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Naive Bidding

George Deltas, Richard Engelbrecht-Wiggans
University of Illinois at Urbana-Champaign, 1206 South Sixth Street, Champaign, Illinois 61820
{deltas@uiuc.edu, http://netfiles.uiuc.edu/deltas/www/index.html, eplus@uiuc.edu}

This paper presents an equilibrium explanation for the persistence of naive bidding. Specifically, we consider a common value auction in which a "naive" bidder (who ignores the winner's curse) competes against a fully rational bidder. We show that the naive bidder earns higher equilibrium profits than the rational bidder when the signal distribution is symmetric and unimodal. We then consider a sequence of such auctions with randomly selected participants from a population of naive and rational bidders, with the proportion of bidder types in the population evolving in response to their relative payoffs in the auctions. We show that the evolutionary equilibrium contains a strictly positive proportion of naive bidders. Finally, we consider more general examples in which a naive bidder matched against a rational bidder does (i) worse than his rational opponent, but (ii) better than a rational bidder matched against another rational bidder. Again, the evolutionary equilibrium contains a strictly positive proportion of naive bidders. The results suggest that overconfident recent entrants in Internet and other low transaction-cost auctions of items that appeal to a mass audience may earn higher payoffs than their experienced competitors and, thus, will not eventually be driven from the market.

Key words: bounded rationality; bidding; winner's curse; common value auctions
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1. Introduction

Consider a common value auction with two bidders, one of whom ignores the "winner's curse," while the other is "fully rational" and knows that her competitor is not.[1] In such a setting, the common wisdom is that the "naive" bidder earns lower profits than his competitor and will eventually learn to account for the information contained in his competitor's signal.[2] As a consequence, in any long-run equilibrium, only fully rational bidding behavior survives.

We show the fallacy of such an argument. First, we demonstrate that the naive bidder earns a higher payoff than the rational bidder under a wide range of conditions. We then show that (under the same conditions) naive bidding survives in an evolutionary equilibrium. Finally, we analyze some illustrative cases in which naive bidding survives in an evolutionary equilibrium *even though* a naive bidder earns a lower payoff than his fully rational competitor.

[1] The winner's curse is a very real phenomenon; Kagel (1995, p. 537) mentions a variety of claims that the winner's curse occurs in the real world. With regard to laboratory experiments, he states, "In fact, the winner's curse has been such a pervasive phenomenon that most of the initial experiments in the area focused on its robustness and the features of the environment that might attenuate its effects."

[2] Throughout, we use "her" for the rational bidder and "him" for the naive bidder.

We consider an oral ascending price common value auction without reserve, in which two bidders compete for an item the value of which is the product of two independent characteristics. Each bidder observes one of these characteristics.[3] One of the bidders is "naive"; he does not realize the other bidder's signal contains useful information about the item's value, and, therefore, bids as if he has a private value (observed with noise). The second bidder is fully rational, i.e., she understands that she competes against a naive bidder and formulates her strategy optimally.

One might expect that in such a setting the naive bidder will be subject to the winner's curse. He is likely to pay too much for the goods he wins and perhaps also to win goods of low value. Over time, he might be expected to learn to "shade" his bid and behave as a rational bidder would. In other words, one might expect that naive bidding is unprofitable and will eventually disappear.[4]

Contrary to such expectations, we show that for all symmetric and unimodal signal distributions the naive bidder's payoff *exceeds* that of his fully rational

[3] The model easily generalizes to an environment in which there are more characteristics than bidders, with each of the two bidders observing only one of the characteristics (but not the same one).

[4] See Capen et al. (1971), Cox and Isaac (1984), and McAfee and McMillan (1987).

opponent. We then repeat this auction with randomly selected participants from a population whose makeup evolves in response to how well each type of bidder does in each auction. We show that for all symmetric unimodal distributions, the evolutionary equilibrium contains a strictly positive proportion of both types of bidders; naive bidding persists rather than disappears.

We also provide two numerical examples using well-known signal distributions that are not symmetric. For the exponential distribution, a naive bidder matched against a rational bidder does worse than the rational bidder. However, a naive bidder matched against a rational bidder does better than a rational bidder matched against another rational bidder. As a result, the evolutionary equilibrium contains a strictly positive proportion of naive bidders; naive bidding can persist *even when* a naive bidder does worse than a rational bidder when the two are matched against each other. For the lognormal distribution, profit rankings are a function of the signal dispersion. In the range of parameter values that we consider, we observe 6 of the 24 possible rankings. This suggests that general results on payoff rankings may not be possible over the space of all distributions. We observe that the relative profitability of naive bidding is a decreasing function of signal dispersion. Indeed, as signal dispersion goes to zero, the proportion of naive bidders in the population goes to one, while for sufficiently high signal dispersion, the proportion of naive bidders in the population becomes zero.

Naive bidding, as we model it, can arise in a number of ways. It may be due to an inability to "do the math" right.[5] It could also arise if the naive bidder erroneously believes that his competitor has no expertise in assessing the signal he observes. In other words, naive bidding could arise from extreme overconfidence in one's own estimate relative to the estimate of the competing bidder. Such overconfidence in people's ability to predict the accuracy of their estimates has been documented in the behavioral literature.[6] In addition, naive bidding could arise from an inability of a bidder to think in the "other person's shoes." Such inability to consider the problem from the point of view of one's competitor has been demonstrated in a number of experiments. (See, for example, Ball et al. 1991, Samuelson and Bazerman 1985, and references therein.) Finally, a bidder will bid "naively" if he (erroneously) believes that his competitor wants to purchase the item for private consumption on the basis of some characteristic that is not valued by the market. In any of these cases, the bidding of the competing bidder is "noise" as far as this bidder is concerned.

All of the above sources of unsophisticated or naive bidding behavior are likely to be more prevalent in Internet auctions, which are characterized by low entry and transaction costs, and which are selling items that appeal to wide audiences. With the advent of such auctions in recent years, many individuals who have no experience in bidding and who are bidding for items for their private collection (likely for the purpose of later resale, but not as part of an organized business) are able to compete alongside more experienced professional bidders. The findings of this paper suggest that these new "naive" entrants are likely to prosper when competing against their more astute counterparts, and even if they do not, that they will not be entirely driven off the market.[7]

Our results on the "profitability" of naive bidding parallel those on the value of information. In decision-theoretic settings, having additional information never hurts. In game-theoretic settings, having additional information can hurt when your competitors know that you have the additional information. In some games, a player who is known to be ignorant of certain information does better than a "better" informed player. For example, Engelbrecht-Wiggans and Weber (1983) present a sequential auction example with one perfectly informed bidder against a less-informed bidder in which the less-informed bidder has the higher expected profit. More broadly, our results also suggest that the maintained assumption of economic models that agents behave rationally and that naive behavior eventually "dies out" may not be applicable in many game-theoretic settings. Rather, in such settings unsophisticated behavior may be advantageous.[8]

The results of this paper may appear, at first sight, paradoxical. How can a boundedly rational bidder earn a higher payoff when competing against a fully rational bidder? After all, the fully rational bidder can always imitate a boundedly rational one. The key insight is that the naive bidder enjoys an advantage in terms of commitment: Nature has picked his type to be naive. Given that the rational bidder is faced with

[5] This may be the underlying cause for the winner's curse in many experiments (e.g., see Kagel and Levin 1986).

