Vrije Universiteit Amsterdam

School of Business and Economics

Time Series Models 2022

Assignment 2 TSM-14

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1 Introduction

This paper will continue the analysis with the application of the Stochastic Volatility (SV) model with the use of linear Kalman filter methods. This analysis was conducted for both the data set of the pound-dollar exchange returns and the S&P500 index returns, which will be elaborated on further in the paper. The analysis was commenced with acknowledging the following background information. Firstly, we denote the closing price at time t to be denoted as P_t and where its return is formulated as follows.

$$r_t = \log\left(P_t/P_{t-1}\right) = \Delta \log P_t = \Delta p_t, \quad t = 1, \ldots, n \tag{1}$$

In this formulation p_t can be seen as the discretized realisation from a continuous-time log-price process $\log P(t)$, which is formulated as follows

$$\mathrm{d} \log P(t) = \mu \mathrm{d}t + \sigma(t) \mathrm{d}W(t) \tag{2}$$

where it can be noted that μ is the mean return, $\sigma(t)$ the continuous volatility process and $W(t)$ is the standardized Brownian motion. With this in mind we then focus on the $\sigma(t)$, where we let $\log\,\sigma(t)^2$ follow a Ornstein-Uhlenbeck process as follows.

$$\log \sigma(t)^2 = \xi + H(t), \quad \mathrm{d}H(t) = -\lambda H(t)\mathrm{d}t + \sigma_\eta \mathrm{d}B(t) \tag{3}$$

where ξ is constant with $0 < \lambda < 1, \sigma_\eta$ can only be positive and is regarded as the so-called volatility of the volatility and $B(t)$ is the standardized Brownian motion, which is independent of $W(t)$.

Using these formulations and applying the Euler-Maruyama discretisation method it is then possible to obtain the SV model for daily returns y_t formulated as follows.

$$y_t = \mu + \sigma_t \varepsilon_t, \quad \log \sigma_t^2 = \xi + H_t, \quad H_{t+1} = \phi H_t + \sigma_\eta \eta_t \tag{4}$$

Noteworthy is that both σ_t and ε_t are stochastic processes, which gives us a non-linear model, where $\phi = 1-\lambda$ and $0 < \phi < 1$. Hence, in order to make the model linear the data is transformed as follows $x_t = \log\left(y_t - \mu\right)^2$. Then with some redefinition's it is possible to get the AR(1) + noise model, as follows.

$$x_t = h_t + u_t, \quad h_{t+1} = \omega + \phi h_t + \sigma_\eta \eta_t \tag{5}$$

where it can be noted that $u_t = \log \varepsilon_t^2$, whereas $\omega = (1-\phi)\xi$ and $h_t = H_t + \xi$. Furthermore, it can also be noted that u_t does not have to be Gaussian, where we can assume ε_t to be Gaussian and apply an adjustment, which results u_t to be generated from a log chi-square distribution, where the mean and variance are well defined. In addition, this analysis can be regarded as an approximate analysis for equation (4) with equation (5), where due to the fact that we consider u_t as Gaussian with mean and variance generated from log chi-square distribution it is a approximation of a linear Gaussian model. This allows us to consider the QML-method in combination with the Kalman filter in order to estimate the unknown parameters.

The paper is divided as follows, it will first present the graphs and descriptives for the pound-dollar exchange return, which will be followed up with going through the basis of the QML-method. Furthermore, parameter estimation will be covered with the combination of the QML-method and the Kalman filter and then the Kalman smoothing. Next, the paper will then consider the S&P500 dataset in the first part of extension (e), where upon the same analysis as before will be conducted together with extending the analysis with looking into and including a Realized Volatility (RV) measure in the model in the second part of extension (e). Lastly, the analysis will be concluded with looking into the original SV model in the non-linear setting without the RV in light of the particle filter in extension (f).

