Efficient Equilibria in a Public Goods Game

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***Abstract*—The "best-shot" public goods game is a network game, defined on a social network. As in most strategic games, it contains a structured tradeoff between stability and efficiency. The present study considers a multi-agent system, in which each agent represents a player in the "best-shot" game. It is demonstrated that any Pure-strategy Nash Equilibrium (PNE) of the "best-shot" game is Pareto efficient and that best-response dynamics converge into a PNE within a linear number of steps. It is also shown that the game is a potential game. The potential function can be utilized for the search of PNEs with certain social properties. In order to improve efficiency beyond the limited set of stable states, a mechanism of side payments is proposed. We prove that by using side payments an outcome that maximizes social welfare can be stabilized. A distributed protocol based on asymmetric distributed constraints optimization, which enables the search for efficient outcomes, is proposed. Finally, an extensive experimental evaluation compares the actual social welfare in outcomes achieved by different search paradigms for common social networks structures.**

I. Introduction

In network games the utilities of players depend both on their own actions and on the actions taken by their neighbors in the network. A well known example of network games are "best-shot" public goods games [1], [2], in which players share common goods. Each player chooses whether to take some action or avoid it. The action is associated with an investment in some local public good. The action might be in the form of performing some computational effort, where computation results can be easily shared locally. It can also be buying a book or some other product that is easily lent from one player to another. Each player wants to have at least one player in her neighborhood taking the action, including herself. However, there is a cost associated with taking the action, so if any of the player's neighbors takes the action then the player would prefer to avoid it.

Typically, there are many Pure-strategy Nash Equilibria (PNEs) in a "best-shot" game, each corresponds to a maximal independent set of vertices in the network [3]. We refer to these outcomes as *stable*. Finding a maximal independent set is trivial, and therefore it is easy to find *some* stable outcome. However, finding a PNE with a maximal social welfare (SW) (i.e., a PNE with minimal number of players taking the action) is computationally intensive [2]. Such PNEs are commonly considered *efficient*. Nevertheless, some outcomes may possess greater SW, but some players may remain unsatisfied in such outcomes. This tradeoff between efficiency and stability motivates the use of an incentive mechanism to promote stability in efficient states.

A possible approach in order to incentivize unsatisfied players to agree on some preferred outcomes, is the *side payments* mechanism. Side payments allow transfers of utility (money) between players, such that players who gain from some outcome may want to pay others that are unsatisfied by it. Side payments can help promote efficiency by providing incentives to some players for seeing more fully the impact of their actions [4]. Regarding the "best-shot" game, one expects that in efficient outcomes "central players" will take the action, while "peripheral players" will free-ride their neighbors [5][1]. Intuitively, side payments allow central players to take the action and be compensated by peripheral players.

Another version of the game proposed a relaxation of the complete information assumption. In this version, the players are only informed of their own degree and do not know the whole network. This results in the existence of a Bayesian Nash Equilibrium (BNE) in which players with low degree choose to take the action, and players with high degree choose to free-ride their neighbors [5]. Since the result is likely to be inefficient, Grubshtein and Meisels [6] proposed a distributed and cooperative approach for finding an outcome that Pareto dominates the BNE outcome. Such outcomes were proved to be Pareto efficient, but not necessarily a PNE.

A centralized probabilistic approach for finding approximation to the most efficient PNEs, using simulated annealing, was proposed by DallAsta *et al.* [7]. However, even the most efficient PNE is not guaranteed to maximize SW among all possible outcomes. DallAsta *et al.* [7] also showed convergence of best-response dynamics to a PNE outcome under some prior conditions.

The present study proves that a procedure of best-response dynamics (starting at any outcome) converges to a PNE within at most $2{\cdot}n$ improving steps (where n is the number of agents). It is also shown that a PNE is Pareto efficient, and that the "best-shot" public goods game is in fact a *potential game*. Furthermore, we prove that an outcome of maximal SW can be stabilized using side payments. An ADCOP representation, together with the mechanism of side payments, creates a distributed framework that enables the agents to compute *a*

[1]For the intuitive meaning of central player or peripheral player on a social network see Figures 1 and 2 in Section III-C.

stable outcome of maximal SW (*stable* in the sense that each of the players does not wish to deviate unilaterally).