[6] See, for instance, Brenner et al. (1996), Block and Harper (1991), Griffin and Tversky (1992), and references therein. Overconfidence may also be a contributing factor in overbidding in the market sports players (Cassing and Douglas 1980) and oil leases (Mead et al. 1983, Hendricks et al. 1987).

[7] However, the model applies more generally to all common value auctions in which unsophisticated bidders are present, including mineral rights auctions (e.g., oil, natural gas, or coal). With minor changes to notation, the model also applies to low-price auctions, such as procurement competitions (e.g., bidding for power plant construction contracts).

[8] Non-game-theoretic models with a results of a similar flavor include Blume and Easley (1992) and Kyle and Wang (1997), among others.

a naive opponent, he does better by bidding rationally rather than naively. Were a rational bidder to face another rational bidder, he would have preferred to have been naive (provided that his competitor perceives the fact that he is naive). Unfortunately (for the rational bidder), nature gives the naive bidder a first-mover advantage.

The paper proceeds as follows. Section 2 formally defines our basic auction model of two bidders competing for a single object. In §3, we derive equilibrium bidding strategies and their corresponding expected profits. In the following section, we consider a sequence of our basic auctions and examine the evolutionary consequences. The paper ends with some discussion on extensions and a few concluding remarks on managerial insights. The appendix contains the proofs.

2. The Model

2.1. Preliminaries

Consider two risk-neutral bidders competing for an object to be sold in an oral ascending auction without reserve.[9] The value of the object is common to both bidders and equals $V = S_n S_r$, where S_n is a signal of the value received by a naive bidder (the exact meaning of "naive" will be defined shortly), and S_r is a signal of the value received by a fully rational bidder.[10] The expected values of the two signals are normalized to 1. Further, S_n and S_r are assumed to take nonnegative values and be independent of each other. For notational simplicity, we assume that the two signals are identically distributed with a nondegenerate density $f(s)$ and associated continuous distribution function $F(s)$. Finally, we assume that $f(s)$ is positive in the interval $[S_{\min}, S_{\max}]$.[11]

This framework captures a number of different bidding environments. For example, consider the competition between two art dealers for a painting that they intend to resell. The signals of the bidders represent different facets of the painting. That is, each dealer is an expert in evaluating a different determinant of the painting's value. In our model, these signals are assumed to be independent of each other and identically distributed, i.e., the two characteristics that describe the item are not correlated and are of equal importance in determining the variability of the item's value. One of the bidders is assumed to be "naive" in the sense that he does not infer any information about the value of the item from the bidding behavior of his opponent. In our art-dealer example, the naive bidder might be a dealer who believes that the other dealer is not an expert in judging the quality of the item, and instead observes a signal that is pure noise. Alternatively, the naive bidder could be a bidder who believes that his competitor wants to purchase the item on the basis of some characteristic that is not valued by the market. In our example, this would correspond to a dealer who believes that the other bidder has idiosyncratic tastes and wants to purchase the painting for personal consumption. More loosely, one could think of the naive bidder as being a bidder who has not thought about the problem and has not formulated any beliefs. As a result, the bidding behavior of the other bidder is pure noise and, therefore, the naive bidder does not condition his evaluation of the item's worth on the bidding behavior of his opponent. Therefore, for the naive bidder, his beliefs about the process that describes the value of the item to himself can be described as $V_n = S_n \eta$, where η is a random variable with mean equal to 1 and independent of the value of the item to his competitor.[12] The above assumptions imply that the naive bidder believes that he is competing in a privately known values auction. In contrast, the second bidder is fully rational. She understands that she competes against a naive bidder and formulates her strategy optimally. Note that the assumption that the rational bidder observes the type of her competitor is crucial. We conjecture that similar results may hold when a rational bidder observes the type of the other bidder with sufficiently high probability. However, formal analysis of this case is complicated by the fact that in the event the rational bidder does not observe her opponents type, she would update her beliefs about the type of her competitor as the auction progresses,

[9] A small reserve will not affect the qualitative nature of the results. Nevertheless, there exist real-world auctions in which the seller cannot credibly commit not to sell (e.g., mineral rights auctions and Department of Treasury auctions).

[10] Similar results are obtained if the value is the average of the two signals. These are easily derived following the development in this paper, and are also available from the authors upon request. The adoption of the multiplicative signals model is a matter of taste.

[11] The results go through even if the signal distribution is discrete, or mixed discrete and continuous.

[12] Notice that the naive bidder is correct about the expected value of the object conditional on his signal alone. The signal of the other bidder is interpreted as ex post uncertainty. If there is no ex post uncertainty in V, then the draws of η would all appear to be negative (as in equilibrium, the naive bidder wins when $S_r < 1$), which would surprise a naive bidder who plays the game a few times. If instead there was substantial ex post uncertainty in V (i.e., if V also depends on signals that no bidder observes), then the draws η could often be positive (though with probability less than 1/2). Because the naive bidder wins items of lower than average value (conditional on his signal), if the auction is played often enough, he will still realize that the mean of η is not 1. We implicitly assume that the auction is not played often enough, or that the naive bidder does not have memory good enough, or is not sophisticated enough to realize this discrepancy. This requires a lesser degree of nonsophistication than the one required for the naive bidder to ignore a string of negative draws. We, nevertheless, suppress ex post randomness in V for notational simplicity.

as the probability that a bidder continues bidding at any price given price differs across the two types.[13]

2.2. Bidder Strategies

Based on his beliefs, the naive bidder has a dominant strategy to stay in the auction until the price reaches the expected value of the object conditional on his signal. This value equals $V_n = S_n$. The rational bidder will stay in the auction as long as the price is such that if she were to win at that price, she would reap positive expected surplus. Given the naive bidder's strategy, winning the object at a price P implies that the signal of the naive bidder is $S_n = P$. Therefore, immediately upon winning the item at a price equal to P, the rational bidder calculates the value of the item to be $S_r P$. Notice that when the signal of the rational bidder is less than 1, she would regret winning the item regardless of the realized price. On the contrary, when her signal is greater than 1 the value of the item, upon winning it, exceeds its price regardless of the price that she wins it at. Hence, an optimal strategy of the rational bidder is to immediately quit the auction if she observes a below-average signal and stay in the auction until she wins if she observes an above-average signal. As a result, the rational bidder wins if and only if she gets an above-average signal.

