

Density estimation, clustering, PCA and principal curves

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Setup

```
data <- read.table('zip.train')  
# get only zeros  
data <- data[which(data[,1]==0), -1]
```

Hierarchical clustering

Euclidean distance:

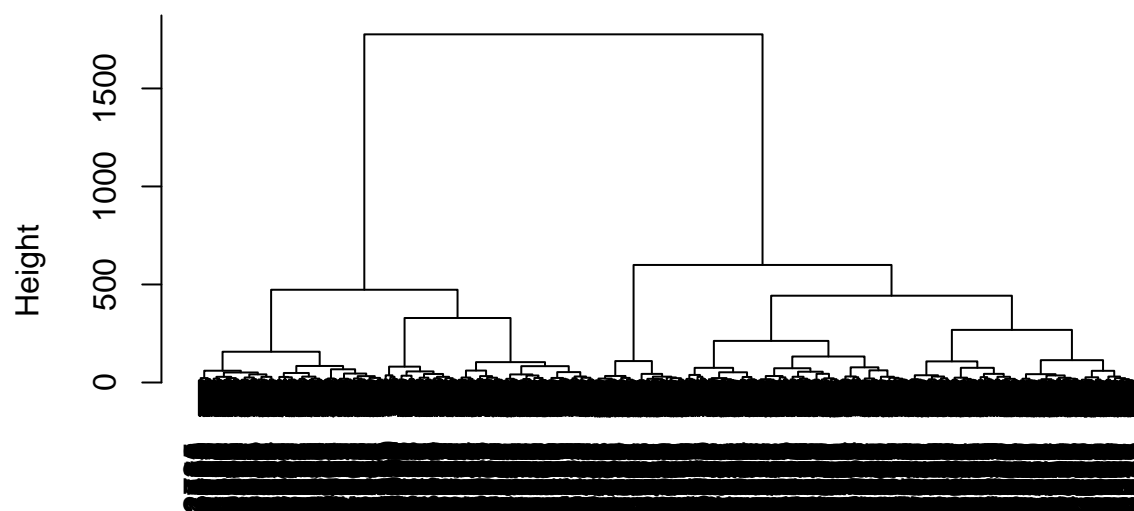
```
d.Eucl <- dist(data)  
summary(d.Eucl)
```

```
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.  
##      3.01   11.89   14.31   14.09   16.46   23.43
```

Clustering:

```
k <- 4  
hcl.Eucl <- hclust(d.Eucl, method="ward.D")  
plot(hcl.Eucl)
```

Cluster Dendrogram



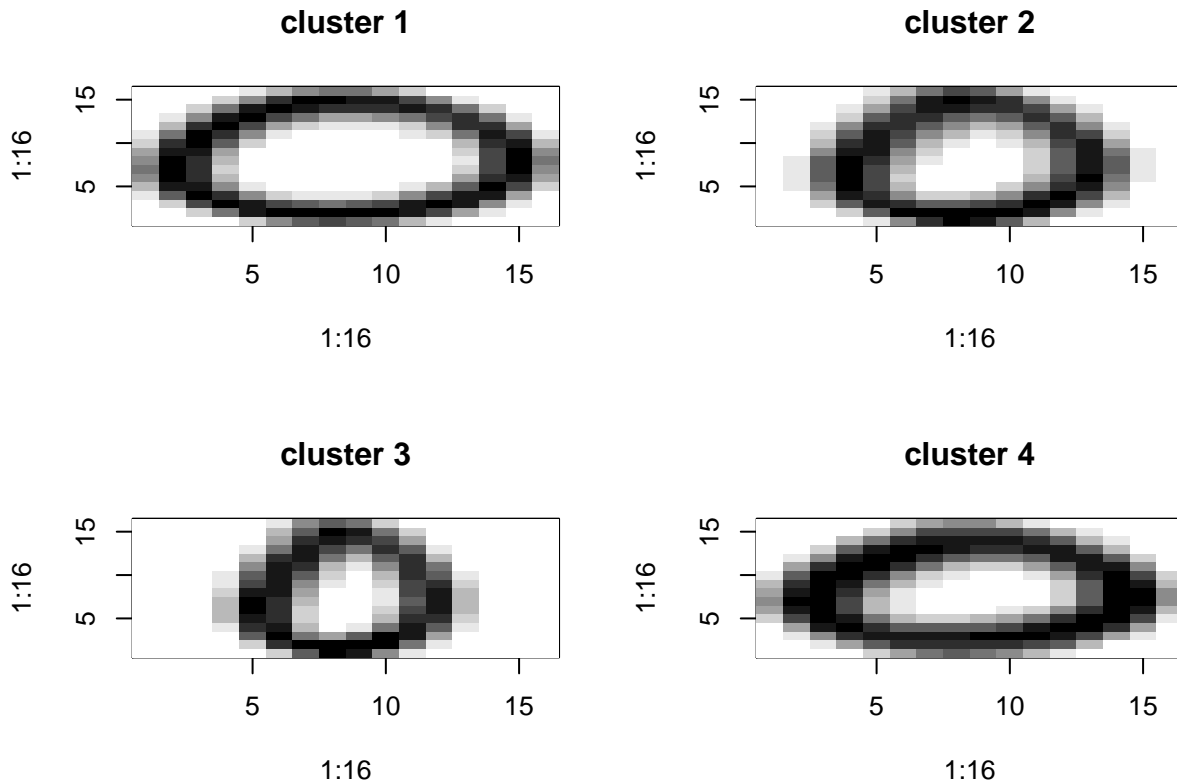
```
d.Eucl
hclust (*, "ward.D")
```

```
hcl.Eucl.k <- cutree(hcl.Eucl,k)
table(hcl.Eucl.k)
```

```
## hcl.Eucl.k
## 1 2 3 4
## 236 574 115 269
```

```
clusters <- hcl.Eucl.k

# Plot the average digit of each cluster
op<-par(mfrow=c(2,2))
for(i in 1:k) {
  mean.digit <- colMeans(data[which(clusters == i), ])
  plot.zip(mean.digit)
  title(paste("cluster", i, sep=" "))
}
```



