Session 3

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0 Topics

1. Pareto Optimality & VMP (vector maximum problem)
2. Inefficiency of equilibrium
3. Common value auctions
4. Revenue maximization

1 Pareto Optimality & VMP

$N = \{1, \dots, n\}$ agents, A set of allocations

$$\begin{aligned} v_i : A \to \mathbb{R} &\qquad \text{valuation function of agent } i. \\ a \mapsto v_i(a) &\qquad \text{valuation of agent } i \text{ for allocation } a. \end{aligned}$$

Pareto Optimality

$a, b \in A$

b Pareto dominates a if $v_i(b) \geq v_i(a)$ for all $i \in N$ with strict inequality for at least one $j \in N$.

a is Pareto undominated (also: Pareto optimal, efficient) if there is no allocation $b \in A$ that Pareto dominates a.

TU (Total Utility)

For $a \in A$

$$TU(a) = \sum_{i \in N} v_i(a)$$

a is TU maximizer if

$$TU(a) \geq TU(b) \quad \text{for all } b \in A$$

Fact: If a is a TU maximizer, then a is efficient.

Proof: by contradiction, assume a is not efficient. There exists a $b \in A$ such that $v_i(b) \geq v_i(a)$ for all $i \in N$ and $v_j(b) > v_j(a)$ for at least one $j \in N$. $\sum_{i \in N} v_i(b) > \sum_{i \in N} v_i(a) \implies TU(b) > TU(a)$, so a is not a TU maximizer, this is a contradiction.

Note: Pareto undominated $\not\Longrightarrow$ TU maximizer, here's an example:

allocation	1	2	3	4	5	TU
a_1	3	2	0	4	2	11
a_2	2	1	4	3	1	11
a_3	4	2	1	4	3	14
a_4	3	2	1	4	2	12

a_2 and a_3 are Pareto undominated

Quasi-linear utility

$$u_i(a,p) = v_i(a) - p$$

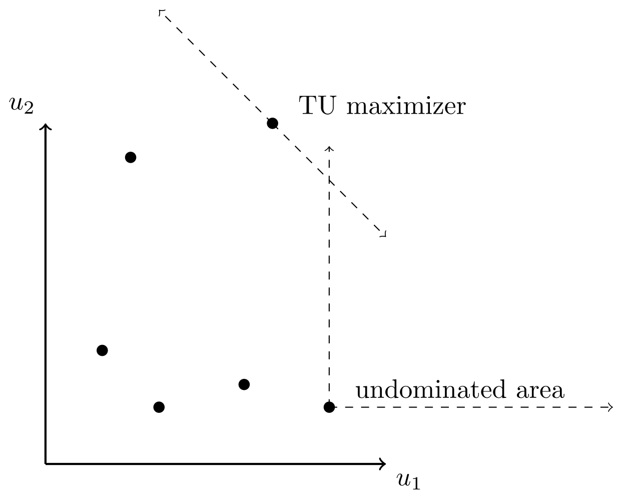
Quasi-linear because it's only linear in p and not in a.

Fact: Under quasi-linear utility, Pareto optimality is equivalent to TU maximization.
This is called the no wealth effect, in the real world there is a wealth effect: you're happier with €100 if you were broke than if you were a millionaire.

In the context of the example above, assume quasi-linear utility and money transfers. Then a_3 + money transfers Pareto dominates a_2. An example: P1 and P5 both pay €1.50 to P3.

player	1	2	3	4	5	TU
a_3	4	2	1	4	3	14
new allocation	2.5	2	4	4	1.5	14
calculations of payments	(4-1.5)		(1+3)		(3-1.5)	

If there is quasi-linear utility and enough money for money transfers, you can always take the TU maximizer and make it Pareto dominate any other allocation (not all other allocations simultaneously though). Here's an example with 2 agents and 6 allocations:



When the TU maximizer with money transfers Pareto dominates the undominated allocation, another allocation, the one at the top left, becomes undominated.

