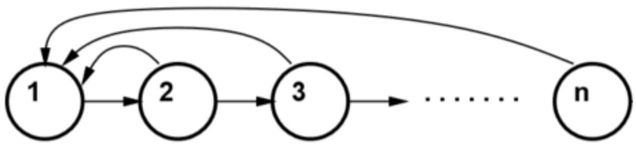
Planning and Learning in Dynamical Systems (046194)
Homework 3, due 3/6/2020

Question 1 – Simple MDP

Consider the n states MDP given in Figure 1. In each state there are two actions - Left which leads to the starting state 1, and Right which takes us one step closer to the rightmost state n. In state n, any action leads to the leftmost state $s=1$ and a reward 1 is given ($r(s=n,a)=1$). The reward for every other state and action is 0. The discount factor is γ, so a delay in reaching the goal will inflict an exponential reduction in the total reward.



1. What is the optimal policy μ?
2. Find the value $V^{\mu}(s)$ of each state.

We apply the uniform exploration policy π: $\pi(a=\text{right} \mid s)=\pi(a=\text{left} \mid s)=0.5$.

3. Find the stationary distribution over the states $d_{\pi}(s)$.
4. Run value iteration starting from $v^0(s)=0$ to find $v^1(s), v^2(s), v^3(s)$. Write $v^{\infty}(s)$ using $V^{\pi}(s)$.
5. Starting from state $s=1$, how many steps in expectation will it take to reach state $s=n$?
6. Starting from $s=1$, under a policy of your choice, what is the minimal number of steps it will take to visit every state-action pair at least once?
7. Let's say we employ an optimistic exploration scheme, where every previously unvisited state-action pair is assumed to have reward 1. Starting from $s=1$, how many steps will it take to visit every state-action pair at least once?
8. Assume $n=3, \gamma=0.5$. Find $V^{\pi}(s)$.

Question 2 – The $c\mu$ rule

Assume N jobs are scheduled to run on a single server. At each time step (t=0,1,2,…), the sever may choose one of the remaining unfinished jobs to process. If job i is chosen, then with probability $\mu_i > 0$ it will be completed, and removed from the system; otherwise the job stays in the system, and remains in an unfinished state. Notice that the job service is memoryless – the probability of a job completion is independent of the number of times it has been chosen.

Each job is associated with a waiting cost $c_i > 0$ that is paid for each time step that the job is still in the system. The server's goal is minimizing the total cost until all jobs leave the system.

a. Describe the problem as a Markov decision process. Write Bellman's equation for this problem.
b. Show that the optimal policy is choosing at each time step $i^* = \arg\max_i c_i\mu_i$ (from the jobs that are still in the system).

Hint: Compute the value function for the proposed policy and show that it satisfies the Bellman equation).

Remark: the $c\mu$ law is a fundamental result in queuing theory, and applies also to more general scenarios.

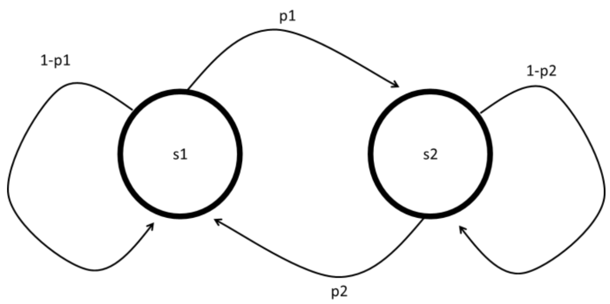
Question 3 - DP operator not contracting in Euclidean norm

Recall the fixed-policy DP operator T^π defined as (see Section 5.4 of the lecture notes)

$$\left(T^\pi(J)\right)(s) = r(s,\pi(s)) + \gamma \sum_{s' \in S} p(s' \mid s,\pi(s))\, J(s'),$$

where $\gamma < 1$. We have seen that T^π is a contraction in the sup-norm. Show that T^π is not necessarily a contraction in the Euclidean norm. We will later see, in the learning part of the course, that the Euclidean norm is related to a an approximation error due to learning, which will require additional discussion.

Hint: one possible approach is to consider the following 2-state MDP, and choose appropriate values for p_1, p_2, γ to obtain a contradiction to the contraction property.



Question 4 – Stochastic Shortest Path

Consider an MDP with $\gamma = 1$, positive cost with a cost free termination state denoted by 0. Once the system reaches that state it remains there at no further cost. We are interested in the optimal policy in such an MDP; the policy with the minimal cost to the terminal state.

- Define the value function v^π as the expected sum of costs until reaching terminations, and similarly v^* as the optimal. Write Bellman equations for v^π, v^*. Note the difference between the terminal state and all other states.
- Write the Bellman operators T, T_π explicitly for the SSP problem.

In the next part we will show that under certain assumptions, T_π is contractions in a suitably defined norm (the contracting property of T follows from the contraction of T_π).

Definition (1): A proper policy. A policy is proper if there exists a positive probability that the termination state, 0, will be reached.

Definition (2): Let $J, \xi \in R^S$. The maximum norm of J is defined as follows.

$$\|J\|_\xi = \max_{s \in S} \frac{|J(s)|}{\xi(s)}$$

- For the SSP problem, it holds $TJ^* = J^*$ for J^* the optimal cost vector and is finite if all stationary policies are proper. Explain why the proper polices assumption is needed?

- Prove: Assume all stationary policies are proper. Then, there exists a vector ξ with positive components such that the mapping T_π, for all stationary policies π, are contraction mappings with respect to the weighted maximum norm.

Guidelines to the proof.

- You first need to define ξ properly. Consider a new SSP with same transitions and costs all equal to -1, except at the termination state, 0. Define $\hat{J}(s)$ as the optimal value from state s in the new SSP problem, and define
$$\xi(s) = -\hat{J}(s)\ .$$
See that $\xi(s) \geq 1$.
- By writing the optimal Bellman equation of the new SSP for $\hat{J}$, and basic analysis, show the following:
  a) For any stationary policy π, $\sum_{s' \in S} p^\pi(s' \mid s)\xi(s') \leq \xi(s) - 1$.
  b) $\xi(s) - 1 \leq \beta\xi(s)$, where $\beta = \max_{s'} \frac{\xi(s')-1}{\xi(s')}$. Explain why $\beta < 1$.
- Prove that, for any π, T_π is a contraction in the max norm and its contraction coefficient is β. Hint: as usual, prove by showing that for any $J_1, J_2 \in R^S$,
$$\frac{|T_\pi J_1(s) - T_\pi J_2(s)|}{\xi} \leq \beta\|J_1 - J_2\|_\xi.$$

Question 5

The Value Iteration produces a sequence of value functions $\{V_i\}$ where V_0 is the initial value function supplied to the algorithm. Denote the greedy policy w.r.t. V_i by π_i. Does $\{\pi_i\}$ is a sequence of improving policies? i.e, does $V^{\pi_i} \geq V^{\pi_{i-1}}$ for all i? Prove or give a counter example.