MSc Computational Methods in Ecology and Evolution: Maths for Biology

Solutions: Differentiation, limits, & Taylor series
Tutorial 2nd Feb 2021

Dr Bhavin S. Khatri
bkhatri@imperial.ac.uk

Information needed for this tutorial:

1. You will be asked to calculate limits of the form $\lim_{x\to a}\{f(x)\}$. Note that limits of the sum of a number of terms is a sum of their limits:

$$\lim_{x\to a}\{f(x)+g(x)\} = \lim_{x\to a}\{f(x)\} + \lim_{x\to a}\{g(x)\}\,.$$

   Using this rule you can calculate limits of products, by expanding out brackets and evaluating the limit of each term.

2. The first principles (limit) definition of a derivative is

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}$$

3. A Taylor series is a series expansion of a function $f(x)$ about some specific value a in terms of monomials of increasing order:

$$f(x-a) = f(a) + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a}(x-a) + \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=a}(x-a)^2 + \left.\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\right|_{x=a}(x-a)^3 + ...$$

   .

   Typically, we want to expand about $x = 0$ ($a = 0$), which gives

$$f(x) = f(0) + \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=0}x + \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=0}x^2 + \left.\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\right|_{x=0}x^3 + ...$$

   .

4. When asked to specify a Taylor series in x to a certain order n the $O(x^m)$ notation is used to signify the remaining terms starting with the highest *non-zero* term of order $m > n$.

5. Useful Taylor series about $x = 0$ to know (specified to 5^{th} order)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + O(x^6)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$$

QUESTION 1

Differentiation of x^2 as change in area

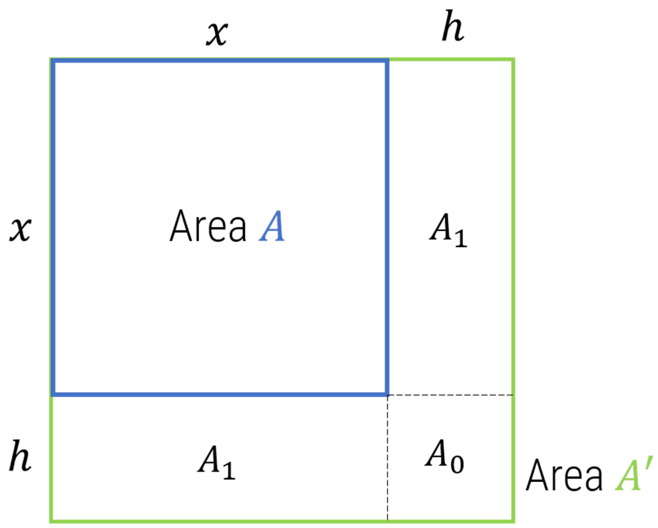

FIG. 1. Differentiating x^2

a) The area of the smaller square is $A = x^2$. What is the area of the bigger square A' in terms of the areas A, A_0, A_1?
$A' = A + 2A_1 + A_0$

b) What is the area of the bigger square A' in terms of x and h?
$A' = (x + h)^2$

c) Expand out this expression for A' and identify each term with areas A, A_0, xA_1.
$A' = x^2 + 2xh + h^2 \Rightarrow A = x^2;\ A_1 = xh;\ A_0 = h^2$

d) Evaluate all these areas for $x = 1$ and $h = 0.3$.
For $h = 0.3$ & $x = 1$:
$A = 1;\ A_1 = 0.3;\ A_0 = 0.3^2 = 0.09 \Rightarrow A' = A + 2A_1 + A_0 = 1 + 0.6 + 0.09 = 1.69$

e) i) Show that the change in area with respect to the change in length is

$$\frac{\Delta A}{\Delta x} = \frac{A' - A}{h} = 2x + h$$

and ii) evaluate $\frac{\Delta A}{\Delta x}$ for $x = 1$ and $h = 0.3$.
i)

$$\frac{\Delta A}{\Delta x} = \frac{A' - A}{h} = \frac{2A_1 + A_0}{h} = \frac{2xh + h^2}{h} = 2x + h$$

ii)

$$\frac{\Delta A}{\Delta x} = 2x + h = 2 + 0.3 = 2.3$$

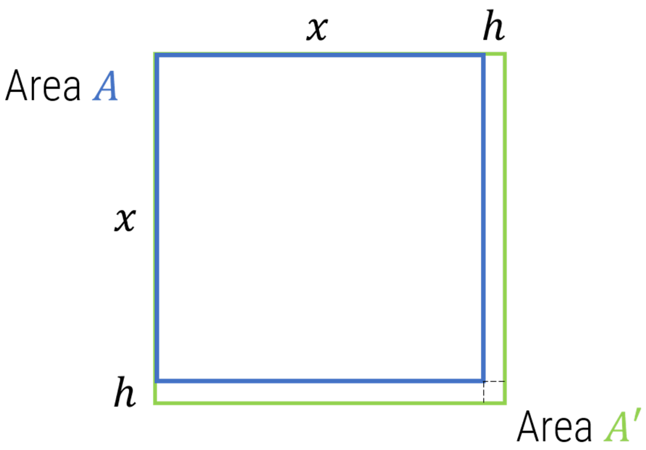

FIG. 2. Smaller h

f) Now imagine the two squares are more similar in size as in diagram above; i) calculate the different areas again and evaluate $\frac{\Delta A}{\Delta x}$ for $x = 1$ and $h = 0.05$. ii) Repeat for $x = 1$ and $h = 10^{-6}$.

