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Mathematical Preliminaries

(Part 2)

Lecture 03

Day 03/31

CS 154

Formal Languages and Computability

Spring 2018

Agenda of Day 03

- Waiting List Enrollment ...
- Summary of Lecture 02
- Lecture 03: Teaching ...
  - Mathematical Preliminaries (Part 2)

Summary of Lecture 02: We learned ...

Sets

- A set is ...
  - ... a collection of distinct elements.
- A list is ...
  - ... a collection of ordered elements.
- A set is known when its boundary is clearly defined.
- Universal set of a set is ...
  - ... the set containing all possible elements under consideration.
- Three methods to represent sets ...
  - Roster method
  - Venn diagram
  - Set builder

- The power set of the set S is ...
  - ... the set of all subsets of S.
  - It is denoted by 2^S.
  - $|2^S| = 2^{|S|}$
- A set is finite if ...
  - ... its size is a natural number.
- A set is infinite if ...
  - ... we cannot express its size by a natural number.

Any question?

Mathematical Preliminaries

Recap from Math 42

Cartesian Products

Motivation

- Recall that in sets, "order of elements" does NOT matter.
- But in practice, we do need "ordered collections".
- As we said before, in computer science we use "Lists" for ordered collections.
- The question is how we can mathematically model lists (ordered collections)?

Introduction

- Mathematicians defined a new mathematical structure called "n-tuple".
- An n-tuple is denoted by $(a_1 , a_2 , \ldots , a_n)$.
  - A special case of n-tuple is 2-tuple aka "ordered pair" (a_1 , a_2).
- We use a mathematical operation called "Cartesian product" to create n-tuples.
- This operation is named after the great French philosopher, mathematician, and physicist René Descartes (1596-1650).

Cartesian Products Definition

Definition

- Let A and B be two sets.
- The Cartesian product of A and B is the set of all ordered pairs (a , b), where $a \in A \wedge b \in B$.
- Cartesian product of A and B is denoted by A x B.

- Definition of Cartesian product by using set builder:

$$A \times B = \{(a , b) : a \in A \wedge b \in B\}$$

Cartesian Products Examples

Example 24

- Let $A = \{0, 1\}$, $B = \{3, 6, 9\}$; $A \times B = ?$
- $\{0, 1\} \times \{3, 6, 9\} = \{(0, 3), (0, 6), (0, 9), (1, 3), (1, 6), (1, 9)\}$

Example 25

- Let $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a , b\}$; $Q \times \Sigma = ?$

Example 26

- Let $Q = \{q_0, q_1\}$, $\Sigma = \{a , b\}$; $Q \times (\Sigma \cup \{\lambda\}) = ?$
  - "λ" is pronounced "lambda".

Cartesian Products Notes

- What is the result of the following Cartesian product?

  $A = \{1, 2\}$, $B = \phi$; $A \times B = ?$

  $A \times B = \phi$

  – In fact, the result of Cartesian product would be ϕ if one of the sets is ϕ.

- Prove it!

- Is this a true statement: $A \times B = B \times A$

  Does Cartesian product have commutative property?

  In general, No!

  It means: $A \times B \neq B \times A$

- But in the following special conditions, they can be equal:

  $A \times B = B \times A$ iff $(A = B) \vee (A = \phi) \vee (B = \phi)$

Cartesian Products Notes

- We can extend the Cartesian product to 3 sets:

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

- But if you associate the sets, we'll get:

$$(A \times B) \times C = \{((a, b), c) : a \in A, b \in B, c \in C\}$$

$$A \times (B \times C) = \{(a, (b, c)) : a \in A, b \in B, c \in C\}$$

- Therefore, Cartesian product does NOT honor association rule.

$$(A \times B) \times C \neq A \times (B \times C)$$

- In this course, we don't associate the sets.

Cartesian Products Extension

- We can extend the idea to n sets and define n-tuple as:

$$S_1 \text{ x } S_2 \text{ x } \ldots \text{ x } S_n = \{(x_1, x_2, \ldots, x_n) : x_1 \in S_1, \ldots, x_n \in S_n\}$$

Homework

- Let $Q = \{q_0, q_1\}$, $\Sigma = \{a , b\}$, $\Gamma = \{x , y\}$
- $Q \text{ x } (\Sigma \cup \{\lambda\}) \text{ x } \Gamma = ?$

