Maximum Likelihood Estimation and Model Fitting, CMEE MSc. Tin-Yu Hui

Practical 2 (23 Feb 2021)

Question 1

i. If X and Y are independent, then the joint pdf of X and Y is the ______ of their marginal pdf.

ii. Given _____ and a statistical model MLE provides estimates to the _______ of interest.

iii. [Circle the correct answer] The likelihood function is a function of parameters/data.

Question 2 [Marginal and conditional distributions]

Given a pair of r.v. X and Y with their joint pdf $f_{XY}(x,y) = y\left(\frac{1}{2} - x\right) + x$, where $0 < x < 1$ and $0 < y < 2$.

i. Show that $f_{XY}(x,y)$ is a valid joint pdf.

ii. Find the two marginal distributions, $f_X(x)$ and $f_Y(y)$. What can you say about the mean and variance of X?

iii. Find the conditional distribution of X given $Y = y$.

iv. Now assume we know $y = 1/2$, what is the conditional distribution of X?

v. Calculate the conditional mean and variance of X, given $y = 1/2$.

Question 3 [Bivariate normal, entirely optional]

Given $\boldsymbol{X} = \binom{X_1}{X_2}$, and $X \sim MVN(\boldsymbol{\mu} = \binom{0}{0}, \boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix})$. The variance-covariance matrix $\boldsymbol{\Sigma}$ suggests that X_1 and X_2 both have variance of 1 and covariance ρ, $-1 < \rho < 1$. If $\rho > 0$ then the pair is positively correlated. If $\rho = 0$, then X_1 and X_2 are uncorrelated. Further, for multivariate normal r.v., $\rho = 0$ also implies independence.

vi. Show that the eigenvalues of $\boldsymbol{\Sigma}$ are $\lambda_1 = 1 + \rho$ and $\lambda_2 = 1 - \rho$. Please also find their associated unit eigenvectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

vii. Let $\boldsymbol{P} = [\boldsymbol{v_1}\ \boldsymbol{v_2}]$, show that the inverse of $\boldsymbol{P}$ is $\boldsymbol{P^T}$.

viii. Show that $\boldsymbol{P\Sigma P^T} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ is a diagonal matrix.

Decomposing $\boldsymbol{\Sigma}$ allows us to generate correlated multivariate normal r.v. from independent normal r.v.. Or, conversely, to "decorrelate" multivariate normal r.v. back into independent ones.

ix. Let $\boldsymbol{Z} = \binom{Z_1}{Z_2}$. $Z_1 \sim N(0, \lambda_1)$, $Z_2 \sim N(0, \lambda_2)$, and Z_1 and Z_2 are independent. Then $\boldsymbol{X} = \boldsymbol{P^T Z}$ will follow multivariate normal distribution with $\boldsymbol{\mu} = \binom{0}{0}$ and $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.

```
# VALUE OF rho
rho<-0.6
# GENERATE z1 AND z2, INDEPENDENTLY
z1<-rnorm(20000, mean=0, sd=sqrt(1+rho))
```

```
z2<-rnorm(20000, mean=0, sd=sqrt(1-rho))

# TRANSFORM (z1, z2) INTO (x1, x2), VIA P-TRANSPOSE
x1<-z1/sqrt(2)+z2/sqrt(2)
x2<-z1/sqrt(2)-z2/sqrt(2)

# SCATTER PLOT OF (x1, x2)? LOOKS LIKE AN ELLIPSE?
plot(x1, x2)

# THE SAMPLE VARIANCE AND COVARIANCE OF x1 AND x2?
var(x1)
var(x2)
cov(x1, x2)
```

x. Conversely, we can back transform the correlated $\boldsymbol{X}$ into independent normal r.v., via $\boldsymbol{P}$ (i.e. $\boldsymbol{Z} = \boldsymbol{P}\boldsymbol{X}$)

```
# LET'S REMOVE z1 AND z2 FROM OUR CURRENT R SESSION
rm(r1); rm(r2);

# WE BACK TRANSFORM (x1, x1) INTO (z1, z2), VIA P
z1<-x1/sqrt(2)+x2/sqrt(2)
z2<-x1/sqrt(2)-x2/sqrt(2)

# THE COVARIANCE BETWEEN z1 AND z2 SHOULD BE CLOSE TO ZERO
var(z1)
var(z2)
cov(z1, z2)
cor.test(z1, z2)
```

Question 4

i. $X_1, X_2, \ldots, X_n$ follow i.i.d. $Exponential(\lambda)$. What is the likelihood function $L(\lambda)$?

ii. Please also find the log-likelihood function $l(\lambda)$.

iii. Find $\lambda = \hat{\lambda}$ such that the log-likelihood function is maximised.

<u>Question 5</u>

i. Let $X_1, X_2, \dots, X_n$ be i.i.d. $Poisson(\lambda)$. Find the MLE for λ.

ii. If I observed 5, 3, 2, and 6 events (independently), all within a fixed period of time, what would be the best guess for λ?

Question 6

The exercise left to you in class. Let $X_1, X_2, \dots, X_n$ be i.i.d. $N(\mu, \sigma^2)$, both μ and σ^2 are not known. Find the MLE for the two parameters. (Hints: Write down the likelihood and log-likelihood function. Differentiate (partially) the log-likelihood with respect to the two parameters. Set the derivatives to zero, and solve for the unknowns, …)

Question 7 [Linear regression exercise]

It is a spin-off exercise from the original marked-recapture experiment for census population size estimation. We measured the difference in their body lengths and how long (in days) they had been hanging around before falling back into our hands. We would like to investigate the relationship between the two variables. There is a short note on the use of `optim()` in today's presentation. You will need the dataset `recapture.csv`