10212 Linear Algebra B

University of Manchester

27 January 2020

Textbook

Students are **strongly** advised to acquire a copy of the Textbook:

D. C. Lay. **Linear Algebra and its Applications**. Pearson, 2006. ISBN 0-521-28713-4. (Or other editions)

Lecture notes serve only as indication of the course content.

Homework:

- Homework x has to be returned for marking before 09:00 on Friday in Week $x - 1$.
- consists of some odd numbered exercises from the Textbook.
- Textbook contains answers to most odd numbered exercises.

Communication

- **Course website:**
  https://personalpages.manchester.ac.uk/staff/alexandre.borovik/math10212.html
  or the same location with the shortened address:
  https://bit.ly/2Ba17GU
- **Email:** Feel free to send questions, etc., to
  alexandre.borovik@manchester.ac.uk
  but only from your university's e-mail account.
  Emails from Gmail, Hotmail, etc. automatically go to spam.

Linear Forms

$\boxed{\textbf{A}}$ The total cost of a purchase of amounts g_1, g_2, g_3 of some goods at unit prices p_1, p_2, p_3 is

$$p_1 g_1 + p_2 g_2 + p_3 g_3 = \sum_{i=1}^{3} p_i g_i.$$

Expressions of this kind,

$$a_1 x_1 + \cdots + a_n x_n$$

are called **linear forms in variables** $x_1, \ldots, x_n$ **with coefficients** $a_1, \ldots, a_n$.

Linear Algebra is mathematics of linear forms

B

- Over the course, we shall develop increasingly compact notation for operations of Linear Algebra.
  In particular,

$$p_1 g_1 + p_2 g_2 + p_3 g_3$$

can be very conveniently written as

$$\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

Compression of notation

$\boxed{\textbf{C}}$

- . . . and then abbreviated

$$\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = P^T G,$$

where

$$P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

Linear Algebra for Physicists

$\boxed{\mathbf{D}}$

- Physicists use even shorter notation and, instead of

$$p_1 g_1 + p_2 g_2 + p_3 g_3 = \sum_{i=1}^{3} p_i g_i$$

write

$$p_1 g^1 + p_2 g^2 + p_3 g^3 = p_i g^i.$$

This notation was introduced by Albert Einstein.
Will not be used in the course.

Warning:

- Increasingly compact notation leads to increasingly compact and abstract language used.
- Linear Algebra focuses on the development of a special mathematics language rather than on procedures.
- This language is used all over mathematics and statistics.

Prerequisites:

More abstract bits of MATH10111:

- Functions: 1–1, onto,bijective.
- Equivalence relations.
- Binary operations and groups.

Linear equation

$\boxed{\textbf{A}}$ A **linear equation** in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the **coefficients** $a_1, \ldots, a_n$ are real numbers. The subscript n can be any natural number.

$\boxed{\textbf{B}}$ A **system of linear equations** is a collection of one or more linear equations involving the same variables, say $x_1, \ldots, x_n$. For example,

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1 \end{aligned}$$

$\boxed{\textbf{C}}$ A **solution** of the system is a list $(s_1, \ldots, s_n)$ of numbers that makes each equation a true identity when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

The set of all possible solutions is called the **solution set** of the linear system.

Two linear systems are **equivalent** if they have the same solution set.

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Two linear systems are **equivalent** if the have the same solution set.

$\boxed{\textbf{B}}$ We shall be use the following **elementary operations** on systems od simultaneous liner equations:

- Replacement: Replace one equation by the sum of itself and a multiple of another equation.
- Interchange: Interchange two equations.
- Scaling: Multiply all terms in a equation by a nonzero constant.

$\boxed{\text{C}}$ Note: The elementary operations are reversible.

$\boxed{\text{D}}$ **Theorem: Elementary operations preserve equivalence.**

If a system of simultaneous linear equations is obtained from another system by elementary operations, then the two systems have the same solution set.

$\boxed{\textbf{E}}$ A system of linear equations has either

- no solution, or
- exactly one solution, or
- infinitely many solutions.

$\boxed{\textbf{F}}$ A system of linear equations has either

- no solution, or
- exactly one solution, or
- infinitely many solutions.

A system of linear equations is said to be **consistent** it if has solutions (either one or infinitely many), and a system in **inconsistent** if it has no solution.

