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Modual Code: CMEE

Modual Name: Computational Methods in Ecology and Evolution

Question Number: A3

Date: 30.3.2021

Question A3 Maximum Likelihood.

(a) $\because \lambda \sim uniform(1,2) \Rightarrow f(\lambda) = \begin{cases} 1 & 1<\lambda<2 \\ 0 & otherwise \end{cases}$

$\because Y|\lambda \sim Exponential(\lambda) \Rightarrow f_{Y|\lambda}(y|\lambda) = \lambda \cdot e^{-\lambda y}$

$\therefore f_Y(y) = \int_{-\infty}^{+\infty} f_{Y|\lambda}(y|\lambda)\, d\lambda = \int_{-\infty}^{+\infty} (\lambda \cdot e^{-\lambda y}) d\lambda$

$= \int_{-\infty}^{1} (\lambda \cdot e^{-\lambda y})\, d\lambda + \int_{1}^{2} (\lambda \cdot e^{-\lambda y})\, d\lambda +$

$\int_{2}^{+\infty} (\lambda \cdot e^{-\lambda y})\, d\lambda$

$= \int_{1}^{2} (\lambda \cdot e^{-\lambda y})\, d\lambda$

$= -e^{-\lambda y} \Big|_{1}^{2}$

$= e^{-y} - e^{-2y}$

$= e^{-y}(1-e^{-y})$

(b) $\because X_i \sim N(0,\sigma^2)$ $f_{X_i}(x_i) = \dfrac{e^{-\frac{x_i^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$

$\therefore L(\sigma^2) = \prod_{i=1}^{n} \dfrac{e^{-\frac{x_i^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$

$= \dfrac{e^{-\frac{(x_1^2+x_2^2+\dots+x_n^2)}{2\sigma^2}}}{(2\pi\sigma^2)^{\frac{n}{2}}}$

$\therefore \ell(\sigma^2) = -\dfrac{\sum_{i=1}^{n} x_i^2}{2\sigma^2} - \dfrac{n}{2}\ln(2\pi\sigma^2)$

let $\ell'(\sigma^2) = 0 \Rightarrow \dfrac{\sum_{i=1}^{n} x_i}{2} \cdot \dfrac{1}{\sigma^4} - \dfrac{n}{2} \cdot \dfrac{2\pi}{2\pi\sigma^2} = 0$

$\Rightarrow \hat{\sigma}^2 = \dfrac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

(c) from (b) $\Rightarrow l(\sigma) = -\dfrac{\sum_{i=1}^{n} x_i^2}{2\sigma^2} - \dfrac{n}{2}\ln(2\pi\sigma^2)$

$\therefore l'(\sigma) = \sigma^{-3} \cdot \sum_{i=1}^{n} x_i - \dfrac{n}{\sigma}$

let $l'(\sigma) = 0 \Rightarrow \sigma^4 = \dfrac{n}{\sum_{i=1}^{n} x_i}$

$\therefore \hat{\sigma} = (\bar{x})^{-\frac{1}{4}}$

(d)

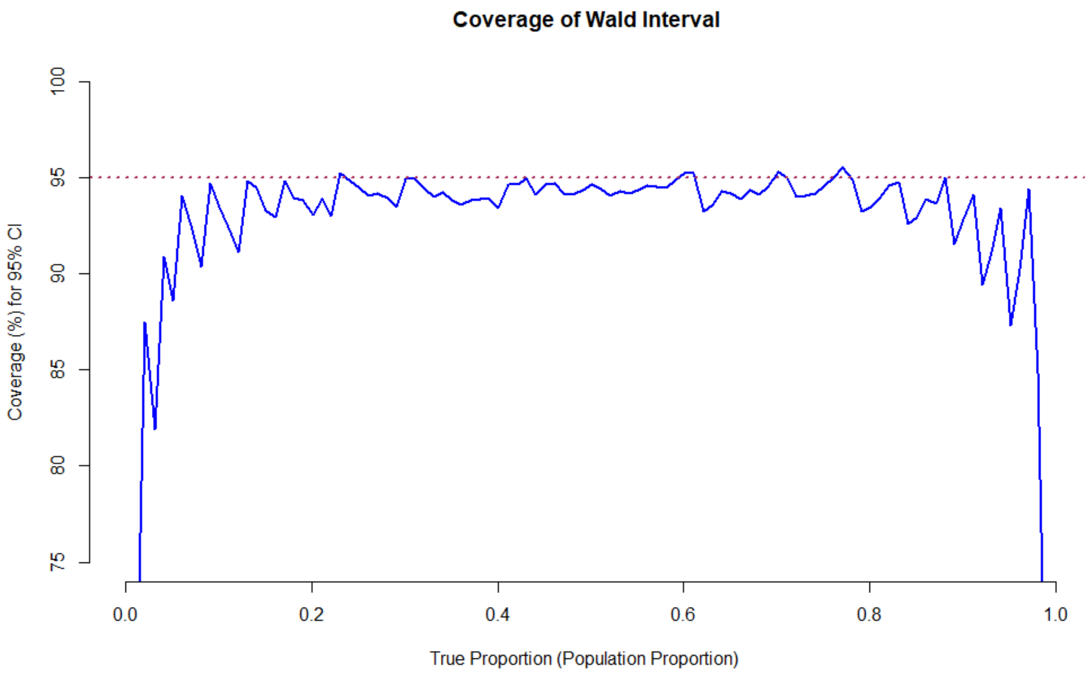
Wald interval relies a lot on normal approximation assumption of binomial distribution and there are no modifications or corrections that are applied. It is the most direct confidence interval that can be constructed from this normal approximation:

$$\theta = \frac{\hat{p}(1-\hat{p})}{n}$$

But this approcimation always returns 95% confidence interval less than the true range. Which can be illustrated in R using various p value.

```
wald <- function(x, n, conf.level = 0.95){
  p <- x/n
  sd <- sqrt(p*((1-p)/n))
  z <- qnorm(c( (1 - conf.level)/2, 1 - (1-conf.level)/2))
  #returns thresholds at which conf.level has to be cut at.
  # normally use 95% CI, which is -1.96 and +1.96
  CI <- p + z*sd
```

```
  return(CI)
}
#example
wald(x = 20, n =40) #this will return 0.345 and 0.655
```



This graph shows the deficiency of the Wald Interval.