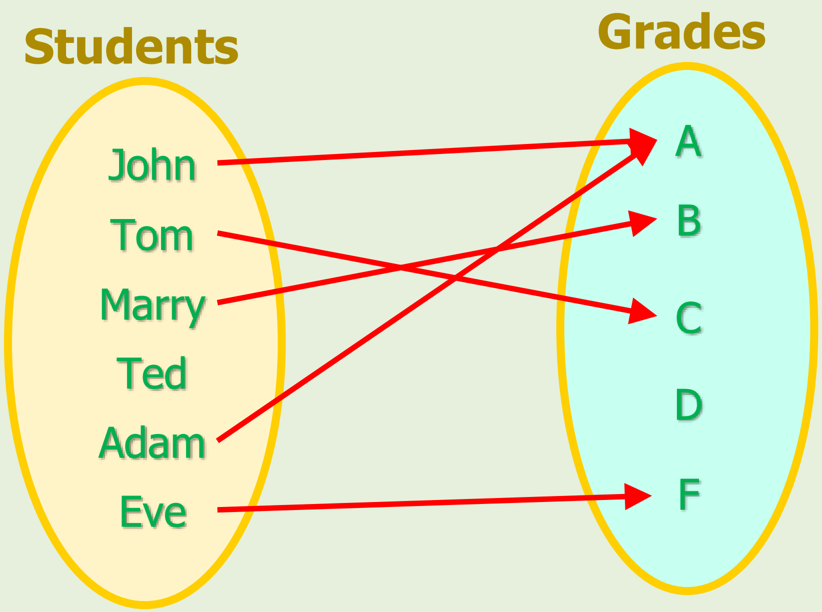
Mathematical Preliminaries

Recap from Math 42

Functions

Introduction

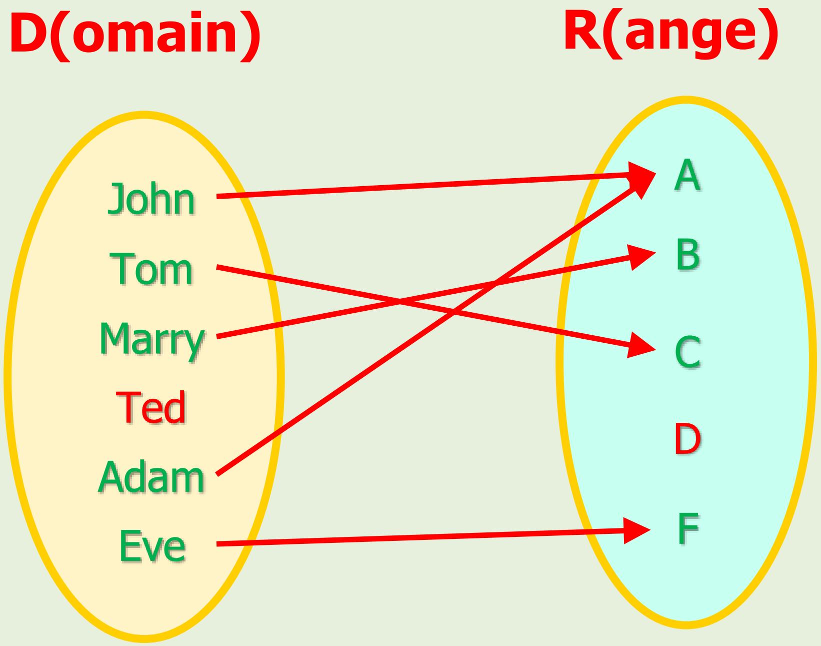
- In many situations in real life, there is a relationship between two sets.
- For example, we assign a letter grade to each student of a class.



- This relationship is an example of the concept of "function".

Functions Definition

- Let D and R be two sets.
- A function from D to R is a rule that assigns (or maps) to some (could be all) elements of D a "unique element" of R.
  - The set D is called the "domain" of the function f.
  - The set R is called the "range" of the function f.



- In the previous example:
- Domain is the set of students
- Range is the set of letter grades

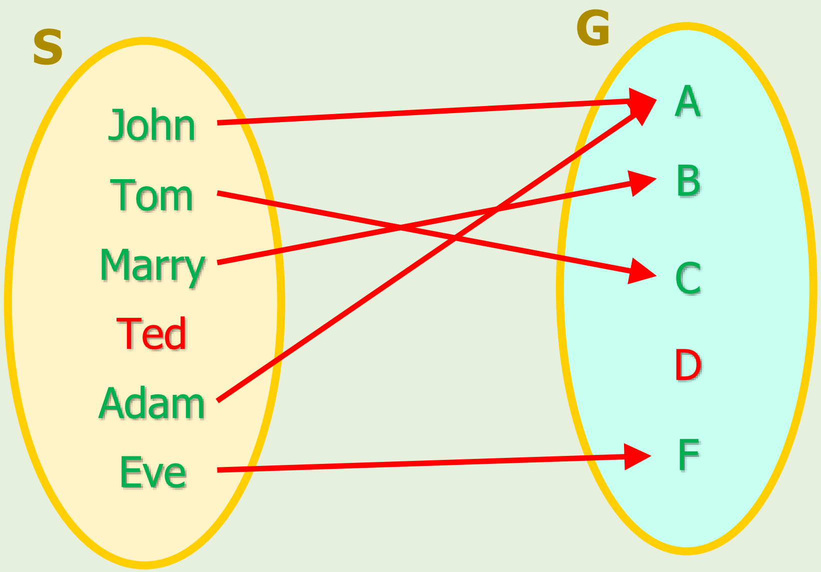
Functions Naming and Notation

- We usually name a function by lower-case letters such as f, h, δ (pronounced "delta"), etc.
- For example, the function δ is denoted by: $\delta : D \rightarrow R$