Matrix notation

$\boxed{\textbf{G}}$ The system

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ x_1 + x_2 \quad\quad &= 2 \\ x_2 + x_3 &= 3 \end{aligned}$$

has the **matrix of coefficients**

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\boxed{\mathbf{H}}$ . . . and the **augmented matrix**

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix};$$

notice how the coefficients are aligned in columns, and how missing coefficients are replaced by 0.

$\boxed{\mathbf{I}}$ A matrix with m rows and n columns is called an $\mathbf{m \times n}$ **matrix**.

Elementary row operations

$\boxed{\textbf{J}}$

Replacement Replace one row by the sum of itself and a multiple of another row.

Interchange Interchange two rows.

Scaling Multiply all entries in a row by a nonzero constant.

The two matrices are **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

$\boxed{\mathbf{K}}$ Note: the row operations are reversible.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Existence and uniqueness questions

$\boxed{\mathbf{A}}$

- Is the system consistent?
- If a solution exist, is it *unique*?

Leading entries

$\boxed{\textbf{B}}$ A **nonzero** row or column of a matrix is a row or column which contains at least one nonzero entry.

$\boxed{\textbf{C}}$ A **leading entry** of a row is the leftmost nonzero entry (in a non-zero row).

Echelon form

$\boxed{\textbf{D}}$ A matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any row of zeroes.
2. Each leading entry of a row is in column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeroes.

Reduced echelon form

$\boxed{\mathbf{E}}$ If, in addition, the following two conditions are satisfied,

4. All leading entries are equal 1.
5. Each leading 1 is the only non-zero entry in its column

then the matrix is in **reduced echelon form**.

Row reduction

$\boxed{\mathbf{F}}$ An **echelon matrix** is a matrix in echelon form.

Any non-zero matrix can be **row reduced** (that, transformed by elementary row transformations) into a matrix in echelon form (but the same matrix can give rise to different echelon forms).

Examples

$\boxed{\textbf{G}}$ The following is a schematic presentation of an echelon matrix:

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

H and this is a reduced echelon matrix:

$$\begin{bmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Uniqueness of the reduced echelon form

$\boxed{\mathbf{I}}$

Theorem: Uniqueness of the reduced echelon form.

Each matrix is row equivalent to one and only one reduced echelon form.

Row equivalence is an equivalence relation on the set on $m \times n$ matrices: it is

- reflexive
- symmetric
- transitive

Every equivalence class contains exactly one matrix in reduced echelon form.

Pivot positions

$\boxed{\textbf{A}}$ A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

The Row Reduction Algorithm

$\boxed{\mathbf{B}}$

$$\begin{bmatrix} 0 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 3 \\ 1 & 1 & 1 & 2 & 2 \end{bmatrix}$$

A **pivot** is a nonzero number in a pivot position which is used to create zeroes in the column below it.

A rule for row reduction:

$\boxed{\mathbf{C}}$

1. Pick the leftmost non-zero column; interchange rows, if needed, to make its topmost entry non-zero; it is a **pivot**.
2. Using **scaling**, make the pivot equal 1.
3. Using **replacement** row operations, kill all non-zero entries in the column below the pivot.
4. Mark the row and column containing the pivot as **pivoted.**
5. Repeat the same with the matrix made of not pivoted yet rows and columns.
6. Using **replacement** row operations, kill all non-zero entries in the column above the pivot entries.

Solution of Linear Systems

$\boxed{\textbf{D}}$

When we converted the augmented matrix of a linear system into its reduced row echelon form, we can write out the entire solution set of the system.

E Let

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

be the augmented matrix of a a linear system; then the system is equivalent to

$$\begin{aligned} x_1 \qquad\quad - 5x_3 &= 1 \\ x_2 + \quad x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

$\boxed{\text{F}}$ The variables x_1 and x_2 correspond to pivot columns in the matrix and a re called **basic variables** (also **leading** or **pivot variables**).

$\boxed{\text{G}}$ The other variable, x_3 is a **free variable**.

Free variables can be assigned arbitrary values and then leading variables expressed in terms of free variables:

$$\begin{array}{lcl} x_1 & = & 1 + 5x_3 \\ x_2 & = & 4 - x_3 \\ x_3 & & \text{is free} \end{array}$$

Theorem: Existence and Uniqueness

$\boxed{\textbf{H}}$ A linear system is consistent if and only if the rightmost column of the augmented matrix is **not** a pivot column—that is, if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix} \text{ with } b \text{ nonzero}$$

$\boxed{\textbf{I}}$ If a linear system is consistent, then the solution set contains either

(i) a unique solution, when there are no free variables, or

(ii) infinitely many solutions, when there is at least one free variable.

