Competitive Bidding in High-Risk Situations

E. C. Capen, Atlantic Richfield Co.
R. V. Clapp, Atlantic Richfield Co.
W. M. Campbell, Atlantic Richfield Co.

Introduction

We would like to share with you our thoughts on the theory of competitive bidding. It is a tough business. We are not sure we understand as much as we ought to about the subject. As in most scientific endeavors, we think there is more knowledge to be gained by talking with others than by keeping quiet.

Our first attempt at actually using a probability model approach to bidding was in 1962. We borrowed heavily from Lawrence Friedman's fine paper on the subject.[1] But the further our studies went, the more problems we noticed for our particular application. We decided to strike out on our own. By 1965 we had our model just as it is today. But having a model and completely understanding its workings are not the same thing. We are still learning.

While we refer to the "model" as though it were some inanimate object, it is not. What we want to describe to you is a system for taking the best judgments of people — properly mixed, of course, with historical evidence — and putting those judgments together in a rational way so they may be used to advantage.

Lest the reader be too casual, thinking that since he is not personally involved in lease sales he need not pay the closest attention, we offer this thought. There is a somewhat subtle interaction between competition and property evaluation, and this phenomenon — this culprit — works quietly within and without the specific lease sale environment. We would

venture that many times when one purchases property it is because someone else has already looked at it and said, "Nix." The sober man must consider, "Was he right? Or am I right?" The *method of analysis* we will describe is strictly for sealed bid competitive lease sales, but the *phenomenon* we will be talking about pervades all competitive situations.

Industry's Record in Competitive Bidding

In recent years, several major companies have taken a rather careful look at their records and those of the industry in areas where sealed competitive bidding is the method of acquiring leases. The most notable of these areas, and perhaps the most interesting, is the Gulf of Mexico. Most analysts turn up with the rather shocking result that, while there seems to be a lot of oil and gas in the region, the industry probably is not making as much return on its investment there as it intended.[2-5] In fact, if one ignores the era before 1950, when land was a good deal cheaper, he finds that the Gulf has paid off at something less than the local credit union.

Why? Have we been poor estimators of hydrocarbon potential? Have our original cost estimates been too conservative? Have we not predicted allowables well? Was our timing off? Or have we just been unlucky?

It is our view that none of the factors these questions suggest has been the major cause of the in-

If it is true, as common sense tells us, that a lease winner tends to be the bidder who most overestimates reserves potential, it follows that the "successful" bidders may not have been so successful after all. Studies of the industry's rate of return support that conclusion. By simulating the bidding game we can increase our understanding and thus decrease our chance for investment error.

dustry's performance, though certainly all may have contributed. Poor luck might affect a few offshore participants. But the whole industry? Not likely. Industry has had enough opportunities in the Gulf to invoke the law of averages — if we may be so loose with mathematics.

We believe that in the competitive bidding environment normal good business sense utterly failed to give people the return they expected. Since many industry folk have not understood the rather complex laws of probability at work in competitive bidding, they have been inclined to make serious errors in arriving at their dollar bid for a particular tract. We are not saying that all of the bids turned out poorly. But enough of them have, throughout the industry, leading to lower rates of return than people planned for.

A new wrinkle appeared in the 1970 Offshore Louisiana wildcat sale (an $850 million sale). Because of a Federal Power Commission order, some of the gas companies assumed they would be able to include their bonus investment in rate base. If they are correct, then their risk in offshore exploration has been effectively removed. They will make their legal return regardless of how much or how little reserves they find. This most recent sale, then, is very different from the others we have mentioned. The bidding model we would like to describe does not apply if lease bonus can be included in rate base.

We want to emphasize that we are not criticizing competitive bonus bidding as a method for acquiring leases from selling authorities. We believe this method is fair for all concerned. If the industry has not performed as well as it hoped, perhaps it is only because the industry has failed to understand the laws of probability that seem to govern the whole estimation-bidding process.

A "Think" Sale

Let us play a little game. Think of yourself as a manager whose task it is to set bids on parcels in an impending sale. On any one of your parcels you have a consensus property value put together by your experts. (We will not worry for the moment about how you handled risk, what your discount rate is, if you have one, or how you arrived at your reserves and costs.) One thing you can be sure of: Your value is either too high or too low; it has no chance of being exactly the true value.

Not to belabor a simple point, there are people in our business who fall in love with a number and fail to recognize the uncertainty associated with it. If a company's estimate happens to be $5 million, who knows what the actual worth might be? If the tract is dry, the owner will have a loss — bonus plus exploration costs. If the tract produces — how much? There are fields discovered 50 years ago where we still do not know the reserves. And the uncertainty in field size before drilling is fantastic. So we repeat: Reserve estimates are either high or low—and maybe not even close.

We will assume, however, that on the average your value estimates are correct. (This does not contradict what we have already said. Most people are aware that they are high on some and low on others, but

over the long haul, they ought to come out about right on their value estimates.) You realize that other managers are going through the same agony you are. You ask yourself, "What do my competitors think these tracts are worth?" You know that some of your opponents may have better information than you, some worse. There will be, on sale date, quite a divergence of opinion as to value among the bidders. If you doubt this, look at the published bids by serious competitors at any recent sale. Bid ratios between the highest and lowest serious competitors range to as much as 100 and are commonly 5 or 10. (See Table 1.)

Implications of Divergence

What are the implications of this divergence of opinion? We could certainly argue that some people may have overestimated the true value of the parcel, and others may have underestimated it. Consider a piece of land that has exactly 10 million bbl of recoverable oil. If you let five different people in your own company interpret the seismic data, logs on nearby wells, and other sundry information, you will get five different estimates of reserves — even though they all use the same basic information. The problem becomes more confounding if we look at reserve estimates (before drilling now) of five different companies. They may each have different seismic data and different logs. Isn't it likely that some companies will come up with more than 10 million bbl? And some less? We have already admitted that while our estimates of reserves may be all right on the average, on any one tract we are going to be either high or low.

In Table 1 we saw evidence of this wide variation in value estimates by different competitors. Perhaps the several bidders had somewhat different exploration information. We all know the difference one properly placed seismic line can make in our mapping. Whatever the reasons, it is clear that different information leads to different value estimates.

Let us look at what different competitors can do given the same basic information. In the 1969 Alaska North Slope Sale, we find Atlantic Richfield and

TABLE 1—BIDS BY SERIOUS COMPETITORS IN RESENT SALES
(All bids in millions of dollars)

Offshore Louisiana, 1967 Tract SS 207	Santa Barbara Channel, 1968 Tract 375	Offshore Texas, 1968 Tract 506	Alaska North Slope, 1969 Tract 059
32.5	43.5	43.5	10.5
17.7	32.1	15.5	5.2
11.1	18.1	11.6	2.1
7.1	10.2	8.5	1.4
5.6	6.3	8.1	0.5
4.1		5.6	0.4
3.3		4.7	
		2.8	
		2.6	
		0.7	
		0.7	
		0.4	
Ratio of Highest to Lowest Bid			
10	7	109	26

Humble bidding independently of each other. Since the two companies are equal partners in much exploration and development, both probably had essentially the same information; but each company took that information and developed its own evaluations without consulting the other. Table 6 shows the ratio of the Humble bid to the Atlantic Richfield bid for 55 tracts on which the companies competed against each other. At one extreme we find Humble making bids of about 0.03 of Atlantic Richfield's bid; at the other, Humble's bid is about 17 times higher than Atlantic Richfield's. And between these two extremes, we find a smooth gradation of ratios.