The plan of the paper is as follows. In Section II the game is defined formally, followed by theoretical properties of the game and its PNEs that are presented in Section III. Section IV presents the mechanism of side payments and proves its ability to secure efficiency, while in Section V an ADCOP representation of the problem is defined so that a solution can be computed in a distributed manner. An extensive experimental evaluation and discussion are given in Sections VI and VII, respectively.

II. The Game Model

A common and compact representation for network games is the graphical model proposed by Kearns *et al.* [8], where a game is based on some underlying graph describing the players' interactions network. Consider a finite set of players $N = \{1, ..., n\}$ who are connected in a network $G = \{N, E\}$. Each vertex in G represents a player, and edges represent the interaction structure in the game. Given a player $i \in N$, denote the set of i's neighbors by N_i. These are the players whose actions impact i's payoff. The *neighborhood* of i is the set $\{i\} \cup N_i$. Each player chooses an action $a_i \in S_i = \{T, F\}$. Denote the choice $a_i = T$ as *taking the action*, and the choice $a_i = F$ as *avoiding* it. The *utility* of player i for some outcome $v \in S = S_1 \times ... \times S_n$, and action-cost c, $0 < c < 1$, is defined as follows:

$$u_i(v) := \begin{cases} 1-c, & \text{if } a_i(v) = T \\ 1, & \text{if } a_i(v) = F, \exists j \in N_i : a_j(v) = T \\ 0, & \text{if } a_i(v) = F, \forall j \in N_i : a_j(v) = F \end{cases}$$

We say that a player i has a *null-neighborhood* if $a_i(v) = F$, and $\forall j \in N_i : a_j(v) = F$.

Given a "best-shot" public goods game on a network $G = \{N, E\}$, and an outcome $v \in S$, the possible *states* of a player $i \in N$ can be divided into the next four distinct types:

1) **TT**: $a_i(v) = T$ and $\exists j \in N(i) : a_j(v) = T$
2) **TF**: $a_i(v) = T$ and $\forall j \in N(i) : a_j(v) = F$
3) **FT**: $a_i(v) = F$ and $\exists j \in N(i) : a_j(v) = T$
4) **FF**: $a_i(v) = F$ and $\forall j \in N(i) : a_j(v) = F$

Note that the states TF, FT are *stable* from a player's point of view. A player in one of these states has no incentive to change her strategy unilaterally. The states TT, FF are not stable. A player in state FF (with a null-neighborhood) would prefer to change her strategy to T, and a player in state TT would prefer to change her strategy to F.

III. Stability of the "Best-Shot" Game

Nash Equilibrium is an important notion in game theory. We say that an outcome is a PNE, if every player does not prefer to change her own strategy, given the strategies of all other players in this outcome. It is considered desirable to arrive at PNE outcomes that are efficient. Common definitions for efficiency are *Pareto Efficiency* (PE) and *Social Welfare* (SW) [9]. The notion of PE is considered first and SW efficiency measures will be discussed later on at Section III-C.

Proposition 1. *A PNE in a "best-shot" public goods game is PE. That is, there is no other outcome which further increases the utility of some players without reducing the utility of at least a single player.*

Proof. Assume by contradiction that some outcome v is a PNE and is not PE. Since v is not PE there exists some outcome v', such that v' dominates v. i.e., for each player i, $u_i(v) \leq u_i(v')$ and there exists a player j such that $u_j(v) < u_j(v')$. In every PNE there are no null-neighborhoods. Hence, for some player j, $u_j(v) < u_j(v')$ if and only if in outcome v player j takes the action, while in outcome v', j *free-rides* one of her neighbors (i.e., i is in state FT). Consider a neighbor k of player j such that $a_k(v') = T$ – if $a_k(v) = T$, then v is not a PNE, in contradiction (since j could choose F and free-ride k). However, if $a_k(v) = F$ then $u_k(v) = 1$ (since $a_j(v) = T$). It is also known that $a_k(v') = T$, so $u_k(v') = 1 - c$. Therefore, $u_k(v') = (1 - c) < 1 = u_k(v)$ in contradiction to Pareto dominance of v' over v. □

A. Convergence of Best-Response

PNEs in a "best-shot" game are in some sense efficient. A question remains whether and how players reach such outcomes. DallAsta *et al.* [7] have shown that simple local search algorithms can find a PNE and from certain types of outcomes best-response dynamics are guaranteed to converge into a PNE. Next, a proof is presented that best-response dynamics converge into a PNE from *any* outcome, in any order of the players, within at most $2 \cdot n$ steps of best response.