Vectors

$\boxed{\textbf{A}}$ A matrix with only one column is called a **column vector**, or simply a **vector**.

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Two vectors are **equal** if and only if they have the same number of rows and their corresponding entries are equal.

The set of al vectors with n entries is denoted $\mathbb{R}^n$.

Operations on vectors

$\boxed{\textbf{D}}$

The **sum** $\mathbf{u} + \mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is obtained by adding corresponding entries in $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Operations on vectors

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The **scalar multiple** $c\mathbf{v}$ of a vector $\mathbf{v}$ and a real number (“**scalar**”) c is the vector obtained by multiplying each entry in $\mathbf{v}$ by c.

$$1.5 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \\ -3 \end{bmatrix}.$$

Operations on vectors

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The vector of all zeroes is called the zero vector and denoted $\mathbf{0}$:

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Algebraic properties of $\mathbb{R}^n$

$\boxed{\textbf{G}}$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and all scalars c and d:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

(Here $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.)

Linear combinations

$\boxed{\textbf{H}}$

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$ and scalars $c_1, c_2, \dots, c_p$, the vector

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ with **weights** $c_1, c_2, \dots, c_p$.

Rewriting a linear system as a vector equation

$\boxed{\text{I}}$

$$\begin{aligned} x_2 + x_3 &= 2 \\ x_1 + x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 2 \end{aligned}$$

can be written as equality of two vectors:

$$\begin{bmatrix} x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

which is the same as

$$x_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

Vector equation

J Denote

$$\mathbf{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix},$$

then the vector equation can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Solution set of a vector equation

$\boxed{\mathbf{K}}$ *Solving a linear system is the same as finding an expression of the vector of the right part of the system as a linear combination of columns in its matrix of coefficients.*

Example

L Write the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

in a way that calls attention to its columns:

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$$

Solution set of a vector equation

M A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}.$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$$

In particular $\mathbf{b}$ can be expressed by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there is a solution of the corresponding linear system.

Span

$\boxed{\textbf{N}}$ If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) by $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

That is, $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors which can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

Span

$\boxed{\mathbf{O}}$

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \{\, \mathbf{b} : x_1\mathbf{v}_1 + \cdots x_p\mathbf{v}_p = \mathbf{b} \text{ has a solution} \,\}$$

Matrix-vector product

A is $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$
$\mathbf{x}$ is in $\mathbb{R}^n$
$\boxed{\mathbf{A}}$ The **product of** A **and** $\mathbf{x}$, denoted $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights:

$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \end{aligned}$$

Example

$\boxed{\mathbf{B}}$ The system

$$\begin{aligned} x_2 + x_3 &= 2 \\ x_1 + x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 2 \end{aligned}$$

was written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

where

$$\mathbf{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}.$$

The same system in the matrix product notation

$\boxed{\mathbf{C}}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

or

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Solution set of a matrix equation

$\boxed{\text{D}}$ **Theorem.** If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} = \begin{bmatrix} A | \mathbf{b} \end{bmatrix}.$$

Existence of solutions

$\boxed{\textbf{E}}$ The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of columns of A.

$\boxed{\textbf{F}}$ **Theorem.** Let A be an $m \times n$ matrix. Then the following statements are equivalent.

(a) For each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b} \in \mathbb{R}^n$ is a linear combination of columns of A.

(c) The columns of A span $\mathbb{R}^n$.

(d) A has a pivot position in every row.

Row-vector rule for computing $A\mathbf{x}$

$\boxed{\mathbf{G}}$

If the product $A\mathbf{x}$ is defined then the ith entry in $A\mathbf{x}$ is the sum of products of corresponding entries from the row i of A and from the vector $\mathbf{x}$.

Properties of the matrix-vector product

$\boxed{\mathbf{H}}$

Theorem.

If A is an $m \times n$ matrix, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and c is a scalar, then

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

(b) $A(c\mathbf{u}) = c(A\mathbf{u})$.

Homogeneous linear systems

I

A linear system is **homogeneous** if it can be written as

$$A\mathbf{x} = \mathbf{0}.$$

A homogeneous system always has at least one solution $\mathbf{x} = \mathbf{0}$ (**trivial** solution).

