Take-home Exam Assignment
02619 Model Predictive Control
Individual Problem
Autumn 2018

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You must hand in a report (Building 303B Office 110) no later than 9:00 January 7, 2019. You must also upload an electronic version (pdf) to CampusNet. This submission should also include a zip-file with all Matlab files that you have used as well as original Word / Latex files. The **short** report that you hand in must be done individually.

1 General problem description

Consider the system

$$Z(s) = G(s)U(s) + H(s)W(s), \tag{1.1a}$$
$$Y(s) = Z(s) + V(s), \tag{1.1b}$$

where $Z(s)$ is the output (CV), $U(s)$ is the manipulated variables (MV), and $Y(s)$ is the measurement. $W(s)$ and $V(s)$ are the process and measurement noise. The transfer functions are

$$G(s) = \frac{K(\beta s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} e^{-\theta s}, \tag{1.2a}$$
$$H(s) = \frac{K_w}{\tau_w s + 1}. \tag{1.2b}$$

Let $W(s) \sim N_{iid}(0, 1)$ and $V(s) \sim N_{iid}(0, \sigma_v^2)$. The parameters are: $K = 1$, $\tau_1 = \tau_2 = 5$, $\beta = 2$, $\theta = 5$, $K_w = 0.1$, $\tau_w = 10$, $\sigma_v = 0.1$.

2 Discrete-time state-space realization

Provide Matlab code for computation of the discrete-time linear state-space model

$$x_{k+1} = Ax_k + Bu_k + Gw_k \tag{2.1a}$$
$$z_k = Cx_k \tag{2.1b}$$
$$y_k = z_k + v_k \tag{2.1c}$$

with ZOH parametrization of the input, u, and a sample time of $T_s = 2.0$. Report the matrices (numerical values) for A, B, G, C, R_{ww}, R_{wv} and R_{vv}.

3 Stationary Kalman filter

Provide equations and Matlab code for computation of the stationary Kalman filter gains, K_{fx} and K_{fw}. You must also report the numerical values for K_{fx} and K_{fw}. Provide Matlab code for a Kalman filter that computes $\hat{x}_{k|k}$ and $\hat{w}_{k|k}$, based on the measurement and the Kalman filter gains (among other things).

4 State space condensing

The outputs based on the linear state space model and the Kalman filter may be predicted by the linear equation

$$Z_k = b_k + \Gamma U_k, \quad b_k = \Phi_x \hat{x}_{k|k} + \Phi_w \hat{w}_{k|k}. \tag{4.1}$$

Derive this equation and provide expressions for Γ, Φ_x and Φ_w.

Implement a function

```
function [Gamma, Phix, Phiw] = CondenseStateSpaceModel(A,B,G,C,N)
```

in Matlab that can compute Γ, Φ_x and Φ_w. Provide the Matlab code and the numerical results for $N = 10$.

5 MPC design (input constraints)

Consider the linear-quadratic input constrained optimization problem

$$\begin{aligned}
\min_{z,x,u} \quad & \phi = \frac{1}{2}\sum_{j=0}^{N-1} \left\|\hat{z}_{k+1+j|k} - \bar{z}_{k+1+j|k}\right\|_{Q_z}^2 + \left\|\Delta\hat{u}_{k+j|k}\right\|_{Q_{\Delta u}}^2 && \text{(5.1a)}\\
s.t. \quad & \hat{x}_{k+1|k} = A\hat{x}_{k|k} + B\hat{u}_{k|k} + G\hat{w}_{k|k} && \text{(5.1b)}\\
& \hat{x}_{k+1+j|k} = A\hat{x}_{k+j|k} + B\hat{u}_{k+j|k}, \quad j = 1, 2, \dots, N-1 && \text{(5.1c)}\\
& \hat{z}_{k+j|k} = C\hat{x}_{k+j|k}, \quad j = 1, 2, \dots, N && \text{(5.1d)}\\
& u_{\min} \le \hat{u}_{k+j|k} \le u_{\max}, \quad j = 0, 1, \dots, N-1 && \text{(5.1e)}\\
& \Delta u_{\min} \le \Delta\hat{u}_{k+j|k} \le \Delta u_{\max}, \quad j = 0, 1, \dots, N-1 && \text{(5.1f)}
\end{aligned}$$

that represents the regulator in the MPC. Let $Q_z = 1.0$ and $Q_{\Delta u} = 0.01$.

Demonstrate that this problem can be expressed as a convex quadratic program in the form

$$\begin{aligned}
\min_{x} \quad & \frac{1}{2}x'Hx + g_k'x && \text{(5.2a)}\\
s.t. \quad & l \le x \le u && \text{(5.2b)}\\
& b_{l,k} \le Ax \le b_{u,k} && \text{(5.2c)}
\end{aligned}$$

where

$$\begin{aligned}
g_k &= M_b b_k + M_z \bar{Z}_k + M_u \hat{u}_{k-1|k} && \text{(5.3a)}\\
b_{l,k} &= \Delta U_{\min} + I_0 \hat{u}_{k-1|k} && \text{(5.3b)}\\
b_{u,k} &= \Delta U_{\max} + I_0 \hat{u}_{k-1|k} && \text{(5.3c)}
\end{aligned}$$

Provide expressions for H, M_b, M_z, M_u, l, u, x, $\Delta U_{\min}$, $\Delta U_{\max}$, and I_0.

Implement the Matlab function

```
function [Hqp,Mbqp,Mzqp,Muqp,lqp,uqp,Aqp,IOqp,dUminqp,dUmaxqp] = ...
       MPCDesign(A,B,G,C,Qz,Qdu,umin,umax,dumin,dumax,N)
```

Report the matrices for $N = 10$.

6 MPC compute

Implement af function

```
function [uk,Uk,Zk,xhat] = MPCCompute(yk,ukm1,Zbar,xhat,...
                               Hqp,Mbqp,Mzqp,Muqp,...
                               lqp,uqp,Aqp,IOqp,dUminqp,dUmaxqp)
```

that computes $\hat{u}_{k|k}$, U_k, Z_k and $\hat{x}_{k+1|k}$. It must consist of 1) a Kalman filter, 2) a constrained regulator, and 3) a one-step predictor.

Discuss and illustrate your code for your reasonable choice of N.

7 Closed-loop simulation

Implement a Matlab function

```
function [U,Z,Zpred,Upred] = ClosedLoopMPC(...
                                 um1, Zbar, xhat, ...
                                 W, V, ...
                                 A,B,G,C,...
                                 Hqp,Mbqp,Mzqp,Muqp,...
                                 lqp,uqp,Aqp,IOqp,dUminqp,dUmaxqp)
```

that can simulate the linear system and the MPC in closed-loop. (U, Z) are the closed-loop sequence $\{u_k, z_k\}$, while (Zpred,Upred) are the corresponding predictions by the MPC.

Report the Matlab code and a simulation demonstrating your MPC. You should also provide Matlab code that can produce a movie of (Zpred,Upred) for a reasonable simulation scenario of your choice.