Example 27

$S = \{John, Tom, Marry, Ted, Adam, Eve\}$, $G = \{A, B, C, D, F\}$

$f : S \rightarrow G$

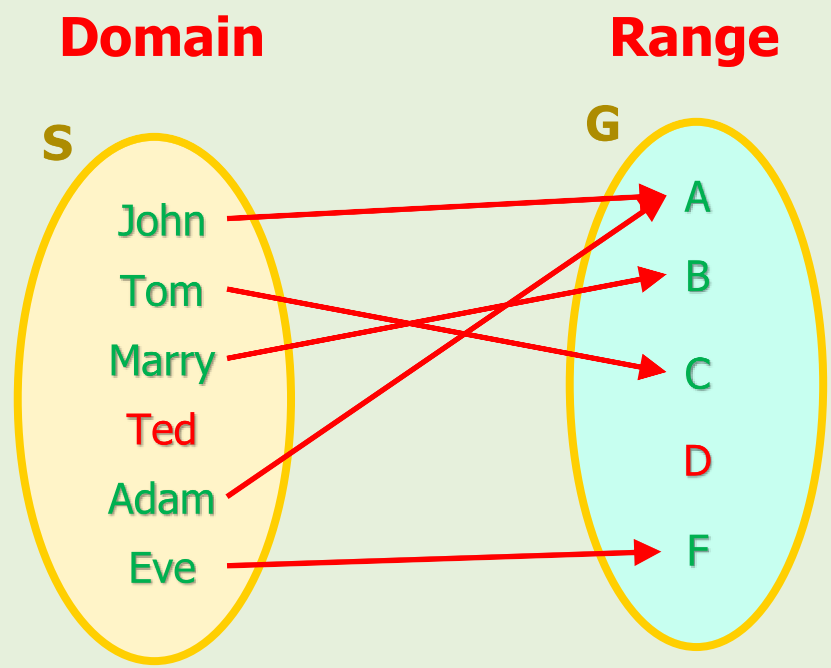


- What is the rule of the function shown by the figure?

$$\begin{cases} f(John) = A \\ f(Tom) = C \\ f(Marry) = B \\ f(Adam) = A \\ f(Eve) = F \end{cases}$$

Functions Notes

- f(Ted) = ?
- Undefined

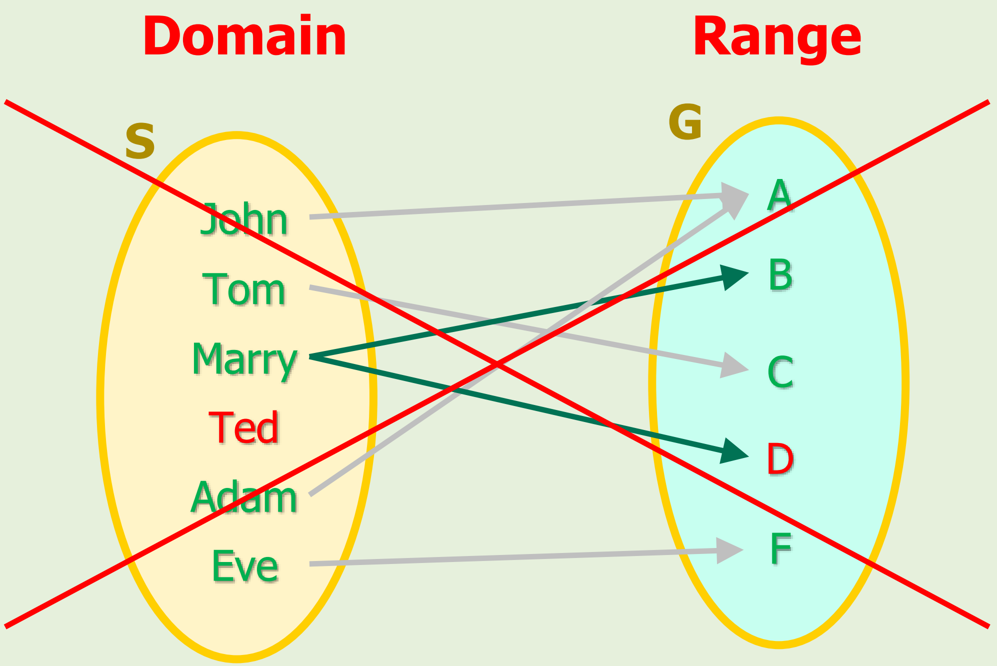


- So, it is possible to have some elements in the domain that is NOT mapped to any value of the range. (e.g. Ted in the domain)
- Also, it is possible to have some elements in the range that is NOT assigned by any value of the domain. (e.g. D in the range)

Functions Notes

- Is it possible for Marry to have two grades at the same time?
- Definitely, NO.

  In this universe, it cannot happen.