Principal components

```
principal_components_obj <- princomp(data)
summary(principal_components_obj)
```

Importance of components:

	Comp.1	Comp.2	Comp.3	Comp.4	Comp.5
## Standard deviation	5.4146472	3.6375417	3.2365429	2.37779476	2.18976881
## Proportion of Variance	0.2819983	0.1272688	0.1007555	0.05438196	0.04612142
## Cumulative Proportion	0.2819983	0.4092671	0.5100227	0.56440464	0.61052606
	Comp.6	Comp.7	Comp.8	Comp.9	Comp.10
## Standard deviation	2.01868699	1.69603163	1.56844440	1.44979302	1.34718509
## Proportion of Variance	0.03919621	0.02766776	0.02366162	0.02021707	0.01745664
## Cumulative Proportion	0.64972227	0.67739004	0.70105165	0.72126872	0.73872536
	Comp.11	Comp.12	Comp.13	Comp.14	Comp.15
## Standard deviation	1.28436659	1.12409757	1.07287822	1.05696480	1.02427987
## Proportion of Variance	0.01586661	0.01215386	0.01107151	0.01074551	0.01009121
## Cumulative Proportion	0.75459197	0.76674583	0.77781734	0.78856286	0.79865407
	Comp.16	Comp.17	Comp.18	Comp.19	
## Standard deviation	0.99457363	0.954461634	0.907175383	0.849861296	
## Proportion of Variance	0.00951437	0.008762401	0.007915688	0.006947079	
## Cumulative Proportion	0.80816844	0.816930843	0.824846531	0.831793611	
	Comp.20	Comp.21	Comp.22	Comp.23	
## Standard deviation	0.829865023	0.80490110	0.789560840	0.739034167	
## Proportion of Variance	0.006624011	0.00623148	0.005996217	0.005253336	
## Cumulative Proportion	0.838417622	0.84464910	0.850645319	0.855898655	

