ROB310

Problem Set 6

Problem 6.1:

a)

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 0 & 0 & \sqrt{3} \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 3, \ \lambda_2 = 2, \ \lambda_3 = 1$$

$$\sigma_1 = \sqrt{3}, \ \sigma_2 = \sqrt{2}, \ \sigma_3 = 1$$

$$\Rightarrow \underline{\Sigma} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{b/c } A^T A = \text{diagonal})$$

$$\Rightarrow \underline{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{I}$$

$$\Rightarrow \underline{A} \underline{v}_i = \sigma_i \underline{u}_i \quad \Rightarrow \underline{U} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\underline{A} = \underline{U} \underline{\Sigma} \underline{V}^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} -5 \\ 3 \end{bmatrix} \qquad A^T A = \begin{bmatrix} -5 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \end{bmatrix} = 34$$

$$= \lambda_1 = 34 \quad \Rightarrow \quad \sigma_1 = \sqrt{34}$$

$$\Rightarrow \quad \Sigma = \begin{bmatrix} \sqrt{34} \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \underline{v}_1 = 1 \quad \Rightarrow \quad \underline{V} = 1$$

$$\Rightarrow \quad \underline{A}\underline{v} = \sigma_1 u_1 \quad \Rightarrow \quad \underline{A} \cdot (1) = (\sqrt{34})\, u_1 \quad \Rightarrow \quad \underline{u}_1 = \frac{1}{\sqrt{34}} \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

orthogonal to $\underline{u}_1$ $\Rightarrow$ $\underline{u}_2 = \frac{1}{\sqrt{34}} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$\therefore \underline{A} = \underline{U}\, \underline{\Sigma}\, \underline{V}^T = \frac{1}{\sqrt{34}} \begin{bmatrix} -5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \sqrt{34} \\ 0 \end{bmatrix} \cdot (1)$$

Problem 6.2:

a) $$\underline{M} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{2\times 1} \underbrace{\begin{bmatrix} 3 & 4 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} 7 \\ 71 \end{bmatrix}}_{2\times 1} \Rightarrow 2\times 1 \text{ matrix} \quad \therefore \text{rank } \underline{1}.$$

b)

$$\underline{A} = \begin{bmatrix} 2 & -2 \\ 2 & 4 \\ -1 & 4 \end{bmatrix}, \qquad \max_{\substack{\underline{x} \in \mathbb{R}^n \setminus \{0\} \\ \|x\| = 1}} \|\underline{A}\underline{x}\|_2 = \|A\|_2 = \max\{\sqrt{\lambda} : \text{there exists } \underline{x} \in \mathbb{R}^n \text{ w/ } \underline{A}^T \underline{A}\underline{x} = \lambda \underline{x}\}$$

$$\underline{A}^T \underline{A} = \begin{bmatrix} 2 & 2 & -1 \\ -2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 2 & 4 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 36 \end{bmatrix} \Rightarrow \max(\sqrt{\lambda}) = \sqrt{36} = 6$$

$\rightarrow$ The maximum value is 6.

c)

$$\|\underline{A}\|_2 = \max_{\substack{\underline{x} \in \mathbb{R}^n \setminus \{0\} \\ \|\underline{x}\| = 1}} \|\underline{A}\underline{x}\|_2 \quad (\ast\ast), \qquad \underline{A} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T$$

$$\Rightarrow \underline{A}\underline{x} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T \underline{x} \Rightarrow \|\underline{A}\underline{x}\|_2^2 = \underline{x}^T \underline{V}\,\underline{\Sigma}^T \underline{U}^T \underline{U}\,\underline{\Sigma}\,\underline{V}^T \underline{x} \Rightarrow \underline{U}^T\underline{U} = \underline{I}$$

$$\Rightarrow \underline{x}^T \underline{V}\,\underline{\Sigma}^T \underline{\Sigma}\,\underline{V}^T \underline{x} \quad (\ast)$$

$\Rightarrow$ w/ $\underline{y} = \underline{V}^T \underline{x}$ & $\underline{y}^T = \underline{x}^T \underline{V}$

$$(\ast) = \underline{y}^T \underline{\Sigma}^T \underline{\Sigma}\,\underline{y} \Rightarrow \|\underline{\Sigma}\,\underline{y}\|_2^2$$

w/ $(\ast\ast)$

$$\Rightarrow \|\underline{A}\|_2 = \max_{\substack{\underline{x} \in \mathbb{R}^n \setminus \{0\} \\ \|\underline{y}\| = 1}} \|\underline{\Sigma}\,\underline{y}\|_2 = \|\underline{\Sigma}\|_2$$

$$\therefore \|\underline{A}\|_2 = \|\underline{\Sigma}\|_2 \quad \boxtimes.$$

d) SVD for $\underline{B}$ (pos. definite & symmetric):

$\underline{B}$ = pos. definite $\Rightarrow \overset{\ast}{(}\underline{B}\underline{v} = \lambda \underline{v}) \Rightarrow \lambda$ = positive.

$$\underline{B}^T \underline{B}\,\underline{v} = \underline{B}\,\underline{B}\,\underline{v} = \underline{B}^2 \underline{v} = \underline{B}\lambda \underline{v} = \lambda \underline{B}\underline{v} \overset{\ast}{=} \lambda\lambda \underline{v} = \lambda^2 \underline{v}$$

↳ $B^T = B$ b/c $\underline{B}$ = symmetric

→ The singular values = eigenvalues of $\underline{B}$

$\Rightarrow$ The eigenvectors of $\underline{B}^T\underline{B}$ are the same as of $\underline{B}$.

$$\underline{B}\underline{v}_i = \sigma_i \underline{u}_i = \lambda_i \underline{u}_i \Rightarrow \underline{B} = \lambda_i \quad \therefore \underline{v}_i = \underline{u}_i$$

$\Rightarrow$ $\underline{v}_i$ = orthogonal.

$$\Rightarrow \underline{B} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T = \underline{V}\,\underline{\Sigma}\,\underline{V}^T$$

$\underline{\Sigma}$ = square matrix w/ eigenvalues of $\underline{B}$ on the diagonal entries.

$\underline{V}$ = a matrix of corresponding eigenvectors of $\underline{B}$ normalized.

=> SVD of $\underline{B}$ = eigenvalue problem of $\underline{B}$

-> for $\underline{B}$ = a pos. definite & symmetric matrix.

Problem 6.3 :

-> SVD application : Recommender systems

=> used as a collaborative filtering algorithm. (CF)

-> user-item rating matrix for CF alg. :

① row = a user

② column = an item

③ entries = ratings given by users to items

-> used to find relevant correlations in data

-> used to reduce the # of features of data set (b/c reduces space dimensions)

-> reduces computation time.

Web page link:
https://medium.com/@m_n_malaeb/singular-value-decomposition-svd-in-recommender-systems-for-non-math-statistics-programming-4a622de653e9

[1]M. Malaeb, "Singular Value decomposition (SVD) in recommender systems for Non-math-statistics-programming…", Medium, 2019. [Online]. Available: https://medium.com/@m_n_malaeb/singular-value-decomposition-svd-in-recommender-systems-for-non-math-statistics-programming-4a622de653e9. [Accessed: 10- Nov- 2019].