We have portrayed the same information a bit differently in Fig. 1. Here you will see a cross-plot of Humble's bids and Atlantic Richfield's bids for the same 55 tracts. No one has yet been able to identify any pattern or hint of correlation in these numbers. Clearly, the fact that companies have much the same seismic lines and well logs does not mean that those companies will come up with similar bids or property values.

On seeing such an exhibit, some ask if the wide range might not be due to differing discount rates or differing market conditions. But those items offset all of a company's bids in the same direction. A lower discount rate by one company, for instance, would force all of its evaluations up in dollars. There still would be large differences in bids.

Now more often than not, he who "sees" the most barrels will "see" the most dollar value. (Again, we recognize the effect of risk, cost estimates, production rates, pricing, discount rates and all that. But for the moment, let us focus on concepts and not clutter the picture with all these other items.) Can we not then conclude that he who thinks he sees the most reserves, will *tend* to win the parcel in competitive bidding? This conclusion leads straightway to another: *In competitive bidding, the winner tends to be the player who most overestimates true tract value.* And yet
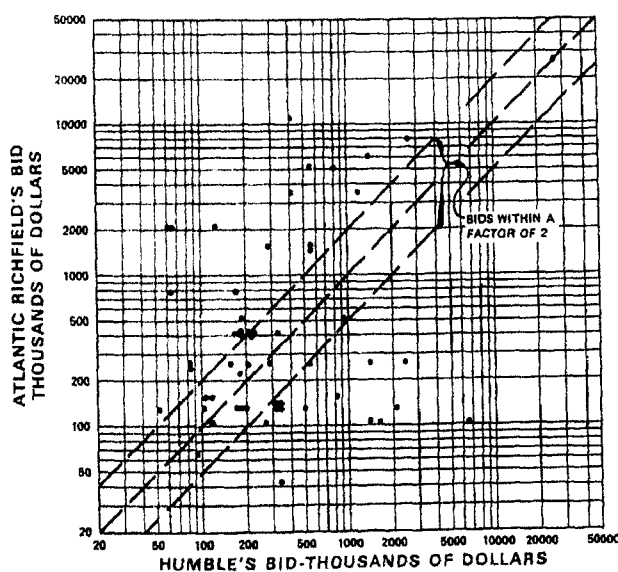

Fig. 1—Atlantic Richfield bids vs Humble bids, 1969 Alaska North Slope Lease Sale.

another: He who bids on a parcel what he thinks it is worth will, in the long run, be taken for a cleaning.

A chorus enters sobbing, "But you told us earlier that our evaluations were correct on the average, albeit high sometimes and low sometimes. Doesn't the law of averages save us from ruin?" First, the so-called law of averages never guaranteed salvation for anyone, though it often gives some courage to act. Second, it is true (or we assume it so) that one's evaluations are correct on the average — but it is not true that one's evaluations on *tracts he wins* are correct on the average. There is a difference. Only in a noncompetitive environment, can one counter his overevaluated parcels with his underevaluated parcels and expect to do well on average. In bidding, however, he has a poor chance of winning when he has underestimated value and has a good chance of winning when he has overestimated it. So we say the player *tends* to win a *biased set* of tracts — namely, those on which he has overestimated value or reserves.

Note that we are talking now about *trends* and *tendencies* — not about what will happen every time one purchases a tract. It is possible that everyone will underestimate the value of a particular parcel. The winner will, under those circumstances, have a very attractive investment. But that is like winning the Irish Sweepstakes on your first ticket and then going around claiming that buying sweepstakes tickets is going to be a great investment for the future. As we make our investment decisions we must distinguish among the lucky event, the unlucky one, and the average of what occurs year after year.

Some may argue that the industry is smarter now — has new exploration techniques — and will not make the same kind of mistakes in the future. It is certainly true that we are better able to make exploration judgments these days; but it still does not mean we are very good. Anyway, even when technology was not so advanced, we were probably still "about right on average".

For example, before the "new technology" one might have expected a particular reservoir to contain 10 million bbl. If he had examined his uncertainties, he would have said the reservoir, if it exists, might have any amount between 2 million and 50 million bbl. With better information, he might still say he expects 10 million bbl, but his uncertainty has decreased and now ranges from 3 million to 35 million bbl. We claim that the effect of new technology only narrows our uncertainties — and does not necessarily change our expected values — again on average.

Bid Strategy

So what is the best bid strategy? We cannot tell you and will not even try. The only thing we can do is show you one approach to the mathematical modeling of competitive sales. The theory, as we interpret it, agrees well with what we perceive has happened in the real world.

For some competitive environments, in order to reach some specified return on investment, the model suggests a lower bid than one might come up with otherwise. What are these environments? The following rules are not without exceptions; but for the nor-

mal level of competition and the large uncertainties underlying our value estimates, the rules seem to apply.

1. The less information one has compared with what his opponents have, the lower he ought to bid.
2. The more uncertain one is about his value estimate, the lower he should bid.
3. The more bidders (above three) that show up on a given parcel, the lower one should bid.

How do we know these rules? Call it simulation. We modeled the competitive bidding process on a computer as closely as we knew how and then sat back to let the machine churn away. We allowed for such things as different numbers of bidders, different value estimates by the opponents, different information positions for the opponents, different bid levels* by the opponents, and the proper ranges of uncertainty about each of these. We let the computer take our estimates of competition (with the associated uncertainties) and play the lease sale game over and over again. After some thousands of runs the computer tells us, for our various bid levels, the probability of our winning the parcel and its value to us. Looking at the results, we simply choose a bid level that assures us (in a probability sense) of not investing incremental dollars at less than some specified rate of return.

We made all kinds of sensitivity tests to see "what if". We examined the effect of low rate of return criteria for opponents and checked on few opponents vs many. We looked into the influence of an opponent's superior information. We varied every significant variable we could identify.

When it was all over, we concluded that the competitive bidding environment is a good place to lose your shirt.

Previously we listed three reasons for lowering one's bid. The first two are easy enough to understand. But the third takes some work. Most people assume that the tougher the competition (i.e., the more serious bidders there are) the more they must bid to stay with the action. What action are they wanting to stay with? If they are trying to maximize the number of acres they buy, they are right. If they would like to maximize the petroleum they find, they are probably right. But if they are trying to invest money at some given rate of return, our model says they are probably wrong.

Although the concept may not be clear to everyone, we are convinced that if one's mistakes tend to be magnified with an increase in number of opponents, then he must bid at lower levels in the face of this stiffer opposition in order to make a given rate of return. Let us reinforce this with an example.

Assume we have a 10-tract sale. Also, for the sake of simplicity, let us assume that all tracts will be productive and that after exploratory drilling costs, each will be worth $10 million at a 10 percent discount rate. Each competitor in this sale correctly estimates the total value of the sale acreage but on any one tract he may be too high or too low. (This assumption

merely means that one tends to be unbiased in his estimate of value. He may not be correct on any one parcel, but he does all right on the average.)

As in the real world, let us have the competitors disagree as to the value of the individual tracts — and let that divergence of opinion be about the same as we see in major lease sales. But let the average of all the competitors' value estimates be very close to the true value. (Here we are saying that when they estimate value the competitors are not misled in the same direction.)