##	Comp.24	Comp.25	Comp.26	Comp.27
## Standard deviation	0.726713484	0.698193574	0.687303306	0.668185865
## Proportion of Variance	0.005079636	0.004688758	0.004543631	0.004294382
## Cumulative Proportion	0.860978291	0.865667050	0.870210680	0.874505063
##	Comp.28	Comp.29	Comp.30	Comp.31
## Standard deviation	0.659304035	0.647999455	0.637373101	0.5939174
## Proportion of Variance	0.004180976	0.004038829	0.003907452	0.0033928
## Cumulative Proportion	0.878686038	0.882724867	0.886632319	0.8900251
##	Comp.32	Comp.33	Comp.34	Comp.35
## Standard deviation	0.587255986	0.573597975	0.567951056	0.555763676
## Proportion of Variance	0.003317119	0.003164619	0.003102616	0.002970889
## Cumulative Proportion	0.893342238	0.896506857	0.899609472	0.902580362
##	Comp.36	Comp.37	Comp.38	Comp.39
## Standard deviation	0.552024997	0.538698345	0.526466829	0.518791171
## Proportion of Variance	0.002931053	0.002791242	0.002665927	0.002588757
## Cumulative Proportion	0.905511414	0.908302656	0.910968583	0.913557340
##	Comp.40	Comp.41	Comp.42	Comp.43
## Standard deviation	0.507826032	0.503375796	0.482878731	0.478021925
## Proportion of Variance	0.002480482	0.002437198	0.002242757	0.002197869
## Cumulative Proportion	0.916037822	0.918475020	0.920717777	0.922915646
##	Comp.44	Comp.45	Comp.46	Comp.47
## Standard deviation	0.465353688	0.45635247	0.445994937	0.436932516
## Proportion of Variance	0.002082919	0.00200312	0.001913225	0.001836263
## Cumulative Proportion	0.924998566	0.92700169	0.928914910	0.930751173
##	Comp.48	Comp.49	Comp.50	Comp.51
## Standard deviation	0.424917547	0.420832896	0.414851340	0.405769129
## Proportion of Variance	0.001736663	0.001703435	0.001655355	0.001583668
## Cumulative Proportion	0.932487836	0.934191271	0.935846626	0.937430294
##	Comp.52	Comp.53	Comp.54	Comp.55
## Standard deviation	0.396333723	0.391326504	0.386534129	0.375530467
## Proportion of Variance	0.001510874	0.001472939	0.001437083	0.001356427
## Cumulative Proportion	0.938941168	0.940414107	0.941851190	0.943207617
##	Comp.56	Comp.57	Comp.58	Comp.59
## Standard deviation	0.372638199	0.366779612	0.358008929	0.35229274
## Proportion of Variance	0.001335614	0.001293947	0.001232804	0.00119375
## Cumulative Proportion	0.944543231	0.945837178	0.947069981	0.94826373
##	Comp.60	Comp.61	Comp.62	Comp.63
## Standard deviation	0.347528746	0.344140744	0.343056131	0.334199237
## Proportion of Variance	0.001161683	0.001139143	0.001131974	0.001074279
## Cumulative Proportion	0.949425415	0.950564558	0.951696532	0.952770811
##	Comp.64	Comp.65	Comp.66	Comp.67
## Standard deviation	0.331720252	0.326452953	0.323800873	0.3189413399
## Proportion of Variance	0.001058401	0.001025055	0.001008468	0.0009784255
## Cumulative Proportion	0.953829212	0.954854267	0.955862735	0.9568411610
##	Comp.68	Comp.69	Comp.70	Comp.71
## Standard deviation	0.3167406817	0.3086003943	0.3058864767	0.2984098895
## Proportion of Variance	0.0009649701	0.0009160077	0.0008999673	0.0008565103
## Cumulative Proportion	0.9578061310	0.9587221387	0.9596221060	0.9604786163
##	Comp.72	Comp.73	Comp.74	Comp.75
## Standard deviation	0.2976113718	0.2911986837	0.2881088948	0.2864809198
## Proportion of Variance	0.0008519325	0.0008156146	0.0007983981	0.0007894008
## Cumulative Proportion	0.9613305488	0.9621461633	0.9629445614	0.9637339622
##	Comp.76	Comp.77	Comp.78	Comp.79
## Standard deviation	0.2847447595	0.2807935876	0.2780327257	0.2731934205

