# ROB310-Problem Set 8

## Problem 8.1

a) $E[X] = \sum_{x \in X} x f(x)$, $f(x) = ax^2$, $a = \frac{1}{6}$

$$\Rightarrow (-1)(\tfrac{1}{6}) + (0)(\tfrac{1}{6}) + (1)(\tfrac{1}{6}) + (2)(\tfrac{2}{3}) = \tfrac{4}{3}$$

$$\begin{aligned} Var[X] &= E[(X-E[X])^2] = E[(X-E[X])(X-E[X])] \\ &= E[X^2 - 2XE[X] + E[X]^2] = E[X^2] - 2E[XE[X]] + E[E[X]^2] \\ &= E[X^2] - E[X]^2 \\ &\Rightarrow \sum_{x \in X} x^2 f(x) - (\tfrac{4}{3})^2 = (1)(\tfrac{1}{6}) + (0)(\tfrac{1}{6}) + (1)(\tfrac{1}{6}) + (4)(\tfrac{2}{3}) - (\tfrac{16}{9}) \\ &= 11/9 \end{aligned}$$

b) $E[X] = \int_X x f(x)\,dx$, $f(x) = ax^2$, $a = \frac{1}{3}$

$$\Rightarrow \tfrac{1}{3}\int_{-1}^{2} x^3 dx = \tfrac{1}{3}\left[\tfrac{x^4}{4}\right]_{-1}^{2} = \tfrac{5}{4}$$

$$Var[X] = E[X^2] - E[X]^2 = \int_{-1}^{2} x^2 f(x)\,dx - (\tfrac{5}{4})^2$$

$$= \tfrac{1}{3}\int_{-1}^{2} x^4 dx - \tfrac{25}{16} = \tfrac{1}{3}\left[\tfrac{x^5}{5}\right]_{-1}^{2} - \tfrac{25}{16} = 51/80$$

c) The probability that $y$ is in a small interval $[\bar{y}, \bar{y} + \Delta y]$.
$(\Delta y)$

$$\Rightarrow Pr(y \in [\bar{y}, \bar{y}+\Delta y]) = \int_{\bar{y}}^{\bar{y}+\Delta y} f_y(y)\,dy \cong f_y(\bar{y})\Delta y \text{ as } \Delta y \to 0.$$

$\rightarrow$ w/ $\bar{x} = g(\bar{y})$, using Taylor's : $g(\bar{y}+\Delta y) \cong g(\bar{y}) + \frac{dg}{dy}(\bar{y})\Delta y = \bar{x} + \Delta x$, $\Delta x < 0$.

$\rightarrow$ $\Delta x < 0$ b/c $g(y)$ = dec.

$\rightarrow$ $y \in [\bar{y}, \bar{y}+\Delta y] \Rightarrow g(y) \in [g(\bar{y}), g(\bar{y}+\Delta y)]$

$$\Rightarrow x \in [\bar{x}+\Delta x, \bar{x}]$$

$\rightarrow$ if $\Delta y$ & $\Delta x$ are small: $x \in [\bar{x}+\Delta x, \bar{x}] = y \in [\bar{y}, \bar{y}+\Delta y]$

$$\Rightarrow Pr(y \in [\bar{y}, \bar{y}+\Delta y]) = f_y(\bar{y})\Delta y$$

$$= Pr(x \in [\bar{x}+\Delta x, \bar{x}]) = -f_x(\bar{x})\Delta x$$

$$\Rightarrow f_y(\bar{y})\Delta y = -f_x(\bar{x})\Delta x \quad \Rightarrow f_x(\bar{x}) = f_y(\bar{y}) \Big/ -\left(\frac{\Delta x}{\Delta y}\right)$$

As $\Delta y \to 0$ & $\Delta x \to 0$ : $\frac{\Delta x}{\Delta y} \to \frac{dg(\bar{y})}{dy}$

$$\therefore \quad f_x(\bar{x}) = f_y(\bar{y}) \Big/ -\left(\frac{dg(\bar{y})}{dy}\right) \quad \Rightarrow f_x(x) = f_y(y) \Big/ -\frac{dg(y)}{dy}$$

$\to$ Change of variables formula.

<u>Problem 8.2</u>

a) $\to$ Change of variables: $f_y(y) = \frac{f_x(x)}{\left(\frac{dg(x)}{dx}\right)}$

$\to$ $g$ = continuously differentiable, strictly monotonically inc. func.

$$\Rightarrow y = g(x) \Rightarrow dy = \frac{dg(x)}{dx}dx$$

$\to$ W/ $\mathcal{Y} = \{ y \mid \exists x \in \mathcal{X} : y = g(x) \}$

$$\Rightarrow E[y] = \int_{\mathcal{Y}} y f_y(y)\,dy = \int_{\mathcal{X}} g(x)\left(\frac{f_x(x)}{\cancel{\left(\frac{dg(x)}{dx}\right)}}\right)\left(\cancel{\frac{dg(x)}{dx}}\,dx\right)$$

$$\Rightarrow E[y] = \int_{\mathcal{X}} g(x) f(x)\,dx$$

b) W/ Law of the unconscious statistician, vector random variable $\begin{bmatrix} x \\ y \end{bmatrix}$ w/ PDF $f_{xy}(x,y)$:

$$\Rightarrow E[g(x)h(y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y) f_{xy}(x,y)\,dx\,dy$$