Finally, assume that to protect himself from the risks and uncertainties of the estimating procedure, each competitor chooses to bid one-half *his* value estimate. What we want to do is check the rate of return of the winners as we increase the number of bidders.

Table 2 reflects the sale as if only Company A bids. Remember, he correctly estimates that the 10 tracts are worth $100 million to him and he bids one-half of his value estimate on each tract. The sum of his 10 bids is then $50 million. He wins all tracts since there is no competition. Since he pays $50 million for what is worth $100 million (at a 10 percent discount rate) his rate of return for the sale will be about 17 percent** after tax. This is his reward even though he has overestimated value on Tracts 2, 6, and 8.

Table 3 examines the consequences of adding one competitor, Company B. Since both companies are unbiased in their estimates, use the same discount rate for calculating value, and bid the same fraction of their respective values, then we would expect each to win half the time. As it turns out, that is exactly what happens. But see what else happens. In Table 1 we saw that Company A won all 10 tracts — on seven of which he had underestimated value and on three of which he had overestimated. Now along comes Company B and wins five of the seven tracts on which Company A had underestimated value. Remember our contention that one *tends* to lose those tracts on which he has underestimated value? Company A has spent more than 70 percent as much money as he spent when he was the only bidder, but now he gets only half as much acreage. The only thing that saves him is his strategy to bid one-half his value estimates. His rate of return drops to 14 percent. The "industry" consisting of the two companies has about the same return.

Now go to Table 4 and see what happens if we raise the number of bidders to four. More and more of Company A's underevaluated tracts have been grabbed off by the competition. Company A is left with only Tract 8, which he evaluated at $35 million. (It is worth only $10 million, remember.) The selling authority's take has climbed to about $92 million — the sum of all the high bids. Company A's return drops to about 5 percent, whereas the industry's return is about 11 percent. Company A turns out to be a little unlucky in that its return is lower than the industry's. Somebody has to be unlucky. That should not detract from our argument. We could pick any

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*Bid level is the fraction of his value estimate a player will bid.

**We estimated this return and others in the example from studies of cash flows from typical projects.

of the competitors and see the same trend toward lower returns.

Table 5 shows the results of eight bidders. Company A still retains its Tract 8. Bidders E through H pick up five of the 10 tracts. The seller gets about $26 million more than he did with the four competitors. Since the tracts did not pick up any more reserves, the additional expenditure must mean a decreased rate of return for the industry. We estimate about 8 percent — even though each bidder is bidding only half his value estimate.

There is no table to show the results for 16 bidders, but the trend continues onward to lower returns. The 16 bidders spent a total of $162.6 million for a return of about 6 percent.

What if the industry had wanted to make about 10 percent on its investment? What percent of value would each competitor have had to bid to accomplish that goal? Just taking the results of our example, the bid levels would have been something like this:

Number of Bidders	Total Value Estimates for Highest Estimators on Each Tract	Bid Level for 10 Percent Return
1	$100 million	1.00
2	$139 million	0.72
4	$184 million	0.54
8	$237 million	0.42
16	$325 million	0.31

(The bid levels that appear in the third column are valid for only the particular example we have just gone through, where everyone uses the same return criterion and everyone uses the same bidding strategy. Companies, in the real world, are not so inclined to play that way. Nevertheless, the phenomenon of decreasing rate of return with increasing numbers of

bidders appears to us a general rule of sealed bidding.)

It is certainly true that the value of the tracts does not change just because there are more bidders. What does change drastically as the number of bidders increases is the set of tracts one wins. Not only does that set get smaller with increasing competition, but also its quality tends to decrease compared with what the winner thought it would be ahead of time.

The more serious bidders we have, the further from true value we expect the top bidder to be. If one wins a tract against two or three others, he may feel fine about his good fortune. But how should he feel if he won against 50 others? Ill. He would wonder why 50 others thought it was worth less. On the average, one misjudges true value much worse when he comes out high against 50 other bidders than when he beats only two or three. Hence, our bidding model usually tells us to move toward lower bids as competition increases in order to protect ourselves from the winner's curse. True, the probability of purchasing property decreases — but so does the chance of losing that shirt.

Some Mathematics

The theory of competitive bidding obviously involves mathematics. For those so inclined, we will lay out here and in the Appendix analytical procedures for examining the effects we have spoken of. (Then we will say, "But the analytical approach is so difficult from the practical side that we must try a simulation.") What we will try for analytically is the *expected value of the winning bid*. We simply compare that value with

TABLE 2—CASE 1—ONLY COMPANY A BIDS ON PROPERTY

Tract Number:	1	2	3	4	5	6	7	8	9	10
A's bid*	1.9	5.6	2.6	3.4	3.7	5.2	1.9	17.5	3.9	4.3

	Company A	Industry
Winning bids*	50	50
Value of acreage won*	100	100
Present-worth profit*	50	50
Investor's rate of return, percent	17	17

*In millions of dollars.

TABLE 3—CASE 2—ONE COMPETITOR ENTERS SALE WITH COMPANY A

Tract Number:	1	2	3	4	5	6	7	8	9	10
A's bid	1.9	5.6*	2.6	3.4	3.7*	5.2*	1.9	17.5*	3.9*	4.3
B's bid	3.8*	5.1	4.0*	4.9*	0.6	4.2	5.9*	4.5	1.8	15.2*

	Company A	Industry
Winning bids**	35.9	69.7
Value of acreage won**	50.0	100.0
Present-worth profit**	14.1	30.3
Investor's rate of return, percent	14	14

*Winning bid.
**Millions of dollars.

TABLE 4—CASE 3—THREE COMPETITORS ENTER SALE WITH COMPANY A

Tract Number:	1	2	3	4	5	6	7	8	9	10
A's bid	1.9	5.6	2.6	3.4	3.7	5.2	1.9	17.5*	3.9	4.3
B's bid	3.8	5.1	4.0	4.9	0.6	4.2	5.9*	4.5	1.8	15.2*
C's bid	5.7	3.1	2.6	6.5*	9.8*	9.8*	4.0	1.5	3.3	3.7
D's bid	6.5*	8.3*	7.8*	6.4	3.3	2.2	3.3	5.0	4.5*	2.7

	Company A	Industry
Winning bids**	17.5	91.8
Value of acreage won**	10.0	100.0
Present-worth profit**	− 7.5	8.2
Investor's rate of return, percent	5	11

*Winning bid.
**Millions of dollars.

TABLE 5—CASE 4—SEVEN COMPETITORS ENTER SALE WITH COMPANY A

Tract Number:	1	2	3	4	5	6	7	8	9	10
A's bid	—	—	—	—	—	—	—	17.5	—	—
B's bid	—	—	—	—	—	—	5.9	—	—	15.2
C's bid	—	—	—	6.5	—	—	—	—	—	—
D's bid	—	—	7.8	—	—	—	—	—	—	—
E's bid	—	10.3	—	—	—	—	—	—	—	—
F's bid	—	—	—	—	14.3	13.0	—	—	—	—
G's bid	23.3	—	—	—	—	—	—	—	4.7	—
H's bid	—	—	—	—	—	—	—	—	—	—

	Company A	Industry
Winning bids*	17.5	118.5
Value of acreage won*	10.0	100.0
Present-worth profit*	− 7.5	− 18.5
Investor's rate of return, percent	5	8

*Millions of dollars.

true parcel value to see whether a particular bidding strategy can lead to trouble.