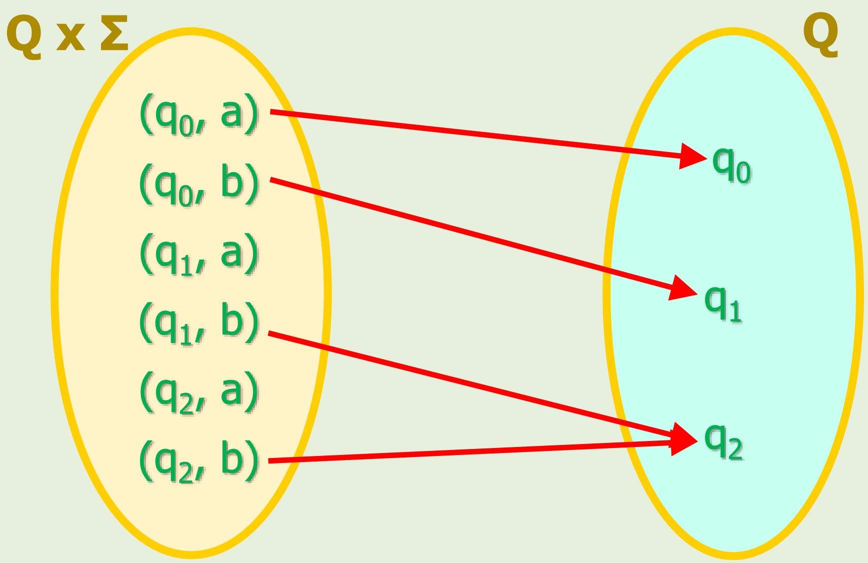


- That's why, in the definition of function, we said some elements of the domain are uniquely mapped to an element of the range.
- In other words, if there is a mapping, it should be unique.

Functions

Example 28: Mixing Cartesian Product and Function

- Let $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a , b\}$, $\delta : Q \times \Sigma \to Q$
- What is the domain and range of δ?
- Domain: $Q \times \Sigma = \{q_0, q_1, q_2\} \times \{a , b\} =$
  $\{(q_0, a), (q_0, b), (q_1, a), (q_1, b), (q_2, a), (q_2, b)\}$
- Range: $\{q_0, q_1, q_2\}$
- The rule of δ is shown in the following figure. Write the rule using algebraic notation.



- Rule of function δ:

$$\begin{cases} \delta(q_0, a) = q_0 \\ \delta(q_0, b) = q_1 \\ \delta(q_1, b) = q_2 \\ \delta(q_2, b) = q_2 \end{cases}$$

Homework

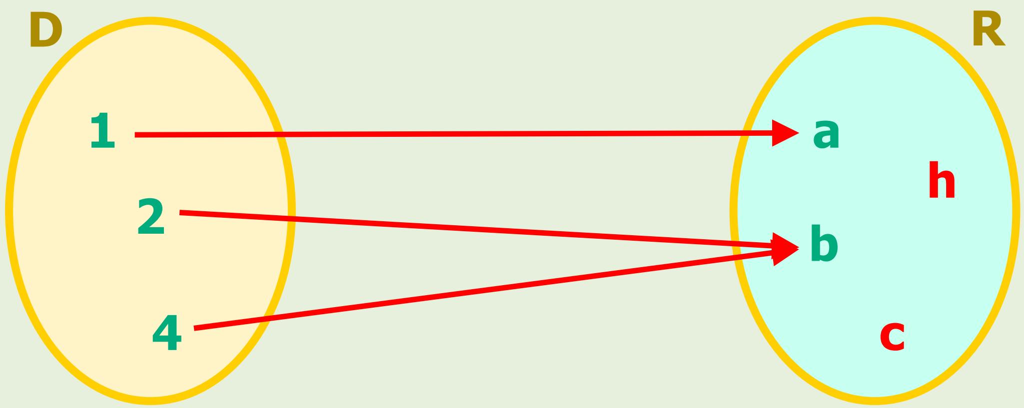
- Let $Q = \{q_0, q_1\}$, $\Sigma = \{a\}$, $\Gamma = \{x\}$, $\delta : Q \times \{\Sigma \cup \{\lambda\}\} \times \Gamma \rightarrow Q$
- What is the domain and range of δ?

Total Function

- A function is called "total function" if all of its domain elements are defined.

Example 29

- The following function is "total function" because all domain elements are defined.

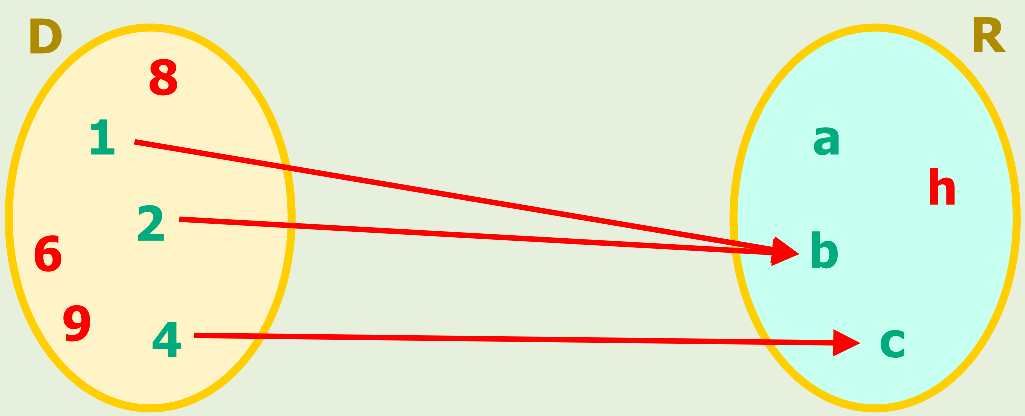


Partial Function

- If there exist at least one element in the domain of a function that is undefined, then the function is called "partial function".

