Exercise 1. (Double auction)

Consider the double auction with one seller and one buyer (Chatterjee and Samuelson (1983)). The valuations v_s and v_b of the seller and the buyer are independently drawn from the uniform distribution on the interval $[0, 1]$. Trade takes place if the bid p_b of the buyer exceeds the bid p_s of the seller. That is, when $p_b \geq p_s$. The price against which trade takes place is set to be $p = \frac{1}{2}(p_b + p_s)$.

Suppose both bidders use linear bid functions,

$$\begin{aligned} p_b(v_b) &= a_b v_b + c_b \\ p_s(v_s) &= a_s v_s + c_s, \end{aligned}$$

where a_b, c_b, a_s and c_s are parameters.

(a) Show that the unique linear BNE of the double auction is given by the bid functions

$$\begin{aligned} p_b(v_b) &= \frac{2}{3} v_b + \frac{1}{12} \\ p_s(v_s) &= \frac{2}{3} v_s + \frac{1}{4}, \end{aligned}$$

(b) Compute for which combinations (v_b, v_s) of valuations trade takes place in the equilibrium. Is the equilibrium efficient? Motivate your answer.

Exercise 2. (Ex post equilibrium in the common value auction)
Suppose a single indivisible object is auctioned off between two bidders, 1 and 2. Bidder 1 has a signal $s_1 \geq 0$ about the value of the object and bidder 2 has a signal $s_2 \geq 0$. Naturally, both bidders know their own signal, but not that of the other bidder. The true value of the object is

$$v = s_1 + s_2.$$

We use the Vickrey auction to sell the common value item. Thus, the bidders simultaneously bid a price for the object. If bidder 1 bids price p_1 and bidder 2 bids price p_2, the bidder that bids the highest price wins the object, but only pays the price bid by the other bidder.

Suppose that bidder 1 bids according to $p_1(s_1) = as_1$ and bidder 2 bids according to $p_2(s_2) = bs_2$, where a and b are positive real numbers with $\frac{1}{a} + \frac{1}{b} = 1$.
Suppose further that, after the bidders made their bid, they discuss what happened. They find out that bidder 1 had signal s_1^*, that bidder 2 had signal s_2^*, and that $p_1^* \leq p_2^*$ (where of course $p_1^* = as_1^*$ and $p_2^* = bs_2^*$).
(a) Compute the profit for bidder 2 in this case in terms of p_1^* and s_2^*.
(b) Show that the bid functions (p_1, p_2) constitute an ex post equilibrium.

Exercise 3. (Pareto optimal division of a cake)

Suppose that two consumers, A and B, want to divide a cake among the two of them. The utility of consumer A for a fraction α of the cake is given by $u_A(\alpha) = 100\alpha$, while the utility of consumer B for the remaining fraction $1 - \alpha$ of the cake is given by $u_B(1 - \alpha) = 100\sqrt{1 - \alpha}$.

(a) What are all possible (=feasible) distributions of the cake? (We assume free disposal.) Draw the region of corresponding feasible utility vectors (u_A, u_B).

(b) Determine all Pareto optimal divisions of the cake in this setting.

Now suppose we add money to the system, and suppose that both consumers have quasi-linear utility functions.

(c) Write down the utility of consumer A for a fraction α of the cake and an amount x of money. Do the same for consumer B.

(d) Suppose the total amount available in the system is 50 euros, and both consumers can borrow and perform any monetary compensation they would wish to make. (Thus, the bidders can give each other an IOU.) Determine all feasible allocations in this model. What are the possible Pareto optimal distributions of the cake in this model? Draw the region of corresponding feasible utility vectors (u_A, u_B).