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Formal Languages

(Part 2)

Lecture 05

Day 05/31

CS 154

Formal Languages and Computability

Spring 2018

Agenda of Day 05

- Summary of Lecture 04
- Quiz 1
- Lecture 05: Teaching ...
  - Formal Languages (Part 2)

Summary of Lecture 04: We learned ...

Alphabets & Strings

- Alphabet is ...
  - ... a nonempty and finite set of symbols, denoted by Σ.
- String is ...
  - ... a finite sequence of symbols from the alphabet.
- Length of string w is ...
  - ... the number of symbols in the string, denoted by $|w|$.
- Empty string is ...
  - ... A string with no symbol, denoted by λ
  - $|\lambda| = 0$

Operations on Strings

- Concatenation of u and v is uv.
  - $\lambda w = w\lambda = w$ (neutral element)
- Reverse of w is denoted by w^R.
- Substring
- Prefix and Suffix
  - $w = uv$, u=prefix, v=suffix
  - λ is suffix and prefix of every string because: $w = \lambda\, w = w\, \lambda$
- Exponent operator
  - $w^n = w\, w\, w \ldots w$
  - $w\, w^n = w^n\, w = w^{n+1}$
  - $w^0 = \lambda$

Summary of Lecture 04: We learned ...

Formal Languages

- Star operator: Σ^*
  - The set of all possible strings obtained by concatenating zero or more symbols from Σ.
  - Universal set of all strings over Σ.
- Plus operator: Σ^+
  - The set of all possible strings obtained by concatenating one or more symbols from Σ.
  - $\Sigma^+ = \Sigma^* - \{\lambda\}$
  - $\Sigma^* = \Sigma^+ \cup \{\lambda\}$
- Formal language is ...
  - ... any subset of Σ^*
- Special cases:
  - $\{\ \}$ and $\{\lambda\}$

- Formal languages are sets, so, they have all sets properties.
- Formal languages can be finite or infinite.

U = All Formal Languages

Any question?

NAME	Alan M. Turing		
SUBJECT	CS 154	TEST NO.	1
DATE	02/08/2018	PERIOD	1, 2, 3

TEST RECORD	
PART 1	123
PART 2	
TOTAL	

Your list # goes here!

Quiz 1

Use Scantron

Operations on Languages

The Regular Set Operations

- Since languages are sets, we can apply all regular set operations on them.

Union

- {a, aa, ab} ∪ {a, ab, bbb, bba, b} = {a, aa, ab, bbb, bba, b}

Intersection

- {a, aa, ab} ∩ {a, ab, bbb, bba, b} = {a, ab}

Minus

- {a, aa, ab} - {a, ab, bbb, bba, b} = {aa}

Complement of Languages

Definition

- Let L be a language over a given alphabet Σ.
- Complement of L, denoted by $\overline{L}$, is defined as:

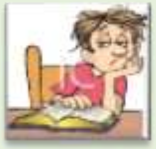
$$\overline{L} = U - L = \Sigma^* - L$$

Example 18

- Let L = {λ, b, aa, aab} over Σ = {a, b} ; $\overline{L} = ?$
- $\overline{L} = \Sigma^* - L$
- $\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$
- $\overline{L} = \{\cancel{\lambda}, a, \cancel{b}, \cancel{aa}, ab, ba, bb, aaa, \cancel{aab}, \ldots\}$

  $= \{a, ab, ba, bb, aaa, \ldots\}$

Homework



- Given the following languages over $\Sigma = \{a, b\}$
  a. Represent L by set builder
  b. Find $\overline{L}$ and represent it by set builder

1. Set of all strings that contains at least one a
2. Set of all strings that contains more than one a
3. Set of all strings that contains exactly one a