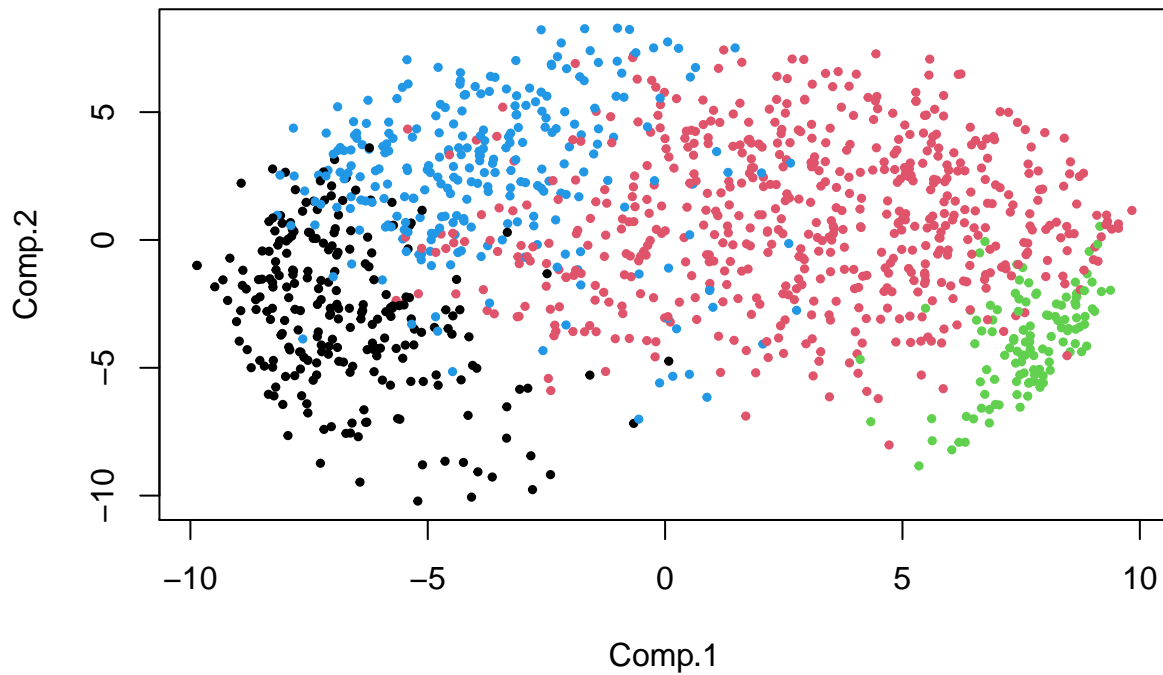
## Proportion of Variance	0.0007798618	0.0007583689	0.0007435291	0.0007178714
## Cumulative Proportion	0.9645138240	0.9652721929	0.9660157221	0.9667335934
##	Comp.80	Comp.81	Comp.82	Comp.83
## Standard deviation	0.269043407	0.2668153522	0.2632005904	0.2601760043
## Proportion of Variance	0.000696227	0.0006847433	0.0006663154	0.0006510894
## Cumulative Proportion	0.967429820	0.9681145637	0.9687808792	0.9694319686
##	Comp.84	Comp.85	Comp.86	Comp.87
## Standard deviation	0.2578378676	0.2542594666	0.2522523161	0.2493000441
## Proportion of Variance	0.0006394396	0.0006218139	0.0006120353	0.0005977931
## Cumulative Proportion	0.9700714082	0.9706932221	0.9713052574	0.9719030505
##	Comp.88	Comp.89	Comp.90	Comp.91
## Standard deviation	0.246262825	0.2451963153	0.2426160997	0.2417065600
## Proportion of Variance	0.000583316	0.0005782745	0.0005661681	0.0005619311
## Cumulative Proportion	0.972486366	0.9730646409	0.9736308090	0.9741927401
##	Comp.92	Comp.93	Comp.94	Comp.95
## Standard deviation	0.2366038035	0.234616065	0.2320289245	0.2282869642
## Proportion of Variance	0.0005384552	0.000529446	0.0005178338	0.0005012662
## Cumulative Proportion	0.9747311953	0.975260641	0.9757784751	0.9762797413
##	Comp.96	Comp.97	Comp.98	Comp.99
## Standard deviation	0.2261632475	0.2247714664	0.2216149963	0.2201608952
## Proportion of Variance	0.0004919831	0.0004859466	0.0004723941	0.0004662153
## Cumulative Proportion	0.9767717244	0.9772576710	0.9777300650	0.9781962803
##	Comp.100	Comp.101	Comp.102	Comp.103
## Standard deviation	0.2164098540	0.2158400379	0.2148681270	0.2114036810
## Proportion of Variance	0.0004504641	0.0004480951	0.0004440687	0.0004298642
## Cumulative Proportion	0.9786467445	0.9790948395	0.9795389082	0.9799687724
##	Comp.104	Comp.105	Comp.106	Comp.107
## Standard