i) For $h = 0.05$, $A = 1;\ A_1 = 0.05;\ A_0 = 0.05^2 = 0.0025$
$A' = A + 2A_1 + A_0 = 1 + 0.1 + 0.0025 = 1.1025$
$\frac{\Delta A}{\Delta x} = 2x + h = 2 + 0.05 = 2.05$

ii) For $h = 10^{-6}$, $A = 1;\ A_1 = 10^{-6};\ A_0 = (10^{-6})^2 = 10^{-12}$
$A' = A + 2A_1 + A_0 = 1 + 2 \times 10^{-6} + 10^{-12} = 1.000002000001 \approx 1$
$\frac{\Delta A}{\Delta x} = 2x + h = 2 + 10^{-6} = 2.000001 \approx 2$

g) This demonstrates that

$$\lim_{h \to 0} \left\{ \frac{\Delta A}{\Delta x} \right\} = \frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}x^2}{\mathrm{d}x} = 2x;$$

which parts of the diagram or which areas does this $2x$ correspond? Why is there a coefficient 2? Why are the other terms not important?

$2x$ corresponds to the sides of the square and the area A_1; there is a 2 because there are 2 sides that contribute to the change in area for a small change in h.

The other area $A_0 = h^2$ is not important, as it becomes vanishingly small compared to $2A_1 = 2xh$, as $h \to 0$.

QUESTION 2

Derivatives of trigonometric functions

a) Use the first principles (limit) definition of a derivative and the Taylor series expansion of the exponential function to show

$$\frac{\mathrm{d}e^{\alpha x}}{\mathrm{d}x} = \lim_{h\to 0}\left\{\frac{e^{\alpha(x+h)} - e^{\alpha x}}{h}\right\} = \alpha e^{\alpha x}$$

$$\begin{aligned}\frac{\mathrm{d}e^{\alpha x}}{\mathrm{d}x} &= \lim_{h\to 0}\left\{\frac{e^{\alpha(x+h)} - e^{\alpha x}}{h}\right\}\\ &= \lim_{h\to 0}\left\{\frac{e^{\alpha x}e^{\alpha h} - e^{\alpha x}}{h}\right\}\\ &= \lim_{h\to 0}\left\{\frac{e^{\alpha x}(e^{\alpha h} - 1)}{h}\right\}\\ &= e^{\alpha x}\lim_{h\to 0}\left\{\frac{e^{\alpha h} - 1}{h}\right\}\end{aligned}$$

Taylor expansion: $e^{\alpha h} = 1 + \alpha h + \frac{(\alpha h)^2}{2} + \ldots$

$$\begin{aligned}\Rightarrow \frac{\mathrm{d}e^{\alpha x}}{\mathrm{d}x} &= e^{\alpha x}\lim_{h\to 0}\left\{\frac{\cancel{1} + \alpha h + \frac{(\alpha h)^2}{2} + \ldots - \cancel{1}}{h}\right\}\\ &= e^{\alpha x}\lim_{h\to 0}\left\{\frac{\alpha h + \frac{(\alpha h)^2}{2} + \ldots}{h}\right\}\\ &= e^{\alpha x}\lim_{h\to 0}\left\{\alpha + \frac{\alpha^2 h}{2} + \ldots\right\}\\ &= \alpha e^{\alpha x}\end{aligned}$$

b) Using Eulers formula it can be shown that

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

Using these results show that

$$\frac{\mathrm{d}\sin x}{\mathrm{d}x} = \cos x$$

$$
\begin{aligned}
\frac{\mathrm{d}\sin x}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2i}(e^{ix}-e^{-ix})\right)\\
&= \frac{1}{2i}\left(\frac{\mathrm{d}e^{ix}}{\mathrm{d}x}-\frac{\mathrm{d}e^{-ix}}{\mathrm{d}x}\right)\\
&= \frac{1}{2i}(ie^{ix}-(-i)e^{-ix})\\
&= \frac{\cancel{i}}{2\cancel{i}}(e^{ix}+e^{-ix})\\
&= \cos x
\end{aligned}
$$

$$
\frac{\mathrm{d}\cos x}{\mathrm{d}x} = -\sin x
$$

(Hint: $\frac{1}{i} = -i$)

