

Efficient Computational Algorithms for Magnetic Equilibrium in a Fusion Reactor

Jiaxing Liang (University of Maryland, College Park)
Advisors: Howard Elman & Tonatiuh Sánchez-Vizuet

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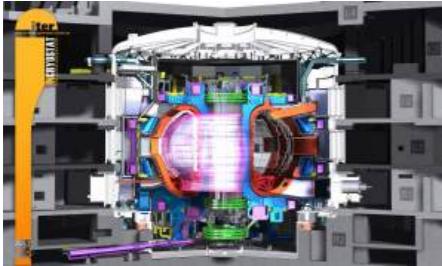


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Introduction

- Equilibrium Computation in magnetic confinement fusion devices Tokamaks.



The Grad-Shafranov Equation:

$$-\nabla \cdot \left(\frac{1}{\mu x} \nabla \psi \right) = \begin{cases} x \frac{d}{d\psi} p(\psi) + \frac{1}{2\mu x} \frac{d}{d\psi} g^2(\psi) & \text{in } \Omega_p(\psi) \\ I_i/S_i & \text{in } \Omega_{C_i} \\ 0 & \text{elsewhere} \end{cases}$$

$$\lim_{\|(x,y)\| \rightarrow \infty} \psi(x, y) = 0.$$

ψ : poloidal flux function, flow in poloidal direction with constant angle in the cylindrical coordinate.

p : hydrodynamic pressure, $p(\psi)$

I_i : current intensity in the coils

$\Omega_p(\psi)$: confinement region within plasma boundary

μ : magnetic permeability

g : toroidal field function, $g(\psi)$

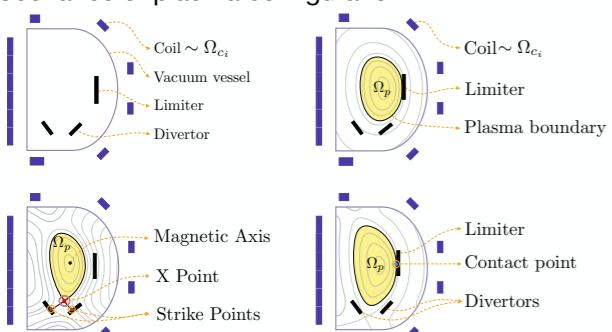
S_i : cross section area of the coils

Ω_{C_i} : regions where the currents locate

• Semilinear elliptic, free boundary problem.

• Uncertainty Quantification: Current intensities of 12 external coils.

• Scenarios of plasma configuration:



Limited configuration: no separatrix - contact point - limiter prevents plasma from touching the vacuum vessel physically.

Diverted configuration: last closed line before limiter (separatrix) - x point & strike points - divertor removes impurities.

Objective

- Quantify the impact of variations in the current intensity on key physical quantities, like x-point & strike point locations etc..
- Approximate the expectation of the equilibrium configuration with sampling methods like Monte Carlo & enhance efficiency with surrogate computations.

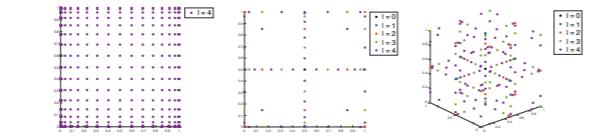
Surrogate function

- Build surrogate function with sparse grid stochastic collocation to replace the nonlinear solve.

Idea: sets of current $\{\xi^{(k)}, k = 1, \dots, n_{sc}\}$ in the parameter space are chosen in a special way. For each ξ , we evaluate the nonlinear equation to get the direct solutions $\{\psi_h^{(k)}\}$. An interpolation (surrogate) is then constructed to mimic the original nonlinear function with

$$\widehat{\psi}_h(\cdot, \xi) = \sum_k \psi_h^{(k)}(\cdot) L_{\xi^{(k)}}(\xi)$$

Reference: V. Barthelmann, E. Novak, and K. Ritter. High dimensional polynomial interpolation on sparse grids. *Advances in Computational Mathematics*, 12:273–288, 2000.



Left to right: Full tensor grid 2d, level 4. Chebyshev sparse grids for 2d and 3d from level 0 to level 4.

Compared to full tensor grid, Sparse Grid Stochastic Collocation requires fewer grids but with satisfying accuracy.