Homogeneous linear systems

$\boxed{\textbf{K}}$

A linear system is **homogeneous** if it can be written as

$$A\mathbf{x} = \mathbf{0}.$$

A homogeneous system always has at least one solution $\mathbf{x} = \mathbf{0}$ (**trivial** solution).

$\boxed{\textbf{L}}$

Therefore for homogeneous systems an important question os existence of a **nontrivial** solution, that is, a nonzero vector $\mathbf{x}$ which satisfies $A\mathbf{x} = \mathbf{0}$:

The homogeneous system $A\mathbf{x} = \mathbf{b}$ has a nontrivial solution if and only if the system has at least one free variable.

Example

$\boxed{\textbf{M}}$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_1 + 3x_3 + x_3 &= 0 \end{aligned}$$

Nonhomogeneous systems

When a nonhomogeneous system has many solutions, the general solution can be written in parametric vector form a one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

$\boxed{\textbf{A}}$

Example.

$$x_1 + x_2 + x_3 = 1.$$

$\boxed{\textbf{B}}$

Example.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_1 + 3x_3 + x_3 &= 5 \end{aligned}$$

Solution of nonhomogeneous system

C

Theorem. Suppose the equation

$$A\mathbf{x} = \mathbf{b}$$

is consistent for some given $\mathbf{b}$, and $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{w} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the homogeneous equation

$$A\mathbf{x} = \mathbf{0}.$$

Linear independence

$\boxed{\mathbf{D}}$

An indexed set of vectors

$$\{\, \mathbf{v}_1, \dots, \mathbf{v}_p \,\}$$

in $\mathbb{R}^n$ is **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only trivial solution.

Linear dependence

$\boxed{\mathbf{E}}$

The set

$$\{\,\mathbf{v}_1, \ldots, \mathbf{v}_p\,\}$$

in $\mathbb{R}^n$ is **linearly dependent** if there exist weights $c_1, \ldots, c_p$, **not all zero**, such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Linear independence of matrix columns

$\boxed{\mathbf{F}}$

The matrix equation

$$A\mathbf{x} = \mathbf{0}$$

where A is made of columns

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$$

can be written as

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Linear independence of matrix columns

$\boxed{\textbf{G}}$

The columns of matrix A are linearly independent **iff** the equation

$$A\mathbf{x} = \mathbf{0}$$

has **only** the trivial solution.

$\boxed{\mathbf{H}}$

A set of one vectors $\{\mathbf{v}_1\}$ is linearly dependent if $\mathbf{v}_1 = \mathbf{0}$.

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.

Theorem: Characterisation of linearly dependent sets

$\boxed{\mathbf{I}}$

An indexed set

$$S = \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \}$$

of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

Theorem: dependence of "big" sets

$\boxed{\textbf{J}}$

If a set contains more vectors than entries in each vector, then the set is linearly dependent.

Thus, any set

$$\{\, \mathbf{v}_1, \ldots, \mathbf{v}_p \,\}$$

in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Transformation

$\boxed{\textbf{A}}$

A **transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

$\mathbb{R}^n$ is the **domain** of T.

$\mathbb{R}^m$ is the **codomain** of T.

Matrix transformations

$\boxed{\mathbf{B}}$

$$\begin{aligned} T : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \mathbf{x} &\mapsto A\mathbf{x} \end{aligned}$$

where A is an $m \times n$ matrix.

In short:

$$T(\mathbf{x}) = A\mathbf{x}.$$

The range of a matrix transformation

$\boxed{\textbf{C}}$

The range of T is the set of all linear combinations of the columns of A.

Indeed, each image $T(\mathbf{x})$ has the form

$$T(\mathbf{x}) = A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$$

Linear transformations

$\boxed{\textbf{D}}$

A transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **linear** if:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$;
- $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ and all scalars c.

Properties of linear transformations

$\boxed{\textbf{E}}$

If T is a linear transformation then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}).$$

The identity matrix

$\boxed{\textbf{F}}$

An $n \times n$ matrix with 1's on the diagonal and 0's elsewhere is called the **identity** matrix I_n:

$$\begin{bmatrix}1\end{bmatrix}, \quad \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

The identity transformation

$\boxed{\mathbf{G}}$

It is easy to check that

$$I_n\mathbf{x} = \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

Therefore the linear transformation associated with the identity matrix is the identity transformation of $\mathbb{R}^n$:

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \mathbf{x} & \mapsto & \mathbf{x} \end{array}$$

The matrix of a linear transformation

H Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation.
Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$ where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

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A is the **standard matrix for the linear transformation** T.

