Basics of auction design

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In this reader we present a bit of general notation and theory regarding auction design. We thoroughly discuss 1^{st} and 2^{nd} price sealed bid auctions for a single indivisible item. We include a discussion of dominant strategy, ex post equilibrium, and BNE. We also discuss ex post efficiency and revenue equivalence.

1 The general setting

The auction design problem we discuss in this introduction can be formulated as follows. Given is a single indivisible item, and a finite set $N = \{1, \ldots, n\}$ of agents. Each agent $i \in N$ has a valuation

$$v_i \in [0, 1]$$

from ownership of the item.

THE TASK. Imagine you are a social planner [1] who wants to choose allocate the item to an agent $i \in N$ with

$$v_i \geq v_j$$

for all $j \in N$. [2] In words, your task as the social planner is to allocate Pareto optimally. In case you manage to do so, we say that the allocation is ex post efficient.

Now the central question here is:

Is it possible to achieve this aim?

In case all information is known, so when you know all valuations, this is easily achieved. However, ususally there is a lack of information on the side of the social planner.

ASYMMETRIC INFORMATION. We assume that each agent i knows his own valuation v_i, but not the valuations of other agents. (This is called the private information setting). In

[1] A social planner is an entity that is assumed to be impartial, and aims to maximize societal welfare.

[2] When all valuations are different, this agent is uniquely identified. However, when several agents have the highest valuation, one of them is selected.

particular, the social planner does not know any of the valuations.

In this case, the social planner needs to extract information from the agents if he wants to allocate efficiently. There are many ways to try to achieve this, such as trading platforms, negotiation protocols, markets, etcetera. We focus in particular on auctions.

QUASI LINEAR UTILITY. In order to use auctions for solving the information problem, we need one additional assumption on utility. We assume that agents have quasi-linear utility. Formally agent i has quasi-linear utility if ownership of the item for a payment of p euros results in utility

$$u_i = v_i - p.$$

The interpretation of this is that the amount p of money paid by agent i has a linear effect on the utility of agent i.

2 Vickrey auction

We consider the 2^{nd} price, sealed bid) auction (also known as the Vickrey auction) for a single indivisible item. As before, there are n bidders. Each bidder is allowed to submit a bid $b_i \in [0,1]$, and bids are made simultaneously. [3] The winner i is selected from the bidders with the highest bid (usually a unique person, but possibly more). The winner i pays

$$p_i = \max\{b_j \mid j \neq i\}. \text{ [4]}$$

Other bidders do not pay anything.

STRATEGIES. A strategy of agent i is a bid function $s_i \colon [0,1] \to [0,1]$ that assigns to each possible valuation $v_i \in [0,1]$ a *reported* valuation (a bid) $s_i(v_i) \in [0,1]$. [5]

DOMINANT STRATEGY. A strategy $s_i \colon [0,1] \to [0,1]$ is called a dominant strategy for the Vickrey auction if, for any realized valuation $v_i \in [0,1]$ and profile $(b_j)_{j\neq i}$ of bids of other agents, the bid $s_i(v_i)$ is a best response of agent i to $(b_j)_{j\neq i}$ in the game described above.

Theorem 2.1 *In the Vickrey auction, it is a dominant strategy to report $s_i(v_i) = v_i$. In the resulting equilibrium, allocation is ex post efficient.*

[3] Therefore the auction is sealed bid. You can think of this as the bidders handing in their bids in sealed envelopes to the auctioneer (social planner).

[4] And hence the name 2^{nd} price.

[5] You can think of this reported valuation as being the answer of agent i to the question: "What is your valuation?" Of course, agent i cannot be forced to report truthfully, since the social planner does not have the information to check the veracity of the reported valuation. So, $s_i(v_i)$ may well be different from v_i.

Proof. We first argue the second part of the above claim. Suppose that every agent reports truthfully. That is $b_i = v_i$ for every i. Then, since the selected winner i has a maximimum bid, he also has a maximimum valuation. Hence, allocation is efficient.

Take an agent i. Take any profile $b_{-i} = (b_j)_{j \neq i}$ of bids of agents other than i. Write $b^* = \max\{b_j \mid j \neq i\}$. Let v_i be the valuation of agent i. We argue that the bid $b_i^* = v_i$ is a best response.

If $v_i > b^*$. Then bidding $b_i \leq b^*$ results in zero payoff. Bidding $b_i > b^*$ results in payoff $u_i = v_i - b^* > 0$. Hence, bidding $b_i^* = v_i$ is a best response.

In the same way, we can check that bidding $b_i^* = v_i$ is a best response in case $v_i \leq b^*$. This concludes the proof. ∎

EX POST EQUILIBIUM. A strategy profile $s = (s_1, \ldots, s_n)$ in an auction is called an ex post equilibrium if, for any realization $v = (v_1, \ldots, v_n)$ of valuations, the resulting profile $(s_1(v_1), \ldots, s_n(v_n))$ of actions is a Nash equilibrium.

