Nonlinear Dyn (2013) 73:1367–1383
DOI 10.1007/s11071-013-0870-3

ORIGINAL PAPER

Design of sampled data state estimator for Markovian jumping neural networks with leakage time-varying delays and discontinuous Lyapunov functional approach

R. Rakkiyappan · Quanxin Zhu · T. Radhika

Received: 15 December 2012 / Accepted: 9 March 2013 / Published online: 29 March 2013

Abstract This paper is concerned with the sampled-data state estimation problem for neural networks with both Markovian jumping parameters and leakage time-varying delays. Instead of the continuous measurement, the sampled measurement is used to estimate the neuron states, and a sampled-data estimator is constructed. In order to make full use of the sawtooth structure characteristic of the sampling input delay, a discontinuous Lyapunov functional is proposed based on the extended Wirtinger inequality. A less conservative delay dependent stability criterion is derived via constructing a new triple-integral Lyapunov–Krasovskii functional and the famous Jenson integral inequality. Based on the Lyapunov–Krasovskii functional approach, a state estimator of the considered neural networks has been achieved by solving some linear matrix inequalities, which can be easily facilitated by using the standard numerical software. Finally, two numerical examples are provided to show the effectiveness of the proposed methods.

R. Rakkiyappan · T. Radhika
Department of Mathematics, Bharathiar University, Coimbatore 641046, Tamilnadu, India

R. Rakkiyappan
e-mail: rakkigru@gmail.com

Q. Zhu (✉)
School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, Jiangsu, China
e-mail: zqx22@126.com

Keywords Sampled data · state estimator · neural network · Lyapunov–Krasovskii functional · leakage time-varying delay

1 Introduction

Neural networks have been more and more popular due to their extensive applications, such as image processing, pattern recognition, associative memory, and optimization problems [1–4]. Recently, a lot of results have been reported on various aspects of neural networks; see, e.g., [5–7]. In such applications, it is of prime importance to ensure that the designed neural network is stable. On the other hand, time-delays are inevitable in the applications of neural networks due to the finite switching speed of amplifiers and communication time [8, 9]. Moreover, in some particular applications, time delays are deliberately introduced. For example, to process moving images, one must introduce time-delays in the signals transmitted among the cells [8]. It has been found that the existence of time delays may lead to instability and oscillation in a neural network. Therefore, the stability analysis of neural networks with time delays has received much attention; see, e.g., [10–16], and the references therein.

State estimation is of great significance to estimate the neuron states through available output measurements of the networks, and then utilizes the estimated neuron states to achieve certain design objectives. As a consequence, the state estimation problem for neural networks has been a hot reach topic that has drawn considerable attention, and a large number of results have been available in the literature [17–27]. The sampled-data control theory had attracted much attention due to its effectiveness in engineering applications. Especially, a new approach to dealing with the sampled-data control problem had been proposed in [28], where the sampling period had been converted into the time-varying delay. As its extension, the authors in [29] investigated the sampled-data state estimation problem for a class of recurrent neural networks, with time-varying delays, where the sampled measurements had been used to estimate the neuron states.

In order to effectively deal with the sampled-data control, the authors in [30, 31] introduced a concept that discontinuous sampled control inputs treat time-varying delayed continuous signals although applied actual control signals are discontinuous. Owing to the works of [30, 31], many types of the sampled-data control scheme by using the concept [30, 31] have been proposed. It is worth mentioning that in [28, 32], a new approach to dealing with the sample-data control problem has been proposed by converting the sampling period into a time-varying but bounded delay. The main purpose is to estimate the neuron states through the output sampled measurement, which can lead to a significant reduction of the information communication burden in the network. In many applications, such as signal processing, control engineering, for large-scale neural networks, it is quite common that only partial information can be accessible from the network outputs [33]. Therefore, it is of great significance to estimate the neuron states through available output measurements of the networks. Due to the pioneering work [33], the state estimation problem for neural networks has received increasing research attention. Therefore, it is of both theoretical and practical importance to study the sampled-data state estimation problem of neural networks.

The Zero–Order–Hold (ZOH) is a mathematical model of the practical signal reconstruction done by a conventional Digital-to-Analog Converter (DAC). It literally holds the digital signal for the sample time, then moves to the next digital sample and holds that signal for the sample time as well, in order to reconstruct the analogue signal. To improve the intersampling performance, hybrid system models with both continuous and discrete-time signals are generally built via a ZOH, which are also named as sampled-data systems [34]. In such systems, the control signals are kept constant during the sampling period and cannot be changed to deal with the nonlinearity of the plant [35]. This remarkable characteristic makes the analysis and design more difficult and complex.

Recently the impulsive model approach has been extended to the case of variable sampling with a known upper bound [34] and to Networked Control Systems (NCSs) [36], where a discontinuous Lyapunov function method was introduced. This method improved the existing time-independent Lyapunov-based results and it inspired a piecewise-continuous (in time) Lyapunov functional approach to sampled-data systems [37]. Extensions of the above discontinuous Lyapunov constructions to NCSs lead to complicated conditions [36, 38]. Moreover, these conditions become conservative if the network induced delay is not small. A novel discontinuous Lyapunov functional is introduced, which is based on the application of the Wirtinger type inequality [39]. Being applied to sampled-data systems, the new method recovers the conditions in [40], and thus is more conservative than that in [41]. However, the new method leads to efficient sufficient conditions for the positive values of network-induced delays. The new stability analysis is applied to the problem of sampled-data stabilization by using the artificial delay [42].

Markovian jump systems, introduced by Krasovskii and Lidskii [43] in 1961, have received increasing attention in the area of control and operations research fields. The neural networks may have finite modes and the mode may switch from one to another at different times and it is shown that the switching between different neural network modes can be governed by a Markov chain. Furthermore, Markov chains have also been widely used as a generic framework for modeling gene networks. A finite state homogeneous Markov chain model has been constructed from microarray data in [44]. It is suggested that Markov chain models incorporating rule-based transitions between states are capable of mimicking biological phenomena. Recently, the authors in [20, 45] have proposed the problem of state estimation for neural networks with Markovian jumping parameters. However, to the best of the authors knowledge, the design of sampled-data state estimator for Markovian jumping neural networks with leakage time-varying delays and discontinuous Lyapunov functional approach has not been fully investigated, and such a situation motivates our current research.

Recently, Gopalsamy [46] pointed out that in real nervous systems, time delay in the stabilizing negative feedback terms has a tendency to destabilize a system (this kind of time delays are known as leakage delays or "forgetting" delays). Moreover, sometimes it has more significant effect on dynamics of neural networks than other kinds of delay. Hence, it is of significant importance to consider the leakage delay effects on dynamics of neural networks. There are very little existing works have been studied for neural networks with leakage delays by the few methods such as M-matrix theory and LMIs. The LMI method is a powerful tool for validating the theoretical results and for finding less conservative results when the upper bound of the leakage delay is very small. However, it is worth noting that in those existing results, the leakage delay considered in the negative feedback terms is usually a constant. The most interesting research on systems with leakage time-varying delay is a central issue and few of the researchers have taken into this and derived several sufficient stability conditions, see for example [47]. However, to the best of the authors knowledge, the design of sampled-data state estimator for Markovian jumping neural networks with leakage time-varying delays and discontinuous Lyapunov functional approach has not been fully investigated, and such a situation motivates our current research.

In this paper, we aim to investigate the sampled-data state estimator for Markovian jumping neural networks with leakage time-varying delays and discontinuous Lyapunov functional approach. By utilizing Lyapunov–Krasovskii functional, we recast the state estimator problem into a numerically solvable problem. By employing a new Lyapunov–Krasovskii functional, Jenson's integral inequalities of the discrete time-varying delays are obtained in terms of LMIs and can be easily solved by using the Matlab LMI toolbox [48]. Finally, two numerical examples are provided to show the usefulness of the proposed stability conditions.

Notations Let $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and the superscript "T" denotes the transpose of a matrix or vector. I denotes the identity matrix with compatible dimensions. For square matrices M_1 and M_2, the notation $M_1 > (\geq, <, \leq)\, M_2$ denotes $M_1 - M_2$ is a positive-definite (positive-semi-definite, negative, negative-semi-definite) matrix. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with a natural filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ and $\mathbf{E}[\cdot]$ stand for the correspondent expectation operator with respect to the given probability measure P. Also, let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of continuously differentiable function ϕ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the uniform norm $\|\phi\|_\tau = \max\{\max_{-\tau\leq\theta\leq 0}|\phi(\theta)|, \max_{-\tau\leq\theta\leq 0}|\phi'(\theta)|\}$.

2 Problem description and preliminaries

Let $\{r(t), t \geq 0\}$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $\mathcal{S} = \{1, 2, \dots, N\}$ with generator $Q = (q_{ij})_{N\times N}$ given by

$$P\big\{r(t+\Delta t) = j | r(t) = i\big\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + q_{ii}\Delta t + o(\Delta t), & i = j, \end{cases}$$

where $\Delta t > 0$ and $\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t} = 0$, $q_{ij} \geq 0$ is the transition rate from i to j, if $i \neq j$ while $q_{ii} = -\sum_{j\neq i} q_{ij}$.

Consider the following neural networks with leakage time-varying delays:

$$\begin{aligned}\dot{x}(t) = {} & -A\big(r(t)\big)x\big(t-\sigma(t)\big) + W_1\big(r(t)\big)g\big(x(t)\big) \\ & + W_2\big(r(t)\big)g\big(x\big(t-\tau(t)\big)\big) + J(t),\end{aligned} \tag{1}$$

where $x(.) = [x_1(.), x_2(.), \dots, x_n(.)]^T \in \mathbb{R}^n$ is neuron state vector; $y(.)$ is the output of the networks; $A = \operatorname{diag}\{a_1, \dots, a_n\} > 0$ is a diagonal matrix with positive entries $a_i > 0$; the matrices W_1 and W_2 represent the connection weight matrix and the delayed connection weight matrix, respectively; $g(x(.)) = [g_1(x_1(.)), \dots, g_n(x_n(.))]^T$ denotes the neuron activation function; $J(t) = [J_1(t), \dots, J_n(t)]^T$ is an external input vector; $\tau(t)$ and $\sigma(t)$ denote time-varying delays and time-varying leakage delays, respectively, and they satisfy

$$\begin{aligned} & 0 \leq \sigma(t) \leq \sigma, \qquad \dot{\sigma}(t) \leq \sigma_\mu, \\ & 0 \leq \tau(t) \leq \tau, \qquad \dot{\tau}(t) \leq \tau_\mu, \end{aligned}$$

where σ, τ, σ_μ, and τ_μ are known constants. Additional, we always assume that each neuron activation function $g_i(.)$ satisfies the following condition:

$$(H_1):\quad l_i^- \le \frac{g_i(x)-g_i(y)}{x-y} \le l_i^+,$$
$$\forall x, y \in \mathbb{R},\ x \ne y,\ i = 1, 2, \dots, n,$$

where l_i^-, l_i^+ are known constant scalars. The network measurement is expressed by