Theorem: Interpolation error bound in L_∞ norm

$$\|f - \mathcal{S}(f)\|_\infty = \mathcal{O}(N^{-k} \cdot \log N^{(k+2)(d-1)+1})$$

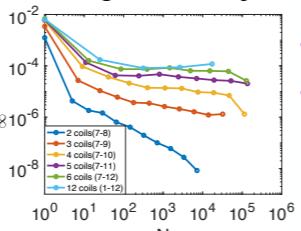
N: # of collocation nodes, d: dimension, k: measure of smoothness of ψ wrt ξ .

Three Approaches

- Monte Carlo + Surrogate
- Multilevel Monte Carlo + Nonlinear solve
- Multilevel Monte Carlo + Surrogate

Monte Carlo + Surrogate

Convergence study



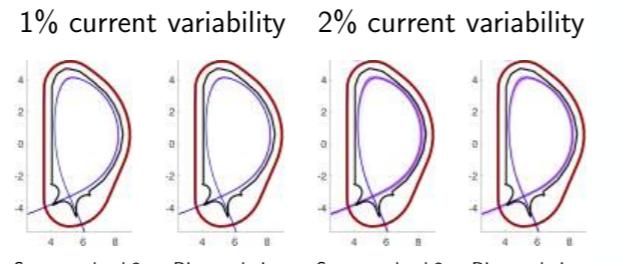
- Fewer coils lead to faster error decay.
- Most errors decrease except for perturbing currents in all 12 coils. Reason: as d increases, need large N to observe error decay.

Efficiency study

	Level 2	Level 3
1% Noise	Surrogate 5100	Surrogate 5000
	Time per Sample (s.) 0.16	Time per Sample (s.) 0.59
Total Time (s.)	831	15141
2% Noise	Surrogate 20000	Surrogate 31000
	Time per Sample (s.) 0.16	Time per Sample (s.) 0.59
Total Time (s.)	3264	94602
	1 hour	4.1 hours
	26 hours	30 hours

- Cost is way low for surrogate than nonlinear solve.

Accuracy study



- Surrogate evaluations and direct solutions behave similarly for separatrix, x points, and strike points.
- Surrogate mimics the non-linear function with a minor accuracy loss.

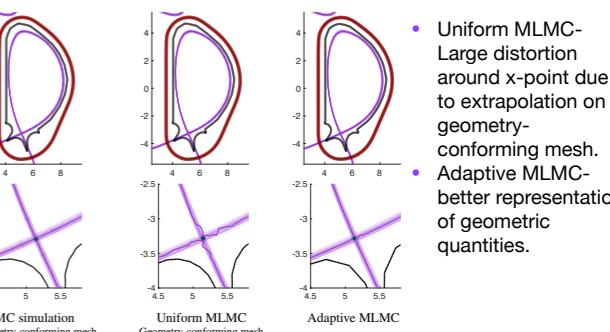
Quantity of interest	Evaluation of surrogate		Direct solver	
	Sample A (2500 realizations)	Variance	Sample B (1900 realizations)	Variance
x point	(5.14,-3.29)	(2.89,-0.4,1.48e-03)	(5.14,-3.29)	(2.95e-04,1.31e-03)
magnetic axis	(6.41,0.61)	(1.53,-0.3,1.14e-03)	(6.41,0.61)	(1.19e-03,6.42e-04)
strike points	(4.16,-3.71)	(2.42e-03,2.38e-03)	(4.16,-3.71)	(8.59e-04,2.09e-03)
inverse aspect ratio	0.32	5.53e-06	0.32	4.85e-06
elongation	1.86	3.00e-04	1.86	3.12e-04
upper triangularity	0.43	2.87e-04	0.43	2.65e-04
lower triangularity	0.53	2.37e-04	0.53	2.15e-04

- Means and variances are comparable for both methods.
- Surrogate requires more samples than nonlinear solve.

Multilevel Monte Carlo + Nonlinear solve

Monte Carlo

- Estimator: $A_{MC} := \mathbb{E}(u_h) = \frac{1}{N} \sum_{i=1}^N u_h^{(i)}$
- Relative MSE: $\mathcal{E}_{AMC}^2 = \frac{\|\mathbb{E}(u) - \mathbb{E}(u_h)\|_Z^2}{\|\mathbb{E}(u)\|_{L^2(W,Z)}^2} + \frac{\mathbb{V}(u_h)}{N \|\mathbb{E}(u)\|_{L^2(W,Z)}^2} = \mathcal{E}_{Bias}^2(h) + \mathcal{E}_{Stat}^2(N)$
- Error splitting: $\begin{cases} \mathcal{E}_{DisErr}^2 \leq (1-\theta)\epsilon^2 & \rightarrow M = M(\epsilon), \\ \mathcal{E}_{Stat}^2 \leq \theta\epsilon^2 & \rightarrow N = N(\epsilon). \end{cases}$
- Sampling cost: $C_{MC} = \mathcal{O}(\epsilon^{-2-\gamma/\alpha})$