deviation	0.2106018075	0.206614161	0.2054938535	0.2046650989
## Proportion of Variance	0.0004266093	0.000410607	0.0004061662	0.0004028967
## Cumulative Proportion	0.9803953817	0.980805989	0.9812121549	0.9816150517
##	Comp.108	Comp.109	Comp.110	Comp.111
## Standard deviation	0.2023181690	0.2013254428	0.1988267239	0.1944103990
## Proportion of Variance	0.0003937095	0.0003898553	0.0003802381	0.0003635341
## Cumulative Proportion	0.9820087612	0.9823986165	0.9827788547	0.9831423888
##	Comp.112	Comp.113	Comp.114	Comp.115
## Standard deviation	0.1940925284	0.1928641710	0.1894963900	0.1878125686
## Proportion of Variance	0.0003623463	0.0003577744	0.0003453886	0.0003392778
## Cumulative Proportion	0.9835047350	0.9838625094	0.9842078981	0.9845471759
##	Comp.116	Comp.117	Comp.118	Comp.119
## Standard deviation	0.1859214843	0.1851664404	0.1824696768	0.1822530287
## Proportion of Variance	0.0003324798	0.0003297849	0.0003202488	0.0003194888
## Cumulative Proportion	0.9848796557	0.9852094406	0.9855296894	0.9858491782
##	Comp.120	Comp.121	Comp.122	Comp.123
## Standard deviation	0.1791788670	0.1771018975	0.1761411053	0.1747534096
## Proportion of Variance	0.0003088017	0.0003016842	0.0002984198	0.0002937362
## Cumulative Proportion	0.9861579800	0.9864596642	0.9867580839	0.9870518201
##	Comp.124	Comp.125	Comp.126	Comp.127
## Standard deviation	0.1742816490	0.1707503537	0.1702123751	0.1682441426
## Proportion of Variance	0.0002921524	0.0002804332	0.0002786688	0.0002722614
## Cumulative Proportion	0.9873439725	0.9876244057	0.9879030746	0.9881753360
##	Comp.128	Comp.129	Comp.130	Comp.131
## Standard deviation	0.1672353048	0.1665194654	0.1642036122	0.1615246860
## Proportion of Variance	0.0002690061	0.0002667081	0.0002593412	0.0002509481
## Cumulative Proportion	0.9884443420	0.9887110501	0.9889703913	0.9892213395