Example 30

- The following function is "partial function" because at least one element of the domain is undefined:
- f(8) = Undefined



Mathematical Preliminaries

Recap from Math 42

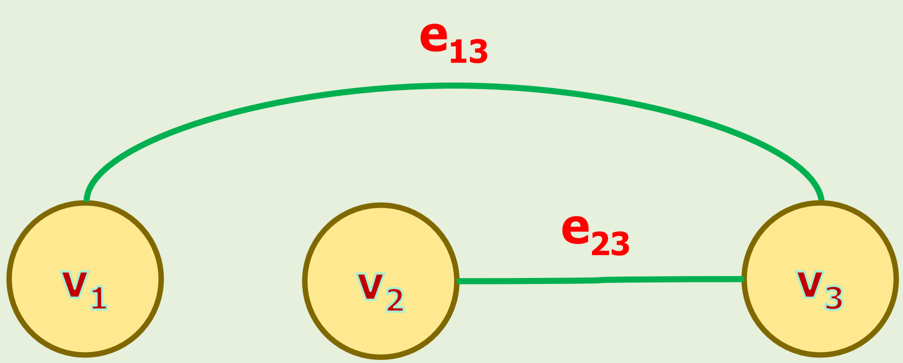
Graphs

Graph Definition

- A graph is a "mathematical construct" consisting of two sets:
  - A non-empty and finite set of vertices $V = \{v_1 , v_2 , \dots , v_n\}$
  - A finite set of edges $E = \{e_1 , e_2 , \dots , e_m\}$
  - Each edge connects two vertices.

Example 31

- $V = \{v_1 , v_2 , v_3\}$
- $E = \{e_{13} , e_{23}\}$

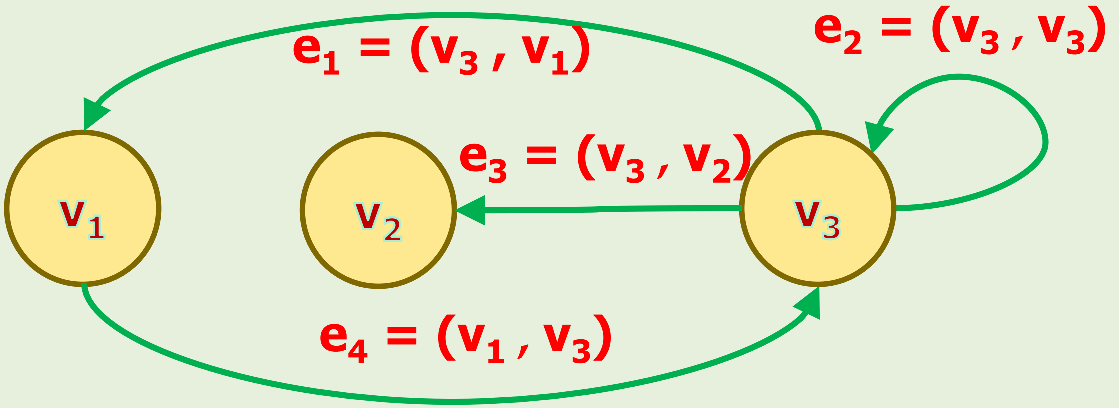


Directed Graph (aka Digraph)

- If the direction of the edges matters, then we call the graph "directed graph".
- The edges are shown by "ordered pairs" (start , end).
  - In this course, we only use directed graphs.

Example 32

- Draw a digraph with the following specifications:
  - $V = \{v_1 , v_2 , v_3\}$, $E = \{(v_1 , v_3), (v_3 , v_1), (v_3 , v_2), (v_3 , v_3)\}$