2 SV-data graphs and descriptives

In this section, the dataset for the pound/dollar daily exchange returns is explored, giving the sample size N = 945. Before this is done the data is transformed as previously noted, as follows $x_t = log(y_t - \mu)^2$, where y_t is scaled by 100. This transformation is conducted in order to make the SV-model linear and hence we can apply Kalman filter and smoother on the data.

Table 2.0.1 shows the descriptive statistics of the log-returns of exchange rate (pound/US) (y_t) and the log demeaned return (x_t). For y_t it can be observed that the data is close to zero due to the small value of standard deviation and mean of y_t. The value of kurtosis for y_t is 4.8941 which is greater than 3, implying that the data is heavy tailed and the presence of outliers. For skewness, y_t is positively skewed, as the value is greater than zero, indicating the tail on the right side of the distribution is longer. The mean of x_t and and its standard deviation are not close to zero. The value of kurtosis for x_t is also greater than 3, it shows a similar result as y_t. x_t, however, is negatively skewed which means the tail on the left side of the distribution is longer. Figure 2.0.1 shows the two values, y_t and x_t, for the pound/dollar series, where it is evident that especially at the end of the data set of y_t high volatility can be observed.

Table 2.0.1: Descriptive Statistics of variables

	y_t	x_t
Mean	-0.0004	-11.4439
Std.Dev	0.0071	2.2386
Min	-0.0330	-28.3677
Max	0.0453	-6.1714
Kurtosis	4.8941	4.1365
Skewness	0.6051	-1.2595
N	945	945

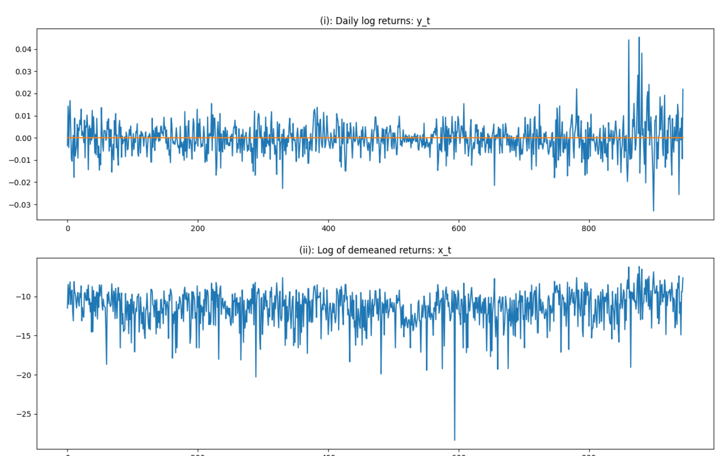


Figure 2.0.1: (i) y_t is the log-returns of exchange rate (pound/US), (ii) $x_t = log(y_t - \mu)^2$ is log of demeaned returns

3 The QML-method and Kalman filter estimation

This section discusses the estimation of the SV model by means of the QML method and Kalman filter method. For it can be noted that $u_t = \log \varepsilon_t^2$ in model 5 follows a chi-squared distribution of mean -1.27 and variance $\frac{\pi^2}{2}$. After arrangement, we can transform u_t to u_t^* as in the following model:

$$\begin{aligned} x_t &= h_t + d_t + u_t^*, \quad u_t^* \sim N(0, H_t) \\ h_{t+1} &= \omega + \phi h_t + \sigma_\eta \eta_t, \quad \eta_t \sim N(0, 1). \end{aligned} \tag{6}$$

In order to estimate the SV model (6), the QML method based on the Kalman filter is used for x_t by treating $u_t^* \sim N(0, H_t)$. Since x_t is not conditionally Gaussian and the Kalman filter yields minimum mean square linear estimators of the state and future observations, we can formulate a quasi-likelihood function by using one-step ahead errors and their variances from the QML approach.
In order to derive the Kalman filter for model (6), let