$$y(t) = C\big(r(t)\big)x(t), \tag{2}$$

where $y(t) \in \mathbb{R}^m$ is the measurement output and $C \in \mathbb{R}^{m\times n}$ is a known constant matrix with appropriate dimensions. In this paper, the measurement output is sampled before it enters the estimator. Denote by t_k the updating instant time of the Zero–Order–Hold (ZOH), and suppose that the updating signal at the instant t_k has experienced a constant signal at the instant t_k has experienced a constant signal transmission delay η. It is assumed that the sampling intervals satisfy

$$t_{k+1} - t_k = h_k \le h, \quad k \ge 0, \tag{3}$$

where h is a positive scalar and represents the largest sampling interval. It is clear that

$$t_{k+1} - t_k + \eta \le h + \eta \le \rho, \quad k \ge 0. \tag{4}$$

Therefore, the network measurement (2) has a form $y(t_k) = C(r(t))x(t_k - \eta)$. Considering the behavior the ZOH, we have

$$y(t) = C\big(r(t)\big)x(t_k - \eta), \quad t_k \le t < t_{k+1}. \tag{5}$$

Based on the available sampled measurement (5), the following full-order state estimation for the delayed neural networks (1) is designed:

$$\begin{aligned}\dot{\widehat{x}}(t) &= -A\big(r(t)\big)\widehat{x}\big(t-\sigma(t)\big) + W_1\big(r(t)\big)g\big(\widehat{x}(t)\big)\\ &\quad + W_2\big(r(t)\big)g\big(\widehat{x}\big(t-\tau(t)\big)\big) + J(t)\\ &\quad + K\big(r(t)\big)\big(C\big(r(t)\big)x(t_k-\eta)\\ &\quad - C\big(r(t)\big)\widehat{x}(t_k-\eta)\big),\end{aligned} \tag{6}$$

where $\widehat{x}(t)$ is the estimation of the neuron state $x(t)$, and $K \in \mathbb{R}^{n\times m}$ is the gain matrix of the estimator to be designed later. Defining $d(t) = t - t_k + \eta$, $t_k \le t \le t_{k+1}$. Therefore, the full-order state estimation of the delayed neural networks (1) can be written as

$$\begin{aligned}\dot{\widehat{x}}(t) &= -A\big(r(t)\big)\widehat{x}\big(t-\sigma(t)\big) + W_1\big(r(t)\big)g\big(\widehat{x}(t)\big)\\ &\quad + W_2\big(r(t)\big)g\big(\widehat{x}\big(t-\tau(t)\big)\big) + J(t)\\ &\quad + K\big(r(t)\big)\big(C\big(r(t)\big)x\big(t-d(t)\big)\\ &\quad - C\big(r(t)\big)\widehat{x}\big(t-d(t)\big)\big), \quad t_k \le t < t_{k+1}.\end{aligned} \tag{7}$$

Define the error vector by $e(t) = x(t) - \widehat{x}(t)$. Then the error dynamics can be directly obtained from (1) and (7),

$$\begin{aligned}\dot{e}(t) &= -A\big(r(t)\big)e\big(t-\sigma(t)\big)\\ &\quad - K\big(r(t)\big)C\big(r(t)\big)e\big(t-d(t)\big)\\ &\quad + W_1\big(r(t)\big)\phi(t) + W_2\big(r(t)\big)\phi\big(t-\tau(t)\big),\end{aligned} \tag{8}$$

or its equivalent form

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\bigg[e(t) - A\big(r(t)\big)\int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s\bigg]\\ &\quad = -A\big(r(t)\big)e(t) - A\big(r(t)\big)e\big(t-\sigma(t)\big)\dot{\sigma}(t)\\ &\qquad - K\big(r(t)\big)C\big(r(t)\big)e\big(t-d(t)\big) + W_1\big(r(t)\big)\Phi(t)\\ &\qquad - W_2\big(r(t)\big)\Phi\big(t-\tau(t)\big),\end{aligned} \tag{9}$$

where $\phi(t) = g(x(t)) - g(\widehat{x}(t))$ and it follows from (4) that $\eta \le \tau(t) < t_{k+1} - t_k + \eta \le \rho$ and $\dot{d}(t) = 1$ for $t \ne t_k$. From assumption H_1, for any $y \in \mathbb{R}$, $y \ne 0$, the function $g_i(t)$ satisfies

$$l_i^- \le \frac{g_i(y)}{y} \le l_i^+, \quad g_i(0) = 0,\ i = 1, 2, \dots, n. \tag{10}$$

For simplicity, let

$$\begin{aligned}&L^- = \operatorname{diag}\big(l_1^-, l_2^-, \dots, l_n^-\big),\\ &L^+ = \operatorname{diag}\big(l_1^+, l_2^+, \dots, l_n^+\big), \quad i = 1, 2. \dots, n,\\ &L = \operatorname{diag}(l_1, l_2, \dots, l_n), \quad \text{where}\\ &\quad l_i = \begin{cases}\min(|l_i^-|, |l_i^+|): & l_i^- \times l_i^+ > 0,\\ 0: & l_i^- \times l_i^+ \le 0.\end{cases}\end{aligned}$$

Lemma 1 (Jensen's inequality) *For any constant matrix $M \in \mathbb{R}^{n\times n}$, a scalar $\gamma > 0$, a vector function $\omega : [0, \gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then*

$$\begin{aligned}&\gamma \int_0^{\gamma} \omega^T(s) M \omega(s)\,\mathrm{d}s\\ &\quad \ge \bigg(\int_0^{\gamma} \omega(s)\,\mathrm{d}s\bigg)^T M \bigg(\int_0^{\gamma} \omega(s)\,\mathrm{d}s\bigg).\end{aligned}$$

Lemma 2 *For any positive definite symmetric constant matrix Q and scalar $\tau > 0$, such that the following integrations are well defined, we have*

$$
\begin{aligned}
&-\int_{-\tau}^{0}\int_{t+\theta}^{t} x^T(s)Qx(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
&\quad\le -\frac{2}{\tau^2}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} x(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)^T \\
&\qquad\times Q\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} x(s)\,\mathrm{d}s\,\mathrm{d}\theta\right).
\end{aligned}
$$

Lemma 3 *For any positive definite symmetric constant matrix Q and scalar $\tau > 0$, such that the following integrations are well defined, we have*

$$
\begin{aligned}
&-\int_{t-\tau}^{t} e^T(s)Qe(s)\,\mathrm{d}s \\
&\quad\le -\frac{2}{\tau^3}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} e(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)^T \\
&\qquad\times Q\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} e(s)\,\mathrm{d}s\,\mathrm{d}\theta\right).
\end{aligned}
$$

Lemma 4 (Schur complement) *Given constant matrix Ω_1, Ω_2 and Ω_3 with appropriate dimensions, where $\Omega_1^T = \Omega_1$ and $\Omega_2^T = \Omega_2 > 0$, then*

$$\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ * & -\Omega_2 \end{bmatrix} < 0, \quad \textit{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{bmatrix} < 0.$$

For the sake of convenience, denote $A(r(t)=i) = A_i$, $C(r(t)=i) = C_i$, $W_1(r(t)=i) = W_{1i}$, $W_2(r(t)=i) = W_{2i}$, respectively.

3 Main results

In this section, we will give our main results in terms of LMIs. For the sake of presentation simplicity, we denote

$$H_1 = \mathrm{diag}\{H_1^- H_1^+, H_2^- H_2^+, \ldots, H_n^- H_n^+\},$$

$$H_2 = \mathrm{diag}\left\{\frac{H_1^- + H_1^+}{2}, \frac{H_2^- + H_2^+}{2}, \ldots, \frac{H_n^- + H_n^+}{2}\right\}.$$

Theorem 3.1 *For given scalars $L^- = \mathrm{diag}(l_1^-, l_2^-, \ldots, l_n^-)$, $L^+ = \mathrm{diag}(l_1^+, l_2^+, \ldots, l_n^+)$, assume that there exist positive diagonal matrices $M = \mathrm{diag}\{m_1, m_2, \ldots, m_n\}$, $\Lambda_1 = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\Lambda_2 = \mathrm{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, symmetric positive definite matrices R_1, R_2, R_3, R_4, R_5, P_i $(i = 1, 2, \ldots, N)$, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, S_1, S_2, S_3, S_4, D, U_1, U_2, and an arbitrary matrix $F = [0, 0, F_1, 0, F_2, F_3, 0, 0, 0, F_4, 0, 0, 0, 0, 0, 0, 0]$, $n \times n$ real matrices K and G, $n \times n$ diagonal matrices $\Gamma_1 > 0$ and $\Gamma_2 > 0$ such that the following LMIs hold:*

$$\mathcal{Y}_i = \begin{bmatrix} \Xi_1 & F^T \\ F & -(\frac{2}{\tau^2}Q_5 + \frac{2}{\sigma^2}Q_6) \end{bmatrix} < 0, \tag{11}$$

where

$$\Xi_1 = (\Xi_{j,k})_{17\times 17},$$

with

$$
\begin{aligned}
\Xi_{1,1} &= -P_iA_i - A_i^TP_i + R_1 + R_3 + R_4 \\
&\quad + \tau^2\Lambda_2(L^+ - L^-) + 2\tau Q_1 + 2\sigma Q_2 + \sigma^2 R_5 \\
&\quad + \frac{\tau^2}{2}Q_3 + \frac{\sigma^2}{2}Q_4 + \frac{d^2}{2}Q_7 + S_1 - S_3 - U_1 \\
&\quad - U_2 - H_1\Gamma_1 - H_1\Gamma_2 + \sum_{j=1}^{N} q_{ij}P_j,
\end{aligned}
$$

$$\Xi_{1,2} = P_i - L^-\Lambda_1 + L^+M,$$

$$\Xi_{1,3} = (1-\sigma_\mu)P_iA_i - \tau F_1 - \sigma F_1,$$

$$\Xi_{1,4} = H_2\Gamma_1, \qquad \Xi_{1,5} = -\tau F_2 - \sigma F_2 + U_1,$$

$$\Xi_{1,6} = -\tau F_3 - \sigma F_3, \qquad \Xi_{1,7} = U_2, \qquad \Xi_{1,8} = S_3,$$

$$\Xi_{1,10} = -\tau F_4 - \sigma F_4,$$

$$\Xi_{1,11} = A_iP_iA_i^T - \sum_{j=1}^{N} q_{ij}P_jA_j, \qquad \Xi_{1,17} = H_2\Gamma_2,$$

$$
\begin{aligned}
\Xi_{2,2} &= \frac{\tau^2}{2}Q_5 + \frac{\sigma^2}{2}Q_6 + \eta^2 S_3 + (\rho-\eta)^2 S_4 \\
&\quad + (\rho-\eta)^2 D + \tau^2 U_1 + \sigma^2 U_2 - P_i - P_i^T,
\end{aligned}
$$