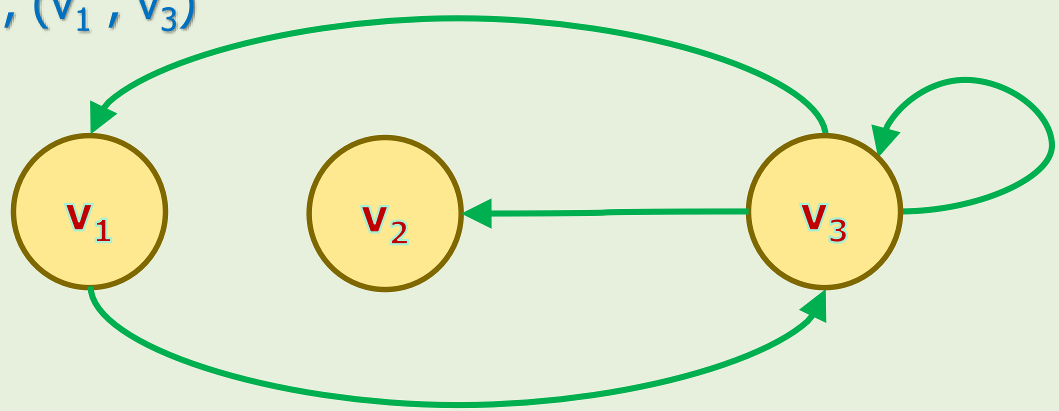
Graphs Terminologies

Walk

- A sequence of edges like (v_i , v_j) , (v_j , v_k) , … , (v_m , v_n) , is called a walk from v_i to v_n.
  - Note that the end vertex of e_i is the start vertex of e_{i+1}.
  - In other words, in a walk we cannot jump!

Example 33

- Each of the following sequences are a walk from v_1 to v_3:
  - Walk 1: (v_1 , v_3)
  - Walk 2: (v_1 , v_3) , (v_3 , v_3)
  - Walk 3: (v_1 , v_3) , (v_3 , v_1) , (v_1 , v_3)
  - …

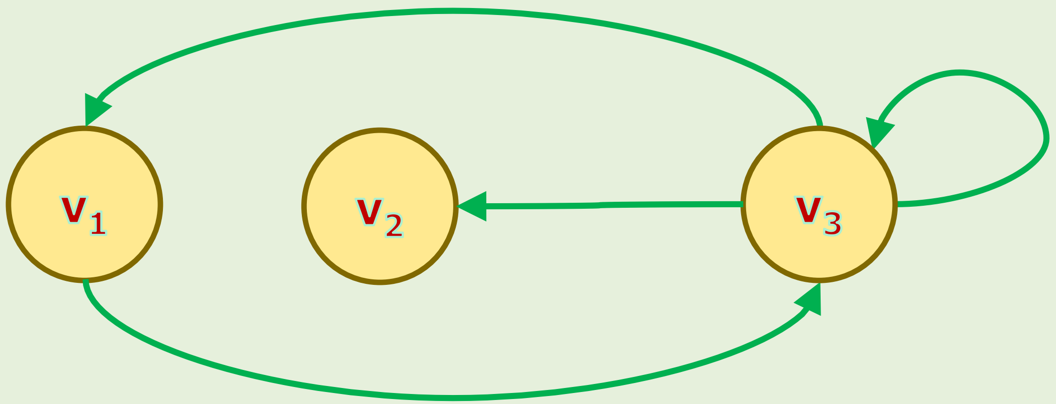


Length of Walks

- The "length of a walk" is the total number of edges traversed.

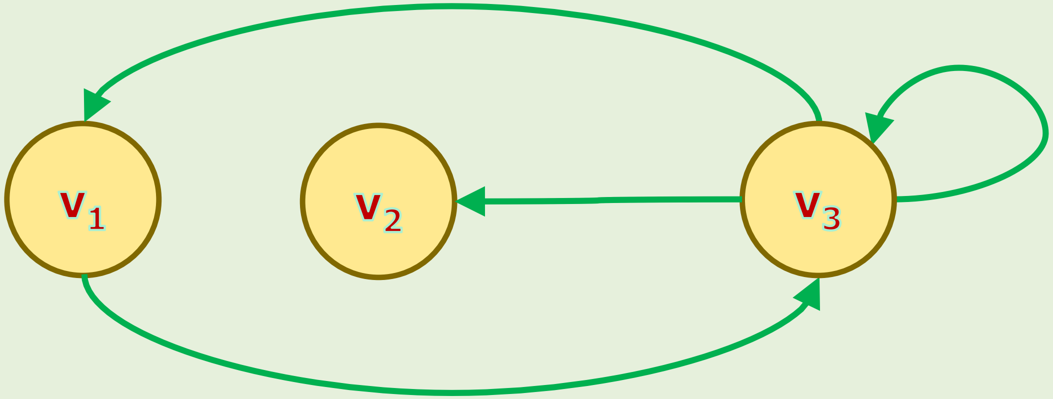
Example 33 (cont'd)

- Walk 1: (v_1, v_3) ; length = 1
- Walk 2: $(v_1, v_3), (v_3, v_3)$; length = 2
- Walk 3: $(v_1, v_3), (v_3, v_1), (v_1, v_3)$; length = 3



Path

- A walk that no edge is repeated.



