San José State University
Department of Computer Science

Ahmad Yazdankhah
ahmad.yazdankhah@sjsu.edu
www.cs.sjsu.edu/~yazdankhah

Formal Languages

(Part 1)

Lecture 04

Day 04/31

CS 154

Formal Languages and Computability

Spring 2018

Agenda of Day 04

- Announcement
- Summary of Lecture 03
- Lecture 04: Teaching ...
  - Formal Languages (Part 1)

Announcement

- Our first quiz will be this Thursday!
  - Some of the questions are multiple choice.
  - So, please have Scantron 882 E.
  - If you forget, no problem at all! I'll sell it at:

$~~2~~0 Each **Just $19.95!**

- My office hours are changed:

Tuesdays and Thursdays, 7:15pm – 9:15pm

Summary of Lecture 03: We learned ...

Cartesian Products

- In many cases, we need ordered collections like $(x_1, x_2, \ldots, x_n)$.
- Cartesian product can produce ordered collections.
- The Cartesian product of A and B is ...
  - ... the set of all ordered pairs (a , b), where $a \in A \wedge b \in B$.

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

- Does Cartesian product have commutative property?
  - In general, no, but in the following special cases, yes:
  - $(A = B) \vee (A = \phi) \vee (B = \phi)$

- We can extend the Cartesian product to 3 sets:

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

- Does Cartesian product have associative property?
- No, because:

$$(A \times B) \times C = \{((a, b), c) : a \in A, b \in B, c \in C\}$$

$$A \times (B \times C) = \{(a, (b, c)) : a \in A, b \in B, c \in C\}$$

Any question?

Summary of Lecture 03: We learned ...

Functions

- A function from D to R is ...
  - ... a rule that assigns to some elements of D (domain) a unique element of R (range).
- A total function is ...
  - ... a function that all of its domain elements are defined.
- A partial function is ...
  - ... a function that at least one member of its domain elements is "undefined".

Any question?

Summary of Lecture 03: We learned ...

Graphs

- A graph is a mathematical construct consisting of two sets:
  - A non-empty and finite set of vertices $V = \{v_1 , v_2 , \ldots , v_n\}$
  - A finite set of edges $E = \{e_1 , e_2 , \ldots , e_m\}$
- A walk is ...
  - ... a sequence of edges from v_i to v_n. $(v_i , v_j) , (v_j , v_k) , \ldots , (v_m , v_n)$
- The length of a walk is ...
  - ... the number of edges traversed.
- A path is ...
  - ... a walk that no edge is repeated.
- A simple path is ...
  - ... a path that no vertex is repeated.
- A loop is ...
  - ... an edge from a vertex to itself.
- A cycle is ...
  - ... a path from a vertex (called base) to itself.
- A simple cycle is ...
  - ... a cycle that no vertex other than base is repeated.

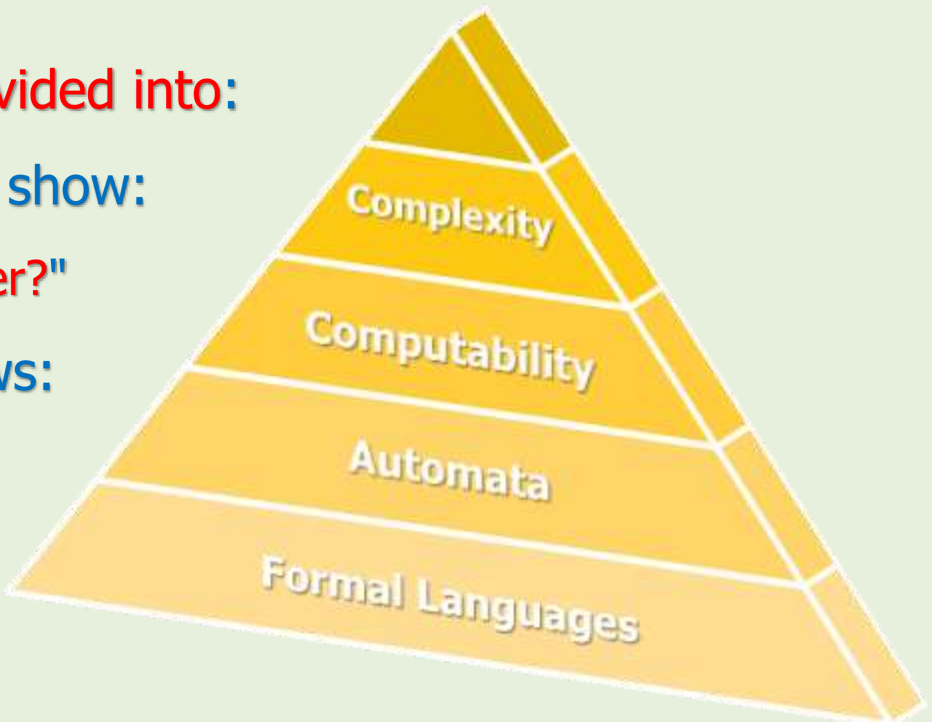
Any question?

