Some Practice Problems for the Probabilistic Method

Zoe Xi

May 16, 2021

1 Problem 1

Let $S_1, S_2, ..., S_{2^{n-1}-1}$ be n-element subsets of $\mathbb{N}$ for some positive integer n. Pleas show that there exists a 2-coloring of $\mathbb{N}$ (i.e., an assignment of red or blue to each element of $\mathbb{N}$) such that each S_i contains both a red and a blue element (i.e., no S_i is monochromatic).

Solution For every $i \in \mathbb{N}$, let X_i be a random indicator of the coloring of i such that X_i has an equal chance of being red or blue. Let A_k be the event that S_k is monochromatic. We have:

$$Pr(A_k) = 1/2^n + 1/2^n = 2^{1-n}$$

which is due to that fact that the chance for all of the n numbers in S_k to be assigned the same color, either red or blue, is $1/2^n$. Let A be the disjunction of all the A_k for $1 \leq k \leq 2^{n-1} - 1$. Then we have

$$Pr(A) \leq \Sigma_k(Pr(A_k)) = (2^{n-1} - 1) * (2^{1-n}) < 1$$

Therefore, the probability is positive that the event A does not occur, which means that all S_k are not monochromatic under certain coloring of the natural numbers. **QED**

2 Problem 3

Show that we can color the elements of the set $\{1, 2, ..., 1987\}$ with 4 colors such that any arithmetic progression of ten terms of the set is not monochromatic.

Solution For every n, let X_n be a random indicator of the coloring of n such that X_n has an equal chance of being 1, 2, 3, or 4, indicating 4 different colors. For each sequence σ, let A_σ be the event that the sequence is monochromatic. Clearly, $Pr(A_\sigma) = 4^{1-n}$, where n is the length of the sequence.

Let B be the event that there is at least 1 monochromatic arithmetic progression of length 10. It is clear that for every given number x, there can be no more than $(1987 - x)/9$ arithmetic progressions of length 10 starting from x. Therefore, the total number of arithmetic progressions of length 10 is less than $1986 * 1987/(2 * 9) < (2/9) * 10^6$. We have:

$$Pr(B) \leq 4^{1-10} * (2/9) * 10^6 < 4{\cdot}10^{-6} * (2/9) * 10^6 = 8/9 < 1$$

where the fact $4^{1-10} < 4{\cdot}10^{-6}$ is used. Therefore, the probability is positive that the event B does not occur, which simply implies that there is a coloring of the numbers in {1,2,...,1987} such that no arithmetic progression of length 10 can be formed with these numbers that is monochromatic. **QED**

3 Problem 5

Let S be a set of n real numbers such that $\Sigma_{x \in S}(x) = 0$. In addition, some of the real numbers in S are non-zeros. Prove that one can label these numbers $a_1, a_2, \ldots, a_n$ in a manner such that $a_1a_2 + a_2a_3 + ... + a_{n-1}a_n + a_na_1 < 0$.

Solution Let S be $\{x_1, x_2, ..., x_n\}$. Let X be a random variable ranging over the $n!$ permutations on S with equal chances. Let X_i refer to element i in X. Clearly, $E(X_i) = (x_1 + x_2 + ... + x_n)/n$. Furthermore, we have

$$E(X_iX_j) = ((\Sigma_i x_i)^2 - \Sigma_i x_i^2)/(n * (n-1)) = -\Sigma_i x_i^2/(n * (n-1)) < 0$$

By linearity of expectation, we have

$$E(X_1X_2 + X_2X_3 + \ldots + X_{n-1}X_n + X_nX_1) < 0$$

Therefore, we can choose a particular permutation $(a_1, a_2, \ldots, a_n)$ on S such that the following inequality holds:

$$a_1a_2 + a_2a_3 + ... + a_{n-1}a_n + a_na_1 < 0$$

QED