Example 34

- Which one is a path?
- Walk from v_1 to v_3:

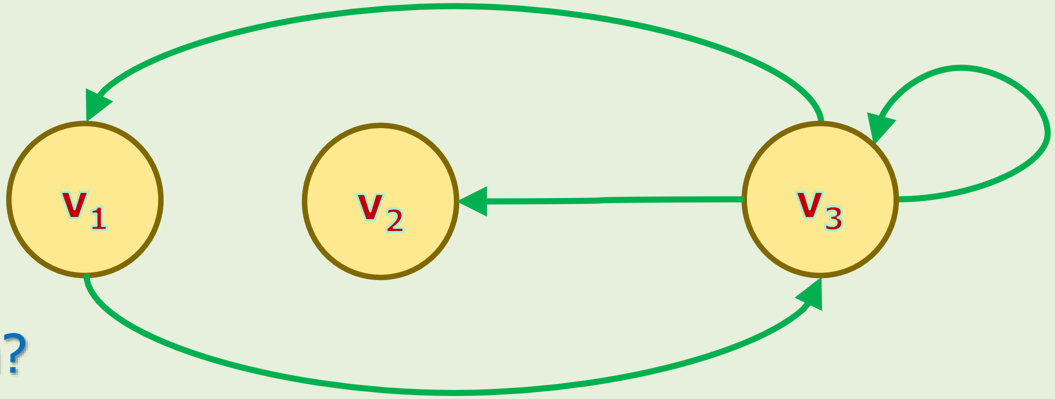
✓ – Walk 1: (v_1, v_3)

✓ – Walk 2: $(v_1, v_3), (v_3, v_3)$

✗ – Walk 3: $(v_1, v_3), (v_3, v_1), (v_1, v_3)$

Simple Path

- A path that no vertex is repeated.
  - In other words, no vertex should be visited more than once.



Example 35

- Which one is a simple path?
- Walk from v_1 to v_3:

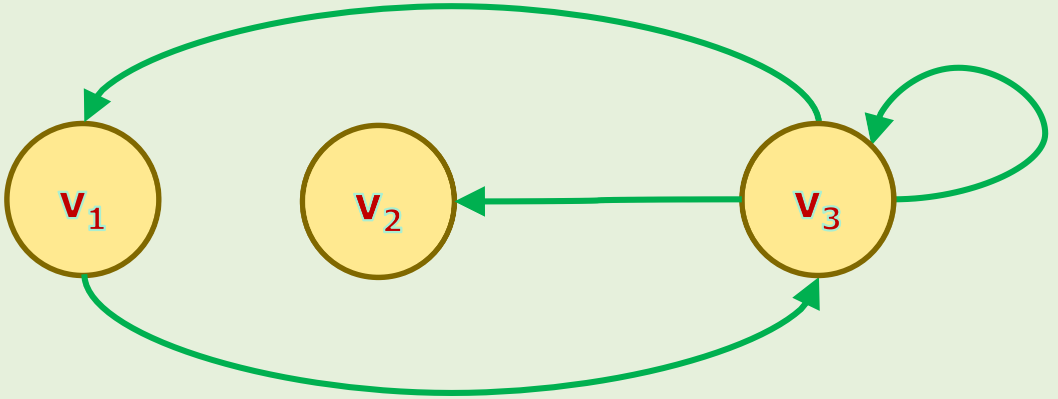
✓ – Walk 1: (v_1 , v_3)

✗ – Walk 2: $(v_1 , v_3) , (v_3 , v_3)$

✗ – Walk 3: $(v_1 , v_3) , (v_3 , v_1) , (v_1 , v_3)$

Loop

- An edge from a vertex to itself is called loop.



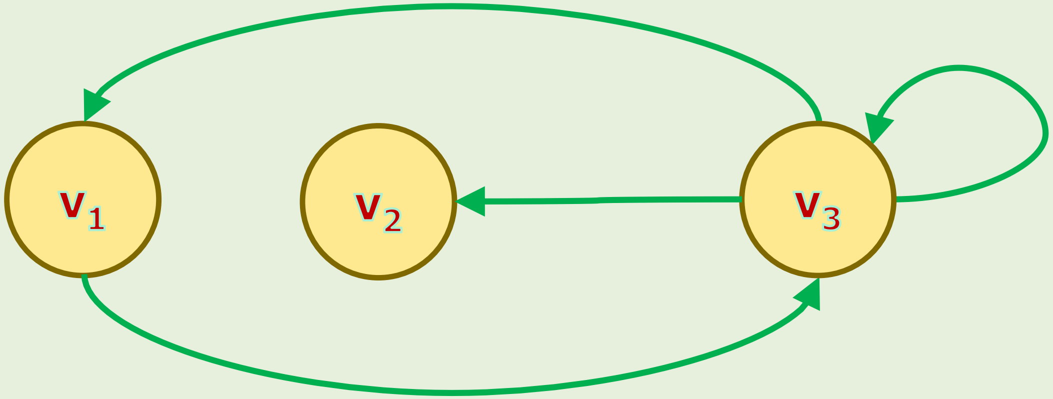
Example 36

- Which one is a loop?
- Walk from v_3 to v_3:
  - ✓ – Walk 1: (v_3, v_3)

- Is there any other loop in this graph?

