MSc Computational Methods in Ecology and Evolution: Maths for Biology

Solutions: Rotations, Oscillations and Complex Numbers
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Dr Bhavin S. Khatri
bkhatri@imperial.ac.uk

This tutorial will explore how oscillating behaviour is intimately related to motion on a circle and how the concept of complex numbers is the most natural way to understand and describe these orbits close to the fixed point.

QUESTION 1

Simple equation for predator prey motion

Oscillations are found throughout many diverse phenomena in ecology and evolution. Predator-prey dynamics are amongst the most well-known as shown in Fig.1a with data on populations of hares and lynxes. Simple versions of these dynamics are represented by the Lotka-Volterra model Fig.1b, which produce oscillating solutions for hares and lynxes Fig.1c&d for different *initial conditions*.

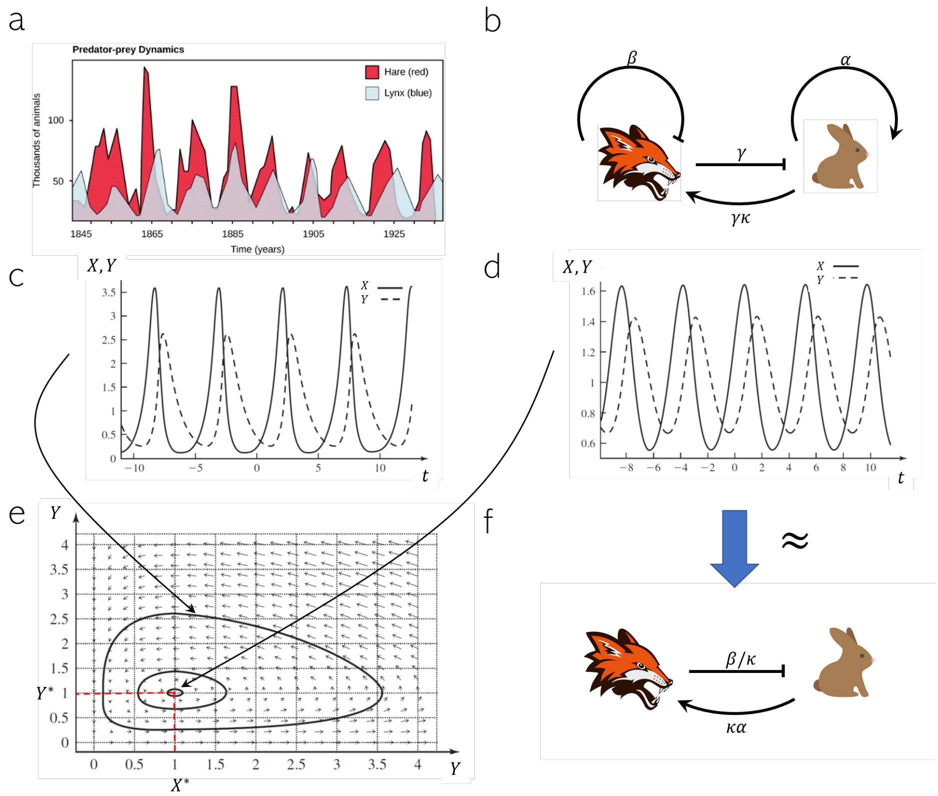

FIG. 1. Predator-Prey oscillations

Plotting these on a *phase plane* (i.e. plot X vs Y) shows that the behaviour taken together can be described by orbits. When the initial conditions are close to the *fixed point* (X^*, Y^*), orbits are *elliptical* and behaviour very well-represented by simple sinusoidal behaviour for prey X and predator Y.

$$X(t) = X^* + R_1 \cos(\omega t - \phi) \quad (1)$$
$$Y(t) = Y^* + R_2 \sin(\omega t - \phi) \quad (2)$$

Lets assume the predator natural rate of death is the same rate of growth of prey ($\omega = \beta = \alpha$) and that the predator is

100% efficient in gaining energy for growth from death of prey due to predation ($\kappa = 1$). In this case the motion of the predator-prey numbers is exactly circular (about the fixed point X^*,Y^*) with $R = R_1 = R_2$. If we let $x(t) = X(t) - X^*$ and $y(t) = Y(t) - Y^*$, be the number of excess prey and predators, then our equation of motion is

$$x(t) = R\cos(\omega t - \phi) \quad (3)$$
$$y(t) = R\sin(\omega t - \phi). \quad (4)$$

a) Plot x, y on an axis for times $t = 0, 1, 2, 3, 8$ years for $\omega = \pi/4$ years^{-1} with $R = 1$ & $\phi = 0$ and label each point with their corresponding value of t. What is the locus on which the points lie?

b) On a different axis plot $x(t)$ vs t & $y(t)$ vs t.

c) Repeat a) & b) for i) $\omega = \pi/4$ & $\phi = \pm\pi/4$ and ii) $\omega = \pi/2$ & $\phi = 0, \pm\pi/4$, remembering to label points with time t. What does doubling the rate ω do? What does changing the phase ϕ do?