$$\begin{aligned} v_t &= x_t - Z_t h_t - d_t, \quad F_t = Z_t P_t Z_t' + H_t, \\ M_t &= P_t Z_t' F_t^{-1}, \quad h_{t|t} = h_t + M_t v_t, \\ P_{t|t} &= P_t - M_t F_t^{-1} Z_t P_t, \quad h_{t+1} = T_t h_{t|t} + \omega, \\ P_{t+1} &= T_t P_{t|t} T_t' + R_t Q_t R_t', \end{aligned} \tag{7}$$

for $t = 1, \cdots, n$. Estimation of model (6) can be carried out by setting

$$d_t = -1.27, \quad H_t = \frac{\pi^2}{2}, \quad T_t = \phi, \quad Z_t = 1, \quad R_t = 1.$$

Therefore, the prediction error decomposition form of the Gaussian likelihood can be obtained by the Kalman filter and the parameters, ω, $Q_t = \sigma_\eta^2$, and $T_t = \phi$, can be estimated by maximizing the quasi log-likelihood function (8) with $\sigma_\eta^2 > 0$ and $\phi \in (0, 1)$ using method SLSQP from python.

$$logL = -\frac{n}{2} log(2\pi) - \frac{1}{2} \sum_{t=2}^{n} (logF_t + \frac{v_t^2}{F_t}). \tag{8}$$

The initial values of parameters for the QML optimization can be achieved by the method of moments.
The initial state h_1 is $N(a_1, P_1)$ where $a_1 = \frac{\omega}{1-\phi}$ and $P_1 = \frac{Q_t}{1-\phi^2}$ with initial values of parameters

$$\phi = 0.9, \quad \sigma_\eta^2 = \frac{cov(x_t, x_{t-1})(1 - \phi^2)}{\phi}, \quad \omega = (E(x_t) - d_t)(1 - \phi).$$

Table 3.0.1 shows the QML estimates of the parameters. The estimates $\hat{\phi} = 0.9913$ and $\hat{\sigma}_\eta^2 = 0.007$ are close to one and zero, implying persistence of past observations h_t, which is the log of volatility. The constant $\hat{\omega}$ has a negative value of -0.0872.

Table 3.0.1: Estimation results for the SV model

	pound/dollar
$\hat{\phi}$	0.9913
$\hat{\sigma}^2_\eta$	0.007
$\hat{\omega}$	-0.0872
$logL$	-2083.6472

4 QML-method estimation smoothing

From the previous section, the QLM estimates were obtained, which will be further used in order to obtain the smoothed values of h_t. This is done by backwards recursion for which the smoothing cumulant r_t is formulated as follows,

$$r_{t-1} = Z_t' F_t^{-1} v_t + L_t' r_t \tag{9}$$

where $L_t = T_t - k_t Z_t$. Furthermore,the smoothed estimates are subsequently obtained as follows.

$$\hat{h}_t = a_t + p_t r_{t-1} \tag{10}$$

where the variables v_t, k_t, F_t, a_t, and p_t are obtained as the outputs from the Kalman filter. In the upper panel of Figure 4.0.1 the $\hat{h}_t$ (red line) can be observed graphically plotted against the transformed data x_t. The time sereis of the smoothed estimates h_t captures the overall level of the transformed data x_t and it is mostly flat with little variation. Noteworthy is that the obtained filtered and smoothed estimates for h_t are saved in order to obtain the filtered and smoothed values for H_t formulated as follows.

$$\begin{aligned} E\left[H_t \mid Y_t\right] &= E\left[h_t \mid Y_t\right] - \xi \\ \mathrm{E}[H_t \mid Y_n] &= E\left[h_t \mid Y_n\right] - \xi \\ \xi &= \hat{\omega}/(1-\hat{\phi}) \end{aligned} \tag{11}$$

In the lower panel of Figure 4.0.1 filtered and smoothed values for H_t are plotted against each other. From this it is clear that the smoothed estimates follow the filtered estimates relatively narrow, which is as expected since the smoothed estimates take into account the whole data set, making it possible to adapt better to shocks. Noteworthy is that it seems that the smoothing estimates take a lower value relative to the filtered estimate when there is a downward trend and a higher value when there is a upward trend.