Reverse of Languages

Definition

- Let L be a language over a given alphabet Σ.
- Reverse of L, denoted by L^R, is defined as:

$$L^R = \{w : w^R \in L\}$$

Example 19

- Let L = {b, ab, aab, abab} ; L^R = ?
- L^R = {b, ba, baa, baba}

Example 20

- Let $L = \{a^n b^n : n \geq 0\}$; L^R = ?
- $L^R = \{b^n a^n : n \geq 0\}$

Concatenation of Languages

Definition

- Let L_1 and L_2 be two languages over two alphabets Σ_1 and Σ_2.
- The concatenation of L_1 and L_2, denoted by L_1L_2, is defined as:

$$L_1L_2 = \{xy : x \in L_1, y \in L_2\} \text{ over } \Sigma = \Sigma_1 \cup \Sigma_2$$

Example 21

- Let $L_1 = \{a, ab\}$ and $L_2 = \{b, ba, baa\}$; $L_1L_2 = ?$
- $L_1L_2 = \{a, ab\} \{b, ba, baa\}$

$$= \{ab, aba, abaa, abb, abba, abbaa\}$$

Concatenation Notes

1. The concatenation of two languages looks like Cartesian product of two sets.
   - Instead of ordered-pair, we concatenate two strings.
2. $\phi\ L = L\ \phi = \phi$ (prove it!)
   - ϕ has the same role as 0 (zero) for multiplication.
3. $\{\lambda\}\ L = L\ \{\lambda\} = L$
   - $\{\lambda\}$ has the same role as 1 (one) for multiplication.
   - $\{\lambda\}$ is the neutral language for concatenation operation of languages.

Exponential Operation

Definition

- For a language L and a natural number n, L^n is defined as:

$$L^n = \underbrace{L\ L\ \ldots\ L}_{\text{n times}}$$

Example 22

Let $L = \{a, ab\}$; $L^2 = ?$; $L^3 = ?$

$L^2 = \{a, ab\}\ \{a, ab\}$

$= \{aa, aab, aba, abab\}$

$L^3 = L\ L^2 = \{a, ab\}\ \{aa, aab, aba, abab\}$

$= \{aaa, aaab, aaba, aabab, abaa, abaab, ababa, ababab\}$

- In general: $L\ L^n = L^n\ L = L^{n+1}$
- Where $n \in \mathbb{N}$ (natural numbers)

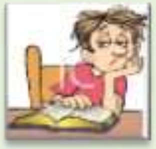
Exponential Operation

Example 23

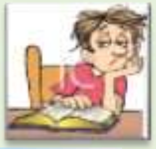
- Let $L = \{a^n b^n : n \geq 0\}$; $L^2 = ?$
- $L^2 = \{a^n b^n a^m b^m : n \geq 0, m \geq 0\}$

- Note that n and m are independent.
- For example ab aabb ($n=1$, $m=2$) belongs to L^2.

- How about $L^3 = ?$



Homework



Special cases

- $L^0 = ?$
- $L^0 = \{\lambda\}$ (prove it!)
- $\phi^0 = ?$

Optional

- For the following problems, we might use "prove by induction".
- $\{\lambda\}^n = \{\lambda\}$
- $\phi^n = ?$ for $n \geq 1$

Star-Closure Operation

Definition

- Let L be a language over a given alphabet Σ.
- Star-closure of L is defined as:

$$L^* = L^0 \cup L^1 \cup L^2 \cup \ldots$$

- Star-closure is also known as Kleene closure (pronounce it clay-knee) or Kleene star.
  - We prefer star-closure.
- The star-closure of a language L consists of all strings that can be formed by concatenating zero or more strings from L.

Star-Closure Operation

Example 24

- Let L = {a, ab} over Σ = {a, b} ; $L^* = ?$

- $L^* = L^0 \cup L^1 \cup L^2 \cup \ldots$
- $L^0 = \{\lambda\}$
- $L^1 = \{a, ab\}$
- $L^2 = \{aa, aab, aba, abab\}$
- $L^3 = \{aaa, aaab, aaba, aabab, abaa, abaab, ababa, ababab\}$
- $L^4 = \ldots$
- $L^* = \{\lambda, a, ab, aa, aab, aba, abab,$
  $aaa, aaab, aaba, aabab, abaa, abaab, ababa, ababab, \ldots\}$