Let

$$f_i(x) = \text{probability density function for } i\text{th opponent's bid.}$$

And let

$$F_i(x) = \text{probability that the } i\text{th opponent bids a value less than } x.$$

Therefore,

$$\prod_{i=1}^{n} F_i(x) = \text{probability that } n \text{ independent opponents all bid a value less than } x.$$

Now let

$$g(x) = \text{probability density function for our bid.}$$

Define

$$h(x) = K_n \left[\prod_{i=1}^{n} F_i(x) \right] g(x) = \text{probability density function for our winning bid,}$$

where

$$K_n = \text{constant to make the integral of that density} = 1$$

$$K_n = 1 \Big/ \int_{-\infty}^{\infty} \left[\prod_{i=1}^{n} F_i(x) \right] g(x) dx.$$

Then it is a simple matter to get the expected value of our winning bid, $E(X_w)$

$$E(X_w) = \int_{-\infty}^{\infty} x h(x) dx$$

$$= \int_{-\infty}^{\infty} x K_n \left[\prod_{i=1}^{n} F_i(x) \right] g(x) dx.$$

Then under some very simple assumptions (too simple for the real world), we can define some $F_i(x)$ and $g(x)$ in such a way that we can evaluate the integral. In fact, we can show that if $f_i(x)$ and $g(x)$ are uniform on the interval of 0 to 2, and all competitors bid their full value estimate, then:

$$K_n = n + 1$$

$$E(X_w) = 2 \left(\frac{n+1}{n+2} \right).$$

These uniform distributions imply a true value of 1 (the mean of each is 1). If there are no opponents ($n = 0$), then:

$$E(X_w) = 2 \left(\frac{1}{2} \right) = 1.$$

That is what we hope if we bid our value estimate against no opposition. On the average, we win tracts at our value. But what if there are five opponents?

$$E(X_w) = 2 \left(\frac{5+1}{5+2} \right) = \frac{12}{7} \approx 1.71.$$

That means that on the average, we would expect to pay 71 percent more than value on the tracts we won. That is not good.

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One might think he could take the reciprocal of 1.71 to get his "break-even" bid level. Not so. The subtleties of competion force the "break-even" bid level to be even lower than that reciprocal, although perhaps not too much lower.

We can set up the mathematics, but for the real world, we cannot solve the equations. Instead, we simulate the whole process. And that is all right, for by simulation, we can do many things we would not even try with strict mathematical analysis.

How Can a Bidding Strategist Win Tracts?

Some will claim he cannot — we believe they are wrong.

An analyst comes in claiming a tract is worth X. The bidding strategist then recommends a bid of, say, $X/2$. A voice from the rear cries, "That bid won't be competitive." The voice is usually forgetting about the large divergence in value estimates by competitors. There is a very good chance some other competitor will see a much larger value than X. We could not be competitive with any bid we would reasonably try. So our chance of winning depends more upon our reserves estimate than upon our particular bid level. The bid level adjustment is primarily for the purpose of achieving a certain profitability criterion.

Some interesting evidence to back up these comments comes from the 1969 Alaska North Slope Sale.[6] Examine the second-high bids for that sale. The sum of those second-high bids was only $370 million compared with the winning bid sum of $900 million. Said another way, the fellow who liked the tract second best was willing to bid, on the average, only 41 percent as much as the winner. In this respect, the sale was not atypical.

If that is not shocking enough, try this one. For 26 percent of the tracts, had the second-high bidder increased his bid by a factor of 4, he still would not have won the tract. A 50-percent increase in bid by the second-high man would not have won 77 percent of the tracts. Turn the idea around. If every tract winner had bid only two-thirds as much as he did, the winners still would have retained 79 percent of the tracts they won. (The apparent discrepancy, 77 percent vs 79 percent, comes from the 15 tracts that drew only one serious bidder.) We therefore conclude, based on historical study, that bid manipulation to achieve desired profitability does not drastically impair one's chances of winning acreage.

TABLE 6—ALASKA LEASE SALE, 1969 —
RATIO OF HUMBLE BID TO ARCO BID
For the 55 tracts on which both companies bid

0.03	0.32	0.50	1.11	2.53
0.03	0.32	0.51	1.13	2.56
0.04	0.33	0.51	1.31	3.82
0.06	0.33	0.60	1.39	5.25
0.08	0.36	0.69	1.39	5.36
0.11	0.36	0.76	1.40	6.14
0.12	0.36	0.77	1.79	7.98
0.16	0.39	0.78	2.02	9.19
0.18	0.41	0.79	2.41	13.32
0.22	0.45	0.82	2.41	15.45
0.24	0.45	1.00	2.50	16.80

How Far Off Might the Winner Be?

We have been saying that the winner of a tract tends to be the one who most overestimates value. You may say, "So, if we win, we wish we hadn't. If we lose, we wish we hadn't. You mathematicians are really saying to stay away from lease sales." That is not what we are saying. The bidding model gives us a bid that we can make with confidence, and be happy with when we win. Yes, we may have overestimated value. But we have bid lower than our value estimate — hedging against expected error. In a probability sense, we "guarantee" that we obtain the rate of return we want.

As to how far off the highest estimator might be, we have resorted to simulation of the estimating process. We perhaps could have got the result through use of extreme value theory, but we chose not to. Also, we want to caution the reader that we are examining what we think will happen on the average — not what will happen on a particular tract. If the wildcat fails, obviously everyone was too high in his value estimate. If the well hits, it is entirely possible everyone was too low. That is not the kind of problem we are talking about. The question is more likely: "If I win 10 parcels at a sale, how many barrels will they all contain compared with my pre-sale estimate?"

Fig. 2 shows the results of our simulations (using log-normal distributions) for various numbers of competitors and degrees of uncertainty. We use the variance of a distribution — measure of its spread — to quantify general uncertainty as to value among competitors. One can get a rough idea of the magnitude of variance by measuring the parameter on sets of bids on tracts in past sales. That variance, however, will be too high since the actual bids contain "noise" items apart from property evaluation — for example, various company discount rates and bid levels. Obviously, there is not so much uncertainty in drainage sales as there is in North Slope-type wildcat sales. We use variance to account for these differences.

Intuition would argue that the greatest potential for large errors in estimating reserves exists on the frontier — Alaska. The simulation agrees wholeheartedly. For 12 serious bidders in an environment of uncertainty such as the North Slope, the one estimating the largest amount of expected reserves can

expect to be off by a factor of 4 on average. In the Louisiana Offshore, facing the same kind of competition, he would expect to miss by a factor of only 2.5.

Nature of the Model

We must choose a probability distribution for the value estimates of various companies. The log-normal seems to us the best. Many writers have documented the variables in our business that seem to follow the log-normal. Here is a partial list of them:

1. Reservoir volume
2. Productive area
3. Net pay thickness
4. Recoverable hydrocarbons
5. Bids on a parcel in a lease sale
6. Land value estimates calculated by companies.