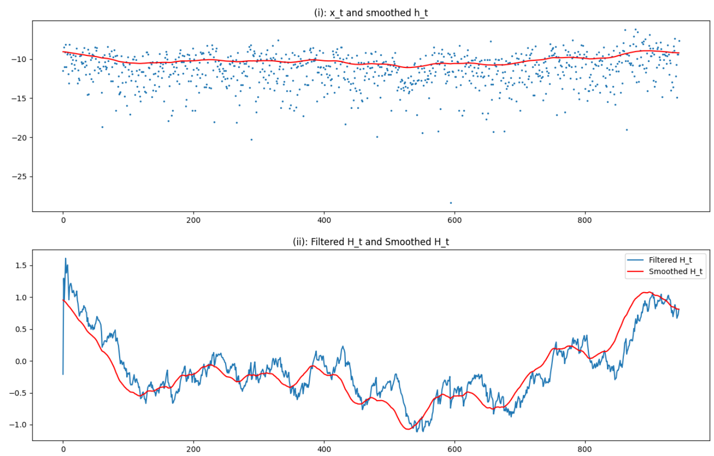


Figure 4.0.1: Filtered and smoothing estimates for SV data: (i) $\hat{h}_t$ (red line) plotted against transformed data x_t; (ii) filtered estimates of H_t (red line) plotted against the smoothed estimates of H_t (blue line).

5 Extension (e)

The first part of this section will focus on revisiting the previous analysis, but now for the data of daily returns for the S&P 500 index, giving the sample size N = 1751. Therefore, we first employ the data transformation for y_t as $x_t = log(y_t - \mu)^2$, where figure 5.0.1 shows y_t, the log-returns of daily returns for the S&P 500 index, and x_t as the log of demeaned returns. From this it is clear that the raw data seems relatively stable over the years, except for the start of 2020, which can be identified as the COVID-19 crash, which shows volatility clustering. Similar observations can be made in the plot for x_t.

Furthermore, in table 5.0.1 the descriptives of the data can be observed, since the mean of y_t is close to zero and its standard deviation is small, the observations of the data are close to the mean. The values of kurtosis and skewness for y_t are 20.771 and -1.0437, respectively. It indicates that the data is heavily tailed and negatively skewed, indicating the presence of extreme values. The observations of x_t are not close to zero because its mean and standard deviation values are not close to zero. The kurtosis x_t implies the distribution is shorter, where tails are thinner than the normal distribution. For skewness, x_t is negatively skewed.

Table 5.0.1: Descriptive Statistics of the data of daily returns for S&P 500 index

	y_t	x_t
Mean	-0.0005	-11.1966
Std.Dev	0.0114	2.5404
Min	-0.1267	-23.0147
Max	0.0896	-4.1243
Kurtosis	20.771	1.1126
Skewness	-1.0437	-0.7695
N	1751	1751

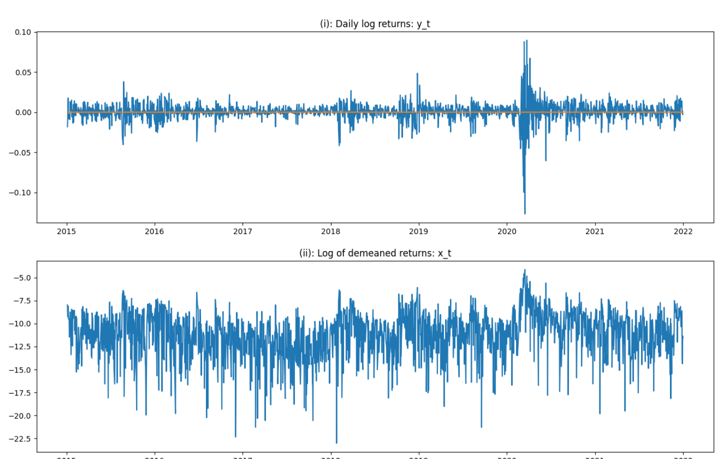


Figure 5.0.1: (i) y_t is the log-returns of daily returns for S&P 500 index (pound/US), (ii) $x_t = log(y_t - \mu)^2$ is log of demeanded returns