Multilevel Monte Carlo + Surrogate

Three types of surrogate

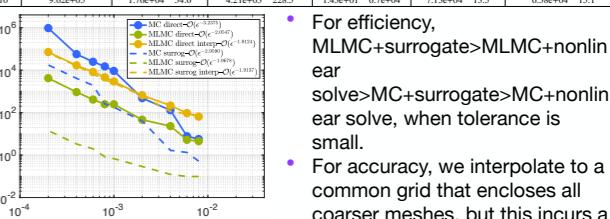
- Single-level sparse grid + single-level spatial grid
- Single-level sparse grid + multilevel spatial grid
- Multilevel in both sparse & spatial grids

Two sampling methods

- Monte Carlo
- Multilevel Monte Carlo

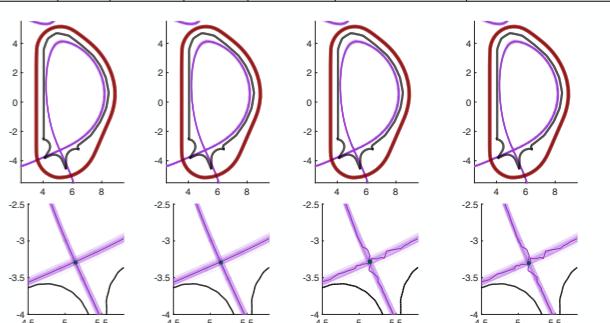
Efficiency study

ϵ	MC-FE Direct solver	MC-FE Surrogate	MLMC-FE Direct solver	MLMC-FE Surrogate	MLMC-FE DS interp	MLMC-FE Surrogate interp
8×10^{-3}	5.67e+00	3.16e+00	1.0	4.52e+00	6.84e+00	0.08
6×10^{-3}	7.69e+00	1.33e+00	5.8	5.22e+00	9.81e+00	8.84e+01
4×10^{-3}	1.30e+02	1.67e+00	78.1	2.32e+01	1.1e+03	1.14e+02
2×10^{-3}	4.83e+02	4.07e+01	11.9	4.62e+02	10.5	2.91e+01
10^{-3}	9.22e+03	1.80e+02	51.3	2.47e+02	37.3	7.76e+01
8×10^{-4}	1.50e+04	4.07e+02	57.8	2.48e+02	60.5	8.36e+01
6×10^{-4}	2.68e+04	1.93e+03	12.9	4.13e+02	60.0	1.97e+00
4×10^{-4}	5.68e+04	4.12e+03	13.8	9.29e+02	61.1	3.34e+00
2×10^{-4}	9.62e+05	1.76e+04	54.6	4.21e+03	228.5	1.43e+01



Accuracy study

x point	MC-FE DS	MC-FE Surrogate	MLMC-FE DS	MLMC-FE Surrogate	MLMC-FE DS interp 2 common	MLMC-FE Surrogate interp 2 common
magnetic axis	(6.41,0.61)	(6.41,0.61)	(6.44,0.56)	(6.44,0.56)	(6.41,0.61)	(6.41,0.61)
strike points	(4.16,-3.71)	(4.16,-3.71)	(5.56,-4.22)	(5.56,-4.22)	(4.16,-3.71)	(5.56,-4.22)
inverse aspect ratio	0.32	0.32	0.32	0.32	0.32	0.32
elongation	1.86	1.86	1.86	1.86	1.86	1.86
upper triangularity	0.43	0.43	0.43	0.43	0.43	0.43
lower triangularity	0.53	0.53	0.53	0.53	0.53	0.53



Reference

- Surrogate approximation of the Grad-Shafranov free boundary problem via stochastic collocation on sparse grids, with Howard Elman and Tonatiuh Sánchez-Vizuet. (*Journal of Computational Physics*, 448 (2022), paper No. 110699.)
- Multilevel Monte Carlo methods for the Grad-Shafranov free boundary problem, with Howard Elman and Tonatiuh Sánchez-Vizuet. (*Computer Physics Communications*, 298 (2024), paper No. 109099.)