MATRIX ALGEBRA REVIEW FOR STATISTICS

LESSON 3 NOTES

Rev. 2.1

by

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XVI. EIGENVALUES & EIGENVECTORS OF A REAL SQUARE MATRIX:

We have seen that one way of characterizing a general n x p matrix is by finding its nullspace. In this section we will characterize more completely the special case of a square n x n matrix **A** with real-valued elements. (The theory also applies with some modifications to complex matrices, but this will not be covered here.)

Consider first an n x n diagonal matrix **A** with elements $d_1 \ldots d_n$:

$$\mathbf{A} \quad = \quad \begin{bmatrix} d1 & 0 & \ldots & 0 \\ 0 & d2 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & dn \end{bmatrix} \qquad \text{(XVI.1)}$$

Suppose $\mathbf{u}_1 = [\ 1\ 0 \ldots 0]^t$. Then

$$\mathbf{A}\,\mathbf{u}_1 \quad = \quad d_1\,\mathbf{u}_1 \qquad \text{(XVI.2)}$$

Multiplying $\mathbf{u}_1$ by **A** then results in $\mathbf{u}_1$ back again, but multiplied by the scalar d_1. There are, in general, n such linearly independent $\mathbf{u}_i$, each associated with a particular d_i. The $\mathbf{u}_i$ are not necessarily unique, as a scalar multiple of, e.g., $\mathbf{u}_1$, still satisfies eq.(XVI.2). In particular, if $\mathbf{u}_1$ satisfies eq.(XVI.2), then so does $-\mathbf{u}_1$. To help resolve this ambiguity, it is common to require each $\mathbf{u}_i$ to have $|\ \mathbf{u}_i\ | = 1$ (or, equivalently, the sum of its squared elements equal to 1). Sometimes there is also the convention that the first element of $\mathbf{u}_1$ is positive. If any $d_i = d_j$ for some $i \neq j$, we can choose any $\mathbf{u}_i$ and $\mathbf{u}_j$ which span the subspace associated with d_i and d_j. A convenient set of $\mathbf{u}_1 \ldots \mathbf{u}_n$ that work for **A** of eq.(XVI.1) are the column vectors of the n x n identity matrix **I**.

The values $\lambda_1 \ldots \lambda_n$ and the vectors $\mathbf{u}_1 \ldots \mathbf{u}_n$ for which

$$\mathbf{A}\,\mathbf{u} \quad = \quad \lambda\,\mathbf{u} \qquad \mathbf{u} \neq \mathbf{0} \quad \text{and } |\ \mathbf{u}\ | = 1 \qquad \text{(XVI.3)}$$

are called the “eigenset” of **A**. The values $\lambda_1 \ldots \lambda_n$ are called the “eigenvalues” or “characteristic values” or “latent roots” of **A**. The associated vectors $\mathbf{u}_1 \ldots \mathbf{u}_n$ are called the (normalized) “eigenvectors” or “characteristic vectors” or “latent vectors” of **A**. Sometimes the collection of eigenvalues $\lambda_1 \ldots \lambda_n$ is called the “spectrum” of **A**. For the **A** of eq.(XVI.1), $\lambda_i = d_i$ and $\mathbf{u}_i$ is the i-th column of the identity matrix.

It should be obvious that the eigenvalues $\lambda_1 \ldots \lambda_n$ and eigenvectors $\mathbf{u}_1 \ldots \mathbf{u}_n$ are intimately connected with **A**, and say something special about **A**, although what exactly they say may not yet completely clear.

Now let’s extend our search for solutions to equations of the type eq.(XVI.3) for more general **A**.

For the 1 x 1 case **A** = [a_{11}], **A** is necessarily diagonal, and

$$\lambda_1 = a_{11} \qquad \text{(XVI.4a)}$$

$$\mathbf{u}_1 = [\ 1\ ] \qquad \text{(XVI.4b)}$$

For the 2 x 2 case, we can rearrange eq.(XVI.3) to give

$$(\mathbf{A} - \lambda\,\mathbf{I}\,)\,\mathbf{u} = \mathbf{B}\,\mathbf{u} = \mathbf{0} \qquad \mathbf{u} \neq \mathbf{0} \qquad \text{(XVI.5)}$$

This is a homogeneous system of linear equations with coefficient matrix

$$\mathbf{B} = (\mathbf{A} - \lambda\,\mathbf{I}\,) \qquad \text{(XVI.6)}$$

We know that **B u** = **0** has only the trivial solution **u** = **0** if **B** is *nonsingular*. So if we wish **u** ≠ **0** solutions of eq.(XVI.5), we need **B** to be *singular*. We also know that B is singular if and only if

$$|\,\mathbf{B}\,| = |\,\mathbf{A} - \lambda\,\mathbf{I}\,| = 0 \qquad \text{(XVI.7)}$$

Eq.(XVI.7) is called the "characteristic equation" for **A**, and is the route by which we find the eigenvalues of **A**.

