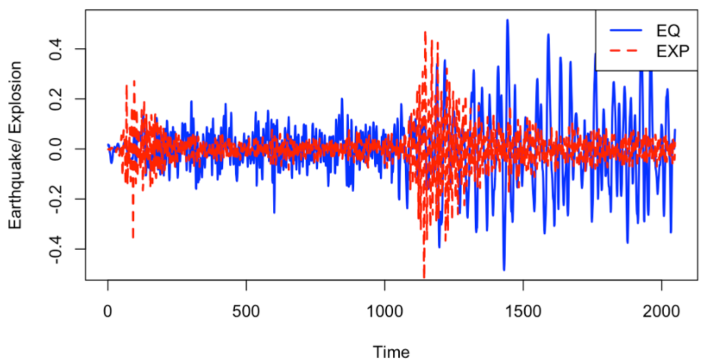
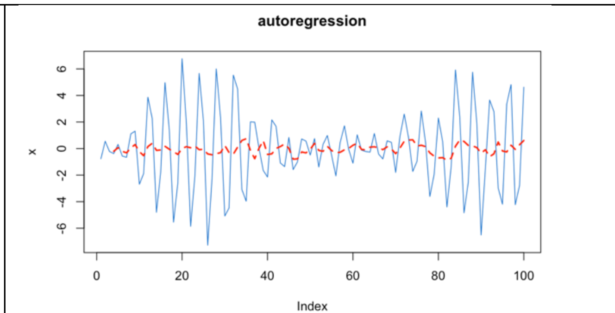
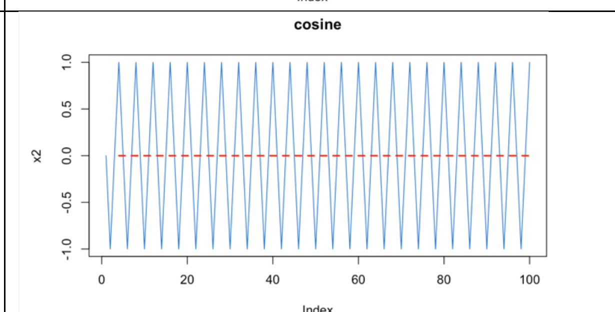
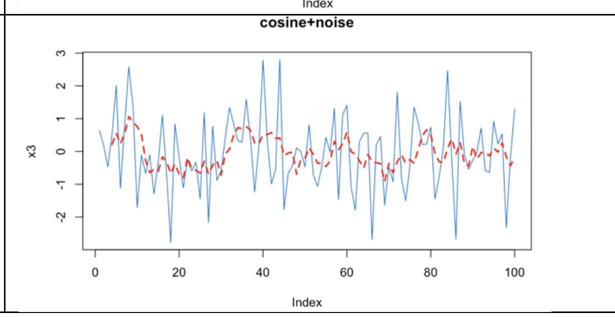
Problems 1.1, 1.3, 1.6, 1.8, 1.9, 1.10, 1.16.

1.1



1.3

The autoregression process shows high variability with sharp fluctuations. After applying 4 point moving average, the fluctuations are smoothed and variance is reduced.	
A perfectly cosine wave with period 4. After applying 4 point moving average, the fluctuations are canceled and turn out to be a flat line near 0.	
A cosine wave with added noise. The moving average eliminates the fluctuations and smooths the noise.	

1.6

a) It is not stationary because the mean is not constant. $E(x)=ß_1+ß_2t$, this formula depends on t, which means it is not stationary unless $ß_2=0$.

b) $y_t= x_t - x_{t-1}= ß_1+ß_2t+w_t-(ß_1+ß_2(t-1)+w_{t-1})= ß_2+(w_t-w_{t-1})$

$E(y_t)= ß_2$, which is a constant.

$\gamma_y(0)=Var (w_t-w_{t-1})= Var(w_t)+Var(w_{t-1})-2Cov(w_t, w_{t-1})=2\sigma_w^2$, where xu$w_t$ is white noise

$\gamma_y (1)= \gamma_y (-1)= Cov(w_t-w_{t-1}, w_{t-1}-w_{t-2})=Cov(w_t, w_{t-1})-Cov(w_t, w_{t-2})-Cov(w_{t-1}, w_{t-1})+Cov(w_{t-1}, w_{t-2})= -\sigma_w^2$,

$Cov(A+B,\ C+D) = Cov(A,C) + Cov(A,D) + Cov(B,C) + Cov(B,D)$

$\gamma_y (h)=0$ for $|h|>1$,

All moments are time-invariant, which is stationary.