The analysis is continued with the parameter estimation by means of the QLM-method and Kalman filter, where the estimates can be found in 5.0.2. From this it is clear that $\hat{\phi}$ is close to one, whereas $\hat{\sigma}^2_\eta$ is relatively small indicating that the effect of h_t is persistent on $h_t + 1$ and does not fade away quickly. Lastly, $\hat{\omega}$ shows that the intercept of AR(1) part of the model is negative. This observation of QML estimates is aligned with the SV data.

Table 5.0.2: Parameter estimates obtained with the use of the QML-method and Kalman filter for S&P 500 index.

	Estimates
$\hat{\phi}$	0.9686
$\hat{\sigma}^2_\eta$	0.0876
$\hat{\omega}$	-0.3104
$\log L$	-3993.8107

Lastly, the analysis is concluded with looking into smoothing for the QML-estimates, where both the x_t is plotted against the smoothed h_t as well as the filtered H_t against the smoothed H_t in Figure 5.0.2. From the upper panel it is clear that in comparison to the first data-set, the smoothed h_t is more volatile, where it it most evident at the start of 2020 with the COVID-19 financial crash. Furthermore, from the lower panel it can noted that the filtered and smoothed H_t closely follow each other and are almost indistinguishable. This is not as expected since it would be expected that the smoothed estimates would show a smoother line in comparison to the filtered estimates.However this can be due to the fact that the S&P 500 index data is more noisy and volatile, which is proved in the higher value of $\hat{\sigma^2_\eta} = 0.0876$ compared to 0.007 from the SV data.

Figure 5.0.2: Filtered and smoothing estimates for SP500 data: (i) $\hat{h}_t$ (red line) plotted against transformed data x_t; (ii) filtered estimates of H_t (red line) plotted against the smoothed estimates of H_t (blue line).

The second part of this section extends the model to include a Realized Volatility measure obtained from the website (`https://realized.oxford-man.ox.ac.uk/data/download`). The choice of the Realized Volatility measure in this assignment will be **Realized Parzen** as elaborated in Shephard [2010], which incorporates the Parzen weight function in the calculation for the realized Volatility. In Figure (5.0.3), the time series of the transformed data x_t and the log of Realized Parzen, namely $\log(RV_t)$ are plotted against each other. Similar trends of upwards and downwards movements are shared between the two time series which motivate the inclusion of the realized measure as the explanatory variable in the model for x_t. In particular, $\log(RV_t)$ is less volatile than x_t during year 2015 to 2022.

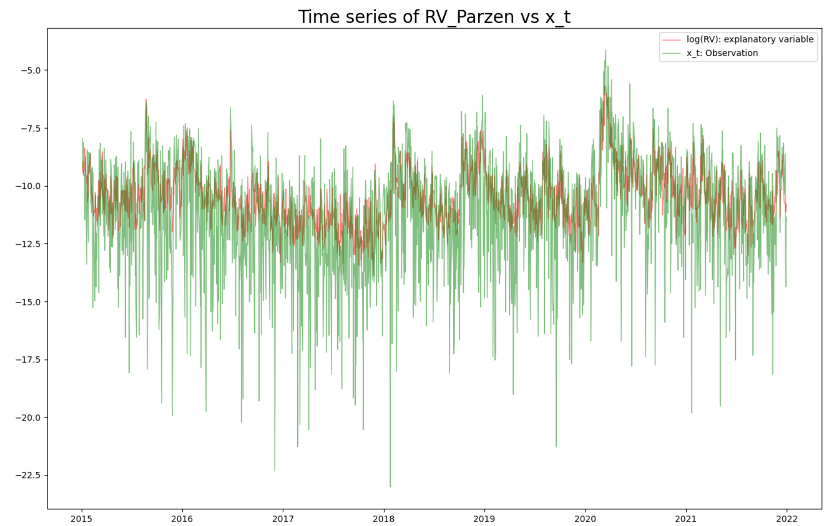


Figure 5.0.3: Time series plot of RV Parzen and x_t

For this purpose, consider the model:

$$\begin{aligned} x_t &= \beta \cdot \log RV_t + h_t + u_t = \beta \cdot \log RV_t + h_t + d_t + u_t^*, \qquad u_t^* \sim N(0, H_t) \\ h_{t+1} &= \omega + \phi h_t + \sigma_\eta \eta_t, \quad \eta_t \sim N(0,1). \end{aligned} \tag{12}$$

The analysis is two-fold, which firstly involves obtaining the β coefficient and next the contribution of including the Realized volatility measure on the filtered and smoothed estimates is investigated. Firstly, we will quantify the coefficient β using the Generalized Least Squares Estimation by running two times the Kalman filters. This technique is called Augmented Kalman filter, which is illustrated as the following.
As the first step, we run the Kalman filter on the model $x_t = h_t + d_t + u_t^*$ as performed previously, and we save the prediction error and its variance v_t^* and F_t. Secondly, the Kalman filter is run on the model:

$$\log RV_t + d_t = h_t + d_t + u_t^*$$

, which simplifies to $RV_t = h_t + u_t^*$ while keeping the state update equation $h_{t+1} = \omega + \phi h_t + \sigma_\eta \eta_t$ unchanged. This time, only the prediction error is saved, denoted by $x_t^* = \log RV_t - h_t$.
From the values of v_t^* and F_t, and x_t^*, the generalised least squares estimate of β and its variance matrix are obtained by:

$$\hat{\beta} = \left(\sum_{t=1}^{n} X_t^{*\prime} F_t^{-1} X_t^* \right)^{-1} \sum_{t=1}^{n} X_t^{*\prime} F_t^{-1} v_t^*, \quad \mathrm{Var}(\hat{\beta}) = \left(\sum_{t=1}^{n} X_t^{*\prime} F_t^{-1} X_t^* \right)^{-1} \tag{13}$$

Equation 13 evaluates the estimated coefficient $\hat{\beta} = 0.9501$ and $\text{Var}(\hat{\beta}) = 0.0044$. The coefficient $\hat{\beta} = 0.9501$ close to 1 is to be expected, because $x_t = h_t + u_t = \log(\sigma_t^2) + \log(\epsilon_t^2)$. And the realized measure RV_t calculates the volatility at a higher frequency, therefore both time series x_t and $\log(RV_t)$ should have around the same level as shown in 5.0.3 with a scale β and other factors present in Equation 12. Next, we subtract $\hat{\beta} \cdot \log(RV_t)$ from x_t to obtained a new time series and let it be the observation update equation:

$$x_t^{new} = x_t - \beta \cdot \log(RV_t) = h_t + d_t + u_t^*$$

, which resembles the models in model (12). QML method is then applied to find the estimates $\left(\hat{\phi}, \hat{\sigma_\eta^2}, \hat{\omega}\right)$, as shown in the second column in Table 5.0.3. Comparing with the QML estimates from the original model for S&P500 in the first column in the same table, it is observed that $\hat{\phi}$ are similar to each other, where including RV_t results in a higher value(closer to 1). As for parameter $\hat{\sigma_\eta^2}$ and $\hat{\omega}$, including RV_t reduces the value of $\hat{\sigma_\eta^2}$ from 0.0876 to 0.0004 and reduces the absolute value of $\hat{\omega}$ from -0.3104 to 0.0002. Lastly, likelihood evaluated at the QML estimates gives a higher value when RV_t is incorporated in the model.