Onto and one-to-one

$\boxed{\mathbf{J}}$

A transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **onto** $\mathbb{R}^m$ if each $\mathbf{b} \in \mathbb{R}^m$ is the image of *at least one* $\mathbf{x} \in \mathbb{R}^n$.

T is **one-to-one** if each $\mathbf{b} \in \mathbb{R}^m$ is the image of *at most one* $\mathbf{x} \in \mathbb{R}^n$.

One-to-one: a criterion

$\boxed{\mathbf{K}}$

A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is one-to-one

iff

the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

One-to-one and onto in terms of matrices

L

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear transformation and let A be the standard matrix for T. Then:

- T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
- T is one-to-one if and only if the columns of A are linearly independent.

Labeling of matrix entries

$\boxed{\mathbf{A}}$

Let A be an $m \times n$ matrix.

$$\begin{aligned} A &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

Diagonal matrices, zero matrices

B

The **diagonal entries** in A are $a_{11}, a_{22}, a_{33}, \ldots$

A **diagonal matrix** is a square matrix whose non-diagonal entries are zeroes.

Matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} \pi & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are all diagonal.

Zero matrix 0 is a $m \times n$ matrix whose entries are all zero.

Sums

$\boxed{\mathbf{C}}$

If

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \text{ and } B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}$$

are $m \times n$ matrices then

$$\begin{aligned} A + B &= \begin{bmatrix} \mathbf{a}_1 + \mathbf{b}_1 & \mathbf{a}_2 + \mathbf{b}_2 & \cdots & \mathbf{a}_n + \mathbf{b}_n \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1j} + b_{1j} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} + b_{i1} & \cdots & a_{ij} + b_{ij} & \cdots & a_{in} + b_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mj} + b_{mj} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \end{aligned}$$

Scalar multiple

If c is a a scalar then

$$
\begin{aligned}
cA &= \begin{bmatrix} c\mathbf{a}_1 & c\mathbf{a}_2 & \cdots & c\mathbf{a}_n \end{bmatrix} \\
&= \begin{bmatrix} ca_{11} & \cdots & ca_{1j} & \cdots & ca_{1n} \\ \vdots & & \vdots & & \vdots \\ ca_{i1} & \cdots & ca_{ij} & \cdots & ca_{in} \\ \vdots & & \vdots & & \vdots \\ ca_{m1} & \cdots & ca_{mj} & \cdots & ca_{mn} \end{bmatrix}
\end{aligned}
$$

Theorem: properties of matrix addition

D

Let A, B, and C be matrices of the same size and r and s be scalars.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$.

Composition of linear transformations

A

Let B be an $m \times n$ matrix, A an $p \times m$ matrix.
They define linear transformations

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad \mathbf{x} \mapsto B\mathbf{x}$$

and

$$S : \mathbb{R}^m \longrightarrow \mathbb{R}^p, \quad \mathbf{y} \mapsto A\mathbf{y}.$$

Their composition

$$(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$$

is a linear transformation

$$S \circ T : \mathbb{R}^n \longrightarrow \mathbb{R}^p.$$

What is its matrix?

Multiplication of matrices

$\boxed{\mathbf{B}}$

We need to compute $A(B\mathbf{x})$ in matrix form. Observe

$$B\mathbf{x} = x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n.$$

Hence

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) \\ &= A(x_1\mathbf{b}_1) + \cdots + A(x_n\mathbf{b}_n) \\ &= x_1A(\mathbf{b}_1) + \cdots + x_nA(\mathbf{b}_n) \\ &= \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{bmatrix}\mathbf{x} \end{aligned}$$

Therefore multiplication by the matrix

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{bmatrix}$$

transforms $\mathbf{x}$ into $A(B\mathbf{x})$.