c)

$$E(V_t) = \frac{1}{2q+1}\sum_{j=-q}^{q}(\beta_1+\beta_2(t-j)) = \frac{1}{2q+1}\left[\sum_{j=-q}^{q}\beta_1 + \sum_{j=-q}^{q}\beta_2 t - \sum_{j=-q}^{q}\beta_2 j\right]$$

$$= \frac{1}{2q+1}\left[(2q+1)\beta_1 + (2q+1)\beta_2 t - \beta_2 \sum_{j=-q}^{q} j\right] \xrightarrow{} 0 \quad \text{symmetry}$$

$$= \frac{1}{2q+1}\cdot\left[(2q+1)(\beta_1+\beta_2 t)\right]$$

$$= \beta_1+\beta_2 t.$$

$$\gamma_v(h) = Cov[(\beta_1+\beta_2 t)+u_t,\ \beta_1+\beta_2(t-h)+u_{t-h}]$$

$$= Cov(u_t, u_{t-h}) \qquad u_t = \frac{1}{2q+1}\sum_{j=-q}^{q} w_{t-j} \qquad u_{t-h} = \frac{1}{2q+1}\sum_{k=-q}^{q} w_{t-h-k}$$

$$= \frac{1}{2q+1}\sum_{j=-q}^{q}\sum_{k=-q}^{q} Cov(w_{t-j}, w_{t-h-k})$$

when $t-j = t-h-k \Rightarrow j = h+k$

1.8

1.8.

a) $X_t = \delta + X_{t-1} + W_t$, with $X_0 = 0$

$X_1 = \delta + X_0 + W_1 = \delta + W_1$

$X_2 = \delta + X_1 + W_2 = \delta + \delta + W_1 + W_2 = 2\delta + W_1 + W_2$

$X_3 = \delta + X_2 + W_3 = 3\delta + W_1 + W_2 + W_3$

$\vdots$

$X_t = \delta t + \sum_{k=1}^{t} W_k$

b) $E X_t = E\left(\delta t + \sum_{k=1}^{t} W_k\right) = \delta t + \sum_{k=1}^{t} E W_k = \delta t.$

$\gamma(s,t) = \operatorname{COV}(X_s, X_t) = E[(X_s - \delta s)(X_t - \delta t)]$

$= E\left(\sum_{j=1}^{s} W_j \sum_{k=1}^{t} W_k\right)$

$= \sum_{j=1}^{\min(s,t)} \sigma_w^2$

W_j is white noise

when $j=k$. $E W_j^2 = \sigma_w^2$

when $j \neq k$ $E W_j W_k = 0$

c) mean function depends on time, t

d) $\rho_x(t-1,t) = \dfrac{(t-1)\sigma_w^2}{\sqrt{(t-1)\sigma_w^2} \cdot \sqrt{t\sigma_w^2}} = \sqrt{\dfrac{t-1}{t}}$ when $t \to \infty$, the result $\to 1$

The adjacent levels are nearly perfectly correlated for large t.

The process is nonstationary and level-based correlations can be misleading.

(mean: δt, variance $= t\sigma_w^2$)

e) $Y_t = X_t - X_{t-1} = \delta + W_t$ $E(Y_t) = \delta$, $Var(Y_t) = \sigma_w^2$, which are both constant

$Cov(Y_t\ Y_{t-h})$ $Cov(\delta + W_t, \delta + W_{t-h}) = 0$ when $h \neq 0$

1.9

1.9. $X_t = U_1 \sin(2\pi w_0 t) + U_2 \cos(2\pi w_0 t)$, let $2\pi w_0 t = a$, $2\pi w_0 (t-h) = b$

with. U_1, U_2 are independent random variables with 0 mean and $E(U_1^2) = E(U_2^2) = \sigma^2 \Rightarrow E(U_1) = E(U_2) = 0$

$E(X_t) = E(U_1)\sin(2\pi w_0 t) + E(U_2)\cos(2\pi w_0 t) = 0$ $\Rightarrow$ mean is constant 0

$\gamma(h) = \operatorname{COV}(X_t, X_{t-h}) = E(X_t \cdot X_{t-h}) - \underbrace{E(X_t)\cdot E(X_{t-h})}_{0}$

$= E(X_t \cdot X_{t-h})$

$= E(U_1 \sin a + U_2 \cos a)(U_1 \sin b + U_2 \cos b)$

$= E(U_1^2 \sin a \sin b + U_1 U_2 \sin a \cos b + U_1 U_2 \sin b \cos a + U_2^2 \cos a \cos b)$