Wolf and sheep. One ex post equilibrium in the Vickrey auction is the Wolf and sheep strategy profile. In this profile, there is one wolf, let us say bidder 1. The wolf has bid strategy $s_1(v_1) = 1$. Thus, the wolf bids 1 regardless of his valuation. All other bidders $j \neq 1$ are sheep, with bid strategies $s_j(v_j) = 0$.

We argue that the resulting profile $(s_1, \ldots, s_n)$ is an ex post equilibrium. Take any realization $v = (v_1, \ldots, v_n)$ of valuations. We argue that the resulting profile $(s_1(v_1), \ldots, s_n(v_n)) = (1, 0, \ldots, 0)$ of actions is a Nash equilibrium.

Best responses wolf. Given the bid vector $(s_2(v_2), \ldots, s_n(v_n)) = (0, \ldots, 0)$, any bid larger than zero is a best response, since that way agent 1 wins the item for a price of zero. In particular $s_1(v_1) = 1$ is a best response.

Best responses sheep. For agent $j \neq 1$. Given the bid vector $(1, 0, \ldots, 0)$, of opponents, any bid strictly smaller than one is a best response, since agent j can only win the item by bidding 1, in which case the price is 1, with a utility of $v_j - 1 \leq 0$. In particular $s_j(v_j) = 0$ is a best response.

3 First price, sealed bid auction

Another way to resolve the informational asymmetry, and achieve an (ex post) efficient allocation is the 1^{st} price, sealed bid auction. The 1^{st} price, sealed bid auction for the sales of a single, indivisible item is executed as follows. Each bidder is allowed to submit a bid $b_i \in [0,1]$. These bids should be made simultaneously. A winner i is appointed among the bidders in such a way that $b_i \geq b_j$ for all j. The winner pays his own bid. [6] Other bidders pay nothing.

Theorem 3.1 *The strategy profile $b = (b_1, \ldots, b_n)$ given by $b_i(v_i) = \frac{n-1}{n} \cdot v_i$ is a BNE of the associated Bayesian game. Moreover, in equilirium, allocation is ex post efficient.*

Proof. Consider bidder i, having valuation v_i, facing bid strategies $b_j(v_j) = \frac{n-1}{n} \cdot v_j$ of bidders $j \neq i$. It suffices to show that the bid $b_i = \frac{n-1}{n} \cdot v_i$ maximizes the expected payoff function $E_i(v_i, b)$ over $b \in [0,1]$.

A. We first compute the payoff function $E_i(v_i, b)$. Suppose that bidder i bids $b \in [0,1]$. Then

$$\begin{aligned}
E_i(v_i, b) &= (v_i - b) \cdot \text{Prob}\,[b \text{ is the winning bid}] = (v_i - b) \cdot \text{Prob}\,[b_j < b \text{ for all } j \neq i] \\
&= (v_i - b) \cdot \text{Prob}\left[\frac{n-1}{n} \cdot v_j < b \text{ for all } j \neq i\right] \\
&= (v_i - b) \cdot \text{Prob}\left[v_j < \frac{n}{n-1} \cdot b \text{ for all } j \neq i\right] \\
&= (v_i - b) \cdot \left(\frac{n}{n-1} \cdot b\right)^{n-1}.
\end{aligned}$$

B. Thus, we want to maximize the above payoff function $E_i(v_i, b)$ over b. The first order condition states that

$$(n-1) \cdot \frac{n}{n-1} \cdot \left(\frac{n}{n-1} \cdot b\right)^{n-2} \cdot (v_i - b) - \left(\frac{n}{n-1} \cdot b\right)^{n-1} = 0.$$

This can be rewritten to

$$n \cdot (v_i - b) - \frac{n}{n-1} \cdot b = 0,$$

which yields $b = \frac{n-1}{n} \cdot v_i$. It can easily be checked that this indeed is a maximum location. This concludes the proof.

We check that, under the BNE, allocation is efficient.

$$b_i > b_j \quad \Leftrightarrow \quad \frac{n-1}{n} \cdot v_i > \frac{n-1}{n} \cdot v_j \quad \Leftrightarrow \quad v_i > v_j.$$

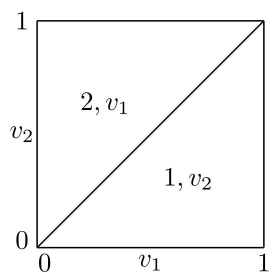
Thus, in the BNE, bidder i wins over bidder j in the auction precisely when $v_i > v_j$. This ensures efficiency of allocation in equilibrium. ∎

[6] And hence the name 1^{st} price.

4 Revenue equivalence

In this section we compute, for $n = 2$, the expected revenue for the auctioneer in the BNE of the 1^{st} price sealed bid auction, and the dominant strategy equilibrium of the 2^{nd} price sealed bid auction. Valuations are supposed to be iid draws from the uniform distribution on the unit interval.