Definition: Matrix multiplication

C

If A is an $p \times m$ matrix and B is an $m \times n$ matrix with columns $\mathbf{b}_1$, ..., $\mathbf{b}_n$ then the **product** AB is the $p \times n$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_n$:

$$AB = A\begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_p\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 & \cdots & A\mathbf{b}_p\end{bmatrix}.$$

Columns of AB

$\boxed{\textbf{D}}$

Each column $A\mathbf{b}_j$ of AB is a linear combination of columns of A with weights taken from the jth column of B:

$$\begin{aligned} A\mathbf{b}_j &= \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} \\ &= b_{1j}\mathbf{a}_1 + \cdots + b_{nj}\mathbf{a}_n \end{aligned}$$

Mnemonic rules

$\boxed{\textbf{E}}$

$$[m \times n \text{ matrix}] \cdot [n \times p \text{ matrix}] = [m \times p \text{ matrix}]$$

$\boxed{\textbf{F}}$

$$\mathrm{column}_j(AB) = A \cdot \mathrm{column}_j(B)$$

$$\mathrm{row}_i(AB) = \mathrm{row}_i(A) \cdot B$$

Theorem: Properties of matrix multiplication

G

Let A be an $m \times n$ matrix and let B and C be matrices for which indicated sums and products are defined.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$ for any scalar r
5. $I_m A = A = A I_n$

Powers of matrix

H

$$A^k = A \cdots A \qquad (k \text{ times})$$

If $A \neq 0$ then we set

$$A^0 = I$$

The transpose of a matrix

I

The **transpose** A^T of an $m \times n$ matrix A is the $n \times m$ matrix whose rows are formed from corresponding columns of A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Theorem: Properties of transpose

J Let A and B denote matrices matching sizes.

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(rA)^T = r(A^T)$ for any scalar r
4. $(AB)^T = B^T A^T$

Invertible matrices

A An $n \times n$ matrix A is **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \text{ and } AC = I$$

C is called the **inverse** of A.

The inverse of A, if exists, is unique (!) and is denoted A^{-1}:

$$A^{-1}A = I \text{ and } AA^{-1} = I.$$

Singular matrices

$\boxed{\mathbf{B}}$

A non-invertible matrix is called a **singular** matrix.

An invertible matrix is **nonsingular**.

Theorem: Inverse of a 2×2 matrix

$\boxed{\mathbf{C}}$

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $ad - bc \neq 0$ then A is invertible and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad - bc$ is called the **determinant** of A:

$$\det A = ad - bc$$

Theorem: Solving matrix equations

$\boxed{\textbf{D}}$

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Quiz

$\boxed{\mathbf{E}}$

- Suppose the second column of B is all zeroes. What can you say about the second column of AB?

Theorem: Properties of invertible matrices

$\boxed{\textbf{F}}$

(a) If A is an invertible matrix, then A^{-1} is also invertible and

$$(A^{-1})^{-1} = A$$

Theorem: Properties of invertible matrices

G

(a) If A is an invertible matrix, then A^{-1} is also invertible and

$$(A^{-1})^{-1} = A$$

(b) If A and B are $n \times n$ invertible matrices, then so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Theorem: Properties of invertible matrices

H

(a) If A is an invertible matrix, then A^{-1} is also invertible and

$$(A^{-1})^{-1} = A$$

(b) If A and B are $n \times n$ invertible matrices, then so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(c) If A is an invertible matrix, then so is A^T, and

$$(A^T)^{-1} = (A^{-1})^T$$

Elementary matrices

$\boxed{\mathbf{I}}$

An **elementary matrix** is obtained by performing a single elementary row operation on an identity matrix.

$\boxed{\mathbf{J}}$ **Theorem.** If an elementary row transformation is performed on an $n \times m$ matrix A, the resulting matrix can be written as EA, where E is made by the same row operations on I_n.

$\boxed{\mathbf{K}}$ **Theorem.** Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Theorem: Characterisation of invertible matrices

$\boxed{\mathbf{L}}$ An $n \times n$ matrix A is invertible

iff

A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Computation of inverses

M

- Form the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$ and row reduce it.
- If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.
- Otherwise A has no inverse.

The Invertible Matrix Theorem 2.3.8:

N

For an $n \times n$ matrix A, the following are equivalent:

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A are linearly independent.

The Invertible Matrix Theorem, continued

$\boxed{\mathbf{O}}$

(f) $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

(g) $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA = I$.

(k) There is an $n \times n$ matrix D such that $AD = I$.

(l) A^T is invertible.

One-sided inverse is the inverse

P

Let A and B be square matrices.

If $AB = I$ then both A and B are invertible and

$$B = A^{-1} \text{ and } A = B^{-1}.$$

Theorem: Invertible linear transformations

Q

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and A its standard matrix.

Then T is invertible **iff** A is an invertible matrix.

In that case, the linear transformation $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the only transformation satisfying

$$\begin{aligned} S(T(\mathbf{x})) &= \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \\ T(S(\mathbf{x})) &= \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \end{aligned}$$