ROB301 - Problem Set 7

Problem 7.1

a)

$$f(x,y) = \frac{2}{11}(x+y)^2$$

$$f_X(x) = \int_{y\in Y} f(x,y)\,dy = \int_{y\in Y} \frac{2}{11}(x^2+2xy+y^2)\,dy$$

$$= \frac{2}{11}\left(x^2y + xy^2 + \frac{y^3}{3}\right)\Big]_0^1 = \frac{2}{11}\left(x^2+x+\frac{1}{3}\right)$$

$$f_Y(y) = \int_{x\in X} \frac{2}{11}(x^2+2xy+y^2)\,dx = \frac{2}{11}\left(\frac{x^3}{3} + x^2y + y^2x\right)\Big]_{-1}^{2}$$

$$= \frac{2}{11}\left(\frac{8}{3} + 4y + 2y^2 + \frac{1}{3} - y + y^2\right) = \frac{6}{11}(1+y+y^2)$$

b)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f(y)} = \frac{\cancel{2}^{1}}{\cancel{11}}\cdot\frac{\cancel{11}}{\cancel{6}_{3}}\,\frac{(x+y)^2}{(1+y+y^2)} = \frac{(x+y)^2}{3(1+y+y^2)}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)} = \frac{\cancel{2}}{\cancel{11}}\cdot\frac{\cancel{11}}{2}\,\frac{(x+y)^2}{(x^2+x+\frac{1}{3})} = \frac{3(x+y)^2}{(3x^2+3x+1)}$$

c)

$$Pr(x>1 \mid y=0.5) = \int_1^2 f(x|y=0.5) = \int_1^2 \frac{4}{21}\left(x+\frac{1}{2}\right)^2$$

$$= \frac{4}{21}\left[\left(2+\frac{1}{2}\right)^2 - \left(1+\frac{1}{2}\right)^2\right] = 7/9$$

Problem 7.2

① $f(x|y) = f(x) \iff f(y|x) = f(x)$

$\rightarrow$ ② $f(x|y,z) = f(x|z) \iff f(x,y|z) = f(x|z)f(y|z)$

Def^n of conditional probability: $f(x|y) = \frac{f(x,y)}{f(y)}$, $f(y|x) = \frac{f(x,y)}{f(x)}$

$$\Rightarrow f(x,y) = f(x|y)f(y)$$
$$= f(y|x)f(x)$$

$$\Rightarrow f(x|y)f(y) = f(y|x)f(x)$$

$\rightarrow$ $f(x) \neq 0$ & $f(y) \neq 0$ else $f(x|y)$ & $f(y|x)$ = undefined.

$f(x|y) = f(x) \Rightarrow f(x)f(y) = f(y|x)f(x)$

$\Rightarrow f(y) = f(y|x) \quad (\text{for } f(x) \neq 0)$

$f(y|x) = f(y) \Rightarrow f(y)f(x) = f(x|y)f(y)$

$\Rightarrow f(x) = f(x|y) \quad (\text{for } f(y) \neq 0)$

$\therefore f(x|y) = f(x) \Leftrightarrow f(y|x) = f(y)$

$f(x, y|z) = f(x|y, z)f(y|z)$

$f(x|y, z) = f(x|z) \Rightarrow f(x, y|z) = f(x|z)f(y|z)$

$f(x, y|z) = f(x|z)f(y|z) \Rightarrow f(x|y, z)f(y|z) = f(x|z)f(y|z)$

$\Rightarrow f(x|y, z) = f(x|z)$

$\rightarrow$ b/c $f(y|z) \neq 0$ $\rightarrow$ ($f(x|y, z)$ will be undefined otherwise).

<u>Problem 7.3</u>

a) $\rightarrow$ continuous random variables: $x, y, z \Rightarrow x, y, z \in [0, 1]$

$\rightarrow$ joint PDF: $f(x, y, z) = C g_x(x, z) g_y(y, z)$, C = const

g_x, g_y = funcs.

$\Rightarrow$ let C be a const. s.t. : $\int_0^1 \int_0^1 \int_0^1 f(x, y, z)\,dx\,dy\,dz$

$$\Rightarrow f(x, y|z) = \frac{f(x, y, z)}{f(z)} = \frac{\cancel{C} g_x(x, z) g_y(y, z)}{\cancel{C} G_x(z) G_y(z)} \quad *$$

$\rightarrow$ w/ $G_x(z) = \int_0^1 g_x(x, z)$

$G_y(z) = \int_0^1 g_y(y, z)$

$$\Rightarrow f(x|z) = \frac{f(x,z)}{f(z)} = \frac{\int_0^1 f(x,y,z)\,dy}{f(z)} = \frac{\cancel{c}\, g_x(x,z)\,\cancel{G_y(z)}}{\cancel{c}\, G_x(z)\,\cancel{G_y(z)}}$$

$$= \frac{g_x(x,z)}{G_x(z)}$$

$$\Rightarrow f(y|z) = \frac{f(y,z)}{f(z)} = \frac{\int_0^1 f(x,y,z)\,dx}{f(z)} = \frac{\cancel{c}\, g_y(y,z)\,\cancel{G_x(z)}}{\cancel{c}\,\cancel{G_x(z)}\, G_y(z)}$$

$$= \frac{g_y(y,z)}{G_y(z)}$$

$\rightarrow$ Plug in ⊛ :

$\Rightarrow f(x,y|z) = f(x|z)\,f(y|z)$ $\quad \therefore$ x & y = conditionally ind.