Objective of This Lecture

- To review alphabets
- To review strings
- To introduce formal languages
- To examine some languages

Introduction: The Big Picture of the Course

- The foundation of the computer science is the "theory of computation".
- The theory of computation is divided into:
- The first three from the bottom show:
  - "What can be done with computer?"
- The forth one, complexity, shows:
  - "What can be done in practice?"



- Let's start with "Formal Languages"!
- First, we need to have a common understanding about "alphabet" and "strings".

Alphabets & Strings

Alphabets

Definition

- A "nonempty" and "finite" set of "symbols"
- An alphabet is denoted by Σ.

- Note that when we say "an alphabet", we mean a set of symbols.

- We usually (in this course) use lowercase letters a, b, c, ... for alphabets' symbols.
- In some cases, we use digits like 0, and 1 or other symbols as alphabets.

Alphabets Example

Example 1

- Σ = {a, b}   This is our celebrity alphabet!
- Σ = {0, 1}
- Σ = {ε, α, β}

- Can the following set be an alphabet?   Σ = {ЯЕ , ДК , Ѭ , پ , Γ }

Strings

Definition

- A "finite" sequence of symbols from the alphabet

Example 2

- Let $\Sigma = \{a, c, d, e, g, h, l, o, p, r, s, t, u\}$.
- Are these valid strings?
- cat , dog , horse , house , apple

Strings Examples

Example 3

- Let Σ = {a, b}.
- Are the following strings valid strings?
- baba , aabb , bbbbbbbbbbba , ...
- Not all of them!
  - "..." is not a valid string because "." is not in the alphabet!
  - Note that in "formal languages" arena, we don't care whether the strings are meaningful or not!

- We use lowercase letters w, u, v, ... for string variables.
- w = baba
- u = bbbbbbbbbbba

Strings Size (aka Length)

Definition

- The number of symbols in the string
- The size of string w is denoted by |w|.

Example 4

- |aaa| = 3
- |babba| = 5
- |aaba| = 4

- In general:

$$|a_1\ a_2\ \ldots\ a_n| = n$$

Empty String

Definition

- A string with no symbol
  - A sequence of zero symbols
- Empty string is denoted by λ (pronounced "lambda").
- The length of λ:

$$|\lambda| = 0$$

  - In some books, empty string might be shown as: ε (epsilon)

Strings Operations

Concatenation

Definition

- Concatenation of two strings u and v is the string uv.

Example 5

- Let u = aaba and v = bb ; uv = ?
- uv = aababb

- The length of concatenation:

$$|uv| = |u| + |v|$$

- λ is the neutral element for concatenation:

$$\lambda w = w\lambda = w$$

Example 6

- λaabb = aabλb = aλabb = aλabbλ = aabb

Reverse

Definition

- Reverse of a string w is obtained by writing the symbols in reverse order.
- Reverse of w is denoted by w^R. (pronounced "w reverse")
  - If $w = a_1 a_2 \ldots a_{n-1} a_n$, then $w^R = a_n a_{n-1} \ldots a_2 a_1$

Example 7

- Let $w = aaba$; $w^R = ?$
- $w^R = abaa$

Homework

- Prove that $(uv)^R = v^R u^R$

Substring

Definition

- Substring of a string w is any string of consecutive symbols of w.

Example 8

String	Substring
aababb	aa
aababb	ab
aababb	bab
aababb	b
aababb	λ

Prefix and Suffix

Definition

- Let w be a sting. If we can write w = uv, then ...
  - u is called "prefix".
  - v is called "suffix".

Example 9

- Let w = aababb
- If we consider u = aa as a prefix for w, then the rest would be the suffix.
  - v = babb.

- Are these the only prefix and suffix?

Prefix and Suffix

Example 9 (cont'd)

- The complete list of all possible prefixes and suffixes of w are:

Prefix = u	Suffix = v
λ	aababb
a	ababb
aa	babb
aab	abb
aaba	bb
aabab	b
aababb	λ

- So, λ is prefix and suffix of every string (NOT at the same time). because: w = λ w = w λ

Exponent Operator

Definition

- For a string w and a natural number n, w^n is defined as concatenation of n w's.

$$w^n = \underbrace{w\ w\ w\ \dots\ w}_{n \text{ times}}$$

Example 10

- Let w = aaba ; $w^2 = ?$
- $w^2 = w\ w$ = aabaaaba
- $w^3 = w\ w\ w$ = aabaaabaaaba
- In general: $w\ w^n = w^n\ w = w^{n+1}$
- Where $n \in \mathbb{N}$ (natural numbers)

Special case

- $w^0 = ?$
- $w^0 = \lambda$
- How can you prove this?

Example 11

- $(aaba)^0 = \lambda$
- Note that $aaba^0 = aab$

Formal Languages

Introduction of Two New Operations on Sets

- Before introducing "formal languages", we need to introduce two new operations on sets.

- We did not mention them because we needed the concept of concatenation.

Star Operator on Alphabets

Definition

- Let Σ be an alphabet.
- Σ^* is the set of "all possible strings" obtained by concatenating "zero or more" symbols from Σ.