##	Comp.132	Comp.133	Comp.134	Comp.135
## Standard deviation	0.1592640275	0.1585407001	0.1576406743	0.157160574
## Proportion of Variance	0.0002439729	0.0002417618	0.0002390247	0.000237571
## Cumulative Proportion	0.9894653123	0.9897070742	0.9899460988	0.990183670
##	Comp.136	Comp.137	Comp.138	Comp.139
## Standard deviation	0.1561689694	0.1533914486	0.1522057732	0.1493433916
## Proportion of Variance	0.0002345825	0.0002263125	0.0002228273	0.0002145251
## Cumulative Proportion	0.9904182523	0.9906445648	0.9908673921	0.9910819172
##	Comp.140	Comp.141	Comp.142	Comp.143
## Standard deviation	0.1484194966	0.1472266246	0.1456170333	0.1445709162
## Proportion of Variance	0.0002118791	0.0002084869	0.0002039532	0.0002010333
## Cumulative Proportion	0.9912937963	0.9915022832	0.9917062364	0.9919072697
##	Comp.144	Comp.145	Comp.146	Comp.147
## Standard deviation	0.1428050446	0.1415669283	0.1412379305	0.1399251920
## Proportion of Variance	0.0001961522	0.0001927657	0.0001918708	0.0001883207
## Cumulative Proportion	0.9921034220	0.9922961877	0.9924880584	0.9926763791
##	Comp.148	Comp.149	Comp.150	Comp.151
## Standard deviation	0.1391200074	0.1356123701	0.1350575285	0.1333023238
## Proportion of Variance	0.0001861596	0.0001768906	0.0001754461	0.0001709156
## Cumulative Proportion	0.9928625387	0.9930394293	0.9932148754	0.9933857909
##	Comp.152	Comp.153	Comp.154	Comp.155
## Standard deviation	0.1329271277	0.1319336187	0.1304982924	0.1291583685
## Proportion of Variance	0.0001699548	0.0001674238	0.0001638007	0.0001604543
## Cumulative Proportion	0.9935557457	0.9937231695	0.9938869702	0.9940474245
##	Comp.156	Comp.157	Comp.158	Comp.159
## Standard deviation	0.1289054778	0.1254975436	0.1246460438	0.1232066092
## Proportion of Variance	0.0001598265	0.0001514874	0.0001494387	0.0001460072
## Cumulative Proportion	0.9942072510	0.9943587385	0.9945081772	0.9946541843
##	Comp.160	Comp.161	Comp.162	Comp.163
## Standard deviation	0.1225148046	0.1218834273	0.1187282063	0.1179410130
## Proportion of Variance	0.0001443721	0.0001428879	0.0001355857	0.0001337938
## Cumulative Proportion	0.9947985565	0.9949414444	0.9950770301	0.9952108238
##	Comp.164	Comp.165	Comp.166	Comp.167
## Standard deviation	0.1171581480	0.1162714740	0.1140422423	0.1136760817
## Proportion of Variance	0.0001320235	0.0001300327	0.0001250943	0.0001242923
## Cumulative Proportion	0.9953428473	0.9954728800	0.9955979743	0.9957222667
##	Comp.168	Comp.169	Comp.170	Comp.171
## Standard deviation	0.1128296715	0.1111635859	0.110626032	0.1097282184
## Proportion of Variance	0.0001224483	0.0001188588	0.000117712	0.0001158091
## Cumulative Proportion	0.9958447150	0.9959635738	0.996081286	0.9961970949
##	Comp.172	Comp.173	Comp.174	Comp.175
## Standard deviation	0.1087237168	0.1076607018	0.1067517898	0.1054908776
## Proportion of Variance	0.0001136985	0.0001114861	0.0001096116	0.0001070375
## Cumulative Proportion	0.9963107934	0.9964222795	0.9965318911	0.9966389286
##	Comp.176	Comp.177	Comp.178	Comp.179
## Standard deviation	0.1041939671	0.1036004532	0.1024508151	1.007169e-01
## Proportion of Variance	0.0001044218	0.0001032356	0.0001009571	9.756871e-05
## Cumulative Proportion	0.9967433504	0.9968465860	0.9969475431	9.970451e-01
##	Comp.180	Comp.181	Comp.182	Comp.183
## Standard deviation	9.967662e-02	9.922998e-02	9.776328e-02	9.691779e-02
## Proportion of Variance	9.556366e-05	9.470916e-05	9.193009e-05	9.034688e-05
## Cumulative Proportion	9.971407e-01	9.972354e-01	9.973273e-01	9.974177e-01
##	Comp.184	Comp.185	Comp.186	Comp.187
## Standard deviation	9.629211e-02	0.0954640238	9.391332e-02	9.317427e-02