The intuition for this is as follows: The naive bidder computes the expected value of the item "as if" the signal of the other bidder is equal to its mean, i.e., equals 1. When the actual value of the rational bidder's signal is less than one, the naive bidder overestimates the value of the item and would, therefore, be willing to push its price above the actual expected value. Realizing this, the rational bidder has no reason to enter the bidding. Conversely, when the rational bidder gets an above-average signal, i.e., $S_r > 1$, the naive bidder will be underestimating the value of the item and would drop out before its price reaches the actual expected value. Realizing this, the rational bidder stays in the auction until she wins.[14] No other strategy yields a higher payoff for the rational bidder.[15]

3. Bidder Payoffs

Having fixed the equilibrium bidding behavior, we now turn to the calculation of the equilibrium payoffs. The naive bidder wins if and only if the rational bidder's signal is less than or equal to 1, in which case the rational bidder bids zero and the naive bidder pays nothing. Therefore, the naive bidder's expected profit is the expected value of the object when the rational bidder's signal is at most 1. The rational bidder only bids if her signal exceeds 1 and stays in the auction until she wins. Because the price at which the naive bidder drops out is independent of the rational bidder's signal and has a mean of 1, the expected payment of the rational bidder conditional on winning equals 1. These observations suggest the following proposition, a formal proof for which (and for each of our other results) appears in the appendix.

Proposition 1. *The* ex ante *payoffs of the naive and rational bidders are positive and given by* $\Pi_n = \Pr[S_r < 1]\mathrm{E}[S_r \mid S_r < 1]$ *and* $\Pi_r = \Pr[S_r > 1]\mathrm{E}[S_r - 1 \mid S_r > 1]$, *respectively.*

Which type of bidder has higher ex ante profits? There are two opposing effects. The average value of the item, conditional on the naive bidder winning the auction, is lower than the average value of the item conditional on the rational bidder winning the auction. However, the naive bidder pays a price of zero for the items he wins, while the rational bidder pays a price equal to the unconditional value of the item. As it turns out, the payoffs of the two bidders cannot be unambiguously ranked: Under some parameterizations the naive bidder earns a higher payoff than the rational bidder, for other parameterizations the reverse is true. However, much insight (and some interesting results) can be obtained by considering signal distributions that are symmetric with respect to their mean.

[13] It is interesting to note that, as the paper shows, naive bidding is profitable and likely to persist when the rational bidders are sufficiently capable so that they can ascertain their competitors' types. Less-capable rational bidders who cannot ascertain the type of their competitors would reduce the payoff of naive bidders (relative to that of rational bidders) and would lead to the extinction of naive bidding.

[14] We note, parenthetically, that this feature of the rational bidder's optimal bidding strategy reduces the information revealed in this auction relative to an auction with two rational bidders. When two rational bidders compete against each other, the signal of the losing bidder (and the fact that the winning bidder had a higher signal) is revealed at the conclusion of the auction. When a naive bidder competes against a rational bidder and the rational bidder has a below average signal, one only observes the fact that one bidder had a below average signal.

[15] However, there are other strategies that are equally good. Observe that the lowest possible price at which the naive bidder will drop out is $P_{\min} = S_{\min}$. This is a positive number unless $S_{\min} = 0$. Therefore, the rational bidder can bid the price up to $P_{\min}$ with impunity: She is at no risk of winning. Doing so will result in no benefit to her compared to the strategy of dropping out at the start of the auction (unless she benefits when her competitor earns a lower payoff, perhaps because a competitor with fewer resources might not be able to aggressively compete with her in a future auction). In fact, if she incurs even infinitesimal costs from bidding she will strictly prefer to drop out at the very start. Results for the case of an aggressive rational bidder (who stays in the bidding until the price reaches $P_{\min}$) are obtained by subtracting from the (ex ante) expected payoff of the naive bidder an amount equal to the probability that he wins times $P_{\min}$.

In particular, when the signal distributions are symmetric, the expected payoffs of either bidder can be written as a function of a single conditional expectation of the signals. This conditional expectation is a measure of signal dispersion. As Proposition 2 shows below, the profits of each bidder are monotonic in this expectation: The rational bidder's payoff increases with signal dispersion, whereas the naive bidder's payoff decreases with signal dispersion.

Proposition 2. *For symmetric signal distributions, the expected payoff of either bidder depends only on signal dispersion as measured by* $z \equiv 1 - \mathrm{E}[S \mid S < 1]$, *i.e., the difference between the mean and the expected value of the signal, conditional on it being lower than the mean. The rational bidder's payoff increases with the dispersion of the signal, while the naive bidder's payoff decreases with the dispersion of the signal.*

Notice that, in our model, an increase in signal dispersion is equivalent to an increase in the variance of the item's value. Then, the intuition for the above result is as follows: The naive bidder's payoff decreases with the dispersion of the signal because (i) in the event that he wins, he pays a price of zero, and (ii) he wins when the signal of his opponent, i.e., one of the components of the item's value, is below average. Therefore, an increase in the signal dispersion reduces the expected value of the item in the events that he wins without reducing the price that he pays for it. Conversely, an increase in the signal dispersion increases the payoff of the rational bidder because (i) the expected price that he pays equals 1 and is independent of his signal, but (ii) he wins when his signal is above average. Therefore, signal dispersion does not increase his expected payment, but increases the expected value of the item in the events that he wins.

Notice that (by construction) the expected value of the item is independent of signal dispersion and equals 1. However, it remains to be determined whether signal dispersion affects only the relative payoff of the two bidders or *also* affects the sum of their payoffs. It follows easily from the proof of Proposition 2, as Corollary 1 states below, that a change in signal dispersion does not affect the bidders' combined surplus (or the seller's surplus): Signal dispersion does not affect the size of the "pie" that the two bidders split between them, but only their shares of that pie.

Corollary 1. *When the signal distribution is symmetric, the seller's expected revenue and the bidders' combined surplus are independent of signal dispersion and equal half the expected value of the item.*

Next we examine how the bidders split their half of the pie. Because for symmetric signal distributions the profit response of the two bidders is monotonic in signal dispersion, but exhibits the opposite response, there may exist some critical level of signal dispersion such that for lower levels of dispersion the naive bidder's payoff exceeds that of the fully rational bidder, while for higher levels of dispersion the converse is true. In principle, this critical dispersion level could be zero, which would imply that the fully rational bidder has higher expected profits for all symmetric signal distributions. However, Proposition 3 below gives a simple condition under which this critical dispersion level is bounded away from zero.

Proposition 3. *When the signal distribution is symmetric, the expected payoff of the naive bidder exceeds that of the rational bidder if* $2\mathrm{E}[S \mid S < 1] > 1$; *i.e., if the expected value of the signal is less than two times the expected value of signal conditional on it being lower than the mean.*

A much stronger result can be obtained if we further reduce the distribution of signals to be unimodal. As Theorem 1 states below, for this important class of signal distributions the expected payoff of the naive bidder always exceeds that of the rational bidder.