$$\Xi_{2,3} = -P_iA_i, \qquad \Xi_{2,4} = \Lambda_1 - M + P_iW_{1i},$$

$$\Xi_{2,6} = P_iW_{2i}, \qquad \Xi_{2,11} = -P_iA_i,$$

$$\Xi_{2,17} = -G_iC_i, \qquad \Xi_{3,3} = -(1-\sigma_\mu)R_3,$$

$$\Xi_{3,11} = -(1-\sigma_\mu)A_iP_iA_i^T, \qquad \Xi_{3,12} = F_1,$$

$$\Xi_{3,13}=F_1,\qquad \Xi_{4,4}=R_2-\Gamma_1,$$
$$\Xi_{5,5}=-(1-\tau_\mu)R_1-\frac{\pi^2}{4}D-U_1,$$
$$\Xi_{5,8}=\frac{\pi^2}{4}D,\qquad \Xi_{5,12}=F_2,\qquad \Xi_{5,13}=F_2,$$
$$\Xi_{6,6}=-(1-\tau_\mu)R_2,\qquad \Xi_{6,12}=F_3,$$
$$\Xi_{6,13}=F_3,\qquad \Xi_{7,7}=-R_4-U_2,$$
$$\Xi_{8,8}=-S_1+S_2-S_3-S_4-\frac{\pi^2}{4}D,$$
$$\Xi_{8,9}=S_4,\qquad \Xi_{9,9}=-S_2-S_4,$$
$$\Xi_{10,10}=-(\rho-\eta)^2(1-\tau_\mu)D,$$
$$\Xi_{10,12}=F_4,\qquad \Xi_{10,13}=F_4,$$
$$\Xi_{11,11}=\sum_{j=1}^{N}q_{ij}A_j^TP_jA_j-R_5,$$
$$\Xi_{12,12}=-\frac{1}{\tau}Q_1,\qquad \Xi_{13,13}=-\frac{1}{\sigma}Q_2,$$
$$\Xi_{14,14}=-\frac{4}{\tau^2}\Lambda_2\big(L^+-L^-\big)-\frac{2}{\tau^3}Q_1-\frac{2}{\tau^2}Q_3,$$
$$\Xi_{15,15}=-\frac{2}{\sigma^3}Q_2-\frac{2}{\sigma^2}Q_4,\qquad \Xi_{16,16}=-\frac{2}{d^2}Q_7,$$
$$\Xi_{17,17}=-\Gamma_2\quad \textit{and other entries of } \Xi \textit{ is } 0.$$

Then the error system (8) is asymptotically stable in the mean square. Moreover, the gain matrix K_i of the state estimator (7) is given by $K_i=P_i^{-1}G_i$.

Proof In order to establish the stability conditions, we introduce the following Lyapunov–Krasovskii functional $V(e(t),t,r(t)=i):=V(e(t),t,i)$ is given by

$$\begin{aligned}V\big(e(t),t,i\big)&=V_1\big(e(t),t,i\big)+V_2\big(e(t),t,i\big)\\&\quad+V_3\big(e(t),t,i\big)+V_4\big(e(t),t,i\big)\\&\quad+V_5\big(e(t),t,i\big)+V_6\big(e(t),t,i\big)\\&\quad+V_7\big(e(t),t,i\big),\end{aligned}\tag{12}$$

where

$$\begin{aligned}&V_1\big(e(t),t,i\big)\\&=\Big[e(t)-A_i\int_{t-\sigma(t)}^{t}e(s)\,\mathrm{d}s\Big]^T\\&\quad\times P_i\Big[e(t)-A_i\int_{t-\sigma(t)}^{t}e(s)\,\mathrm{d}s\Big]\\&\quad+2\sum_{i=1}^{n}\Big\{\int_0^{e_i(t)}\lambda_i\big(\Phi_i(s)-l_i^-(s)\big)\,\mathrm{d}s\\&\quad+\int_0^{e_i(t)}m_i\big(l_i^+(s)-\Phi_i(s)\big)\,\mathrm{d}s\Big\}\\&\quad+\int_{t-\tau(t)}^{t}e^T(s)R_1e(s)\,\mathrm{d}s\\&\quad+\int_{t-\tau(t)}^{t}\Phi^T\big(e(s)\big)R_2\Phi\big(e(s)\big)\,\mathrm{d}s\\&\quad+\int_{t-\sigma(t)}^{t}e^T(s)R_3e(s)\,\mathrm{d}s+\int_{t-\sigma}^{t}e^T(s)R_4e(s)\,\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}&V_2\big(e(t),t,i\big)\\&=2\sum_{i=1}^{n}\Big\{\int_{-\tau}^{0}\int_{\theta}^{0}\int_{t+\mu}^{t}\alpha_ie_i(s)\big[\Phi_i\big(e_i(s)\big)\\&\quad-l_i^-\big(e_i(s)\big)\big]\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\Big\}\\&\quad+2\sum_{i=1}^{n}\Big\{\int_{-\tau}^{0}\int_{\theta}^{0}\int_{t+\mu}^{t}\alpha_ie_i(s)\big[l_i^+\big(e_i(s)\big)\\&\quad-\Phi_i\big(e_i(s)\big)\big]\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\Big\}\\&\quad+2\int_{-\tau}^{0}\int_{t+\theta}^{t}e^T(s)Q_1e(s)\,\mathrm{d}s\,\mathrm{d}\theta\\&\quad+2\int_{-\sigma}^{0}\int_{t+\theta}^{t}e^T(s)Q_2e(s)\,\mathrm{d}s\,\mathrm{d}\theta\\&\quad+\sigma\int_{t-\sigma}^{t}\int_{\theta}^{t}e^T(s)R_5e(s)\,\mathrm{d}s\,\mathrm{d}\theta,\end{aligned}$$

$$\begin{aligned}V_3\big(e(t),t,i\big)&=\int_{-\tau}^{0}\int_{\theta}^{0}\int_{t+\mu}^{t}e^T(s)Q_3e(s)\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\\&\quad+\int_{-\sigma}^{0}\int_{\theta}^{0}\int_{t+\mu}^{t}e^T(s)Q_4e(s)\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\\&\quad+\int_{-d}^{0}\int_{\theta}^{0}\int_{t+\mu}^{t}e^T(s)Q_7e(s)\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta,\end{aligned}$$

$$\begin{aligned}V_4\big(e(t),t,i\big)&=\int_{-\tau}^{0}\int_{\theta}^{0}\int_{t+\mu}^{t}\dot e^T(s)Q_5\dot e(s)\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\\&\quad+\int_{-\sigma}^{0}\int_{\theta}^{0}\int_{t+\mu}^{t}\dot e^T(s)Q_6\dot e(s)\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta,\end{aligned}$$

$$\begin{aligned} V_5\big(e(t),t,i\big) &= \int_{t-\eta}^{t} e^T(s)S_1e(s)\,\mathrm{d}s + \int_{t-\rho}^{t-\eta} e^T(s)S_2e(s)\,\mathrm{d}s \\ &\quad + \eta\int_{-\eta}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)S_3\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ &\quad + (\rho-\eta)\int_{-\rho}^{-\eta}\int_{t+\theta}^{t} \dot{e}^T(s)S_4\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta, \end{aligned}$$

$$\begin{aligned} V_6\big(e(t),t,i\big) &= (\rho-\eta)^2\int_{t_k}^{t} \dot{e}^T(s)D\dot{e}(s)\,\mathrm{d}s \\ &\quad - \frac{\pi^2}{4}\int_{t_k-\eta}^{t-\eta}\big(e(s)-e(t_k-\eta)\big)^T D\big(e(s) \\ &\quad - e(t_k-\eta)\big)\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned} V_7\big(e(t),t,i\big) = \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)U_1\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ + \sigma\int_{-\sigma}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)U_2\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta. \end{aligned}$$

Define infinitesimal generator (denoted by $\mathcal{L}$) of the Markov process acting on $V(e(t),t,i)$ as follows:

$$\begin{aligned} &\mathcal{L}V\big(e(t),t,i\big) \\ &:= \lim_{\Delta\to 0^+}\frac{1}{\Delta}\sup\big[\mathbb{E}\big\{V\big(e(t+\Delta),t+\Delta, \\ &\qquad r(t+\Delta)\big)|e(t),r(t)=i\big\} \\ &\qquad - V\big(e(t),t,r(t)=i\big)\big]. \end{aligned}$$

Then, a direct computation yields

$$\begin{aligned} &\mathcal{L}V_1\big(e(t),t,i\big) \\ &= 2\bigg[e(t)-A_i\int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s\bigg]^T \\ &\quad \times P_i\big[\dot{e}(t)-A_ie(t)+(1-\sigma_\mu)A_ie\big(t-\sigma(t)\big)\big] \\ &\quad + \sum_{j=1}^{N} q_{ij}\bigg[e(t)-A_j\int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s\bigg]^T \\ &\quad \times P_j\bigg[e(t)-A_j\int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s\bigg] \\ &\quad + 2\big[\Phi^T\big(e(t)\big)-e^T(t)L^-\big]\Lambda_1\dot{e}(t) \\ &\quad + 2\big[e^T(t)L^+-\Phi^T\big(e(t)\big)\big]M\dot{e}(t) \\ &\quad + e^T(t)R_1e(t)-(1-\tau_\mu)e^T\big(t-\tau(t)\big) \\ &\quad \times R_1e\big(t-\tau(t)\big) \\ &\quad + \Phi^T\big(e(t)\big)R_2\Phi\big(e(t)\big) \\ &\quad - (1-\tau_\mu)\Phi^T\big(e\big(t-\tau(t)\big)\big)R_2\Phi\big(e\big(t-\tau(t)\big)\big) \\ &\quad + e^T(t)R_3e(t)-(1-\sigma_\mu)e^T\big(t-\sigma(t)\big) \\ &\quad \times R_3e\big(t-\sigma(t)\big) \\ &\quad + e^T(t)R_4e(t)-e^T(t-\sigma)R_4e(t-\sigma), \end{aligned} \tag{13}$$

$$\begin{aligned} \mathcal{L}V_2\big(e(t),t,i\big) &= 2\sum_{i=1}^{n}\bigg\{\frac{\alpha_i\tau^2}{2}e_i^T(t)\big[l_i^+-l_i^-\big]e_i(t) \\ &\quad - \int_{-\tau}^{0}\int_{t+\theta}^{t}\alpha_ie_i(s)\big[l_i^+-l_i^-\big]e_i(s)\,\mathrm{d}s\,\mathrm{d}\theta\bigg\} \\ &\quad + 2\tau e^T(t)Q_1e(t) - 2\int_{t-\tau}^{t} e^T(s)Q_1e(s)\,\mathrm{d}s \\ &\quad + 2\sigma e^T(t)Q_2e(t) - 2\int_{t-\sigma}^{t} e^T(s)Q_2e(s)\,\mathrm{d}s \\ &\quad + \sigma^2e^T(t)R_5e(t) \\ &\quad - \bigg(\int_{t-\sigma(t)}^{t} e^T(s)\,\mathrm{d}s\bigg)R_5\bigg(\int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s\bigg). \end{aligned} \tag{14}$$