$\rightarrow$ if $g_x(x,z) = x+z$ & $g_y(y,z) = y+z$

$$\Rightarrow^{✡} f(x,y) = \int_0^1 f(x,y,z)\,dz = c\int_0^1 (x+z)(y+z)\,dz$$

$$= c\int_0^1 xy + xz + yz + z^2\,dz = c\left[xyz + \frac{(x+y)z^2}{2} + \frac{z^3}{3}\right]_0^1$$

$$= c\left[xy + \frac{x+y}{2} + \frac{1}{3}\right]$$

$$\Rightarrow f(x) = \int_0^1 f(x,y)\,dy = c\int_0^1 \left(xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{3}\right) dy$$

$$= c\left[\frac{xy^2}{2} + \frac{xy}{2} + \frac{y^2}{4} + \frac{y}{3}\right]_0^1 = c\left[x + \frac{7}{12}\right]$$

$\Rightarrow f(y) \rightarrow$ similarly $\Rightarrow c\left[y + \frac{7}{12}\right]$

$$\Rightarrow f(x)f(y) = c^2\left[xy + \frac{7x}{12} + \frac{7y}{12} + \frac{49}{144}\right]$$

$$\rightarrow \text{w/ ✡}\quad f(x,y) = c\left[xy + \frac{x+y}{2} + \frac{1}{3}\right]$$

$\neq$ (not equal!)

$\therefore$ x & y $\neq$ ind.

b) Similarly in part (a): $f(x,y,z) = C g_x(x,z) g_y(y,z)$

which: $f(x,y|z) = f(x|z) f(y|z)$

$\to$ $x, y, z \in \{0, 1\}$ $\to$ discrete random variables.

$\to$ definitions of g_x & g_y:

x	z	$g_x(x,z)$
0	0	2
0	1	0
1	0	0
1	1	2

y	z	$g_y(y,z)$
0	0	0
0	1	2
1	0	2
1	1	0

$\to C = \frac{1}{2}$

x	y	z	$C g_x(x,z) g_y(y,z)$
0	0	0	0
0	0	1	0
0	1	0	2
0	1	1	0
1	0	0	0
1	0	1	2
1	1	0	0
1	1	1	0

$\circledast$

x	y	$f(x,y)$
0	0	0
0	1	2
1	0	2
1	1	0

x	$f(x)$
0	2
1	2

y	$f(y)$
0	2
1	2

$\circledast$

x	y	$f(x) f(y)$
0	0	4
0	1	4
1	0	4
1	1	4

$\therefore$ $f(x,y) \neq f(x) f(y)$

Problem 7.4

a) $\to$ either lose (\$0) or win (\$1000): $S_1 \in \{0, 1000\}$

$\to$ $Pr(S_1 = 1000) = \frac{1}{100} \Rightarrow Pr(S_1 = 0) = 1 - \frac{1}{100} = 0.99$

$\Rightarrow E(S_1) = 1000\left(\frac{1}{100}\right) + 0(0.99) = 10$ $\to$ same amount as a ticket.

∴ Since the ticket has another chance to win (after the expense for the ticket is covered by the expected profit by the first chance to win), the probabilistically correct answer is to join the game.

→ Assume the ticket drawn in first round is put back into the basket:

→ Similarly: $E(S_2) = 10000000 \left(\frac{1}{1000000}\right) + 0\left(1 - \frac{1}{1000000}\right)$
$= 10.$

→ the expected profit of a ticket is double of its expense (... if you buy enough :P)

b) → the person can answer correct / incorrect from a probabilistic point of view: $S_1 \in \{\text{correct, incorrect}\}$.
$\Rightarrow S_1 \in \{c, i\}$

→ the person could have attended ROB310 or not:
$S_2 \in \{\text{attended, not attended}\}$
$\Rightarrow S_2 \in \{a, n\}$.

$Pr(S_1 = i) = f_{S_1}(i) = \frac{80000}{100000} = 0.8$, $f_{S_1}(c) = 1 - 0.8 = 0.2$.

$f_{S_2}(a) = \frac{40}{100000} = 0.0004$, $f_{S_1|S_2}(c|a) = 1$

→ $f_{S_2|S_1}(a|c) = \frac{f_{S_1|S_2}(c|a)\, f_{S_2}(a)}{f_{S_1}(c)} = \frac{40}{20000} = 0.002.$

→ The probability of finding someone that took ROB 310 in everyone that answered correctly is 0.2%.