Theorem Proofing

Let $S = \sum_{i=0}^{N} \lambda_i$

$$\lambda_0 |V_0|^2 + \sum_{i=1}^{N} \lambda_i |V_i|^2 = \sum_{i=1}^{N} \frac{\lambda_0 \lambda_i}{S} |V_i + V_0|^2 + \sum_{1 \le j < k \le N} \frac{\lambda_j \lambda_k}{S} |V_j - V_k|^2$$

$$\left(\sum_{j=0}^{N} \lambda_j\right) \cdot \lambda_0 \left| \sum_{j=1}^{N} \lambda_j V_j \big/ \lambda_0 \right|^2 + \left(\sum_{j=0}^{N} \lambda_j\right) \cdot \left(\sum_{i=1}^{N} \frac{1}{\lambda_i} |\lambda_i V_i|^2 \right) = \sum_{j=1}^{N} \lambda_0 \lambda_j \left[V_j^2 + 2V_j \frac{\sum_{i=1}^{N} \lambda_i V_i}{\lambda_0} + \left(\frac{\sum_{i=1}^{N} \lambda_i V_i}{\lambda_0} \right)^2 \right] + \sum_{1 \le j < k \le N} \lambda_j \lambda_k |V_j - V_k$$

$$\left(\sum_{j=0}^{N} \lambda_j\right) \left(\sum_{i=1}^{N} \frac{1}{\lambda_i} (\lambda_i V_i)^2 \right) + \left| \sum_{j=1}^{N} \lambda_j V_j \right|^2 = \sum_{j=1}^{N} \frac{\lambda_0 (\lambda_j V_j^2)}{\lambda_j} + \sum_{j=1}^{N} \lambda_j \cdot 2V_j \cdot \sum_{i=1}^{N} \lambda_i V_i + \sum_{1 \le j < k \le N} \lambda_j \lambda_k (V_j^2 + V_k^2 - 2V_j V_k)$$

$$\left(\sum_{j=1}^{N} \lambda_j\right) \left(\sum_{i=1}^{N} \frac{1}{\lambda_i} (\lambda_i V_i)^2 \right) + \left| \sum_{j=1}^{N} \lambda_j V_j \right|^2 = 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j V_i V_j + \sum_{1 \le j < k \le N} \lambda_k \lambda_j (V_j^2 + V_k^2) - 2 \sum_{1 \le j < k \le N} (V_j V_k \lambda_j \lambda_k)$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j V_i^2 + \left| \sum_{j=1}^{N} \lambda_j V_j \right|^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} 2 \lambda_i \lambda_j V_i V_j + \sum_{1 \le j < k \le N} \lambda_k \lambda_j V_j^2 + \lambda_k \lambda_j V_k^2 - 2 \sum_{1 \le j < k \le N} (\lambda_j \lambda_k V_j V_k)$$

Note that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j V_i^2 = \sum_{i=1}^{N} \lambda_i^2 V_i^2 + \sum_{1 \le j < k \le N} \lambda_j \lambda_k V_j^2 + \lambda_j \lambda_k V_k^2$$

$$\left| \sum_{j=1}^{N} \lambda_j V_j \right|^2 = \sum_{i=1}^{N} \lambda_i^2 V_i^2 + \sum_{1 \le j < k \le N} 2 \lambda_j \lambda_k V_j V_k$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} 2 \lambda_i \lambda_j V_i V_j = 2 \left[\sum_{i=1}^{N} \lambda_i^2 V_i^2 + 2 \sum_{1 \le j < k \le N} \lambda_j \lambda_k V_j V_k \right]$$

Based on above observations:

$$LHS = 2\sum_{i=1}^{N} \lambda_i^2 V_i^2 + \sum_{1\le j<k\le N} \lambda_j \lambda_k (V_j^2 + V_k^2) + \sum_{1\le j<k\le N} 2\lambda_j \lambda_k V_j V_k$$

$$RHS = 2\sum_{i=1}^{N} \lambda_i^2 V_i^2 + \sum_{1\le j<k\le N} \lambda_j \lambda_k (V_j^2 + V_k^2) + \sum_{1\le j<k\le N} 2\lambda_j \lambda_k V_j V_k$$

As we can see, LHS and RHS are exactly the same. $\square$