Theorem 1. *When the signal distribution is symmetric and unimodal, the expected payoff of the naive bidder is higher than that of the rational bidder.*

The results of the previous section were all derived under the assumption that the signal distribution is symmetric. However, it is easy to see that the expected profits of the "naive" bidder exceed those of the fully rational bidder for all distributions with sufficiently low dispersion.

4. The Evolutionary Stability of Naive Bidding

In this section we investigate whether naive bidding persists in the long run when the signal distribution is symmetric and unimodal. We show that it does, if persistence is defined as a nonzero proportion of types in an evolutionary equilibrium.[16] In addition, we investigate whether rational bidders will also survive in positive proportion in such an equilibrium. We show that they do. Finally, we examine whether naive bidding can survive when the signal distribution is such that the naive bidder's payoff is *lower* than that of a fully rational opponent when the two compete against each other. We show that it can. Relative bidder payoff rankings and expected prices are obtained as intermediate results.

[16] Persistence is not necessarily guaranteed by the fact that (for such signal distributions) a naive bidder obtains a higher payoff than his fully rational competitor: A naive bidder must also earn a higher payoff than two rational bidders competing against each other.

Consider a sequence of oral ascending common value auctions without reserve in which two bidders compete for an item the value of which is given by $V = S_1 S_2$. One of the bidders observes S_1, the other bidder observes S_2. The distributional assumptions on the other components of value are identical to those described in §2.1. The two bidders are drawn randomly from a population of bidder types that includes a proportion q of naive bidders (the rest of the bidders being fully rational), i.e., each bidder, independent of the others, is naive with probability q and rational with probability $1-q$. A bidder who is fully rational knows the type of her competitor. We assume that the proportion of naive bidders increases if their expected profits exceed those of the rational bidders, with the expectation taken over all the possible bidder matches. Conversely, the proportion of naive bidders decreases if their expected profits are lower than those of the rational bidders. This type of growth dynamics are based on the observation that, if the two types of bidders represent two different types of dealers, then the more successful type will have greater financial resources and will be able to participate in more auctions. The less successful type will have fewer resources and will participate in fewer auctions or will drop out of the market altogether. Notice that past experience enters the growth dynamics only through relative profits of the two bidders; the strategic types of the bidders are fixed.[17] A proportion, q_{eq}, of naive bidders is an evolutionary equilibrium if the expected profits of the two bidder types are equal.

We first examine the case of symmetric and unimodal signal distributions, then consider distributions outside this class.

4.1. Symmetric and Unimodal Signal Distributions

Suppose that S_1 and S_2 are both drawn from a distribution that is symmetric and unimodal. In this case, we show below that there is a unique proportion $q_{eq} \in (0, 1)$ of naive bidders such that the expected profits of a naive bidder equal the expected profits of a rational bidder, that is, there is a unique interior proportion of naive types that constitutes an evolutionary equilibrium. Furthermore, populations that consist of one type only (i.e, either of all rational or of all naive bidders) are not stable, i.e., they are not robust to mutations.

To show these results, we first need to derive the expected price and bidder profits when the bidders are of the same type and compare them with the expected price and bidder profits when the bidders are of different types. Throughout this section, we denote by $P_{i,j}$ the price in an auction in which a bidder of type i competes with a bidder of type j. (Note that $P_{i,j} \equiv P_{j,i}$.) We also denote by $\Pi_{i|j}$ the expected payoff of a bidder of type i given that his competitor is a bidder of type j. (Note that, in general, $\Pi_{i|j} \neq \Pi_{j|i}$.) The expected price and payoffs in auctions with bidders of the same type are given by Proposition 4 below.

Proposition 4. *When two naive bidders compete for the item, the expected price and bidders' payoffs equal*

$$E[P_{n,n}] = E[\min(S_1, S_2)]$$

and

$$\Pi_{n|n} = (1 - E[\min(S_1, S_2)])/2,$$

respectively. When two fully rational bidders compete for the item, the expected price and bidders' payoffs equal

$$E[P_{r,r}] = E[\min(S_1, S_2)^2]$$

and

$$\Pi_{r|r} = (1 - E[\min(S_1, S_2)^2])/2,$$

respectively.

It turns out that expected prices under the three possible bidder configurations can be ranked when the signal distributions are symmetric and unimodal. In particular, the expected price is highest when two naive bidders compete against each other. This tends to confirm the intuition of the "winner's curse." However, the expected price is lowest when a naive bidder competes with a rational bidder: The winner's curse only helps the seller if *both* bidders ignore it! In particular, we have as follows.

Proposition 5. *When the signal distribution is symmetric and unimodal, the expected price in an auction with two naive bidders is (weakly) higher than the expected price in an auction with two rational bidders, which in turn is higher than the expected price in an auction with one naive and one rational bidder. That is,* $E[P_{n,n}] \geq E[P_{r,r}] > E[P_{n,r}]$.

An interesting feature of the price ranking is that seller revenue is not monotonic in the number of naive bidders in the auction; it is this feature that leads to a positive equilibrium proportion of both bidder types when signal distributions are symmetric and unimodal. The price ranking also allows us to rank the profits of the bidders in each configuration, with one exception: One cannot rank the profits of a rational bidder when she competes against a naive

[17] In particular, naive bidders do not learn from the pattern of wins and losses (they tend to lose contested auctions more often than they would expect) and the realized values of the objects they win (they tend, on average, to be of lesser value conditional on their signal than they would expect) that their view of the world is flawed.

bidder relative to the profits of a rational bidder when she competes against another rational bidder.[18]

Corollary 2. *When signal distributions are symmetric and unimodal, a partial ordering of bidder payoffs is given by* $\Pi_{n|r} > \{\Pi_{r|r}, \Pi_{r|n}\} \geq \Pi_{n|n}$.

Note that it may sometimes be to a rational bidder's advantage to educate her naive opponent about the winner's curse.[19] We are now ready to state the main result of this section.

Theorem 2. *When the signal distribution is symmetric and unimodal, there is a unique stable evolutionary equilibrium in which the proportion of naive bidders,* q_{eq}, *is positive and strictly less than* 1.

Theorem 2 implies that rational bidders will coexist in a stable equilibrium with naive bidders, even though in any match-up between the two types, the naive bidder earns higher profits. The intuition for this can be seen by observing that a population that consists entirely of naive bidders is not robust to mutation: A rational bidder in such a population would be matched against a naive bidder with probability 1 and earn $\Pi_{r|n}$, while the naive bidders will be matched against each other almost with probability 1 and earn $\Pi_{n|n}$. Because by Corollary 1 $\Pi_{r|n} > \Pi_{n|n}$, the rational bidder will earn higher expected profits.

4.2. Extensions to Nonsymmetric Distributions: Some Surprises

The assumption of symmetry and unimodality of the signal distribution allows us to obtain very strong results: For all distributions in this class, the naive bidder earns greater or equal profits than a rational bidder. For all distributions in this class, both naive and rational bidding will persist in an evolutionary equilibrium. However, while symmetry and unimodality are sufficient conditions, they are not necessary conditions for these results. There are numerous examples of other distributions for which they are also true. Of course, this fact should not surprise the reader. What may be surprising is that naive bidding can persist in an evolutionary equilibrium *even* in distributions in which a naive bidder does worse than a rational bidder when the two are competing against each other. A detailed analysis of two examples is instructive.