$$
\begin{aligned}
\frac{\mathrm{d}\cos x}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}(e^{ix}+e^{-ix})\right)\\
&= \frac{1}{2}\left(\frac{\mathrm{d}e^{ix}}{\mathrm{d}x}+\frac{\mathrm{d}e^{-ix}}{\mathrm{d}x}\right)\\
&= \frac{1}{2}(ie^{ix}+(-i)e^{-ix})\\
&= \frac{i}{2}(e^{ix}-e^{-ix})\\
&= -\frac{1}{2i}(e^{ix}-e^{-ix})\\
&= -\sin x
\end{aligned}
$$

c) Use the product (or quotient) rule and chain rule to show that

$$
\frac{\mathrm{d}\cot x}{\mathrm{d}x} = -\csc^2 x
$$

where $\cot x = 1/\tan x$ and $\csc x = \frac{1}{\sin x}$ (N.B. Not the same as $\sin^{-1} x = \arcsin x$).
(Hint: $\sin^2 x + \cos^2 x = 1$)

Need to use the product rule: $\frac{\mathrm{d}uv}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

$$
\begin{aligned}
\frac{\mathrm{d}\cot x}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{\sin x}\right)\\
&= \frac{1}{\sin x}\frac{\mathrm{d}\cos x}{\mathrm{d}x} + \cos x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sin x}\right)
\end{aligned}
$$

Let $w = \sin x \Rightarrow$ by chain rule $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sin x}\right) = \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{1}{w}\right) = \frac{\cos x}{-w^2} = -\frac{\cos x}{\sin^2 x}$

$$
\begin{aligned}
\Rightarrow \frac{\mathrm{d}\cot x}{\mathrm{d}x} &= -\frac{\cancel{\sin x}}{\cancel{\sin x}} + \cos x \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sin x}\right) \\
&= -1 - \frac{\cos^2 x}{\sin^2 x} \\
&= -\frac{\cos^2 x + \sin^2 x}{\sin^2 x} \\
&= -\frac{1}{\sin^2 x} \\
&= -\csc^2 x
\end{aligned}
$$

QUESTION 3

Properties of the Gaussian function

The Gaussian function $y = e^{-x^2}$ is ubiquitous in statistics and probability theory, as it describes the Normal/Gaussian distribution.

a) Plot/sketch this function and indicate values x^* for which $\frac{dy}{dx} = 0$ (there is one obvious value of x^* where $\frac{dy}{dx} = 0$, and two less obvious values).

b) Sketch $\frac{dy}{dx}$ qualitatively. (Hint: The Gaussian is a symmetric/even function ($f(-x) = f(x)$) and so its derivative will be an anti-symmetric/odd function ($f'(-x) = -f'(x)$) — in other words the derivative of a Gaussian should be equal and opposite in sign when reflected about the y-axis)

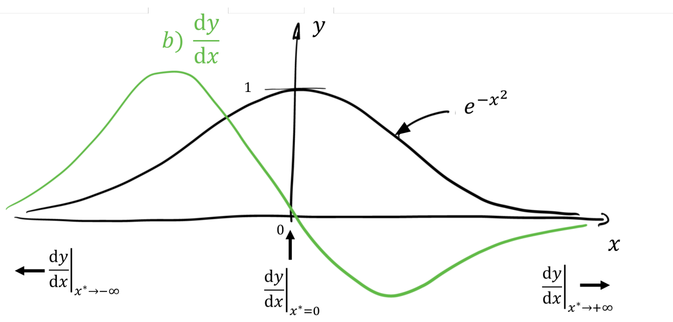

FIG. 3. a) & b)

By inspection it is clear that $y = e^{-x^2}$ has zero slope ($\frac{dy}{dx} = 0$) for $x^* = 0$, but the less obvious values are $x^* = \pm\infty$.

c) Using the chain rule, show $\frac{dy}{dx} = -2xe^{-x^2}$, and plot your result; verify that your sketch in b) is qualitatively accurate.

Let $y = e^z$, with $z = -x^2$:

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = -2xe^z = -2xe^{-x^2}$$

d) Solve $\frac{dy}{dx} = 0$ and verify your answer agrees with the answer from a).

$\frac{dy}{dx}\Big|_{x=x^*} = -2x^*e^{-(x^*)^2} = 0$ This can be zero if either 1) $x^* = 0$ or 2) when $e^{-(x^*)^2} = 0$, which gives $x^* = \pm\infty$.

e) Which solution x^* of $\frac{dy}{dx} = 0$ corresponds to where y is at maximum. Argue qualitatively why this must be a maximum from the plot of $\frac{dy}{dx}$.

$x^* = 0$ corresponds to the maximum; as x increases through $x^* = 0$, $\frac{dy}{dx}$ goes from +ve to -ve & $\Rightarrow \frac{dy}{dx}$ is decreasing through the turning point/stationary point, as must occur for a maximum occurring at x^* (for $x < x^*$ $\frac{dy}{dx} > 0$, while for $x < x^*$ $\frac{dy}{dx} < 0$).

f) We can show this mathematically, by examining the *curvature* of y, which is defined to be the 2nd derivative $\frac{d^2y}{dx^2}$. Show that $\frac{d^2y}{dx^2} = 2e^{-x^2}(x^2 - 1)$ and that the curvature is negative at the point x^* where y is maximum; why does this mean y is at maximum?