The first four items have been ordained by Nature. The last two are man-made. Why should they perform like Nature? There is an amazing theorem in mathematics — the Central Limit Theorem — that says if you take sums of random samples from any distribution with finite mean and variance, the sums will tend toward a normal or Gaussian distribution. The tendency will be stronger the more numbers there are in each sum. If the original numbers come from a normal distribution, the sum is guaranteed to be normal. If we insert the word "product" for "sum" we can then insert the word "log-normal" for "normal." Since we arrive at value through a series of multiplications of uncertain parameters (reservoir length $\times$ reservoir width $\times$ net pay $\times$ recovery $\times$ after-tax value per barrel), it is not surprising that bids and land-value estimates seem to take on this log-normal characteristic.[7]

There are certain problems in applying the theorem. Negative dollars (a loss or lower-than-criterion rate of return) will not fit the log-normal distribution. No one knows how to take the logarithm of a negative number. And we all know that the value calculation involves more than simple multiplication. Even so, the error in our assumption does not appear to be great, and we happily use the log-normal distribution in our computer simulation.

The evaluation of a potential cash flow stream by different investment criteria has been the subject of much study. We believe that methods involving the discounting of the cash flow stream are effective for the decision maker. The criterion we prefer is present worth or present value (PW), using as the discount rate the Internal or Investor's Rate of Return (IRR) expected to be earned by the investor in the future.[8] The very essence of PW is that it is the value or *worth* we place on an investment opportunity at the *present* time. In a situation where the future cash flow is known with certainty, we can discount this cash flow to the present.

We do not know the future cash flow with certainty, however, and resort to using the *expected value* concept. Expected value can mean different things to different people, but we use it in the accepted probabilistic sense: Expected value is the sum of all possible events multiplied by their chance of occurrence. Arithmetic mean is a common term for expected
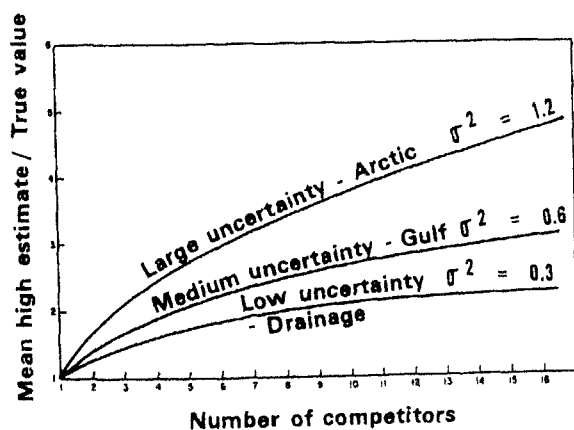

Fig. 2—Relation of mean high estimate to true value under various conditions of uncertainty.

value. Expected value is *not* necessarily the mode (most probable value), nor the median (the value that is exceeded half the time). We do not specify all the possible events, since this would be an outrageous number. But we do try to specify enough possible events so that the calculations with these relatively few discrete values will yield a good value. The "good" value should be close to that expected from a consideration of *all* possible events.

The tract value plays a much smaller role in our model than one might think. We essentially normalize everything to value = 1.0. The model tells us what *fraction* of our value (bid level) to bid in order to maximize expected present worth for the competition we put in. The bid level can change only if our idea of the competition somehow changes. If we think the degree of competition is independent of tract value, then value need never be discussed. But sometimes there are tracts that, because of their potential, may cause competitors to deviate from past or expected performance. We allow for this by considering the competition the way we think it will be for a given tract. In that sense, then, value gets into the model.

Our model differs from some other models that have been discussed. An earlier philosophy reasoned thus: "Our value may be incorrect on a given tract, but it is correct on the average. So let our value estimate serve as the mean of the distribution from which our opponents draw." We think that tack can lead to trouble. It is inconsistent with the idea that when we win, our estimated value was probably higher than true value. Instead, we let the true value of a tract be 1.0 and simply take our value estimate from a distribution with mean = 1, the same as everyone else. We treat all value estimates as independent random variables. Our model is similar in this respect to Rothkopf's.[9] The variance of our distribution may be the same or different from our opponents' — depending on the relative quality of our information.

Model Input Data

Some believe that the input requirements for a competitive bidding model are quite severe — that reliable input is impossible to obtain. We do not think so. Unless one successfully engages in espionage, he is not going to know his opponent's bid. But he does not need to. We have found that by studying the behavior of companies in past sales, we can get a fair clue as to what they will do in the future — close enough to make the model results meaningful.

Here is the information we think is necessary to make an intelligent bid. Keep in mind that each bit of input is an uncertain quantity. We treat it as uncertain by using probabilities and probability distributions. That, after all, is the way the world is.

We believe that the input data are best determined by a combination of historical data and the judgment of explorationists. To illustrate the use of our model, we will develop a set of input data for a purely hypothetical example.

What sort of data do we need? Primarily, we need information about the competition we are likely to face. We try to identify companies that are likely to bid on the parcel. This allows us to use any specific

knowledge we have about a competitor or his exploration activities. For each of the potential competitors, we then try to estimate the probability that he will bid. To the competitors specifically named, we can add some "other bidders" in order to make the expected number of bidders consistent with our beliefs:

Company	Probability of Bidding
A	0.8
B	0.7
C	0.5
Other bidder	0.5
Other bidder	0.5
Expected number of bidders	3.0

In this example, we expect three competitors, but we acknowledge that there could be as few as none or as many as five. In the simulation performed by our model, the number of competitors will vary, from trial to trial, from a low of zero to a high of five. The proportion of trials on which a given bidder appears will be approximately equal to the probability we have assigned above.

The next item we require is usually the most difficult to estimate: the bid level of each potential competitor. If he calculates a value of X for the property, what fraction of that value is he likely to bid? To further complicate the matter, we need to estimate this fraction as if the X value were based on our own rate of return criterion. In other words, the bid level is used to adjust for differences in evaluation criteria and for the fraction of value that a given competitor will bid.

We believe that historical data can be of help in estimating bid levels. We can go back to a previous sale or sales and compare a given competitor's bids with the value estimates we made on the same tracts. At first we were tempted to compute the ratio of a competitor's bid to our value on each tract and then average these ratios over all tracts. We discovered that under the assumptions of our model of the bidding process this gives a biased estimate of the competitor's bid level. We can show that to get an unbiased estimate of his bid level on a tract we need to divide the ratio of his bid to our value by the quantity e^{σ^2}. Here σ^2 is the variance of the natural logarithm of our value estimate on the tract. (Our value estimate, remember, is considered a random variable. Estimates of σ^2 are not easy to come by, but again historical data can be of help.) We can then calculate an average bid level for the competitor from these unbiased estimates on all the tracts. This bid level estimate incorporates differences in evaluation criteria, as well as the fraction of value that the com-

TABLE 7—INPUT DATA FOR COMPETITION

Company	Probability Of Bidding	Bid Level	Variance
A	0.8	0.6	0.6
B	0.7	0.6	0.6
C	0.5	0.4	0.6
Other bidder	0.5	0.3	0.8
Other bidder	0.5	0.3	0.8

petitor bids, on average. We then modify this according to our explorationists' judgment about the current sale and the particular tract in question to add another column to our hypothetical input data:

Company	Bid Level
A	0.6
B	0.6
C	0.4
Other bidder	0.3
Other bidder	0.3

We also need to specify how much variation we think is possible in each competitor's bid. Even if we knew that the bid levels specified above were precisely correct, we still would be uncertain as to the actual bids because we do not know the value that each competitor places on the tract. We must try to estimate the variability in each competitor's value estimate. We do this by specifying the variance of the estimate. (Actually, we specify the variance of the natural logarithm of the estimate. Hereafter, when we mention variance, we will be referring to the variance of the logarithm of a quantity, because this is a useful parameter in the log-normal distribution.)

