Trajectory Optimization of Hypersonic Vehicles with Uncertainty

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Physical problem

▷ Initial value problem:

$$\begin{cases} \dot{x}(t) = f(t, x(t), \omega) & t \in D \\ x(t_0) = x_0 \end{cases}$$

- t : time, $t \in D = [t_0, t_f] \subset \mathbb{R},$ bounded interval,
- x : unknown function $D \to \mathbb{R}$,
- ω : uncertain variable $\omega \in \Omega$,

▷ Solve for a stochastic solution $x = x(t, \omega) : D \times \Omega \to \mathbb{R}$.

Trajectory Optimization of Hypersonic Vehicles

- Lift and drag coefficients generally depend on factors like:
  - Angle of attack, vehicle shape, and velocity, etc.
- Simplified dependency on angle of attack $\alpha[^\circ]$:

$$C_L = -0.04 + 0.8\alpha, \qquad C_D = 0.012 - 0.01\alpha + 0.6\alpha^2.$$

S.Tauqeer ul Islam Rizvi, L.He, and D.Xu. Optimal trajectory analysis of hypersonic boost-glide waverider with heat load constraint. Aircraft Engineering and Aerospace Technology, 87(1):67–78, 2015.

- Our project focuses on studying the **impact of variations in the lift and drag coefficients** on the physical properties of trajectory.

$$\text{Variations} \begin{cases} \text{Imperfect calibration of the wind tunnel} \\ \text{Turbulence in the airflow} \\ \text{discrepancies between the reference and actual exposed area} \end{cases}$$

Trajectory Optimization of Hypersonic Vehicles

$$
\begin{aligned}
\text{Let} \quad & D = [t_0, t_f]. \\
\min_{x \in W_{1,\infty}^{n_x}(I),\ u \in L_{\infty}^{n_u}(I)} \quad & J = \varphi(x(t_0), x(t_f)) + \int_{t_0}^{t_f} L(t, x(t), u(t))\, dt \\
\text{s.t.} \quad & \dot{x}(t) = f(t, x(t), u(t), \omega), \\
& b(x(t_0), x(t_f)) = 0_{\mathbb{R}^{n_b}}, \\
& p(t, x(t), u(t)) \leq 0_{\mathbb{R}^{n_p}}, \\
& s(t, x(t)) \leq 0_{\mathbb{R}^{n_s}}, \\
& u(t) \in \mathcal{U}.
\end{aligned}
$$

Space and norm

Three Banach spaces:

state and control: $(Z, \|\cdot\|_Z)$

$Z := W_{1,\infty}^{n_x}(D) \times L_{\infty}^{n_u}(D), \quad \|(x, z)\|_Z := \max\{\|x\|_{1,\infty}, \|u\|_\infty\},$

Let $z = (x, u) \in Z$

Equality constraints: dynamics and boundary conditions $(V, \|\cdot\|_V)$

$V := L_\infty^{n_x}(D) \times \mathbb{R}^{n_\varphi}, \quad \|(v_1, v_2)\|_V := \max\{\|v_1\|_\infty, \|v_2\|_2\},$

$$H(x, u) = 0_V = \begin{bmatrix} f(\cdot, x, u) - \dot{x} \\ -b(x(t_0), x(t_f)) \end{bmatrix}, \quad H : Z \to V$$

Inequality constraints: mixed control-state constraint & pure state $(W, \|\cdot\|_W)$

$W := L_\infty^{n_p}(D) \times C^{n_s}, \quad \|(w_1, w_2)\|_W := \max\{\|w_1\|_\infty, \|w_2\|_\infty\},$

$$G(x, u) = 0_W = \begin{bmatrix} -p(\cdot, x, u) \\ -s(\cdot, x) \end{bmatrix}, \quad G : Z \to W$$

K : a closed convex cone with vertex at 0_W

Set of admissible controls

$S := W_{1,\infty}^{n_x}(D) \times U_{\text{ad}}, \quad \text{with} \quad U_{\text{ad}} := \{u \in L_\infty^{n_u}(D) | u(t) \in \mathcal{U} \text{ a.e. in } D\},$

S is closed and convex with non-empty interior, if $\mathcal{U}$ is closed and convex with non-em

$J : Z \to \mathbb{R}$

Bochner space and norm

- **The Bochner space:**
  - For any Banach space Z of real-valued functions on the physical domain D with norm $\|\cdot\|_Z$, the set of strongly measurable r-summable mappings $v : \Omega \to Z$ is in the space

$$L^r(\Omega, Z) := \{v : \Omega \to Z \big| \; v \text{ strongly measurable}, \|v\|_{L^r(\Omega,Z)} < \infty\}.$$

For $0 < r \leq \infty$, the **Bochner norm** is defined as

$$\|v\|_{L^r(\Omega,Z)} = \begin{cases} \left(\int_\Omega \|v(\cdot,\omega)\|_Z^r \, d\mathbb{P}(\omega)\right)^{1/r} & \text{if } 0 < r < \infty, \\ \operatorname{ess\,sup}_{\omega\in\Omega} \|v(\cdot,\omega)\|_Z & \text{if } r = \infty. \end{cases}$$

Eg. $L^2(\Omega, Z)$ consists of Banach-valued functions with finite second moments.
Rk: for our problem $Z = W_{1,\infty}(D)$.