See plots below.

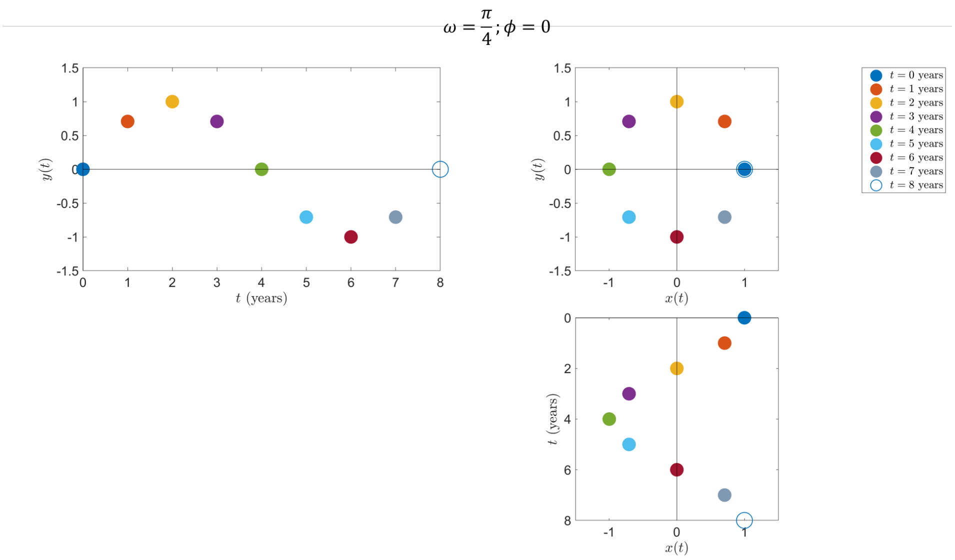

FIG. 2. $\omega = \pi/4, \phi = 0$

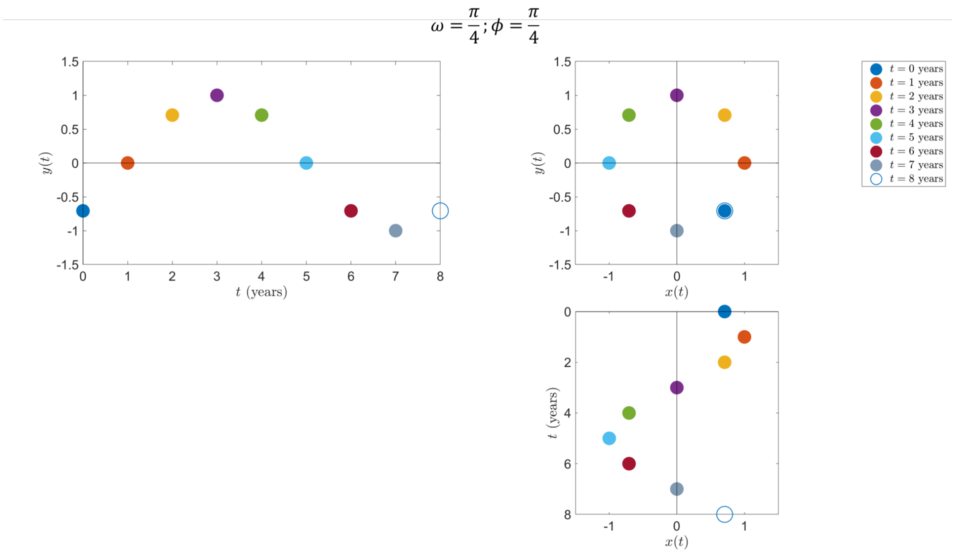

FIG. 3. $\omega = \pi/4, \phi = \pi/4$

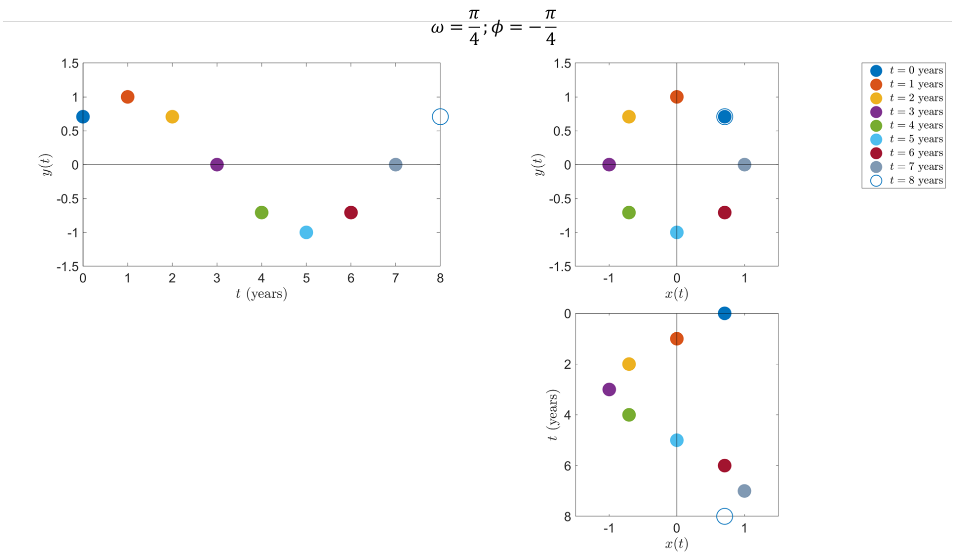

FIG. 4. $\omega = \pi/4, \phi = -\pi/4$

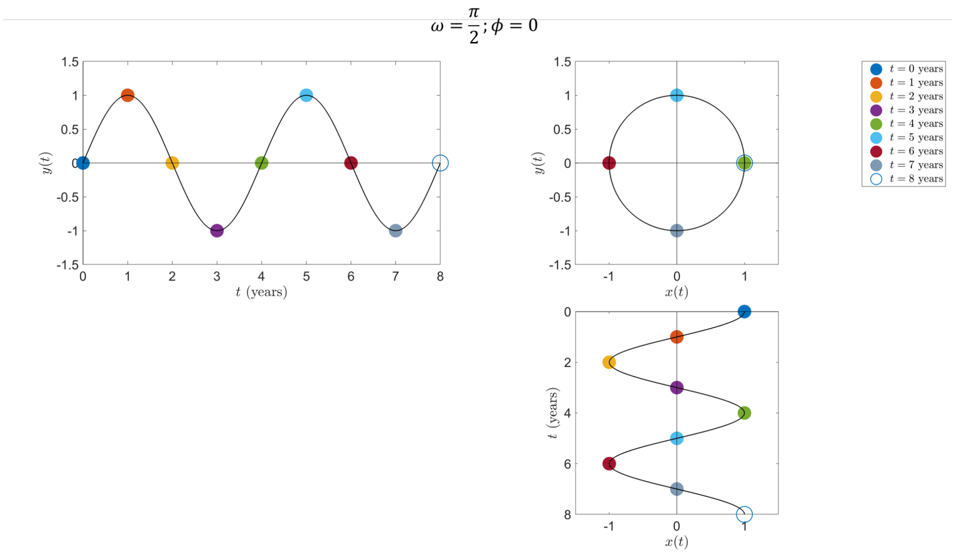

FIG. 5. $\omega = \pi/2, \phi = 0$

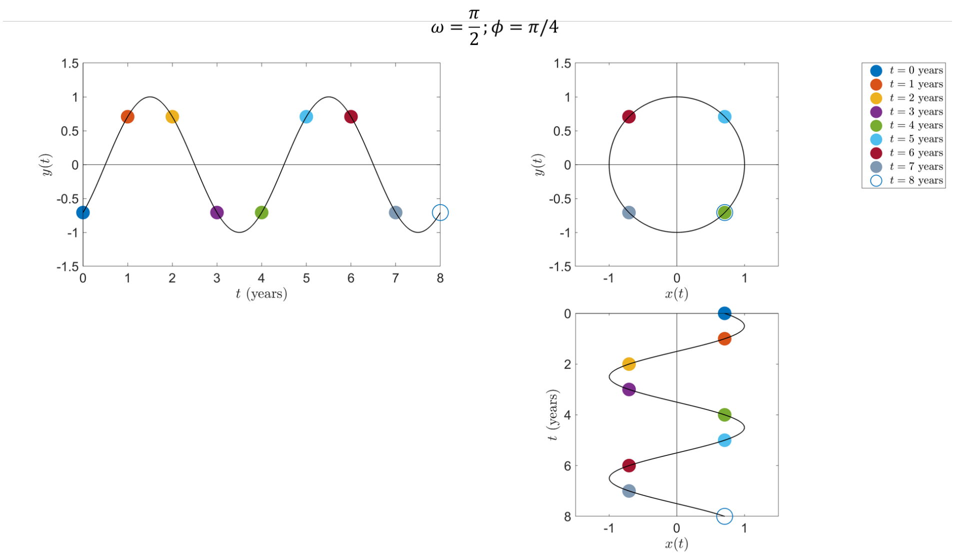

FIG. 6. $\omega = \pi/2, \phi = \pi/4$

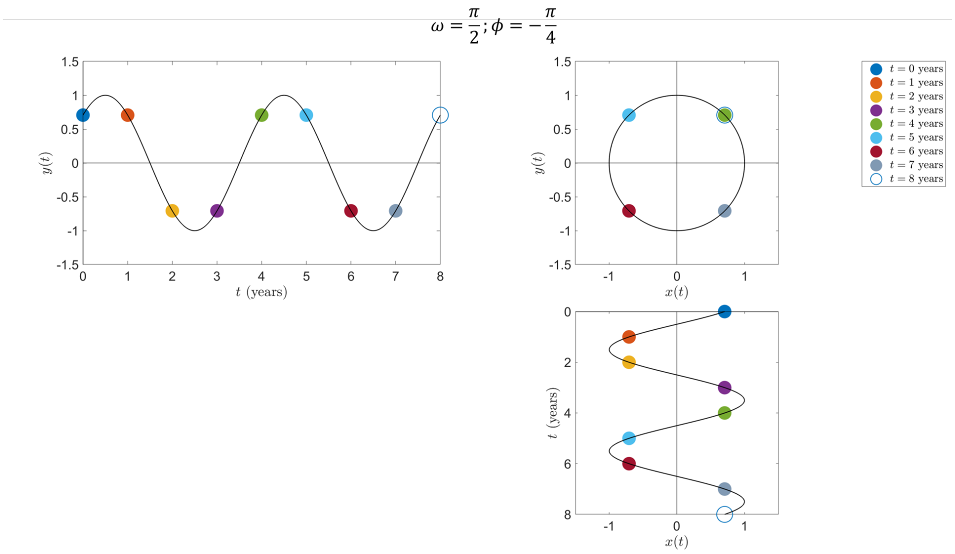

FIG. 7. $\omega = \pi/2, \phi = -\pi/4$

The points of x vs y lie on a locus of a circle or radius 1.

Doubling the rate ω decreases the period of the oscillations by half, as indicated by a halving of the time between peaks in $x(t)$ and $y(t)$.

