# Session 2
## Auctions and Electronic Markets

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## Economics

Sale of 1 indivisible item, $N = \{1, \ldots, n\}$ agents, buyers, bidders, players.
Each agent $i \in N$ has a valuation $v_i \in [0, 1]$ for ownership of the item.

## Topics

- auction formats
  - $1^{\text{st}}$ price sealed bid
  - $2^{\text{nd}}$ price sealed bid
- equilibrium concepts
  - DSE (Dominant Strategy Equilibrium)
  - ex-post/no-regret equilibrium
  - BNE (Bayesian Nash Equilibrium)

## Efficiency/Pareto optimality

**Aim:** try to allocate to an agent $i \in N$ such that $v_i \geq v_j$ for all $j \in N$.

**Problem:** information asymmetry
Private values: agent $i$ knows $v_i$, but not the valuations of the others. Auctioneer does not know any $v_i$'s.

**Task:** can we design

- a market
- a negotiation protocol
- a trading platform
- an auction

in such a way that *in equilibrium* the resulting allocation is efficient?

**Answer:** yes.

## Auction

An auction consists of:

1. Legal moves (bids)
2. Allocation
3. Payments

# Auction Design: Vickrey auction/2$^{\text{nd}}$ price, sealed bid auction

1. Bids: each agent submits a single bid $b_i \in [0,1]$ or $b_i(v_i)$ ex-ante (before $v_i$ is known).
2. Allocate item to an agent $i \in N$ with $b_i \geq b_j$ for any $j \in N$, this is the winner.
3. Winner $i$ pays $p_i = \max_j\{b_j | j \neq i\}$, for the others, $j \neq i$, $p_j = 0$.

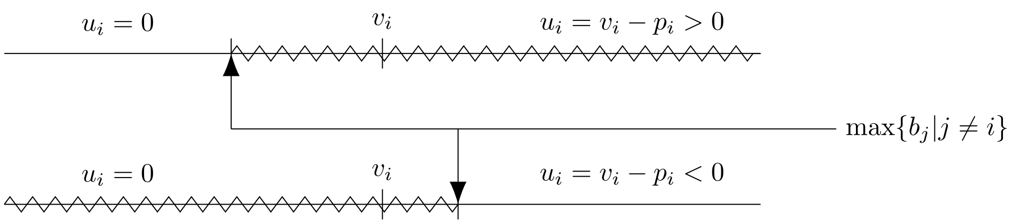
# Theorem (Vickrey)

Bidding truthfully, $b_i(v_i) = v_i$, is a *dominant strategy* in the Vickrey auction. (dominant strategy: no matter what the other people do, a dominant strategy is a best response to that). In the resulting DSE, the allocation is efficient.

2 remarks:

- Quasi-linear utility: $u_i = v_i - p_i$.
- $b_i(v_i) = v_i$ is a dominant strategy; maximizes utility regardless of other bidders' bids.

## Sketch of proof




signifies the best response for bidder $i$. $b_i(v_i) = v_i$ is *always* a best response, and is therefore a dominant strategy.

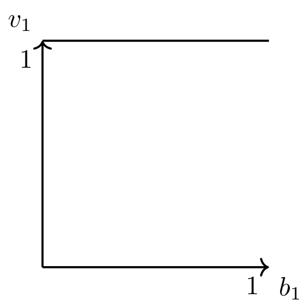
# Wolf and Sheep

## Ex-post/no-regret equilibrium

$b_i(v_i)$ defines a *bid strategy*; $v_i \mapsto b_i(v_i)$.
$b = (b_1, \ldots, b_n)$ defines a *bid profile.*

Profile $b$ is an ex-post equilibrium if for every realization $(v_1, \ldots, v_n)$ of valuations the bid profile $(b_1(v_1), \ldots, b_n(v_n))$ is a Nash Equilibrium in the auction game.

**Wolf:** Bidder 1 is the wolf, $b_1(v_1) = 1$. The bid function looks like this:




**Sheep:** Bidder $j \neq i$, $b_j(v_j) = 0$.

**Claim:** Profile $(b_1, \ldots, b_n)$ is an ex-post equilibrium.

**Proof:** Take any realization $(v_1, \dots, v_n)$. Is the resulting bid profile $(b_1(v_1), \dots, b_n(v_n)) = (1, 0, \dots, 0)$ a Nash Equilibrium?

**Wolf:** Bidding against $(0, \dots, 0)$, possible utility levels: $u_1 = 0$ or $u_i = v_1 - p_1 = v_1$
**Sheep:** Bidding against $(1, 0. \dots, 0)$, possible utility levels for $j \neq 1$: $u_j = 0$ or $u_j = v_j - 1 \leq 0$.