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-2xe^{-x^2}\right)\\ &= -2x\frac{\mathrm{d}e^{-x^2}}{\mathrm{d}x} - e^{-x^2}\frac{\mathrm{d}2x}{\mathrm{d}x}\\ &= -2x(-2xe^{-x^2}) - 2e^{-x^2}\\ &= 2e^{-x^2}(2x^2-1)\end{aligned}$$

At $x^* = 0$:

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x^*=0} = -2$$

,
which means the curvature is negative. Curvature is the rate of change of the derivative $\Rightarrow$ negative curvature means the derivative is *decreasing* through $x^* = 0$, as discussed in the previous part of this question, is what we would expect at a maximum.

QUESTION 4

Taylor series

a) i) Use the Taylor series of $\sin\theta$ to show

$$\lim_{\theta\to 0}\left\{\frac{\sin\theta}{\theta}\right\} = 1$$

Taylor series:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...$$

$$\begin{aligned}\Rightarrow \lim_{\theta\to 1}\left\{\frac{\sin\theta}{\theta}\right\} &= \Rightarrow \lim_{\theta\to 1}\left\{\frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...}{\theta}\right\}\\ &= \Rightarrow \lim_{\theta\to 1}\left\{1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + ...\right\}\\ &= 1\end{aligned}$$

Compare this to simply evaluating $\frac{\sin\theta}{\theta}$ at $\theta = 0$ is it even possible to evaluate this?

No it cannot be evaluated as

$$\left.\frac{\sin\theta}{\theta}\right|_{\theta=0} = \frac{0}{0}$$

which is indeterminant or undefined.

ii) Plot $\frac{\sin\theta}{\theta}$ and verify that it asymptotes to 1 as $\theta \to 0$.

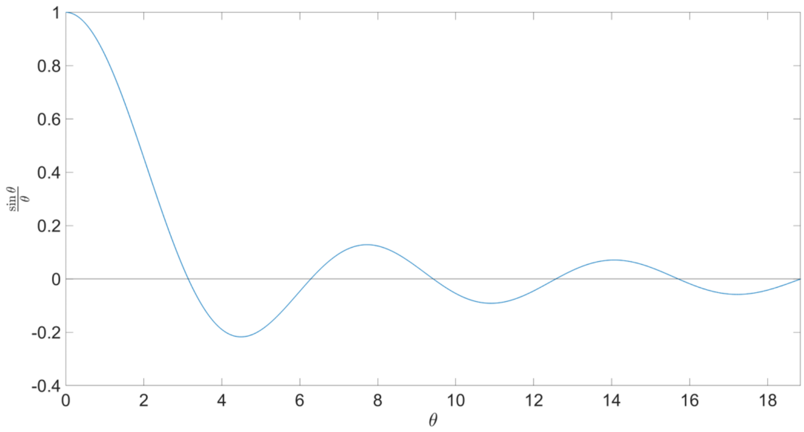

FIG. 4. $\frac{\sin\theta}{\theta}$

b) The following formula is the probability of fixation of a mutant with selective advantage s and initial frequency $1/N$ in a population of N haploid individuals (Kimura, Genetics, 1962)

$$p_{fix} = \frac{1-e^{-2s}}{1-e^{-2Ns}}.$$

i) Show using the Taylor expansion of the exponential function that

$$\lim_{s\to 0}\{p_{fix}\} = \frac{1}{N},$$

which is the probability of fixation of a neutral mutant.

$$\begin{aligned}
\lim_{s\to 0}\{p_{fix}\} &= \lim_{s\to 0}\left\{\frac{1-e^{-2s}}{1-e^{-2Ns}}\right\} \\
&= \lim_{s\to 0}\left\{\frac{\cancel{1}-(\cancel{1}-2s+\frac{(2s)^2}{2!}+...)}{\cancel{1}-(\cancel{1}-2Ns+\frac{(2Ns)^2}{2!}+...)}\right\} \\
&= \lim_{s\to 0}\left\{\frac{2s-\frac{(2s)^2}{2!}+...}{2Ns-\frac{(2Ns)^2}{2!}+...}\right\} \\
&= \lim_{s\to 0}\left\{\frac{2\cancel{s}(1-\frac{2s}{2!}+...)}{2N\cancel{s}(1-\frac{2Ns}{2!}+...)}\right\} \\
&= \frac{1}{N}\lim_{s\to 0}\left\{\frac{1-\frac{2s}{2!}+...}{1-\frac{2Ns}{2!}+...}\right\} \\
&= \frac{1}{N}
\end{aligned}$$

ii) Plot p_{fix} vs s for $0 < s \leq 0.1$ on a log-linear scale with $N = 10$ and $N = 100$ and verify that the y-intercept is $1/N$.