2 Inefficiency of equilibrium

Bilateral trade

2 agents: buyer and seller. 1 indivisible item currently owned by seller.

$$v_B \in [0,1]$$
$$v_S \in [0,1]$$

The valuations are assumed to be iid draws from $U[0,1]$.

Efficiency: Trade $\iff$ $v_B \geq v_S$.
Information asymmetry: v_B, v_S are private.

Theorem: Myerson
There is no mechanism with an equilibrium in which allocation is efficient (Pareto optimal).
Proof: not going to prove this.

Double auction

Bid of buyer $p_B \in [0,1]$, bid of seller $p_S \in [0,1]$. Think of these as functions of the valuations; $p_B(v_B)$ and $p_S(v_S)$. Trade $\iff$ $p_B \geq p_S$. The payment from buyer to seller is determined by

$$p = \frac{p_B + p_S}{2}$$

Example: ex post equilibrium

Given some exogenous threshold, $x \in [0,1]$, the bid profiles are

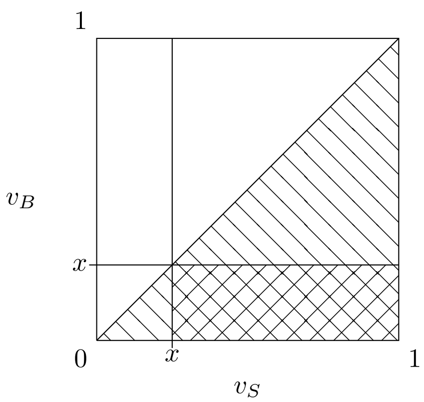
$$p_B(v_B) = \begin{cases} x & \text{if } v_B \geq x \\ 0 & \text{if } v_B < x \end{cases}, \quad p_S(v_S) = \begin{cases} 1 & \text{if } v_S > x \\ x & \text{if } v_S \leq x \end{cases}$$

Claim: This is an ex post equilibrium
Proof:
If $v_B \geq x$ and $v_S \leq x$, the buyer is facing $p_S = x$, the buyer faces the following scenario:

$$\begin{aligned} &\text{if } p_B < x, \quad u_B = 0 \\ &\text{if } p_B \geq x, \quad u_B = v_B - \frac{p_B + x}{2} \end{aligned}$$

The buyer will bid $p_B = x$ to maximize u_B.
Etc. There are in total 8 cases, 4 for the buyer and 4 for the seller.

Efficiency



Efficiency dictates trade should happen in the dashed area, but it does not. Trades only happen where the cross pattern is.

3 Common value auctions

Economics

$n = 2$, 1 indivisible item, v common value. Private signals, $s_1 \in [0,1]$ and $s_2 \in [0,1]$ iid draws from $U[0,1]$. There is a common valuation:

$$v = s_1 + s_2$$

It's a 1st price, sealed bid auction.

Bayesian Nash Equilibrium

$$b_1(s_1) = c \cdot s_1$$
$$b_2(s_2) = c \cdot s_2$$

Bidder 1 has signal s_1 and knows bidder 2 bids according to $b_2(s_2) = c \cdot s_2$. Bidder 1 bids b, what should c be? Maximize expected utility:

$$\begin{aligned}
\mathbb{E}[\text{utility}] &= \mathbb{E}[v - b|\text{you win}] \cdot \mathbb{P}[\text{you win}] \\
&= \mathbb{E}[s_1 + s_2 - b|b > c \cdot s_2] \cdot \mathbb{P}[b > c \cdot s_2] \\
&= \left(s_1 + \mathbb{E}\left[s_2|s_2 < \frac{b}{c}\right] - b\right) \cdot \mathbb{P}\left[s_2 < \frac{b}{c}\right] \\
&= \left(s_1 + \frac{b}{2c} - b\right) \cdot \frac{b}{c}
\end{aligned}$$

