Session 5

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0 Topics

1. VCG (Vickrey-Clarke-Groves) Mechanism
2. Competitive Equilibrium and the Welfare Theorem

1 VCG Mechanism

Economics

$N = \{1, \ldots, n\}$: set of agents
A: set of allocations

$$v_i : A \to \mathbb{R} \quad \text{for all } i$$

For $a \in A$, $v_i(a)$ is the valuation agent i assigns to allocation a.

Assume quasi-linear utility.
Task: Choose $a^* \in A$ such that

$$\sum_{i \in N} v_i(a^*) \geq \sum_{i \in N} v_i(a) \quad \text{for all } a \in A.$$

This is TU maximization.

Assume private information: each agent knows only their own valuation, auctioneer knows none of the valuations.

Question: Can we design or construct a

- market
- auction
- trading platform

such that, *in equilibrium*, allocation is efficient? (TU is maximized)

Answer: yes.

VCG

(1) Strategies

One shot $\equiv$ sealed bid.

Each agent i submits report $r_i : A \to \mathbb{R}$. (reported valuation)

(2) Allocation

Choose allocation $b^* \in A$, such that

$$\sum_{i \in N} r_i(b^*) \geq \sum_{i \in N} r_i(b) \quad \text{for all } b \in A.$$

This maximizes total reported valuation.

(3) Payments (with Externality)

Externality: damage you inflict by participating; like throwing a party at 3AM and waking the whole neighborhood.

Agent i is going to pay

$$p_i = \underbrace{\sum_{j \neq i} r_j(b^{**})}_{\text{TU if you're not there}} - \underbrace{\sum_{j \neq i} r_j(b^*)}_{\text{what the others currently get}}$$

where

$$b^{**} = \arg\max_{b \in A} \sum_{j \neq i} r_j(b).$$

Theorem

In the VCG mechanism, reporting truthfully, so $r_i = v_i$, is a dominant strategy. In the resulting DSE, allocation is efficient.

Proof: 2$^{\text{nd}}$ statement is obvious, only prove the 1$^{\text{st}}$ statement.

Consider agent i, other agents' reports $(r_j)_{j \neq i}$. Take report r_i of agent i knowing v_i.

$$\begin{aligned}
u_i(b^*) &= v_i(b^*) - p_i \\
&= v_i(b^*) - \left[\sum_{j \neq i} r_j(b^{**}) - \sum_{j \neq i} r_j(b^*) \right] \\
&= \underbrace{v_i(b^*) + \sum_{j \neq i} r_j(b^*)}_{b^* \text{ depends on } r_i} - \underbrace{\sum_{j \neq i} r_j(b^{**})}_{\text{does not depend on } r_i}
\end{aligned}$$

The mechanism is going to pick a b^* such that $r_i(b^*) + \sum_{j \neq i} r_j(b^*)$ is maximal.

Agent i wants to maximize $v_i(b^*) + \sum_{j \neq i} r_j(b^*)$, this is done via choosing $r_i = v_i$. $\square$

Usage in practice

This is not used in real life, why? Suppose there are 10 bidders and 100 regional licenses (e.g. mobile telephone companies bidding for regional network licenses).

There are 10^{100} possible allocations, so in VCG every company has to report 10^{100} valuations. This is practically impossible.

Example

5 identical items
9 agents
unit demand: there is only a valuation for 1 item, marginal valuation for a 2$^{\text{nd}}$ item is 0.

$$
\begin{array}{rlr}
 & v_1 = 15 & * \\
** & v_2 = 7 & \\
** & v_3 = 10 & * \\
** & v_4 = 12 & * \\
 & v_5 = 5 & \\
 & v_6 = 6 & \\
 & v_7 = 3 & \\
** & v_8 = 13 & * \\
** & v_9 = 8 & *
\end{array}
$$

$p_1 = 13 + 12 + 10 + 8 + 7 - (13 + 12 + 10 + 8) = 7$, pay the highest excluded bid. Similarly, $p_3 = p_4 = p_8 = p_9 = 7$.

Note: with 1 item this is equivalent to the Vickrey auction.

2 Competitive Equilibrium and the Welfare Theorem

Exchange economy: trade without production
Consider economy with trade and production.
2 consumers, 1 firm
2 goods: wheat and beer

Firm 1 produces beer from wheat: $f(w) = \sqrt{w}$

Consumer 1 has utility $u(w, b) = w \cdot b$
Initial endowment: None, owns firm

Consumer 2 has utility $v(w, b) = w \cdot b^2$
Initial endowment: owns 2 units of wheat

Prices: 1 for wheat, q for beer.