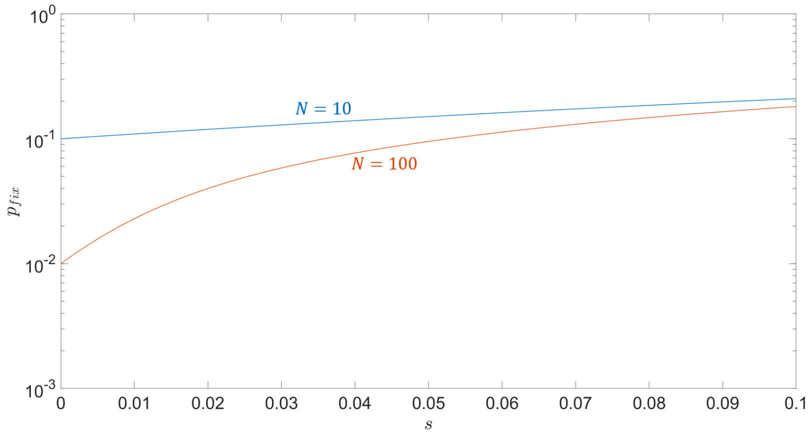

FIG. 5. p_{fix}

c) Discrete time and continuous time evolutionary models use two different definitions of fitness, the *Wrightian* fitness w and *Malthusian* fitness f, which are related by $f = \ln(w)$. The Wrightian fitness is often described in terms of the selective advantage s of a mutant, where $w = 1 + s$.

i) Show the Taylor series expansion of $f(s) = \ln(1+s)$ to 3rd order about $s = 0$ is

$$\ln(1+s) = s - \frac{1}{2}s^2 + \frac{1}{3}s^3 + O(s^4)$$

Taylor expansion of $f(s)$ to 3rd order:

$$f(s) = f(0) + \left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{s=0} s + \left.\frac{\mathrm{d}^2 f}{\mathrm{d}s^2}\right|_{s=0} \frac{s^2}{2!} + \left.\frac{\mathrm{d}^3 f}{\mathrm{d}s^3}\right|_{s=0} \frac{s^3}{3!} + ...$$

$\Rightarrow f(s=0) = \ln(1) = 0$

$\left.\frac{\mathrm{d}f}{\mathrm{d}s}\right|_{s=0} = \left.\frac{1}{1+s}\right|_{s=0} = 1$

$\left.\frac{\mathrm{d}^2 f}{\mathrm{d}s^2}\right|_{s=0} = -\left.\frac{1}{(1+s)^2}\right|_{s=0} = -1$

$\left.\frac{\mathrm{d}^3 f}{\mathrm{d}s^3}\right|_{s=0} = \left.\frac{2}{(1+s)^3}\right|_{s=0} = 2$

$$\Rightarrow f(s) = \ln(1+s) = 0 + s - \frac{s^2}{2} + 2\frac{s^3}{6} + ... = s - \frac{1}{2}s^2 + \frac{1}{3}s^3 + ...$$

ii) Using this result, show that the Malthusian fitness difference (compared to wild type) is equivalent to the selective advantage s, when $|s| \ll 1$, by showing

$$f(s) = \ln(w) = \ln(1+s) \approx s$$

(Hint: when $s \ll 1$, consider how big is s^2 compared to s, and how big is s^3 compared to s^2, and so on)

For $s \ll 1: \ s \gg s^2 \gg s^3 ...$

$$\Rightarrow f(s) \approx s$$

So Malthusian fitness *differences* are equivalent to the Wrightian selective advantage, when $|s| \ll 1$.

iii) Plot $\ln(1+s)$ and s for $-0.5 \leq s \leq 0.5$ to verify this approximation works well for $|s| \ll 1$.

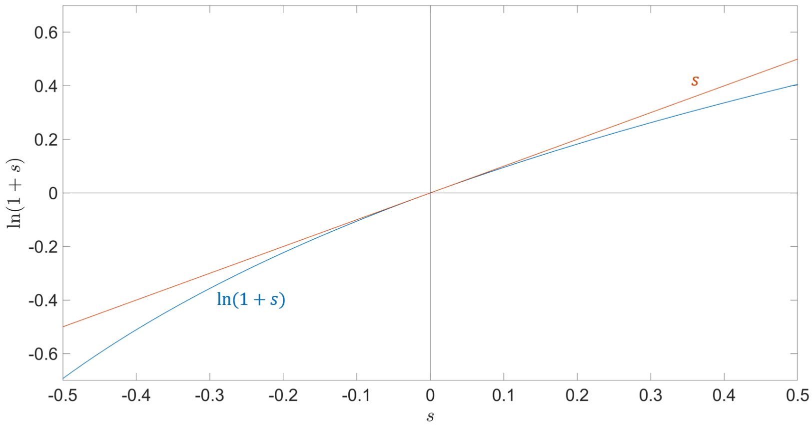

FIG. 6. $f = \ln(1+s) \approx s$

d) The hyperbolic sine and cosine are part of a group of functions (hyperbolic functions), which are analogous to the trigonometric sine and cosine, but for geometry on the unit hyperbola $x^2 - y^2 = 1$, instead of on the unit circle $(x^2 + y^2 = 1)$. They have many applications including solutions to certain differential equations. They are defined by