Table 5.0.3: Estimation results for the S&P500 with and without the Realized measure

	Original	With Realized Parzen
$\hat{\phi}$	0.9686	0.9972
$\hat{\sigma}_\eta^2$	0.0876	0.0004
$\hat{\omega}$	-0.3104	0.0002
$logL$	-3993.8107	-3847.027

After obtaining the QML-estimates, we proceed to compute the smoothed values of h_t, the filtered estimates $E[H_t|Y_t]$, and the smoothed estimated $E[H_t|Y_n]$ by Kalman filter and smoother based on x_t^{new} using the formulas described in Equations 9, 10 and 11. These values are plotted in Figure 5.0.4 and some insights can be derived from comparing with Figure 5.0.2 in the case of the original model.

In the first panel in Figure 5.0.4, the time series smoothed h_t is flatter than in the case of the original model. In the second panel, the spike in the Filtered H_t in the beginning is again caused by the initial values and it takes about 10 observations to adjust to a reasonable level. It is seen that both the time series of filtered and smoothed H_t when including RV_t in the model are flatter and less volatile, compared with the original model of x_t.

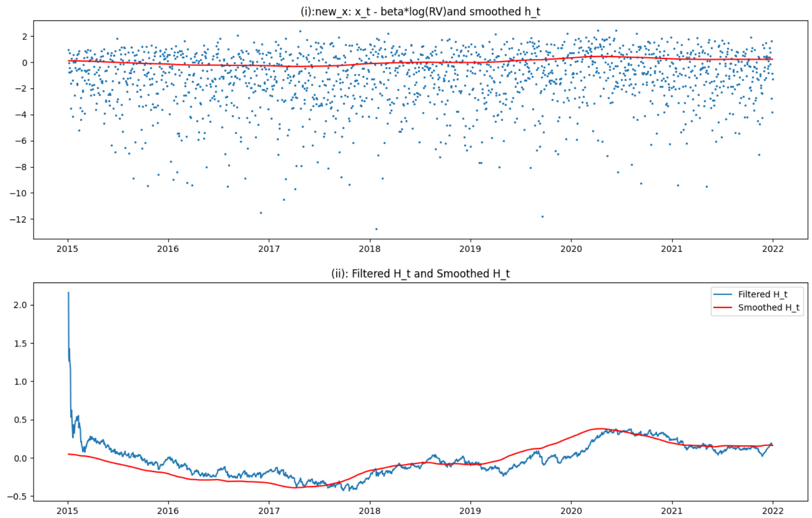


Figure 5.0.4: Filtered and smoothing estimates for x_t^{new}: (i) $\hat{h}_t$ (red line) plotted against transformed data x_t; (ii) filtered estimates of H_t (red line) plotted against the smoothed estimates of H_t (blue line).

6 Extension (f)

In this exercise, we return to the model with the nonlinear specification:

$$y_t = \mu + \sigma_t \varepsilon_t, \quad \log \sigma_t^2 = \xi + H_t, \quad H_{t+1} = \phi H_t + \sigma_\eta \eta_t$$

For this we compute the filtered estimates of H_t in the above equation by use of the particle filter for both the SV data and S&P500 data. In particular, bootstrap filtering with resampling technique is deployed for this exercise for simplicity and efficiency purposes.
Bootstrap filter procedure is explained in the following steps, while noting that we fix the random seeds in the Python program and set N = 10000 in the Monte Carlo simulation to obtain more accurate estimates. In addition, the state space parameter $\alpha_t = H_t$.

1. We draw N values of $\tilde{\alpha_1}^{(i)} \sim N\big(0, \frac{\sigma_\eta^2}{1-\phi^2}\big)$ for $i = 1, ..., N$, where the unconditional mean and variance is obtained from the equation $H_{t+1} = \phi H_t + \sigma_\eta \eta_t$.

2. A recursion is performed on the time index t to draw N values of $\tilde{\alpha_t}^{(i)} \sim N\big(\phi \cdot \alpha_{t-1}^{(i)}, \sigma_\eta^2\big)$ for $t = 1, ..., T$, where T is the length of the time series.