A positive phase angle $\phi = \pi/4$ causes an advance of the sinusoidal variation of $x(t)$ and $y(t)$, since the total phase angle $\theta = \omega t - \phi$ is *decreased*, which can be seen in the phase plots, where the 1st time point $(t = 0)$ is at an angle of $\theta = -\phi = -\pi/4$, which effectively "grabs" values from behind (compared to $\phi = 0$) in the cycle.

A negative phase angle $\phi = -\pi/4$ causes an retardation of the sinusoidal variation of $x(t)$ and $y(t)$, since the total phase angle $\theta = \omega t - \phi$ is *increased*, which can be seen in the phase plots, where the 1st time point $(t = 0)$ is at an angle of $\theta = -\phi = - - \pi/4 = \pi/4$, which effectively "grabs" values from ahead (compared to $\phi = 0$)in the cycle.

d) By setting $t = 0$ in Eqns.(3) and (4) write down the initial condition $x_0 = x(0)$ & $y_0 = y(0)$ and compare to your phase plots for different values of ϕ.

$x_0 = x(0) = R\cos(-\phi) = R\cos(\phi)$ (cos is an even function: $\cos(-\theta) = \cos(\theta)$)

$y_0 = y(0) = R\sin(-\phi) = -R\sin(\phi)$ (sin is an odd function: $\sin(-\theta) = -\sin(\theta)$)

Referring to the phase plot for $\omega = \pi/4$ and $\phi = \pi/4$, we have

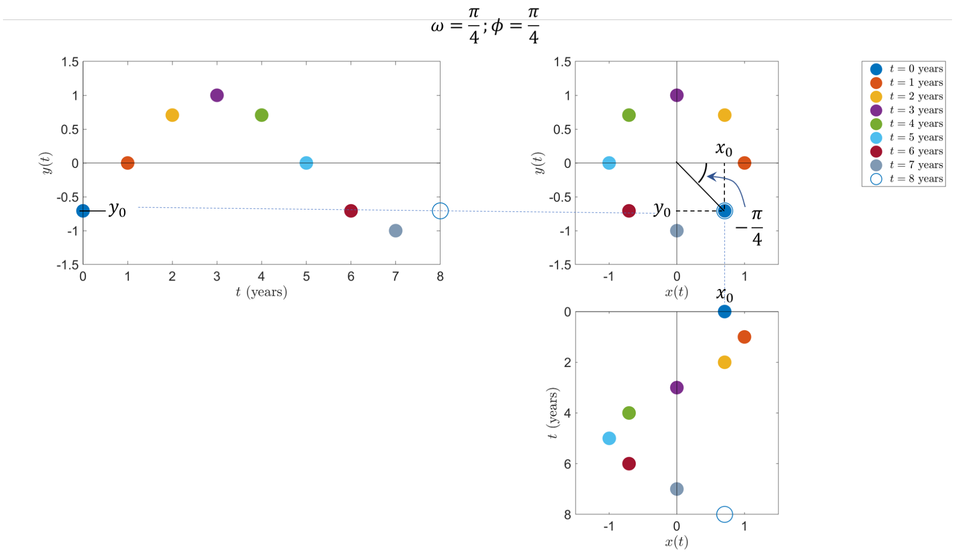

FIG. 8. Initial conditions x_0 and y_0: $\omega = \pi/2, \phi = \pi/4$

e) Divide y_0 by x_0 to show that

$$\phi = -\tan^{-1}\left(\frac{y_0}{x_0}\right) \tag{5}$$

where $\tan^{-1}$ is the inverse of the tangent function $\tan$, such that $\theta = \tan^{-1}(\tan(\theta))$.