Notice that

$$\begin{aligned} &2\sum_{i=1}^{n}\bigg\{\frac{\alpha_i\tau^2}{2}e_i^T(t)\big[l_i^+-l_i^-\big]e_i(t)\bigg\} \\ &\quad = \tau^2e^T(t)\Lambda_2\big[L^+-L^-\big]e(t). \end{aligned} \tag{15}$$

It follows from Lemma 2 that

$$\begin{aligned} &2\sum_{i=1}^{n}\bigg\{-\int_{-\tau}^{0}\int_{t+\theta}^{t}\alpha_ie_i(s)\big[l_i^+-l_i^-\big]e_i(s)\,\mathrm{d}s\,\mathrm{d}\theta\bigg\} \\ &\quad = -2\int_{-\tau}^{0}\int_{t+\theta}^{t} e^T(s)\Lambda_2\big[L^+-L^-\big]e(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ &\quad \le -\frac{4}{\tau^2}\bigg(\int_{-\tau}^{0}\int_{t+\theta}^{t} e^T(s)\,\mathrm{d}s\,\mathrm{d}\theta\bigg)\Lambda_2\big[L^+-L^-\big] \\ &\qquad \times\bigg(\int_{-\tau}^{0}\int_{t+\theta}^{t} e(s)\,\mathrm{d}s\,\mathrm{d}\theta\bigg). \end{aligned} \tag{16}$$

By Lemmas 2 and 3, we have

$$
\begin{aligned}
-2\int_{t-\tau}^{t} e^T(s)Q_1e(s)\,\mathrm{d}s
&\leq -\frac{2}{\tau^3}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} e^T(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\quad\times Q_1\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} e(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\quad-\frac{1}{\tau}\left(\int_{t-\tau}^{t} e^T(s)\,\mathrm{d}s\right)Q_1\left(\int_{t-\tau}^{t} e(s)\,\mathrm{d}s\right),
\end{aligned}\tag{17}
$$

$$
\begin{aligned}
-2\int_{t-\sigma}^{t} e^T(s)Q_2e(s)\,\mathrm{d}s
&\leq -\frac{2}{\sigma^3}\left(\int_{-\sigma}^{0}\int_{t+\theta}^{t} e^T(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\quad\times Q_2\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} e(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\quad-\frac{1}{\sigma}\left(\int_{t-\sigma}^{t} e^T(s)\,\mathrm{d}s\right)Q_1\left(\int_{t-\sigma}^{t} e(s)\,\mathrm{d}s\right).
\end{aligned}\tag{18}
$$

Thus, we obtain

$$
\begin{aligned}
&\mathcal{L}V_2\big(e(t),t,i\big)\\
&\quad\leq e^T(t)\big[\tau^2\Lambda_2\big(L^+-L^-\big)+2\tau Q_1\\
&\qquad+2\sigma Q_2+\sigma^2R_5\big]e(t)\\
&\qquad-\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} e^T(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\qquad\times\left[\frac{4}{\tau^2}\Lambda_2\big(L^+-L^-\big)+\frac{2}{\tau^3}Q_1\right]\\
&\qquad\times\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} e(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\qquad-\frac{1}{\tau}\left(\int_{t-\tau}^{t} e^T(s)\,\mathrm{d}s\right)Q_1\left(\int_{t-\tau}^{t} e(s)\,\mathrm{d}s\right)\\
&\qquad-\left(\int_{-\sigma}^{0}\int_{t+\theta}^{t} e^T(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\qquad\times\left[\frac{2}{\sigma^3}Q_2\right]\left(\int_{-\sigma}^{0}\int_{t+\theta}^{t} e(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\qquad-\frac{1}{\sigma}\left(\int_{t-\sigma}^{t} e^T(s)\,\mathrm{d}s\right)Q_2\left(\int_{t-\sigma}^{t} e(s)\,\mathrm{d}s\right)\\
&\qquad-\left(\int_{t-\sigma(t)}^{t} e^T(s)\,\mathrm{d}s\right)R_5\left(\int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s\right).
\end{aligned}\tag{19}
$$

Similarly, we have

$$
\begin{aligned}
&\mathcal{L}V_3\big(e(t),t,i\big)\\
&\quad\leq\frac{\tau^2}{2}e^T(t)Q_3e(t)-\frac{2}{\tau^2}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} e^T(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\qquad\times Q_3\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} e(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\qquad+\frac{\sigma^2}{2}e^T(t)Q_4e(t)-\frac{2}{\sigma^2}\left(\int_{-\sigma}^{0}\int_{t+\theta}^{t} e^T(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\qquad\times Q_4\left(\int_{-\sigma}^{0}\int_{t+\theta}^{t} e(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\qquad+\frac{d^2}{2}e^T(t)Q_7e(t)-\frac{2}{d^2}\left(\int_{-d}^{0}\int_{t+\theta}^{t} e^T(s)\,\mathrm{d}s\,\mathrm{d}\theta\right)\\
&\qquad\times Q_7\left(\int_{-d}^{0}\int_{t+\theta}^{t} e(s)\,\mathrm{d}s\,\mathrm{d}\theta\right),
\end{aligned}\tag{20}
$$

$$
\begin{aligned}
&\mathcal{L}V_4\big(e(t),t,i\big)\\
&\quad\leq\frac{\tau^2}{2}\dot{e}^T(t)Q_5\dot{e}(t)-\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{e}^T(s)Q_5\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\qquad+\frac{\sigma^2}{2}\dot{e}^T(t)Q_6\dot{e}(t)\\
&\qquad-\int_{-\sigma}^{0}\int_{t+\theta}^{t}\dot{e}^T(s)Q_6\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta.
\end{aligned}\tag{21}
$$

On the other hand, for an arbitrary matrix $F=[0,0,F_1,0,F_2,F_3,0,0,0,F_4,0,0,0,0,0,0,0]$ with appropriate dimensions, let

$$
\begin{aligned}
\xi^T(t)=\Big[&e(t),\dot{e}(t),e\big(t-\sigma(t)\big),\Phi\big(e(t)\big),e\big(t-\tau(t)\big),\\
&\Phi\big(e\big(t-\tau(t)\big)\big),e(t-\sigma),e(t-\eta),e(t-\rho),\\
&\dot{e}\big(t-\tau(t)\big),\int_{t-\sigma(t)}^{t}e(s)\,\mathrm{d}s,\int_{t-\tau}^{t}e(s)\,\mathrm{d}s,\\
&\int_{t-\sigma}^{t}e(s)\,\mathrm{d}s,\int_{-\tau}^{0}\int_{t+\theta}^{t}e(s)\,\mathrm{d}s\,\mathrm{d}\theta,\\
&\int_{-\sigma}^{0}\int_{t+\theta}^{t}e(s)\,\mathrm{d}s\,\mathrm{d}\theta,\int_{-d}^{0}\int_{t+\theta}^{t}e(s)\,\mathrm{d}s\,\mathrm{d}\theta,\\
&e\big(t-d(t)\big)\Big].
\end{aligned}
$$

Then, we have

$$-\int_{t+\theta}^{t}\dot{e}^T(s)Q_5\dot{e}(s)\,\mathrm{d}s \le -2\xi^T(t)F\int_{t+\theta}^{t}\dot{e}(s)\,\mathrm{d}s + \int_{t+\theta}^{t}\xi^T(t)FQ_5^{-1}F\xi(t)\,\mathrm{d}s, \tag{22}$$

$$-\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{e}^T(s)Q_5\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta \le -2\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{e}^T(s)\,\mathrm{d}s\,\mathrm{d}\theta F\xi(t) + \frac{\tau^2}{2}\xi^T(t)F^TQ_5^{-1}F\xi(t), \tag{23}$$

$$-\int_{-\sigma}^{0}\int_{t+\theta}^{t}\dot{e}^T(s)Q_6\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta \le -2\int_{-\sigma}^{0}\int_{t+\theta}^{t}\dot{e}^T(s)\,\mathrm{d}s\,\mathrm{d}\theta F\xi(t) + \frac{\sigma^2}{2}\xi^T(t)F^TQ_6^{-1}F\xi(t). \tag{24}$$

Submitting (22)–(24) into (21), we have

$$\begin{aligned}\mathcal{L}V_4\big(e(t),t,i\big) &\le \frac{\tau^2}{2}\dot{e}^T(t)Q_5\dot{e}(t) + \frac{\sigma^2}{2}\dot{e}^T(t)Q_6\dot{e}(t) - 2\tau e^T(t)F\xi(t) + 2\int_{t-\tau}^{t}e^T(s)\,\mathrm{d}s\,F\xi(t) \\ &\quad + \frac{\tau^2}{2}\xi^T(t)F^TQ_5^{-1}F\xi(t) - 2\sigma e^T(t)F\xi(t) + 2\int_{t-\sigma}^{t}e^T(s)\,\mathrm{d}s\,F\xi(t) \\ &\quad + \frac{\sigma^2}{2}\xi^T(t)F^TQ_6^{-1}F\xi(t).\end{aligned} \tag{25}$$

Similarly, a direct computation yields

$$\begin{aligned}\mathcal{L}V_5\big(e(t),t,i\big) &\le e^T(t)S_1e(t) - e^T(t-\eta)S_1e(t-\eta) + e^T(t-\eta)S_2e(t-\eta) \\ &\quad - e^T(t-\rho)S_2e(t-\rho) + \eta^2\dot{e}^T(t)S_3\dot{e}(t) - \big[e(t)-e(t-\eta)\big]^TS_3\big[e(t)-e(t-\eta)\big] \\ &\quad + (\rho-\eta)^2\dot{e}^T(t)S_4\dot{e}(t) - \big[e(t-\eta)-e(t-\rho)\big]^T \times S_4\big[e(t-\eta)-e(t-\rho)\big],\end{aligned} \tag{26}$$

$$\begin{aligned}\mathcal{L}V_6\big(e(t),t,i\big) &= (\rho-\eta)^2\dot{e}^T(t)D\dot{e}(t) - (\rho-\eta)^2(1-\tau_\mu)\dot{e}^T\big(t-\tau(t)\big)D\dot{e}\big(t-\tau(t)\big) \\ &\quad - \frac{\pi^2}{4}e^T(t-\eta)De(t-\eta) + \frac{\pi^2}{4}e^T\big(t-\tau(t)\big)De(t-\eta) \\ &\quad + \frac{\pi^2}{4}e^T(t-\eta)De\big(t-\tau(t)\big) - \frac{\pi^2}{4}e^T\big(t-\tau(t)\big)De\big(t-\tau(t)\big),\end{aligned} \tag{27}$$

$$\begin{aligned}\mathcal{L}V_7\big(e(t),t,i\big) &= \tau^2\dot{e}^T(t)U_1\dot{e}(t) - \big[e(t)-e\big(t-\tau(t)\big)\big]^T \times U_1\big[e(t)-e\big(t-\tau(t)\big)\big] \\ &\quad + \sigma^2\dot{e}^T(t)U_2\dot{e}(t) - \big[e(t)-e(t-\sigma)\big]^T \times U_1\big[e(t)-e(t-\sigma)\big].\end{aligned} \tag{28}$$