We can again get some help from data on past sales. On individual tracts about 1.2 has been the average variance of the bids.[10] This includes more than just the variation in value estimates, though. It also includes differences in bid levels and evaluation criteria among competitors. The variance in value estimates for a single company would average something less — we have guessed about 0.6.

Another way to estimate this variance, if we assume it is constant over all tracts, is to compare an individual competitor's bids with our values on the tracts in a given sale. This should eliminate variation due to differences in evaluation criteria, assuming a company uses the same criterion in all of its evaluations. If we measure the variance of the ratio of a competitor's bid to our value, there are three components to this variance:

1. Variance of our value estimate (Y)
2. Variance of the competitor's value estimate (X)
3. Variance of the competitor's bid level (K) from tract to tract.

We can show that these components are additive. The variable whose variance we are measuring is $\log_e(KX/Y)$. We can write

$$\log_e(KX/Y) = \log_e(K) + \log_e(X) - \log_e(Y).$$

If K, X, and Y are independent,

$$\text{Var}\,[\log_e(KX/Y)] = \text{Var}\,[\log_e(K)]$$
$$+ \text{Var}\,[\log_e(X)] + \text{Var}\,[\log_e(Y)].$$

By assuming that the last two components are equal and the first is about 0.15, we calculated an average variance for our opponents' value estimates in several sales. The values were not far from the 0.6 estimated above.

We feel free to modify this estimate in accordance with the nature of the sale and the tract in question.

For example, we felt that the 1969 North Slope Sale was characterized by more uncertainty than the typical offshore Louisiana sale. Thus, we generally assigned higher variances to value estimates. In drainage situations, we use lower variances to reflect the fact that the value estimates should be closer to the true values. We also try to differentiate among competitors. Those we feel have better information about a tract are given lower variances and those with poorer information, higher variances. So we shall add another column to our input data:

Company	Variance
A	0.6
B	0.6
C	0.6
Other bidder	0.8
Other bidder	0.8

bids against at most 5 bidders.

Table 7 shows a complete set of the input data on competition.

We add another component, Var $\log_e(K)$ mentioned above (usually about 0.15), to these variances to reflect our uncertainty about our competitors' bid levels. Finally, we estimate the variance in our value estimate by assessing the quality of our information relative to that of our opponents'.

Mechanics of the Model

The parameters for the log-normal distributions assigned to the value estimates of the various bidders (including us) come directly from the data given above. We usually run the model thousands of times to simulate the competitive and evaluation possibilities on a single tract. (See flow chart, Fig. 3.) On each trial, a value is drawn for each random variable, which results in a set of bids by the participating companies. The results of the "sale" are then recorded and the whole process is repeated. After enough trials have been run, the expected results are calculated and printed.

Model Output

The output of the model includes expected results for 15 different bid levels, from 0.1 to 1.5 times our value estimate. Results from our hypothetical example are shown in Table 8. The values in the first column indicate possible bidding levels as fractions of our value estimate. The second column gives the amount of our bid at each level. We have assumed that our estimate of the value of this tract is $10 million. The next column shows the probability of winning, as calculated by the model, for each bidding level. This is useful in estimating the amount of acreage, reserves, etc., we expect to win. The expected amount of our expenditure is shown in the fourth column. In the next column we have the expected present worth for each bidding level. The last column indicates how high we can expect our value estimate to be if we win. If we bid full value (bid level of 1.0) and win on tracts such as this, our value estimate will, on the average, be 1.35 times the true value. It is again obvious that we have to bid less than full value just to break even.

Optimization of Bids

The expected present worth of the submitted bid we will designate as EPW_{Bid}. Given all our usual information about the tract and other bidders, what bid should we submit? What is our optimum bid for the example above?

We can use a graph of EPW_{Bid} vs bid level to consider this problem (Fig. 4). First, what happens if we do not bid? The bid level is zero. No expenditures will be made, and the EPW_{Bid} is zero. Second, what happens if we bid our estimate of the tract value? For the tracts we win, we tend to overestimate value. Hence, the average value of the tracts we win is less than our original estimates. Thus in the example we have a negative EPW_{Bid} of $1.9 million. Third, what happens if we bid less than our estimate? This strategy really provides the only chance we have to get a positive EPW_{Bid}. We must bid somewhere between the one extreme of a very low bid (which means very low chance of winning a big positive value) and the other extreme of a very high bid (which means a high chance of winning a big negative value).

What then is the optimum bid? For the single tract illustration above, and for our investment criterion of maximizing the EPW_{Bid} rather than maximizing reserves or some other goal, we would choose a bid level of 0.35. There may not always be a positive value of EPW_{Bid}, in which case we would not bid.

Usually, however, there is a positive maximum value. It is not always at the same bid level. The maximum shifts along the bid level axis with changes in the number of bidders, their bid levels, and the variances of their estimates.

Deviation from the optimum bid level in either direction will decrease the EPW_{Bid}. If someone "feels" we should bid higher or lower, we can show what this feel costs in terms of EPW. Any bid giving a positive EPW_{Bid} will, of course, give an expected IRR greater than the discount rate. Suppose the discount rate used is the marginal acceptable IRR. Going to a larger bid level than that giving maximum EPW gives a lower EPW. Therefore, that marginal increase in bid has a negative EPW associated with it. Look at Table 8. Going from a bid of 0.5 to 0.6 costs $283 thousand in EPW. Taking an action that decreases the EPW is the same as taking an action that invests money at less than the acceptable IRR. According to the model, then, he who would go above his optimum bid level to gain probability of win advantage can expect to invest part of his money at a return lower than the minimum he said he would accept.

Before leaving the subject of bid optimization, we will comment on another frequently mentioned criterion. Under the existing conditions of uncertainty, there will be "money left on the table" (difference between the winning bid and second-high bid) and rightly so. We can minimize the money left on the
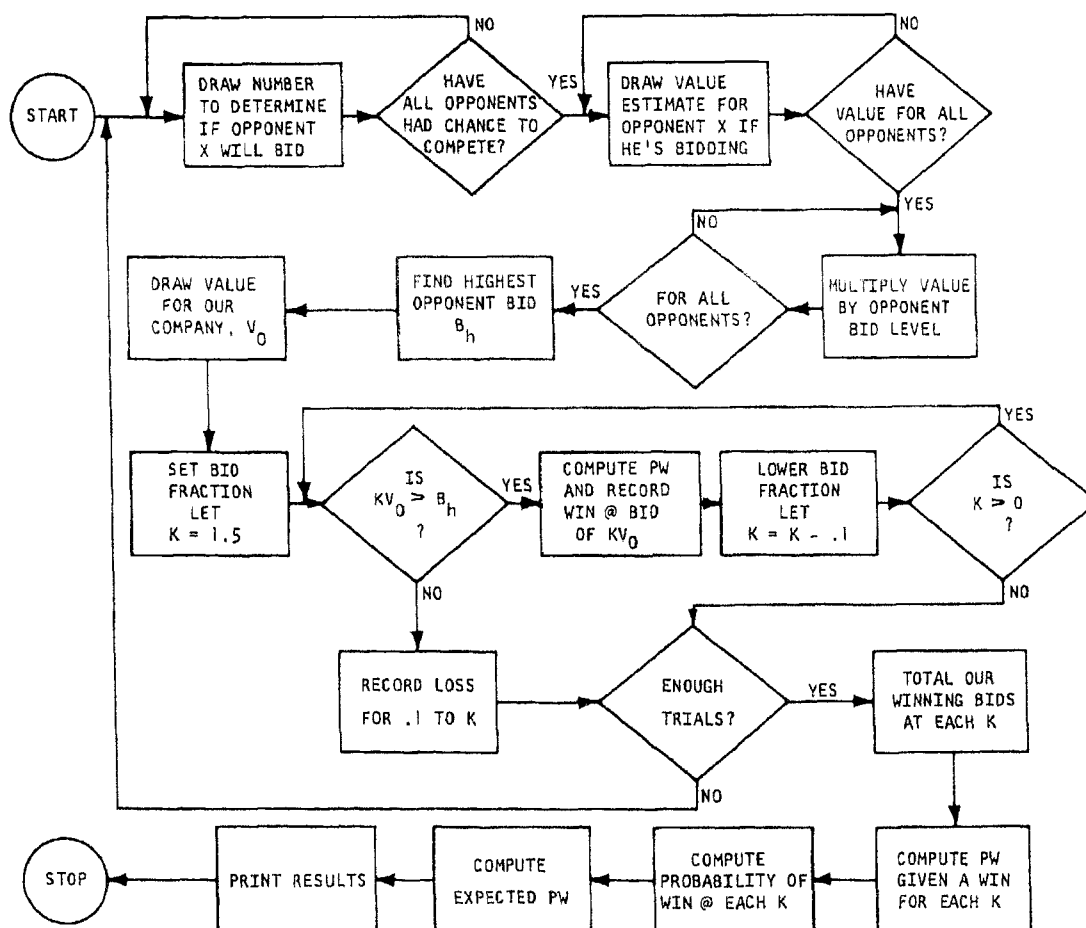