$$\frac{y_0}{x_0} = \frac{-R\sin(\phi)}{R\cos(\phi)} = -\frac{\sin(\phi)}{\cos(\phi)} = -\tan\phi$$

$$\Rightarrow \phi = \tan^{-1}\left(-\frac{y_0}{x_0}\right) = -\tan^{-1}\left(\frac{y_0}{x_0}\right)$$

($\tan^{-1}$ is an odd function)

f) Now for population dynamics, θ represents the phase of the predator-prey cycle, which for a pure sinusoidal behaviour (i.e. close to fixed point) increases linearly in time, $\theta = \omega t$. i) How is ω related to the period of the oscillation T; ii) estimate T from the graph of Lynx-Hare dynamics (to nearest year), calculate ω and state the units of ω.
i) $T = \frac{2\pi}{\omega}$; ii) $T \approx 10$years $\Rightarrow \omega \approx \frac{6}{10}$ years^{-1} [More precise answer is $\omega = \pi/5 \approx 0.628$ years^{-1}, but given the large uncertainty of the estimate of T, such precision is likely unwarranted]

g) Show that the equations of motion obey the implicit equation for a circle

$$x^2(t) + y^2(t) = R^2 \tag{6}$$

(Hint: $\sin^2(\theta) + \cos^2(\theta) = 1$). What is the significance of this result?

$$x^2(t) + y^2(t) = (R\cos(\omega t - \phi))^2 + (R\sin(\omega t - \phi))^2 = R^2(\cos^2(\omega t - \phi) + \sin^2(\omega t - \phi)) = R^2$$

The significance is that this is true for all times t, i.e. the sum of squares of the number of hares and lynxes (predators and prey) is a constant.

h) If at $t = 0$ there are an excess of 80 hares and 60 lynxes, calculate a prediction of the number of excess hares and lynxes (to the nearest integer) at time $t = 15$ years, using the above equations. Your answer should be negative — why is that ok?

For $\omega = 0.6$ $x_0 = 80$ and $y_0 = 60$

$$x_0^2 + y_0^2 = R^2 \Rightarrow R = \sqrt{80^2 + 60^2} = 100$$

$$\phi = -\tan^{-1}\left(\frac{y_0}{x_0}\right) = -\tan^{-1}\left(\frac{60}{80}\right) \approx -0.64\text{radians}$$

$$\Rightarrow x(t = 15 \text{ years}) = R\cos(\omega t - \phi) = 100 \times \cos(0.6 \times 15 + 0.64) = -97.618 = -98 \text{ hares}$$

$$\Rightarrow y(t = 15 \text{ years}) = R\sin(\omega t - \phi) = 100 \times \sin(0.6 \times 15 + 0.64) = -21.698 = -22 \text{ lynxes}$$

For $\omega = \pi/5$ $x_0 = 80$ and $y_0 = 60$

$$x_0^2 + y_0^2 = R^2 \Rightarrow R = \sqrt{80^2 + 60^2} = 100$$

$$\phi = -\tan^{-1}\left(\frac{y_0}{x_0}\right) = -\tan^{-1}\left(\frac{60}{80}\right) \approx -0.64\text{radians}$$

$$\Rightarrow x(t = 15 \text{ years}) = R\cos(\omega t - \phi) = 100 \times \cos(\pi/5 \times 15 + 0.64) = -80.0 = -80 \text{ hares}$$

$$\Rightarrow y(t = 15 \text{ years}) = R\sin(\omega t - \phi) = 100 \times \sin(\pi/5 \times 15 + 0.64) = -60.0 = -60 \text{ lynxes}$$

(All to the nearest whole hare or lynx)

The numbers can be negative since they are relative numbers of individuals compared to the fixed point number X^* and Y^*; $X^* + x(t)$ and $Y^* + y(t)$ should always ≥ 0.

i) If at some time t the excess number of lynxes is 10 calculate the possible number of excess hares (to the nearest integer) assuming this circular model is correct. Can you calculate from this information the time t? Discuss what extra information you would need to determine the number of hares and the time of observation (Hint: your diagrams of the *phase portrait* (part a) should provide a clue).

$y(t) = 10 \Rightarrow x(t) = \pm\sqrt{R^2 - y^2(t)} = \pm\sqrt{100^2 - 10^2(t)} = \pm 99.499 = \pm 99 \text{ hares}$

There are two reasons we cannot calculate the time of observation t: 1) we do not know whether to take the plus or minus sign in the above equation and 2) with a period of T years we will observe exactly the same values of $y = 10$ and $x = \pm 99$.