$$\cos i\theta = \cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$$

$$\sin i\theta = \sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$$

i) Using these equations show that

$$\frac{\mathrm{d}\sinh\theta}{\mathrm{d}\theta} = \cosh\theta$$

$$\frac{\mathrm{d}\cosh\theta}{\mathrm{d}\theta} = \sinh\theta$$

$$\frac{\mathrm{d}\sinh\theta}{\mathrm{d}\theta} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}(e^{\theta} - e^{-\theta}) = \frac{1}{2}(e^{\theta} + e^{-\theta}) = \cosh\theta$$

$$\frac{\mathrm{d}\cosh\theta}{\mathrm{d}\theta} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}(e^{\theta} + e^{-\theta}) = \frac{1}{2}(e^{\theta} - e^{-\theta}) = \sinh\theta$$

ii) Show that the Taylor series for $\sinh\theta$ to 3rd order is

$$\sinh\theta = \theta + \frac{\theta^3}{6} + O(\theta^5)$$

$\Rightarrow \sinh(0) = 0$

$\left.\frac{\mathrm{d}\sinh\theta}{\mathrm{d}\theta}\right|_{\theta=0} = \cosh\theta\Big|_{\theta=0} = 1$

$$
\left.\frac{\mathrm{d}^2 \sinh\theta}{\mathrm{d}\theta^2}\right|_{\theta=0} = \left.\sinh\theta\right|_{\theta=0} = 0
$$
$$
\left.\frac{\mathrm{d}^3 \sinh\theta}{\mathrm{d}\theta^3}\right|_{\theta=0} = \left.\cosh\theta\right|_{\theta=0} = 1
$$

Note all even terms will be zero, including 4th order term:

$$
\begin{aligned}
\Rightarrow \sinh\theta &= 0 + 1\times\theta + 0\times\frac{\theta^2}{2!} + 1\times\frac{\theta^3}{3!} + O(\theta^5) \\
&= \theta + \frac{1}{6}\theta^3 + O(\theta^5)
\end{aligned}
$$

iii) Now consider $f(\theta) = \sinh\theta - \sin\theta$. For small θ, to lowest order their respective Taylor series are

$$
sin\theta = \theta + O(\theta^3)
$$

$$
\sinh\theta = \theta + O(\theta^3)
$$

from this can we conclude that the $f(\theta) = \sinh\theta - \sin\theta = 0$ as $\theta \ll 1$?

No we cannot conclude that $f(\theta) = \sinh\theta - \sin\theta = 0$ for $\theta \ll 1$; what this says is that to the lowest order expansion of each of $\sinh\theta$ and $\sin\theta$ that in $\sinh\theta - \sin\theta$ they cancel, but we do not know whether the higher order terms will also cancel $\Rightarrow$ we look for the next higher order term in the Taylor expansion where the terms θ^n does not cancel — if $\theta \ll 1$ then relative to the lowest n^{th} term all higher order terms will be smaller by at least a factor of θ, which is small and so we can ignore.

Use the Taylor expansion of $\sinh\theta$ and $\sin\theta$ to third order to show that the Taylor series expansion of $f(\theta)$ to lowest order is

$$
f(\theta) = \sinh\theta - \sin\theta = \frac{\theta^3}{3}
$$

We have already calculated the Taylor series of $\sinh\theta$ to 3rd order above, and we know that the Taylor series (to 3rd order) for $\sin\theta = \theta - \frac{\theta^3}{6} + O(\theta^5)$:

$$
\begin{aligned}
\sinh\theta - \sin\theta &= \left(\cancel{\theta} + \frac{\theta^3}{6} + O(\theta^5) +\right)\left(\cancel{\theta} - \frac{\theta^3}{6} + O(\theta^5)\right) \\
&= \frac{\theta^3}{6} + \frac{\theta^3}{6} \\
&= \frac{\theta^3}{3}
\end{aligned}
$$

iv) Plot $f(\theta) = \sinh\theta - \sin\theta$, and $\frac{\theta^3}{3}$ for $0 < \theta < 3$ and verify that they are the same for small θ.