Cycle

- A walk from a vertex (called base) to itself with no repeated edges.
- Remember that: Walk + No repeated edges = path
- Rewording: A cycle is a path from a vertex (called base) to itself.



Example 37

- Which one is a cycle?
- Walk from v_1 to v_1:

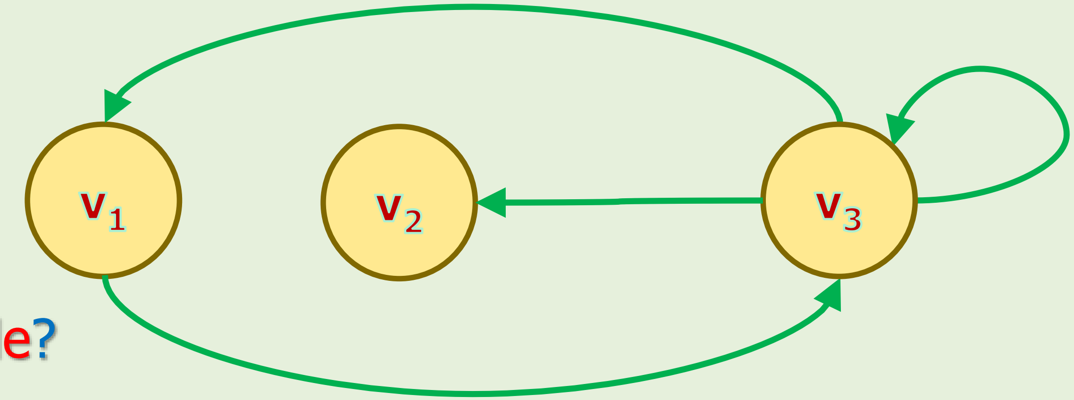
✗ – Walk 1: $(v_1, v_3), (v_3, v_1), (v_1, v_3), (v_3, v_1)$

✓ – Walk 2: $(v_1, v_3), (v_3, v_1)$

✓ – Walk 3: $(v_1, v_3), (v_3, v_3), (v_3, v_1)$

Simple Cycle

- A cycle that no vertex other than the base is repeated.
  - Note that the walk starts from the base and ends to the base.
  - During the walk, the base should not be repeated.

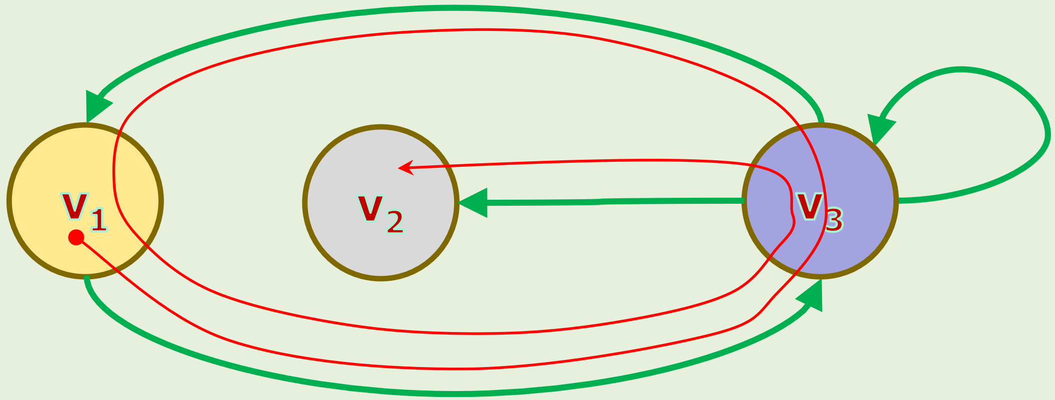


Example 38

- Which one is a simple cycle?
- Walk from v1 to v1:
  - ✗ Walk 1: $(v_1, v_3), (v_3, v_1), (v_1, v_3), (v_3, v_1)$
  - ✓ Walk 2: $(v_1, v_3), (v_3, v_1)$
  - ✗ Walk 3: $(v_1, v_3), (v_3, v_3), (v_3, v_1)$

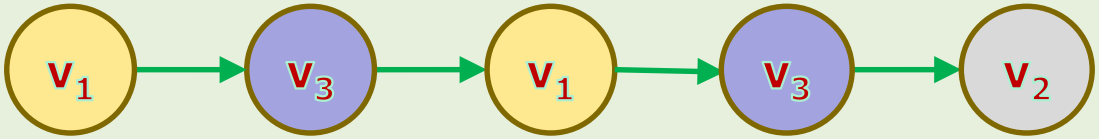
One-Dimensional Projection of a Walk

- "One-dimensional projection" (or just projection) is another way of representing a walk.



Example 39

- Represent the following walk as a one-dimensional projection.
- Walk from v_1 to v_2: (v_1, v_3), (v_3, v_1), (v_1, v_3), (v_3, v_2)



- The length of this walk (= the number of edges) is clearly shown.

References

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3. Sipser, Michael, "Introduction to the Theory of Computation, 3rd ed.," CENGAGE Learning, United States, 2013 ISBN-13: 978-1133187790