## Auction Design: 1<sup>st</sup> price, sealed bid auction

1. Bids: each agent submits a bid $b_i \in [0, 1]$, $b_i(v_i)$ ex-post.
2. Allocate item to an agent $i \in N$ (winner) with $b_i \geq b_j$ for any $j \in N$.
3. Winner $i$ pays $p_i = b_i$.

## Bayesian Nash Equilibrium

## Intermezzo

**Assumptions:** valuations are i.i.d. draws from a uniform distribution on the unit interval [0,1]. The *copula* (contains the dependence structure between random variables):

$$\begin{array}{ccc} & & v_2 \\ & & \begin{array}{cc} \mathrm{L} & \mathrm{H} \end{array} \\ v_1 & \begin{array}{c} \mathrm{L} \\ \mathrm{H} \end{array} & \begin{bmatrix} \frac{3}{12} & \frac{1}{12} \\ \frac{6}{12} & \frac{2}{12} \end{bmatrix} \end{array}$$

$$\mathbb{P}[v_1 = \mathrm{H}, v_2 = \mathrm{L}] = \frac{6}{12}$$

Are $v_1$ and $v_2$ independent? Multiply the marginals to find out:

$$\begin{aligned} \mathbb{P}[v_1 = \mathrm{H}] \cdot \mathbb{P}[v_2 = \mathrm{L}] &= \left(\frac{6}{12} + \frac{2}{12}\right) \cdot \left(\frac{3}{12} + \frac{6}{12}\right) \\ &= \frac{8}{12} \cdot \frac{9}{12} \\ &= \frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12} \end{aligned}$$

So yes, they are independent.

### Bayes' Rule

$$\mathbb{P}[v_1 = \mathrm{H} | v_2 = \mathrm{L}] = \frac{\mathbb{P}[v_2 = \mathrm{L} | v_1 = \mathrm{H}] \cdot \mathbb{P}[v_1 = \mathrm{H}]}{\mathbb{P}[v_2 = \mathrm{L}]}$$

## Interim analysis

**Claim:** Profile $b = (b_1, \dots, b_n)$ with $b_i(v_i) = \frac{n-1}{n} \cdot v_i$ is a BNE.
**Proof:** Take bidder $i$, you know $v_i$ (interim analysis), other bidders $j$ bid $b_j(v_j) = \frac{n-1}{n} \cdot v_j$. Suppose you bid some bid $b$.

$$\begin{aligned} \mathbb{E}(u_i) = (v_i - b) \cdot \mathbb{P}[i \text{ wins}] &= (v_i - b) \cdot \mathbb{P}\left[b > \frac{n-1}{n} v_j \text{ for all } j \neq i\right] \\ &= (v_i - b) \cdot \mathbb{P}\left[v_j < \frac{n}{n-1} b \text{ for all } j \neq i\right] \\ &= (v_i - b) \cdot \left(\frac{n}{n-1} b\right)^{n-1} \end{aligned}$$

To maximize, take derivative with respect to $b$, the first order condition is:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}b}\mathbb{E}(u_i) = -1 \cdot \left(\frac{n}{n-1}b\right)^{n-1} &+ (n-1)\left(\frac{n}{n-1}b\right)^{n-2} \cdot \frac{n}{n-1} \cdot (v_i - b) = 0 \\
\iff (v_i - b) \cdot n &= \frac{n}{n-1}b \quad \left[\text{divide both sides by } \left(\frac{n}{n-1}b\right)^{n-2}\right] \\
\iff v_i - b &= \frac{1}{n-1}b \\
\iff \frac{n}{n-1}b &= v_i \\
\iff b &= \frac{n-1}{n}v_i
\end{aligned}$$

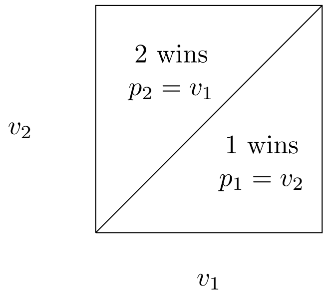
This is ex-post efficient. Even though no one is truthful, the item goes to the bidder with the highest valuation.
For $b_i = a \cdot v_i + c \implies a = \frac{n-1}{n}$ and $c = 0$. This is the *only* BNE.

## Revenue calculation

Calculate the revenue for the $2^{\text{nd}}$ price sealed bid auction with 2 players.

$$\begin{aligned}
b_1(v_1) &= v_1 \\
b_2(v_2) &= v_2
\end{aligned}$$

The payoff diagram to the auctioneer looks like:




$$\begin{aligned}
\mathbb{E}(\text{revenue}) &= 2 \cdot \int_0^1 \int_{v_2}^1 v_2 \,\mathrm{d}v_1 \,\mathrm{d}v_2 \\
&= 2 \cdot \int_0^1 (1 - v_2) v_2 \,\mathrm{d}v_2 \\
&= 2\left[\frac{1}{2}v_2^2 - \frac{1}{3}v_2^3\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}
\end{aligned}$$