Fig. 3—A bidding model.

TABLE 8—MODEL OUTPUT

| Bidding Level	Bid*	Probability of Winning	Expected Bonus Spent*	Expected (PW) of Bid*	Expected Ratio of Our Estimate To True Value, Given We Win
0.0	0	0	0	0	—
0.1	1,000	0.03	30	232	2.32
0.2	2,000	0.09	180	512	2.16
0.3	3,000	0.16	490	670	1.97
0.4	4,000	0.23	933	664	1.79
0.5	5,000	0.29	1,472	477	1.68
0.6	6,000	0.36	2,136	194	1.58
0.7	7,000	0.41	2,878	−212	1.50
0.8	8,000	0.46	3,675	−707	1.44
0.9	9,000	0.50	4,523	−1,288	1.39
1.0	10,000	0.54	5,407	−1,918	1.35
1.1	11,000	0.57	6,324	−2,607	1.32
1.2	12,000	0.61	7,321	−3,313	1.29
1.3	13,000	0.64	8,342	−4,076	1.26
1.4	14,000	0.67	9,330	−4,868	1.24
1.5	15,000	0.69	10,348	−5,682	1.22

*Thousands of dollars.

table by not bidding at all; or, if we have positive knowledge of others' bids, we can bid one cent more. Obviously both approaches are impractical. The fact is that there *will* be money left on the table, so we have chosen the objective of maximizing EPW — not minimizing money left on the table.

Possible Weaknesses in the Approach

If we thought there were any serious flaws in our ideas on bidding, we would not want to waste your time. On the other hand, while we have gathered considerable evidence to support our theory, some chance always remains that we have simply been fooled by the data.

The *log-normal distribution* does not look as if it can work very well as a model for describing uncertainty on tracts of small value. People who use discounted cash flows to arrive at present worths can get negative values even though they expect the tracts to make some, though small, positive rate of return. The log-normal probability distribution allows for no negative values. The effects of differences among company discount rates become highly magnified on the less valuable acreage.
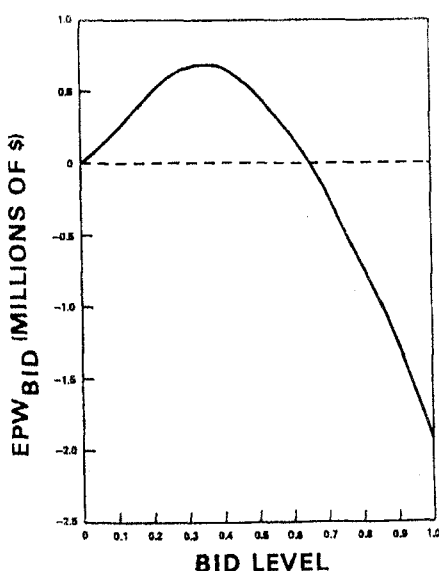
Fig. 4—EPW$_{bid}$ vs bid level.

The reason our model can suggest such a low bid level as a reasonable strategy is the *magnitude of the uncertainty* that we believe is associated with the reserves-value estimating process. We had occasion to compare our independent reserves estimates with those of a partner and found the disagreement to be quite large, though there was no bias by either party. We were as likely to be high as he. If you look at published bids, you can, indirectly, get the same results.

In Fig. 2 we showed that the highest estimator would be off, on the average, by a factor of 2.5 in his expected reserves estimates if he were competing against 11 other independent estimators. Anyone who feels his own reserves estimates are never off by more than 50 percent will feel severe pains swallowing our factor of 2.5.

Of course the amount of uncertainty is just an input parameter for the model. One can put in whatever he likes.

Another problem is our assumption that reserves and value as reflected in final bid estimates *tend to be unbiased.* If we did not make this assumption we would change our ways. No manager is going to submit a bid based on value estimates that he knows are too high or too low. He will enter a multiplier with the intention of being correct on the average. But that tactic does not necessarily guarantee he will be.

We have recognized another weakness without finding much of a solution. How do we account for the competitor who does not bid at all on a particular lease? Does he think it worthless? Has he no interest? Or has he run out of funds? One might argue forcefully that in a major sale he always faces 15 to 20 competitors, whether all of them bid or not.

Conclusions

It is still said that, after many years of exploration, many barrels of oil found, many cubic feet of gas found, and after much red ink, the outlook for future offshore potential is bright. Maybe it is.

Unexpectedly low rates of return, however, follow the industry into competitive lease sale environments year after year. This must mean that by and large industry is paying more for the property than it ultimately is worth. But each competitor thinks he is play-

ing a reasonable strategy. How can the industry go astray? Our sojourn into competitive bidding theory tells us to expect exactly what has happened. It is, then, a theory not only that is mathematically sound, but also that fits reality. Even though each bidder estimates his values properly on average, he tends to win at the worst times — namely when he most over-estimates value. The error is not the fault of the explorationists. They are doing creditable work on a tough job. The problem is simply a quirk of the competitive bidding environment.

Acknowledgment

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APPENDIX

Mathematical arguments leave most people cold. On the other hand, it is nice to know that the logic of English has the solid support of mathematics — especially when we try to explain why a bid level that maximizes present worth should often go down as the number of competitors increases. Of course some of you may have learned long ago to beware of the English language and to trust naught but mathematical rigor. For you, we offer this Appendix.

In the main text we said that we could not carry out the necessary integrations if we used the log-normal distribution. We can, however, analyze a probability distribution that has properties similar to the log-normal. The exponential distribution is our candidate. It is properly skewed. It is defined on the interval 0 to ∞. And if we choose an exponential distribution whose mean is 1.0 (corresponding to true

value normalized to 1.0), we have a spread not too unlike that log-normal whose variance describes the kind of uncertainties faced in the Gulf of Mexico.