For the 2 x 2 case,

$$|\,\mathbf{A} - \lambda\,\mathbf{I}\,| = (a_{11}-\lambda)(a_{22}-\lambda) - a_{21}\,a_{12} \qquad \text{(XVI.8a)}$$

$$= \lambda^2 - (a_{11} + a_{22})\,\lambda + (a_{11}\,a_{22} - a_{21}\,a_{12}) \qquad \text{(XVI.8b)}$$

$$= \lambda^2 - \mathrm{tr}(\mathbf{A})\,\lambda + |\,\mathbf{A}\,| \qquad \text{(XVI.8c)}$$

Note that eq.(18) is a polynomial of degree 2 in λ (the "characteristic polynomial"), the constant is the determinant of **A**, the coefficient of λ is the minus the trace of **A** (i.e., sum of the diagonal elements of **A**), and the coefficient of λ^2 is 1.

The characteristic equation for A can now be written

$$|\,\mathbf{A} - \lambda\,\mathbf{I}\,| = \lambda^2 - (a_{11} + a_{22})\,\lambda + (a_{11}\,a_{22} - a_{21}\,a_{12}) = 0 \qquad \text{(XVI.9)}$$

with general solution

$$\lambda_1\,,\,\lambda_2 = \frac{(a_{11} + a_{22}) \pm \sqrt{\Delta}}{2} \qquad \text{(XVI.10)}$$

where the discriminant Δ is given by

$$\Delta = (a_{11} + a_{22})^2 - 4(a_{11}\, a_{22} - a_{21}\, a_{12}) \qquad \text{(XVI.11a)}$$

$$= (a_{11} - a_{22})^2 + 4\, a_{21}\, a_{12} \qquad \text{(XVI.11b)}$$

The solutions λ_1 and λ_2 to the quadratic eq.(XVI.9) can be

a. Two real roots ($\Delta > 0$)
b. One real multiple root ($\Delta = 0$) (XVI.12)
c. Two complex roots ($\Delta < 0$)

As a numerical example, consider

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{(XVI.13)}$$

The characteristic equation is

$$|\,\mathbf{A} - \lambda\, \mathbf{I}\,| = (2-\lambda)\,(1-\lambda) - 0 = 0 \qquad \text{(XVI.14a)}$$

$$= (2-\lambda)\,(1-\lambda) = 0 \qquad \text{(XVI.14b)}$$

with solutions $\lambda_1 = 2$ and $\lambda_2 = 1$.

The eigenvector for $\lambda_1 = 2$ is found by starting with

$$(\mathbf{A} - \lambda\, \mathbf{I})\, \mathbf{u} = \mathbf{0} \qquad \text{(XVI.15a)}$$

$$\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0} \qquad \text{(XVI.15b)}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(XVI.15c)}$$

These two linear equations are inconsistent unless y = 0. The value of x can be anything, say some constant c. The eigenvector corresponding to $\lambda_1 = 2$ is then $\mathbf{u}_1 = [c\ 0]^t$ for any $c \neq 0$, or $[1\ 0]^t$ if $|\,\mathbf{u}_1\,| = 1$.

The eigenvector for $\lambda_2 = 1$ is found similarly by

$$\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0} \qquad \text{(XVI.16a)}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(XVI.16b)}$$

The first equation requires x = - y, and the second equation contributes no information. Starting with $\mathbf{u}_2 = [-1\ 1]^t$ and then normalizing (i.e., dividing by | **u** |), we obtain the unit eigenvector $\mathbf{u}_2 = [-1/\sqrt{2}\ \ 1/\sqrt{2}]^t \approx [\ -0.7071\ \ 0.7071]^t$.