Consider first signals that are exponentially distributed. The bidder payoffs are, then, given by $\Pi_{r|r} = \Pi_{n|n} = 0.25$, $\Pi_{r|n} = 0.368$, and $\Pi_{n|r} = 0.264$. When a naive bidder competes against a rational bidder, the naive bidder fares *worse* than his competitor. However, bidder competition in such a match-up is less aggressive relative to when two bidders of the same type are competing against each other. In fact, the expected price in an auction with a naive and rational bidder is sufficiently low that the naive bidder's payoff exceeds the payoff of a rational bidder competing against another rational bidder. As a consequence, a population consisting of only rational bidders is not stable to "mutation." Indeed, the proportion of naive bidders in the only stable evolutionary equilibrium is approximately equal to 10%. The driving force behind this result is the negative externality that naive bidding confers to the seller by reducing the intensity of bidder competition. If this externality is sufficiently large, then naive bidding can persist in equilibrium even if naive bidders do less well when matched against rational opponents.

Consider next the case of signals that are lognormally distributed. Denote the standard deviation of the log signal by s and set the mean of the log signal such that the expected value of the signal equals 1. The bidder payoffs and expected price is a function of s. The results show that 6 of the 24 possible rankings of bidder payoffs are observed for $s \in [0, 1.4]$ (and this within the confines of a single distribution!). This suggests the difficulties of establishing general theorems when one considers signal distributions that are not symmetric. Further, the equilibrium proportion of naive bidders decreases monotonically with signal dispersion, as expected from Proposition 2, from 1 (as s approaches 0) to zero as s nears 1.1.

5. Extensions to Auctions with More than Two Bidders

The auctions considered in this paper consist of two bidders. However, in many, if not most, auctions (including Internet auctions) the number of bidders is larger. We believe that the qualitative nature of the results does not change if there are N bidders. We elaborate below using an example with two rational bidders and one naive bidder. Among auctions with three bidders, this breakdown of bidder types is the interesting one to study. If naive bidders were to become "extinct," then the relevant question is what happens to their payoffs as they become relatively rare, in which case it is much more likely that there will be a single naive bidder in any auction than that there will be more than one. Therefore, the most relevant question is whether the results are qualitatively robust to increasing the number of rational bidders. As the three-bidder illustration below shows, the results remain valid in this case (indeed, some actually appear

[18] The relative ranking of these two profits depends on the distribution of signals even within the class of symmetric unimodal signal distributions. For instance, if the signal distributions are uniform in the interval [0.2, 1.8], then $\Pi_{r|r} = 0.16$ and $\Pi_{r|n} = 0.20$. In contrast, if the signal distributions are uniform in the interval [0.8, 1.2], then $\Pi_{r|r} = 0.06$ and $\Pi_{r|n} = 0.05$.

[19] This has, in fact, been conjectured in Thaler (1992, p. 62).

to be stronger). Therefore, the two-bidder model we analyze in detail in this paper is not an exception to the rule, but is likely to hold more broadly.

Consider an auction with two rational bidders and one naive bidder. The value of the item is given by $V = S_{r1}S_{r2}S_n$, where S_{r1} is the signal of the first rational bidder, S_{r2} is the signal of the second rational bidder, and S_n is the signal of the naive bidder. All signals are independently distributed with a uniform distribution on the interval $(0, 2)$. (This is an open interval so that we do not have to worry about ties, which simplifies the proof.) Notice that this is our worst-case scenario in terms of symmetric unimodal distributions (in the two bidder case, as the dispersion of the signals decreases, the payoff of the rational bidder decreases, while that of the naive bidder increases).

The following statements describe an equilibrium of the bidding game: (a) The naive bidder drops out when the price reaches his signal. (b) A rational bidder j stays out of the auction if $S_j < 1$. If $S_j > 1$ and the other rational bidder is observed to drop out immediately, then bidder j also drops out. (c) If $S_j > 1$ and the other rational bidder starts bidding, then bidder j stays in the auction until the naive bidder drops out (at which point bidder j infers the naive bidder's signal). After the naive bidder drops out, bidder j continues to stay in the auction until he wins or the price reaches $S_n S_j^2$.

The proof that the above indeed forms an equilibrium is as follows. (a) The naive bidder's perception of the world is equivalent to setting $S_{r1} = S_{r2} = 1$. This immediately implies that his dominant strategy is to stay in the auction until $P = S_n$. (b) We consider the optimality of the strategy of the rational bidders from the point of view of Bidder 1 (the first rational bidder). If Bidder 2 (the other rational bidder) stays out, then $S_{r2} < 1 \Rightarrow \mathrm{E}[S_{r2}] = \frac{1}{2}$. Therefore, conditional on the departure of Bidder 2, the value of the item from the point of view of Bidder 1 is $V = \frac{1}{2}S_{r1}S_n$. If Bidder 1 enters the bidding and beats the naive bidder at a price P_{win}, which would imply that $S_n = P_{\text{win}}$, he gets an item of value $\frac{1}{2}S_{r1}P_{\text{win}}$, which is less than P_{win} for all $S_{r1} < 2$. Therefore, if Bidder 2 does not enter the bidding, Bidder 1 is better-off not entering either. Suppose instead that Bidder 2 enters and $S_{r1} < 1$. Then, if Bidder 1 enters the bidding, and Bidder 2 bids according to the equilibrium, Bidder 1 would pay $S_n S_{r2}^2$ to win the item, which would be worth $S_n S_{r1} S_{r2}$. Observe that the price would exceed the value because $S_{r2} > 1 > S_{r1}$. Therefore, if Bidder 1 has a below average draw, he is better-off staying out. (c) Finally, if both rational bidders have signals greater than 1, they will both enter the bidding. Because under this event $S_{r1}S_{r2} > 1$, the rational bidders will find it optimal to stay in the auction until the naive bidder drops out, at which time they will be able to infer his signal. After the naive bidder drops out, Bidder 1 will continue to stay in the auction as long as the price is such that if Bidder 2 were to drop out, Bidder 1's payoff of winning at that price would be positive, i.e., as long as $S_n S_{r1} S_{r2} > P_{\text{win}}$. If Bidder 2 were to drop out at P_{win}, this would imply (from part c of the proposed equilibrium) that his signal was $(P_{\text{win}}/S_n)^{1/2}$. Substituting into the above inequality, we get $S_n S_{r1} P_{\text{win}}^{1/2} > P_{\text{win}} S_n^{1/2} \Rightarrow P_{\text{win}} < S_n S_{r1}^2$. This shows that the drop-out price of a rational bidder when the second rational bidder enters the bidding is indeed as given by part (c) of the proposed equilibrium. This completes the proof.