The level of mathematics we use is not difficult — a little calculus and a little probability theory. We want to derive an equation that will tell us EPW (Expected Present Worth) as a function of our bid level, the opponent's bid levels, and the number of opponents. By solving that equation, we will show that the bid level for which we get the largest EPW peaks out at two or three opponents and then falls.

$h(x) = \lambda e^{-\lambda x}$, probability density function for value estimate for each bidder

$\dfrac{1}{\lambda}$ = mean of value distribution for ourselves and our opponents, assumed equal to true value

c_o = fraction of our value estimate we choose to bid

$g(x) = \dfrac{\lambda}{c_o} e^{-\lambda x/c_o}$, probability density function for our bid

c_i = fraction of his value estimate that Company i chooses to bid

$f_i(x) = \dfrac{\lambda}{c_i} e^{-\lambda x/c_i}$, probability density function for bid of Company i

$F_i(x) = 1 - e^{-\lambda x/c_i}$, cumulative bid distribution for Company i

n = number of bidding opponents

$\dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$, notation for combinations

$\prod\limits_{i=1}^{n} F_i(x)$ = probability that all opponents will bid less than x, or the probability that we win if we bid x

To get our EPW we multiply 3 terms:

PW if we bid x and win
Probability of winning if we bid x
Probability of our bidding x.

Then we integrate or sum up over all possible values of x.

$$ \text{EPW} = \int_0^\infty \left(\frac{1}{\lambda} - x\right) \left[\prod_{i=1}^{n} F_i(x) \right] \frac{\lambda}{c_o} e^{-\lambda x/c_o} \, dx. $$

$$ \dots \dots \dots \text{(A-1)} $$

Let us simplify by assuming that all opponents will use the same bid levels. Then

$$ \prod_{i=1}^{n} F_i(x) = [F_i(x)]^n $$

$$ = (1 - e^{-\lambda x/c_i})^n, $$

which we expand binomially

$$ = \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{-\lambda k x/c_i}. \quad \dots \text{(A-2)} $$

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Then

$$\text{EPW} = \int_0^\infty \left(\frac{1}{\lambda} - x\right)\left[\sum_{k=0}^n (-1)^k \binom{n}{k} e^{-\lambda k x/c_i}\right]$$

$$\bullet \ \frac{\lambda}{c_0} e^{-\lambda x/c_0} \, dx$$

$$= \int_0^\infty \left(\frac{1}{\lambda} - x\right)\frac{\lambda}{c_0}\left[\sum_{k=0}^n (-1)^k \binom{n}{k} e^{-\lambda x\left(\frac{k}{c_i} + \frac{1}{c_0}\right)}\right] dx$$

$$= \int_0^\infty \frac{1}{c_0}\left[\sum_{k=0}^n (-1)^k \binom{n}{k} e^{-\lambda x\left(\frac{k}{c_i} + \frac{1}{c_0}\right)}\right] dx$$

$$- \int_0^\infty \frac{x\lambda}{c_0}\left[\sum_{k=0}^n (-1)^k \binom{n}{k} e^{-\lambda x\left(\frac{k}{c_i} + \frac{1}{c_0}\right)}\right] dx$$

$$= \frac{1}{\lambda}\sum_{k=0}^n \frac{(-1)^k \binom{n}{k}}{\left(\frac{kc_0}{c_i} + 1\right)} - \frac{c_0}{\lambda}\sum_{k=0}^n \frac{(-1)^k \binom{n}{k}}{\left(\frac{kc_0}{c_i} + 1\right)^2}$$

$$= \frac{1}{\lambda}\sum_{k=0}^n \left\{\frac{(-1)^k \binom{n}{k}}{\left(\frac{kc_0}{c_i} + 1\right)}\left[1 - \frac{c_0}{\left(\frac{kc_0}{c_i}\right) + 1}\right]\right\}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{(A-3)}$$

In the computations we will normalize by setting

$$\frac{1}{\lambda} = 1.0.$$

Mathematics does not interpret anything. People have to do that. Look at Fig. 5, which shows the results of the computations. For purposes of this example, we have chosen to let all opponents use exactly the same strategy: each bids one-half of his particular value estimate. We consider that all opponents have information of equal quality and that the mean of the distribution from which their value estimate comes is the true tract value. We plot our optimum bid level (bid level that maximizes our EPW) vs the number of opponents we face.

At the left of the graph you see that for no opposition the mathematics says to bid a penny. That will be the highest bid and will win. In reality that will not work. The selling authority may set a minimum bid. It may also choose, for one reason or another, not to honor the highest bid. But then no one seriously proposes the use of a competitive bidding model when there is no competition.

The optimum bid level goes up (maximum of 0.28) until the number of opponents reaches two, whereupon it begins its descent. We interpret the curve to be saying that we should bid fairly low if the number of opponents is very small (like one) because there is a good chance that we will be able to pick up a bargain or two. The mathematics appears to be telling us that if we bid any higher, we will just be leaving money on the table. The more competitors we have,
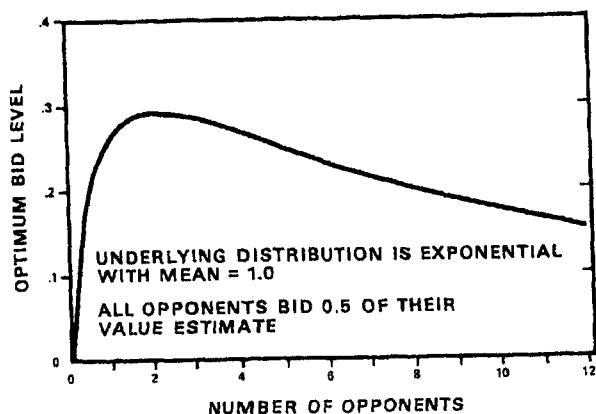
Fig. 5—Optimum bid level vs number of opponents.

the less chance there is for bargains and the higher we must bid to get the property (make our investment). This is the kind of influence of increasing competition that most people see immediately. We call it competitive influence of the first kind.

But we see that after the second opponent the optimum bid level begins to fall. For 12 opponents it has dropped to only 0.15 — about half the maximum it achieved for two opponents. A counter-influence has begun to dominate. The tracts we win tend to be those on which we have overestimated value. The more opponents, the worse our error on the average when we win. We call this competitive influence of the second kind.

Both competitive influences are always present. They do not, however, always "weigh" the same. For most competitive situations, we think competitive influence of the second kind is more important.

The purist may be unhappy that we have drawn a curve through our computed points, giving values for such impossibilities as 3.33 opponents. In setting up a strategy, however, we are never certain of how many competitors we will face on a given parcel. If we thought there was a one-third chance of facing four opponents and a two-thirds chance of facing three opponents, then we would be justified in "expecting" 3.33 opponents. For actual computing with the formula just derived, we should be able to switch from factorials to gamma functions if we expect fractional opponents.

We would get somewhat different pictures if we altered the strategies of our opponents, but the principal characteristics that we used to illustrate the two kinds of competitive influences would remain.

Our simulations using the log-normal distribution show results similar to the ones in this analysis. That is not too surprising. As we pointed out earlier, the log-normal and the exponential have some important similarities. Furthermore, the simulation we carry out is really a numerical integration of the kinds of factors we have examined analytically in this Appendix.

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