Proposition 2. *Best-response dynamics in a "best-shot" public goods game converge into a PNE within at most $2 \cdot n$ steps.*

Proof. Consider a process of best-response dynamics. Each player i, in turn, chooses her best strategy given the strategy profile of all other players. The process continues until no player can improve her utility by changing strategy unilaterally. Note that this type of process is not necessarily guaranteed to converge in every game. In a "best-shot" game, there are two possible improving responses for a player:

1) moving from FF to TF
2) moving from TT to FT

Lemma 3. *No player reaches a TT state as a result of a best-response step (of any player).*

Proof. In every unilateral deviation, the only players whose state is affected are the deviating player and her neighbors. If a player is at state FF and changes strategy to TF (move 1), the deviating player clearly will not reach a TT state. All neighbors of the deviating player have strategy F. Hence, they are all in state FT after the deviation. If a player is at state TT and changes strategy to FT (move 2), clearly, the deviating player does not reach a TT state. Let i denote the deviating player, and j denote some neighbor of i. Before deviation, i was at state TT and j could have been at state TT or FT. If j was at state TT before, it is not possible that i's deviation moved her to TT (because it was j's state a priori). If j was at state FT her new state is now FT or FF. In both cases, i's deviation could not move players to state TT from any other state. $\square$

Lemma 4. *In best-response dynamics, each type of improving response can occur only once for some player i.*

Proof. Once a player made a move of type 1, she is in state TF and remains in this state since: (a) No other player reaches a TT state as a result of a best-response play of some player, and (b) A player in state TF will not change strategy, so states FF or FT are not possible. Once a player made a move of type 2, she is in state FT and will not be in state TT again. Hence, it is not possible for i to deviate once again from TT to FT. $\square$

To conclude the proof of Proposition 2, each player in the game can perform at most 2 improving responses in best-response dynamics, before her state becomes stable (FT or TF). After at most $2 \cdot n$ plays, all players are stable, therefore they have reached a PNE. $\square$

B. A Potential-Game Perspective

The concept of a potential game is used to define games in which the change in players' utility corresponds to the change in some global *Potential Function* [10], [11]. A game is an (ordinal) potential game if for every change in a player's strategy, the sign of the change in the player's utility is equal to the sign of the change in the global potential function. The potential function can be a useful tool to analyze equilibrium attributes of the game, as will be demonstrated below.

Proposition 5. *The "best-shot" public goods game is an ordinal potential game.*

Proof. The potential function (for a single player i) $f_i : S_1 \times ... \times S_n \longrightarrow \mathbb{R}^+$ is defined as follows:

$$f_i(v) := \begin{cases} 0 & \text{if } a_i(v) = T, \exists j \in N_i : a_j(v) = T \\ c & \text{if } a_i(v) = F \\ 1 & \text{if } a_i(v) = T, \forall j \in N_i : a_j(v) = F \end{cases} \tag{1}$$

A global potential function $F : S_1 \times ... \times S_n \longrightarrow \mathbb{R}^+$ is: $F(v) := \sum_{i \in N} f_i(v)$. One needs to show that for every $i \in N$, every $v_{-i} \in S_{-i}$, and every $v_i^1, v_i^2 \in S_i$ it holds that $u_i(v_i^1, v_{-i}) - u_i(v_i^2, v_{-i}) > 0$ iff $F(v_i^1, v_{-i}) - F(v_i^2, v_{-i}) > 0$.

Consider some player $i \in N$. For the first direction, we assume that $u_i(v_i^1, v_{-i}) - u_i(v_i^2, v_{-i}) > 0$ and want to show that $F(v_i^1, v_{-i}) - F(v_i^2, v_{-i}) > 0$. The relation $u_i(v_i^1, v_{-i}) - u_i(v_i^2, v_{-i}) > 0$ holds when player i performs an improving step (moves from v_i^2 to v_i^1). When a player moves from TT to FT the change in her utility is c, and the change in the potential function is $(c + M)$, where M is the number of other players that as a result of the change in i's strategy moved from state TT to state TF. Therefore, $F(v_i^1, v_{-i}) - F(v_i^2, v_{-i}) > 0$ as required. When a player moves from state FF to state TF the change in her utility is $(1 - c)$, and the change in the potential function is also $(1 - c)$ since every one of i's neighbors chooses F, so their potential remains the same. The only change is in i's potential, from c to 1.

