x, y) = (2, 0.5)$

c) As x gets larger all graphs approach the value 1.

d) We have:

$$f(b) = \frac{b^n}{b^n + b^n} = \frac{b^n}{2b^n} = \frac{1}{2}$$

Since the saturation of all graphs is at 1, we see that $x = b$ gives the value of the independent variable for which the image of the function assumes half of the value of the saturation. Therefore, b is called half-saturation.

6. a) $N(0) = 40 \cdot 2^0 = 40 \cdot 1 = 40$

b) Since $2^t = e^{\ln 2^t} = e^{t \cdot \ln 2}$, we have:

$$N(t) = 40 \cdot e^{t \cdot \ln 2}$$

c)
$$N(t) = 1000$$
$$40 \cdot e^{t \ln 2} = 1000$$
$$e^{t \ln 2} = \frac{1000}{40}$$
$$e^{t \ln 2} = 25$$
$$\ln(e^{t \ln 2}) = \ln 25$$
$$t \ln 2 = \ln 25$$

$$t = \frac{\ln 25}{\ln 2}$$
$$t \cong 4.64$$

5 The file `words_moby_dick.csv` contains data on the cumulative distribution of the number of times specific words occur in the text of the novel *Moby Dick*, by Herman Melville. Assume that this distribution is a power law of the form

$$y = ax^D$$

where a = 100.

(a) Using the provided data, and R/Python, estimate the exponent of this power law. (*Hint:* If you plot this data in log-log scale, what should correspond to the exponent D?)

(b) Can you think of a better way to estimate D?

A rough estimate of the exponent can be found by solving for D and averaging:

$$\log(y) = \log(ax^D) \rightarrow D = \frac{\log(y) - \log(a)}{\log(x)} \tag{3}$$

Plugging the data of word occurrences y_f and word label x_l, we can calculate one D per point. Averaging across all data points it would give us a first rough estimate of D

$$D_{av} = \frac{1}{\#points} \sum_{points} \frac{\log(y_f) - \log(a)}{\log(x_l)} \approx -0.41 \tag{4}$$

However, there is a more accurate way to get D; i.e., as the slope of the linear fit of y vs x in log-log space (see figure 1). Using this method, we get a more accurate estimate of $D = -1.114$

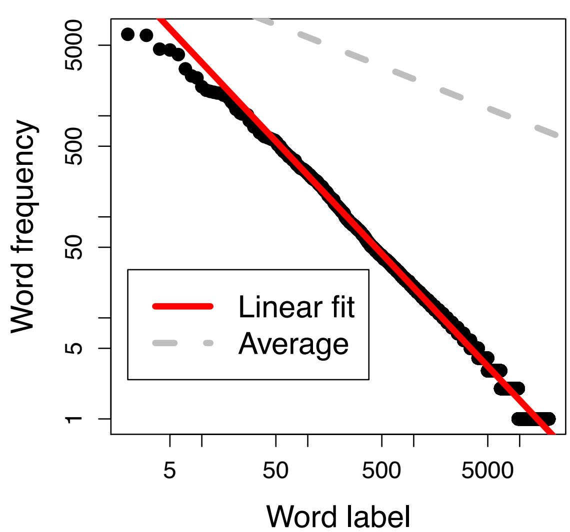
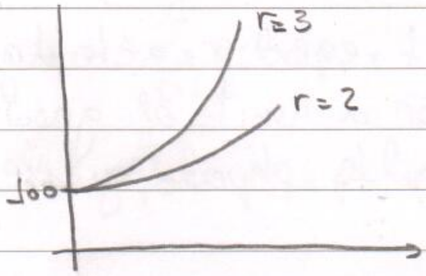


Figure 1: Log-log plot of word label versus word frequency and linear fit

7 a)



The population with r=3 grows faster than the population with r=2

b) $\frac{N(t+1)}{N(t)} = \frac{250}{200}$

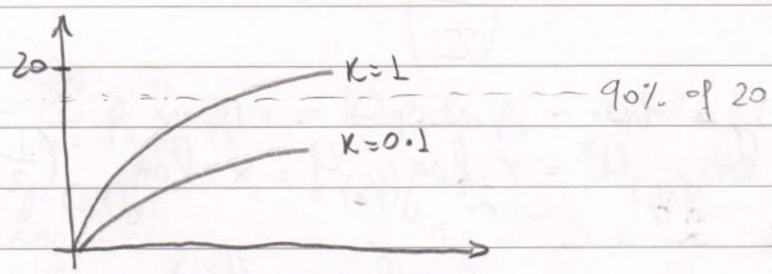
$\frac{\cancel{N_0} e^{r(t+1)}}{\cancel{N_0} e^{rt}} = \frac{250}{200}$