→ Since x and y are independent:

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\, h(y)\, f_x(x)\, f_y(y)\, dx\, dy$$

$$\Rightarrow \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx \int_{-\infty}^{\infty} h(y)\, f_y(y)\, dy = E[g(x)]\, E[h(y)]$$

→ Similarly for discrete random variables:

$$\Rightarrow \sum_{x\in X}\sum_{y\in Y} g(x)\, h(y)\, f_{xy}(x,y) \;\ldots\; \underbrace{\sum_{x\in X} g(x)\, f_x(x) \sum_{y\in Y} h(y)\, f_y(y)}$$

c) w/ $\bar{x} = E[x]$

$$Var[x] = \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\, dx = \int_{-\infty}^{\infty} (x^2 - 2x\bar{x} + \bar{x})\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - 2\bar{x}\int_{-\infty}^{\infty} x f(x)\, dx + \bar{x}^2 \int_{-\infty}^{\infty} f(x)\, dx$$

$$Var[x] = E[x^2] - 2\bar{x}^2 + \bar{x}^2 = E[x^2] - \bar{x}^2$$

→ $Var[x] \geq 0 \quad \therefore E[x^2] = Var[x] + \bar{x}^2 \geq \bar{x}^2$

↳ will be equality if $Var[x] = 0$.

Problem 8.3

$$Pr(5 < x < 15) = Pr(|x-10| < 5)$$

$$\Rightarrow 1 - Pr(|x-10| \geq 5)$$

→ Chebyshev's inequality: $Pr(|x-\mu| \geq a) \leq \dfrac{\sigma^2}{a^2}$

w/ $\mu = 10$, $\sigma^2 = 15$

$$\Rightarrow Pr(|x-10| \geq 5) \leq \frac{15}{25} = \frac{3}{5}$$

$$\Rightarrow Pr(5 < x < 15) = 1 - Pr(|x-10| \geq 5) = 1 - \frac{3}{5} = \frac{2}{5}$$

## Problem 8.4

→ 2 random variables: ① box selected $x \in \{1, 2\}$.

→ $f_x(1) = f_x(2) = 1/2$

② black or white marble $y \in \{b, w\}$.

→ $f_{y|x}(b|1) = f_{y|x}(w|1) = 1/2$

→ $f_{y|x}(b|2) = \frac{2}{3}$

→ $f_{y|x}(w|2) = \frac{1}{3}$

→ w/ total probability theorem:

$$f_y(b) = f_{y|x}(b|1) f_x(1) + f_{y|x}(b|2) f_x(2)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right) = 7/12$$

## Problem 8.5

a) → Integration of PDF = 1

$$\Rightarrow \int_{-\infty}^{\infty} f_{n_1}(\bar{n}_1)\, d\bar{n}_1 = \alpha_1\left(\int_{-1}^{0}(1+\bar{n}_1)\, d\bar{n}_1 + \int_0^1 (1-\bar{n}_1)\, d\bar{n}_1\right)$$

$$= \alpha_1\left(1 - \tfrac{1}{2} + 1 - \tfrac{1}{2}\right) = 1 \quad \Rightarrow \boxed{\alpha_1 = 1}$$

$$\Rightarrow \int_{-\infty}^{\infty} f_{n_2}(\bar{n}_2)\, d\bar{n}_2 = \alpha_2\left(\int_{-2}^{0}(1+\tfrac{1}{2}\bar{n}_2)\, dn_2 + \int_0^2 (1-\tfrac{1}{2}\bar{n}_2)\, d\bar{n}_2\right)$$

$$= \alpha_2 (2 - 1 + 2 - 1) = 1 \quad \Rightarrow \boxed{\alpha_2 = \tfrac{1}{2}}$$

b) → $x$ = uniformly distributed $\Rightarrow f_x(x) = \begin{cases} \frac{1}{10} & \text{for } -5 \le x \le 5 \\ 0 & \text{otherwise.} \end{cases}$