We next turn to the payoff comparison. The naive bidder wins when both rational bidders have below average signals (which happens with probability $\frac{1}{4}$ and results in conditional expected payoff of $\frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}$), or when one of the rational bidders has a below average signal and the other an above average signal (which happens with probability $\frac{2}{4}$ and results in conditional payoff of $\frac{1}{2} \cdot \frac{3}{2} \cdot 1 = \frac{3}{4}$). Therefore, $\Pi_n = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4} = \frac{7}{16}$. Because the expected value of the item is 1, this leaves $\frac{9}{16}$ to be divided among the seller and the two rational bidders. It immediately follows that $\Pi_r < \frac{9}{32} < \Pi_n$. Observe that this strengthens the result with only two bidders, because when there is one rational bidder and one naive bidder and signals are uniformly distributed on $[0, 2]$, the payoffs of the two bidders are equal.

Finally, to verify that a zero percentage of naive bidders is not evolutionary stable, we need to show that the payoff of a naive bidder competing against two rational bidders exceeds the payoff of a rational bidder competing against two other rational bidders. This is easy to show. Because the expected value of the item is 1, the expected payoff of each of the three rational bidders cannot exceed $\frac{1}{3}$, which is less than the $\frac{7}{16}$ a naive bidder gets competing against two rational bidders. This proves the evolutionary stability of naive bidding.

6. Concluding Remarks

We conclude by offering a number of managerial insights that are suggested by the results of this paper. The first arises from the observation that the worst combination of bidder types from the point of view of the seller is a mix of naive and rational bidders. Given that there will always be sophisticated (rational) bidders in auctions (including online auctions) and given that the seller cannot turn a rational bidder into a naive one, it is a good strategy for the seller to educate the naive bidders about the complexity of common value auctions. Further, this education process must be observable to the rational bidders. Efforts by bidding sites to educate online bidders are not uncommon. For example, eBay provides a small discussion of why bidders should submit proxy (2nd price) bids that equal their willingness to pay for an item and advise against submitting a low proxy bid only to revise it upwards, in effect educating bidders about dominant strategies. This advice is predicated on the (often reasonable) presumption that valuations are private. When values are likely to be common, online sellers (like eBay) might do better educating bidders that their competitors may privately observe information that is relevant to the value of auctioned item.

Second, our paper suggests that a naive bidder who is understood to be naive does better than a naive bidder who is perceived to be rational. A related implication for rational bidders is that, though a strategic announcement of naivete may not be credible (because it would presuppose that the bidder is rational enough to understand the implications of this announcement), delegating decisions to an unsophisticated agent may be a good strategy (provided that this delegation of bidding to an agent is observed and credible, and that his lack of sophistication is also observed).

Finally, sellers may be wise to use a positive reserve. Though this recommendation runs counter to what would be advised in private value auctions in which sellers compete for participants, it has certain benefits in the environment we study. A low reserve does not reduce by much the expected payoff of a rational bidder, and thus is not likely to substantially affect his participation decision. With regards to the naive bidder, a small reserve decreases his *perceived* expected payoff by less than it decreases his *actual* expected payoff. Therefore, any adverse effects on entry could be lower than the increased revenue conditional on entry.

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Appendix

Proof of Proposition 1. We first consider the payoff of the naive bidder. If S_r is less than 1 and the naive bidder's signal equals S_n, the naive bidder's payoff, $\Pi_n^{cc}(S_n, S_r)$, equals S_nS_r because he wins and pays a price of zero. If S_r exceeds 1, the naive bidder loses the auction and his payoff equals zero. Therefore, the expected payoff of the naive bidder with a signal S_r, $\Pi_n^c(S_n)$ is given by

$$\Pi_n^c(S_n) = \int_0^1 f(S_r)S_nS_r\,dS_r.$$

Integrating over the distribution of the naive bidder's signals yields his ex ante payoff, Π_n,

$$\begin{aligned}\Pi_n &= \int_0^{S_{\max}}\int_0^1 f(S_r)S_nS_r\,dS_r f(S_n)\,dS_n\\ &= \Pr[S_r<1]\mathrm{E}[S_r\mid S_r<1]\int_0^{S_{\max}} f(S_n)S_n\,dS_n.\end{aligned}$$

Given that $\mathrm{E}[S_i]=1$, the ex ante payoff of the naive bidder can be written as

$$\Pi_n = \Pr[S_r<1]\mathrm{E}[S_r\mid S_r<1].$$

We next turn to the calculation of the rational bidder's payoff. If the rational bidder's signal S_r is less than 1 and the naive bidder's signal equals S_n, the payoff of the rational bidder, $\Pi_r^{cc}(S_n, S_r)$, is zero, because she will drop out of the auction immediately. Next, observe that if the rational bidder's signal, S_r, exceeds 1 and the naive bidder's signal is S_n, the rational bidder wins the auction (at a price equal to the naive bidder's signal), and her payoff equals

$$\Pi_r^{cc}(S_n, S_r) = S_n(S_r-1).$$

The expected payoff of the rational bidder with signal S_r, $\Pi_r^c(S_r)$, then equals

$$\Pi_r^c(S_r) = \int_0^{S_{\max}} f(S_n)S_n(S_r-1)\,dS_n.$$

This integrates to $\Pi_r^c(S_r) = (S_r-1)$ because $\mathrm{E}[S_n]=1$. The ex ante payoff of the rational bidder, Π_r, i.e., his payoff before he obtains any information on the item, is obtained by integrating his conditional profit over the distribution of his signals for signals that exceed 1. This yields

$$\Pi_r = \Pr[S_r>1]\mathrm{E}[S_r-1\mid S_r>1]. \quad \square$$

Proof of Proposition 2. When the signal distribution is symmetric, the probability that the rational bidder's signal exceeds the mean equals the probability that it is lower than the mean, i.e.,

$$\Pr[S_r>1] = \Pr[S_r<1] = \tfrac{1}{2}.$$

Let $\mathrm{E}[S_r\mid S_r<1] = 1-z$ where $z\in[0,1]$ is a measure of the dispersion of the signal distribution. Then, $\mathrm{E}[S_r\mid S_r>1] = 1+z$. Therefore, (using the results in Proposition 1) the ex ante profits of the two bidders become $\Pi_n = (1-z)/2$ and $\Pi_r = z/2$. Note that Π_r rises with the dispersion of the signal while Π_n falls. $\square$

Proof of Proposition 3. Using the expressions for bidder payoffs obtained in the proof of Proposition 2 above, we can see that the naive bidder has a higher payoff than the rational bidder when $z<\frac{1}{2}$. Because $z\equiv 1-\mathrm{E}[S\mid S<1]$, the above implies $1-\mathrm{E}[S\mid S<1] < 1/2 \Rightarrow \mathrm{E}[S\mid S<1] > \frac{1}{2}$. $\square$