Furthermore, we have

$$2\dot{e}^T(t)P_i\big[-\dot{e}(t) - A_ie\big(t-\sigma(t)\big) - K_iC_ie\big(t-d(t)\big) + W_{1i}\Phi\big(e(t)\big) + W_{2i}\Phi\big(e\big(t-\tau(t)\big)\big)\big] = 0. \tag{29}$$

It is well known that for any $n\times n$ diagonal matrix $\Gamma_1 > 0$, the following inequalities hold:

$$\begin{bmatrix} e(t) \\ \Phi(e(t)) \end{bmatrix}^T \begin{bmatrix} -H_1\Gamma_1 & H_2\Gamma_1 \\ * & -\Gamma_1 \end{bmatrix} \times \begin{bmatrix} e(t) \\ \Phi(e(t)) \end{bmatrix} \ge 0, \tag{30}$$

$$\begin{bmatrix} e(t) \\ e(t-d(t)) \end{bmatrix}^T \begin{bmatrix} -H_1\Gamma_2 & H_2\Gamma_2 \\ * & -\Gamma_2 \end{bmatrix} \times \begin{bmatrix} e(t) \\ e(t-d(t)) \end{bmatrix} \ge 0. \tag{31}$$

Adding the left hand side of (13)–(31) to $\mathcal{L}V(e(t), t, i)$, we get

$$\mathcal{L}V\big(e(t), t, i\big) \le \xi^T(t)\mathcal{Y}_i\xi(t). \tag{32}$$

Thus, it follows from (11) and the generalizes Itô's formula that

$$\begin{aligned}
&\mathbf{E}V\big(e(t), t, i\big) - \mathbf{E}V\big(e(0), 0, r(0)\big)\\
&\quad = \int_0^t \mathbf{E}\mathcal{L}V\big(e(s), s, r(s)\big)\,\mathrm{d}s \le 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathbf{E}V\big(e(t), t, i\big) \le \mathbf{E}V\big(e(0), 0, r(0)\big) < \infty,\\
&\quad t \ge 0,
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
&\mathbf{E}V\big(e(0), 0, r(0)\big)\\
&\quad = \mathbf{E}\bigg[e(0) - A\big(r(0)\big)\int_{-\sigma(0)}^0 e(s)\,\mathrm{d}s\bigg]^T\\
&\qquad \times P\big(r(0)\big)\bigg[e(0) - A\big(r(0)\big)\int_{-\sigma(0)}^0 e(s)\,\mathrm{d}s\bigg]\\
&\qquad + \int_{-\tau(0)}^0 \mathbf{E}e^T(s)R_1e(s)\,\mathrm{d}s\\
&\qquad + \int_{-\tau(0)}^0 \mathbf{E}\Phi^T\big(e(s)\big)R_2\Phi\big(e(s)\big)\,\mathrm{d}s\\
&\qquad + \int_{-\sigma(0)}^0 \mathbf{E}e^T(s)R_3e(s)\,\mathrm{d}s\\
&\qquad + \int_{-\sigma}^0 \mathbf{E}e^T(s)R_4e(s)\,\mathrm{d}s\\
&\qquad + 2\sum_{i=1}^n\bigg\{\int_{-\tau}^0\int_\theta^0\int_\mu^0 \mathbf{E}\alpha_ie_i(s)\big[\Phi_i\big(e_i(s)\big)\\
&\qquad - l_i^-\big(e_i(s)\big)\big]\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\bigg\}\\
&\qquad + 2\sum_{i=1}^n\bigg\{\int_{-\tau}^0\int_\theta^0\int_\mu^0 \mathbf{E}\alpha_ie_i(s)\big[l_i^+\big(e_i(s)\big)\\
&\qquad - \Phi_i\big(e_i(s)\big)\big]\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\bigg\}\\
&\qquad + 2\int_{-\tau}^0\int_\theta^0 \mathbf{E}e^T(s)Q_1e(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\qquad + 2\int_{-\sigma}^0\int_\theta^0 \mathbf{E}e^T(s)Q_2e(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\qquad + \sigma\int_{-\sigma}^0\int_\theta^0 \mathbf{E}e^T(s)R_5e(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\qquad + \int_{-\tau}^0\int_\theta^0\int_\mu^0 \mathbf{E}e^T(s)Q_3e(s)\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\\
&\qquad + \int_{-\sigma}^0\int_\theta^0\int_\mu^0 \mathbf{E}e^T(s)Q_4e(s)\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\\
&\qquad + \int_{-d}^0\int_\theta^0\int_\mu^0 \mathbf{E}e^T(s)Q_7e(s)\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\\
&\qquad + \int_{-\tau}^0\int_\theta^0\int_\mu^0 \mathbf{E}\dot{e}^T(s)Q_5\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\\
&\qquad + \int_{-\sigma}^0\int_\theta^0\int_\mu^0 \mathbf{E}\dot{e}^T(s)Q_6\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\mu\,\mathrm{d}\theta\\
&\qquad + \int_{-\eta}^0 \mathbf{E}e^T(s)S_1e(s)\,\mathrm{d}s + \int_{-\rho}^{-\eta} \mathbf{E}e^T(s)S_2e(s)\,\mathrm{d}s\\
&\qquad + \eta\int_{-\eta}^0\int_\theta^0 \mathbf{E}\dot{e}^T(s)S_3\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\qquad + (\rho-\eta)\int_{-\rho}^{-\eta}\int_\theta^0 \mathbf{E}\dot{e}^T(s)S_4\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\qquad + \tau\int_{-\tau}^0\int_\theta^0 \mathbf{E}\dot{e}^T(s)U_1\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\qquad + \sigma\int_{-\sigma}^0\int_\theta^0 \mathbf{E}\dot{e}^T(s)U_2\dot{e}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\quad \le \bigg\{2\lambda_{\max_{i\in S}}(P_i)\Big(1 + \sigma^2\max_{i\in S}a_i\Big) + \tau\lambda_{\max}(R_1)\\
&\qquad + \tau\lambda_{\max}(R_2) + \sigma\lambda_{\max}(R_3) + \sigma\lambda_{\max}(R_4)\\
&\qquad + \tau^3\big[L^+ - L^-\big]\lambda_{\max}(\Lambda_2) + 2\tau^2\lambda_{\max}(Q_1)\\
&\qquad + 2\sigma^2\lambda_{\max}(Q_2) + \sigma^3\lambda_{\max}(R_5) + \frac{\tau^3}{2}\lambda_{\max}(Q_3)\\
&\qquad + \frac{\sigma^3}{2}\lambda_{\max}(Q_4) + \frac{d^3}{2}\lambda_{\max}(Q_7) + \frac{\tau^3}{2}\lambda_{\max}(Q_5)\\
&\qquad + \frac{\sigma^3}{2}\lambda_{\max}(Q_6) + \mu\lambda_{\max}(S_1)\\
&\qquad + (\rho-\eta)\lambda_{\max}(S_2) + \eta^3\lambda_{\max}(S_3)\\
&\qquad + (\rho-\eta)^3\lambda_{\max}(S_4) + \tau^3\lambda_{\max}(U_1)\\
&\qquad + \sigma^3\lambda_{\max}(U_2)\bigg\}\mathbf{E}\|\phi\|_\tau^2 < \infty,
\end{aligned}$$

where $\overline{\tau} = \max\{\tau, \sigma, \eta, \rho\}$. Hence, by Lemma 1, we obtain

$$
\begin{aligned}
\mathbf{E}\left\| A_i \int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right\|^2
&= \mathbf{E}\left[A_i \int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right]^T \left[A_i \int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right] \\
&\le \lambda_{\max}\big(A_i^2\big)\mathbf{E}\left[\int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right]^T \left[\int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right] \\
&\le \frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_3)} \left[\int_{t-\sigma(t)}^{t} \mathbf{E}e(s)\,\mathrm{d}s \right]^T \\
&\quad \times R_3 \left[\int_{t-\sigma(t)}^{t} \mathbf{E}e(s)\,\mathrm{d}s \right] \\
&\le \sigma(t)\frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_3)} \left\{ \int_{t-\sigma(t)}^{t} \mathbf{E}e^T(s)R_3e(s)\,\mathrm{d}s \right\} \\
&\le \sigma\frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_3)} \left\{ \int_{t-\sigma}^{t} \mathbf{E}e^T(s)R_3e(s)\,\mathrm{d}s \right\} \\
&\le \sigma\frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_3)}\mathbf{E}V_1\big(e(t), t, i\big) \\
&\le \sigma\frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_3)}\mathbf{E}V\big(e(t), t, i\big) \\
&\le \sigma\frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_3)}\mathbf{E}V\big(e(0), 0, r(0)\big), \qquad t \ge 0.
\end{aligned} \tag{34}
$$

Similarly, it follows from the definition of $V_1(e(t), t, i)$ that

$$
\begin{aligned}
\mathbf{E}\left\| e(t) - A_i \int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right\|^2
&= \mathbf{E}\left[A_i \int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right]^T \left[A_i \int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right] \\
&\le \frac{\mathbf{E}V_1(e(t), t, i)}{\lambda_{\min}(P_i)} \\
&\le \frac{\mathbf{E}V(e(t), t, i)}{\lambda_{\min}(P_i)} \\
&\le \frac{\mathbf{E}V(e(0), 0, r(0))}{\lambda_{\min}(P_i)},
\end{aligned}
$$

which together with (34) yields

$$
\begin{aligned}
\mathbf{E}\big\| e(t) \big\|^2
&= \mathbf{E}\left\| e(t) - A_i \int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s + A_i \int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right\|^2 \\
&\le 2\mathbf{E}\left\| A_i \int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right\|^2 \\
&\quad + 2\mathbf{E}\left\| e(t) - A_i \int_{t-\sigma(t)}^{t} e(s)\,\mathrm{d}s \right\|^2 \\
&\le 2\sigma\frac{\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_3)}\mathbf{E}V\big(e(0), 0, r(0)\big) \\
&\quad + 2\frac{\mathbf{E}V(e(0), 0, r(0))}{\lambda_{\min}(P_i)}.
\end{aligned} \tag{35}
$$