The first order condition is:

$$\begin{aligned}
\frac{1}{c}\left(s_1 + \frac{b}{2c} - b\right) + \left(\frac{1}{2c} - 1\right) \cdot \frac{b}{c} &= 0 \\
(2cs_1 + b - 2cb) + (1 - 2c) \cdot b &= 0 \quad \text{(multiplied with } 2c^2\text{)} \\
2cs_1 &= -b + 2cb - b + 2cb \\
cs_1 &= -b + 2cb \\
b &= \frac{c}{2c - 1} \cdot s_1
\end{aligned}$$

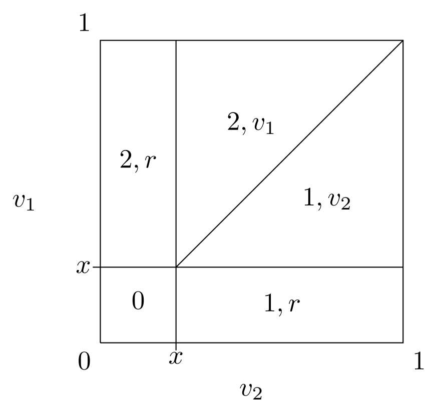
Because of symmetry:

$$\frac{c}{2c-1} = c$$
$$c = 0 \quad \text{or} \quad c = 1$$

4 Revenue maximization

$n = 2$, 1 indivisible item, $v_1 \in [0, 1]$ and $v_2 \in [0, 1]$ are iid draws from $U[0, 1]$, r reservation value. It is a 2nd price, sealed bid plus reserve auction. If both bids are above r, it's a regular Vickrey auction. If one of the bids is lower, the winner pays r. If both are lower, the item doesn't get allocated. The idea is to sacrifice efficiency to increase expected revenue.

In this auction, $b_1(v_1) = v_1$ and $b_2(v_2) = v_2$ is a DSE (dominant strategy equilibrium). The proof is easy, view the reserve price as the auctioneer playing as an extra bidder who always bids r. The DSE is still $b_i(v_i) = v_i$ for all $i \in N$. The payoff diagram for the auctioneer:



$$
\begin{aligned}
\mathbb{E}[\text{revenue}] &= 2r(1-r)\cdot r + 2\cdot \int_r^1 \int_{v_1}^1 v_1 \,\mathrm{d}v_2 \,\mathrm{d}v_1 \\
&= 2r^2(1-r) + 2\cdot \int_r^1 v_2(1-v_2)\,\mathrm{d}v_1 \\
&= 2r^2(1-r) + \left[\frac{1}{2}v_1^2 - \frac{1}{3}v_1^3\right]_r^1 \\
&= 2r^2 - 2r^3 + 2\left(\frac{1}{6} - \frac{1}{2}r^2 + \frac{1}{3}r^3\right) \\
&= \frac{1}{3} + r^2 - \frac{4}{3}r^3
\end{aligned}
$$

Sanity check: for $r=0$, $\mathbb{E}[\text{revenue}] = \frac{1}{3}$, which is the same as the expected revenue of the Vickrey auction (2^{nd} price, sealed bid auction). For $r=1$, $\mathbb{E}[\text{revenue}] = 0$, which is logical, no one will win because the auctioneer bids the maximum bid.

First order condition

$$
\begin{aligned}
2r - 4r^2 &= 0 \\
2r(1-2r) &= 0 \\
r = 0 \quad \text{or} \quad r &= \frac{1}{2}
\end{aligned}
$$

For $r=0$, $\mathbb{E}[\text{revenue}] = \frac{1}{3}$, for $r = \frac{1}{2}$, $\mathbb{E}[\text{revenue}] = \frac{1}{3} + \left(\frac{1}{2}\right)^2 - \frac{4}{3}\left(\frac{1}{2}\right)^3 = \frac{5}{12}$. As $\frac{5}{12} > \frac{1}{3}$, the auctioneer can expect more revenue by setting $r = \frac{1}{2}$.