Proof of Theorem 1. Consider a symmetric unimodal signal distribution, $f(s)$, and denote the expectation of S conditional on it being lower than the mean by $\mathrm{E}_f[S\mid S<1]$. Note that the probability mass of $f(s)$ can be spread evenly over its support to yield a uniform distribution with the same support. For this corresponding uniform distribution, denote the expectation of S, conditional on it being lower than the mean, by $\mathrm{E}_u^f[S \mid S < 1]$. It is clear that

$$\mathrm{E}_u^f[S \mid S<1] < \mathrm{E}_f[S \mid S<1] \Rightarrow 1-\mathrm{E}_u^f[S \mid S<1] > 1-\mathrm{E}_f[S \mid S<1].$$

Observe that for all mean 1 uniform distributions with support exclusively on the nonnegative numbers $\mathrm{E}[S \mid S < 1] \geq 1/2$. The result follows from Proposition 3 and this inequality. $\square$

Proof of Proposition 4. We first consider the case of two naive bidders competing against each other. Regardless of the type of his opponent, a naive bidder will stay in the bidding until the price reaches the value of his signal. The realized price will be the lower of the two signals. Therefore,

$$\mathrm{E}[P_{n,n}] = \mathrm{E}[\min(S_1, S_2)].$$

Because the expected value of the item equals 1 and the expected bidder surplus equals the expected value of the item minus the expected price, it follows that the expected per-bidder profit equals

$$\Pi_{n|n} = \tfrac{1}{2}(1 - \mathrm{E}[\min(S_1, S_2)]).$$

Now turn to the competition between two rational bidders. The symmetric equilibrium in the bidding competition between two such bidders is for a bidder with signal S_i to stay in the auction until the price reaches S_i^2. To verify this, consider the expected payoff of Bidder 1 with signal S_1 if he wins the item at a price P. Under the equilibrium bidding strategies, this price equals S_2^2 and his profit is

$$\Pi_1 = S_1 S_2 - S_2^2.$$

Notice that this payoff is positive if $S_1 > S_2$ and negative otherwise. Then there are no profitable deviations from the symmetric equilibrium strategy: Overbidding means that Bidder 1 may win when his value is lower than that of his competitor (and thus incur a loss upon wining). Conversely, underbidding means that Bidder 1 may not win the item even when winning it (under the equilibrium strategy) would have yielded positive profits. The above implies that the expected price equals

$$\mathrm{E}[P_{r,r}] = \mathrm{E}[\min(S_1, S_2)^2].$$

Hence, the expected payoff of each of the two rational bidders equals

$$\Pi_{r|r} = \tfrac{1}{2}(1 - \mathrm{E}[\min(S_1, S_2)^2]). \quad \square$$

Proof of Proposition 5. First compare the expected price when both bidders are naive with the expected price when both are rational. Symmetry of the signal distribution and signal nonnegativity imply that the signal distribution must have zero density outside the $[0, 2]$ interval. Symmetry also implies that $f(x) = f(2-x)$, $F(x) = 1 - F(2-x)$, and $F(1) = \frac{1}{2}$. Unimodality implies that $f(x)$ is a nondecreasing function of x in $[0, 1]$ and, therefore, that $F(x)$ is convex in $[0, 1]$. Finally, symmetry and the convexity of $F(x)$ in $[0, 1]$ imply that $F(x) \leq x/2\ \forall x \in [0, 1]$. The expected price of an auction with two naive bidders exceeds that of an auction with two rational bidders if

$$\begin{aligned}
&\mathrm{E}[\min(S_1, S_2)] \geq \mathrm{E}[\min(S_1, S_2)^2] \\
&\quad \Leftrightarrow \int_0^1 x f(x)[1-F(x)]\,dx + \int_1^2 x f(x)[1-F(x)]\,dx \\
&\qquad \geq \int_0^1 x^2 f(x)[1-F(x)]\,dx + \int_1^2 x^2 f(x)[1-F(x)]\,dx.
\end{aligned}$$

Notice that we have broken the range of integration at the mean of the signal distribution. We can rewrite the second and fourth integrals of this inequality so that all four integrals have support in the $[0, 1]$ interval. In particular, using symmetry and the change of variable $x = 2 - y$ we can write:

$$\begin{aligned}
\int_1^2 x f(x)[1-F(x)]\,dx &= -\int_1^0 (2-y) f(2-y)[1-F(2-y)]\,dy \\
&= -\int_1^0 (2-y) f(y) F(y)\,dy \\
&= \int_0^1 (2-y) f(y) F(y)\,dy.
\end{aligned}$$

Following similar steps, we can write

$$\int_1^2 x^2 f(x)[1-F(x)]\,dx = \int_0^1 (2-y)^2 f(y) F(y)\,dy.$$

Using another (trivial) change of variable $y = x$ and substituting these two integrals in the above inequality, we obtain

$$\begin{aligned}
&\int_0^1 x f(x)[1-F(x)]\,dx + \int_0^1 (2-x) f(x) F(x)\,dx \\
&\quad \geq \int_0^1 x^2 f(x)[1-F(x)]\,dx + \int_0^1 (2-x)^2 f(x) F(x)\,dx.
\end{aligned}$$

Moving everything to the right-hand side and factoring out terms appropriately, we get

$$\int_0^1 [(x-x^2)(1-F(x)) + (2-x-(2-x)^2)F(x)] f(x)\,dx \geq 0,$$

which after some manipulation becomes

$$\int_0^1 (1-x)(x-2F(x)) f(x)\,dx \geq 0.$$

This inequality is true because all terms in the integrand are nonnegative in the $[0, 1]$ interval, and it is strict for all symmetric unimodal distributions except the uniform with support on $[0, 2]$.

We turn now to the comparison of the expected price in an auction with two rational bidders and the expected price in an auction with one rational and one naive bidder. Corollary 1 shows that the expected price when a rational bidder competes against a naive bidder equals 1/2. Therefore, the expected price in an auction with two rational bidders exceeds the price in an auction with one bidder of each type if

$$\begin{aligned}
\mathrm{E}[\min(S_1, S_2)^2] > \tfrac{1}{2} \;\Leftrightarrow\; &\int_0^1 x^2 f(x)[1-F(x)]\,dx \\
&+ \int_1^2 x^2 f(x)[1-F(x)]\,dx > \tfrac{1}{4}.
\end{aligned}$$

Employing symmetry and the appropriate change of variable substitutions (as above), we have

$$\int_0^1 x^2 f(x)[1-F(x)]\,dx + \int_0^1 (2-x)^2 f(x)F(x)\,dx > \tfrac{1}{4}.$$

Successively simplifying and factoring terms, we obtain

$$\int_0^1 f(x)[x^2(1-F(x)) + (2-x)^2 F(x)]\,dx > \tfrac{1}{4}$$

$$\Leftrightarrow \int_0^1 f(x)[x(x-2F(x)) + 4F(x) - 2xF(x)]\,dx > \tfrac{1}{4}$$

$$\Leftrightarrow \int_0^1 f(x)x(x-2F(x))\,dx + \int_0^1 4f(x)F(x)\,dx - \int_0^1 2xf(x)F(x)\,dx > \tfrac{1}{4}. \qquad \text{(P5-1)}$$