Example 12

- Let $\Sigma = \{a\}$; $\Sigma^* = ?$
- $\Sigma^* = \{a\}^* = \{\lambda, a, aa, aaa, aaaa, \ldots\}$

Star Operator on Alphabets

Example 13

- Let $\Sigma = \{a, b\}$; $\Sigma^* = ?$
- $\Sigma^* = \{a, b\}^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, ...\}$

- Note what strategy we used to enumerate all combinations?

- Let $\Sigma = \{a, b, c\}$; $\Sigma^* = ?$

Plus Operator on Alphabets

Definition

- Let Σ be an alphabet.
- Σ^+ is the set of "all possible strings" obtained by concatenating "one or more" symbols from Σ.

Example 14

- Let $\Sigma = \{a\}$; $\Sigma^+ = ?$
- $\Sigma^+ = \{a\}^+ = \{a, aa, aaa, aaaa, ...\}$

Plus Operator on Alphabets

Example 15

- Let $\Sigma = \{a, b\}$; $\Sigma^+ = ?$
- $\Sigma^+ = \{a, b\}^+ = \{a, b, aa, ab, ba, bb, aaa, aab, ...\}$

- Note that the only difference between Σ^+ and Σ^* is that Σ^+ does NOT contain λ.
- Hence:

$$\Sigma^+ = \Sigma^* - \{\lambda\}$$

$$\Sigma^* = \Sigma^+ \cup \{\lambda\}$$

- Also note that Σ is finite but both Σ^+ and Σ^* are infinite.

Formal Languages Definition

Definition

- Let Σ be an alphabet.
- Any subset of Σ^* is called a "formal language" over Σ.

- Σ^* is called the "universal formal language" over Σ.

Formal Languages Definition

Example 16

- Let $\Sigma = \{a, b\}$ be an alphabet.
- Then:
- $\Sigma^* = \{a, b\}^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, ...\}$

- The following sets are examples of formal languages.
- $L_1 = \{a, b, aa, aab\}$ because $L_1 \subseteq \Sigma^*$
- $L_2 = \{\lambda, ba, bb, bbb, aaa, aab\}$ because $L_2 \subseteq \Sigma^*$

Formal Languages

Example 16 (cont'd)

- How about the following sets? Are they languages? Why?
- $L_3 = \phi = \{\ \}$
- $L_4 = \{\lambda\}$

Two Special Languages

- Empty Language : $\{\ \}$ or ϕ
- Language with Empty String : $\{\lambda\}$

Formal Languages Notes

1. For simplicity, we use "language" to refer to the "formal language".
   – To refer natural languages, we specifically mention "natural" word.

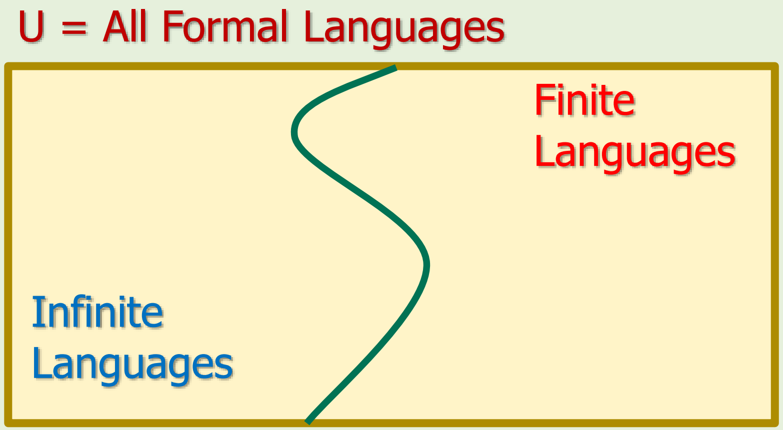
2. A language is a "set". So, it has all properties of sets.

3. $\{\lambda\}$ is a language while λ is a string.
   – $|\lambda| = 0$; This is the size of the string λ.
   – $|\phi| = |\{\ \}| = 0$; $|\{\lambda\}| = 1$; These are the sizes of languages.

Formal Languages Notes

4. In some books, "strings" are called "sentences" to analogize the formal languages with the natural languages.
   – In this course, we mostly use "strings"!

5. Like sets, we have both "finite" and "infinite" languages.
   – This is our first categorization of formal languages.



Formal Languages Exercises

Example 17

- Given the following languages by set-builder method over $\Sigma = \{a, b\}$.
- Represent them by using roster method (enumerate the strings):

- $L_1 = \{a^n b^n : n \geq 0\}$

This is our celebrity language!

- $L_2 = \{a^n b^{2n} : n \geq 0\}$
- $L_3 = \{a^{n+2} b^n : n \geq 0\}$
- $L_4 = \{a^n b^m : n \geq 0, m \geq 0\}$

References

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3. Michael Sipser, "Introduction to the Theory of Computation, 3rd ed.," CENGAGE Learning, United States, 2013 ISBN-13: 978-1133187790