We could determine the time if we knew if y was increasing or decreasing (up to a period T of the predator-prey cycle).

QUESTION 2

Complex worms

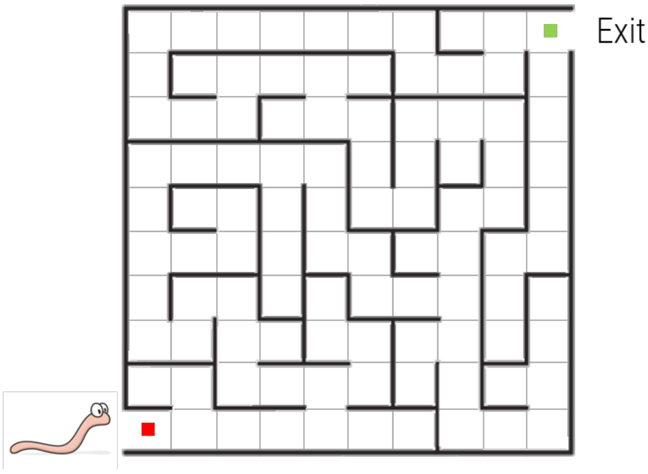

FIG. 9. Guide the worm out of the maze!

Imagine a worm placed in a 2D maze they are intelligent worms that happen to understand just two mathematical instructions for each move at a time: crawl a distance x then rotate $90°$ or $\pi/2$ radians (anti-clockwise) and then crawl a distance y in other words they move like the Knight chess piece, except with arbitrary lengths x and y of the capital L.

a) Starting at the red square and ending at the green square, your task is to write down the list of instructions for each step in turn, which will navigate the worm through this maze, where for each step you specify $x + iy$ where i indicates "rotate by $90°$". For a given path you want to specify the minimal number of steps. (N.B. assume the worm only moves in whole squares (x and y are integers) and that they can move left/right and up/down(i.e. x and y can take negative values). (Hint: 1st move I would suggest is $4 + i \times 1 = 4 + i$).

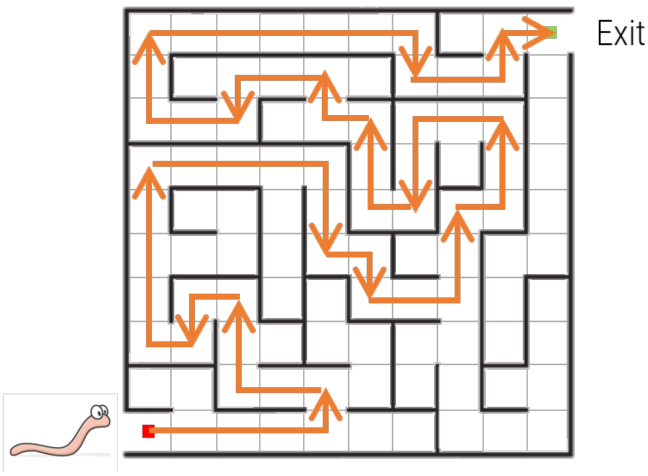

FIG. 10. Solution

Step no.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a)	$4+i$	$-2+2i$	$-1-i$	$-1+4i$	$4-2i$	$1-i$	$2+2i$	$1+2i$	$-2-2i$	$-1+2i$	$-1+i$	$-2-i$	$-2+2i$	$6-i$	$2+i$	1
d)	$4.1e^{0.2i}$	$2.8e^{-0.8i}$	$1.4e^{0.8i}$	$4.1e^{-1.3i}$	$4.5e^{-0.5i}$	$1.4e^{-0.8i}$	$2.8e^{0.8i}$	$2.2e^{1.1i}$	$2.8e^{0.8i}$	$2.2e^{-1.1i}$	$1.4e^{-0.8i}$	$2.2e^{0.5i}$	$2.8e^{-0.8i}$	$6.1e^{-0.2i}$	$2.2e^{0.5i}$	$1e^{0i}$

b) Given your solution, what do you notice if you sum any two adjacent steps by the rule

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

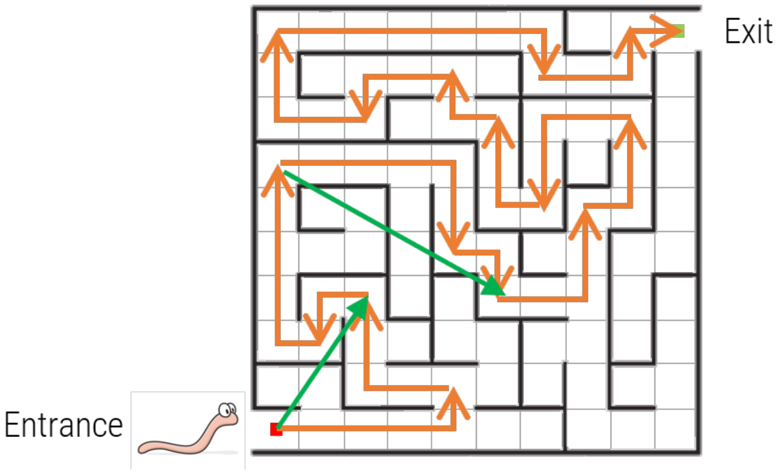

FIG. 11. Solution

Taking the sum of two consecutive moves, would result in moving the net distance in one move (except it being illegal!), as shown for the two green vectors shown:

$$(4+i)+(-2+2i)=2+3i$$

$$(4-2i)+(1-i)=5-3i$$

c) Add up all the steps using this addition rule – what do you find?

The sum of all the steps should result in $9+9i$, which is the single move to go directly from entrance to exit.

d) A new species of worm is discovered that, amazingly, can jump a certain distance r at an angle θ, which is communicated by the mathematical instruction $re^{i\theta}$ (This species of worm only understands radians, not degrees!). If this worm follows the same path as the worm of the previous species, what are the set of instructions you would need to supply? (Hint: use your trigonometric skills! And you need only provide answer to 1 d.p.)

See table above: for each move in part a) which is $x+iy$, $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}(y/x)$

e) If the worm jumps to the exit in 1 step what instruction would you provide? $r=\sqrt{9^2+9^2}=9\sqrt{2}$ and $\theta=\tan^{-1}(9/9)=\pi/4=0.8$ radians, so the instruction would be $9\sqrt{2}e^{i\pi/4}=12.7e^{0.8i}$.

QUESTION 3

Multiplication & rotation using complex numbers

a) Given intuition from the previous question (Q2) that i represents a rotation 90° ($\pi/2$ radians), what is $i\times i=i^2=?$ Hence, what is $\sqrt{-1}$?

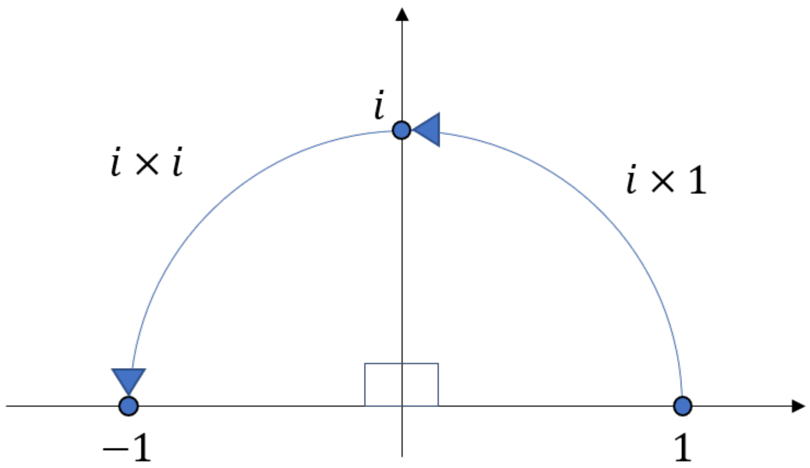

FIG. 12. i^2

If i rotates numbers by $\pi/2$ (i.e. $i = i \times 1$) then multiplying by i again should result in another rotation by $pi/2$ giving the number -1. $\Rightarrow i^2 = -1$ and hence $\sqrt{-1} = i$.

b) Given two complex numbers $z = a + ib$ and $w = c + id$, multiply them by using the normal rules of multiplying out brackets, but in addition, using the rule just discovered that $i^2 = -1$ to show

$$zw = ac - bd + i(ad + bc).$$

$$zw = (a + ib)(c + id) = ac + iad + ibc + i^2bd = ac - bd + i(ad + bc)$$

c) Let the complex number $u(\theta) = \cos\theta + i\sin\theta$. Calculate uz,uw & uv for $\theta = \pi/4$, using the complex number representation of each of the vectors below. What is the length of each vector and how does it change after multiplication by u? Sketch where uz, uw & uv lie on the x - y plane:

$$z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The complex number representation of each vector above is $z = 1$, $w = i$ and $v = \frac{1}{\sqrt{2}}(1 + i)$. Hence, multiplying by $u(\theta = \pi/4) = \cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}}(1 + i)$ gives:

$$uz = 1 \times \frac{1}{\sqrt{2}}(1 + i) = \frac{1}{\sqrt{2}}(1 + i)$$

(blue dot)

$$uw = i \times \frac{1}{\sqrt{2}}(1 + i) = \frac{1}{\sqrt{2}}(i + i^2) = \frac{1}{\sqrt{2}}(-1 + i)$$

(red dot)

$$uv = \frac{1}{\sqrt{2}}(1 + i) \times \frac{1}{\sqrt{2}}(1 + i) = \frac{1}{2}(1 + i)^2 = \frac{1}{2}(1 + 2i + i^2) = \frac{1}{2}(1 + 2i - 1) = i$$