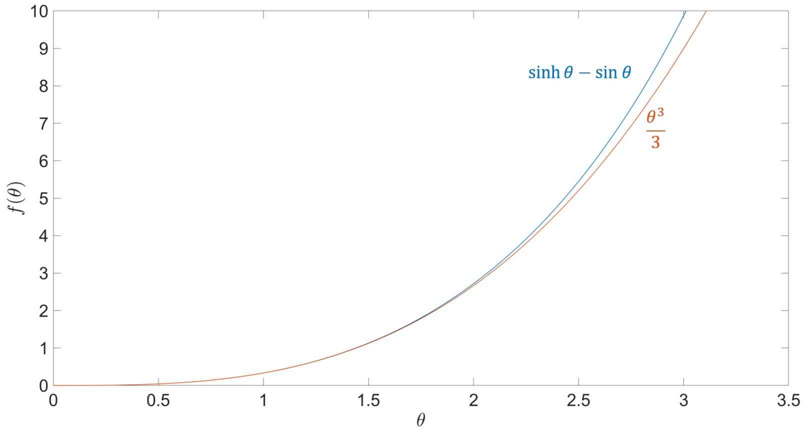

FIG. 7. $\sinh\theta - \sin\theta$

e) *Derive Eulers formula:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

using the Taylor series expansion of $e^{i\theta}$ and comparing to the Taylor expansion of $\cos\theta$ and $\sin\theta$.
(Hint: $i^2 = -1$; $i^3 = i^2 \times i = -i$; $i^4 = i^2 \times i^2 = 1$; $i^5 = i \times i^4 = i$; $i^6 = i^4 \times i^2 = -1$; $i^7 = i^4 \times i^3 = -i...$)

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + ... \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + ... \end{aligned}$$

Now series expansion of $\cos\theta$ and $\sin\theta$ are

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ...$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...$$

All even terms of the Taylor series expansion of $e^{i\theta}$, can be identified with $\cos\theta$, and all odd terms with $i\sin\theta$, and hence

$$e^{i\theta} = \cos\theta + i\sin\theta$$

.

Supplementary (completely optional) Questions

QUESTION 5

Differentiation of x^3 as change in volume

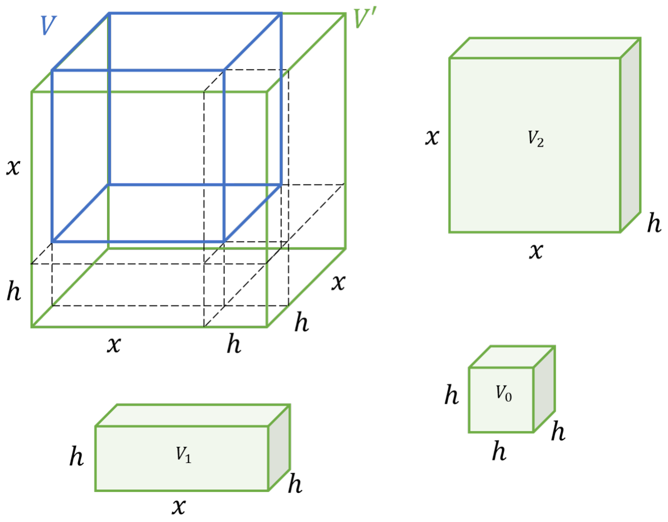

FIG. 8. Differentiating x^3

a) What is the volume of the bigger cube V' in terms of the volumes V,V_0,V_1,V_2?
$V' = V + 3V_2 + 3V_1 + V_0$

b) What is the volume of the bigger cube V' in terms of x and h?
$V' = (x+h)^3$

c) Expand out this expression for V' and identify each term with volumes V,V_0,V_1,V_2.
$V' = x^3 + 3x^2h + 3xh^2 + h^3 \Rightarrow V = x^3;\ V_2 = x^2h;\ V_1 = xh^2;\ A_0 = h^3$

d) Evaluate all these volumes for $x = 1$ and $h = 0.3$.
For $h = 0.3$ & $x = 1$:
$V = 1;\ V_2 = 0.3;\ A_1 = (0.3)^2 = 0.09;\ A_0 = 0.3^3 = 0.027$
$\Rightarrow V' = V + 3V_2 + 3V_1 + V_0 = 1 + 0.9 + 0.27 + 0.027 = 2.197$

e) i) Show that the change in volume with respect to the change in length is

$$\frac{\Delta V}{\Delta x} = \frac{V' - V}{h} = 3x^2 + 3xh + h^2$$

And ii) evaluate $\frac{\Delta V}{\Delta x}$ for $x = 1$ and $h = 0.3$.
i)

$$\frac{\Delta V}{\Delta x} = \frac{V' - V}{h} = \frac{3A_2 + 3V_1 + V_0}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2$$

ii)

$$\frac{\Delta V}{\Delta x} = 3x^2 + 3xh + h^2 = 3 + 0.9 + 0.09 = 3.99$$

f) Repeat d) and e)ii) for $x = 1$ and $h = 0.05$ and $h = 10^{-6}$.