For the second direction, assume $u_i(v_i^1, v_{-i}) - u_i(v_i^2, v_{-i}) \leq 0$ and show that $F(v_i^1, v_{-i}) - F(v_i^2, v_{-i}) \leq 0$ similarly to the first direction. $\square$

The potential function defined above has an interesting feature: its global maximum is not only a PNE (as every local maximum of the potential function), but it is also the PNE with the lowest SW of all PNEs. This holds since in each local maximum all players are at state FT or state TF and according to Equation 1 the global maximum of the potential function is the outcome with most possible players that choose T, between all stable outcomes. In contrast, the PNE with the highest SW is the outcome giving the minimal local maximum of the potential function.

C. Equilibrium Efficiency vs. Stability

PNEs in "best-shot" games were proven to be PE. However, consider the example in Figure 1, which has two possible PNEs. In one PNE, the central player chooses T and all others free ride. In the other PNE the central player chooses F and all others choose T. From a global perspective, the network is better off in the first outcome, that maximizes SW.

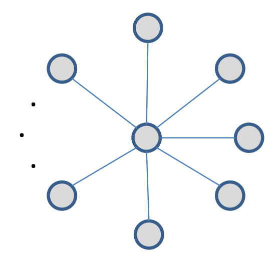

Fig. 1. Pareto Efficiency vs. Social Welfare

To emphasize the tradeoff between efficiency and stability, consider the game instance described in Figure 2. The outcome that maximizes SW is the one in which players a and b choose T and the rest of the players choose F. This outcome is clearly not a PNE since both a and b are in state TT. In the best PNE, 5 players choose T. It can be achieved for example when player a and all peripheral players around b choose T, and the rest of the players choose F. The PNE with the worst SW is when all peripheral players choose T and the central players, a and b, choose F. The existence of such instances of the game, with inefficient stable states, motivates the pursue of a mechanism that can secure efficiency.

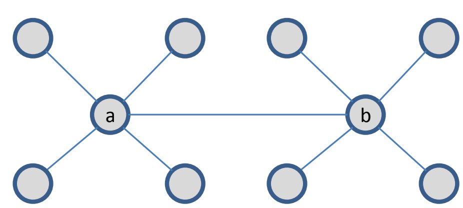

Fig. 2. Efficiency vs. Stability

IV. Efficiency with Side Payments

Side payments are defined as a transfer function between players, where each player may pay (or receive payment from) each one of its neighbors [4].

Definition 1. *Given a "best-shot" game $G = \{N, E\}$, side payments are defined by a transfer function $\tau : N \times N \times S_1 \times ... \times S_n \longrightarrow \mathbb{R}^+$, where $\tau_{i,j}(v)$ denotes the payment being transferred from player i to player j in outcome v. We restrict our attention to transfer functions such that if $j \notin N_i$ then $\forall v : \tau_{i,j}(v) = 0$.*

Definition 2. *Given some outcome v in game G*

- *The incoming transfer of each player i in outcome v is $\tau_i^{in}(v) := \sum_{j \in N_i} \tau_{j,i}(v)$*
- *The outbound transfer of each player i in outcome v is $\tau_i^{out}(v) := \sum_{j \in N_i} \tau_{i,j}(v)$*

Definition 3. *We say that some outcome v in G is Side Payments Enforceable (SPE) if there exists a transfer function τ defining side payments in G, such that: $\forall i \in N \forall v'_i \in S_i : u_i(v) + \tau_i^{in}(v) - \tau_i^{out}(v) \geq u_i(v'_i, v_{-i})$*

A. Securing Maximal Social Welfare

Proposition 6. *For every "best-shot" public goods game, a maximal SW outcome is SPE.*

Proof. Consider an outcome v^* of maximal SW. We want to show that v^* is SPE. Since v^* maximizes SW, obviously there are no players in state FF. (A player in state FF could change its strategy to T and the SW of the resulting outcome would increase, in contradiction). Moreover, players in states TF, FT do not require any incentives to keep their chosen strategy. One only needs to show that players in state TT can receive enough incentives so they would keep their choice (T).