It is conventional to summarize the eigenset of a matrix by listing the eigenvalues and then combining the unit eigenvectors into a matrix **U** whose *columns* are the eigenvectors in the order of the eigenvalues given. I.e.,

$$\lambda_1, \lambda_2 = 2 \text{ and } 1$$

$$\mathbf{U} = \begin{bmatrix} 1 & -0.7071 \\ 0 & 0.7071 \end{bmatrix} \qquad \text{(XVI.17)}$$

For comparison, using R:

```
> A<- matrix(c(2,1, 0,1), ncol=2, byrow=TRUE)
> A
     [,1] [,2]
[1,]    2    1
[2,]    0    1

> eigen(A)
$values
[1] 2 1

$vectors
     [,1]       [,2]
[1,]    1 -0.7071068
[2,]    0  0.7071068
```

For the general case of eq.(XVI.7) where order(**A**) = n, the characteristic polynomial p(λ) = | **A** – λ **I** | has a coefficient of λ^n of $(-1)^n$ and the signs of the terms in the polynomial alternate. We know from algebra (by expanding the factored form of the polynomial) that the constant term is the *product* of the roots of the polynomial, but is also equal here to |**A**|, so *in general the product of the eigenvalues (the roots of the characteristic polynomial) equals | **A** |*. We also know form algebra that the coefficient of λ^{n-1} is $(-1)^{n-1}$ times the *sum* of the roots of the polynomial, which is the sum of the eigenvalues, and here is also the trace of **A**. So *in general the sum of the eigenvalues of **A** equals tr(**A**).*

A more general method of finding the eigenvector **u** associated with an eigenvalue λ is by realizing we already know the general solution of a homogeneous system of equations using the generalized inverse. So

$$\mathbf{u} = (\mathbf{B}^- \mathbf{B} - \mathbf{I})\,\boldsymbol{\theta} \qquad \text{(XVI.18)}$$

where **B** is given by eq.(XVI.6) for a particular eigenvalue λ, and $\boldsymbol{\theta}$ is a vector of arbitrary parameters. For the $\lambda_1 = 2$ case above,

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad \lambda = 2 \qquad \text{(XVI.19)}$$

A generalized inverse $\mathbf{B}^-$ of **B** is

$$\mathbf{B}^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \text{(XVI.20)}$$

so

$$\mathbf{u} = (\, \mathbf{B}^- \mathbf{B} - \mathbf{I} \,)\, \boldsymbol{\theta} \qquad \text{(XVI.21a)}$$

$$= (\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{(XVI.21b)}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{(XVI.21c)}$$

$$= \begin{bmatrix} -x \\ 0 \end{bmatrix} \qquad \text{(XVI.21d)}$$

The choice x = -1 results in the same $\mathbf{u}_1 = [1\ 0]^t$ as before.

For a 3 x 3 matrix **A**, finding the eigenset becomes much more difficult. The characteristic equation is cubic and the eigenvectors are determined by 3 equations in 3 unknowns. For a numerical example of a 3 x 3 matrix, consider

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \qquad \text{(XVI.22)}$$

This matrix is clearly of rank less than 3, because the last row is the sum of the first two.

The characteristic equation is

$$|\, \mathbf{A} - \lambda\, \mathbf{I} \,| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 3 & 2 & 1-\lambda \end{vmatrix} = 0 \qquad \text{(XVI.23a)}$$

$$= \quad (2-\lambda)\,[\,(1-\lambda)\,(1-\lambda)-2\,] - (1)\,[\,(1)(1-\lambda)-3\,] + 0 \quad = 0 \qquad \text{(XVI.23b)}$$

$$= \quad -\lambda\,(\lambda^2 - 4\,\lambda + 2) \quad = 0 \qquad \text{(XVI.23c)}$$

The solutions to eq.(XVI.23c) are $\lambda = 0$, $2 + \sqrt{2}$ and $2 - \sqrt{2}$. Note that zero is a valid value for an eigenvalue, in which case its corresponding eigenvector is in the nullspace of **A**. (Any singular matrix **A** will have at least one eigenvalue equal to zero.) For $\lambda_1 = 0$, the corresponding eigenvector $\mathbf{u}_1 = [x\ y\ z]^t$ is determined from

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \mathbf{u}_1 = \mathbf{0} \qquad \text{(XVI.24)}$$

One solution is x = 1, y = -2 and z = 1. Normalizing, the unit eigenvector is

$$\mathbf{u}_1 \quad = \quad \begin{bmatrix} 0.4082 \\ -0.8165 \\ 0.4082 \end{bmatrix} \qquad \text{(XVI.25)}$$