By (35), we have

$$
\begin{aligned}
\mathbf{E}\big\| e(t) \big\|^2 &\le 2\left(\frac{\sigma\lambda_{\max}(A_i^2)}{\lambda_{\min}(R_3)} + \frac{1}{\lambda_{\min}(P_i)} \right)\mathcal{M}\mathbf{E}\|\phi\|_{\overline{\tau}}^2 \\
&\le 2\left(\frac{\sigma \max_{1\le i\le N}[\lambda_{\max}(A_i^2)]}{\lambda_{\min}(R_3)} \right. \\
&\qquad \left. + \frac{1}{\min_{1\le i\le N}[\lambda_{\min}(P_i)]} \right)\mathcal{M}\mathbf{E}\|\phi\|_{\overline{\tau}}^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{M} = \Big\{ & 2\lambda_{\max_{i\in S}}(P_i)\Big(1 + \sigma^2 \max_{i\in S} a_i\Big) + \tau\lambda_{\max}(R_1) \\
& + \tau\lambda_{\max}(R_2) + \sigma\lambda_{\max}(R_3) + \sigma\lambda_{\max}(R_4) \\
& + \tau^3\big[L^+ - L^-\big]\lambda_{\max}(\Lambda_2) + 2\tau^2\lambda_{\max}(Q_1) \\
& + 2\sigma^2\lambda_{\max}(Q_2) + \sigma^3\lambda_{\max}(R_5) \\
& + \frac{\tau^3}{2}\lambda_{\max}(Q_3) + \frac{\sigma^3}{2}\lambda_{\max}(Q_4) \\
& + \frac{d^3}{2}\lambda_{\max}(Q_7) + \frac{\tau^3}{2}\lambda_{\max}(Q_5) \\
& + \frac{\sigma^3}{2}\lambda_{\max}(Q_6) + \mu\lambda_{\max}(S_1) \\
& + (\rho - \eta)\lambda_{\max}(S_2) + \eta^3\lambda_{\max}(S_3) \\
& + (\rho - \eta)^3\lambda_{\max}(S_4) + \tau^3\lambda_{\max}(U_1) \\
& + \sigma^3\lambda_{\max}(U_2) \Big\}.
\end{aligned}
$$

It implies that the equilibrium point of system (8) or (9) is locally asymptotically stable. In what follows, we shall prove that $\mathbf{E}\|e(t)\|^2 \to 0$ as $t \to \infty$. It follows from (32), (33) and Lemma 1 that for any constant $\theta \in [0, 1]$,

$$\begin{aligned}
&\big\|e(t+\theta) - e(t)\big\|^2 \\
&\quad = \left[\int_t^{t+\theta} \dot{e}(s)\,\mathrm{d}s\right]^T \left[\int_t^{t+\theta} \dot{e}(s)\,\mathrm{d}s\right] \\
&\quad \le \theta \int_t^{t+\theta} \dot{e}(s)\dot{e}(s)\,\mathrm{d}s \\
&\quad \le \int_t^{t+1} \dot{e}(s)\dot{e}(s)\,\mathrm{d}s \\
&\quad \le \frac{1}{\lambda_{\min}(\mathcal{Y}_i)} \int_t^{t+1} \xi^T(s)\mathcal{Y}_i\xi(s)\,\mathrm{d}s \to 0 \quad \text{as } t \to \infty,
\end{aligned}$$

which implies that for any $\epsilon > 0$, there exists a $T_1 = T_1(\epsilon) > 0$ such that

$$\mathbf{E}\big\|e(t+\theta) - e(t)\big\|^2 < \frac{\epsilon}{2}, \qquad t > T_1,\ \theta \in [0, 1]. \tag{36}$$

On the other hand, (33) yields

$$\begin{aligned}
&\left\|\int_t^{t+1} e(s)\,\mathrm{d}s\right\|^2 \\
&\quad = \left[\int_t^{t+\theta} e(s)\,\mathrm{d}s\right]^T \left[\int_t^{t+\theta} e(s)\,\mathrm{d}s\right] \\
&\quad \le \int_t^{t+1} e^T(s)e(s)\,\mathrm{d}s \\
&\quad \le \frac{1}{\lambda_{\min}(\mathcal{Y}_i)} \int_t^{t+1} \xi^T(s)\mathcal{Y}_i\xi(s)\,\mathrm{d}s \to 0 \quad \text{as } t \to \infty,
\end{aligned}$$

which implies that for any $\epsilon > 0$, there exists a $T_2 = T_2(\epsilon) > 0$ such that

$$\left\|\int_t^{t+1} e(s)\,\mathrm{d}s\right\|^2 < \frac{\epsilon}{2}, \qquad t > T_2.$$

It should be pointed out that $e(s)$ is continuous on $[t, t+1]$, $t > 0$. Applying the integral mean value theorem, we know there exists a vector $\zeta_t = (\zeta_{t1}, \dots, \zeta_{tn})^T \in \mathbb{R}^n$, $\zeta_{tj} \in [t, t+1]$ $(j = 1, \dots, n)$, such that

$$\mathbf{E}\big\|e(\zeta_t)\big\|^2 = \mathbf{E}\left\|\int_t^{t+1} e(s)\,\mathrm{d}s\right\|^2 < \frac{\epsilon}{2}, \qquad t > T_2. \tag{37}$$

By (36) and (37), we find that for any $\epsilon > 0$, there exists a $T = \max\{T_1, T_2\} > 0$ such that $t > T$ implies

$$\begin{aligned}
\mathbf{E}\big\|e(t)\big\|^2 &\le \mathbf{E}\big\|e(t) - e(\zeta_t)\big\|^2 + \mathbf{E}\big\|e(\zeta_t)\big\|^2 \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore, we conclude that system (8) or (9) has a unique equilibrium point which is globally asymptotically stable in the mean square. This completes the proof. □

Remark 3.2 The Markov chain $\{r(t), t \ge 0\}$ only takes a unique value 1 (i.e., $S = \{1\}$), the system (8) will be reduced to the following leakage time-varying delay recurrent neural network:

$$\begin{aligned}
\dot{e}(t) = {}& -Ae\big(t - \sigma(t)\big) - KCe\big(t - d(t)\big) + W_1\phi(t) \\
& + W_2\phi\big(t - \tau(t)\big),
\end{aligned} \tag{38}$$

or its equivalent form

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[e(t) - A\int_{t-\sigma(t)}^t e(s)\,\mathrm{d}s\right] \\
&\quad = -Ae(t) - Ae\big(t - \sigma(t)\big)\dot{\sigma}(t) \\
&\qquad - KCe\big(t - d(t)\big) + W_1\Phi(t) \\
&\qquad - W_2\Phi\big(t - \tau(t)\big),
\end{aligned} \tag{39}$$

where A, C, W_1, W_2 denote A_1, C_1, W_{11}, W_{21}, respectively.

For system (33), we have the following result by Theorem 3.1.

Corollary 3.3 *For given scalars $L^- = \operatorname{diag}(l_1^-, l_2^-, \dots, l_n^-)$, $L^+ = \operatorname{diag}(l_1^+, l_2^+, \dots, l_n^+)$, assume that there exist positive diagonal matrices $M = \operatorname{diag}\{m_1, m_2, \dots, m_n\}$, $\Lambda_1 = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, $\Lambda_2 = \operatorname{diag}\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, symmetric positive definite matrices P, R_1, R_2, R_3, R_4, R_5, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, S_1, S_2, S_3, S_4, D, U_1, U_2 and an arbitrary matrix $F = [0, 0, F_1, 0, F_2, F_3, 0, 0, 0, F_4, 0, 0, 0, 0, 0, 0, 0]$, $n \times n$ real matrices K and G, $n \times n$ diagonal matrices $\Gamma_1 > 0$ and $\Gamma_2 > 0$ such that the following LMI holds*:

$$\mathcal{Y} = \begin{bmatrix} \Xi_1 & F^T \\ F & -(\frac{2}{\tau^2}Q_5 + \frac{2}{\sigma^2}Q_6) \end{bmatrix} < 0, \tag{40}$$

where

$$\Xi_1 = (\Xi_{j,k})_{17\times 17},$$

with

$$\begin{aligned}
\Xi_{1,1} &= -PA - PA^T + R_1 + R_3 + R_4 \\
&\quad + \tau^2 \Lambda_2 (L^+ - L^-) + 2\tau Q_1 + 2\sigma Q_2 + \sigma^2 R_5 \\
&\quad + \frac{\tau^2}{2} Q_3 + \frac{\sigma^2}{2} Q_4 + \frac{d^2}{2} Q_7 + S_1 - S_3 \\
&\quad - U_1 - U_2 - H_1 \Gamma_1 - H_1 \Gamma_2,
\end{aligned}$$

$$\Xi_{1,2} = P - L^- \Lambda_1 + L^+ M,$$

$$\Xi_{1,3} = (1 - \sigma_\mu) PA - \tau F_1 - \sigma F_1,$$

$$\Xi_{1,4} = H_2 \Gamma_1, \qquad \Xi_{1,5} = -\tau F_2 - \sigma F_2 + U_1,$$

$$\Xi_{1,6} = -\tau F_3 - \sigma F_3, \qquad \Xi_{1,7} = U_2,$$

$$\Xi_{1,8} = S_3, \qquad \Xi_{1,10} = -\tau F_4 - \sigma F_4,$$

$$\Xi_{1,11} = APA^T, \qquad \Xi_{1,17} = H_2 \Gamma_2,$$

$$\begin{aligned}
\Xi_{2,2} &= \frac{\tau^2}{2} Q_5 + \frac{\sigma^2}{2} Q_6 + \eta^2 S_3 + (\rho - \eta)^2 S_4 \\
&\quad + (\rho - \eta)^2 D + \tau^2 U_1 + \sigma^2 U_2 - P - P^T,
\end{aligned}$$

$$\Xi_{2,3} = -PA, \qquad \Xi_{2,4} = \Lambda_1 - M + PW_1,$$

$$\Xi_{2,6} = PW_2, \qquad \Xi_{2,11} = -PA,$$

$$\Xi_{2,17} = -GC, \qquad \Xi_{3,3} = -(1 - \sigma_\mu) R_3,$$

$$\Xi_{3,11} = -(1 - \sigma_\mu) APA^T,$$

$$\Xi_{3,12} = F_1, \qquad \Xi_{3,13} = F_1, \qquad \Xi_{4,4} = R_2 - \Gamma_1,$$

$$\Xi_{5,5} = -(1 - \tau_\mu) R_1 - \frac{\pi^2}{4} D - U_1,$$

$$\Xi_{5,8} = \frac{\pi^2}{4} D, \qquad \Xi_{5,12} = F_2, \qquad \Xi_{5,13} = F_2,$$

$$\Xi_{6,6} = -(1 - \tau_\mu) R_2, \qquad \Xi_{6,12} = F_3,$$

$$\Xi_{6,13} = F_3, \qquad \Xi_{7,7} = -R_4 - U_2,$$

$$\Xi_{8,8} = -S_1 + S_2 - S_3 - S_4 - \frac{\pi^2}{4} D,$$

$$\Xi_{8,9} = S_4, \qquad \Xi_{9,9} = -S_2 - S_4,$$

$$\Xi_{10,10} = -(\rho - \eta)^2 (1 - \tau_\mu) D,$$

$$\Xi_{10,12} = F_4, \qquad \Xi_{10,13} = F_4, \qquad \Xi_{11,11} = -R_5,$$

$$\Xi_{12,12} = -\frac{1}{\tau} Q_1, \qquad \Xi_{13,13} = -\frac{1}{\sigma} Q_2,$$

$$\Xi_{14,14} = -\frac{4}{\tau^2} \Lambda_2 (L^+ - L^-) - \frac{2}{\tau^3} Q_1 - \frac{2}{\tau^2} Q_3,$$

$$\Xi_{15,15} = -\frac{2}{\sigma^3} Q_2 - \frac{2}{\sigma^2} Q_4,$$

$$\Xi_{16,16} = -\frac{2}{d^2} Q_7, \qquad \Xi_{17,17} = -\Gamma_2,$$

and other entries of Ξ is 0*. Then the error system* (33) *is globally asymptotically stable in the mean square. Moreover, the gain matrix K of the state estimator* (7) *is given by $K = P^{-1} G$.*

Remark 3.4 To obtain the stability criterion for the Theorem 3.1, the usage of Lemma 3 is introduced. The relation between the integral term $\int_{t-\tau}^{t} e^T(s) Q e(s)\, ds$ and $(\int_{-\tau}^{0} \int_{t+\theta}^{t} e(s)\, ds\, d\theta)^T Q (\int_{-\tau}^{0} \int_{t+\theta}^{t} e(s)\, ds\, d\theta)$ provides a new approach to solve the appropriate Lyapunov–Krasovskii functional, which reduces the conservative results. In other aspect, applying Lemma 3 again, $\xi(t)$ has the form as by introducing the term $\int_{-\tau}^{0} \int_{t+\theta}^{t} e(s)\, ds\, d\theta$, which furthermore reduces the conservatism.