3. Next, we compute the weight in the form of the conditional density of y_t given $\tilde{\alpha_t}^{(i)}$ for each draw by the following equation:

$$\tilde{w_t}^{(i)} = P(y_t|\tilde{\alpha_t}^{(i)}), \quad \text{where } y_t = \mu + \sigma_t \varepsilon_t, \quad \log \sigma_t^2 = \xi + H_t$$

   By use of the fact that $H_t = \tilde{\alpha_t}^{(i)}$ from each draw in the previous step, we also take the QML estimate xi as given calculated from the previous section, we can derive that $\log \sigma_t^2$ and find $\sigma_t = \sqrt{exp(\xi + \tilde{\alpha_t}^{(i)})}$.

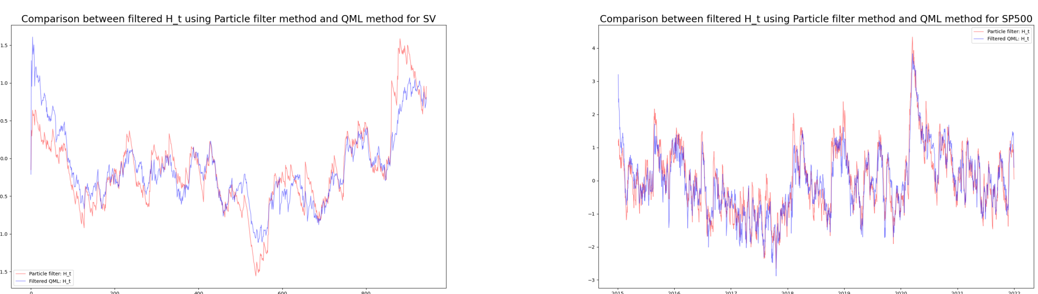
Therefore, it is shown that $Y_t \sim N\big(\mu, \sigma_t^2\big)$, and weight $\tilde{w_t}^{(i)}$ can be computed and normalized as: $w_t^{(i)} = \frac{\tilde{w}_t^{(i)}}{\sum_{j=1}^{N} \tilde{w}_t^{(j)}}$.

4. We compute the filtered states $\hat{a_{t|t}} = \sum_{i=1}^{N} w_t^{(i)} \cdot \tilde{\alpha_t}^{(i)}$ as an weighted average of the total N draws.

5. The resampling is performed to prevent degeneracy, and it is done by drawing N values of new $\alpha_t^{new(i)}$ from $\hat{\alpha_t}^{(i)}$ for $i = 1, ..., N$ with replacement and with the corresponding weights a vector of $w_t^{(i)}$.

Now, we set $t = t + 1$ and let these newly drawn $\alpha_t^{new(i)}$ replace $\alpha_{t-1}^{(i)}$ in the mean of the distribution for $\tilde{\alpha_t}^{(i)} \sim N\big(\phi \cdot \alpha_{t-1}^{(i)}, \sigma_\eta^2\big)$ in step 2.

The filtered estimates of $H_t = \hat{a_{t|t}}$ are plotted for both SV data and S&P500 data, as shown in Figure 6.0.1. In the left panel for the SV data, there are periods of times where the Particle filtered H_t are above and also below the QML filtered H_t, while the differences are more present for extreme high and low values at around 1.5 and -1.5 on the y-axis. As for the S&P500 data in the right panel, the two filtered estimates H_t are more similar and close to each other for all the time period from the beginning of year 2015 to the end of year 2021. Overall, based on the Figure 6.0.1, it is concluded that the Particle filtered H_t and the QML filtered H_t have the same trends of upwards and downwards movements.



(a) SV data

(b) S&P500 data

Figure 6.0.1: Comparison of filtered estimates H_t: Particle filtering using Bootstrap filtering technique (red line) and filtered QML estimates (blue line)

References

Sheppard K. Shephard, N. Realising the future: forecasting with high-frequency-based volatility (heavy) models. *Journal of Applied Econometrics*, 25(2):197–231, 2010.