$\frac{\cancel{e^{rt}} \cdot e^{r \cdot 1}}{\cancel{e^{rt}}} = \frac{5}{4}$

$e^r = \frac{5}{4}$

$r = \ln\left(\frac{5}{4}\right) \cong 0.22$

8 a)



b) $L(x) = 0.9 \cdot L_\infty$

$\cancel{L_\infty}(1-e^{-x}) = 0.9 \cancel{L_\infty}$

$1-e^{-x} = 0.9$

$e^{-x} = 0.1$

$-x = \ln(0.1)$

$x = -\ln(0.1)$

$x \cong 2.30$

$L(x) = 0.99 L_\infty$

$\cancel{L_\infty}(1-e^{-x}) = 0.99 \cancel{L_\infty}$

$1-e^{-x} = 0.99$

$e^{-x} = 0.01$

$-x = \ln(0.01)$

$x = -\ln(0.01)$

$x \cong 4.60$

The parameter L_∞ is the horizontal asymtote of $L(x)$ and the graph approaches this value as x increases, without ever reaching it. Biologically, $L\infty$ stands for a limit of growth for the fish, due possibly to physiological or developmental constraints.

c) The growth curve with $K=1$ reaches 90% of $L\infty$ faster than the growth curve with $K=0.1$. The larger the value of K, the faster the curve approaches $L\infty$.

9 For the following pairs of functions, plot the ratio (quotient) between the two using R or Python. Based on the behaviour of the ratio when $x \to \infty$ (vary large values of x), how does each of these functions compare to the other in the velocity that they grow?

(a) $f(x) = e^x$ and $g(x) = x^2$

(b) $f(x) = e^x$ and $g(x) = x^{10}$

(c) $f(x) = \log(x)$ and $g(x) = x$

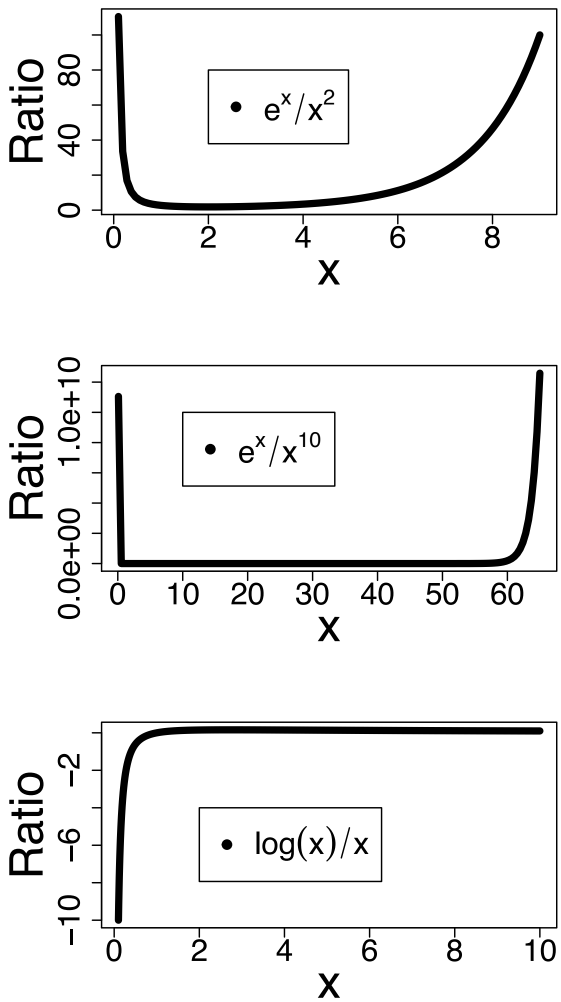


Figure 2: Ratios of the different functions.

Based on figure 2, I conclude that

$$e^x >> x^{10} > x^2 > x >> \log(x) \tag{5}$$

10 a) $H = -\sum_{i=1}^{5} p_i \ln p_i =$

$$= -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 - p_4 \ln p_4 - p_5 \ln p_5$$

Since $p_1 = p_2 = p_3 = \cdots = p_5$, we have:

$$H = -5 p_1 \ln p_1$$

Also, since all species are equally abundant:

$p_1 = p_2 = p_3 = \cdots = p_5 = \frac{1}{5}$, Thus:

$$H = -\cancel{5} \cdot \frac{1}{\cancel{5}} \ln\left(\frac{1}{5}\right) = -\ln 5^{-1} = \ln 5$$

b) Since $p_1 = p_2 = \cdots = p_9 = p_{10} = \frac{1}{10}$

$$H = -\sum_{i=1}^{10} p_i \ln p_i = -10 p_1 \ln p_1 = -\cancel{10} \cdot \frac{1}{\cancel{10}} \ln\left(\frac{1}{10}\right)$$
$$= \ln 10$$

c) $S = 5$: $H = \ln 5 \Rightarrow \frac{H}{\ln 5} = \frac{\ln 5}{\ln 5} = 1$

$S = 10$: $H = \ln 10 \Rightarrow \frac{H}{\ln S} = \frac{\ln 10}{\ln 10} = 1$

d) If there are N equally abundant species:

$$p_1 = p_2 = \cdots = p_{N-1} = p_N = \frac{1}{N}$$

Then:

$$H = -\sum_{i=1}^{N} p_i \ln p_i = -p_1 \ln p_1 - p_2 \ln p_2 - \cdots - p_N \ln p_N$$

$$= -N p_1 \ln p_1 = -N \cdot \left(\frac{1}{N}\right) \ln\left(\frac{1}{N}\right)$$

$$= -\ln N^{-1} = \ln N$$

Thus:

$$\frac{H}{\ln S} = \frac{\ln N}{\ln N} = 1$$