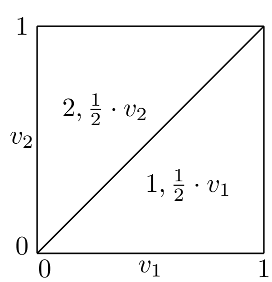
SECOND PRICE SEALED BID. Consider the 2^{nd} price sealed bid auction with the dominant strategy equilibrium. For $n = 2$ we thus have $b_1(v_1) = v_1$ and $b_2(v_2) = v_2$. The revenues for the auctioneer are displayed in the diagram below.



Notice that bidder 2 wins the auction precisely when $v_2 \geq v_1$. So, indeed, allocation is ex post efficient. The payment for bidder 2 is $b_1 = v_1$. The expected revenue can now be computed by

$$\mathbb{E}\left[\text{revenue}\right] = 2 \cdot \int_0^1 \int_{v_1}^1 v_1 dv_2 dv_1 = 2 \cdot \int_0^1 v_1(1 - v_1)dv_1 = 2 \cdot \left[\tfrac{1}{2} \cdot v_1^2 - \tfrac{1}{3}v_1^3\right]_0^1 = 2 \cdot \tfrac{1}{6} = \tfrac{1}{3}.$$

FIRST PRICE SEALED BID. The same computation can now be done for symmetric BNE in the 1^{st} price sealed bid auction. For $n = 2$ we have $b_1(v_1) = \frac{1}{2} \cdot v_1$ and $b_2(v_2) = \frac{1}{2} \cdot v_2$. The revenues for the auctioneer are displayed in the diagram below.



Again, notice that bidder 2 wins the auction precisely when $v_2 \geq v_1$. So, indeed, allocation is ex post efficient. The payment for bidder 2 is $b_2 = \frac{1}{2} \cdot v_2$. The expected revenue can then be computed by

$$\mathbb{E}\,[\text{revenue}] = 2 \cdot \int_0^1 \int_0^{v_2} \tfrac{1}{2} \cdot v_2 dv_1 dv_2 = \int_0^1 v_2^2 dv_2 = \left[\tfrac{1}{3} \cdot v_2^3\right]_0^1 = \tfrac{1}{3}.$$

Notice that the result is the same as for the dominant strategy equilibrium in the Vickrey auction. This is not just a coincidence, but a consequence of a general fact on the payoffs for any ex post efficient BNE, known as the Revenue Equivalence Theorem (RET). We state the RET, without proof, for our setting of sales of a single indivisible item for n bidders with valuations that are iid draws from the uniform distribution. An auction is called individually rational if each bidder has a strategy that guarantees a utility of at least zero.

Theorem 4.1 (RET). *Given any auction A, and any BNE s in that auction such that*

[1] A is individually rational

[2] payments in A are non-negative

[3] s is ex post efficient.

Then $\mathbb{E}\,[\text{revenue}] = \frac{n-1}{n+1}$.

5 Exercises

Exercise 1. (Dutch auction)

Show that the 1^{st} price, sealed bid auction does not have an ex post equilibrium.

Exercise 2. (Revenue Equivalence Theorem)

Consider the case where $n = 2$. Show that the expected profit of bidder 1 in the Vickrey auction under truthful reporting equals his expected profit under the BNE in the 1^{st} price, sealed bid auction. (Hint: when you do this correctly, the result for both cases should be $\frac{1}{6}$.)

Exercise 3. (Ex post equilibrium in the Vickrey auction)

Consider the Vickrey auction with two bidders. Let

$$b_1(v_1) = \begin{cases} 0 & \text{if } v_1 \leq \frac{1}{4} \\ \\ \frac{1}{2} & \text{if } v_1 > \frac{1}{4} \end{cases} \quad \text{and} \quad b_2(v_2) = \begin{cases} \frac{1}{4} & \text{if } v_2 \leq \frac{1}{2} \\ \\ 1 & \text{if } v_2 > \frac{1}{2} \end{cases}$$

be bidding strategies for bidders 1 and 2, respectively.

(a) Show that (b_1, b_2) is an ex post equilibrium in the Vickrey auction.

(b) Is the resulting allocation of the item efficient? Motivate your answer.

(c) Which condition of the RET is violated?

(d) Draw a diagram to represent the revenues, and compute the average revenue for the auctioneer.

Exercise 4. (Comparison of equilibrium notions)

Argue that every dominant strategy equilibrium is also an ex post equilibrium. Also argue the every ex post equilibrium is a BNE. Argue that the Wolf and sheep equilibrium is not a dominant strategy equilibrium. Also argue that the BNE in the 1^{st} price sealed bid auction is not an ex post equilibrium.

Exercise 5. (Ex post equilibrium in the Vickrey auction)

Determine the expected revenue for the auctioneer in the Wolf and sheep ex post equilibrium of the 2^{nd} price sealed bid auction. Which condition for the RET is violated in this equilibrium?