## Proportion of Variance	8.918413e-05	0.0000876568	8.483216e-05	8.350225e-05
## Cumulative Proportion	9.975068e-01	0.9975945026	9.976793e-01	9.977628e-01
##	Comp.188	Comp.189	Comp.190	Comp.191
## Standard deviation	9.059901e-02	9.033555e-02	8.928564e-02	8.921304e-02
## Proportion of Variance	7.895017e-05	7.849166e-05	7.667775e-05	7.655311e-05
## Cumulative Proportion	9.978418e-01	9.979203e-01	9.979970e-01	9.980735e-01
##	Comp.192	Comp.193	Comp.194	Comp.195
## Standard deviation	8.697419e-02	0.0855516982	0.085076736	8.434026e-02
## Proportion of Variance	7.275903e-05	0.0000703985	0.000069619	6.841889e-05
## Cumulative Proportion	9.981463e-01	0.9982166672	0.998286286	9.983547e-01
##	Comp.196	Comp.197	Comp.198	Comp.199
## Standard deviation	0.0835673969	8.212212e-02	8.167815e-02	8.069939e-02
## Proportion of Variance	0.0000671707	6.486739e-05	6.416792e-05	6.263927e-05
## Cumulative Proportion	0.9984218758	9.984867e-01	9.985509e-01	9.986136e-01
##	Comp.200	Comp.201	Comp.202	Comp.203
## Standard deviation	7.957112e-02	7.822277e-02	7.789702e-02	7.677664e-02
## Proportion of Variance	6.089997e-05	5.885353e-05	5.836437e-05	5.669756e-05
## Cumulative Proportion	9.986745e-01	9.987333e-01	9.987917e-01	9.988484e-01
##	Comp.204	Comp.205	Comp.206	Comp.207
## Standard deviation	0.0762778739	0.0748997015	7.425422e-02	7.344300e-02
## Proportion of Variance	0.0000559633	0.0000539593	5.303328e-05	5.188084e-05
## Cumulative Proportion	0.9989043291	0.9989582884	9.990113e-01	9.990632e-01
##	Comp.208	Comp.209	Comp.210	Comp.211
## Standard deviation	7.242563e-02	7.206277e-02	7.103112e-02	7.034090e-02
## Proportion of Variance	5.045344e-05	4.994915e-05	4.852923e-05	4.759069e-05
## Cumulative Proportion	9.991137e-01	9.991636e-01	9.992121e-01	9.992597e-01
##	Comp.212	Comp.213	Comp.214	Comp.215
## Standard deviation	6.878580e-02	6.745533e-02	6.677591e-02	6.480702e-02
## Proportion of Variance	4.550967e-05	4.376619e-05	4.288899e-05	4.039711e-05
## Cumulative Proportion	9.993052e-01	9.993490e-01	9.993919e-01	9.994323e-01
##	Comp.216	Comp.217	Comp.218	Comp.219
## Standard deviation	6.424773e-02	6.263331e-02	6.033088e-02	5.853141e-02
## Proportion of Variance	3.970286e-05	3.773261e-05	3.500947e-05	3.295218e-05
## Cumulative Proportion	9.994720e-01	9.995097e-01	9.995447e-01	9.995777e-01
##	Comp.220	Comp.221	Comp.222	Comp.223
## Standard deviation	5.703652e-02	0.0559823254	0.0555226601	5.411284e-02
## Proportion of Variance	3.129048e-05	0.0000301445	0.0000296515	2.816481e-05
## Cumulative Proportion	9.996090e-01	0.9996391191	0.9996687706	9.996969e-01
##	Comp.224	Comp.225	Comp.226	Comp.227
## Standard deviation	5.241599e-02	5.145110e-02	4.964427e-02	4.870931e-02
## Proportion of Variance	2.642614e-05	2.546217e-05	2.370525e-05	2.282076e-05
## Cumulative Proportion	9.997234e-01	9.997488e-01	9.997725e-01	9.997953e-01
##	Comp.228	Comp.229	Comp.230	Comp.231
## Standard deviation	0.0478724030	4.684441e-02	0.0459254831	4.298732e-02
## Proportion of Variance	0.0000220433	2.110676e-05	0.0000202868	1.777407e-05
## Cumulative Proportion	0.9998173931	9.998385e-01	0.9998587866	9.998766e-01
##	Comp.232	Comp.233	Comp.234	Comp.235
## Standard deviation	4.208325e-02	4.089236e-02	3.651817e-02	3.343271e-02
## Proportion of Variance	1.703432e-05	1.608387e-05	1.282697e-05	1.075101e-05
## Cumulative Proportion	9.998936e-01	9.999097e-01	9.999225e-01	9.999333e-01
##	Comp.236	Comp.237	Comp.238	Comp.239
## Standard deviation	3.315996e-02	3.162968e-02	2.882956e-02	2.743623e-02
## Proportion of Variance	1.057631e-05	9.622671e-06	7.994332e-06	7.240276e-06
## Cumulative Proportion	9.999438e-01	9.999535e-01	9.999615e-01	9.999687e-01

```
##               Comp.240      Comp.241      Comp.242      Comp.243
## Standard deviation  2.686977e-02 2.431290e-02 2.218992e-02 2.123335e-02
## Proportion of Variance 6.944387e-06 5.685645e-06 4.736065e-06 4.336537e-06
## Cumulative Proportion 9.999756e-01 9.999813e-01 9.999861e-01 9.999904e-01
##               Comp.244      Comp.245      Comp.246      Comp.247
## Standard deviation  1.832865e-02 1.604184e-02 1.351387e-02 1.159116e-02
## Proportion of Variance 3.231225e-06 2.475225e-06 1.756572e-06 1.292290e-06
## Cumulative Proportion 9.999936e-01 9.999961e-01 9.999979e-01 9.999991e-01
##               Comp.248      Comp.249      Comp.250      Comp.251
## Standard deviation  7.942824e-03 4.754518e-03 1.417905e-03 7.173093e-04
## Proportion of Variance 6.068147e-07 2.174299e-07 1.933749e-08 4.949019e-09
## Cumulative Proportion 9.999998e-01 1.000000e+00 1.000000e+00 1.000000e+00
##               Comp.252      Comp.253      Comp.254      Comp.255
## Standard deviation  5.341281e-04 1.864795e-04 1.445722e-08 9.999422e-09
## Proportion of Variance 2.744082e-09 3.344786e-10 2.010369e-18 9.617362e-19
## Cumulative Proportion 1.000000e+00 1.000000e+00 1.000000e+00 1.000000e+00
##               Comp.256
## Standard deviation  0
## Proportion of Variance 0
## Cumulative Proportion 1
```

```
principal_components <- principal_components_obj$scores[, 1:2]
plot(principal_components, pch=19, cex=.5, col=clusters)
```



