Optimal control with uncertainty quantification write-up

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1. The initial value problem with random parameters

Let $I = [t_0, t_f] \subseteq \mathbb{R},\ t_0 < t_f$, be the time interval, $\boldsymbol{\xi} = [\xi_1, \cdots, \xi_d], d \geq 1$, be a set of parameters characterizing the uncertainty random inputs. We treat them as mutually independent random variables such that

$$\boldsymbol{\xi} : \Omega \to W \subseteq \mathbb{R}^d, \quad \boldsymbol{\xi}(\omega) = [\xi_1(\omega), \cdots, \xi_d(\omega)],$$

where Ω is the set of possible outcomes in the probabilistic description and W is the parameter space. Our problem involves solving a stochastic initial value problem with the form

$$\begin{aligned} &\frac{d}{dt}x(t) = f(t, x(t), \boldsymbol{\xi}), && t \in I, \\ &x(t_0) = x_0, && t = t_0. \end{aligned} \tag{1}$$

The solution space to the initial value problem (1) is $H_0^1(I)$. We seek a random function $x : I \times \Omega \to H_0^1(I)$ such that (1) holds almost surely.

1.1. Bochnor space and norm

The fully specified problem (1) is given on a space defined in terms of both the spatial space Z (for our problem is $H_0^1(I)$) and a parameter space W. This space is known as a *Bochner space*. Given a complete and separable probability space $(W, \Sigma, \mathbb{P})$ and a Banach space $(Z, \|\cdot\|_Z)$, the Bochner space contains the set of strongly measurable r-summable mappings $u(\cdot, \boldsymbol{\omega}) : W \to Z$ such that the corresponding norm is finite:

$$L^r(W, Z) := \{u(\cdot, \boldsymbol{\omega}) : W \to Z \,\big|\, u(\cdot, \boldsymbol{\omega}) \text{ strongly measurable},\ \|u\|_{L^r(W,Z)} < \infty\}.$$

This space is equipped with a norm

$$\|u\|_{L^r(W,Z)} = \begin{cases} \left(\int_W \|u(\cdot, \boldsymbol{\omega})\|_Z^r \, d\mathbb{P}(\boldsymbol{\omega})\right)^{1/r} = \left(\mathbb{E}\left[\|u(\cdot, \boldsymbol{\omega})\|_Z^r\right]\right)^{1/r} & \text{if } 0 < r < \infty, \\ \operatorname{ess\,sup}_{\boldsymbol{\omega} \in W} \|u(\cdot, \boldsymbol{\omega})\|_Z & \text{if } r = \infty. \end{cases}$$

1.2. Stochastic collocation method

Let $\{\boldsymbol{\xi}^{(i)}\}_{i=1}^N$ be a set of nodes in the random space with N nodes, and let $\{x(\cdot, \boldsymbol{\xi}^{(i)})\}$ represent the realizations of (1) for each sample i. We seek an approximation $\widehat{x}(\cdot, \boldsymbol{\xi})$ in a proper polynomial space such that $\widehat{x}(t, \boldsymbol{\xi}^{(i)}) = x(t, \boldsymbol{\xi}^{(i)})$ for all $i = 1, \cdots, N$ and $\widehat{x}(\cdot, \boldsymbol{\xi})$ is an approximation to the true solution $x(\cdot, \boldsymbol{\xi})$ in the sense that the difference measured in L^p norm is sufficiently small. Using a Lagrangian interpolation approach, we express the interpolant as

$$\widehat{x}(t, \boldsymbol{\xi}) = \sum_{i=1}^{N} x(t, \boldsymbol{\xi}^{(i)}) \ell_i(\boldsymbol{\xi}),$$

where ℓ_i are the Lagrangian interpolating polynomials satisfying $\ell_i(\boldsymbol{\xi}^{(j)}) = \delta_{i,j}$, the Kronecker delta.

2. Construction of the surrogate function

We now present a brief overview of the sparse grid stochastic collocation method [1, 3, 4, 6] for approximating the solution to (1) with stochastic parameters. We will describe the method in terms of a generic solution u. This method uses a set of m_i nested nodes $X^i = \left\{x_1^i, \ldots, x_{m_i}^i\right\}$ within $[0, 1]$ such that $X^i \subset X^{i+1}$. This setup allows for the generation of *sparse grid nodes* for dimension d and *level* q (with $q \geq d$) as

$$H(q, d) = \bigcup_{q-d+1 \leq |\boldsymbol{i}| \leq q} \left(X^{i_1} \times \cdots \times X^{i_d}\right) \in [0, 1]^d, \tag{2}$$

where $|\boldsymbol{i}| = i_1 + \ldots + i_d$. This set of interpolation points is a direct consequence of the construction of the Smolyak quadrature formula, defined by

$$\mathscr{S}_{q,d}[u] = \sum_{q+1 \leq |\boldsymbol{i}| \leq q+d} (-1)^{q+d-|\boldsymbol{i}|} \binom{d-1}{q+d-|\boldsymbol{i}|} \cdot \left(\mathrm{I}_{X^{i_1}} \otimes \cdots \otimes \mathrm{I}_{X^{i_d}}\right)[u]. \tag{3}$$

where $I_{X^i}[u] := \sum_{j=1}^{m_i} u(x_j^i, \cdot)\phi_j$ represents the univariate quadrature, and the basis functions $\phi_k(x_j^i)$ is Kronecker delta, equating to 1 when $k = j$.

For our model problem, the sparse grid stochastic collocation method constructs the surrogate function $\widehat{u}$ as per (3) by computing the direct solution of the discrete version of (1) at isotropic sparse grid nodes (2) with the Clenshaw-Curtis quadrature abscissa [1, 2].

2.1. Interpolation error

As stated in [5, 7], consider $u \in C^0(W, H_0^1(I))$. Let the single-dimensional interval along the k-th dimension be

$$W_k = \left[I_k - \tau\,|I_k|\,, I_k + \tau\,|I_k|\right],$$

with its corresponding complementary multi-dimensional parameter space being

$$W_k^c = \prod_{i=1, i\neq k}^{d} W_i.$$

Given an arbitrary element ω_k^c in W_k^c, and for each ω_k in W_k, we assume the function $u(\cdot, \omega_k, \omega_k^c) : W_k \to C^0(W_k^c; Z)$ admits an analytic extension $u(\cdot, z, \omega_k^c)$ in the region

$$W_k^* := \{z \in \mathbb{C} : \mathrm{dist}(z, W_k) \leq \iota_k \;\text{ for some } \iota_k > 0\}$$

of the complex plane, then the interpolation error exhibits an algebraic convergence rate

$$\left\|u - \widehat{u}\right\|_\infty \leq C P^{-\nu}, \tag{4}$$

where P denotes the number of nodes in the sparse grid ($P = \dim(H(q, d))$), C is a positive constant and the power ν is an increasing function of the size of the domain of definition of the function's analytic extension in the complex plane. On the other hand, if the mapping is less regular with respect to the stochastic parameter $\boldsymbol{\xi}$, for instance of class $C^k(\boldsymbol{W}, Z)$, then [1, Theorem 8]

$$\left\|u - \widehat{u}\right\|_\infty \leq C P^{-k} |\log P|^{(k+2)(d-1)+1}\,, \tag{5}$$

where $\mathcal{I}_d$ is the identity operator in a d-dimensional space.

3. Appendix

References

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