→ b/c r.v.'s are independent:

$$f_{z_1, z_2|x}(z_1, z_2 | x) = f_{z_1|x}(z_1|x)\, f_{z_2|x}(z_2|x) \quad *$$

$\rightarrow z_1 - x = n_1, \quad z_2 - x = n_2$ :

$$\Rightarrow f_{z_1|x}(z_1|x) = \begin{cases} \alpha_1 (1 - z_1 + x) & \text{for } 0 \le z_1 - x \le 1 \\ \alpha_1 (1 + z_1 - x) & \text{for } -1 \le z_1 - x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow f_{z_2|x}(z_2|x) = \begin{cases} \alpha_2 (1 - \frac{1}{2} z_2 + \frac{1}{2} x) & \text{for } 0 \le z_2 - x \le 2 \\ \alpha_2 (1 + \frac{1}{2} z_1 - \frac{1}{2} x) & \text{for } -2 \le z_2 - x \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\rightarrow$ W/ Bayes' Theorem :

$$f_{x|z_1,z_2}(x|z_1,z_2) = \frac{\overbrace{f_{z_1|x}(z_1|x)\, f_{z_2|x}(z_2|x)\, f_x(x)}^{f_{z_1,z_2|x}(z_1,z_2|x) \text{ w/ } \circledast}}{\circledast \left( \int_{-5}^{5} f_{z_1|x}(z_1|\bar{x})\, f_{z_2|x}(z_2|\bar{x})\, f_x(\bar{x})\, d\bar{x} \right)}$$

normalization
$n(z_1, z_2)$

$\rightarrow$ W/ $z_1 = 0$, $z_2 = 0$ :

$\circledast \Rightarrow \frac{1}{10} f_{z_1|x}(0|x)\, f_{z_2|x}(0|x) \Rightarrow nor(x)$.

$\rightarrow$ W/ $x \in [-5, -1]$ : $nor(x) = 0$.

$\rightarrow$ W/ $x \in [-1, 0]$ : $nor(x) = \frac{1}{10} \alpha_1 (1 + x) \alpha_2 (1 + \frac{x}{2}) = \frac{1}{20}(1 + x)(1 + \frac{x}{2})$

$\rightarrow$ W/ $x \in [0, 1]$ : $nor(x) = \frac{1}{10} \alpha_1 (1 - x) \alpha_2 (1 - \frac{x}{2}) = \frac{1}{20}(1 - x)(1 - \frac{x}{2})$

$\rightarrow$ W/ $x \in [1, 5]$ : $nor(x) = 0$.

$$\Rightarrow f_{x|z_1,z_2}(x|0,0) = \frac{1}{n(0,0)} nor(x)$$

$$\rightarrow n(0,0) = \int_{-1}^{1} nor(x)\,dx = \frac{1}{20} \left[ \int_{-1}^{0} (1 + x)(1 + \frac{x}{2})\,dx + \int_{0}^{1} (1 - x)(1 - \frac{x}{2})\,dx \right]$$

$$= \frac{1}{20} \left( \frac{5}{12} + \frac{5}{12} \right) = \frac{1}{24}$$

$$\therefore f_{X|Z_1,Z_2}(x|0,0) = \begin{cases} 0 & \text{for } -5 \le x \le -1,\ 1 \le x \le 5 \\ {}^{12}/_{10}\,(1+x)(1+\frac{x}{2}) & \text{for } -1 \le x \le 0 \\ {}^{12}/_{10}\,(1-x)(1-\frac{x}{2}) & \text{for } 0 \le x \le 1. \end{cases}$$

c) W/ $z_1 = 0$, $z_2 = 1$:

$$nor(x) = \frac{1}{10} f_{z_1|x}(0|x)\, f_{z_2|x}(1|x)$$

$\rightarrow$ W/ ① $x \in [-5,-1]$ ② $x \in [1,5]$ $\Rightarrow$ $nor(x) = 0$.

$\rightarrow$ W/ $x \in [-1, 0]$:

$$nor(x) = \frac{1}{10}\alpha_1(1+x)\alpha_2\left(\frac{1}{2}+\frac{x}{2}\right) = \frac{1}{40}(1+x)^2$$

$\rightarrow$ W/ $x \in [0,1]$:

$$nor(x) = \frac{1}{10}\alpha_1(1-x)\alpha_2\left(\frac{1}{2}+\frac{x}{2}\right) = \frac{1}{40}(1-x^2)$$

$$\rightarrow n(0,1) = \int_{-1}^{1} nor(x)\,dx$$

$$= \frac{1}{40}\left[\int_{-1}^{0}(1+x)^2\,dx + \int_{0}^{1}(1-x^2)\,dx\right]$$

$$= \frac{1}{40}\left(\frac{1}{3}+\frac{2}{3}\right) = \frac{1}{40}$$

$$\therefore f_{X|Z_1,Z_2}(x|0,1) = \begin{cases} 0 & \text{for } -5 \le x \le -1,\ 1 \le x \le 5 \\ (1+x)^2 & \text{for } -1 \le x \le 0 \\ 1-x^2 & \text{for } 0 \le x \le 1. \end{cases}$$

d) W/ $z_1 = 0$, $z_2 = 3$:

$$nor(x) = \frac{1}{10} f_{z_1|x}(0|x)\, f_{z_2|x}(3|x)$$

$\to$ w/ $x \in [-5, 5] \Rightarrow nor(x) = 0$.

$\to$ this is b/c the intervals of pos. probability of $f_{Z_1|X}(0|x)$ and $f_{Z_2|X}(3|x)$ do NOT overlap.

$\therefore f_{X|Z_1,Z_2}(x|0,3) = 0$.