Three equilibrium concepts that are the same:

- competitive equilibrium
- Walrasian equilibrium
- price equilibrium

Assume a price-taker model. We are going to check whether these prices are equilibrium prices.

Currently both consumers have 0 utility, so they both have incentive to start trading.

C2 sells his wheat and buys beer with that money. Assume he sells all of his wheat. His budget is now 2.

$$
\begin{aligned}
\max \quad & v(w, b) = w \cdot b^2 \\
\text{subject to:} \quad & w + q \cdot b = 2 \quad \text{(non-satiation, so equality)}
\end{aligned}
$$

Using Lagrange: $\mathcal{L} = w \cdot b^2 - \lambda(w + q \cdot b - 2)$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial w} &= b^2 - \lambda &&= 0 \\ \\ \frac{\partial \mathcal{L}}{\partial b} &= 2wb - \lambda q &&= 0 \end{aligned}\right] \times q \qquad \Longrightarrow \begin{aligned} q \cdot b^2 &= 2wb \\ q \cdot b &= 2w \end{aligned}$$

Using the constraint $w + q \cdot b = 2$ we get for consumer 2 $b = \frac{4}{3q}$ and $w = \frac{2}{3}$. This is his demand function.

C1 has only technology, no initial endowments. He operates the firm, wants to maximize profit

$$\begin{aligned} \max \quad & \Pi_f = q \cdot b - w \\ \text{subject to:} \quad & b = \sqrt{w} \quad \text{(non-satiation)} \end{aligned}$$

By substitution,

$$\max \quad q \cdot \sqrt{w} - w$$

First order condition:

$$\frac{\mathrm{d}\Pi_f}{\mathrm{d}w} = -\frac{q}{2\sqrt{w}} - 1 = 0 \iff w = \frac{q^2}{4}$$

Then, $b = \sqrt{\frac{q^2}{4}} = \frac{q}{2}$. The resulting profit is then $\Pi_f = q \cdot b - w = \frac{q^2}{2} - \frac{q^2}{4} = \frac{q^2}{4}$. This is the budget consumer 1 goes to market with.

He wishes to maximize utility

$$\begin{aligned} \max \quad & u(w, b) = w \cdot b \\ \text{subject to:} \quad & w + q \cdot b = \frac{q^2}{4} \end{aligned}$$

This is equivalent to

$$\max_b \quad \left(\frac{q^2}{4} - q \cdot b\right) \cdot b$$

The maximum is halfway between the roots, so $b = \frac{1}{2} \cdot \frac{q}{4} = \frac{q}{8}$. Then $w = \frac{q^2}{4} - \frac{q^2}{8} = \frac{q^2}{8}$.

Competitive Equilibrium

$(1, q)$ is a competitive equilibrium if the market clears, i.e. total supply must equal to total demand for all goods.

$$\begin{aligned} TS_w &= TD_w \\ TS_b &= TD_b \end{aligned}$$

For beer:

$$\begin{aligned} TS_b &= TD_b \\ \frac{q}{2} &= \frac{q}{8} + \frac{4}{3q} \\ \frac{3q^2}{8} &= \frac{4}{3} \\ q^2 &= \frac{32}{9} \\ q &= \frac{4}{3} \cdot \sqrt{2} \end{aligned}$$

For wheat:

$$\begin{aligned} TS_w &= TD_w \\ 2 &= \frac{2}{3} + \frac{q^2}{8} + \frac{q^2}{4} \\ q &= \frac{4}{3} \cdot \sqrt{2} \end{aligned}$$

The market clears for this q. We get the same answer for q for the market clearing condition with both beer and wheat.

Welfare Theorem

In the competitive equilibrium, the outcomes (utilities) are Pareto efficient.

$$\begin{aligned}
&\text{C1, in equilibrium:} \quad w = \frac{4}{9}, \quad b = \frac{\sqrt{2}}{6} \text{ gives } u = \frac{4}{9} \cdot \frac{\sqrt{2}}{6} = \frac{2}{27}\sqrt{2} \\
&\text{C2, in equilibrium:} \quad w = \frac{2}{3}, \quad b = \frac{1}{\sqrt{2}} \text{ gives } v = \frac{2}{3} \cdot \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{3}
\end{aligned}$$

Note

This is completely different from the TU maximizer, if we wanted to do that it would look like

$$\begin{aligned}
\text{max} \quad & w_1 \cdot b_1 + w_2 \cdot b_2^2 \\
\text{such that:} \quad & b_1 + b_2 = b_f \\
& b_f = \sqrt{w_f} \\
& w_1 + w_2 + w_f = 2
\end{aligned}$$

There are no prices, only the distribution is a problem. This is "communism".