For some outcome v if a player j with strategy F has exactly one neighbor i with strategy T, we say that j *depends on* i. We argue that in outcome v^*, for every player i at state TT, there exists at least one player j that depends on i. Assume by contradiction that for some player i with state TT, there is no player j that depends on i. Then, if i would change her strategy to F, all players would still have at least one neighbor that chooses T, and the aggregated cost would be smaller than in v^* (i does not pay, all others stay with the same strategy). That contradicts the assumption of maximum SW in v^*.

For each player in state TT, the loss from choosing T instead of F is exactly c. It was shown that for each player i in state TT, there exists at least one player j that depends on i. j's gain from i's choice is also c since $u_j(v^*) = 1$ and j's maximal utility given i chooses F is $1 - c$. Define the transfer function τ:

$$\tau_{j,i}(v^*) := \begin{cases} \frac{c}{d_i(v^*)} & \text{if } \bigstar \\ 0 & \text{otherwise} \end{cases}$$

where $d_i(v^*)$ denotes the number of players in N that depend on i in outcome v^*, and $\bigstar$ requires:

- $a_i(v^*) = T$
- $a_j(v^*) = F$
- $\nexists k \in N_j, k \neq i : a_k(v^*) = T$

In addition, note that for all outcomes $v \neq v^*$ it holds that $\forall j, i \in N : \tau_{j,i}(v) = 0$.

Altogether, each player at state TT receives exactly c from the neighbors that depend on her, and do not want to deviate. Each player j that depends on some neighbor i, transfers payments only to i (since she depends on only one player by definition) and the payments are at most c. Hence, for each paying player j: $u_j(v^*) + \tau_j^{in}(v^*) - \tau_j^{out}(v^*) = 1 - \tau_j^{out}(v^*) \geq (1 - c) = u_j(T, v^*_{-j})$ as required. $\square$

Given an outcome v^* that maximizes SW, one can infer from the above proof a trivial distributed procedure for computing the side payments transfers. Each player i in state TT demands a payment of c, and each neighbor of i that depends on her, performs a transfer of $\frac{c}{d_i(v^*)}$ to i. In the following we concentrate on the search method for the required outcome and ignore the side payments mechanism that can be easily computed by the above result.

V. ADCOP-based Search

Given a "best-shot" public goods game one wants to find outcomes with special attributes. Our interest is in outcomes maximizing SW and in the optimal PNE outcomes (when side payments are not allowed), as well as in the worst PNE outcomes (like the *Price of Anarchy* and *Price of Stability* measures [12]). Search can be performed by modeling the "best-shot" game as an asymmetric distributed constraints optimization problem (ADCOP) [13], similarly to Grubshtein and Meisels [6]. Once an ADCOP is constructed, it can be solved by an appropriate algorithm such as *K-ary SyncABB-1ph* [14].

An ADCOP is a tuple $< A, X, D, R >$ where $A = \{A_1, A_2, ..., A_n\}$ is a finite set of agents. $X = \{X_1, X_2, ..., X_m\}$ is a finite set of variables. Each variable is held by a single agent. $D = \{D_1, D_2, ..., D_m\}$ is a set of domains. Each domain D_i consists of the finite set of values that can be assigned to variable X_i. R is the set of relations (constraints). Each constraint $C \in R$ is a function $C : D_{i_1} \times D_{i_2} \times ... \times D_{i_k} \rightarrow \prod_{j=1}^{k} \mathbb{R}_{\geq}$ that defines a non-negative cost for every participant in every value combination of a set of variables. The *asymmetry* of constraints in an ADCOP relates to the different costs for every participant. An *assignment* is a pair including a variable, and a value from it's domain. A *complete assignment* consists of assignments to all variables in X. A *solution* to an ADCOP is a complete assignment of minimal cost.

A. Maximal Social Welfare Search

Given a public goods game $G = \{N, E\}$, one defines an ADCOP as follows:

- Each agent $A_i \in A = \{A_1, ..., A_n\}$ corresponds to a single player and holds a single variable X_i.
- Each variable X_i has a domain $D_i = \{T, F\}$.
- For every agent A_i construct a constraint over players in A_i's neighborhood. The cost incurred is $(1 - u_i)$, where

u_i is the utility of player $i \in N$ given the choices of i's neighbors.