The second integral evaluates to $2F(x)^2|_{x=0}^{x=1} = 1/2$. The third integral can be written (using integration by parts)

$$\int_0^1 2xf(x)F(x)\,dx = xF(x)^2|_{x=0}^{x=1} - \int_0^1 F(y)^2\,dy = \tfrac{1}{4} - \int_0^1 F(y)^2\,dy.$$

Substituting into Inequality (P5-1) and simplifying, we get

$$\int_0^1 f(x)x(x-2F(x))\,dx + \int_0^1 F(y)^2\,dy > 0.$$

Observe that the first integral is nonnegative for all symmetric unimodal distributions and zero only for a uniform distribution with support on the interval $[0, 2]$. The second integral is nonnegative for all symmetric unimodal distributions and equals zero only for the degenerate distribution with probability mass concentrated at 1. Therefore, the left-hand side of the above inequality is strictly positive for all (nonnegative) symmetric-unimodal distributions. □

Proof of Corollary 2. Note that $\Pi_{n|n} = (1 - \mathrm{E}[P_{n,n}])/2$ and $\Pi_{r|r} = (1 - \mathrm{E}[P_{r,r}])/2$. It then directly follows from Proposition 5 that $\Pi_{r|r} \geq \Pi_{n|n}$ with equality payoffs holding only when signals are distributed uniformly with support in the $[0, 2]$ interval. Next note that $\Pi_{n|r} + \Pi_{r|n} = (1 - \mathrm{E}[P_{n,r}])$. It then directly follows from Proposition 5 that $\Pi_{n|r} + \Pi_{r|n} > 2\Pi_{r|r}$. Because $\Pi_{n|r} \geq \Pi_{r|n}$ (from Theorem 1), we have $\Pi_{n|r} > \Pi_{r|r}$. Finally, observe that the naive bidder's strategy is independent of his competitor's type. It then follows that a fully rational bidder would obtain a higher payoff when competing against a naive bidder than a naive bidder competing against another naive bidder. □

Proof of Theorem 2. The expected profits of naive and rational bidders as a function of the fraction of bidders who are naive are $\mathrm{E}[\Pi_{naive}] = q\Pi_{n|n} + (1-q)\Pi_{n|r}$ and $\mathrm{E}[\Pi_{rational}] = q\Pi_{r|n} + (1-q)\Pi_{r|r}$. At the equilibrium proportion of naive bidders the expected profits of naive bidders must equal the expected profits of rational bidders. Therefore, q_{eq} satisfies

$$q_{eq}\Pi_{n|n} + (1-q_{eq})\Pi_{n|r} = q_{eq}\Pi_{r|n} + (1-q_{eq})\Pi_{r|r}$$

$$\Rightarrow\ q_{eq} = \frac{\Pi_{n|r} - \Pi_{r|r}}{[\Pi_{r|n} - \Pi_{n|n}] + [\Pi_{n|r} - \Pi_{r|r}]}.$$

A sufficient set of conditions for $q_{eq} \in (0, 1)$ is $\Pi_{n|r} - \Pi_{r|r} > 0$ and $\Pi_{r|n} - \Pi_{n|n} > 0$. These conditions have been shown to hold from Corollary 2.

For this evolutionary equilibrium to be stable it is required that a small increase q results in higher expected profits for the rational bidders relative to those for the naive bidders. If that is the case, then the proportion of naive bidders will tend to decline back towards q_{eq}. Otherwise, the proportion of naive bidders will increase further and diverge from q_{eq}. Stability, then, requires

$$\frac{d(\mathrm{E}[\Pi_{rational}] - \mathrm{E}[\Pi_{naive}])}{dq} > 0$$

$$\Rightarrow\ [\Pi_{r|n} - \Pi_{n|n}] + [\Pi_{n|r} - \Pi_{r|r}] > 0.$$

However, both bracketed terms of the above inequality are positive for signal distributions that are symmetric unimodal. Therefore, the interior evolutionary equilibrium, q_{eq}, is stable. □

References

Ball, S. B., M. H. Bazerman, J. S. Carroll. 1991. An evaluation of learning in the bilateral winner's curse. *Organ. Behavior Human Decision Processes* **48** 1–22.

Block, R. A., D. R. Harper. 1991. Overconfidence in estimation: Testing the anchoring-and-adjustment hypothesis. *Organ. Behavior Human Decision Processes* **49** 188–207.

Blume, L., D. Easley. 1992. Evolution and market behavior. *J. Econom. Theory* **58** 9–40.

Brenner, L. A., D. J. Koehler, V. Liberman, A. Tversky. 1996. Overconfidence in probability and frequency judgements: A critical examination. *Organ. Behavior Human Decision Processes* **65** 212–219.

Capen, E. C., R. V. Clapp, W. M. Campbell. 1971. Competitive bidding in high-risk situations. *J. Petroleum Tech.* **23** 641–653.

Cassing, J., R. W. Douglas. 1980. Implications of the auction mechanism in baseball's free agent draft. *Southern Econom. J.* **47** 110–121.

Cox, J. C., R. M. Isaac. 1984. In search of the winner's curse. *Econom. Inquiry* **22** 579–592.

Engelbrecht-Wiggans, R., R. J. Weber. 1983. A sequential auction involving asymmetrically-informed bidders. *Internat. J. Game Theory* **12** 123–127.

Griffin, D., A. Tversky. 1992. The weighting of evidence and the determinants of confidence. *Cognitive Psych.* **24** 411–435.

Hendricks, K., R. H. Porter, B. Boudreau. 1987. Information, returns, and bidding behavior in OCS auctions: 1954–1969. *J. Indust. Econom.* **35** 517–542.

Kagel, J. H. 1995. Auctions: A survey of experimental research. J. H. Kagel, A. E. Roth, eds. *Handbook of Experimental Economics*. Princeton University Press, Princeton, NJ, 501–586.

Kagel, J. H., D. Levin. 1986. The winner's curse and public information in common value auctions. *Amer. Econom. Rev.* **76** 894–920.

Kyle, A. S., F. A. Wang. 1997. Speculation duopoly with agreement to disagree: Can overconfidence survive the market test? *J. Finance* **52** 2073–2090.

McAfee, R. P., J. McMillan. 1987. Auctions and bidding. *J. Econom. Lit.* **25** 699–738.

Mead, W. J., A. Moseidjord, P. E. Sorensen. 1983. The rate of return earned by lessees under cash bonus bidding of OCS oil and gas leases. *Energy J.* **4** 37–52.

Samuelson, W. F., M. H. Bazerman. 1985. The winner's curse in bilateral negotiations. *Res. Experiment. Econom.* **3** 105–137.

Thaler, R. H. 1992. *The Winner's Curse: Paradoxes and Anomalies of Economic Life*. Princeton University Press, Princeton, NJ.