Uncertainty Quantification (UQ)

- **Source of uncertainty:** Uncertainties in the lift (C_L) and drag (C_D) coefficient:

$$C_L = -0.04 + 0.8\alpha, \qquad C_D = 0.012 - 0.01\alpha + 0.6\alpha^2, \quad \alpha : \text{angle of attack } [^\circ].$$

- **Baseline lift and drag coefficients:**

$$\omega_1 : -0.04 \quad \omega_2 : 0.8 \quad \omega_3 : 0.012 \quad \omega_4 : -0.01 \quad \omega_5 : 0.6$$

- **Uncertainty Quantification:**
  - Independent variation in each entry of the baseline coefficients. Each coefficient is uniformly distributed.
    τ: perturbation level ($\tau = 1\%$ & 2%). ω_k: coefficient in the k-th entry.
    - Joint density: $\pi(\boldsymbol{\omega}) = \prod_{k=1}^{d} \pi_k(\omega_k) = \prod_{k=1}^{d} \frac{1}{2\tau|\omega_k|}$.
    - 5-d parameter space: $W := \prod_{k=1}^{d} [\omega_k - \tau|\omega_k|, \omega_k + \tau|\omega_k|]$.
  - Each entry in the array is considered a parameter, so we have a 5-dimensional parameter space.

Objectives

- **Uncertainty Quantification:**
  Quantify the **impact of variations in the lift and drag coefficient** on determining
  1. The expectation of trajectories of hypersonic vehicles $\mathbb{E}\left[x(t,\boldsymbol{\omega})\right] = \int_W x(t,\boldsymbol{\omega})\pi(\boldsymbol{\omega})d\boldsymbol{\omega}$.
  2.
- **Improve sampling efficiency:**
  Use **sampling methods** to approximate the expectation. To improve efficiency
  1. Build a **surrogate function** for the optimal control problem and replace the expensive solution of the high-fidelity model to perform Monte Carlo samplings.

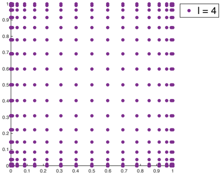
Sparse Grid Stochastic Collocation

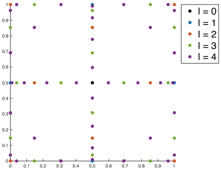
- **Idea**: Select a set of points $\{\boldsymbol{\omega}_k\}_{k=1}^{n_s}$ in the parameter space in a special way. For each $\boldsymbol{\omega}_k \in W$, we evaluate the solver to get the realization $\{x_h(\cdot, \boldsymbol{\omega}_k)\}_{k=1}^{n_s}$. A global interpolation (surrogate) $\widehat{x}_h \in C^0(W, Z_h)$ is then constructed to mimic the solution x by linear combinations of the point values,

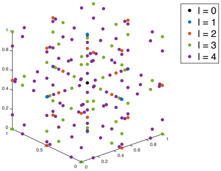
$$\widehat{x}_h(\cdot, \boldsymbol{\omega}) = \sum_{k=1}^{n_s} x_h(\cdot, \boldsymbol{\omega}_k)\ell_k(\boldsymbol{\omega}).$$

- **Reference**: V. Barthelmann and his collaborators.

V. Barthelmann, E. Novak, and K. Ritter. High dimensional polynomial interpolation on sparse grids. Advances in Computational Mathematics, 12:273–288, 2000.







Left to right: Full tensor grid 2d, level 4. Chebyshev sparse grids for 2d and 3d from level 0 to level 4.

Sparse Grid Stochastic Collocation

- **Software:** MATLAB `SPINTERP` package.
- **Developer and reference:** A.Klimke.
  SPINTERP, piecewise multilinear hierarchical sparse grid interpolation, http://people.math.sc.edu/burkardt/msrc/spinterp/spinterp.html, 2007.

Theorem: Interpolation error bound in L_∞ norm

$$\|f - \mathscr{S}(f)\|_\infty = \mathscr{O}\left(N^{-k} \cdot |\log N|^{(k+2)(d-1)+1}\right)$$

N: # of collocation nodes, d: dimension, k: measure of smoothness of ψ wrt $\boldsymbol{\omega}$.

Numerical Results

- Noise: 2%.
- Quadrature nodes: Chebyshev Gauss-Lobatto nodes.
- Number of sparse grid nodes used: 61.

Level q	0	1	2	3	4	5
# of sparse grid nodes P_q	1	11	61	241	801	2433

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Numerical results

Thank You |¬_¬|