Density estimation

```
library('ks')
```

```
## Warning: package 'ks' was built under R version 4.0.3
```

```
# clustered.PC$scores[,1:2]
```

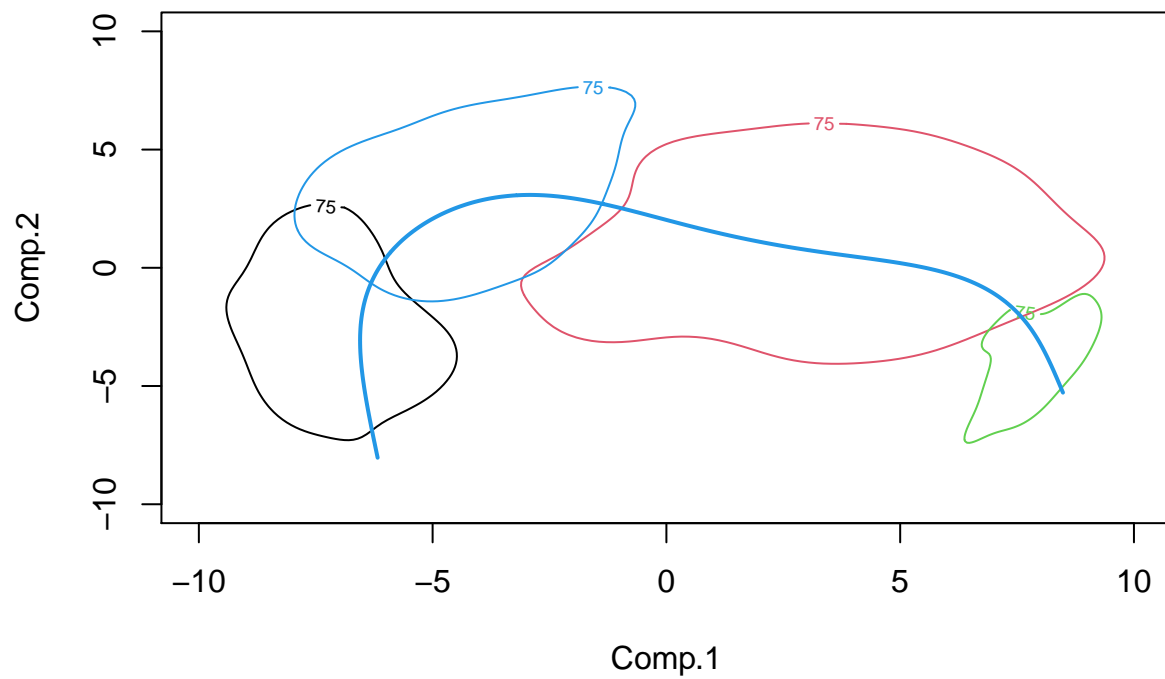
```
kde_1 = ks::kde(x = principal_components[which(clusters==1), 1:2])  
kde_2 = ks::kde(x = principal_components[which(clusters==2), 1:2])  
kde_3 = ks::kde(x = principal_components[which(clusters==3), 1:2])  
kde_4 = ks::kde(x = principal_components[which(clusters==4), 1:2])
```

```
plot(kde_1, cont=75, col=1, xlim=c(-10, 10), ylim=c(-10, 10))  
plot(kde_2, cont=75, col=2, add=TRUE)  
plot(kde_3, cont=75, col=3, add=TRUE)  
plot(kde_4, cont=75, col=4, add=TRUE)
```

```
library(princurve)
```

```
## Warning: package 'princurve' was built under R version 4.0.3
```

```
principal_curve <- principal_curve(as.matrix(data))  
principal_curve_projection <- predict(principal_components_obj, principal_curve$s)  
  
lines(principal_curve_projection[principal_curve$ord, 1:2], col=4, lwd=2)
```



```
kde_lambda_1 = ks::kde(x = principal_curve$lambda[which(clusters==1)])
kde_lambda_2 = ks::kde(x = principal_curve$lambda[which(clusters==2)])
kde_lambda_3 = ks::kde(x = principal_curve$lambda[which(clusters==3)])
kde_lambda_4 = ks::kde(x = principal_curve$lambda[which(clusters==4)])

plot(kde_lambda_1, col=1, xlim=c(-5, 40), ylim=c(0, 0.2), xlab='lambda')
plot(kde_lambda_2, col=2, add=TRUE)
plot(kde_lambda_3, col=3, add=TRUE)
plot(kde_lambda_4, col=4, add=TRUE)
```