Remark 3.5 In this paper, Theorem 3.1 provides a stability criterion for the state estimator neural networks with Markovian jumping parameters and leakage time varying delays. Such stability criterion is derived based on the assumption that the leakage time varying delays are differentiable and the values of σ_μ are known. Meanwhile, by using a free weighting matrix is so impactful and it plays an essential role for getting feasible solutions. To the best of our knowledge, the result discussed in this paper are new and extended by constructing a new Lyapunov–Krasovskii functional with discontinuous Lyapunov functional and triple integral terms with time-varying delays as well as leakage time-varying delays.

Remark 3.6 In this paper, Theorem 3.1 provides a delay-dependent condition to ensure the existence of a desired sampled-data state estimator for delayed Markovian jumping neural networks with leakage time-varying delays by using discontinuous Lyapunov functional approach. Triple integral terms is introduced in $V_2(e(t), t, i)$ which originates from [49], and a new improved delay-dependent stability criterion is derived. In order to make full use of the sawtooth structure characteristic of the sampling input delay, a discontinuous Lyapunov functional is proposed based on the extended Wirtinger inequality of the vector form. In addition, more triple integral terms are introduced for the upper bound of the discrete time-varying delay τ, leakage time-varying delay σ and sampling input delay d. Recently, authors studied in [20] state estimation results for Markovian jumping neural networks with time-varying delays. State estimation results for Markovian jumping neural networks based on the passivity theory have been studied in [27]. Still there is no paper have not been studied state estimation results for Markovian jumping neural networks based on sampled-data control. We hope that this paper filled this gap and proposing new error estimation conditions for Markovian jumping neural networks based on sampled-data control. Based on proper construction of the Lyapunov–Krasovskii functional of this paper gives better result than those studied in the existing literature [20, 27].

4 Numerical examples

Example 1 Consider a two-dimensional Markovian jump neural networks with leakage time-varying delays:

$$\begin{aligned}\dot{x}(t) &= -A\big(r(t)\big)x\big(t-\sigma(t)\big) + W_1\big(r(t)\big)g\big(x(t)\big)\\ &\quad + W_2\big(r(t)\big)g(x\big(t-\tau(t)\big) + J(t),\end{aligned} \tag{41}$$

where $x(t) = (x_1(t), x_2(t))^T$ and the Markov process $\{r(t), t \geq 0\}$ taking values in $S = \{1, 2\}$ with generator $Q = [\begin{smallmatrix}-2 & 2\\ 3 & -3\end{smallmatrix}]$.

Other parameters of the network (36) are given as follows:

$$A_1 = \begin{bmatrix}1.5 & 0\\ 0 & 2\end{bmatrix}, \qquad A_2 = \begin{bmatrix}1.4 & 0\\ 0 & 1.9\end{bmatrix},$$

$$W_{11} = \begin{bmatrix}0.3 & -0.4\\ 0.4 & 0.3\end{bmatrix}, \qquad W_{12} = \begin{bmatrix}0.2 & -0.3\\ 0.3 & 0.2\end{bmatrix},$$

$$W_{21} = \begin{bmatrix}0.3 & 0.3\\ 0.3 & 0.3\end{bmatrix}, \qquad W_{22} = \begin{bmatrix}0.2 & 0.2\\ 0.2 & 0.2\end{bmatrix},$$

$$C_1 = \begin{bmatrix}0.9 & 0.8\\ 0.7 & 0.5\end{bmatrix}, \qquad C_2 = \begin{bmatrix}0.8 & 0.7\\ 0.6 & 0.4\end{bmatrix}.$$

It can be verified that $H_1^- = H_2^- = 0$ and $H_1^+ = H_2^+ = 1$. Thus

$$H_1 = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \qquad H_2 = \begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix},$$

and the activation functions are taken as follows:

$$g_1\big(x(t)\big) = -0.1137 \tanh x(t),$$
$$g_2\big(x(t)\big) = -0.1279 \tanh x(t).$$

Obviously, $l_1^- = l_2^- = 0, l_1^+ = 0.1137, l_2^+ = 0.1279$ and the state estimation gain K can be designed

$$K_1 = \begin{bmatrix}-1.4356 & 1.8125\\ -1.1734 & 1.8897\end{bmatrix},$$

$$K_2 = \begin{bmatrix}-1.6815 & 2.4231\\ -1.2222 & 2.0216\end{bmatrix}.$$

For the given values of $\tau(t) = 0.7 + 0.6\cos(t)$, $\sigma(t) = 0.75 + 0.6\cos(t)$, $d(t) = 0.9 + 0.1\sin(t)$, $\tau = 0.99$, $\sigma = 0.05$, $\sigma_\mu = 0.1$, $\tau_\mu = 0.1$ and the sampling interval $h = 0.19$, we obtained the feasible solution. When $\tau > 1$, the feasible solution of LMI is not possible. By using the above values, we can easily find the gain matrices K_1 and K_2 for the error system (8). The simulation results are provided in Fig. 1, which shows the effectiveness of the proposed methods. Figure 1 shows the trajectories of x_1 and x_2 along with the error signals. For simplicity, we use xx_1 and xx_2 to denote $\widehat{x}_1$ and $\widehat{x}_2$, respectively in Fig. 1.

Example 2 Consider a two-dimensional neural networks with leakage time-varying delays:

$$\begin{aligned}\dot{x}(t) &= -Ax\big(t-\sigma(t)\big) + W_1 g\big(x(t)\big)\\ &\quad + W_2 g(x\big(t-\tau(t)\big) + J(t),\end{aligned} \tag{42}$$

where

$$A_1 = \begin{bmatrix}1 & 0\\ 0 & 1.5\end{bmatrix}, \qquad W_1 = \begin{bmatrix}0.4 & -0.5\\ 0.6 & 0.3\end{bmatrix},$$

$$W_2 = \begin{bmatrix}0.4 & 0.4\\ 0.4 & 0.4\end{bmatrix}, \qquad C = \begin{bmatrix}0.5 & 0.6\\ 0.4 & 0.3\end{bmatrix}.$$

It can be verified that $H_1^- = H_2^- = 0$ and $H_1^+ = H_2^+ = 1$. Thus

$$H_1 = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \qquad H_2 = \begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix}.$$

In this example, the nonlinear activation function is taken in the form

$$g_1\big(x(t)\big) = -0.1137 \tanh x(t),$$
$$g_2\big(x(t)\big) = -0.1279 \tanh x(t).$$

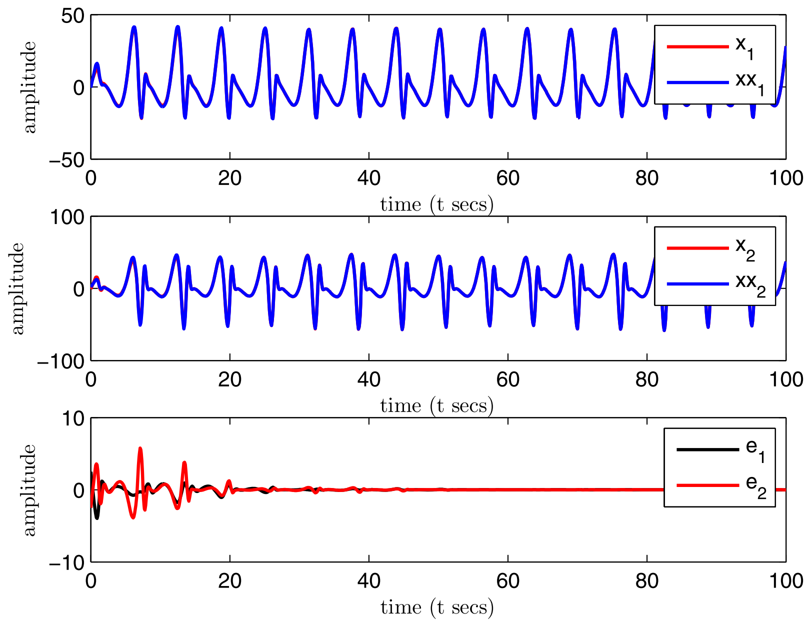

Fig. 1 The state trajectories of systems with leakage delay $\sigma = 0.1$

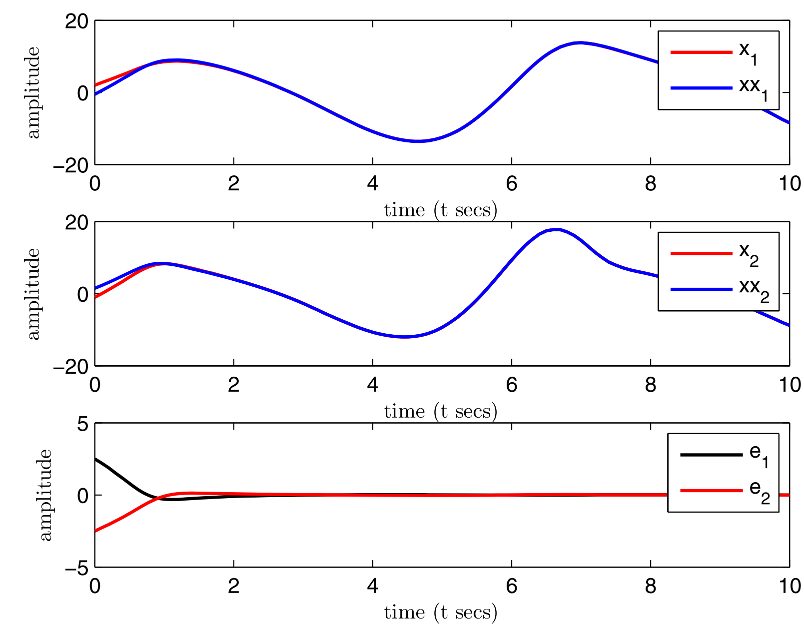

Fig. 2 The state trajectories of systems with leakage delay $\sigma = 0.1$

Obviously, Assumption (H_1) holds with $l_1^- = l_2^- = 0$, $l_1^+ = 0.1137$, $l_2^+ = 0.1279$. Thus, we can get the following parameters:

$$L^- = \mathrm{diag}\{0, 0\}, \qquad L^+ = \mathrm{diag}\{0.1137, 0.1279\}.$$

Using the Matlab LMI toolbox and Corollary 3.3, we obtain the feasible solution for the given values of $\tau = 0.99$, $\sigma = 0.05$, $\sigma_\mu = 0.1$, $\tau_\mu = 0.1$ and the sampling interval $h = 0.19$. Here, in order to show to get a state estimator for above network, let us consider the case of having the time-varying delay $\tau(t) = 0.7 + 0.6\cos(t)$, $\sigma(t) = 0.7 + 0.6\cos(t)$, $d(t) = 0.2 + 0.1\sin(t)$, which means that the delay is so effective and it plays an important role for getting feasible solutions. By solving the LMIs in Corollary 3.3, the corresponding gain matrix is given as

$$K = \begin{bmatrix} -1.3868 & 1.8738 \\ -1.2257 & 2.4685 \end{bmatrix}.$$

Therefore, it follows from Corollary 3.3 that the error-state system (33) is globally asymptotically stable. The responses of the state dynamics for the error-state system (33) which converges to zero asymptotically are given in Fig. 2, where xx_1 and xx_2 denote $\widehat{x}_1$ and $\widehat{x}_2$, respectively.