The optimal solution is an assignment that corresponds to a maximum-SW outcome in the underlying "best-shot" game.

B. Potential-Based Search

Max-Potential Search: The agents, variables and domains are the same as above. The change is in the constraints. For every agent A_i construct a constraint over players in i's neighborhood and the cost incurred is $(1 - f_i)$ if the outcome produces a local maximum in the potential function, and $n+1$ otherwise. (Where f_i is i's potential, from Equation 1). The optimal solution is a global maximum of the potential function, which is a PNE that minimizes the SW.

Min-Max-Potential Search: The only change in the constraints construction is that the cost incurred is f_i if the outcome produces a local maximum in the potential function, and $n + 1$ otherwise. Since there exists at least one local maximum, the returned solution will be a local maximum. Due to the cost of f_i, the solution will be a minimal local maximum, which corresponds to a PNE that maximizes the SW, as required.

VI. Experimental Evaluation

It was shown in Section IV that a Maximal SW outcome can become stable when using a side payments mechanism. The example in Figure 2 demonstrates an outcome maximizing SW that is more efficient than the best PNE outcome. An empirical evaluation is required in order to examine the actual improvement in SW of a PNE when cooperation and side payments are allowed.

In the evaluation four different types of outcomes were examined:

- Max-SW – An outcome of maximal SW, not necessarily a PNE. Such an outcome can easily become a PNE if we use the side payments approach of Section IV.
- Best PNE – A PNE outcome that maximizes SW (without using side payments). This outcome is reached by searching a Min-Max-Potential outcome.
- Worst PNE – A PNE outcome that minimizes the SW. This outcome is reached by searching a Max-Potential outcome.
- Best-Response PNE – An outcome reached by the use of best-response dynamics (we choose an outcome randomly, choose a random order of players, and let each player choose her best response in turn).

It is important to note that best-response PNEs are found in linear time, due to convergence within at most $2 \cdot n$ steps. The other three types require solving an ADCOP. Therefore, run time measures are compared only between these three approaches. The best-response run time is negligible in comparison to the run time required for solving an ADCOP, but does not guarantee any type of optimality.

A. Problem Generation

Two types of networks are used: Erdös-Rényi random networks and Barabási-Albert scale-free networks. For each setting, 100 random network instances were generated and the different search processes and outcomes were evaluated.

Erdös-Rényi random networks (ER Networks) are defined by 2 parameters: the number of vertices n, and an edge probability ep [15]. Instances of such networks were constructed by generating n vertices, and for each pair of vertices i,j the edge $\{i, j\}$ was added with a probability ep. Results for $n = \{10, ..., 20\}$ and $ep = 0.2$ are presented in Figure 3.

Barabási-Albert scale-free networks (BASF Networks) are defined by 4 parameters: n, m_0, ep and m. n is the number of vertices, m_0 and ep define the basic network, and $m \leq m_0$ defines the number of links created from each new vertex to an existing vertex. The process creates random scale-free networks, that are widely observed in social networks [16]. Results for $m_0 = 4$, $m = 1$, $ep = 0.75$, $c = 0.9$ are presented in Figure 4.

B. Experimental Results

It is known that SW(Max-SW) $\geq$ SW(Best PNE) $\geq$ SW(Best-Response PNE) $\geq$ SW(Worst PNE). It is also known that the Max-Potential solution is a worst PNE and the Min-Max-Potential solution is a best PNE. Figure 3 shows an interesting result for Erdös-Rényi random graphs: the best PNE is almost optimal, on average. One can also observe that an average best-response process converges to a PNE with SW that is far worse than the best PNE and is much better than the worst PNE.

For Barabási-Albert scale-free networks the results are different (Figure 4). Here, the SW of the best PNE becomes worse in comparison to the optimal SW as the number of agents n increases. Another interesting result is that an average best-response process in these networks reaches almost the worst PNE outcome. Note that it is typical for scale-free networks to have some central players with many peripheral neighbors. In order to understand the difference between the optimal SW and the SW of the best PNE, recall the network in Figure 2. In the optimal outcome, the central players (a and b) choose T, while peripheral players choose F. This is clearly not a PNE since the central players are connected. The same intuition holds for scale-free networks, that are likely to have such central players. In order to explain the convergence of best-response processes to a "bad" PNE, consider the network in Figure 1. In order for the central player to choose T in a best-response step, all other players' choices need to be F when her turn arrives. Since the probability is very low for such a coincidence when we have few central players and many peripheral players, one expects central players to choose F and peripheral players to choose T. Hence, we get low SW for best-response dynamics.