i) For $h = 0.05$, $V = 1;\ V_2 = 0.05;\ V_1 = 0.05^2 = 0.0025;\ V_0 = 0.05^3 = 0.000125$
$V' = V + 3V_2 + 3V_1 + V_0 = 1 + 0.15 + 0.0075 + 0.000125 = 1.157625$
$\frac{\Delta V}{\Delta x} = 3x^2 + 3xh + h^2 = 3 + 0.15 + 0.0025 = 3.1525$
ii) For $h = 10^{-6}$, $V = 1;\ V_2 = 10^{-6};\ V_1 = 10^{-12};\ V_0 = 10^{-18}$
$V' = V + 3V_2 + 3V_1 + V_0 = 1 + 3 \times 10^{-6} + 3 \times 10^{-12} + 10^{-18} = 1.00000300.... \approx 1$
$\frac{\Delta V}{\Delta x} = 3x^2 + 3xh + h^2 = 3 + 3 \times 10^{-6} + \times 10^{-12} = 3.00000300000... \approx 3$

g) This demonstrates that

$$\lim_{h\to 0}\left\{\frac{\Delta V}{\Delta x}\right\} = \frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\mathrm{d}x^3}{\mathrm{d}x} = 3x^2;$$

which parts of the diagram or which volumes does this $3x^2$ correspond? Why is there a coefficient 3? Why are the other terms not important?

$3x^2$ corresponds to the faces of the cube and the volumes V_2; there is a 3 because there are 3 faces that contribute to the change in volume for a small change in h.

The other volumes $V_1 = 3xh^2$ and $V_0 = h^3$ is not important, as it becomes vanishingly small compared to $3V_2 = 3x^2h$, as $h \to 0$.

h) *Using the binomial theorem, show that the change in (hyper)volume of a hypercube of dimension n, $\Delta\Omega = x^n$ is dominated by n hypersurfaces of area x^{n-1} and hence in the limit of an infinitesimal change h

$$\lim_{h\to 0}\left\{\frac{\Delta\Omega}{\Delta x}\right\} = \frac{\mathrm{d}\Omega}{\mathrm{d}x} = \frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1};$$

(Hint: Binomial theorem is

$$\begin{aligned}(x+h)^n &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}h^k \\ &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + ...\end{aligned}$$

)

The "volume" of a hypercube in n-dimensions, where each side is length x is $\Omega = x^n$. The "volume" of a hypercube in n-dimensions, where each side is length $x + h$ is $\Omega' = (x + h)^n$. Hence, using the binomial theorem

$$\begin{aligned}\Omega' &= (x+h)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}h^k \\ &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + ... \\ &= \Omega + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-3)}{3!}x^{n-3}h^3 + ...\end{aligned}$$

$$\frac{\Delta\Omega}{\Delta x} = \frac{\Omega' - \Omega}{h} = \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-3)}{3!}x^{n-3}h^3 + \ldots}{h}$$
$$= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \frac{n(n-1)(n-2)}{3!}x^{n-3}h^2 + \ldots$$

$$\begin{aligned}
\Rightarrow \frac{\mathrm{d}\Omega}{\mathrm{d}x} &= \lim_{\Delta x\to 0}\left\{\frac{\Delta\Omega}{\Delta x}\right\} \\
&= \lim_{h\to 0}\left\{\frac{\Omega'-\Omega}{h}\right\} \\
&= \lim_{h\to 0}\left\{nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \frac{n(n-1)(n-2)}{3!}x^{n-3}h^2 + \ldots\right\} \\
&= nx^{n-1}
\end{aligned}$$

QUESTION 6

Derivative of $\sqrt[n]{x} = x^{1/n}$

There isnt a nice (conventional) geometric interpretation of what the quantity $\sqrt[n]{x} = x^{1/n}$ represents if x represents a length along a line.

a) However, if $\Omega = x^n$ is the “volume” of a hypercube in n-dimensions, then what does $\sqrt[n]{\Omega} = \Omega^{1/n}$ represent? If $\Omega = x^n$ then $\sqrt[n]{\Omega} = \Omega^{1/n} = x$, the length of a side of the hypercube.

b) With $x = \Omega^{1/n}$ and using fact that $\frac{\mathrm{d}x}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\Omega}{\mathrm{d}x}\right)^{-1}$ show that

$$\frac{\mathrm{d}x^{1/n}}{\mathrm{d}x} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

$\Omega = x^n$
$\frac{\mathrm{d}x}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\Omega}{\mathrm{d}x}\right)^{-1}$

$$\Rightarrow \frac{\mathrm{d}\Omega}{\mathrm{d}x} = nx^{n-1}$$

.

$$\Rightarrow \frac{\mathrm{d}\Omega^{1/n}}{\mathrm{d}\Omega} = \frac{\mathrm{d}x}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\Omega}{\mathrm{d}x}\right)^{-1} = \left(nx^{n-1}\right)^{-1}$$

. Now we substitute in $x = \Omega^{1/n}$:

$$\Rightarrow \frac{\mathrm{d}\Omega^{1/n}}{\mathrm{d}\Omega} = \left(n(\Omega^{1/n})^{n-1}\right)^{-1} = \frac{1}{n}\Omega^{-\frac{n-1}{n}} = \frac{1}{n}\Omega^{\frac{1}{n}-1}$$

Ω is of course just any variable, so

$$\frac{\mathrm{d}x^{1/n}}{\mathrm{d}x} = \frac{1}{n}x^{\frac{1}{n}-1}$$