(yellow dot)

The length of a vector with components x & y ($z = x + iy$) in complex number form) is $|z| = (x+iy)(x-iy) = \sqrt{x^2+y^2}$ (where the operation $z(i \to -i) = x - iy$ is called complex conjugation); using this we see that the length does not change under operation of uz:

$$|z| = |1| = 1$$

$$|w| = |i| = i \times -i = 1$$

$$|v| = |\frac{1}{\sqrt{2}}(1+i)| = \frac{1}{2}(1+i)(1-i) = \frac{1+1}{2} = 1$$

$$|uz| = |\frac{1}{\sqrt{2}}(1+i)| = \frac{1}{2}(1+i)(1-i) = \frac{1+1}{2} = 1$$

$$|uw| = |\frac{1}{\sqrt{2}}(-1+i)| = \frac{1}{2}(-1+i)(-1-i) = \frac{1+1}{2} = 1$$

$$|uv| = |i| = i \times -i = 1$$

(Note it is perfectly fine to calculate this using $|z| = \sqrt{x^2+y^2}$ by directly identifying the x and y components of the complex number representation).

Hence all the lengths are unchanged by multiplying by u.

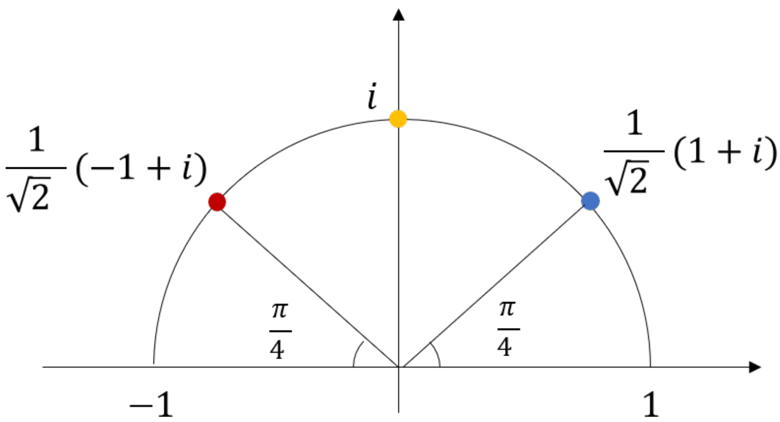

FIG. 13. uz

What does the complex number $u(\theta = \pi/4)$ do?

We see that $u(\theta = \pi/4)$ rotates points in a plane by the angle $\pi/4$ anticlockwise without changing it's distance from the origin. In general, $u(\theta)$ rotates points in the plane by the angle θ.

d) Complex numbers provide a simple and compact way of describing rotations, in particular in their polar-form $re^{i\theta}$; the connection between the polar form of a complex number of it's Cartesian form is found by Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{7}$$

Use Euler's formula to show that Eqn.3 can be expressed as follows

$$z(t) = Re^{i(\omega t - \phi)} \tag{8}$$

where $z(t) = x(t) + iy(t)$, and ϕ is as given by Eqn.5.

Treating $x(t)$ and $y(t)$ as components of a vector, its complex number representation is $z(t) = x(t) + iy(t) = R\cos(\omega t - \phi) + iR\sin(\omega t - \phi)$. Using Euler's formula, by direct correspondence, we get

$$z(t) = R(\cos(\omega t - \phi) + i\sin(\omega t - \phi)) = Re^{i(\omega t - \phi)}$$

e) By setting $t = 0$ show that this can be expressed explicitly in terms of the initial condition $z(0) = z_0 = x_0 + iy_0$, using your expressions for x_0 and y_0 derived in Q1d:

$$z(t) = z_0 e^{i\omega t} \tag{9}$$

$$z_0 = z(0) = R(\cos(\phi) + i\sin(-\phi)) = x_0 + iy_0$$

But using the polar representation:

$$z_0 = z(0) = Re^{-i\phi}$$

hence,

$$z(t) = Re^{i(\omega t - \phi)} = Re^{-i\phi}e^{i\omega t} = z_0 e^{i\omega t}$$

Note that this is equivalent to the following matrix equation, which will be studied in the next tutorial:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

f) Using your estimate of ω from Q1 use this last equation to predict the excess number of hares and lynxes at time $t = 15$ years, given at $t = 0$ there are an excess of 80 hares and 60 lynxes. Your answer should agree with Q1g.

$x_0 = 80$, $y_0 = 60$:

$$\Rightarrow z(t) = (80 + 60i)e^{0.6\times 15i} = (80 + 60i)(\cos(0.6 \times 15) + i\sin(0.6 \times 15)) = -97.618 - 21.698i \approx -98 - 22i$$

That is an excess of -98 hares and -22 lynxes, which agrees with the answer of Q1g.