Regarding the run time measures, we used the standard measures for ADCOP and DCOP algorithms (cf. [17]). Figure 5 presents the average number of Non-Concurrent Constraint Checks (NCCC) for each of the approaches. Note that the

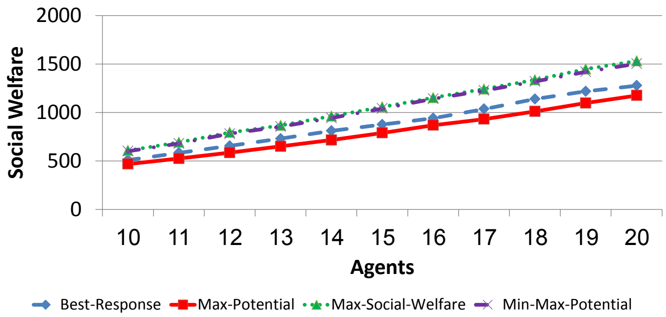

Fig. 3. SW in ER Networks

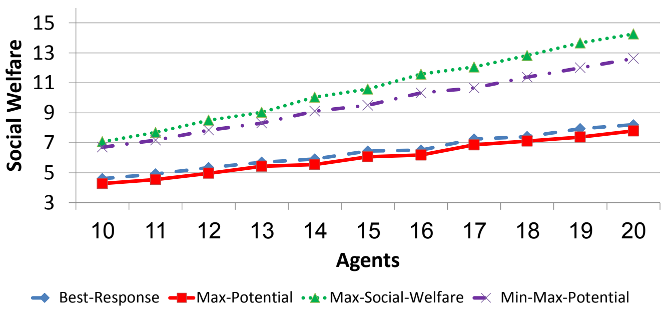

Fig. 4. SW for BASF Networks

graph uses logarithmic scale. It is clear that the search for a Max-SW outcome is much more computationally intensive than searching for a best/worst PNE. This is expected since whenever one searches for a PNE the stability of each local neighborhood is required. Therefore, a massive pruning is enabled – any partial assignment of choices that includes unstable local neighborhoods cannot be extended into a valid solution. When one searches for a Max-SW outcome such kind of pruning is not possible.

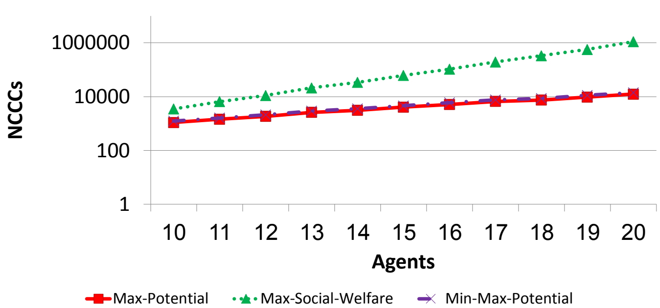

Fig. 5. NCCCs for BASF Networks

VII. Conclusions

The "best-shot" public goods game is investigated. Every PNE was shown to be Pareto efficient (PE) and any best-response dynamic converges into a PNE within at most $2 \cdot n$ steps. The "best-shot" game was shown to be a potential game, and the potential enables the finding of PNEs with minimal or maximal social welfare (SW).

A new approach to secure efficient Nash equilibrium by the use of side payments in network games is proposed. For the "best-shot" public goods game, the proposed approach guarantees a maximal-SW PNE by using the side payments mechanism. Both the search for such an outcome and the search for the required side payments are performed in a distributed manner, by modeling the problem as an ADCOP.

An extensive experimental evaluation demonstrates the effectiveness of the proposed side payments approach in Barabási-Albert scale-free networks. For Erdös-Rényi random networks the improvement in SW is small. Hence, it might be sufficient to search for the best PNE outcome which according to the evaluation requires less computational effort. In all cases, a random best-response process usually converges into a PNE with relatively low SW.

One may also want to extend the side payments approach to other network games, or to more generalized classes of games. In order to do so, a complete characterization is required for games in which maximal SW outcomes are *side payments enforceable*.

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