5 Conclusion

In this paper, the sampled-data state estimation problem has been considered for neural networks with both Markovian jumping parameters and leakage time-varying delays. Based on the extended Wirtinger inequality, a discontinuous Lyapunov functional which gives full information of sawtooth structure characteristic of the sampling delay has been proposed. The results have shown that the application of the discontinuous Lyapunov functional gets less conservatism than those used in the continuous Lyapunov functional. Based on some integral inequalities and a new Lyapunov–Krasovskii functional containing triple integral terms, new delay-dependent stability criteria for designing a state estimator of the considered neural networks have been established in terms of LMIs. The gain matrix of the proposed state estimator has been determined by solving the LMI problem. Finally, two numerical examples have been used to demonstrate the usefulness of the main results.

Acknowledgements The work of R. Rakkiyappan was supported by NBHM Research Project under the sanctioned No: 2/48(7)/2012/NBHM(R.P.)/R and D II/12669 and Quanxin Zhu's work was jointly supported by the National Natural Science Foundation of China (10801056), the Natural Science Foundation of Zhejiang Province (LY12F03010) and the Natural Science Foundation of Ningbo (2012A610032).

References

1. Hagan, M.T., Demuth, H.B., Beale, M.: Neural Network Design. PWS Publishing Company, Boston (1996)
2. Gupta, M.M., Jin, L., Homma, N.: Static and Dynamic Neural Networks: from Fundamentals to Advanced Theory. Wiley, New York (2003)
3. Cichoki, A., Unbehauen, R.: Neural Networks for Optimization and Signal Processing. Wiley, Chichester (1993)
4. Haykin, S.: Neural Networks: a Comprehensive Foundation. Prentice Hall, New York (1998)
5. Ahn, C.K.: Robust stability of recurrent neural networks with ISS learning algorithm. Nonlinear Dyn. **65**, 413–419 (2011)
6. Haken, H.: Pattern recognition and synchronization in pulse-coupled neural networks. Nonlinear Dyn. **44**, 269–276 (2006)
7. Hendzel, Z.: An adaptive critic neural network for motion control of a wheeled mobile robot. Nonlinear Dyn. **50**, 849–855 (2007)
8. Civalleri, P.P., Gilli, M., Pandolfi, L.: On stability of cellular neural networks with delay. IEEE Trans. Circuits Syst. I, Fundam. Theory Appl. **40**, 157–165 (1993)
9. Marcus, C.M., Westervelt, R.M.: Stability of analog neural networks with delay. Phys. Rev. A **39**, 347–359 (1989)
10. Cao, J.: Global stability conditions for delayed CNNs. IEEE Trans. Circuits Syst. I **48**, 1330–1333 (2001)
11. Liao, T.L., Wang, F.C.: Global stability for cellular neural networks with time delay. IEEE Trans. Neural Netw. **11**, 1481–1484 (2000)
12. Li, X., Cao, J.: Delay-dependent stability of neural networks of neutral-type with time delay in the leakage term. Nonlinearity **23**, 1709–1726 (2010)
13. Fu, X., Li, X., Akca, H.: Exponential state estimation for impulsive neural networks with time delay in the leakage term. Arab. J. Math. (2012). doi:10.1007/s40065-012-0045-y
14. Zhu, Q., Cao, J.: Robust exponential stability of Markovian jump impulsive stochastic Cohen–Grossberg neural networks with mixed time delays. IEEE Trans. Neural Netw. **21**, 1314–1325 (2010)
15. Zhu, Q., Cao, J.: Stability analysis of Markovian jump stochastic BAM neural networks with impulsive control and mixed time delays. IEEE Trans. Neural Netw. Learn. Syst. **23**, 467–479 (2012)
16. Van Den Driessche, P., Zou, X.: Global attractivity in delayed Hopfield neural network models. SIAM J. Appl. Math. **6**, 1878–1890 (1998)
17. Park, J., Kwon, O.: Design of state estimator for neural networks of neutral-type. Appl. Math. Comput. **202**, 360–369 (2008)
18. Park, J., Kwon, O.: Further results on state estimation for neural networks of neutral-type with time-varying delay. Appl. Math. Comput. **208**, 69–75 (2009)
19. Liu, Y., Wang, Z., Liu, X.: State estimation for discrete-time Markovian jumping neural networks with mixed mode-dependent delays. Phys. Lett. A **372**, 7147–7155 (2008)
20. Balasubramaniam, P., Lakshmanan, S., Jeeva, S.: Sathya theesar, state estimation for Markovian jumping recurrent neural networks with interval time-varying delays. Nonlinear Dyn. **60**, 661–675 (2010)
21. Liu, Y., Wang, Z., Liang, J., Liu, X.: Synchronization and state estimation for discrete-time complex networks with distributed delays. IEEE Trans. Syst. Man Cybern., Part B, Cybern. **38**, 1314–1325 (2008)
22. Shen, B., Wang, Z., Liu, X.: Bounded synchronization and state estimation for discrete time-varying stochastic complex networks over a finite horizon. IEEE Trans. Neural Netw. **22**, 145–157 (2011)

23. Huang, H., Feng, G., Cao, J.: Robust state estimation for uncertain neural networks with time-varying delay. IEEE Trans. Neural Netw. **19**, 1329–1339 (2008)
24. Liu, X., Cao, J.: Robust state estimation for neural networks with discontinuous activations. IEEE Trans. Syst. Man Cybern., Part B, Cybern. **40**, 1425–1437 (2010)
25. Bao, H., Cao, J.: Delay-distribution-dependent state estimation for discrete-time stochastic neural networks with random delay. Neural Netw. **24**, 19–28 (2011)
26. Huang, H., Feng, G., Cao, J.: Guaranteed performance state estimation of static neural networks with time-varying delay. Neurocomputing **74**, 606–616 (2011)
27. Lakshmanan, S., Park, J.H., Ji, D.H., Jung, H.Y., Nagamani, G.: State estimation of neural networks with time-varying delays and Markovian jumping parameter based on passivity theory. Nonlinear Dyn. **70**, 1421–1434 (2012)
28. Fridman, E., Shaked, U., Suplin, V.: Input/output delay approach to robust sampled-data H_∞ control. Syst. Control Lett. **54**, 271–282 (2005)
29. Li, N., Hu, J., Hu, J., Li, L.: Exponential state estimation for delayed recurrent neural networks with sampled-data. Nonlinear Dyn. **69**, 555–564 (2012)
30. Mikheev, Y., Sobolev, V., Fridman, E.: Asymptotic analysis of digital control systems. Autom. Remote Control **49**, 1175–1180 (1988)
31. Astrom, K., Wittenmark, B.: Adaptive Control. Addison-Wesley, Reading (1989)
32. Fridman, E., Seuret, A., Richard, J.P.: Robust sampled-data stabilization of linear systems: an input delay approach. Automatica **40**, 1441–1446 (2004)
33. Wang, Z., Ho, D.W.C., Liu, X.: State estimation for delayed neural networks. IEEE Trans. Neural Netw. **16**, 279–284 (2005)
34. Naghshtabrizi, P., Hespanha, J., Teel, A.: Exponential stability of impulsive systems with application to uncertain sampled-data systems. Syst. Control Lett. **57**, 378–385 (2008)
35. Zhu, X.: Stabilization for sampled-data neural network based control systems. IEEE Trans. Syst. Man Cybern., Part B, Cybern. **41**, 210–221 (2011)
36. Naghshtabrizi, P., Hespanha, J., Teel, A.: Stability of delay impulsive systems with application to networked control systems. In: Proceedings of the 26th American Control Conference, New York, USA, July 2007
37. Fridman, E.: A refined input delay approach to sampled-data control. Automatica **46**, 421–427 (2010)
38. Liu, K., Fridman, E.: Stability analysis of networked sontrol systems: a discontinuous Lyapunov functional approach. In: Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, China, December 2009
39. Hardy, G., Littlewood, J.E., Polya, G.: Inequalities. Cambridge University Press, Cambridge (1934)
40. Mirkin, L.: Some remarks on the use of time-varying delay to model sample and hold circuits. IEEE Trans. Autom. Control **52**, 1109–1112 (2007)
41. Fridman, E.: A refined input delay approach to sampled-data control. Automatica **46**, 421–427 (2010)
42. Kharitonov, V., Niculescu, S.I., Moreno, J., Michiels, M.: Static output feedback stabilization: necessary conditions for multiple delay controllers. IEEE Trans. Autom. Control **52**, 1109–1112 (2007)
43. Krasovskii, N.N., Lidskii, E.A.: Analysis and design of controllers in systems with random attributes. Autom. Remote Control **22**, 1021–1025 (1961)
44. Kim, S., Li, H., Dougherty, E.R., Chao, N., Chen, Y., Bittner, M.L., Suh, E.B.: Can Markov chain models mimic biological regulation? J. Biol. Syst. **10**, 337–357 (2002)
45. Wang, Z., Liu, Y., Liu, X.: State estimation for jumping recurrent neural networks with discrete and distributed delays. Neural Netw. **22**, 41–48 (2009)
46. Gopalsamy, K.: Stability and Oscillations in Delay Differential Equations of Population Dynamics. Kluwer Academic, Dordrecht (1992)
47. Li, X., Fu, X.: Effects of leakage time-varying delay on stability of nonlinear differential systems. Journal of Franklin Institute. doi:10.2016/j.jfranklin.2012.04.007
48. Gahinet, P., Nemirovski, A., Laub, A.J., Chilali, M.: LMI Control Toolbox. The Mathworks, Natick (1995)
49. Liu, Z., Yu, J., Xu, D., Peng, D.: Triple-integral method for the stability analysis of delayed neural networks. Neurocomputing **99**, 283–289 (2013)