Using R,

```
> A<- matrix(c(2,1,0, 1,1,1, 3,2,1), ncol=3, byrow=TRUE)
> A
     [,1] [,2] [,3]
[1,]    2    1    0
[2,]    1    1    1
[3,]    3    2    1

> det(A)  #determinant
[1] 1.110223e-16

> eigen(A)  #eigenset
$values
[1]  3.414214e+00  5.857864e-01 -8.139755e-17

$vectors
          [,1]       [,2]       [,3]
[1,] 0.3365568  0.5615167  0.4082483
[2,] 0.4759631 -0.7941045 -0.8164966
[3,] 0.8125199 -0.2325878  0.4082483
```

The third eigenvalue is zero, and its eigenvector corresponds to eq.(XVI.25).

For the n x n matrix case **A**, note that

$$\mathbf{A\,U} = \mathbf{U\,\Lambda} \qquad \text{(XVI.26)}$$

where **U** is the matrix whose columns are the eigenvectors and **Λ** is the diagonal matrix of (possibly repeated) eigenvalues. Assuming **U** is invertible, multiply both sides on the right by $\mathbf{U}^{-1}$:

$$\mathbf{A\,U\,U}^{-1} = \mathbf{U\,\Lambda\,U}^{-1} \qquad \text{(XVI.27a)}$$

$$\mathbf{A} = \mathbf{U\,\Lambda\,U}^{-1} \qquad \text{(XVI.27b)}$$

So, if the matrix **U** of eigenvectors is invertible, then **A** can be represented by a so-called "similarity" transform of a diagonal matrix **Λ** of its eigenvalues. By rearrangement,

$$\mathbf{U}^{-1}\,\mathbf{A\,U} = \mathbf{\Lambda} \qquad \text{(XVI.28)}$$

and the matrix **U** is said to "diagonalize" the matrix **A**.

One of the benefits of the transformation given in eq.(XVI.27b) is that powers of the matrix A can be developed economically. E.g.,

$$\mathbf{A}^2 = \mathbf{A\,A} \qquad \text{(XVI.29a)}$$

$$= \mathbf{U\,\Lambda\,U}^{-1}\,\mathbf{U\,\Lambda\,U}^{-1} \qquad \text{(XVI.29b)}$$

$$= \mathbf{U\,\Lambda\,I\,\Lambda\,U}^{-1} \qquad \text{(XVI.29c)}$$

$$= \mathbf{U\,\Lambda}^2\,\mathbf{U}^{-1} \qquad \text{(XVI.29d)}$$

where the elements of $\mathbf{\Lambda}^2$ are λ_i^2.

Similarly,

$$\mathbf{A}^k = \mathbf{U\,\Lambda}^k\,\mathbf{U}^{-1} \qquad \text{(XVI.30)}$$

EXERCISES:

In each case, find the eigenvalues and eigenvectors of the matrix given.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

A1. $\lambda_1 = 1$, $\mathbf{e}_1 = [1, 0]^t$; $\lambda_2 = 0$, $\mathbf{e}_2 = [0, 1]^t$

2. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

A2. $\lambda_1 = 1$, $\mathbf{e}_1 = [\sqrt{2}/2, \sqrt{2}/2]^t$; $\lambda_2 = 0$, $\mathbf{e}_2 = [0\ 1]^t$

3. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

A3. $\lambda_1 = (1+\sqrt{5})/2$, $\mathbf{e}_1 = [0.851, 0.526]^t$

$\lambda_2 = (1-\sqrt{5})/2$, $\mathbf{e}_2 = [-0.526, 0.851]^t$

4. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

A4. $\lambda_1 = 1$, $\mathbf{e}_1 = [0, 1]^t$; $\lambda_2 = 1$, $\mathbf{e}_2 = [0, 1]^t$

5. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

A5. $\lambda_1 = 2$, $\mathbf{e}_1 = [\sqrt{2}/2, \sqrt{2}/2]^t$; $\lambda_2 = 0$, $\mathbf{e}_2 = [\sqrt{2}/2, -\sqrt{2}/2]^t$

6. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

A6. $\lambda_1 = (3+\sqrt{5})/2$, $\mathbf{e}_1 = [0.526, 0.851]^t$

$\lambda_2 = (3-\sqrt{5})/2$, $\mathbf{e}_2 = [-0.851, 0.526]^t$

7. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

A7. $\lambda_1 = 3$, $\mathbf{e}_1 = [\sqrt{2}/2, \sqrt{2}/2]^t$; $\lambda_2 = 1$, $\mathbf{e}_2 = [\sqrt{2}/2, -\sqrt{2}/2]^t$

8. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

A8. $\lambda_1 = 4$, $\mathbf{e}_1 = [\sqrt{2}/2, \sqrt{2}/2]^t$; $\lambda_2 = 2$, $\mathbf{e}_2 = [\sqrt{2}/2, -\sqrt{2}/2]^t$

9. $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

A9. $\lambda_1 = 5$, $\mathbf{e}_1 = [\sqrt{2}/2, \sqrt{2}/2]^t$; $\lambda_2 = 3$, $\mathbf{e}_2 = [\sqrt{2}/2, -\sqrt{2}/2]^t$

10. $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

A10. $\lambda_1 = 4+\sqrt{3}$; $\lambda_2 = 4-\sqrt{3}$; $\lambda_3 = 1$

R LANGUAGE SOLUTIONS TO EXERCISES:

```
> A = matrix(c(1,0, 0,0), ncol=2, byrow=TRUE)  #Q1
> A
     [,1] [,2]
[1,]    1    0
[2,]    0    0
> eigen(A)
$values
[1] 1 0

$vectors
     [,1] [,2]
[1,]   -1    0
[2,]    0   -1


> A = matrix(c(1,0, 1,0), ncol=2, byrow=TRUE)  #Q2
> A
     [,1] [,2]
[1,]    1    0
[2,]    1    0
> eigen(A)
$values
[1] 1 0

$vectors
          [,1] [,2]
[1,] 0.7071068    0
[2,] 0.7071068    1

> A = matrix(c(1,1, 1,0), ncol=2, byrow=TRUE)  #Q3
> A
     [,1] [,2]
[1,]    1    1
[2,]    1    0
> eigen(A)
$values
[1]  1.618034 -0.618034

$vectors
           [,1]       [,2]
[1,] -0.8506508  0.5257311
[2,] -0.5257311 -0.8506508

> A = matrix(c(1,0, 1,1), ncol=2, byrow=TRUE)  #Q4
> A
     [,1] [,2]
[1,]    1    0
[2,]    1    1
> eigen(A)
$values
[1] 1 1

$vectors
     [,1]          [,2]
[1,]    0  2.220446e-16
[2,]    1 -1.000000e+00

> A = matrix(c(1,1, 1,1), ncol=2, byrow=TRUE)  #Q5
> A
```

```
     [,1] [,2]
[1,]    1    1
[2,]    1    1
> eigen(A)
$values
[1] 2 0

$vectors
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068

> A = matrix(c(1,1, 1,2), ncol=2, byrow=TRUE)  #Q6
> A
     [,1] [,2]
[1,]    1    1
[2,]    1    2
> eigen(A)
$values
[1] 2.618034 0.381966

$vectors
          [,1]       [,2]
[1,] 0.5257311 -0.8506508
[2,] 0.8506508  0.5257311

> A = matrix(c(2,1, 1,2), ncol=2, byrow=TRUE)  #Q7
> A
     [,1] [,2]
[1,]    2    1
[2,]    1    2
> eigen(A)
$values
[1] 3 1

$vectors
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068

> A = matrix(c(3,1, 1,3), ncol=2, byrow=TRUE)  #Q8
> A
     [,1] [,2]
[1,]    3    1
[2,]    1    3
> eigen(A)
$values
[1] 4 2

$vectors
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068
```

```
> A = matrix(c(4,1, 1,4), ncol=2, byrow=TRUE)  #Q9
> A
     [,1] [,2]
[1,]    4    1
[2,]    1    4
> eigen(A)
$values
[1] 5 3

$vectors
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068

> A = matrix(c(3,2,1, 2,3,1, 1,1,3), ncol=3, byrow=TRUE)    #Q10
> A
     [,1] [,2] [,3]
[1,]    3    2    1
[2,]    2    3    1
[3,]    1    1    3
> eigen(A)
$values
[1] 5.732051 2.267949 1.000000

$vectors
          [,1]       [,2]          [,3]
[1,] 0.6279630 -0.3250576  7.071068e-01
[2,] 0.6279630 -0.3250576 -7.071068e-01
[3,] 0.4597008  0.8880738 -1.819942e-15
```