$= E(U_1^2 \sin a \sin b + U_2^2 \cos a \cos b)$

$= E(U_1^2)\sin a \sin b + E(U_2^2)\cos a \cos b$

$= \sigma^2(\sin a \sin b + \cos a \cos b)$

$= \sigma^2 \cos(a-b)$

$= \sigma^2 \cos(2\pi w_0 h)$

Therefore, this series is weekly stationary with auto covariance function $\gamma(h) = \sigma^2 \cos(2\pi w_0 h)$

1.10

1.10.

a) $MSE(A) = E[(X_{t+l} - AX_t)^2]$, X_t with zero mean

$= E[X_{t+l}^2] + A^2 E[X_t^2] - 2A E[X_{t+l}\cdot X_t]$

$= \gamma(0) + A^2(\gamma_0) - 2A\gamma(l)$

$\frac{d}{dA} MSE(A) = 2A\gamma(0) - 2\gamma(l) = 0$

$\therefore A = \frac{\gamma(l)}{\gamma(0)} = \rho(l)$

$\because A' = 0$

$\therefore \rho(l)$ is the minimum value.

b) $MSE_{min} = \gamma(0) + \rho(l)^2\gamma(0) - 2\rho(l)\gamma(l)$

$= \gamma(0)[1+\rho(l)^2 - \frac{2\rho(l)\gamma(l)}{\gamma(0)}]$ $\Rightarrow \rho(l)$

$= \gamma(0)[1+\rho(l)^2 - 2\rho(l)^2]$

$= \gamma(0)\cdot[1-\rho(l)^2]$

c) if $X_{t+l} = AX_t$

$Cov(cX, Y) = c\,Cov(X, Y)$
$Cov(X, X) = Var(X)$

for zero mean, weakly stationary series

Then. $Cov(X_{t+l}, X_t) = Cov(AX_t, X_t) = A\,Cov(X_t, X_t) = A\,Var(X_t) = A\gamma(0)$

$Var(X_{t+l}) = Var(AX_t) = E[AX_t - E[AX_t]]^2 = E[(A(X_t - E[X_t]))^2] = A^2[E X_t - E(X_t)^2] = A^2 Var(X_t)$

$\therefore Var(X_{t+l}) = A^2\gamma(0)$

$$\therefore \rho(l) = \frac{Cov(AX_t, X_t)}{\sqrt{Var(X_t)\cdot Var(X_{t+l})}} = \frac{A\gamma(0)}{\sqrt{A^2 Var(X_t)\cdot Var(X_t)}}$$

$$= \frac{A\gamma(0)}{|A\,Var(X_t)|}$$

$$= \frac{A}{|A|} = \begin{cases} -1 & A<0 \\ 1 & A>0 \end{cases}$$

1.16

1.16.

$t = 1, 2, \cdots$ $\quad U \sim Unif(0,1)$

a) $E(X_t) = \int_0^1 \sin(2\pi ut)\,du = [-\frac{\cos(2\pi ut)}{2\pi t}]_0^1 = \frac{\cos(2\pi t)}{-2\pi t} + 1$

$= 1 - 1 = 0.$

$\cos a\cos b + \sin a\sin b$ $\quad \Rightarrow \cos a\cos b - \sin a\sin b$

$2\sin a\sin b = \cos(a-b) - \cos(a+b)$

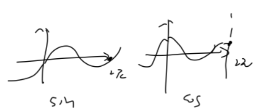
$\gamma(h) = E[X_t\cdot X_{t-h}] = \int_0^1 \sin(2\pi ut)\sin(2\pi(t-h)u)\,du$

$= \frac{1}{2}\int_0^1 \{\cos(2\pi ut - 2\pi tu + 2\pi hu) - \cos(2\pi ut + 2\pi tu - 2\pi hu)\}\,du$

$= \frac{1}{2}\int_0^1 \{\cos(2\pi hu) - \cos(2\pi(2t-h)u)\}\,du = 0 \quad (h \neq 0)$

$\gamma(0) = E[\sin^2(2\pi ut)]$, let $x = 2\pi ut$, $du = \frac{dx}{2\pi t}$

$= \int_0^1 \sin^2(2\pi ut)\,du$



$= \int_0^1 \sin^2 x\cdot\frac{dx}{2\pi t}$

$= \frac{1}{2\pi t}\int_0^{2\pi t} \sin^2 x\cdot dx$

$= \frac{1}{2\pi t}\int_0^{2\pi t} \frac{1-\cos 2x}{2}\,dx$

$= \frac{1}{2\pi t}\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{2\pi t}$

$= \frac{1}{2\pi t}\cdot(\pi t - \sin 4\pi t)$

$= \frac{1}{2} \quad (h=0)$

$\therefore X_t$ is weakly stationary.