Positive-Closure Operation

Definition

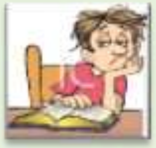
- Let L be a language over a given alphabet.
- Positive-closure of L is defined as:

$$L^{+} = L^{1} \cup L^{2} \cup \ldots$$

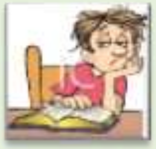
- L^{+} does not contain L^{0}. So, if we add L^{0} to L^{+}, we get L^{*}:

$$L^{*} = L^{0} \cup L^{+}$$

- Is the following statement correct?
- $L^{+} = L \; L^{*}$
- How can you support your answer?



Homework



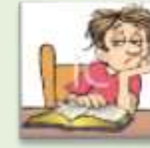
- Let L = {a, ab, bb} over Σ = {a, b}.
- $L^* = ?$
- $L^+ = ?$

- Let L = {λ, a, ab} over Σ = {a, b}.
- $L^* = ?$
- $L^+ = ?$

Special Cases

- $\phi^* = ?$
- $\{\lambda\}^* = ?$

Homework



- Enumerate at least 5 elements of the following languages:
  1. $L = \{w \in \{a, b\}^+\}$
  2. $L = \{w \in \{a, b\}^+ : |w| = 2k, K \geq 0\}$
  3. $L = \{w \in \{a, b\}^+ : |w| = 2k+1, K \geq 0\}$
  4. $L = \{1^{2k} : k \geq 1\}$ over $\Sigma = \{1\}$
  5. $L = \{w \in \{a, b\}^+ : n_a(w) = n_b(w)\}$ //number of a's = number of b's
  6. $L = \{a^n b^n c^n : n \geq 1\}$
  7. $L = \{a^n b^m c^{nm} : n, m \geq 1\}$
  8. $L = \{w\#w : w \in \{a, b\}^+\}$
  9. $L = \{w \in \{a, b\}^+ : |w| = 2k+1, K \geq 0$, w contains at least one a$\}$
  10. $L = \{ww : w \in \{a, b\}^+\}$

Surprising Languages

Surprising Languages Examples

In computer Science, all data are strings!

Example 25 : Natural Numbers

- $\mathbb{N}$ = {0, 1, ..., 123, ..., 456, ..., 5908764, ...}
- Σ = {0, 1, ..., 9}

Example 26 : Binary⁺ Numbers

- Σ = {0, 1}
- B = { 0, 1, ..., 1010, ..., 10000001, ..., 111100001, ...}

Example 27 : Unary Numbers

This is our celebrity numbers!

- Σ = {1}
- A = {1, 11, 111, 1111, 11111, ...}
- Equivalent integer numbers: 1, 2, 3, 4, 5, ...

Surprising Languages Examples

Example 28: Prime Numbers

- $\Sigma = \{0, 1, 2, ..., 9\}$
- $L = \{2, 3, 5, 7, 11, 13, 17, ...\}$

Example 29: Even and Odd Numbers

- $\Sigma = \{0, 1, 2, ..., 9\}$
- $L_1 = \{0, 2, 4, 6, 8, ...\}$
- $L_2 = \{1, 3, 5, 7, 9, ...\}$

Surprising Languages Examples

Example 30: Addition of Unary Numbers

- $\Sigma = \{1, +, =\}$
- $L = \{1^n+1^m=1^{n+m} : n \geq 1, m \geq 1\}$

- **Membership:** L contains strings such as:

  1+11=111

  11+111=11111

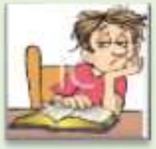
  …

- **Not Membership:** L doesn't contain strings such as:

  $1+11=1 \notin L$

  $11+111=11 \notin L$

Homework



Square of Unary Numbers

- $\Sigma = \{1, \#, =\}$
- $L = \{1^n\#{=}1^k : k = n^2, n \geq 1\}$

- **Membership:** L contains strings such as:

  ??

- **Not Membership:** L doesn't contain strings such as:

  ??

References

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